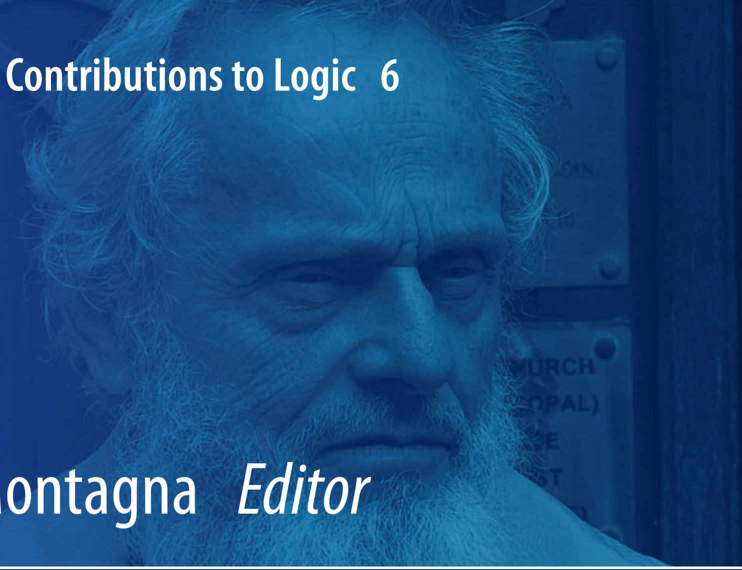
Outstanding Contributions to Logic 6



Franco Montagna *Editor*

Petr Hájek on Mathematical Fuzzy Logic

Springer

Outstanding Contributions to Logic

Volume 6

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Petr Hájek on Mathematical Fuzzy Logic

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To Petr Hájek

To the memory of Marie Hájková

Preface

This volume is about Petr Hájek's contribution to Mathematical Fuzzy Logic. Petr Hájek is not only a great scientist, but also a wonderful human being, and hence it is a great honor for me to take care of this volume. However, commenting on his scientific work is not an easy job: although his scientific contribution is by no means limited to Mathematical Fuzzy Logic, his production in this field is so wide and so important that it is almost impossible to present a complete description of it. Hence, when I began to work on the volume, I started doubting about its success. After Petr's monograph *Metamathematics of Fuzzy Logic* and after the various books on Fuzzy Logic, including Gottwald's *A Treatise on Many-Valued Logics*, two more books, one about the work of Petr Hájek, entitled *Witnessed Years*, and one devoted to Mathematical Fuzzy Logic, the *Handbook of Mathematical Fuzzy Logic*, in which Petr is one of the Editors and one of the main authors, have been written. Moreover, when I told Hájek that we were going to write another volume for him, he replied: Too many honors! And although he added no comments to his response, I had the feeling that what he would really need now is not another volume in his honor, but rather some more health for himself and for his wife.

However, I am absolutely convinced that a new volume on Petr Hájek's work will be very useful, if not for himself, at least for the scientific community. Indeed, Petr's influence on the community of Mathematical Fuzzy Logic was simply great, and the best way we have to celebrate him is to continue his work writing good new papers, possibly developing his ideas. The invited authors of this volume are all prominent scientists, and spent many energies to make their papers as good as possible. Moreover, all papers in this volume discuss some problems that have been previously discussed by Petr and offer original contributions to them. These considerations make me optimistic about the success of the volume.

The volume begins with an Introduction, in which Esteva, Godo, Gottwald, and myself present and comment on Hájek's contribution to Mathematical Fuzzy Logic, and by a scientific biography by Haniková. The remainder of the volume is divided into five parts, with a final appendix containing a bibliography of Petr Hájek.

The second part deals with foundations of many-valued logic, and contains three papers, one by Běhounek and Haniková on Arithmetic and Set Theory over many-valued logic, another by Gottwald on theories of Fuzzy Sets, and yet another by Fermüller and Roschger about the connections between Fuzzy Logic and vagueness.

The third part deals with semantics, and consists of three papers. The first one, by Font, is about the semantics of preservation of truth degrees, which is alternative both to the algebraic semantics and to the standard semantics. With this new semantics, validity remains unchanged, but the consequence relation changes in a significant way. The second paper, by Mundici, proposes another alternative to the standard semantics for which the author is able to prove strong standard completeness, a property which fails for the usual standard semantics. The third paper on semantics, by Aguzzoli and Marra, discusses some general semantic principles and characterizes the three main fuzzy logics, Łukasiewicz, Gödel, and product logics, in terms of them.

The fourth part deals with the algebraic aspects of many-valued logics. In this chapter, algebraic tools are used. This part consists of two papers. The first paper, by Dvurečenskij, deals with the connections between many-valued logic and ℓ-groups, and the second paper, by Ledda, Paoli and Tsinakis, deals with another important property of algebras for many-valued logic, namely, prelinearity, and relates varieties of algebras for substructural logics to varieties of algebras for fuzzy logic.

The fifth part contains two papers, one by Bou, Esteva and Godo, and another by Cintula, Horčik and Noguera, and deals with some more recent developments, namely modal fuzzy logics and weak fuzzy logics. Modal fuzzy logics are discussed in one of the last chapters of Hájek's book, *Metamathematics of Fuzzy Logic*, but although the book presents many very interesting general ideas, it does not contain a complete development of this subject, which seems to be left to the future research. The second subject, weak many-valued logics, was begun already in Hájek's book, in which the author proposed BL as the basic fuzzy logic. But after the publication of the book, several weaker fuzzy logics (for instance, the monoidal t-norm-based logic MTL by Esteva and Godo), were investigated, and hence it makes sense to look for the really basic fuzzy logic.

I conclude this Preface by thanking several researchers, without whom this volume would have not existed. First of all, Petr Hájek, the scientist to whom the volume is dedicated; then Daniele Mundici, who suggested the idea for the first time; then, all the authors of the volume, who accepted to present their results here and to devote them to Petr Hájek; finally, special thanks are due to (in alphabetical order) Libor Běhounek, Petr Cintula, Francesc Esteva, Lluis Godo, Siegfried Gottwald, Zuzana Haniková, and Vincenzo Marra, who helped me either to collect the scientific material of Petr Hájek or to improve the format of the volume. In particular, Lluis Godo's assistance with the LaTex was extremely useful.

All these people deserve special mention, and credits for this volume should be given to them more than to myself.

Siena, Italy Franco Montagna

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Part I
Introduction

Chapter 1
Introduction

Francesc Esteva, Lluís Godo, Siegfried Gottwald and Franco Montagna

1.1 Mathematical Fuzzy Logic

Since Petr Hájek, the scientist we are going to celebrate, is the main contributor to Mathematical Fuzzy Logic, we will first spend a few words about this subject.

Mathematical Fuzzy Logic is a mathematical study of logical systems whose algebraic semantics involve some notion of truth degree. The origins of the discipline are both philosophical (modeling correct reasoning in some particular contexts like the treatment of vague predicates, for which classical logic may appear not adequate), as well as more technical: Zadeh's Fuzzy Set Theory, which has been widely applied, and many-valued logics, which are logics with intermediate truth degrees, whose order is often assumed to be linear. Unlike Fuzzy Set Theory, which is mainly devoted to concrete applications, Mathematical Fuzzy Logic is a subdiscipline of Mathematical Logic, and hence it aims at a mathematical treatment of reasoning with intermediate truth degrees. Hence, as all known logics, Mathematical Fuzzy Logic deals with propositional and first-order formulas (and, in some cases, even with second-order formulas), and it has several semantics, an algebraic semantics, a semantics given by chains, a semantics based on [0, 1], and also a game-theoretical

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semantics. It also deals with such problems as (un)decidability and computational complexity. Although it is questionable whether or not Mathematical Fuzzy Logic can satisfactorily capture vague concepts (and we tend to believe that it is not the case), for their treatment it seems at least more suitable than classical logic and than other non-classical logics. Finally, although Fuzzy Logic is different from probability, it is formally possible to treat probability (and hence, uncertainty) inside Fuzzy Logic enriched with a modality representing *Probably*. Hence, Mathematical Fuzzy Logic is a very beautiful mathematical theory with concrete applications. For more information, one can consult the *Handbook of Mathematical Fuzzy Logic* (Cintula et al. 2011).

1.2 The Beginning

When Petr Hájek begun his work on Mathematical Fuzzy Logic, he and his collaborators immediately realized that several important fuzzy logics, like Łukasiewicz logic and Gödel logic, were already present in the literature. At the same time, the wide literature on t-norms suggested to him to associate to each continuous t-norm a logic, in which conjunction and implication are interpreted as the t-norm and its residuum, respectively. In particular, his attention was attracted by the logic of a very natural continuous t-norm, namely, the product t-norm. With F. Esteva and L. Godo, in the paper *A complete many-valued logic with product-conjunction* (Hájek et al. 1996), the authors offered an axiom system for this product logic and proved that it is (sound and) complete with respect to the standard semantics on [0, 1]. To get this completeness result they introduced an algebraic semantics based upon product algebras in a way similar to the completeness proof which C. C. Chang gave for (the infinite valued) Łukasiewicz logic via MV-algebras (Chang 1959).

The interest of product logic is also emphasized in the paper *Embedding logics into product logic* (Baaz et al. 1998). In that paper, the authors construct a faithful interpretation of Łukasiewicz's logic in product logic (both propositional and predicate), as well as a faithful interpretation of Gödel logic into product logic with the Monteiro-Baaz projection connective Δ. As a consequence, they prove that the set of standard first-order product tautologies is not recursively axiomatizable, and that the set of propositional formulas satisfiable in product logic (resp., in Gödel logic), is NP-complete.

A controversial problem in fuzzy logic is the notion of negation. Indeed, in the theory of fuzzy sets negation is always involutive. But if one defines $\neg\varphi$ as $\varphi \to \bot$, as in intuitionistic logic, then the negation of several fuzzy logics like Gödel and product logic, is not involutive: over [0, 1] it is a function which exchanges 0 and 1 and sends to 0 any other value. Hence, in the paper *Residuated fuzzy logics with an involutive negation* (Esteva et al. 2000) by Esteva, Godo, Hájek and Navara, the authors describe the logic arising from a residuated fuzzy logic with such a kind of negation by the addition of an involutive negation. In these logics, one has two negations: a classical (involutive) negation and the (strict) negation arising from residuation. Interestingly, for the case of usual product logic, while one has standard completeness with respect to the product usual connectives on [0, 1] and the class of all involutive negations, we do not have standard completeness with respect to the usual negation $1 - x$ alone.

1.3 The Monograph "Metamathematics of Fuzzy Logic"

All the above mentioned logics are treated in Hájek's monograph *Metamathematics of Fuzzy Logic* (Hájek 1998). This book has played a fundamental role in the recent development of Mathematical Fuzzy Logic.

It is impossible to summarize the whole content of this book without overlooking something important. For example, the book contains an interesting preliminary discussion about the motivations of fuzzy logic and about their general semantic principles, which will not be reported here. However, in our opinion the main ideas contained in the book are the following:

1. Fuzzy logics are presented as logics of continuous t-norms and their residuals.
2. Since every continuous t-norm is the ordinal sum of Łukasiewicz, Gödel and product t-norms, the corresponding logics (Łukasiewicz, Gödel and product logics) are of fundamental importance.
3. One can look for a common fragment of the three fundamental fuzzy logics, as well as for the logic of all continuous t-norms. Then Hájek proposed a logic, called Basic (Fuzzy) Logic (in symbols, BL), which later on turned out to be the logic of all continuous t-norms and of their residuals.
4. Fuzzy logics are considered as logics of a comparatively graded notion of truth, indeed a formula $\varphi \to \psi$ is 1-true whenever the degree of truth of ψ is greater or equal to that of φ. The ability of explicitly reasoning about truth-degrees motivates the study of the so called Rational Pavelka Logic, which has constants for all rational truth-values.
5. The general semantics of fuzzy logics is constituted by totally ordered commutative, integral and divisible residuated lattices, BL-chains for short. As noted by Baaz in his article in the volume *Witnessed years* (Cintula et al. 2009), Hájek raised the problem of the independence of the axiom $(\varphi \& (\varphi \to \psi)) \leftrightarrow (\psi \& (\psi \to \varphi))$, corresponding to divisibility. This axiom turns out to be independent, but interestingly, if we remove it, we get another interesting logic, namely, the Monoidal T-norm-based Logic MTL of Esteva and Godo.
6. Every schematic extension L of BL has a first-order expansion L∀, which is strongly complete with respect to the class of all safe interpretations on L-chains. The idea is that the existential quantifier and the universal quantifier are interpreted by suprema and infima, and an interpretation on an L-chain is said to be *safe* when all suprema and infima needed to interpret quantifiers exist in the L-chain. Interestingly, Hájek didn't require the L-chains to be complete. Indeed, with the remarkable exception of Gödel logic, for every continuous t-norm logic L,

the set of first-order formulas which are valid in all complete L-chains is not recursively axiomatizable, while the set of formulas which are valid in all safe interpretation over arbitrary L-chains is axiomatizable over L by a finite set of axiom schemata. Yet another interesting feature of this book is the discovery of the axiom $\forall x(\varphi(x) \vee \psi) \to ((\forall x\varphi(x)) \vee \psi)$, which in the case of intuitionistic first-order logic characterizes Kripke models with constant domain. It turns out that in the case of fuzzy logic, this axiom characterizes the semantics by chains.

7. The last part of the book deals with application aspects: e.g., fuzzy modal logics, a logical understanding of fuzzy if-then rules and fuzzy quantifiers like *many* and *probably* are discussed. Interestingly, although Hájek emphasizes the differences between fuzzy logic and probability theory (the former is truth functional, the latter is not, the former deals with vague concepts that may have an intermediate truth degree, while the latter deals with events which are unknown now but will be either completely true or completely false later), the author introduces an interpretation of the logic of probability into fuzzy logic enriched with the modality *Probably*. In this way, the probability of an event φ becomes the truth value of the sentence *Probably* φ.

Although the book is full of interesting results, it doesn't exhaust Petr's research in Mathematical Fuzzy Logic. Here below, we list some problems which are somehow addressed in the book and which have been further investigated by Petr and by his coauthors:

1. First-order fuzzy logics, and in particular: supersound logics, complexity of standard tautologies or of standardly satisfiable formulas and witnessed models.
2. Computational complexity of propositional fuzzy logics.
3. Logics weaker than BL (MTL, hoop logics, ps-BL, flea-logics).
4. Logics with truth constants for the rationals.
5. Logics of probability, of possibility and of belief.
6. Logics with truth-hedges.
7. Fuzzy modal logics.
8. Fuzzy description logic.
9. Mathematical theories (arithmetic, set theory) over fuzzy logic.

1.4 First-Order Fuzzy Logics

As said before, an important contribution by Petr Hájek to first-order fuzzy logic is the discovery of the right semantics for it. Indeed, the first-order version of any schematic extension L of BL (denoted in the sequel by L∀) is strongly complete with respect to the class of all safe interpretations on L-chains (totally ordered models of L), and the same can be easily proved, essentially by the same proof, for extensions of first-order MTL. In general, we do not have completeness with respect to interpretations over completely ordered L-chains. That is, the class of all structures on completely ordered L-chains is a too narrow class to get completeness. One may try to do the opposite way, that is, to enlarge the class of interpretations, and to define a formula *valid* if it is true in all (possibly unsafe) interpretations in L-chains in which its truth value is defined. But in this way we may lose correctness. A predicate fuzzy logic L∀ is said to be *supersound* if every theorem φ of L∀ is valid in all (possibly unsafe) interpretations on any L-chain in which its truth-value is defined.

In the paper *A note on the notion of truth in fuzzy logic* (Hájek and Shepherdson 2001), Hájek and Shepherdson show that among the logics given by continuous t-norms, Gödel logic is the only one that is supersound. All other continuous t-norm logics are (sound but) not supersound. This supports the view that the usual restriction of semantics to safe interpretations (in which the truth assignment is total) is very natural.

