Chapter 6
Consequence and Degrees of Truth in Many-Valued Logic

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6.1 Introduction

Let me begin by calling your attention to one of the main points made by Petr Hájek in the introductory, vindicating section of his influential book (Hájek 1998) (the italics are his):

> *Logic studies the notion(s) of consequence*. It deals with propositions (sentences), sets of propositions and the relation of consequence among them. [page 1]
> [...]
> *Fuzzy logic is a logic*. It has its syntax and semantics and notion of consequence. It is a study of consequence. [page 5]

Petr's book contains no discussion on how consequence in mathematical fuzzy logic should be defined, or why. He simply defines his consequences either by a Hilbert-style axiomatization or semantically by the *truth-preserving* paradigm, which takes 1 as the only designated truth value in the real interval [0, 1] or in other algebraic structures which are ordered and have a maximum value 1. That is, if Γ is a set of formulas and φ is a formula, then[1]

$$\Gamma \vdash \varphi \iff \begin{array}{l} e(\varphi) = 1 \text{ whenever } e(\alpha) = 1 \text{ for all } \alpha \in \Gamma, \\ \text{for any evaluation } e \text{ in the model.} \end{array} \tag{6.1}$$

I would also like to call your attention to a result about propositional Gödel-Dummett logic G, whose consequence $\vdash_{G}$ is defined axiomatically on p. 97 of Hájek

[1] In this chapter I will represent logics as consequences by the symbol $\vdash$, independently of the way they are defined, be it of semantical or syntactical origin, and will add sub- or superscripts when needed. The symbol $\vDash$ will only be used for satisfaction of equations in (classes of) algebras.

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(1998). G is proved in Theorem 4.2.17 to be strongly complete with respect to the standard Gödel algebra over [0, 1] taking the minimum as the t-norm whose residuum interprets the implication. Then Theorem 4.2.18 reads:

Theorem 6.1 *For each theory T over* G*, each formula φ and each rational r such that $0 < r \leqslant 1$, $T \vdash_{\mathrm{G}} \varphi$ iff each evaluation e such that $e(\alpha) \geqslant r$ for each axiom α of T satisfies $e(\varphi) \geqslant r$.*

The same holds if we take all reals in [0, 1] instead of the rationals (by the density of $\mathbb{Q}$ inside $\mathbb{R}$), but Petr establishes this only to give a relation with "partial truth", which has been previously discussed in the book in the framework of the Rational Pavelka Logic.

This result, which in Hájek (1998) appears to be an anecdotal result on the standard semantics of G, has an alternative view when it is reformulated as the coincidence of two consequences: if we define

$$\Gamma \vdash^{\leqslant} \varphi \iff \begin{array}{l} e(\varphi) \geqslant r \text{ whenever } e(\alpha) \geqslant r \text{ for all } \alpha \in \Gamma, \\ \text{for any evaluation } e \text{ and any value } r \text{ in the model,} \end{array} \tag{6.2}$$

then Theorem 6.1 says that $\vdash_{\mathrm{G}}$ and $\vdash^{\leqslant}$ coincide when the model at hand is the Gödel algebra of rationals, or equivalently of the reals, in [0, 1]. This is a more interesting perspective, and it is then natural to wonder whether it holds for other many-valued logics, and why, and whether it is just a technical result or whether it hides some deeper insights.

For future reference let me say now that when considering the definition (6.2) in general, if the model has a complete lattice structure, then it can be equivalently put in the form

$$\Gamma \vdash^{\leqslant} \varphi \iff e(\varphi) \geqslant \bigwedge\{e(\alpha) : \alpha \in \Gamma\} \text{ for all evaluations } e. \tag{6.3}$$

We will see that this setting has also been popular. When 1 is the maximum of the ordered model set, either (6.1), (6.2) or (6.3) yield the same set of *theorems*:

$$\emptyset \vdash \varphi \iff \emptyset \vdash^{\leqslant} \varphi \iff e(\varphi) = 1 \text{ for all evaluations } e. \tag{6.4}$$

Note that, while this is clearly included in (6.1) and (6.2), for this to follow from (6.3) the implicit assumption that the infimum of an empty set is the maximum of the order is needed.

Investigating these and similar issues we discover a connection with the area of *logics preserving degrees of truth*, which has been gaining momentum recently; see Bou (2008, 2012), Bou et al. (2009), Font (2007, 2009), Font et al. (2006). So I will begin by discussing this idea in general (Sect. 6.2); then I will review the results obtained so far in the literature for the case of Łukasiewicz's infinite-valued logic (Sect. 6.3) and for a larger family ofsubstructural logics (Sect. 6.4). The resulting logics are particularly interesting for *abstract algebraic logic* (Sect. 6.5). I will briefly review some results on the Deduction Theorem (Sect. 6.6) and on their axiomatization (Sect. 6.7). The chapter ends with some research proposals.

6.2 Some Motivation and Some History

Speaking generally, logics defined like (6.1) are called *truth-preserving*, while logics defined like (6.2) or (6.3) are called logics that *preserve degrees of truth*. Just a few words to argue why I think that the latter reflects the semantical idea of many-valued logic better than the former; for a lengthier discussion in a wider context, see Font (2009).

The idea of logical consequence as a truth-preserving one, firmly established from Bolzano to Tarski and beyond, is reasonably unproblematic when there is a single notion of truth in the models, and even more when there is a single model. However, it is at least surprising that it has not raised any significant debate in the context of many-valued logic.

Phrases such as "*Truth comes in degrees*" (Cintula et al. 2011, p. v) or "*Truth of a fuzzy proposition is a matter of degree*" (Hájek 1998, p. 2) appear as a starting justification in many papers and books on fuzzy logic or many-valued logic. One may discuss the meaning of these degrees of truth, their philosophical significance, whether they adequately reflect the phenomenon of *vagueness*, and so on, and for those wanting to do this Smith (2008) is a very enlightening exposition. But I think that for the (mathematical) logician the important thing is not to discuss what they *are* or *should be*, but how they are *used* (to define a logic).

Now, if logic dealt only with *tautologies*, then it would be natural to define them as those propositions that are always true, that is, their truth value always attains the maximum degree, as in (6.4). However, if it is *consequence* that matters, then it seems more natural to demand that consequence preserves truth not only in its maximum degree, but in all the available degrees. Thus, the usage of (6.1) in many-valued contexts raises some dissatisfaction: it seems as if, while all points in the model are considered as truth values when the task is to determine the truth value of a complex formula from the truth values of the atomic formulas,[2] only 1 is really treated as a truth bearer when the task is to establish consequence. Under this view, the other points in the model seem to be treated rather as expressing *degrees of falsity*.[3]

Scheme (6.2) can even be considered as an alternative rendering of the same idea of preservation of truth: not of absolute truth, but of that truth that comes in degrees and characterizes the many-valued landscape. While individual points in a model V may still be regarded as *truth values* in that they are the values assigned to propositions by each of the evaluations, (6.2) suggests identifying *degrees of truth* with the sets $\uparrow r = \{s \in V : r \leqslant s\}$, and then implements the idea that consequence

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[2] In whatever mechanism; one need not assume truth-functionality for this discussion to make sense.

[3] Scott in Scott (1974, p. 421) calls them "degrees of error", see below. Gottwald (2001, Sect. 3.1) seems to be sympathetic with this idea as well.

is the relation that *preserves* all these sets; it is in this sense that it is called "preserving degrees of truth". Since the two schemes produce the same set of tautologies (6.4), separate consideration of the logics obtained by the two paradigms is only of interest when assigning the central rôle in logic to consequence. The second paradigm is potentially as general as the first one; it can be applied to any semantics where truth values are ordered and there is a maximum one, which is indeed a very reasonable and common assumption.[4] It may also be taken to justify interpreting generalized matrices as the most general *structures of degrees of truth*, but this is another issue, discussed in Font (2009).

Logics of the form $\vdash^{\leqslant}$ appeared in the literature much earlier than Hájek (1998), but were only thoroughly studied much more recently. The idea seems to have sprung up independently, but sporadically, in several circles in the early 1970s.