Another semantics for first-order fuzzy logics for which completeness in general fails is the standard semantics on [0, 1]. In some cases, the failure is obtained in a very strong sense: for instance, for product logic, both the set of 1-tautologies and the set of 1-satisfiable formulas are not arithmetical. The arithmetical complexity of the standardly satisfiable formulas or of standard tautologies of the most prominent fuzzy logics is summarized in P. Hájek's paper *Arithmetical complexity of fuzzy predicate logics-a survey, II* (Hájek 2009).

Among all logics of continuous t-norms, Gödel first-order logic is the only logic which is complete with respect to the standard semantics on [0, 1]. However, Gödel first-order logic is no longer complete if instead of [0, 1] we take an arbitrary closed subset of [0, 1] containing 0 and 1. Now in P. Hájek's paper *A non-arithmetical Gödel logic* (Hájek 2005c), the following surprising result is proved: Let G↓ denote the first-order Gödel logic with truth degree set $V\!\downarrow = \{0\} \cup \left\{\frac{1}{n} : n = 1, 2, \ldots\right\}$. Then the sets of satisfiable formulas as well as of tautologies of G↓ are non-arithmetical. This is in contrast with the similar system G↑ with truth degree set $V\!\uparrow = \{1\} \cup \left\{\frac{n}{n+1} : n = 0, 1, \ldots\right\}$, whose set of tautologies is shown to be Π_2-complete.

Several new and original ideas about the semantics of first-order fuzzy logics are presented in P. Hájek and P. Cintula's paper *On theories and models in fuzzy predicate logics* (Hájek and Cintula 2006b). There, a general model theory is presented for predicate logics, and a more general version of the completeness theorem is proved, using doubly Henkin theories. Moreover, the (very interesting) concept of witnessed model is introduced. These are models in which suprema and infima used to interpret existential and universal quantifiers are actually maxima and minima. The logic of witnessed models is obtained by adding the axioms $\exists x(P(x) \leftarrow \forall y P(y))$ and $\exists x(\exists y P(y) \rightarrow P(x))$. Interestingly, although these axioms are valid in classical logic, they are not intuitively valid. For instance, the first axiom says that there is an individual x such that if x gets drunk, then everybody gets drunk.

Although the paper by P. Hájek and F. Montagna, *A note on the first-order logic of complete BL-chains* (Hájek and Montagna 2008), is probably not one of the most important papers by Petr, we will mention it because it has a nice story. The paper discusses an error in another paper by Sacchetti and Montagna. The error was based on the wrong assumption that in a complete BL-chain, the fusion operator distributes over arbitrary infima. This property clearly holds in any standard BL-algebra, but is not true in general (Felix Bou found a counterexample). As a consequence of that error, Montagna and Sacchetti claimed that the predicate logics of all complete BL-chains and of all standard BL-chains coincide. During a meeting, Petr told Montagna that he was going to do the same error. Then Petr and Montagna discussed this problem by e-mail, and arrived to the following result: a complete BL-chain **B** satisfies all standard BL-tautologies iff for any transfinite sequence $(a_i : i \in I)$ of elements of **B**, the condition $\bigvee_{i \in I} a_i^2 = (\bigvee_{i \in I} a_i)^2$ holds in **B**. It is nice to observe that Montagna was going to repeat the error in another paper, but fortunately he noticed it before submitting the paper for publication.

1.5 Computational Complexity of Fuzzy Logics

Propositional logics may have quite different complexities. For instance, classical logic is coNP-complete, intuitionistic logic is PSPACE-complete, as well as many modal logics, and linear logic is even undecidable. The most important many-valued logics extending BL are coNP-complete, and Hájek greatly contributed to the proof of this general claim. The book *Metamathematics of Fuzzy Logic* already contains a proof of coNP-completeness of Łukasiewicz, Gödel and product logics. The first result has been proved by Mundici (1987), and then, by different techniques, by (Hähnle 1994). The coNP-completeness of Gödel logic is easy and the coNP-completeness of product logic follows from the above mentioned paper (Baaz et al. 1998).

Another important result about computational complexity of fuzzy logis is the coNP-completeness of BL, which was proved by M. Baaz, P. Hájek, F. Montagna and H. Veith in the paper *Complexity of t-tautologies* (Baaz et al. 2002).

In P. Hájek's paper *Computational complexity of t-norm based propositional fuzzy logics with rational truth constants* (Hájek 2006a), the author discusses the complexity of Gödel logic, Łukasiewicz logic, and product logic added with constants for the rational numbers in [0, 1] along with bookkeeping axioms. For these logics the complexity remains the same as for their fragments without the constants. However, there are t-norms such that the complexity when one adds the rational constants may fall outside the arithmetical hierarchy.

Finally, in the paper *Complexity issues in axiomatic extensions of Łukasiewicz logic* (Cintula and Hájek 2009) P. Cintula and P. Hájek show that all axiomatic extensions of propositional Łukasiewicz logic are coNP-complete.

It is worth noticing that Zuzana Haniková in the paper *A note on the complexity of propositional tautologies of individual t-algebras* (Haniková 2002) proved that all logics of continuous t-norms on [0, 1] are coNP-complete.

1.6 Logics Weaker than BL

There are three types of fragments of BL, namely, the logics in a weaker language which are extended by BL conservatively, the logics in the language of BL whose axiom set is properly included in the axiom set of BL, and the logics which have a weaker language than BL and are extended by BL, but not conservatively. Remarkable examples of fragments in the first sense are the logic BH of basic hoops, which has been investigated by F. Esteva, L. Godo, P. Hájek, and F. Montagna in the paper *Hoops and fuzzy logic* (Esteva et al. 2003) and the logic BHBCK of basic hoop BCK-algebras, investigated by Aglianò, Ferreirim and Montagna in Aglianò et al. (2007). The first logic is the fragment of BL in the language $\{\&, \rightarrow, \top\}$, while the latter logic is the fragment of BL in the language $\{\rightarrow, \top\}$.

The most interesting fragment of the second type is probably the Monoidal t-norm Logic MTL by Esteva and Godo (2001). These authors, having in mind that in t-norm algebras the existence of the residual already yields the left continuity of the t-norm, conjectured that deleting the essential part $a \wedge b \leq a * (a \rightarrow b)$ of the continuity condition, but maintaining the prelinearity condition, should yield the logic of all left continuous t-norms.[1] Although this interesting logic was not due to him, Hájek showed interest in this logic and in his paper *Observations on the monoidal t-norm logic* (Hájek 2002a), he investigates some extensions of MTL. The leading idea was the following: BL has three well-known extensions: Łukasiewicz logic, Gödel logic, and product logic, which are axiomatized over BL by the axioms $\neg\neg\varphi \rightarrow \varphi$, $\varphi \rightarrow (\varphi \& \varphi)$ and $\neg\psi \vee (((\psi \rightarrow (\varphi \& \psi)) \rightarrow \varphi)$, respectively. Then it is natural to investigate the analogous extensions of MTL, namely MTL plus $\neg\neg\varphi \rightarrow \varphi$, denoted by IMTL, MTL plus $\varphi \rightarrow (\varphi \& \varphi)$ and MTL plus $\neg\psi \vee (((\psi \rightarrow (\varphi \& \psi)) \rightarrow \varphi)$, which is denoted by ΠMTL. While MTL plus $\varphi \rightarrow (\varphi \& \varphi)$ is just Gödel logic, IMTL is weaker that Łukasiewicz logic, and MTL plus $\neg\psi \vee (((\psi \rightarrow (\varphi \& \psi)) \rightarrow \varphi)$ is weaker than product logic.

While MTL is obtained from BL by removing divisibility, one may wonder what happens if one removes commutativity of the conjunction. BL deprived of commutativity has been investigated e.g. by Georgescu and Iorgulescu (2001) and by Flondor et al. (2001), see also the book by S. Gottwald, *A treatise on many-valued logics* (Gottwald 2001). In his paper *Fuzzy logics with noncommutative conjunctions* (Hájek 2003b), Hájek finds adequate axiomatizations for these logics and proves a completeness theorem for them. Moreover in his paper *Embedding standard BL-algebras into non-commutative pseudo-BL-algebras* (Hájek 2003a), Hájek proves that each BL-algebra given by a continuous t-norm is a subalgebra of a non-commutative pseudo-BL-algebra on a 'non-standard' interval $[0, 1]^*$.

The logic BL was already an attempt to generalize the three main fuzzy logics, that is, Łukasiewicz, Gödel and product logics. Hence, probably Hájek didn't imagine such an amount of generalizations obtained by removing either connectives or the

[1] Deleting even the prelinearity condition had given the monoidal logic of Höhle (1994, 1995). This logic is characterized by the class of all residuated lattices, but seems to be too general as a logic for t-norms.

divisibility axiom, or the commutativity axiom. In his paper *Fleas and fuzzy logic* (Hájek 2005a), Hájek finds a common generalization of the logic of basic hoops and the logic psMTL of noncommutative pseudo-t-norms. He presents a general completeness theorem and he discusses the relations to the logic of pseudo-BCK algebras. The reference to fleas in the title is due to the following story:

Some scientists make experiments on a flea: they remove one of its legs and tell it: *Jump!*. The flea can still jump. Then they repeat the experiment over and over again, and, although with some difficulty, the flea still jumps. But once all legs are removed, the flea is no longer able to jump. Then the doctors come to the conclusion that a flea without legs becomes deaf. Now the attitude of logicians who remove more and more axioms and symbols and still expect to be able to derive interesting properties, is compared to the attitude of the scientists of the story.

Another interesting paper about fragments is the one by P. Cintula, P. Hájek, R. Horčik, *Formal systems of fuzzy logic and their fragments* (Cintula et al. 2007). There, the authors investigate expansions of the logic BCK with the axiom of prelinearity which come about by the addition of further connectives, which are chosen in such a way that the resulting systems become fragments of well-known mathematical fuzzy logics. These logics are usually characterized by quasivarieties of lattice based algebraic structures, and in some cases by varieties. The authors give adequate axiomatizations for most of them.

1.7 Further Logics Related to BL

1.7.1 Rational Pavelka Logic

Besides the purely logical interest in mathematical fuzzy logics their consideration is motivated by the problem to search for suitable logics for fuzzy sets.

In this context it is natural to ask whether it is possible to generalize the standard entailment as well as provability considerations in logical systems to the case that one starts from *fuzzy sets of formulas*, and that one gets from them as consequence hulls again fuzzy sets of formulas. This problem was first treated by Jan Pavelka in 1979 in his three papers *On fuzzy logic* I, II and III (Pavelka 1979). Accordingly such approaches are sometimes called *Pavelka-style*, but they have also been coined approaches with *evaluated syntax*.

Such an approach has to deal with fuzzy sets $\Sigma^{\sim}$ of formulas, i.e. besides formulas φ also their membership degrees $\Sigma^{\sim}(\varphi)$ in $\Sigma^{\sim}$. And these membership degrees are just the truth degrees of the corresponding logic. This is an easy matter as long as the entailment relationship is considered. An evaluation e is a *model* of $\Sigma^{\sim}$ iff $\Sigma^{\sim}(\varphi) \leq e(\varphi)$ holds for each formula φ. Hence the semantic consequence hull of $\Sigma^{\sim}$ should be characterized by the membership degrees $\mathscr{C}^{\mathsf{sem}}(\Sigma^{\sim})(\psi) = \bigwedge\{e(\psi) \mid e \text{ model of } \Sigma^{\sim}\}$.

For a syntactic characterization of this entailment relation it is necessary to treat *evaluated formulas*, i.e. ordered pairs consisting of a truth degree symbol and a formula in a logical calculus $\mathbb{K}$. Also the rules of inference have to deal with evaluated formulas. Each derivation of an evaluated formula $(\overline{a}, \varphi)$ counts as a derivation of φ *to the degree* a. The *provability degree* of φ from $\Sigma^\sim$ in $\mathbb{K}$ is the supremum over all these degrees. The syntactic consequence hull of $\Sigma^\sim$ is the fuzzy set $\mathscr{C}^{\mathsf{syn}}_{\mathbb{K}}$ of formulas characterized by the membership function $\mathscr{C}^{\mathsf{syn}}_{\mathbb{K}}(\Sigma^\sim)(\psi) = \bigvee\{a \mid \mathbb{K}$ derives $(\overline{a}, \psi)$ out of $\Sigma^\sim\}$.

Already Pavelka proved soundness and completeness saying $\mathscr{C}^{\mathsf{sem}}(\Sigma^\sim) = \mathscr{C}^{\mathsf{syn}}_{\mathsf{L}}(\Sigma^\sim)$, but only for the case that the many-valued logic under consideration here is the (infinite valued) Łukasiewicz logic L. (This restriction comes from the fact that the completeness proof needs the continuity of the residuation operation.) Because the truth degree symbols have to be part of the derivations, here one needs to refer to an uncountable language with constants for all the reals of the unit interval.

Petr Hájek realized the following important facts: (i) it is sufficient to have constants for the rationals from the unit interval; (ii) instead of working with evaluated formulas one can consider implications of the forms $\overline{r} \to \varphi$ and $\varphi \to \overline{r}$; (iii) the semantic degree $\mathscr{C}^{\mathsf{sem}}(\Sigma^\sim)(\psi)$ is the infimum of all rationals r such that $\overline{r} \to \psi$ is satisfiable in all the models of $\Sigma^\sim$, and the provability degree $\mathscr{C}^{\mathsf{syn}}_{\mathsf{L}}(\Sigma^\sim)(\psi)$ is the supremum of all rationals r such that $\overline{r} \to \psi$ is provable from $\Sigma^\sim$. All together this led him to an expanded version of L, expanded by truth degree constants for the rationals from the unit interval and by corresponding bookkeeping axioms to treat these constants well, which he coined *Rational Pavelka Logic*. Hence, in a certain sense, Rational Pavelka Logic is equally powerful as the original Pavelka style extension of Łukasiewicz logic.

One may wonder what is the relationship between the Rational Pavelka Logic and other mathematical fuzzy logics, and in particular, whether Rational Pavelka Logic is conservative over Łukasiewicz logic. In the paper *Rational Pavelka Logic is a conservative extension of Łukasiewicz logic* by Hájek et al. (2000), this last question is solved affirmatively. Besides this result, it is shown that the provability degree of a formula can also be defined within the framework of Łukasiewicz logic, i.e. without truth-constants in the language.

1.7.2 Logics of Probability, of Possibility and of Belief

Already in a 1994, Hájek and Harmancová (1995) noticed that one can safely interpret a probability degree on a Boolean proposition φ as a truth degree, not of φ itself but of another (modal) formula $P\varphi$, read as "φ is *probable*". The point is that "being probable" is actually a fuzzy predicate, which can be more or less true, depending on how much probable is φ. Hence, it is meaningful to take the truth-degree of $P\varphi$ as the probability degree of φ. The second important observation is the fact that the standard Łukasiewicz logic connectives provide a proper modelling of the

Kolmogorov axioms of finitely additive probabilities. For instance, the following axiom

$$P(\varphi \vee \psi) \leftrightarrow ((P\varphi \rightarrow P(\varphi \wedge \psi)) \rightarrow P\psi)$$

faithfully captures the finite-additive property when $\rightarrow$ is interpreted by the standard Łukasiewicz logic implication. Indeed, these were the key issues that are behind the first probability logic defined as a theory over Rational Pavelka logic in the paper by Hájek, Esteva and Godo, *Fuzzy Logic and Probability* (Hájek et al. 1995). This was later described with an improved presentation in Hájek's monograph (Hájek 1998) where P is introduced as a (fuzzy) modality. Exactly the same approach works to capture uncertainty reasoning with necessity measures, replacing the above axiom by $N\varphi \wedge N\psi \rightarrow N(\varphi \wedge \psi)$. More interesting was the generalization of the approach to deal with Dempster-Shafer belief functions proposed in the paper by Godo, Hájek and Esteva, *A fuzzy modal logic for belief functions* (Godo et al. 2003). There, to get a complete axiomatization, the authors use one of possible definitions of Dempster-Shafer belief functions in terms of probability of knowing (in the epistemic sense), and hence they combine the above approach to probabilistic reasoning with the modal logic S5 to introduce a modality B for belief such that $B\varphi$ is defined as $P\Box\varphi$, where $\Box$ is a S5 modality and φ is a propositional modality-free formula. The complexity of the fuzzy probability logics over Łukasiewicz and ŁΠ logics was studied by Hájek and Tulipani (2001).