The best motivated precedent is found in the well-known papers by Scott (1973, 1974). These papers contain a critical view on the interpretation of many-valued logic in general, and particularly on the usage of schemes similar to (6.1), perhaps with more than one element playing the rôle of 1, the "designated elements" in the theory of logical matrices (the italics are his):

> One quirk of many-valued logic that always puzzled me was the distribution of *designated elements*. They were somehow "truer" than the others. [...] On the one hand we were denying bivalence by contemplating multivalued systems; but on the other, a return to bivalence was provided by the scheme of designation. Scott (1973, p. 266)

Scott wants to find an interpretation of the non-classical truth values that justifies both the truth tables and the rules of Łukasiewicz logic, and eventually proves completeness. He first interprets the truth values as "types of propositions" and later on as "degrees of error in deviation from the truth", see Scott (1973, p. 271, 1974, p. 421). He then makes a proposal, summarized in the phrase "to replace many values by many valuations", which actually amounts to considering not just a single matrix but a set of n matrices for each n-valued logic, the designated sets being the principal filters of the n-element Łukasiewicz chain; therefore, this proposal turns out to be essentially scheme (6.2). That such an idea leads to a definitely different "conditional assertion" (i.e., consequence relation) is already observed by realizing that *Modus Ponens* in its usual form would fail, but would still hold in the restricted form

$$\text{if } \vdash^{\leqslant} \alpha \to \beta \text{ then } \alpha \vdash^{\leqslant} \beta, \tag{6.5}$$

which will re-appear in Theorems 6.12 and 6.13. What Scott does explicitly for this consequence in Scott (1974) is to define a set of Gentzen-style rules (of the "multiple conclusion" kind) and to prove its completeness, in the sense that the derivable sequents of this calculus coincide with the entailments of the consequence $\vdash^{\leqslant}$ (extended to be of the "multiple conclusion" kind as well). Surprisingly, this calculus does not contain the fusion connective, nor any rules expressing its residuated

[4] That the truth degrees can be compared (i.e., ordered) seems to be another essential ingredient motivating fuzzy logic: "*We shall understand [fuzzy logic in the narrow sense] as a logic with a comparative notion of truth*" (Hájek 1998, p. 2).

character with respect to implication. In any case, only the completeness part of Scott (1974) seems to have had some impact on the evolution of research on many-valued logic in the following years; the proposal of a different consequence relation seems to have passed unnoticed.

Another mathematically clear though philosophically less motivated precedent of the idea is found in the contemporary Cleave (1974), where the author studies Körner's reinterpretation of Kleene's strong 3-valued logic as a logic (and an algebra) of "inexact predicates". He defines a first-order logic, in a language without implication, as a consequence relation, and chooses to do so by explicitly using (6.3) from Kleene's truth tables, justifying this move only in that it is a generalization of the classical case.[5] The associated relation of logical equivalence, which here coincides with interderivability, turns out to be the identity of truth functions, and so the corresponding Lindenbaum-Tarski construction can be easily performed. Algebraic structures related to this logic are just presented as the De Morgan algebras, but this is wrong; actually they should be the Kleene algebras (i.e., the De Morgan algebras satisfying the inequality $x \wedge \neg x \preccurlyeq y \vee \neg y$), see Balbes and Dwinger (1974, Sect. XI.3) and Font (1997, Sect. 5.1). The main goal of the paper, though, is to present a Gentzen-style axiomatization and to prove its completeness by Schütte-style methods.

At the end of the seventies Pavelka (1979) incorporates degrees of truth to the landscape of many-valued logic in a novel way, but not in the sense of preservation of degrees of truth as we are considering. Inspired by Goguen (1969), he introduces fuzzy logics as fuzzy consequence relations between fuzzy sets of formulas and formulas, with membership degrees coinciding with truth degrees. Moreover, he represents each truth degree $r \in [0, 1]$ as a constant $\overline{r}$ of the language.[6] He considers an axiomatic system where each inference rule is coupled with a rule to calculate provability degrees (degrees of truth of statements saying that something follows from something), and proves what has since been termed "Pavelka-style completeness", which is the coincidence of the degree of membership of a formula φ to the consequences of a fuzzy set of formulas $\widetilde{\Gamma}$ with the degree of provability of φ from $\widetilde{\Gamma}$. Later on this proposal was reformulated in Hájek (1995) by taking only constants for the rationals in [0, 1] (hence the name "Rational Pavelka Logic") and considering "graded formulas", i.e., pairs (φ, r) intended to mean "proposition φ has truth degree at least r", so that the syntax is a calculus of these graded formulas. In Hájek (1998, Sect. 3.3) these pairs are finally taken to be aliases for the formulas $\overline{r} \to \varphi$, because in an evaluation e in the unit interval, $e(\overline{r} \to \varphi) = 1$ if and only if $r \leqslant e(\varphi)$. In this way, Pavelka's idea can be studied in an expansion of the ordinary truth-preserving logic of Łukasiewicz; however, while degrees of truth *seem* to play a more proper rôle in it and in other more recent works in the same line (see Esteva et al. (2007)

[5] As we now know, coincidence of this way of expressing semantical consequence with the truth-preserving one also holds in other, non-classical cases, see Theorems 6.1, 6.2, 6.3 and 6.5.

[6] Pavelka develops his proposal for **L**-valued fuzzy sets, where **L** is an arbitrary complete residuated lattice, but proves his completeness result for the cases where **L** is [**0**, **1**] and all its finite subalgebras.

and references therein), the intended semantics is still truth-preserving, as there is no quantification over all truth degrees when considering consequence.[7]

At the end of the eighties the first major paper where the expression "logics preserving degrees of truth" was coined as having a technical, semantical meaning was published; this was Nowak (1990), preceded by the shorter Nowak (1987). There three schemes implementing the same idea are compared in an abstract algebraic context; the one which amounts to (6.3) is given the same name, and two other variants are called "weakly preserving degrees of truth" and "strongly preserving degrees of truth". The "weakly preserving" and the standard cases are characterized in Nowak (1990, Theorems 3.2 and 4.6) in terms of the abstract properties of selfextensionality (see Sect. 6.5) and projective generation. In this paper the expression "structures of degrees of truth" is proposed to denote any algebraic structure with an ordering relation; finally Theorem 7.6 is obtained:

Theorem 6.2 *Let* $\vdash$ *and* $\vdash^{\leqslant}$ *be the logics defined according to schemes* (6.1) *and* (6.3) *with respect to some algebra with a complete lattice reduct. If the algebra is a complete linear Heyting algebra then* $\vdash \, = \, \vdash^{\leqslant}$.

This does not apply directly to Theorem 6.1, for $[0, 1] \cap \mathbb{Q}$ is not a complete lattice; however, the version of Theorem 6.1 for real numbers, which is also true, is clearly equivalent to the particular case of Theorem 6.2 for the Heyting algebra structure of [0, 1].

Some years later the implication of Theorem 6.2 was refined and shown to be an equivalence: see Theorem 6.5 below.

Another, independent appearance of essentially the same property is found in the conference paper Baaz and Zach (1998), contemporary to Hájek (1998), in a study of Gödel-Dummett logic in the modern framework of fuzzy logics. Here the two logics defined from the two schemes (6.1) and (6.3) are considered when evaluations are restricted to a subset $V \subseteq [0, 1]$, with Gödel's operations; let us denote them as $\vdash_V$ and $\vdash_V^{\leqslant}$ respectively. Then Proposition 2.2 of Baaz and Zach (1998) reads:

Theorem 6.3 *For each closed* $V \subseteq [0, 1]$, $\vdash_V \, = \, \vdash_V^{\leqslant}$.

The same setting and result appear again in Baaz et al. (2007, Proposition 2.15), and after it the authors remark that this is "a unique feature of Gödel logics", but support this claim only with an example showing that it does not hold in Łukasiewicz logic, namely, the failure of *Modus Ponens* for the logic $\vdash_V^{\leqslant}$ when V is the Łukasiewicz algebra $[\mathbf{0}, \mathbf{1}]$ instead of the Gödel algebra on [0, 1]. This statement of uniqueness can be considered correct if understood as referring to the basic fuzzy logics, as has been extended and made precise later on: see Theorem 6.5 and property 5 after Theorem 6.7.

While the discussion in Scott (1973, 1974) is obviously centred on the issue of how to define entailment or consequence in many-valued logic, the discussion in Nowak (1987, 1990) is related to a more general problem, considered by the first

[7] For other approaches to graded consequence, even farther removed from preservation of degrees of truth, see (Chakraborty and Dutta 2010; Gerla 2001).

time in Suszko (1961, Sects. 10, 11) and dealt with in more depth in Wójcicki (1984, Chap. III) and in Wójcicki (1988, Sects. 1.6, 2.10). The central, non-trivial issue is *the relation between logical truth and logical consequence*, technically formulated as the problem of whether and when there is a logic (i.e., a consequence relation) $\vdash_L$ having a given set of formulas L as the set of its theorems, and, if so, how it should be defined in a natural way. Wójcicki's proposal, slightly different in his two studies, amounts to assuming that two binary connectives $\wedge$ and $\to$ exist so that one can define

$$\begin{aligned} \alpha_1, \ldots, \alpha_n \vdash_L \varphi &\iff \alpha_1 \wedge \cdots \wedge \alpha_n \to \varphi \in L, \\ \emptyset \vdash_L \varphi &\iff \varphi \in L. \end{aligned} \tag{6.6}$$

Moreover, the consequence $\vdash_L$ is assumed to be finitary. Of course these stipulations will define a consequence only when L satisfies certain conditions relative to $\wedge$ and $\to$, which Wójcicki determines. His motivation is clearly twofold, for he is aware of the general problem, which he discusses at length, but also of the fact that in the case of Łukasiewicz many-valued logics it is perfectly natural to consider several consequences; actually in 1973 he already published one of the few early papers on this topic, Wójcicki (1973), devoted to comparing the truth-preserving and the *Modus Ponens*-based consequences[8] based on several subalgebras of Łukasiewicz's algebra $[\mathbf{0}, \mathbf{1}]$. This paper, however, does still not consider the consequences preserving degrees of truth; these appear first in the lecture notes Wójcicki (1984), and the quotations in Nowak (1987) make it clear that it was this particular case what inspired Nowak's general study.

6.3 The Łukasiewicz Case

Thus, in Wójcicki (1984, 1988) several consequences defined from each of the sets of tautologies of Łukasiewicz ξ-valued logics are studied, for $\xi \leqslant \aleph_1$[9]; the two we are interested in now are an axiomatically defined one (which I denote here by $\vdash_\xi$) where the axioms are all these tautologies and the only rule is *Modus Ponens*, and one defined by scheme (6.6), denoted in Wójcicki (1988) by $\mathscr{L}_\xi^{(\leqslant\models}$. Concerning the latter, it is only shown that for each finite n the logic $\mathscr{L}_n^{(\leqslant\models}$ coincides with the consequence $\vdash_n^{\leqslant}$ that preserves degrees of truth from the n-element subalgebra of $[\mathbf{0}, \mathbf{1}]$ in the sense of (6.3), and that $\vdash_n^{\leqslant}$ is strictly weaker than $\vdash_n$.

[8] Among the main results, he proved that the two consequences coincide for the finite subalgebras but not for the denumerable one or for the whole interval, in which cases the truth-preserving consequences are not finitary. However, they do coincide on finite sets of assumptions; thus, if one considers only the associated finitary consequences, then the two fully coincide.

[9] For $\xi \leqslant \aleph_0$, ξ is the cardinality of the subalgebra of $[\mathbf{0}, \mathbf{1}]$ taken as the model; in the $\aleph_0$ case, it is the rational subalgebra. $\aleph_1$ is used to refer to the whole algebra $[\mathbf{0}, \mathbf{1}]$.

Explicitly continuing the work in Wójcicki (1988), in the mid-1990s an algebraic study of the logics $\vdash_n^{\leqslant}$ was begun[10] in Gil (1996, Sect. 5.5). The logics, which appear here only marginally in the study of n-sided Gentzen systems, are shown not to be algebraizable but to be finitely equivalential, and to satisfy a Deduction Theorem for the formula $(x \to y)^n \vee y$; the (unpublished) work does not go any further.

The totally general case of logics preserving degrees of truth from arbitrary subalgebras of $[\mathbf{0}, \mathbf{1}]$ appears only in the twenty-first century, namely in Font and Jansana (2001, Sect. C), where two consequences $\vdash_{\mathbf{S}}$ and $\vdash_{\mathbf{S}}^{\leqslant}$, defined by (6.1) and (6.2) respectively, are associated with each subalgebra $\mathbf{S}$ of $[\mathbf{0}, \mathbf{1}]$. These logics, again, appear only marginally in this paper, as examples for some points of abstract algebraic logic, and only the following basic properties are of interest here:

1. For each $\mathbf{S}$ except the 2-element algebra, $\vdash_{\mathbf{S}}$ is a proper extension of $\vdash_{\mathbf{S}}^{\leqslant}$, but the two logics have the same theorems.
2. For all infinite $\mathbf{S}$, all the logics $\vdash_{\mathbf{S}}$ and all the logics $\vdash_{\mathbf{S}}^{\leqslant}$ have the same theorems: the tautologies of $\vdash_{[\mathbf{0},\mathbf{1}]}$, i.e., of ordinary Łukasiewicz logic.
3. If $\mathbf{S}_1 \neq \mathbf{S}_2$ then $\vdash_{\mathbf{S}_1} \neq \vdash_{\mathbf{S}_2}$ and $\vdash_{\mathbf{S}_1}^{\leqslant} \neq \vdash_{\mathbf{S}_2}^{\leqslant}$.
4. The logics $\vdash_{\mathbf{S}}$ and $\vdash_{\mathbf{S}}^{\leqslant}$ are finitary if and only if the algebra $\mathbf{S}$ is finite.

Point 1 extends the already mentioned remark of Baaz et al. (2007) about the failure of Theorems 6.1 and 6.3: they fail in the Łukasiewicz case, not only for the whole algebra on [0, 1] but for any subalgebra. Point 2 reinforces the idea that the logics preserving degrees of truth are only interesting when considering the consequence relation, as even for different (infinite) subalgebras of $[\mathbf{0}, \mathbf{1}]$ they yield the same tautologies.

The result in point 4, which extends a result from Wójcicki (1973),[11] suggests that one move to create a more uniform setting admitting a smoother treatment inside the framework of abstract algebraic logic might be to force all logics under discussion to be finitary.[12] This is done in Font (2003), where the schemes (6.1) and (6.2) are used to define the consequences only of finite Γ; the same symbols $\vdash$ and $\vdash^{\leqslant}$ will be used from now on. Moreover, since all models under consideration have a lattice structure,[13] (6.2) can be replaced by the conjunction of the two conditions

$$\begin{aligned} \alpha_1, \ldots, \alpha_n \vdash^{\leqslant} \varphi &\iff e(\varphi) \geqslant e(\alpha_1) \wedge \cdots \wedge e(\alpha_n) \text{ for all } e, \\ \emptyset \vdash^{\leqslant} \varphi &\iff e(\varphi) = 1 \text{ for all } e, \end{aligned} \tag{6.7}$$

[10] Later results, see Theorem 6.4 and the comments before Theorem 6.5, will make it clear that the logics presented in Gil et al. (1993) also coincide with $\vdash_n^{\leqslant}$, but this was not explicit at the time of its publication.

[11] In Wójcicki (1973) only the "if" part is proved, and only for $\vdash_{\mathbf{S}}$.

[12] As a matter of fact, finitarity is part of the definition of a logic in most studies outside abstract algebraic logic, and also in some inside it.

[13] Clearly a meet-semilattice structure is enough; some recent, purely abstract studies of logics defined by (6.7) in a context beyond fuzzy, many-valued or substructural logics, such as Font (2011) or Jansana (2012), build on this fact.

where e ranges over all evaluations in the universe of the algebra taken as the truth structure defining the logic. Since order is equationally definable through the lattice operations, it is clear that this definition only depends on the equations satisfied by the model algebra. This is used in Font (2003) to observe that all the finitary logics so defined from infinite subalgebras of $[\mathbf{0}, \mathbf{1}]$ will coincide, that is, that *there is only one finitary logic preserving an infinity of degrees of truth from Łukasiewicz algebra*; it will be denoted by $\vdash_{\infty}^{\leqslant}$. By contrast, the finitary logics defined from the same subalgebras by (6.1) depend on the quasi-equations that hold in the subalgebra, and using a result on quasi-varieties of MV-algebras from Gispert and Torrens (1998) it was proved in Font (2003, Theorem 21) that these quasi-equations depend only on the rationals contained in the subalgebra.

The paper Font et al. (2006) is devoted to a more systematic and complete study of the unique finitary logic $\vdash_{\infty}^{\leqslant}$ that preserves an infinity of degrees of truth from $[\mathbf{0}, \mathbf{1}]$. The main results are several characterizations of its algebraic counterparts and its full generalized models, its classification in the hierarchies of abstract algebraic logic, the presentation of a Gentzen system adequate for it, which is also related to the ordinary truth-preserving logic of Łukasiewicz, and its characterization through Tarski-style conditions (i.e., abstract conditions on its consequence operator). However, most of the results in Font et al. (2006) were extended considerably in Bou et al. (2009), so it is better to review this paper here.