This line of research has been followed in a number of papers where analogs of these uncertainty logics have been extended over different fuzzy logics, mainly Łukasiewicz and Gödel logics, see e.g. Flaminio and Godo (2007), Flaminio et al. (2011), Flaminio and Montagna (2011), Flaminio et al. (2013). Hájek himself wrote another very interesting paper (Hájek 2007a), generalising Hájek and Tulipani (2001), about the complexity of general fuzzy probability logics defined over what he calls *suitable* fuzzy logics, i.e. logics whose standard set of truth values is the real unit interval [0, 1] and the truth functions of its (finitely many) connectives are definable by open formulas in the ordered field of reals.

1.7.3 Fuzzy Modal Logics

Another related field where Petr Hájek has made significant contributions is on the study of modal extensions of fuzzy logics and where he has also paved the way for further studies in this field. Inspired by the pioneer work of Fitting (1992a, b) on many-valued modal logic valued on finite Heyting algebras, in a 1996 conference paper with Dagmar Harmancová (Hájek and Harmancová 1996) there is already a first study of a generalization of the modal logic S5 over Łukasiewicz logic. This topic is later developed in Hájek's monograph (Hájek 1998), where he considers modal logics $S5(\mathscr{C})$, where $\mathscr{C}$ stands for any recursively axiomatized fuzzy propositional logic extending BL. The language of $S5(\mathscr{C})$ is that of fuzzy propositional calculus (the language of $\mathscr{C}$) extended by modalities $\Box$ and $\Diamond$. The semantics is given by

Kripke models of the form $K = (W, e, A)$ where W is a set of possible worlds, A is a BL-chain and $e(\cdot, w)$ is an evaluation of propositional variables in A, for each possible world $w \in W$. As usual, $e(\cdot, w)$ extends to arbitrary formulas interpreting propositional connectives by the corresponding operations in A, and to modal formulas as $\Box\varphi$ and $\Diamond\varphi$ as universal and existential quantifiers over possible worlds, that is, $e(\Box\varphi, w) = \inf_{v \in W} e(\varphi, v)$, and $e(\Diamond\varphi, w) = \sup_{v \in W} e(\varphi, v)$. This is clearly a fuzzy variant of classical S5 modal semantics with total accessibility relations. In his book Hájek (1998), Hájek proposes a set of axioms but leaves open the problem of proving its completeness. This problem is positively solved in his 2010 paper (Hájek 2010) where he relates $S5(\mathscr{C})$ to the monadic fragment $m\mathscr{C}\forall$ with just one variable (but with possibly countably-many constants) of the first order logic $\mathscr{C}\forall$, and shows that the monadic axioms of $\mathscr{C}\forall$ provide an axiomatization of $m\mathscr{C}\forall$ that is strongly complete with respect to the general semantics. In Hájek (1998) it is shown that, for $\mathscr{C}$ being Łukasiewicz (Ł) or Gödel (G) logics, $S5(\mathscr{C})$ standard tautologies coincide with the general tautologies. Therefore one gets as a direct consequence the standard completeness of the $S5(\text{Ł})$ and $S5(G)$ logics (the problem is left open for other choices of $\mathscr{C}$). In this paper Petr Hájek also considers other kinds of Kripke models, namely witnessed and interval-valued models, besides some complexity results.

Petr Hájek has also studied other systems of fuzzy (or many-valued) modal logic (Hájek et al. 1994, 1995; Hájek 2002). In particular, in Hájek et al. (1994) a logic called MVKD45 is defined to provide a modal account of a certain notion of necessity and possibility of fuzzy events. MVKD45 is developed over a finitely-valued Łukasiewicz logic Ł_k expanded with some unary operators to deal with truth-constants and its semantics is given by Kripke models of the form $K = (W, e, \pi)$, where W and e are as above (but evaluations are now over the $(k + 1)$-valued Łukasiewicz chain S_k, and $\pi : W \to S_k$ is a possibility distribution on possible worlds. This semantics can be thus considered as a many-valued variant of the classical KD45 modal semantics.

As it has happened in other areas, Hájek ideas have been the seed for further investigations on fuzzy modal logics. Particular relevant are the papers by Caicedo and Rodríguez (2010, 2012) and by Metcalfe and Olivetti (2011) on general modal logics over Gödel logics, the paper by Hansoul and Teheux (2013) on modal logics over Łukasiewicz logic, and the paper by Bou et al. (2011) on minimal modal logics over a finite residuated lattice.

1.7.4 Fuzzy Description Logic

Computer scientists in Artificial Intelligence are interested in weakened but tractable versions of first-order logics. Description Logics (DLs) (Baader et al. 2003) are knowledge representation languages particularly suited to specify formal ontologies. DLs are indeed a family of formalisms describing a domain through a knowledge base (KB) where relevant concepts of the domain are defined (terminology, TBox) and where these defined concepts can be used to specify properties of certain elements of the domain (description of the world, ABox). The vocabulary of DLs consists of *concepts*, which denote sets of individuals, and *roles*, which denote binary relations among individuals and could be interpreted both in a multi-modal system and in first order logic: concepts as formulas and roles as accessibility relations in the modal setting and concepts as unary predicates and roles as binary predicates in the first order setting. A first approach toward fuzzified versions of description logics (FDLs from now on), i.e. versions referring to fuzzy logics instead of classical logic, was introduced in several papers, for instance in Yen (1991), Tresp and Molitor (1998), Straccia (1998), Stoilos et al. (2006), Sánchez and Tettamanzi (2006), Łukasiewicz and Straccia 2008. However, the logic framework behind these initial works is very limited. The fuzzy logic context consisted essentially only of the min-conjunction, the max-disjunction, and the Łukasiewicz negation.

In his 2005 paper *Making fuzzy description logic more general* (Hájek 2005b), Petr Hájek proposes to deal with FDLs taking as basis t-norm based fuzzy logics with the aim of enriching their expressive possibilities (see also Hájek 2006a). This change of view gives rise to a wide number of choices on which a FDL can be based: for every particular problem we can consider the fuzzy logic that seems to be more adequate. As an example, Hájek studies an $\mathscr{ALC}$-style description logic as a suitable fragment of $\mathsf{BL}\forall$. He proves, e.g. that the satisfiability of a concept when taking Łukasiewicz infinite-valued logic as background logic is decidable. The proof makes use of the fact that Łukasiewicz infinite-valued logic is complete w.r.t. witnessed models and it is based on a reduction of the satisfiability problem of a concept in description logic (or modal formula) to a satisfiability problem of a family of formulas of propositional logic, which is a decidable problem. In fact the result is valid for any description logic over any axiomatic extension of BL that satisfies the witnessed axioms, which is proved to be equivalent to the finite model property. But the main interest of Hájek's work was to bring a new view into Fuzzy description logics that took advantage of the recent advances of Mathematical Fuzzy logic, giving birth to a large family of FDLs.

From then, several papers on FDLs have followed Hájek ideas, for instance, García-Cerdaña et al. (2010), Bobillo et al. (2009), Borgwardt and Peñaloza (2011), Cerami et al. (2010), García-Cerdaña et al. (2010), Cerami and Straccia (2013), Borgwardt et al. (2012).

1.7.5 Logics with Truth Hedges

Truth hedges are clauses which directly refer to the truth of some sentence like *it is very true that*, *it is quite true that*, *it is more or less true that*, *it is slightly true that*, etc. In this formulation, after Zadeh, they have been represented in fuzzy logic systems (in broad sense) as functions from the set of truth values (typically the real unit interval) into itself, that modify the meaning of a proposition by applying them over the membership function of the fuzzy set underlying the proposition. In the setting of mathematical fuzzy logic, Petr Hájek proposes in a series of three papers

Hájek (2001, 2002b), Hájek and Harmancová (2000) to understand them as truth functions of new unary connectives called *truth-stressing* or *truth-depressing hedges*, depending on whether they reinforce or weaken the meaning of the proposition they apply over. The intuitive interpretation of a truth-stressing hedge on a chain of truth-values is a subdiagonal non-decreasing function preserving 0 and 1.

In his paper *On very true* (Hájek 2001), Petr Hájek axiomatizes the truth-stresser *very true* as an expansion of BL logic (and of some of their prominent extensions like Łukasiewicz or Gödel logics) by a new unary connective vt satisfying the above mentioned conditions together with the K-axiom $vt(\varphi \to \psi) \to (vt\,\varphi \to vt\,\psi)$ and the rule of necessitation for vt. The logics he defines are shown to be algebraizable and to be complete with respect to the classes of chains of their corresponding varieties, and in the case of the logic over Gödel logic he proves standard completeness. This approach was later followed by Vychodil (2006) in order to deal with *truth depressers* as well. Finally Esteva, Godo and Noguera have given in Esteva et al. (2013) a more general approach containing as particular cases those of Hájek and Vichodyl.

1.8 Mathematical Theories Over Fuzzy Logic

Two particular elementary theories have found the interest of Petr Hájek: an axiomatic set theory FST for fuzzy sets, and formalized arithmetic.

A ZF-like axiomatic theory FST, based upon the first-order logic $\mathsf{BL}\forall\,\Delta$, is discussed by Petr and Z. Haniková in the paper *A development of set theory in fuzzy logic* (Hájek and Haniková 2003). Its first-order language has the equality symbol $=$ as a logical symbol, and $\in$ as its only non-logical primitive predicate. The axioms are suitable versions of the usual ZF-axioms together with an axiom stating the existence of the support of each fuzzy set.

A kind of "standard" model $V^{\mathbf{L}} = \bigcup_{\alpha\in\mathsf{On}} V^{\mathbf{L}}_{\alpha}$ for this theory FST is formed, w.r.t. some complete BL-chain $\mathbf{L}$, completely similar to the construction of Boolean valued models for ZF, i.e. with the crucial iteration step $V^{\mathbf{L}}_{\alpha+1} = \big\{f \in {}^{\mathsf{dom}\,(u)}L \mid \mathsf{dom}\,(u) \subseteq V^{\mathbf{L}}_{\alpha}\big\}$.

For the primitive predicate $\in$ the truth degree $[\![x \in y]\!]$ is defined as $[\![x \in y]\!] = y(x)$ for $x \in \mathsf{dom}\,(y)$ and as 0 otherwise. And $=$ has the truth degree $[\![x = y]\!] = 1$ for $x = y$ and 0 otherwise.

The main results are that the structure $V^{\mathbf{L}}$ is a model of all of the authors' axioms, and that ZF is interpretable in FST.

Another generalized set theory Petr is interested in is *Cantorian set theory* CL_0 over Łukasiewicz logic L_∞. In the background there is an older approach toward a consistency proof for naive set theory, i.e. set theory with *comprehension* and *extensionality* only, via L_∞ initiated by Skolem (1957). This approach resulted — after a series of intermediate steps mentioned e.g. in Gottwald (2001)— in a proof theoretic proof (in the realm of L_∞) of the consistency of naive set theory with *comprehension only* by White (1979) (There are doubts whether this proof is fully correct.).

In this context, Petr's goal is to study the arithmetics of natural numbers. In his paper *On arithmetic in the Cantor-Łukasiewicz fuzzy set theory* (Hájek 2005d), he finds out that this is a rather delicate matter.

Two equality predicates come into consideration here—so called Leibniz equality $x =_l y =_{def} \forall z(x \in z \leftrightarrow y \in z)$ and the usual extensional equality $x =_e y =_{def} \forall z(z \in x \leftrightarrow z \in y)$. Leibniz equality is shown to be a *crisp* predicate, but extensional equality is *not*.

CL_0 becomes *inconsistent* adding the coincidence assumption $x =_l y \leftrightarrow x =_e y$. A constant ω can be introduced to denote a suitably defined crisp set of natural numbers such that $\mathsf{CL}_0(\omega)$ is a conservative extension of CL_0. Even a weak form of induction might be added to $\mathsf{CL}_0(\omega)$ saving consistency, viz. the rule

$$\frac{\varphi(0) \quad \forall x(\varphi(x)) \leftrightarrow \varphi(S(x))}{(\forall x \in \omega)\varphi(x)}$$

for formulas φ which do *not* contain the constant ω.

This restriction on the induction formulas is crucial, however: deleting this restriction makes the system *inconsistent*.

Yet another approach toward arithmetics within mathematical fuzzy logic is offered in Petr Hájek's papers *Mathematical fuzzy logic and natural numbers* (Hájek 2007b), and *Towards metamathematics of weak arithmetics over fuzzy logic* (Hájek 2010). The starting point is a slightly modified form $Q^{\sim}$ of a weakened version Q^{-} of the Robinson arithmetic Q, designed by A. Grzegorczyk, and introducing addition and multiplication as ternary relations. Seen as an elementary theory over $\mathsf{BL}\forall$ this theory is denoted $FQ^{\sim}$. The main results are that $Q^{\sim}$ as a theory over Gödel logic (or also over intuitionistic logic) is essentially incomplete and essentially undecidable, and that $FQ^{\sim}$ is essentially undecidable too.

1.9 Petr's Failures

As noted by Matthias Baaz in the book *Witnessed years* (Cintula et al. 2009), Petr Hájek had a special skill to obtain interesting results also from his failures. Here are some examples. After he invented his logic BL, Petr tried to prove that it is standard complete, that is, that BL is complete with respect to the class of continuous t-norms and their residuals. He didn't succeed (the result was proved by Cignoli, Esteva, Godo and Torrens in the paper *Basic fuzzy logic is the logic of continuous t-norms and their residua* (Cignoli et al. 2000), but he proved something which is very close to the desired result. Namely, he proved that BL added with two axioms which are sound in any continuous t-norm algebra is standard complete. Then Cignoli, Esteva, Godo and Torrens proved that these axioms are redundant, i.e., they are provable in BL, and got the result.

Another example was Petr's attempt to extend the Mostert and Shield's decomposition of a continuous t-norm as an ordinal sum of Łukasiewicz, Gödel and product t-norms. In his paper *Basic fuzzy logic and BL-algebras* (Hájek 1998), Petr did not get the full result, but he proposed a method which was crucial in the proof of Aglianò-Montagna's decomposition of a BL-chain as an ordinal sum of MV-algebras and negative cones of abelian ℓ-groups. That is, he suggested to take a maximal decomposition, that is, a decomposition in which each component can no longer be decomposed as an ordinal sum. To conclude the proof of the Aglianò-Montagna decomposition it is sufficient to prove that any indecomposable component is either an MV-algebra or a negative cone of an abelian ℓ-group.

Finally, Petr failed to invent MTL-algebras, which are due to Esteva and Godo (2001), but he conjectured the independence of the axiom $(\varphi\&(\varphi \to \psi)) \to (\psi\&(\psi \to \varphi))$, which separates BL from MTL, as an open problem. The independence of this axiom from the other axioms of BL may have suggested the investigation of BL deprived of it (and with the obvious axioms for $\wedge$), that is, of MTL.

Finally, Petr tried to prove the redundancy of the axiom $\forall x(\varphi(x) \vee \psi) \to ((\forall x\varphi(x)) \vee \psi)$. It turned out that this axiom is not redundant, for a proof see for instance Esteva et al. (2003). However, a first-order fuzzy logic with this axiom is sound and complete with respect to its chains, while first-order fuzzy logic deprived of this axiom is sound and complete with respect to the class of its (possibly not linearly ordered) algebras.