6.4 Widening the Scope: Fuzzy and Substructural Logics

In the last two decades the study of mathematical fuzzy logic, and particularly its algebraic study, has enormously widened its scope thanks to the work of many people around the world, above all Petr and his collaborators. Hájek (1998) draws a framework where all extensions of his basic logical system BL are encompassed. This logic was later characterized as the logic of all continuous t-norms on [0, 1] and their residua, and it was soon superseded as a ground foundation for the universe of fuzzy logics by MTL, the logic of all left-continuous t-norms and their residua; in turn, MTL was soon identified to be an axiomatic extension of FL_{ew}, the canonical contractionless substructural logic, associated with the class of residuated lattices. Thus, the algebraic study of many-valued logics found its natural environment in the realm of substructural logics; this was to be expected, because the residuation property had been recognized very early as one of the key properties characterizing the behaviour of implication, as in Goguen (1969). It should be noted, however, that when moving from extensions of MTL to substructural logics in general we drop what is considered by some to be an essential ingredient of fuzzy logics, namely their *linearity*. In any case, the dominant paradigm is still truth preservation; good, encyclopaedic overviews of these trends are Chapters I and II of Cintula et al. (2011) for fuzzy logics *stricto sensu* and Galatos et al. (2007) for the wider panorama of substructural logics.

The *residuated lattices* relevant to this discussion are always assumed to be *commutative* and *integral*[14]; the latter property means that the unit 1 of the monoidal structure is also the maximum of the order structure. Therefore each variety K of residuated lattices gives rise to what can be considered a truth-preserving logic $\vdash_{\mathsf{K}}$ defined by (6.1) applied to all algebras in K. In each case, the lattice structure naturally induces a companion logic $\vdash_{\mathsf{K}}^{\leqslant}$ defined by applying (6.7) to all algebras in K. As already observed, this definition depends only on the equations that hold in the models, so in this case a convenient way of highlighting this is to define $\vdash_{\mathsf{K}}^{\leqslant}$ as the finitary logic satisfying

$$\begin{aligned} \alpha_1, \ldots, \alpha_n \vdash_{\mathsf{K}}^{\leqslant} \varphi &\iff \mathsf{K} \vDash \alpha_1 \wedge \cdots \wedge \alpha_n \preccurlyeq \varphi \,, \\ \emptyset \vdash_{\mathsf{K}}^{\leqslant} \varphi &\iff \mathsf{K} \vDash \varphi \approx 1 \,, \end{aligned} \tag{6.8}$$

where $\preccurlyeq$ and $\approx$ are formal symbols for the ordering[15] and the identity relations. When the variety K is generated by a single algebra, then (6.8) can be stated equivalently with this algebra as a unique model, which approaches it to (6.2), (6.3) and (6.4), thus making the interpretation of $\vdash_{\mathsf{K}}^{\leqslant}$ as a logic that preserves degrees of truth from a single model more natural.

The logics $\vdash_{\mathsf{K}}^{\leqslant}$ have been collectively studied in Bou (2008, 2012); and in Bou et al. (2009)[16] in some depth, touching on all aspects already listed at the end of Sect. 6.3, and particularly considering the relations with their companion logics $\vdash_{\mathsf{K}}$; the previous results concerning the latter are systematized in Galatos et al. (2007). It is not possible to summarize the contents of those papers in full, so I will just highlight some points and especially those with some relation with previous work. First, the basic properties and relations match those already found in the Łukasiewicz case:

Theorem 6.4 *For each variety K of residuated lattices the following hold:*

1. *The logic $\vdash_{\mathsf{K}}$ is the extension of $\vdash_{\mathsf{K}}^{\leqslant}$ with either the rule of Modus Ponens or the rule of $\star$-Adjunction (i.e., from φ and ψ to infer $\varphi \star \psi$).*
2. $\alpha_1, \ldots, \alpha_n \vdash_{\mathsf{K}}^{\leqslant} \varphi \iff \emptyset \vdash_{\mathsf{K}} \alpha_1 \wedge \cdots \wedge \alpha_n \to \varphi$, *and* $\emptyset \vdash_{\mathsf{K}}^{\leqslant} \varphi \iff \emptyset \vdash_{\mathsf{K}} \varphi$, *for all* $\alpha_1, \ldots, \alpha_n, \varphi$.

[14] In accordance with most of the literature starting with Ward and Dilworth (1939), here *residuated lattices* are algebras of similarity type $(\wedge, \vee, \star, \to, 1, 0)$ such that $\wedge$, $\vee$ are lattice operations, $\star$ is a commutative monoidal operation (usually called "fusion", "intensional conjunction" or "multiplicative conjunction") with unit 1 also being the maximum of the lattice, and $\to$ is its residuum. A constant 0 is included in the type but in the general case there is no need to postulate anything about it; so these residuated lattices coincide with the FL_{ei}-algebras of Galatos et al. (2007), where the term "residuated lattice" denotes in turn a much larger class. The smaller class of FL_{ew}-algebras is found when postulating that 0 is the minimum of the order, and includes the algebras of most well-known substructural logics such as MTL, BL, Ł_∞, G, Π, etc.

[15] Observe that, in a lattice, an order relation $\alpha \preccurlyeq \beta$ holds if and only if the equation $\alpha \wedge \beta \approx \alpha$ holds; thus, using $\preccurlyeq$ is just a more intuitive way of writing identities of that particular form.

[16] An important error in the proof of Theorem 4.4 in Bou et al. (2009) has been corrected in Bou and Font (2012).

3. *The algebraic counterpart of both logics is the variety* K, *and for each* $\mathbf{A} \in \mathsf{K}$ *the filters of* $\vdash^{\leqslant}_{\mathsf{K}}$ *on* $\mathbf{A}$ *are its lattice filters, while the filters of* $\vdash_{\mathsf{K}}$ *on* $\mathbf{A}$ *are its implicative filters, which are the lattice filters closed under* $\star$.

Point 2 tells us that the relation found by Wójcicki in (6.6) for the finite Łukasiewicz logics extends to all varieties of residuated lattices; in terms of Wójcicki (1984), this says that $\vdash^{\leqslant}_{\mathsf{K}}$ is "the well-determined logic" associated with the theorems (tautologies) of $\vdash_{\mathsf{K}}$. This property is often viewed as justifying that it is not necessary to consider the logics $\vdash^{\leqslant}_{\mathsf{K}}$; it would say that the implication connective of $\vdash_{\mathsf{K}}$ already reflects the notion of a consequence preserving degrees of truth. Admittedly, this is a serious objection, but I think it actually rests on a more basic issue, that of whether the implication connective adequately represents consequence or entailment. This is usually made to depend on the kind of Deduction Theorem the logic satisfies, and it will be shown in Theorem 6.9 that either of the two logics satisfies the ordinary Deduction Theorem for the connective $\rightarrow$ if and only if they actually coincide. Thus, when put in this context, the objection appears to be much weaker.

For the first statement of point 3 to make sense, the notion of *algebraic counterpart* mentioned there has necessarily to be defined in a non-*ad hoc* way, in the context of some general theory of the algebraization of logic, and this is indeed the case, as will be explained after Theorem 6.6. In any event, a technical consequence of the second part of point 3 is that the logic $\vdash^{\leqslant}_{\mathsf{K}}$ coincides with the logic defined by the class of matrices $\big\{\langle \mathbf{A}, F\rangle : \mathbf{A} \in \mathsf{K},\ \ F \text{ a lattice filter of } \mathbf{A}\big\}$. In the particular case of the class MV of MV-algebras, this shows that the logic $\vdash^{\leqslant}_{\mathsf{MV}}$ coincides with the logic studied in Gil et al. (1993), called there "lattice-like Łukasiewicz logic"; moreover, since MV is the variety generated by the Łukasiewicz algebra $[\mathbf{0}, \mathbf{1}]$, the logic $\vdash^{\leqslant}_{\mathsf{MV}}$ is actually the finitary logic $\vdash^{\leqslant}_{\infty}$ preserving an infinity of degrees of truth from the Łukasiewicz algebra mentioned before, a fact probably known to the authors of Gil et al. (1993) but not mentioned there. Thus, some of the results stated (without proof) in this paper anticipate for this particular case the more general ones obtained in Bou et al. (2009); some will be mentioned later on.

The issue of the precise formulation and the scope of Theorems 6.1, 6.2 and 6.3 is settled in Bou et al. (2009, Theorem 4.12) as follows.

Theorem 6.5 *Let* K *be a variety of residuated lattices. Then the two logics* $\vdash^{\leqslant}_{\mathsf{K}}$ *and* $\vdash_{\mathsf{K}}$ *coincide if and only if* K *is a variety of (generalized) Heyting algebras.*

The qualifier "generalized" appears here to cover the case where 0 is not postulated to be the minimum of the order[17]; when it is, that is, when K is actually a variety of FL_{ew}-algebras, the "generalized" can be deleted. Thus, Theorems 6.1, 6.2 and 6.3 cover the case of all the intermediate logics (the axiomatic extensions of intuitionistic logic). Moreover, the converse of the implication in Theorem 6.2 and the claim in Baaz et al. (2007) that "the coincidence of the two entailment relations [i.e., the

[17] Generalized Heyting algebras can be described informally as "Heyting algebras without minimum"; a residuated lattice is a generalized Heyting algebra if and only if the fusion operation $\star$ coincides with the lattice conjunction $\wedge$.

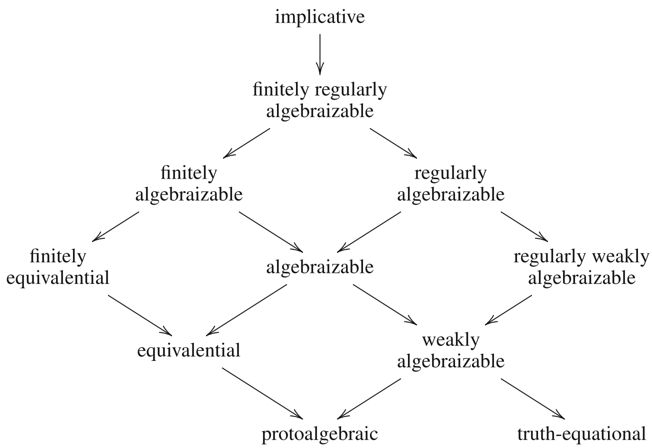

Fig. 6.1 The most important classes of logics in the Leibniz hierarchy; for a finer classification see Cintula and Noguera (2010). "→" means "included in" or "implies"

property of Theorem 6.3] is a unique feature of Gödel's logics" are seen to hold if adequately formulated. The conclusion is that the fuzzy logics preserving degrees of truth may have some interest only for logics that are *not* extensions of Gödel-Dummett logic.

6.5 Abstract algebraic logic classification

In the last two decades, abstract algebraic logic has emerged as an elaborate framework for the study of the algebraic semantics of propositional logics and the relations between metalogical properties of the logics and purely algebraic or model-theoretic properties of their classes of algebraic models; see (Czelakowski 2001; Font and Jansana 2009; Font et al. 2009). The advances in abstract algebraic logic have been partly motivated by advances in the study of many different non-classical logics, and as explained in Cintula et al. (2011, p. 104), they in turn have provided tools for the systematization of the landscape of mathematical fuzzy logic.

One of the main goals of abstract algebraic logic has been to develop methods to classify logics according to some abstract criteria and to study the relations between a logic and its algebraic models in each of the "levels" created by the classification. This has originated two *hierarchies* of a very different character, each with its advantages and its disadvantages.

The *Leibniz hierarchy* (Fig. 6.1) is the more complicated and developed of the two, and can be described in several ways; the one giving it its name is by the behaviour of the so-called Leibniz operator on the theories of the logic, or on the lattices of its filters on arbitrary algebras. As the diagram shows, almost all its members belong to the large class of *protoalgebraic logics*, which can be characterized in several ways: the simplest one is probably by the existence of a set[18] of binary formulas $\Delta(x, y)$ satisfying the basic properties

$$\begin{aligned} \text{Reflexivity}: &\quad \emptyset \vdash \delta(x, x) \text{ for every } \delta \in \Delta \\ \textit{Modus Ponens}: &\quad x, \Delta(x, y) \vdash y\,. \end{aligned} \tag{6.9}$$

The importance of belonging to this hierarchy is that for the logics in these classes the machinery of the theory of matrices can be used in full strength, far beyond the general completeness theorems that hold for all logics whatsoever; many techniques adapted from universal algebra and lattice theory give important, profound results relating a logic with its algebraic models, in particular with the lattices of its filters on arbitrary algebras. The central part of the hierarchy comprises several variants of *algebraizable logics*, all arising from the class introduced by Blok and Pigozzi in their seminal monograph Blok and Pigozzi (1989); these logics enjoy the highest degree of equivalence between a logic and the equational consequence relative to a class of algebras, an equivalence expressible by a pair of mutually inverse definable transformers, whose paradigm is the relation between classical logic and the variety of Boolean algebras, or between intuitionistic logic and Heyting algebras. At the top of this diagram lies the more restricted but still large class of *implicative logics*. These logics slightly generalize those studied by Rasiowa in her highly influential book Rasiowa (1974), and are algebraizable in a very simple and standard way; many of the logics algebraically studied in the past belong to this class.

The *Frege hierarchy* (Fig. 6.2) is less complicated and has also been less studied. Its classifying principle is based on several replacement properties that the logics and some of their generalized models may have. Its weakest, largest level is the class of *selfextensional* logics, originally defined by Wójcicki (1988) as those whose interderivability relation $\dashv\vdash$ is a congruence of the formula algebra.

There are some important theorems connecting the two hierarchies; for instance, every protoalgebraic and Fregean logic with theorems is regularly algebraizable (Font and Jansana 2009, Theorem 3.18). But in general the two hierarchies are orthogonal in the sense that there are logics in the topmost level of each which do not belong even to the lowest level of the other (Theorem 6.6 provides a proper class of examples). As is to be expected, logics belonging to higher levels in both hierarchies enjoy a very nice algebraic behaviour.

The general classification of the two logics associated with each variety K of residuated lattices, determined in Bou et al. (2009, Corollary 4.2) and Galatos et al. (2007, Theorem 2.29), is as follows. The logics $\vdash_{\mathsf{K}}$ have a very good location in the Leibniz hierarchy but in general not in the Frege hierarchy, while the logics $\vdash_{\mathsf{K}}^{\leqslant}$ are in a so-to-speak dual situation.

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[18] For a finitary logic this set can be taken finite; if moreover the logic has a conjunction, then the set can be reduced to a single formula.

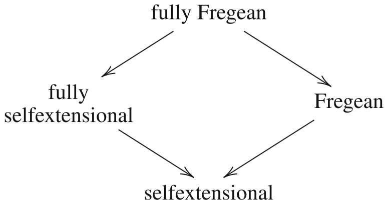

Fig. 6.2 The classes in the Frege hierarchy. "→" means "included in" or "implies"

Theorem 6.6 *Let* K *be any variety of residuated lattices. Then:*

1. *The logic* $\vdash_{\mathsf{K}}$ *is implicative, but need not be even selfextensional.*
2. *The logic* $\vdash_{\mathsf{K}}^{\leqslant}$ *is fully selfextensional, but need not be even protoalgebraic.*

Abstract algebraic logic provides a canonical definition of *the algebraic counterpart*, which applies to an arbitrary logic. The definition uses the notion of a generalized model of a logic, but for restricted classes in the hierarchies the general definition may have a more workable equivalent characterization. In the case of algebraizable logics, this is the notion of *equivalent algebraic semantics*. Thus from point 1 above, point 3 in Theorem 6.4 for $\vdash_{\mathsf{K}}$ means that the equivalent algebraic semantics of $\vdash_{\mathsf{K}}$ is exactly K; for a proof of these facts see Galatos et al. (2007). For selfextensional logics with a conjunction, it is proved in Font and Jansana (2009) that the algebraic counterpart coincides with the notion of the *intrinsic variety* of a logic, which in this case is the variety defined by the set of equations $\{\varphi \approx \psi : \varphi \dashv\vdash \psi\}$. Since the logics $\vdash_{\mathsf{K}}^{\leqslant}$ have a conjunction and by point 2 above they are in particular selfextensional, this can be applied to them; but (6.8) implies that $\varphi \dashv\vdash_{\mathsf{K}}^{\leqslant} \psi \iff \mathsf{K} \vDash \varphi \approx \psi$, and therefore the intrinsic variety of $\vdash_{\mathsf{K}}^{\leqslant}$ is exactly K, which justifies point 3 of Theorem 6.4 regarding $\vdash_{\mathsf{K}}^{\leqslant}$; all this is proved in Bou et al. (2009).

In principle no better classification in the Leibniz hierarchy is possible for the logics $\vdash_{\mathsf{K}}^{\leqslant}$ in general, because many of them fail to be protoalgebraic; for instance $\vdash_{\infty}^{\leqslant}$ has been known not to be protoalgebraic since 1993, see Gil et al. (1993) and Font (2003) for a proof. Those that are protoalgebraic are characterized in several ways in Bou et al. (2009, Theorem 4.6 and Corollary 4.11), and it happens that they are not just protoalgebraic, but automatically finitely equivalential (here the standard notation $x^n = x \star \ldots^{(n} \star x$ is used):

Theorem 6.7 *Let* K *be any variety of residuated lattices. Then the following conditions are equivalent:*

(i) *The logic* $\vdash_{\mathsf{K}}^{\leqslant}$ *is protoalgebraic.*
(ii) *The logic* $\vdash_{\mathsf{K}}^{\leqslant}$ *is equivalential.*
(iii) *The logic* $\vdash_{\mathsf{K}}^{\leqslant}$ *is finitely equivalential, with* $(x \to y)^n \star (y \to x)^n$ *as equivalence formula, for some* $n \in \omega$.
(iv) $\mathsf{K} \vDash x \wedge \big((x \to y)^n \star (y \to x)^n\big) \preccurlyeq y$, *for some* $n \in \omega$.