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Chapter 2
Petr Hájek: A Scientific Biography

Zuzana Haniková

2.1 Introduction

Petr Hájek is a renowned Czech logician, whose record in mathematical logic spans half a century. His results leave a permanent imprint in all of his research areas, which can be delimited roughly as set theory, arithmetic, fuzzy logic and reasoning under uncertainty, and information retrieval; some of his results have enjoyed successful applications. He has, throughout his career, worked at the Academy of Sciences of the Czech Republic,[1] starting as a postgraduate student at the Institute of Mathematics in 1962. At present, he is a senior researcher at the Institute of Computer Science.

Petr's scientific career is well captured by the books he (co)authored:

- P. Vopěnka, P. Hájek: *The Theory of Semisets*. Academia Praha/North Holland Publishing Company, 1972.
- P. Hájek, T. Havránek: *Mechanizing Hypothesis Formation: Mathematical Foundations of a General Theory*. Springer, Berlin, 1978.
- P. Hájek, T. Havránek, M. Chytil: Metoda GUHA: automatická tvorba hypotéz, Academia, Praha, 1983. (in Czech).
- P. Hájek, T. Havránek, R. Jiroušek: *Uncertain Information Processing in Expert Systems*. CRC Press, Boca Raton, 1992.
- P. Hájek, P. Pudlák. *Metamathematics of First-Order Arithmetic*. Springer Verlag, 1993.
- P. Hájek: *Metamathematics of Fuzzy Logic*. Kluwer, Dordrecht, 1998.
- P. Cintula, P. Hájek, C. Noguera (eds.): *Handbook of Mathematical Fuzzy Logic*. College Publications, London, 2011.

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Apart from these books, Petr Hájek is the (co)author of more than 350 research papers, textbooks and popular articles; his works are frequently cited with the number of citations approaching 3,000. He taught logic at the Faculty of Mathematics and Physics, Charles University in Prague, where he was appointed full professor in 1997, and at the Faculty of Nuclear Sciences and Physical Engineering, Czech Technical University. He also taught at the Vienna University of Technology, where he was appointed honorary professor in 1994. For the timespan of four decades, he has been running a weekly seminar of applied mathematical logic, and he co-founded another seminar on mathematical logic that is still being run at the Institute of Mathematics.

He has served as a member of committees and editorial boards and has been a long-time member of the Union of Czech Mathematicians and Physicists. Since 1993, he has been a member of the Association for Symbolic Logic. During 1999–2003 he was the President of Kurt Gödel Society; he was reelected in 2009 and is currently serving his second term. Since 1996 he has been a member of the Learned Society of the Czech Republic. During 1993–2005 he was a member of the Scientific Council of the Academy of Sciences of the Czech Republic. His awards include the Bolzano medal from the Academy of Sciences in 2000, a medal of the Minister of Education of the Czech Republic in 2002, the *De scientiae et humanitate optime meritis* medal from the Academy of Sciences in 2006, the Medal of Merit from the President of the Czech Republic in 2006, the Josef Hlávka medal in 2009, and the EUSFLAT Scientific Excellence Award in 2013.

Apart from the pursuit of mathematics, Petr Hájek is an organist. He graduated from the Academy of Performing Arts in Prague and was, for a considerable period of time, organist on Sundays at the protestant St. Clemens Church in Prague; since childhood years he has been a member of the Evangelical Church of Czech Brethren. He is married, has two children and a grandson. He is fluent in several languages, including German, English, and Polish.

Petr Hájek is generally viewed as a very friendly and modest person, known for his readiness to help and listen to others. Many colleagues consider him their teacher. He is respected for his principles, not least among these, his stands during the totalitarian era, when he would not enter the Communist Party of Czechoslovakia nor cooperate with the State Security[2] when asked to. For considerable periods of time, he was prevented from advancing his career or travelling abroad.

The few above paragraphs condense Petr Hájek's life to a very modest space, collecting the highlights of his professional career. This may be sufficient for many readers. Still, in this biographical essay, I will try to offer somewhat more: to record an appropriate context for events; to mention people that Petr encountered; and to answer some why-questions. I must emphasize that, though I can contribute a knowledge of Petr based on personal acquaintance, being younger I have only met him in his "fuzzy period". Thus in the earlier periods I rely on documents and recollections of others. By nature this is a professional biography, thus it will not delve into Petr's private life.

[2] Known under the acronym 'StB'.

2.2 Early Years and Set Theory

Petr Hájek was born in Prague on February 6, 1940; after him, two girls were subsequently born into the family. His mother was a private language teacher and his father worked in *Papirografia Praha;* the family lived in the Prague quarter of Žižkov. In supplement to the usual education, Petr received a musical one: he took piano lessons in a public school of arts. The family were religious, being members of the Evangelical Church of Czech Brethren and frequenting a church near their home; it was a natural decision for young Petr to start to study the organ, with a view of, one day, being able to play it at services, thus contributing his skill to the community.

In June 1957 Petr completed his secondary education by graduating from a local high school, namely, *Jedenáctiletá střední škola v Praze, Sladkovského náměstí*.[3] At that time, Petr was deliberating his future, deciding between mathematics and music.

The final decision was to make mathematics his main pursuit, and the young Petr commenced his studies at the newly established Faculty of Mathematics and Physics of Charles University in Prague. He finished in 1962, submitting a master thesis in algebra, written under the guidance of Vladimír Kořínek, a well known algebraist. Even though Petr was an excellent student, it was out of the question for him to get a position at the Faculty: authorities declared it undesirable that a religious person such as himself have any contact with students. At that time, upon graduating from the University, students were "assigned" employment roughly in the area of study. The exact process of assignation varied, but its results were often cumbersome: it was not uncommon for Prague residents to be assigned to the outskirts of the country. This time, however, Petr was lucky: in 1962, he obtained a position at the Institute of Mathematics of the Czechoslovak Academy of Sciences. This was also the commencement of his postgraduate training, which, at that time in our country, was called *aspirantura*, and those who successfully completed it were honoured by a *candidatus scientiarum* (CSc) degree.

Petr started his studies under the guidance of Ladislav Rieger, a professor at the Czech Technical University in Prague and a distinguished logician. He introduced Petr to contemporary results in mathematical logic and recommended some essential reading. To appreciate what Rieger's agenda was like, see for example Rieger (1960). He also conducted a seminar in mathematical logic; one of the attendees was Petr Vopěnka. Unfortunately, Rieger passed away in 1963. In his essay *Prague set theory seminar* (Vopěnka 2009), Petr Vopěnka writes: "…Then [after Rieger's death], I decided to start a new seminar in axiomatic set theory, intended mainly for students. The students who enlisted were (in alphabetical order) Bohuslav Balcar, Tomáš Jech, Karel Hrbáček, Karel Příkrý, Antonín Sochor, Petr Štěpánek and some others. We were joined by Lev Bukovský from Bratislava, and, last but not least, Rieger's doctoral student, Petr Hájek. The main target of the seminar was to study non-standard models of Gödel–Bernays set theory". The seminar took place at the Faculty of Mathematics and Physics, where Vopěnka worked throughout.

[3] Currently, *Gymnázium Karla Sladkovského.*

Petr Vopěnka is often considered to have been Petr Hájek's thesis advisor. While there is no doubt that Vopěnka actually advised Petr Hájek's in many respects, and was his teacher, it was Karel Čulík who was appointed the advisor after Rieger's death. Čulík, at that time employed in the Institute of Mathematics, was an excellent mathematician with a broad scope of interests, and, like so many of his colleagues, not in grace of the authorities; he finally left Czechoslovakia in 1976 (see Hájek 2002). Petr Hájek submitted his thesis, 'Models of set theory with individuals', in 1964 (see Hájek 1965), and defended it a year later.

Subsequently to his thesis, Petr Hájek published a considerable number of papers on set theory; many of them were about the role of the axiom of foundation. Some were coauthored by colleagues from Vopěnka's seminar. Some favourite publishing options included *Commentationes Mathematicae Universitatis Carolinae*, a mathematical journal published by Charles University since 1960; *Časopis pro pěstování matematiky* ('Journal for the Fostering of Mathematics'), published by the Union of Czech Mathematicians and Physicists; or *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, where many papers of Vopěnka's group were published in English or in German. Meanwhile, Petr was not neglecting music, and continued to study the organ, under the guidance of Jaroslava Potměšilová, a distinguished Czech organist.

Vopěnka's set theory seminar was a great success: it brought together a group of young researchers[4] who shared a common topic of interest and who contributed substantially to the set-theoretical agenda of the period. Even today, there are very few students of logic in Prague who have never heard about Vopěnka's seminar and are not aware of many of the participants' contributions to mathematical logic, given in the course of their lives. Still, even though Vopěnka himself achieved lasting results in (what he refers to as) Cantor's set theory, he was rather uncomfortable with its progress. In particular, the independence results of the late nineteen sixties seemed for Vopěnka to highlight an element of arbitrariness in choosing set-theoretic axioms which was beyond his endurance (see Vopěnka 2009). Vopěnka is, primarily, a mathematician. For him, investigations of formal theories and relations inbetween them (the term 'metamathematics' is often used) is an interesting, but secondary pursuit; a formal theory does not constitute the objects that form the subject matter of mathematics, but merely tries to capture them, more or less conveniently. He has always had strong preconceptions of the universe of mathematical discourse; in particular, his concern was the phenomenon of infinity. Vopěnka's view was that Cantor's set theory was cumbersome in capturing this phenomenon, having closed many doors that should have remained open.

The Theory of Semisets, written by Petr Vopěnka and Petr Hájek (neither of the authors was fluent in English at that time and the book was translated from Czech by T. Jech and G. Rousseau), was published simultaneously by North Holland Publishing Company and by Academia in Prague in 1972 (see Vopěnka and Hájek 1972). This book is a result of an intense study of the construction of models for set theory, to which Vopěnka contributed significantly during the sixties. A *semiset* is a subclass

[4] In 1963, Petr Vopěnka was twenty-eight, and most of the attendants were undergraduate students.

of a set; the theory of semisets is formally obtained by modifying the axioms of NBG in such a way that they admit (but do not prove) the existence of proper semisets. The theory of sets extends the theory of semisets by simply positing that all semisets are sets; this extension is conservative in the sense that it does not add any new statements about sets. The book develops both theories (i.e., of semisets and of sets) along each other, exploring their mathematics and presenting many results on them, highlighting the differences. It sets great store by *interpretations* (also called 'syntactical models' in the text), typically sought as a means of obtaining relative consistency statements; interpretability later—during his arithmetic years—became the flagship of Petr Hájek's research.

Perhaps it is worth stressing at this point that, while Vopěnka and Hájek joined forces to make a significant step aside from the mainstream of research in mathematical logic, both were, at the same time, excellent and very active researchers in the classical line. Interestingly, the mindsets of these two researchers seem to be very different: with a little exaggeration, one might say that from Vopěnka's view, Petr Hájek is a formalist, whereas from Hájek's view, Petr Vopěnka is a foundationist. Looking at Petr Hájek's works, one notices that very early on he gives a set of axioms and rules; without these, it would be unthinkable to continue. In Petr Vopěnka's works, some axioms will, reluctantly and almost apologetically, be given halfway through the text. From this aspect, the book on semisets is an interesting synthesis of these two approaches operating together. Although excellently thought of and docilely written, the book never attracted a wide audience.

Some years later, Petr Vopěnka wrote another book and brought up another generation of students. This book, called 'An Introduction to Mathematics in an Alternative Set Theory' (see Vopěnka 1979), was published in Bratislava in 1979, having been translated into Slovak language by Pavol Zlatoš. While Vopěnka's alternative set theory can be seen as a continuation of some ideas present in *The Theory of Semisets*, it departs much further from the classical line and, one may say, offers a remedy to some of its alleged misconceptions. A notorious example of a semiset in alternative set theory is the collection of natural numbers n such that n grains of wheat do not form a heap; this property delimits a class within a fixed set, but the class itself is not considered a set. Perhaps this example may sketch how semisets, among other things, can model the vagueness phenomenon. Prior to this publication, Vopěnka had been running a second installment of his set-theoretic seminar, which was dedicated to developing and working in the alternative set theory. Again the seminar was very popular among its contemporaries.[5] Among the former attendants of the seminar, and researchers who contributed to the development later, one can find Karel Čuda, Josef Mlček, Jiří Sgall, Antonín Sochor, Kateřina Trlifajová, Alena Vencovská, Blanka Vojtášková and Jiří Witzany. While Vopěnka's alternative set theory is still a popular concept among Czech logicians, from a more global point of view it seems to have shared the fate of many other hitherto proposed alternatives to the mainstream conception of mathematics: it was trampled underfoot the crowd that pursued the classical direction.

[5] The first installment of Vopěnka's seminar dispersed after 1968.

A focused view of Petr Vopěnka's personality and achievements can be found in Sochor (2001), an introductory paper to a special issue of *Annals of Pure and Applied Logic* dedicated to himself.

The years spent with Vopěnka's group at the Faculty of Mathematics and Physics brought another major change into Petr's life: he met his second wife, Marie, among the people who frequented the seminar. They were married in 1969, after Petr had spent a semester visiting his colleague and lifelong friend, Gert Müller, in Heidelberg. Petr Vopěnka was a witness at the wedding. Petr cooperated with Marie and they coauthored several papers; a glimpse into their life together can be found in Hájková (2009).

2.3 Arithmetic

In the beginning of the seventies, Petr Hájek was still deeply engaged in set theory; however, he also seemed open to starting a new line of research. Alluring new topics presented themselves at that time; in particular, computational complexity was established as a new research area. A bit later, exciting new incompleteness results appeared in the form of natural combinatorial statements independent of Peano arithmetic. A first-hand account of the echoes these great currents had in Prague, and a lot more, is presented in the essay (Pudlák 2009).

During this busy period, Petr also enlisted as a student[6] at the Music Faculty of the Academy of Performing Arts in Prague, where his subject was the organ and his tutor was Jiří Reinberger, a Czech organ virtuoso, teacher and composer. Petr obtained his degree, and continued his engagement as an organist in the St. Clemens Church.

Pavel Pudlák became Petr's student in mid seventies, in particular, he wrote his master thesis under Petr's supervision, on a subject in finite model theory. The scope of Pudlák's interests was rather broad, ranging over algebra, combinatorics, and computational complexity. After some time elapsed, and some deliberation, he and Petr arrived at a decision to make arithmetic the object of their joint study, in the late seventies. Petr had had a previous acquaintance with Andrzej Mostowski in Warsaw, with whom the topic had a long tradition and around whom a working group formed itself gradually (including Zofia Adamowicz and Roman Kossak, see Adamowicz 2009). Poland is a neighbouring country and it was relatively easy to travel there; this was a lucky circumstance, owing to which Polish and Czech logicians were able to meet frequently and share knowledge.

Another person with whom Petr shared his interest in arithmetic was his wife, Marie. She was a member of Petr Vopěnka's department, and her thesis, defended in 1969, concerned binumerations of arithmetic, extending earlier results (Feferman 1960). This inspired Petr to give a course for students on the topic at the Faculty of Mathematics and Physics, in the early seventies.

In the late seventies, Petr gained another student, Vítězslav Švejdar, who was at that time working on his master thesis on interpretability; later, in 1982, he defended a

[6] Because of his employment, the form was a distance study.

dissertation 'Modal Logic and Interpretability' (see Švejdar 1982, 1983). As already remarked, interpretability was a key topic of Petr Hájek's research; Švejdar's work explored interpretability as a modality on arithmetical sentences, in a manner analogous to that of provability.

A mini-seminar on arithmetic was started in the Hájeks' flat around 1978, in which Marie also participated. Gradually a working group on arithmetic formed itself at the Institute of Mathematics; somewhat later on, this group would include Jan Krajíček (then a student of Pavel Pudlák). Shortly before 1980, a regular seminar was started at the Institute. It would meet weekly in long, lively sessions to discuss the group's own results or to present interesting papers; at the especially busy period when Hájek and Pudlák were working on *Metamathematics of First-Order Arithmetic*, reportedly two hours were not sufficient, so there were two sessions; often one was dedicated to what Petr was writing, the other occupied by topic of the attendants' choice. The seminar is still alive at the Institute of Mathematics; after Petr Hájek left, it has been run by Jan Krajíček and Pavel Pudlák for a long period of time; currently, it is run by the joint effort of Pavel Pudlák and Neil Thapen.