This result somehow generalizes facts already known to hold for Łukasiewicz logics. That the $\vdash_n^{\leqslant}$ are finitely equivalential, with these equivalence formulas, is already proved in Gil (1996); and the equivalence between the first three conditions is already stated (without proof) in Gil et al. (1993) for logics that are extensions of $\vdash_\infty^{\leqslant}$.

The equivalence between (i) and (iv) suggests that the denumerable family of varieties of residuated lattices defined by the equations in point (iv) may have both an algebraic and a logical interest. This family turns out to be related to other denumerable families of varieties (some already known) which are studied in Bou et al. (2009); here are some of the consequences of the relations found there:

1. If $\vdash_{\mathsf{K}}^{\leqslant}$ is protoalgebraic, then there is some $n \in \omega$ such that all algebras in K are n-contractive.[19]
2. It follows from 1. that for the majority of best-known fuzzy logics, their companion preserving degrees of truth is not protoalgebraic. This concerns Łukasiewicz logic, product logic, MTL, BL, FL_{ew}, etc. It is important for the general theory of abstract algebraic logic that natural examples of non-protoalgebraic logics are found, because at the time of their introduction in Blok and Pigozzi (1986) it was believed that only pathological logics could fail to be protoalgebraic.
3. If K is a variety of MTL-algebras, then $\vdash_{\mathsf{K}}^{\leqslant}$ is protoalgebraic if and only if there is some $n \in \omega$ such that all chains in K are ordinal sums of simple n-contractive MTL-chains (Horčík et al. 2007). Observe that not all finite MTL-chains satisfy this condition.
4. When K is a variety of BL-algebras, the implication in 1. is an equivalence. In contrast with the MTL case, this implies that the logic preserving degrees of truth with respect to any finite BL-chain is protoalgebraic. In particular, this confirms that the finite-valued Łukasiewicz logics preserving degrees of truth (the $\vdash_n^{\leqslant}$ of Sect. 6.3) are protoalgebraic, hence finitely equivalential.
5. There is only one variety K generated by a family of continuous t-norms over $[0, 1]$ such that $\vdash_{\mathsf{K}}^{\leqslant}$ is protoalgebraic, namely the variety G of Gödel algebras. Here what is new is the uniqueness, because by Theorems 6.3 and 6.5 we already know that $\vdash_{\mathsf{G}}^{\leqslant} = \vdash_{\mathsf{G}}$, and hence $\vdash_{\mathsf{G}}^{\leqslant}$ is not just protoalgebraic but implicative. This unique feature of Gödel-Dummett logic adds to the already mentioned statement in Baaz et al. (2007) concerning Theorem 6.3.

Comparing with Theorem 6.6, which states the good classifications of the logics $\vdash_{\mathsf{K}}$ and $\vdash_{\mathsf{K}}^{\leqslant}$ in the Leibniz and the Frege hierarchies respectively, it seems it is not possible for each logic in the pair to go further in the hierarchy where the other one is well placed without so-to-speak trivializing the situation, due to Bou et al. (2009, Proposition 4.3 and Theorem 4.12):

[19] A residuated lattice is *n-contractive*, also called "n-potent" in the literature, when it satisfies the equation $x^n \approx x^{n+1}$. The associated logics are also called "n-contractive".

Theorem 6.8 *Let* K *be any variety of residuated lattices. Then the following conditions are equivalent:*

(i) *The logic* $\vdash^{\leqslant}_{\mathsf{K}}$ *is weakly algebraizable.*
(ii) *The logic* $\vdash^{\leqslant}_{\mathsf{K}}$ *is Fregean.*
(iii) *The logic* $\vdash_{\mathsf{K}}$ *is selfextensional.*
(iv) *The logics* $\vdash_{\mathsf{K}}$ *and* $\vdash^{\leqslant}_{\mathsf{K}}$ *coincide.*

Moreover, when these conditions hold, the (unique) logic is both implicative and fully Fregean.

Together with Theorem 6.6, this implies that the logics preserving degrees of truth, when they are properly so (i.e., when they are not truth-preserving) are fully selfextensional but not Fregean.

Also, this gives another view on the possibility of extending Theorems 6.1, 6.2 and 6.3 to other logics: this is only possible for logics placed in the highest levels of both hierarchies.

6.6 The Deduction Theorem

The research on different forms of the *Deduction Theorem* (DDT) and its algebraic counterparts is at the core of abstract algebraic logic, and is one of its best developed and best understood areas. However, its results hold only inside the Leibniz hierarchy (because all logics with the DDT are protoalgebraic), and hence it may happen that its investigation for the generality of the logics $\vdash^{\leqslant}_{\mathsf{K}}$ (some of which are protoalgebraic while some aren't) is more difficult and less standardized than that for the logics $\vdash_{\mathsf{K}}$.

It is well known (Galatos et al. 2007, Corollary 2.15) that all the logics $\vdash_{\mathsf{K}}$ satisfy the *Local Deduction Theorem* (LDDT) for the family $\{x^n \to y : n \in \omega\}$; that is, they satisfy, for all Γ, α, β:

$$\Gamma, \alpha \vdash_{\mathsf{K}} \beta \iff \text{there is some } n \in \omega \text{ such that } \Gamma \vdash_{\mathsf{K}} \alpha^n \to \beta. \tag{6.10}$$

This extends the result for $\vdash_{\infty}$, known at least since 1964, see Pogorzelski (1964, Thesis T3.3) and also Wójcicki (1973, Lemma 2) for a detailed proof. It is shown in Bou (2008), using Theorem 11.2 of Galatos et al. (2007) plus the well-known equivalence (Font et al. 2009, Theorem 3.10) between the DDT for an algebraizable logic and the property of having equationally definable principal congruences for its equivalent algebraic semantics, that $\vdash_{\mathsf{K}}$ has the DDT for some implication[20] $\delta(x, y)$, that is, it satisfies, for all Γ, α, β,

$$\Gamma, \alpha \vdash_{\mathsf{K}} \beta \iff \Gamma \vdash_{\mathsf{K}} \delta(\alpha, \beta), \tag{6.11}$$

[20] In principle the general theorem concerns an arbitrary set of formulas acting collectively as an implication, but since in the present case all logics are finitary and have a conjunction, one can directly speak about a single formula.

if and only if all algebras in K are n-contractive for some n, and moreover in such a case the implication formula can be taken to be $x^n \to y$. This determines exactly the scope of the Deduction Theorem for the logics preserving truth, and extends the result for the finite-valued logics $\vdash_n$, also known from Pogorzelski (1964) and Wójcicki (1973). In particular it follows from point 1 after Theorem 6.7 that whenever the logic $\vdash^{\leqslant}_{\mathsf{K}}$ is protoalgebraic, the logic $\vdash_{\mathsf{K}}$ has this DDT.

Is the situation for the logics $\vdash^{\leqslant}_{\mathsf{K}}$ comparable? Is there an exact determination of the scope of the DDT, or one of its variants, for this family of logics? The fact that in general they are not even protoalgebraic means there are no general techniques and makes this investigation more difficult. However, some partial results have already been obtained. Concerning the "classical" DDT, that is, when the operation is the "real" implication $x \to y$ itself, the results in Bou et al. (2009, Proposition 2.8 and Theorem 4.12) remove the possibility that it may hold in any other case than those already known and expected:

Theorem 6.9 *Let* K *be any variety of residuated lattices. Then the following conditions are equivalent:*

(i) *The logic* $\vdash^{\leqslant}_{\mathsf{K}}$ *satisfies the* DDT (6.11) *for* $\delta(x, y) = x \to y$.
(ii) *The logic* $\vdash_{\mathsf{K}}$ *satisfies the* DDT (6.11) *for* $\delta(x, y) = x \to y$.
(iii) *The logics* $\vdash_{\mathsf{K}}$ *and* $\vdash^{\leqslant}_{\mathsf{K}}$ *coincide.*

Observe that one half of the DDT is the rule of *Modus Ponens*. Its failure for the logics $\vdash^{\leqslant}_{n}$ was already observed in Scott (1973, 1974), as recalled in Sect. 6.2; this is also observed in the comments after Lemma 2.17 in Baaz et al. (2007), which establishes necessary and sufficient conditions for a binary function on [0, 1] to be the Gödel conditional; one of them is that the associated binary operation satisfies the DDT. The conjunction of Theorems 6.5 and 6.9 adds some further explanation for this: It is well-known that Gödel's conditional is the residuum of the maximum t-norm, which is the only t-norm turning a residuated lattice structure on [0, 1] into a (generalized) Heyting algebra, and by Theorem 6.5 this is equivalent to point (iii) of Theorem 6.9.

The proceedings paper Bou (2008) determines the cases where the DDT holds for two large classes of cases:

Theorem 6.10 *Let* K *be a variety of MTL-algebras. Then the logic* $\vdash^{\leqslant}_{\mathsf{K}}$ *satisfies the* DDT (6.11) *for some implication* $\delta(x, y)$ *if and only if it is protoalgebraic, that is (see point 3 above) if and only if there is some* $n \in \omega$ *such that all chains in* K *are ordinal sums of simple* n*-contractive MTL-chains. In such a case, the formula* $\delta(x, y) = (x \to y)^n \vee y$ *can be taken as the implication satisfying the* DDT.

The presence of protoalgebraicity in relation with the DDT is expected, because the properties (6.9) follow easily from (6.11), so that every logic with the DDT is protoalgebraic; the interesting part is the converse implication. The more restricted case of the extensions of $\vdash^{\leqslant}_{\infty}$ had already been considered in Gil (1996), Gil et al. (1993); in the first work it is proved that the logics $\vdash^{\leqslant}_{n}$ satisfy the DDT for the same formula δ, and in the second it is stated (without proof) that one such extension satisfies the DDT for some δ if and only if there is some $n \in \omega$ such that the associated algebras are n-contractive, and that the formula δ can be taken as above.

The other case studied in Bou (2008), with a not so neat but still useful conclusion, is the following.

Theorem 6.11 *Assume that* $\mathbf{A}$ *is a finite residuated lattice satisfying the same equations with at most 3 variables as the variety* K. *Then the logic* $\vdash_{\mathsf{K}}^{\leqslant}$ *satisfies the* DDT (6.11) *for some implication* δ *if and only if the following conditions are satisfied:*

1. *The logic* $\vdash_{\mathsf{K}}^{\leqslant}$ *is protoalgebraic.*
2. $\mathbf{A}$ *is distributive as a lattice.*
3. *The lattice operation* $\wedge$ *is residuated in* $\mathbf{A}$, *and every subalgebra of* $\mathbf{A}$ *is closed under the map* $\langle a, b\rangle \longmapsto \max\{c \in A : a \wedge c \leqslant b\}$.

Observe that the second part of condition 3 is a weakened form of the property that the residuum operation is term-definable; thus, these three conditions are close to (but weaker than) saying that the algebra $\mathbf{A}$ is a (generalized) Heyting algebra.

The interest of this result is that its assumption covers in particular the simpler case where the algebra $\mathbf{A}$ generates the variety K, a situation that may be common in applications where one wants to consider the logic preserving degrees of truth from a single truth structure. Observe also that, due to the general assumption on $\mathbf{A}$, condition 2 implies that all the members of K are distributive as lattices. In contrast with Theorem 6.10, here a general form of the formula δ satisfying the DDT has not been determined, but it is known that it cannot be the formula found in Theorem 6.10. It is also known that neither of the three conditions is superfluous.

6.7 Axiomatizations

6.7.1 In the Gentzen style

As explained in Sect. 6.2, Scott (1974) presented a multiple-conclusion Gentzen-style calculus and used it to prove completeness for what is actually a multiple-conclusion version of the logic $\vdash_{[\mathbf{0},\mathbf{1}]}^{\leqslant}$, which coincides with $\vdash_{\mathsf{MV}}^{\leqslant}$, where MV is the variety of MV-algebras. This calculus leaves little room for generalization to other logics of the form $\vdash_{\mathsf{K}}^{\leqslant}$, and anyway this idea has not been followed in the literature.

Consequence in the logics $\vdash_{\mathsf{K}}^{\leqslant}$ reflects the properties of order in K, and these can be expressed by properties of the closure operator of lattice-filter-generation in the algebras in K. These properties, in turn, can be expressed in an abstract form yielding the so-called *Tarski-style* conditions, and in a syntactic form as Gentzen-style rules. The case where $\mathsf{K} = \mathsf{RL}$, the variety of all residuated lattices, is treated in Theorem 5.9 and Corollary 5.10 of Bou et al. (2009), where the following is proved (we assume we deal with sequents of the form $\Gamma \rhd \varphi$ where Γ is a finite set of formulas):

Theorem 6.12 *Let $\mathfrak{G}$ be the Gentzen calculus that has the structural axiom, all the structural rules, the following logical axioms*

$$\begin{array}{rclcrcl} \emptyset & \rhd & 1 & \qquad & \varphi\,,\psi & \rhd & \psi \\ \emptyset & \rhd & \varphi \to \varphi & & \varphi \wedge \psi & \rhd & \varphi \\ \varphi \to (\psi \to \xi) & \rhd & \psi \to (\varphi \to \xi) & & \varphi \wedge \psi & \rhd & \psi \end{array}$$

and the following logical rules

$$\frac{\varphi \;\rhd\; \xi \qquad \psi \;\rhd\; \xi}{\varphi \vee \psi \;\rhd\; \xi}\,(\mathrm{r1}) \qquad \frac{\emptyset \;\rhd\; \varphi \to \psi}{\varphi \;\rhd\; \psi}\,(\mathrm{r2})$$

$$\frac{\varphi \vee \psi \;\rhd\; \xi}{\varphi \;\rhd\; \xi} \qquad \frac{\varphi \star \psi \;\rhd\; \xi}{\varphi \;\rhd\; \psi \to \xi}$$

$$\frac{\varphi \vee \psi \;\rhd\; \xi}{\psi \;\rhd\; \xi} \qquad \frac{\varphi \;\rhd\; \psi \to \xi}{\varphi \star \psi \;\rhd\; \xi}\,.$$

Then the calculus $\mathfrak{G}$ axiomatizes the logic $\vdash^{\leqslant}_{\mathsf{RL}}$ ψ in the following sense: For any formulas $\varphi_1, \ldots, \varphi_n, \psi$, it holds that $\varphi_1, \ldots, \varphi_n \vdash^{\leqslant}_{\mathsf{RL}} \psi$ if and only if the sequent $\varphi_1, \ldots, \varphi_n \rhd \psi$ is derivable in $\mathfrak{G}$.

If $\rhd$ is read as $\preccurlyeq$, there are few surprises in the formulation of this calculus. Notice rule (r2), which corresponds to the already mentioned weak form of *Modus Ponens* (6.5), and rule (r1), which corresponds to the rule of *Proof by Cases*, but in a weak form where no side assumptions appear; this reflects the fact that the lattices in RL need not be distributive.

This base calculus can be extended to obtain a calculus for the logic $\vdash^{\leqslant}_{\mathsf{K}}$ when an equational presentation of the variety K is known. In such a case, every equation $\varphi \approx \psi$ is re-written as the pair of sequents $\varphi \rhd \psi$ and $\psi \rhd \varphi$, and these are added to the logical axioms of the calculus; it is straightforward that the resulting calculus axiomatizes the logic $\vdash^{\leqslant}_{\mathsf{K}}$ in the same sense as in Theorem 6.12.

However, these Gentzen calculi seem not to have interesting properties from a proof-theoretic point of view.

6.7.2 In the Hilbert style

The property in point 2 of Theorem 6.4 might suggest that any axiomatization of $\vdash_{\mathsf{K}}$ provides one of $\vdash^{\leqslant}_{\mathsf{K}}$ just by looking at the theorems of the former logic having the form $\alpha_1 \wedge \cdots \wedge \alpha_n \to \varphi$; but this is hardly satisfactory as an axiomatic presentation of the real relation of consequence of $\vdash^{\leqslant}_{\mathsf{K}}$, for we cannot recognize neither the axioms nor the rules of inference that it satisfies.

A first result is found in Bou et al. (2009, Theorem 2.12):

Theorem 6.13 *The logic* $\vdash^{\leqslant}_{\mathsf{K}}$ *can be presented by the axiomatic system having the set* $\mathrm{Taut}(\mathsf{K}) = \{\varphi : \mathsf{K} \vDash \varphi \approx 1\}$ *as its set of axioms, and the rules*[21]

$$\begin{aligned} \textit{Adjunction for} \wedge :&\quad \varphi\,,\psi \;\rhd\; \varphi \wedge \psi \\ \textit{Restricted Modus Ponens} :&\quad \varphi \;\rhd\; \psi \;\; \textit{provided that}\; \varphi \to \psi \in \mathrm{Taut}(\mathsf{K}) \end{aligned}$$

as its rules of inference. If an axiomatization of $\vdash_{\mathsf{K}}$ *is known, with axioms* $\mathrm{Ax}(\mathsf{K})$ *and Modus Ponens as the only rule, then the set* $\mathrm{Ax}(\mathsf{K})$ *can replace the set* $\mathrm{Taut}(\mathsf{K})$ *in the list of axioms of* $\vdash^{\leqslant}_{\mathsf{K}}$.