The arithmetic group (within the Department of Numerical Algebra, Graph Theory and Mathematical Logic, headed by Miroslav Fiedler) cooperated with other groups, especially set theorists and recursion theorists in Prague, organizing workshops in Alšovice in the Czech mountains of Jizerské hory. The workshops were quite popular, enjoying a warm, informal atmosphere; occasionally the Czech community would be able to welcome distinguished guests, such as Jeff Paris, Per Lindström, or Alex Wilkie. Otherwise, travelling options of Czech logicians, and hence also their chance of meeting researchers from abroad, were limited.

It was a great honour for logicians in Prague to be entrusted with organizing the Logic Colloquium 1980. Petr Vopěnka was appointed chair of the programme committee. Petr Hájek was chair of the organizing committee, and the whole working group at the Institute of Mathematics was involved in the preparations, alongside other Prague logicians. The preliminary list of participants counted nearly 400 heads from all over the world. Before the conference, in the spring of 1980, there was some deal of perplexity among the foreign researchers who were about to take part, regarding whether and how to express their views on the totalitarian regime then in full swing in Czechoslovakia. Particular regard was paid to Václav Benda, a Czech mathematician, a *Charter 77* signatory and the father of five small children, who was at that time imprisoned for political reasons (a so-called "prisoner of conscience"). His wife, Kamila Bendová, was a member of the logic group at the Institute of Mathematics, involved in the organization of the event. The general idea was that a focused effort of many mathematicians might help a fellow mathematician to lessen the pressure of authorities on himself. However, before these intentions were allowed to take a concrete direction, the State Security, in fear of any kind of trouble (the term "provocations" is used in their files), set things in motion so that the Colloquium had to be cancelled. Petr Hájek was obliged to personally send out letters of apology, stating a fictitious reason for cancellation. The affair hit him deeply; moreover, he was, for a time, prevented from travelling abroad.

Despite limitations in contact, Prague came to be considered an important member of the European arithmetic community; apart from the already mentioned researchers in Warsaw, the arithmetic group at the Institute of Mathematics enjoyed longterm, fruitful cooperation with Manchester (Peter Clote, Richard Kaye, Jeff Paris, Alex Wilkie), Amsterdam and Utrecht (Dick de Jongh, Rineke Verbrugge, Albert Visser), Siena (Franco Montagna) and other researchers; many people considered it worth their while to come and stay (see Baaz 2009). In the summer of 1991, Prague hosted a month-long workshop and an associated conference on proof theory, arithmetic and complexity, complementing a similar event in San Diego a year earlier; see Clote and Krajíček (1993) for papers from the meeting.

In arithmetic, Petr applied his craft especially to studying *conservativity* and *interpretability*: given that a consistent, recursively axiomatizable theory T containing arithmetic is incomplete, for each φ independent of T one may ask how conservative it is over T, and whether $T \cup \{\varphi\}$ has an interpretation in T. The notions are studied in the context of arithmetical hierarchy of formulas; particular attention is paid to fragments of arithmetic obtained by setting an upper bound on arithmetical complexity of formulas used in the induction schema. In Petr Hájek's treatment, these notions became a rather neat way of capturing the strength of theories of arithmetic. These topics are extensively covered in Petr's dissertation submitted in 1988 for the *doctor scientiarum* (DrSc) degree. The dissertation is called 'Metamathematics of First-Order Arithmetic' (Hájek 1990), and it is a direct predecessor of Petr's part of the famous book on arithmetic bearing the same title, written jointly with Pavel Pudlák a couple of years later. The dissertation is typewritten in lovely, docile Czech, with handwritten formulas and symbols. Based on this work, Petr became doctor of sciences in 1990.

Around 1990, the Ω-Group, through one of its members, Gert Müller, approached Petr Hájek with the question whether he would be willing to write a monograph on arithmetic. Petr agreed, inviting Pavel Pudlák as a coauthor. *Metamathematics of First-Order Arithmetic* was published by Springer in 1993, in the 'Perspectives in Mathematical Logic' series (Hájek and Pudlák 1993). The book has three parts. The first one investigates fragments of Peano Arithmetic obtained by bounding the arithmetical complexity of formulas used in the induction axiom, showing them sufficient for some parts of mathematics (e.g., combinatorial principles) and developing some technical tools. The second part is devoted to the incompleteness phenomenon and the study of various notions of relative strength of theories, such as the above. The third part, written by Pavel Pudlák, studies bounded arithmetic, reflecting the tumultuous development of this area during the eighties.

2.4 Logic Applied to Computer Science

A prevailing trait of Petr Hájek's personality is his strong desire to offer his service. This desire has many facets, and we shall not be exploring all of them; in this section, we shall look into Petr's efforts to offer the services of logic to other scientific disciplines, mainly computer science, and through it also to medicine, biology, humanities, etc. Characteristically, Petr was always keen to help and employ his skill in interdisciplinary research, but never willing to make one step down from the high standards on clarity and rigour that he maintained.

Very soon after he finished his postgraduate training, a challenge to apply a rather nice portion of logic presented itself. It was initiated by Metoděj Chytil from the Institute of Physiology of the Czechoslovak Academy of Sciences; he proposed some ideas that initiated the development of the General Unary Hypotheses Automaton (GUHA) method. The idea of GUHA rested in listing exhaustively all valid universally quantified implications about a given data matrix, where lines represent objects and columns represent their Boolean properties. A suggested usage was to perform an exhaustive search for valid statements on a small sample of data, thus obtaining all valid statements within reasonable time; then conceiving the "most interesting" statements as hypotheses to be tested on a larger dataset.

The authors of the method were Petr Hájek (who contributed the element of logic), Ivan Havel (who implemented the algorithm) and Metoděj Chytil; it was first presented in 1965 and published as Hájek et al. (1966). The first implementation was running on a MINSK 22 machine.

This pioneering work grounded a new area of applied research in Prague, and much effort was devoted to enhancement of the GUHA method; part of the effort naturally went to implementing and applying GUHA, and to collaborating with intended users, mainly researchers in medicine, biology, and social sciences. The word 'user' is perhaps too laden with recent connotations to convey what it was like to use the early implementation (or, one may say, any implementation) of GUHA; a small interdisciplinary team was usually needed, to collect and prepare the data, to correctly define the parameters of each run, to actually run the program, and to cope with the results.

However, GUHA also lent itself to theoretical endeavours. Obviously, if any operation on data is costly, then time can be saved with applying deduction wherever possible and refraining from testing the validity of deducible statements in the data. Petr Hájek spoke about *observational calculi*, and these form his main contribution to publications about the theoretical aspects of GUHA.

The GUHA team included Kamila Bendová from the Institute of Mathematics, Zdeněk Renc from the Faculty of Mathematics and Physics, Dan Pokorný from Mathematical centre of Biological Institute of the Czech Academy of Sciences, and many other people.

The method benefited considerably from the arrival of Tomáš Havránek on the team. Havránek was a statistician, and under his guidance, statistical quantifiers were introduced to GUHA in addition to a logical implication: moreover, he supervised the employ of the statistical paradigm in the whole approach.

Petr Hájek and Tomáš Havránek wrote a very comprehensive book about GUHA: *Mechanizing Hypothesis Formation: Mathematical Foundations of a General Theory*, published by Springer (Hájek and Havránek 1978). The book contained the full thitherto developed theory, and also many methodological and historical remarks.

A Czech book about GUHA, targeting mainly its potential users, was published by Academia in Prague in 1983 (see Hájek et al. 1983).

Petr gained two successful doctoral students in the GUHA line: Jiří Ivánek and Jan Rauch (Ivánek 1984; Rauch 1986). Both of them have retained an interest in the development of the method, and have continued their work on the method or related issues. The GUHA research continued naturally at the Institute of Computer Science, before and after Petr became its director (in 1992); perhaps we can say that this line of Petr's research played a major role in eventually bringing him into the Institute. The research group there included Anna Sochorová, Dagmar Harmancová, Jana Zvárová, Martin Holeňa and David Coufal.

GUHA never enjoyed a large-scale application or the interest of software-developing companies. Its limitations are easy to grasp: it was designed at a time and place where any kind of commercial enterprise was hardly thinkable; its theoretical aspects were too formidable for a user from a different background; it only operated on binary data; there was little demand for exploratory data analysis. However, it remained an interesting subject of study, a tool for academic applications, and a ground for interdisciplinary cooperation.

Around 1980, Petr Hájek became interested in expert systems, then very popular artificial intelligence tools. Apart from viewing expert systems as a possible application of logic and a stimulation for its development, the interest was due to a practical need for such system, to complement the existing GUHA procedures. In particular, it was hoped that such a system might guide a nonexpert user through the advanced options offered by GUHA implementations, especially its many quantifiers; the ultimate target was a fully automated GUHA. This target provided a name for the earliest version of the expert system—it was called G-QUANT ('G' for 'GUHA' and 'QUANT' for 'quantifiers').

Petr Hájek and his colleagues focused on *rule-based* systems, i.e., those using the architecture of a knowledge base and rules. A knowledge base is a set of propositions. Rules of the form $A \rightarrow S(w)$ express the fact that knowing A contributes to knowing S with some weight w. The weights are taken from a chosen set endowed with some mathematical structure, allowing for comparison and combining weights. Weights intuitively represent how *certain* the given individual is of validity of the given information. Moreover, uncertainty may be present in the form of missing information, inherent vagueness, imprecision, etc.

Dempster–Shafer theory of evidence is a generalization of Bayesian probability theory; it is based on assigning beliefs masses to subsets of events. During the eighties, Petr acted as advisor to a graduate student from Cuba, Julio Valdés. Together they undertook an algebraic analysis of the system of assignations developed by Dempster and Shafer. The structure is that of the Dempster semigroup, an ordered Abelian semigroup with the operation of Dempster's rule of belief combination; their results are collected in the dissertation (Valdés 1987). Also Milan Daniel, originally a student of Tomáš Havránek (who passed away in 1991) wrote his dissertation under the guidance of Petr Hájek (Daniel 1993). David Harmanec, Petr Hájek's doctoral student, finished his studies in the United States under supervision of George Klir.

On a practical line, Petr and his colleagues, mostly based at the Institute of Computer Science—Marie Hájková, Milan Daniel, and Tomáš Havránek—developed and implemented an expert system shell, called EQUANT, in Prolog. 'E' stands for 'empty'—the system has no fixed knowledge base, but concerns itself with combining the assigned weights and the propagation of uncertainty. The system developed over time, and several implementations existed. However, the dream did not come quite true: GUHA never became fully automated.

Theoretical issues on processing uncertainty gave rise to a book, *Uncertain Information Processing in Expert Systems*, written by Petr Hájek, Tomáš Havránek, and Ivan Jiroušek, published in 1992 by CRC Press (Hájek et al. 1992). The issues discussed in the book attracted a wider community; Ivan Kramosil, previously at the Institute of Information Theory and Automation, joined the group at the Institute of Computer Science in 1992.

In the late sixties, Petr Hájek founded a seminar to pursue the GUHA issues; it is customarily referred to as "seminar of applied mathematical logic" or simply "Hájek's seminar". The seminar would meet weekly, at first at the Faculty of Mathematics and Physics in Karlín, then in a Czech Technical University building in Albertov, later also at the Institute of Mathematics. As time passed, the scope of the seminar widened, and it attracted many people from the mathematical logic and computer science communities in Prague. It later moved with Petr to the Institute of Computer Science, and changed contents according to the shift of Petr's interests—recently, a lot of time has been devoted to fuzzy logic. The seminar is still being run by the joint efforts of Petr Hájek and Petr Cintula.

The difficulty in travelling abroad and maintaining contact with researchers from other countries perhaps contributed to bringing local and regional conferences to rather high standards. There was a lot of meetings and workshops, on regular and irregular basis; some of them grew into a tradition and are still continued nowadays. Distinguished speakers from abroad were invited where possible, and the possibility to meet them was regarded as a treat. Let us recall two of the regular events.

MFCS (Mathematical Foundations of Computer Science) is an annual conference started in 1972. The conference is organized in turns in Czech Republic, Slovakia, and Poland, in summertime; it remains a major regional event in theoretical computer science in each of these countries. Petr Hájek would be frequently a member of the programme committee of MFCS, and also a speaker there.

SOFSEM (Software Seminar) is held annually since 1974; intended for the Czechoslovak computer science community, it usually took place in the mountains in wintertime, and until 1994, a meeting would last two weeks, resembling a school more than a conference. The SOFSEM meetings had a warm, lively atmosphere and were extremely popular; at the height of their glory, they were so crowded that it was difficult to secure a place there. As time passed, the SOFSEMs grew more and more international, now being regular international conferences, held in Czech Republic or in Slovakia. Petr Hájek was invited as a speaker there several times, contributing topics discussed in this section.

In the beginning of the 1990s, big changes were in order both for Petr Hájek and for his homeland, Czechoslovakia. The country had just seen the Velvet Revolution, and the fall of the totalitarian regime had splashed away a lot of repression. Many people who had been barely tolerated by the regime, for their political stands, class origin, religious beliefs, or family ties, and consequently had been prevented from developing their careers, travelling abroad, and doing many other things that human spirit longs to do, were free at last. Petr Hájek was, to a considerable degree, such a person.

In 1991, Tomáš Havránek, director of the Institute of Computer Science and Petr Hájek's coauthor and friend, passed away at the bloom of his scientific powers. Soon after, it was proposed to Petr to consider himself a candidate for the position of director. The link to Petr consisted in his longterm engagement in the scientific agenda of the Institute. It was felt that Petr was able to contribute not only his scientific excellence on an international scale, but also an unblemished personal record; at the particular time at the particular place, the second quality was to be appreciated as much as the first one. Petr considered and accepted the idea, he was elected and appointed director of the Institute, and assumed office in March 1992.

The Institute of Computer Science[7] has an interesting history. It was established in 1975 as a General Computing Centre of the Czechoslovak Academy of Sciences, relatively well equipped to provide computing services on demand of the institutes of the Academy. During the 1980s, it was transformed into a scientific institute in its own right. At that time, and especially in the 1990s, the Institute strove to establish itself as a fully fledged academic organization. By being appointed its director, Petr Hájek became an important partaker in the effort.

With the change of political regime, it was also possible for Petr to extend his activities by starting teaching students on a regular basis. In 1993, he became associate professor at the Faculty of Mathematics and Physics, Charles University in Prague; in 1997, he was appointed full professor of mathematics there. He taught a comprehensive course in first-order logic. At the Faculty of Nuclear Sciences and Physical Engineering, he later taught fuzzy logic.[8] He also taught logic at the Vienna University of Technology, being fluent in German, and was appointed honorary professor there in 1994.

2.5 Fuzzy Logic

The monograph *Metamathematics of First-Order Arithmetic* brought both its authors a worldwide recognition. Arithmetic was a subject well in the mainstream of mathematical logic. On the other hand, fuzzy logic, even now, after a continued effort of many researchers spanning more than two decades, still seems to stand slightly in need of defence, or at least, an explanation. Petr has always been a person capable of providing very convincing explanations. We will try to retrace his path, exploring the

[7] The name 'Institute of Computer Science' was established in 1997, but for simplicity we use it also for the earlier period.

[8] It was there that he met Petr Cintula.

interaction between Petr and fuzzy logic, tracing the shift of meaning of the phrase over time.

Fuzzy logic is based on the conviction that the truth of a proposition is a matter of degree, that truth degrees of propositions can be compared, and that the truth degree of a compound proposition can be computed from those of its constituents. This leads to the concept of an algebra of truth degrees; key examples of fuzzy logics have emerged as formal deductive counterparts of some desirable algebraic semantics.

In 1965 Lotfi Zadeh introduced fuzziness in his keynote paper (Zadeh 1965), dealing with fuzzy sets. A fuzzy set was an object of classical set theory, being modelled by its characteristic function on a fixed universe, taking values in some algebra of truth degrees (typically the real unit interval endowed with suitable operations). The concept turned out to be extremely helpful in applications and also intrigued many theoretical researchers, spreading rapidly and giving rise to a fast-growing research area, perhaps best labelled 'theory of fuzzy sets' (though, quite often, the terms 'fuzzy set theory' or even 'fuzzy logic' are used to denote it).

One of the persons who pursued Zadeh's ideas on fuzziness was his doctoral student, Joseph Goguen. His paper (Goguen 1969) remains a source of inspiration for generations of readers; among other things, he distinguishes various kinds of imprecision (e.g., vagueness or ambiguity), he points out the difference between fuzziness and probability, he implicitly introduces a residuated product algebra, and he also sets the challenge to develop a formal deductive system for partially true propositions.

Zadeh's and Goguen's works on fuzziness did not pass unnoticed in the Czech Republic. First one must mention (Pultr 1976), where Aleš Pultr analyzed the concept of fuzziness mainly from a categorical point of view (as Goguen also did). Pultr's doctoral student Jan Pavelka, in his thesis defended in 1976, developed a formal deductive system of fuzzy logic introducing truth constants for elements of the algebra in the language. Pavelka was intrigued by the challenge posed by Goguen; most researchers in fuzzy logic will have heard about Pavelka's logic, as a propositional system conservatively expanding Łukasiewicz logic, allowing for inference among partially true statements, using the values from the standard Łukasiewicz algebra as labels. In fact Pavelka's work is much more comprehensive (Pavelka 1979).

Petr Hájek was the reviewer of Pavelka's thesis; thus he had, quite early on, a direct contact with results obtained in our country and the works they referred to. Many years later, in his monograph (Hájek 1998), he continued the ideas of Pavelka and designed what he called a "rational Pavelka's logic", a system expanding Łukasiewicz logic with constants for rationals within [0, 1] (thus in a countable language).

A bit later, in 1988, a somewhat similar situation recurred: Petr Hájek was the reviewer of the thesis of Vilém Novák, who, like Jan Pavelka many years before him, was a student of Aleš Pultr working on fuzzy logic in language expanded with constants. It was his endeavour to extend Pavelka's results to the first-order case.

In 1991, Gaisi Takeuti visited Prague to attend the already mentioned workshop on proof theory, arithmetic and computational complexity. It was just then that Takeuti had finished a joint paper with Satoko Titani, called *Fuzzy logic and fuzzy* *set theory* (Takeuti and Titani 1992). In this comprehensive piece of work, the terms 'fuzzy logic' and 'fuzzy set theory' acquired a new meaning: the paper contains an axiomatization (with an infinitary rule) of a Gödel logic enriched with Łukasiewicz connectives and the product conjunction, and the constant 1/2 (a predecessor of the logic $Ł\Pi\frac{1}{2}$). The appeal of this system is plain to see: it is a semantically rich logic, subsuming several other already existing systems (such as Łukasiewicz logic or Gödel logic), and it has standard completeness (at the cost of decidability). However it may be argued that the real beauty of the paper lies in the set theory developed in this logic; a first-order theory, the axioms mimicking the Zermelo-Fraenkel ones, governed by the laws of fuzzy logic. The paper leans back on well-established results on set theory in intuitionistic logic, exploiting the fact that Gödel logic is a semilinear extension thereof. Petr Hájek must have been captivated by the paper, because he later contributed both to the logic, rephrasing it in his monograph (Hájek 1998), and to the set theory, recasting the ZF-style theory into the setting of his basic logic (Hájek and Haniková 2003).

In the early 1990s, learning from others, Petr clarified to himself the traits that distinguished fuzzy logic among dozens of other approaches that could be labelled "reasoning under uncertainty"; he gradually started to clarify the distinction to others, and did so with the unrelenting determination of a true missionary. He argued that fuzzy logic, like many-valued logic, has a purely formal deductive facet; he stressed the distinction between *degrees of truth* (involving vague notions, such as 'beautiful') *degrees of belief* (involving the subject's views on potentially crisp notions), and *probability* (Hájek 1994); he ventured to seek the ties of fuzziness to natural language semantics, and to philosophical treatment of the vagueness phenomenon.

Quite importantly, Petr was not alone in his efforts: he was able to pursue some previously made bonds and acquaintances, since many researchers shared his interest in fuzzy logic. At the time, our country's boundaries were open, so it was possible to go abroad and receive guests. Petr knew Franco Montagna, Matthias Baaz, and Jeff Paris from his arithmetic years. He also enjoyed a longterm cooperation with Francesc Esteva and Lluís Godo, initiated in the early nineties. He also knew Siegfried Gottwald. He knew, and was on visiting terms with, researchers in Italy pioneering many-valued and fuzzy logic, such as Daniele Mundici, Antonio Di Nola and Giangiacomo Gerla. He was aware of Ulrich Höhle's work. Moreover, fuzzy logic had had a continuing tradition in the Czech Republic.

In mid 1990s, a group of researchers from fourteen European countries applied successfully for a COST (European Cooperation in Science and Technology) project. The project *Many Valued Logics for Computer Science Applications* was approved and initiated in 1995. The countries (managers) involved in the project were Austria (Matthias Baaz and Erich Peter Klement), Belgium (Etienne Kerre and Marc Roubens), the Czech Republic (Petr Hájek), Finland (Esko Turunen), France (Luisa Itturioz and Guy Tassart), Germany (Peter H. Schmitt and Siegfried Gottwald), Greece (Costas Drossos), Italy (Daniele Mundici and Antonio Di Nola), Poland (Ewa Orłowska and Janusz Kacprzyk), Portugal (Isabel M. A. Ferreirim), Slovakia (Radko Mesiar), Spain (Ventura Verdú Solans and Immaculada P. de Guzmán Molina), Sweden (Patrik Eklund), Turkey (Aydan M. Erkmen and Ismet Erkmen) and the United

Kingdom (Dov Gabbay and Hans Jürgen Ohlbach). The scope of the grant was rather broad; however, among other things, for the 5 years of its duration, it continued to promote cooperation among European researchers who focused on fuzzy logic as a rigorous mathematical discipline. This grant was a milestone in that it established the fuzzy logic community in Europe (however vaguely defined and subject to change in time); in analogously broad terms, a major part of the agenda of this group of researchers can be (and is, nowadays) labelled *mathematical fuzzy logic*. Within the community, many loose ends were tied together, many different perspectives united, and fuzzy logic saw a rapid development, with close ties to already existing many-valued logics, residuated lattices, intuitionistic theories, philosophy of vagueness, and other areas.

Starting in 1992, Petr Hájek served two four-year terms as director of the Institute of Computer Science. He did not mitigate his research during the period of his appointment; quite on the contrary. After an initial phase of searching and sorting the territory, the mid nineties saw him developing a new formal system, intended to capture the logic of continuous t-norms and their residua. This system, since it was a common fragment of some already existing logics describing particular examples of continuous t-norms, was named the 'basic logic' (abbreviated BL). At the time, it may have indeed seemed basic and rather weak; nowadays, when both Petr and his peers have delved much deeper and brought to light many weaker systems, the term 'basic logic' (even 'basic fuzzy logic') seems a bit awkward, so many people choose to call it 'Hájek's basic logic'.

The monograph *Metamathematics of Fuzzy Logic* was published in 1998, the fourth volume of the 'Trends of Logic' series of Kluwer Academic Publishers (Hájek 1998). It offered a thorough development of the basic logic BL (propositional and first-order), which provided the subject of fuzzy logic with a much needed formal treatment meeting the standards of a subarea of mathematical logic. The book also includes an explanation of how these results project back to applications and some neighbouring areas. The monograph was a product of several years' continued effort, evolving from lecture notes for tutorials given on the new and captivating topic. It roughly marks the end of an era that can be viewed as pioneering work in mathematical fuzzy logic for Petr Hájek. The next decade would see mathematical fuzzy logic in full bloom.

Other books on closely related topics emerged at about the same time as Petr's monograph. To start with, Siegfried Gottwald published the English translation (Gottwald 2001) of his earlier monograph in German. Roberto Cignoli, Itala M. L. D'Ottaviano, and Daniele Mundici wrote a book on MV-algebras (Cignoli et al. 1999). Vilém Novák, Irina Perfilieva, and Jiří Močkoř prepared a book covering the evaluated-syntax approach of the group (Novák et al. 2000).

In 2000 Petr's term in office as director of the Institute of Computer Science elapsed; his successor was Jiří Wiedermann. Petr was appointed head of the Department of Theoretical Computer Science, a position he held for several years. Currently, he holds the position of a senior researcher.

A publication of a monograph is a good step in spreading the knowledge and involving other people in the topic. With the publication of Hájek's book and some of the above, more people became involved in fuzzy logic: Petr was active in evangelizing people, gaining the attention of some of his former colleagues in arithmetic for example. Jeff Paris joined efforts with Petr in several papers about fuzzy logic, and Franco Montagna made fuzzy logic his primary research topic. Moreover, a group of students gradually formed around Petr: these included Petr Cintula, Rostislav Horčík, Libor Běhounek and myself; a working group on fuzzy logic was formed. I wrote a dissertation under Petr's supervision (Haniková 2004) and another one (Cintula 2005) appeared a year later. Together with people already working with Petr, such as Ivan Kramosil, Dagmar Harmancová, Peter Vojtáš, Martin Holeňa, Milan Daniel, and some regular visitors, such as Mirko Navara, we saw some very active years, meeting at the seminar of applied mathematical logic, going to conferences, reading papers, and broadening our perspective. Importantly, we were also more and more able to recognize the role of fuzzy logic among other nonclassical logics, in the philosophy of vagueness, as a ground for developing fuzzy mathematics, etc.

The first decade of the new millennium has also been a marked success for mathematical fuzzy logic on an international scale. Though still not quite accepted by the mainstream of mathematical logic, the discipline attracted the attention of more and more researchers, including those who did not work in it, but saw it as relevant for or related to their research. Many young people became involved. In particular, Prague continued the fruitful cooperation with the Barcelona group and with the Vienna group, and with many researchers in Italy. A *MathFuzzLog* working group of EUSFLAT has been established in 2007. The amount of results gathered by the community through the decade called for a new book that would encompass all the new material. In fact, several books were published, but, from the point of view of Petr Hájek, a key moment was the decision to prepare not another monograph, but a handbook with chapters written by people who closely pursued the particular subareas. The *Handbook of Mathematical Fuzzy Logic* was edited by Petr Cintula, Petr Hájek and Carles Noguera; eleven chapters were agreed upon, roughly covering the main areas, and authors started writing their chapters around the middle or 2009. The book was published in 2011, comprising nearly one thousand pages. Apart from editorship, Petr coauthored the introductory chapter and the chapter on arithmetical complexity of fuzzy logics. The main import of this book is that it collects current knowledge in key areas of mathematical fuzzy logic, offering it to interested readers.

2.6 Sources and Acknowledgements

Some of Petr's older papers may be available online through Czech Digital Mathematics Library at www.dml.cz. His full bibliography is maintained by the library of Institute of Computer Science, and is also available online.

In 2009, the volume *Witnessed Years: Essays in Honour of Petr Hájek*, dedicated to Petr Hájek on the occasion of his 70th birthday, was edited by Petr Cintula, Vítězslav Švejdar and myself and published by College Publications. Many of Petr's friends and colleagues contributed and the book contains a lot of information about Petr and his scientific interests.

I am indebted to a number of people for their willingness to share their recollections with me, and for finding time to actually do so; without them, writing this biography would not have been possible. They include (in no particular order) Dagmar Harmancová, Pavel Pudlák, Vítězslav Švejdar, Kamila Bendová, Milan Daniel, Petr Cintula, Aleš Pultr, Miroslav Tůma, Petr Vopěnka, Jiří Ivánek, Marie Hájková, Daniele Mundici, and Franco Montagna. Moreover, a few people read drafts of this text and suggested many improvements; these include Jirka Hanika, Vítězslav Švejdar, Milan Daniel, Miroslav Tůma, and Daniele Mundici. Our librarian, Ludmila Nývltová, has been miraculous in retrieving literature (especially various people's dissertations) and other information. Last but not least, Petr Hájek has borne the fact that his biography is being written, and my repeated questioning him, with a degree of patience usually only found in saints, and he was so kind as to read a draft of the biography as well. Shortcomings in the text remain, of course, my own.

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Part II
Foundational Aspects of Mathematical Fuzzy Logic

This part is devoted to the foundational aspects of Mathematical Fuzzy Logic, and discusses problems like: *How can we apply Fuzzy Logic to the foundations of mathematics, or Is it possible to treat vagueness inside Mathematical Fuzzy Logic*. The papers *Set theory and arithmetic in fuzzy logic*, by Libor Běhounek and Zuzana Haniková and *The logic of fuzzy set theory: a historical approach*, by Siegfried Gottwald discuss the first aspect, namely Hájek's contribution to foundations of mathematics inside Fuzzy Logic. As it might be expected, Set Theory plays a basic role in the foundations of mathematics, even in the context of many-valued logic. It may be understood in several different ways: for instance, one might investigate the usual set theory, but with a many-valued background logic, for instance, MTL, instead of classical logic. Alternatively, one might study set theories, which are inconsistent in a classical context (like Cantor's naive set theory), but become consistent when based on many-valued logic. Finally, one may investigate Zadeh's fuzzy sets. Clearly, these approaches are related to each other, but there are also differences: for instance, fuzzy sets might also be invesigated inside classical logic.

For a discussion about the relationship between Fuzzy Sets and Fuzzy Logic, I warmly invite the reader to consult Gottwald's chapter *The logic of fuzzy set theory: a historical approach*, in which the history of fuzzy sets and their relationship with fuzzy logic is widely discussed, and Hájek's contribution is explained in detail. The chapter also touches on another interesting aspect of fuzzy logic, namely, Giles' intepretation in terms of games.

The chapter *Set theory and arithmetic in fuzzy logic*, by Běhounek and Haniková investigates the other approach, namely axiomatic set theories with a many-valued logic as a background logic. Hájek considered set theories over BL∀, while the authors work in the more general logic MTL∀.

But in my opinion, the most interesting part of this chapter is the investigation of theories, like arithmetic plus a truth predicate, or Cantor's naive set theory, which are classically inconsistent, but are (probably)[1] consistent if the background logic is not classical, but many-valued.

The chapter *Bridges Between Contextual Linguistic Models of Vagueness and TNorm Based Fuzzy Logic*, by Christian G. Fermüller and Christoph Roschger, focuses on another foundational aspect of fuzzy logic, namely the treatment of vagueness. This problem was the source of interesting discussions between researchers from Fuzzy Logic and philosophers, and perhaps it would deserve a whole chapter.

Philosophers and linguists observed that truth degrees are not sufficient for a satisfactory treatment of vagueness, and proposed some alternative approaches. The chapter constitutes a bridge between the approaches proposed by linguists and fuzzy logic, and shows that fuzzy sets can be extracted systematically from the meaning of predicates in a given context and that one can reconstruct a corresponding degree-based semantics of logical connectives in various ways. In particular, the three fundamental t-norms, Łukasiewicz t-norm, minimum, and product, naturally appear in different ways as limits of degrees extracted from contexts.

[1] As the authors remark in this chapter, Terui found an error in White's consistency proof of Cantor's naive set theory over Łukasiewicz logic.

Chapter 3
The Logic of Fuzzy Set Theory: A Historical Approach

Siegfried Gottwald

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3.1 Introduction

The notion of fuzzy set is a technical tool to mathematically grasp the use and the effect of "vague" notions, more precisely: of not sharply delimited notions, in a manner completely different from the way classical mathematics is treating them, if unavoidable: viz. by "precisifying" them into crisp notions. Formally, each fuzzy set A is a fuzzy subset of a given universe of discourse U, characterized by its membership function $\mu_A : U \to [0, 1]$. The value $\mu_A(x)$ is the membership degree of x with respect to the fuzzy set A.