It is interesting to notice how this restricted form of *Modus Ponens* corresponds to the fact (6.5) already observed in 1973, and was also a derived rule in the axiomatization of Scott (1974, Theorem 3.2). This presentation, however, is not very satisfactory. Both its axioms and one of its rules depend on determination of the set $\mathrm{Taut}(\mathsf{K})$, which is in principle infinite; when it is decidable, then both the axioms and rules of this system will be decidable, so that it can be more properly called "in the Hilbert style". This set is the set of theorems of the logic $\vdash_{\mathsf{K}}$, which due to the LDDT (6.10) can in theory be axiomatized with *Modus Ponens* as its only rule; in the cases where such an axiomatization is known, then the set of axioms can replace the set $\mathrm{Taut}(\mathsf{K})$ in the list of axioms of the above presentation of $\vdash^{\leqslant}_{\mathsf{K}}$, but it is still not possible to do the same in the restricted rule of *Modus Ponens*, so in principle this never gives an axiomatic presentation of $\vdash^{\leqslant}_{\mathsf{K}}$ by a finite set of rule schemes.

This last difficulty is solved in Bou (2012, Corollary 2.4):

Theorem 6.14 *Assume* $\mathrm{Ax}(\mathsf{K})$ *is a set of axioms which, together with the only rule of Modus Ponens, axiomatizes the logic* $\vdash_{\mathsf{K}}$. *Then the logic* $\vdash^{\leqslant}_{\mathsf{K}}$ *can be presented by the axiomatic system having the formula* 1 *as its only axiom, and the following sets of schemes* ($\alpha\,,\varphi\,,\psi$ *are arbitrary formulas) as its inference rules:*

$$\begin{aligned} \mathsf{K}\text{-specific rule} :&\quad \alpha \;\rhd\; \alpha \star \varphi \;\textit{for every}\; \varphi \in \mathrm{Ax}(\mathsf{K}) \\ \text{Adjunction for} \wedge :&\quad \varphi\,,\psi \;\rhd\; \varphi \wedge \psi \\ \textit{Modus Ponens}\text{ for} \star :&\quad \alpha \star \big(\varphi \star (\varphi \to \psi)\big) \;\rhd\; \alpha \star \psi \\ \text{Weakening for} \star :&\quad \varphi \star \psi \;\rhd\; \varphi \\ \text{Associativity for} \star :&\quad (\varphi \star \psi) \star \alpha \;\rhd\; \varphi \star (\psi \star \alpha) \\ \text{Commutativity for} \star :&\quad \varphi \star \psi \;\rhd\; \psi \star \varphi \end{aligned}$$

This is in principle applicable to all the logics $\vdash^{\leqslant}_{\mathsf{K}}$, for we might take $\mathrm{Taut}(\mathsf{K})$ as the set $\mathrm{Ax}(\mathsf{K})$; however this might result in a non-recursive axiomatization (some would even refuse to call such a system a "Hilbert-style axiomatization"). The interesting thing is that if some *finite* axiomatization of $\vdash_{\mathsf{K}}$ is known, then the above procedure

[21] The symbol $\rhd$ is here just a neutral replacement for other symbols like $\vdash$ or $\Rightarrow$, which might lead to misunderstanding if used to describe sequents or rules in the present context.

turns it into a finite axiomatization of $\vdash_{\mathsf{K}}^{\leqslant}$, because for each axiom schema φ of $\vdash_{\mathsf{K}}$ one can put a variable that does not appear in φ in the place of α in the K-specific rule and one obtains a single rule (schema) for $\vdash_{\mathsf{K}}^{\leqslant}$. The majority of the well-known fuzzy logics fall under this assumption, so this provides finite axiomatizations of the logics preserving degrees of truth with respect to the most common many-valued truth structures.

6.8 Conclusions

Logics preserving degrees of truth, in the technical sense established in the Introduction and in Sect. 6.2, seem to formalize a notion of consequence for many-valued logics that treats all truth values on an equal footing, i.e., by considering that all these values express a certain degree of truth without designating one of them (or a subset) as "the truth", and giving them the same rôle in a truth-preserving definition of consequence. This possibility has hardly been explored at all in the literature on many-valued logic, save for a short proposal by Scott (1973) and a few other scattered results. The recent systematic study of logics preserving degrees of truth inside the large group of substructural logics has prompted a more technical approach using the tools of abstract algebraic logic. The cases of the two logics $\vdash_{\mathsf{K}}$ and $\vdash_{\mathsf{K}}^{\leqslant}$ associated with each variety K of (commutative, integral) residuated lattices, the first one preserving truth as represented by 1 and the second one preserving degrees of truth, have been reviewed, in particular their classification in the Leibniz and the Frege hierarchies of abstract algebraic logic. It appears that, from the point of view of abstract algebraic logic, the theory of the logics preserving degrees of truth is much richer and diverse than that of the logics preserving truth; in particular some properties of the former seem to depend heavily on those of the associated variety K, while the latter seem to show a more uniform and predictable behaviour.

The survey in this chapter has been limited to published work. While arising from motivations around many-valued and fuzzy logic, the study of logics preserving degrees of truth has been progressively extended, as witnessed by the most recent research reported on in Sects. 6.4–6.7. In hindsight it is now clear that some of the restrictions adopted in the present study (in order to produce a reasonably smooth and powerful development and results) are not essential to its motivations, and may seem *ad hoc* to some readers. Actually, the basic idea (6.2) of a logic preserving degrees of truth requires very little for its application. Thus one can see several natural **directions for future research** in this area. Let me end the chapter by commenting on some of them:

- The results in Sect. 6.5 about the classification of the logics under study in the Leibniz hierarchy of abstract algebraic logic consider only the traditional classes of protoalgebraic, equivalential and algebraizable logics. However, after Raftery (2006) the new class of *truth-equational logics* has been added to this hierarchy (since it is defined by conditions on the Leibniz operator) without being a subclass of protoalgebraic logics. The logics $\vdash_{\mathsf{K}}^{\leqslant}$ are not protoalgebraic in general; therefore investigating whether they are truth-equational or not makes sense, but as far as I know this has not been done.

- As already noted in footnote 13, most of the theoretical background for the study and classification of the logics $\vdash_{\mathsf{K}}^{\leqslant}$ is dependent upon only the connective of lattice conjunction $\wedge$, i.e., it can be developed for *meet-semilattices* rather than lattices. Thus, the investigation of fragments still containing $\wedge$ but perhaps not some of the other connectives may produce interesting results.
- The pairs of companion logics $\left(\vdash_{\mathsf{K}}, \vdash_{\mathsf{K}}^{\leqslant}\right)$ have been studied when K is a variety of residuated lattices. However, it is well known that the natural algebraic counterparts of finitary, finitely algebraizable logics are *quasivarieties*. One of the reasons for the restriction to varieties may be that the logics $\vdash_{\mathsf{K}}$, the algebraizable members of the pairs, have been studied in Galatos et al. (2007) only in this case (corresponding to axiomatic extensions of the basic substructural logic FL). But I think the main reason is the fact that the companion logics $\vdash_{\mathsf{K}}^{\leqslant}$ defined as in (6.8) are determined by varieties, i.e., if defined from an arbitrary class of algebras, the resulting logic coincides with that defined by the variety it generates. However, this is due to the presence of a meet-semilattice conjunction $\wedge$, while the more general definition (6.2) makes sense in quasi-varieties and requires only an ordering relation $\leqslant$. Hence, it would make sense to extend this research to other quasivarieties of algebras corresponding to special substructural logics with a more limited language, such as BCK logic.
- Another restriction has been the assumption of *integrality*, that is, that the unit 1 of the monoidal structure is also the maximum of the lattice structure. Again, there is nothing leading specifically to this choice in the basic idea of a logic preserving degrees of truth. From (6.2) it follows that the theorems of such logics will be the formulas that are always evaluated as the maximum of the order; hence when the algebraic structures need not have a maximum the resulting logic will have no theorems, and in particular will not have the same theorems as the logic preserving truth. A case study of this situation, concerning relevance logic R, is Font and Rodríguez (1994). Removing integrality in general, however, raises several unexpected fundamental questions, both from the motivational side and from the technical one, and has been discussed in Font (2007). To highlight only two: It is not clear that in this case it makes sense to preserve all degrees of truth, and it is not clear that the theoretical support of (Font 2011; Jansana 2012) on selfextensional logics with conjunction can still be used.

The best conclusion is that there is still a great deal of room for research in this area.

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