Fuzzy sets have been introduced into the mathematical discourse in 1965 in a paper Zadeh (1965) by the US-American system scientist Lotfi A. Zadeh. The intention came from applications, particularly from ideas related to the modeling of large scale systems, as explained e.g. in the historical study Seising (2007).

In parallel, and independent of the approach by Zadeh, the German mathematician Dieter Klaua presented two versions Klaua (1965, 1966b) for a cumulative hierarchy

Modified and extended version of a chapter which was published in the volume "Witnessed Years" honoring Petr Hájek's 70th birthday.

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of so-called *many-valued* sets.[1] These many-valued sets had the fuzzy sets of Zadeh as a particular case.

Historically, Zadeh's approach proved to be much more influential than that of Klaua, so we adopt the name fuzzy sets for both these types of objects here.

This chapter intends to sketch the way which led from the introduction of these kinds of non-traditional sets to the development of logics particularly designed to serve as suitable logics to develop the theory of fuzzy sets.

This has not been an obvious development. Even philosophically oriented predecessors of Zadeh in the discussion of vague notions, like Black (1937) and Hempel (1939), did refer only to classical logic, even in those parts of these papers in which they discuss the problem of some incompatibilities of the naively correct use of vague notions and principles of classical logic, e.g., concerning the treatment of negation-like statements.

3.2 The "Fuzzy Sets" of Zadeh

As Zadeh introduced fuzzy sets in his seminal paper Zadeh (1965) he essentially did not relate them, or at least their suitable treatment, to non-classical logics. There was, however, a minor exception: in discussing the meaning of the membership degrees he mentioned—in a "comment" pp. 341–342; and with reference to Kleene's monograph Kleene (1952) and Kleene's three valued logic—with respect to two thresholds $0 < \beta < \alpha < 1$ that one may interpret the case $\mu_A(x) \geq \alpha$ as saying that x belongs to the fuzzy set A, that one may interpret $\mu_A(x) \leq \beta$ as saying that x does not belong to the fuzzy set A, and leaving the case $\beta < \mu_A(x) < \alpha$ as an indeterminate status for the membership of x in A.

This indicates a certain internal three-valuedness of the considerations on fuzzy sets, a topic which essentially remained hidden up to now.

Nevertheless, the overwhelming majority of fuzzy set papers that followed Zadeh (1965) and the other early Zadeh papers on fuzzy sets treated fuzzy sets in the standard mathematical context, i.e. with an implicit reference to a naively understood classical logic as argumentation structure.

Formally, however, it was important that Zadeh not only proposed to define union $A \cup B$ and intersection $A \cap B$ of fuzzy sets A, B by the well known formulas

$$\mu_{A \cup B}(x) = \max\{\mu_A(x), \mu_B(x)\}, \tag{3.1}$$

$$\mu_{A \cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}, \tag{3.2}$$

[1] The German language name for these objects was "mehrwertige Mengen". The stimulus for these investigations came from discussions following a colloquium talk which Karl Menger had given in Berlin (East) in the first half of the 1960s. (Personal communication to this author by D. Klaua.)

but that he also introduced in Zadeh (1965) other operations for fuzzy sets, called "algebraic" by him, as, e.g., an algebraic product AB and an algebraic sum $A + B$ defined via the equations

$$\mu_{AB}(x) = \mu_A(x) \cdot \mu_B(x), \tag{3.3}$$

$$\mu_{A+B}(x) = \min\{\mu_A(x) + \mu_B(x), 1\}. \tag{3.4}$$

The core point here is that it is mathematically more or less obvious that these two additional operations are particular cases of further generalized intersection and union operations for fuzzy sets besides the "standard" versions (3.2) and (3.1), and that they are non-idempotent operations.

3.2.1 Relating the Zadeh Approach to Non-classical Logics

It was Goguen who, starting only from Zadeh's approach, was the first to emphasize an intimate relationship to non-classical logics. In his 1969 paper Goguen (1968–69), he considers membership degrees as generalized truth values, i.e. as truth degrees. Additionally he sketches a "solution" of the sorites paradox, i.e. the heap paradox, using—but only implicitly—the ordinary product $*$ in $[0, 1]$ as a generalized conjunction operation. Based upon these ideas, and having in mind suitable analogies to the situation for intuitionistic logic, he proposes completely distributive lattice ordered monoids, called *closg*'s by him, enriched with a (right) residuation operation $\rightarrow$ characterized by the well known adjointness condition

$$a * b \leq c \quad \Leftrightarrow \quad b \leq a \rightarrow c, \tag{3.5}$$

and with the "implies falsum"-negation, as suitable structures for the membership degrees of fuzzy sets. He introduces in this context the notion of tautology, with the neutral element of the monoid as the only designated truth degree. He defines a graded notion of inclusion in the same natural way as Klaua (3.6) did, of course with the residual implication $\rightarrow$ instead of the implication $\rightarrow_{\mathrm{L}}$ of the Łukasiewicz systems. But he does not mention any results for this graded implication.

Additionally, because of an inadequate understanding of logical calculi, he does not see a possibility to develop a suitable formalized logic of closg's, as may be seen from his statement:

> Tautologies have the advantage of independence of truth set, but no list of tautologies can encompass the entire system because we want to perform calculations with degrees of validity between 0 and 1. In this sense the logic of inexact concepts does not have a *purely* syntactic form. Semantics, in the form of specific truth values of certain assertions, is sometimes required.

So, the question was what structural consequences the acceptance of definitions like (3.3) and (3.4) would have for generalized intersections and unions. A particular, somehow "reverse" question was which structural conditions, besides (3.2) and (3.1), could eliminate such generalizations. An answer to this "reverse" question was given in Bellman and Giertz (1973), cf. also Gaines (1976): some rather natural "boundary conditions" together with the inclusion maximality of the standard intersection w.r.t. each other generalized intersection, with the inclusion minimality of the standard union w.r.t. each other generalized union, with commutativity and associativity, and with the mutual distributivity of the generalized union and intersection force a restriction to the "standard" case (3.2) and (3.1).

However, the set of all these structural restrictions from Bellman and Giertz (1973) seems to be very restrictive, and hence it did not really look convincing. Therefore the restriction to the "standard" operations (3.2) and (3.1) was never accepted by the majority of the mathematically oriented people of the fuzzy community.

As a consequence, a group of authors, a lot of them from the Spanish fuzzy community, discussed what might be suitable choices of such "fuzzy" connectives which might be used to define unions and intersections for fuzzy sets different from (3.2) and (3.1). One of the leading ideas in their considerations was to look at the types of restrictive conditions discussed in Bellman and Giertz (1973) as functional equations or functional inequalities, to reduce this set of functional conditions, to look also at other conditions, and to discuss the solutions of suitable sets of such functional conditions. The paper Alsina et al. (1983) is a typical example, its focus is on pairs of generalized conjunctions and disjunctions. Other papers, with emphasis on generalized implication operations are, e.g., Trillas and Valverde (1985) and Bandler and Kohout (1980).

Almost from the very beginning it was, however, clear from the mathematical point of view that set-algebraic operations for fuzzy sets can be reduced, in a many-valued setting, to generalized connectives in essentially the same way as standard set-algebraic operations for crisp, i.e. classical sets can be reduced to connectives of classical logic.

3.3 The "Many-Valued Sets" of Klaua

In Klaua's two versions Klaua (1965, 1966b) for a cumulative hierarchy of fuzzy sets he considered as membership degrees the real unit interval $\mathscr{W}_\infty = [0, 1]$ or a finite, m-element set $\mathscr{W}_m = \left\{\frac{k}{m-1} \mid 0 \leq k < m\right\}$ of equidistant points of [0, 1]. He also started his cumulative hierarchies from sets U of urelements. The infinite-valued case with membership degree set $\mathscr{W}_\infty = [0, 1]$ gives, in both cases, on the first level of these hierarchies just the fuzzy sets over the universe of discourse U in the sense of Zadeh.

So it is reasonable to identify the many-valued sets of Klaua with the fuzzy sets of Zadeh, as shall be done further on in this chapter.

Furthermore Klaua understood the membership degrees as the truth degrees of the corresponding Łukasiewicz systems L_∞ or L_m, respectively.

The first one of these hierarchies, presented in Klaua (1965, 1967) in 1965, offered an interesting simultaneous definition of a graded membership and a graded equality predicate, but did not work well and was almost immediately abandoned. The main reason for this failure, cf. Gottwald (2010), was that the class of objects that was intended to act as many-valued sets was not well chosen.

The second one of these hierarchies, presented in 1966 in Klaua (1966a, b), had as its objects A functions into the truth degree set $\mathscr{W}$, the values $A(x)$ being the membership degrees of the object x in the generalized set A.

Therefore the 1966 approach by Klaua offered immediately the Łukasiewicz systems of many-valued logic as the suitable logics to develop fuzzy set theory within their realm.

And indeed, the majority of results in Klaua (1966a, b) were presented using the language of these Łukasiewicz systems. Some examples are:

$$\begin{aligned} &\models \quad A \subseteqq B \;\&\; B \subseteqq C \rightarrow_{\mathsf{L}} A \subseteqq C, \\ &\models \quad a\, \varepsilon\, B \;\&\; B \subseteqq C \rightarrow_{\mathsf{L}} a\, \varepsilon\, C, \\ &\models \quad A \equiv B \;\&\; B \subseteqq C \rightarrow_{\mathsf{L}} A \subseteqq C. \end{aligned}$$

Here $\rightarrow_{\mathsf{L}}$ is the Łukasiewicz implication, & the strong (or: arithmetical) conjunction with truth degree function $(u, v) \mapsto \max\{0, u + v - 1\}$, ε the graded membership predicate, and $\models \varphi$ means that the formula φ of the language of Łukasiewicz logic is logically valid, i.e. assumes always truth degree 1.

A graded inclusion relation $\subseteqq$ is defined (for fuzzy sets of the same level in the hierarchy) as

$$A \subseteqq B \;=_{\text{def}}\; \forall x (x\, \varepsilon\, A \rightarrow_{\mathsf{L}} x\, \varepsilon\, B), \tag{3.6}$$

and a graded equality $\equiv$ for fuzzy sets is defined as

$$A \equiv B \;=_{\text{def}}\; A \subseteqq B \wedge B \subseteqq A. \tag{3.7}$$

These are prototypical examples for fuzzy, i.e. graded relationships which appear quite naturally in a fuzzy sets context.

This line of approach was continued in the early 1970s, e.g., in this author's papers Gottwald (1974, 1976). The topic of Gottwald (1974) is the formulation of (crisp) properties of fuzzy relations. The natural continuation, to consider graded properties of fuzzy relations, was realized for the particular cases of the graded uniqueness of fuzzy relations and the graded equipollence of fuzzy sets in Gottwald (1980). A more general approach toward graded properties of fuzzy relations was sketched in the 1991 paper Gottwald (1991).

The topic of Gottwald (1976) was the formulation of generalized versions of the standard ZF axioms valid in a modified version of Klaua's second hierarchy of fuzzy sets. All this happened in the context of the Łukasiewicz logics, even if there was a kind of vague awareness that only few properties of these Łukasiewicz systems really had to be used.

3.4 A Betting Approach

Another author who pointed out a strong relationship between fuzzy sets and many-valued logic is Giles. Starting in 1975, he proposed in a series of papers Giles (1975, 1976, 1979), and again in Giles (1988), a general treatment of reasoning with vague predicates by means of a formal system based upon a convenient dialogue interpretation. This dialogue interpretation he had already used in other papers, like Giles (1974), dealing with subjective belief and the foundations of physics. The main idea is to let "a sentence represent a belief by expressing it tangibly in the form of a bet". In this setting then a "sentence ψ is considered to follow from sentences $\varphi_1, \dots, \varphi_n$ just when he who accepts the bets $\varphi_1, \dots, \varphi_n$ can at the same time bet ψ without fear of loss".

The (formal) language obtained in this way is closely related to Łukasiewicz's infinite-valued logic L_∞: in fact the two systems coincide if one assigns to a sentence φ the truth value $1 - \langle\varphi\rangle$, with $\langle\varphi\rangle$ for the risk value of asserting φ. And he even adds the remark "that, with this dialogue interpretation, Łukasiewicz logic is exactly appropriate for the formulation of the 'fuzzy set theory' first described by Zadeh Zadeh (1965); indeed, it is not too much to claim that L_∞ is related to fuzzy set theory exactly as classical logic is related to ordinary set theory".

3.5 Invoking T-Norms

It the beginning 1980s it became common use in the mathematical fuzzy community to consider t-norms as suitable candidates for connectives upon which generalized intersection operations for fuzzy sets should be based, see (Alsina et al. 1980; Dubois 1980; Prade 1980) or a bit later (Klement 1982; Weber 1983). These t-norms, a shorthand for "triangular norms", first became important in discussions of the triangle inequality within probabilistic metric spaces, see (Schweizer and Sklar 1983; Klement et al. 2000). They are binary operations in the real unit interval which make this interval into an ordered abelian monoid with 1 as unit element of the monoid.

The most basic examples of t-norms for the present context are the Łukasiewicz t-norm T_{L}, the Gödel t-norm T_{G}, and the product t-norm T_P defined by the equations

$$T_{\mathsf{L}}(u, v) = \max\{u + v - 1, 0\},$$
$$T_{\mathsf{G}}(u, v) = \min\{u, v\},$$
$$T_P(u, v) = u \cdot v.$$

The general understanding in the context of fuzzy connectives is that t-norms form a suitable class of generalized conjunction operators.

For logical considerations the class of left-continuous t-norms is of particular interest. Here left-continuity for a t-norm $T : [0, 1]^2 \to [0, 1]$ means that for each $a \in [0, 1]$ the unary function $T_a(x) = T(a, x)$ is left-continuous. The core result, which motivates the interest in left-continuous t-norms, is the fact that just for left-continuous t-norms $*$ a suitable implication function, usually called R-implication, is uniquely determined via the adjointness condition (3.5). Suitability of an implication function here means that it allows for a corresponding sound detachment, or *modus ponens* rule: to infer a formula ψ from formulas $\varphi \to \psi$ and φ *salva veritate*. In the present context this means for the truth degrees the inequality $[\![\varphi]\!] * [\![\varphi \to \psi]\!] \leq [\![\psi]\!]$ and hence it means the logical validity

$$\models \varphi \,\&\, (\varphi \to \psi) \to \psi. \tag{3.8}$$

It was almost immediately clear that a propositional language with connectives $\wedge, \vee$ for the truth degree functions min, max, and with connectives $\&, \to$ for a left-continuous t-norm T and its residuation operation offered a suitable framework to do fuzzy set theory within—at least as long as the complementation of fuzzy sets remains out of scope.

With this limitation, i.e. disregarding complementation, this framework offers a suitable extension of Zadeh's standard set-algebraic operations.

Additionally, this framework, with the "implies falsum" construction, yields a natural way to define a negation, i.e. to introduce a t-norm related complementation operation for fuzzy sets, via the definition $-_T\varphi =_{\text{def}} \varphi \to \overline{0}$ using a truth degree constant $\overline{0}$ for the truth degree 0. However, this particular complementation operation does not always become the standard complementation of Zadeh's approach.

This t-norm based construction gives the infinite-valued Łukasiewicz system L_∞ if one starts from the t-norm T_{L}, and thus the right negation for Zadeh's complementation. This construction gives the infinite-valued Gödel system G_∞ if one starts with the t-norm T_{G}, and it gives the product logic Hájek et al. (1996) if one starts with the t-norm T_P. The "implies falsum" negations of the latter two systems coincide, but are different from the negation operation of the Łukasiewicz system L_∞. So these two cases do not offer Zadeh's complementation. But this can be reached if one adds the Łukasiewicz negation to these systems, as done in Esteva et al. (2000).

It was essentially a routine matter to develop this type of t-norm based logic to some suitable extent, as was done 1984 in this author's paper Gottwald (1984). Also the development of fuzzy set theory on this basis did not offer problems, and it was done in Gottwald (1986), including essential parts of fuzzy set algebra, some fuzzy relation theory up to a fuzzified version of the Szpilrajn order extension theorem, and some solvability considerations for systems of fuzzy relation equations (All these considerations have later been included into the monograph (Gottwald 1993)).[2]

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[2] *Personal reminiscence*: In these papers appears the notion of a φ-operator of a t-norm (so in Gottwald (1984, 1986, 1993) called Φ-operator instead). I learned this notion from Pedrycz. He used it in his PhD work on fuzzy relation equations. In Pedrycz (1985) it is called Ψ-operator, and in Pedrycz (1983) a particular case appears as τ-operator.

Clearly this was a suitable implication operation, and it is just the R-implication for the given t-norm. But in that time I was unaware of the equivalent characterizability of the φ-operator by the adjointness condition (3.5).

There is, however, also another way to develop t-norm based logics for fuzzy set theory. This way avoids the introduction of the R-implications via the residuation operation—and so it does not need the restriction to left-continuous t-norms. Instead it uses additionally negation functions, i.e. unary functions $N : [0, 1] \to [0, 1]$ which are at least order reversing and satisfy $N(0) = 1$ as well as $N(1) = 0$. The strategy to introduce an implication function $I_{T,N}$ in this setting is to define

$$I_{T,N}(u, v) = N(T(u, N(v))). \tag{3.9}$$

The implication connectives defined in this way usually are called S-implications. A prominent paper which studies this type of approach is Butnariu et al. (1995).

But the fact that S-implications do not necessarily satisfy (3.8) means that the corresponding rule of detachment is not always correct, i.e. does not guarantee inferences *salva veritate*. And this seems to be the main reason that this type of approach never became popular among logicians interested in fuzzy set matters.

3.6 Logics of T-Norms

What was missing in all the previously mentioned approaches toward a suitable logic for fuzzy set theory, as long as this logic should be different from the infinite-valued Łukasiewicz system L_∞ or from the infinite-valued Gödel system G_∞,[3] that was an adequate axiomatization of such a logic. All these approaches offered interesting semantics, but did not provide suitable logical calculi—neither for the propositional nor for the first-order level.

The first proposal to fill in this gap was made by Höhle (1994, 1995, 1996) who offered his *monoidal logic*. This common generalization of the Łukasiewicz logic L_∞, the intuitionistic logic, and Girard's integral, commutative linear logic Girard (1987) was determined by an algebraic semantics, viz. the class $\mathsf{M\text{-}alg}$ of all integral residuated abelian lattice-ordered monoids with the unit element of the monoid, i.e. the universal upper bound of the lattice, as the only designated element. So this monoidal logic was determined by a particular subclass of Goguen's closg's, indeed by a variety of algebras. And adequate axiomatizations for the propositional as well as for the first-order version of this logic were given in Höhle (1994, 1996).

Of course, this monoidal logic had the whole matter of the relationship of fuzzy set theory and the t-norm basedness of their set-algebraic operations in the background. But it was not really strongly tied with this background.

3.6.1 The Logic of all Continuous T-Norms

The use of t-norm based logics in fuzzy set theory, particularly those ones based upon left-continuous t-norms, happened throughout the 1980s and beginning 1990s in a

[3] In 1996 the product logic Hájek et al. (1996) was added to this list.

naive way: there was only the naive semantics available, but in general any logical calculus was missing.

To discuss the case of a single corresponding logic based upon an arbitrary left-continuous t-norm seemed to be a very hard problem.

Different from Höhle's quite general approach, and guided by the idea that it would be sufficiently general to restrict the considerations to the case of continuous t-norms, instead of allowing also non-continuous but left-continuous ones, it was the idea of Petr Hájek to ask for the *common part* of all those t-norm based logics which refer to a continuous t-norm: in short, to ask for the logic of all continuous t-norms.

This logic was called *basic logic* by Hájek, later he used also *basic fuzzy logic* or *basic t-norm logic*.[4] This logic is usually denoted BL. It is based upon an algebraic semantics.

There are two crucial observations which pave the way to the original, and still mainly used algebraic semantics for BL. The first one is that for any t-norm $*$ and their residuation operation $\rightarrow$ one has

$$(u \rightarrow v) \vee (v \rightarrow u) = 1, \tag{3.10}$$

with $\vee$ to denote the lattice join here, i.e. the max-operation for a linearly ordered carrier. This *prelinearity* condition (3.10) is a first restriction on the variety M-alg which determines the monoidal logic, and it yields the variety MTL-alg of all MTL-algebras—now with $*$ denoting the semigroup operation.

Moreover, by the way, if this condition is imposed upon the Heyting algebras, which form an adequate algebraic semantics for intuitionistic logic, the resulting class of prelinear Heyting algebras is an adequate algebraic semantics for the infinite-valued Gödel logic.

The second observation is that the continuity condition can be given in algebraic terms: for any t-norm $*$ and its residuum $\rightarrow$ one has that the *divisibility* condition

$$u * (u \rightarrow v) = u \wedge v \tag{3.11}$$

is satisfied if and only if $*$ is a continuous t-norm, see Höhle (1995). Condition (3.11), again with $*$ denoting the semigroup operation and $\wedge$ the lattice meet, is the second restriction here. The subclass of all those algebras from MTL-alg which satisfy this divisibility condition (3.11) is the subvariety BL-alg of all BL-algebras.

Hájek characterized his basic fuzzy logic by this class BL-alg as algebraic semantics—again with the universal upper bound of the lattice as the only designated element. And he gave adequate axiomatizations for the propositional version BL as well as for the first-order version BL∀ of this basic fuzzy logic in his highly influential monograph Hájek (1998).

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4 In the fuzzy logic community e.g. also "Hájek's basic logic" is in use. The simple name "basic logic" has a certain disadvantage because it is also in use in a completely different sense: as some weakening of the standard system of intuitionistic logic, e.g. in (Ardeshir and Ruitenburg 1998; Ruitenburg 1998).

Despite the fact—caused by the properties of the R-implications—that the set $\mathscr{E}$ of equations, which characterizes the algebras of the variety $\mathsf{BL\text{-}alg}$ as the model class of $\mathscr{E}$, could routinely be rewritten as a set $\mathscr{E}^*$ of implications such that $\mathsf{BL\text{-}alg}$ is also the model class of $\mathscr{E}^*$, Petr Hájek offered in Hájek (1998) a much shorter and considerably more compact axiomatic basis for the propositional system BL:

($\mathsf{Ax_{BL}}1$) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$,
($\mathsf{Ax_{BL}}2$) $\varphi \,\&\, \psi \to \varphi$,
($\mathsf{Ax_{BL}}3$) $\varphi \,\&\, \psi \to \psi \,\&\, \varphi$,
($\mathsf{Ax_{BL}}4$) $(\varphi \to (\psi \to \chi)) \to (\varphi \,\&\, \psi \to \chi)$,
($\mathsf{Ax_{BL}}5$) $(\varphi \,\&\, \psi \to \chi) \to (\varphi \to (\psi \to \chi))$,
($\mathsf{Ax_{BL}}6$) $\varphi \,\&\, (\varphi \to \psi) \to \psi \,\&\, (\psi \to \varphi)$,
($\mathsf{Ax_{BL}}7$) $((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$,
($\mathsf{Ax_{BL}}8$) $\overline{0} \to \varphi$,

with the rule of detachment as its (only) inference rule.

Routine calculations show that the axioms $\mathsf{Ax_{BL}}4$ and $\mathsf{Ax_{BL}}5$ essentially code the adjointess condition (3.5). Also by elementary calculations one can show that $\mathsf{Ax_{BL}}7$ codes the prelinearity condition (3.10). This was one of the interesting reformulations Hájek gave to the standard algebraic properties. Another one was that he recognized that the weak disjunction, i.e. the connective which corresponds to the lattice join operation in the truth degree structures, could be defined as

$$\varphi \vee \psi \;=_{\mathrm{def}}\; ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi). \tag{3.12}$$

Here $\wedge$ is the weak conjunction with the lattice meet as truth degree function which can, according to the divisibility condition, be defined as

$$\varphi \wedge \psi \;=_{\mathrm{def}}\; \varphi \,\&\, (\varphi \to \psi). \tag{3.13}$$

A feeling for the compactness of this system may come from the hint that Höhle's axiom system for the monoidal logic consisted of 14 axioms, and did not have to state the prelinearity and the divisibility conditions. Nevertheless, also in this axiom system the axioms ($\mathsf{Ax_{BL}}2$) and ($\mathsf{Ax_{BL}}3$) are redundant, i.e. can be proved from the other ones. Even more, the remaining axioms then are mutually independent, as shown in Chvalovský (2012).

But Hájek's presentation of the basic fuzzy logic BL was only a partial realization of the plan to give the logic of all continuous t-norms. The most natural, somehow standard algebraic semantics for such a logic of all continuous t-norms would be the subclass $\mathsf{T\text{-}alg}$ of $\mathsf{BL\text{-}alg}$ consisting of all BL-algebras with carrier [0, 1], i.e. the subclass of all T-algebras.[5]

It was the guess of Petr Hájek that this standard semantics, determined by the class $\mathsf{T\text{-}alg}$ of all T-algebras, should be an adequate semantics for the fuzzy logic

[5] If a BL-algebra has carrier [0, 1] with its natural ordering then its semigroup operation is automatically a continuous t-norm.

BL too. He was able to reduce the problem to the BL-provability of two particular formulas Hájek (1998a), but the final adequacy result was proved by Cignoli et al. (2000).

And yet another fundamental property of BL could be proved by Esteva et al. (2004): all the t-norm based residuated many-valued logics with *one* continuous t-norm algebra as their standard semantics can be adequately axiomatized as finite extensions of BL. The proof comes by algebraic methods, viz. through a study of the variety of all BL-algebras and their subvarieties which are generated by continuous t-norm algebras: for each one of these subvarieties a finite system of defining equations is algorithmically determined.

3.6.2 The Logic of all Left-Continuous T-Norms

Only a short time after Hájek's axiomatization of the logic of continuous t-norms also the logic of all left-continuous t-norms was adequately axiomatized. It was the guess of Esteva and Godo (2001) that the class MTL-alg should give an adequate semantics for this logic. First they offered an adequate axiomatization of the logic MTL, a shorthand for *monoidal t-norm logic*, which is determined by the class MTL-alg. And later on Jenei and Montagna (2002) proved that MTL is really the logic of all left-continuous t-norms: the logical calculus MTL has an adequate algebraic semantics formed by the subclass of MTL-alg consisting of all MTL-algebras with carrier [0, 1].[6]

3.6.3 First-Order Logics

The extensions of these propositional logics to first-order ones follows the standard lines of approach: one has to start from a first-order language[7] $\mathscr{L}$ and a suitable residuated lattice **A**, and has to define **A**-interpretations **M** by fixing a nonempty domain $M = |\mathbf{M}|$ and by assigning to each predicate symbol of $\mathscr{L}$ an **A**-valued relation in M (of suitable arity) and to each constant an element from (the carrier of) **A**.

The satisfaction relation is also defined in the standard way. The quantifiers ∀ and ∃ are interpreted as taking the infimum or supremum, respectively, of all the values of the relevant instances.

In order to show that this approach worked well one had either to suppose that the underlying lattices of the interpretations are complete lattices, or at least that all the

[6] If an MTL-algebra has carrier [0, 1] with its natural ordering then its semigroup operation is automatically a left-continuous t-norm.

[7] With the two standard quantifiers ∀, ∃, but without function symbols for the present considerations.

necessary infima and suprema do exist in these lattices. Interpretations over lattices which satisfy this last condition are called *safe* by Hájek (1998).

For the logic BL of continuous t-norms, Hájek (1998) added the axioms

($\forall$1) $(\forall x)\varphi(x) \to \varphi(t)$, where t is substitutable for x in φ,
($\exists$1) $\varphi(t) \to (\exists x)\varphi(x)$, where t is substitutable for x in φ,
($\forall$2) $(\forall x)(\chi \to \varphi) \to (\chi \to (\forall x)\varphi)$, where x is not free in χ,
($\exists$2) $(\forall x)(\varphi \to \chi) \to ((\exists x)\varphi \to \chi)$, where x is not free in χ,
($\forall$3) $(\forall x)(\chi \vee \varphi) \to \chi \vee (\forall x)\varphi$, where x is not free in χ,

and the rule of generalization to the propositional system BL yielding the system BL∀.

Then he was able to prove the following general *chain completeness theorem*: A first-order formula φ is BL∀-provable iff it is valid in all safe interpretations over BL-chains.

This result can be extended to finite theories as well as to a lot of other first-order fuzzy logics, e.g. to MTL∀.

We will not discuss further completeness results here but refer to the survey paper Cintula and Hájek (2010) or the more recent extended survey Běhounek et al. (2011).

But it should be mentioned that, as suprema are not always maxima and infima not always minima, the truth degree of an existentially/universally quantified formula may not be the maximum/minimum of the truth degrees of the instances. It is, however, interesting to have conditions which characterize models in which the truth degrees of each existentially/universally quantified formula is witnessed as the truth degree of an instance. Cintula and Hájek (2006) study this problem, Běhounek et al. (2011) surveys it too.

In general, the *Handbook of Mathematical Fuzzy Logic* Cintula et al. (2011) offers extended discussions of all the main developments in the field of mathematical fuzzy logics and shows impressively the wealth of new results which came out up to approximately 2011 from Hájek's ideas.

3.6.4 Some More Recent Extensions

3.6.4.1 Uninorm Based Logics

In their core role as generalized conjunction operations, t-norms are also particular cases of aggregation operators Grabisch et al. (2009). Other types of aggregation operators, introduced for fuzzy sets applications, have been the OWA operators of Yager (1988), and also the *uninorms*, cf. Yager and Rybalov (1996); Metcalfe et al. (2009).

A uninorm $*$ is a binary operation in the unit interval such that $([0, 1], *, e_*)$ becomes an ordered monoid for some suitable unit element $e_* \in [0, 1]$. Hence t-norms as well as t-conorms[8] are uninorms. Another example is the so-called *cross ratio uninorm*:

$$x *_C y = \begin{cases} \frac{xy}{xy+(1-x)(a-y)} & \text{if } \{x, y\} \neq \{0, 1\} \\ 0, & \text{otherwise.} \end{cases}$$

Instead of basing a fuzzy logic upon a t-norm, as discussed up to now, one can also try to start from some uninorm. Such approaches have e.g. been discussed in Gabbay and Metcalfe (2007), Metcalfe and Montagna (2007).

To have a residuation operation $\rightarrowtail_*$ available via the corresponding adjointness condition, one has to restrict the considerations to residuated uninorms $*$, i.e. SS to uninorms which are left-continuous and satisfy additionally the condition $0 * x = x * 0 = 0$. Such residuated uninorms determine *uninorm algebras* $([0, 1], \max, \min, *, \rightarrowtail_*, e_*)$ which are pointed residuated lattices.

The most basic uninorm logic UL is determined by its standard semantics consisting of the class of all uninorm algebras. As for t-norm based logics there are different schematic extensions, completeness results, hypersequent proof systems, and complexity results. The interested reader is referred to Gabbay and Metcalfe (2007), Metcalfe and Montagna (2007), Metcalfe et al. (2009).

3.6.4.2 Equivalence Based Logics

Forming the residual implication to a given left-continuous t-norm opens the way to the basic connectives of the t-norm based mathematical fuzzy logics.

The uninorm logics show, as already the non-commutative fuzzy logics did e.g. in Hájek (2003a, b), that one can start from other binary operations in [0, 1] too. And these other binary operations need not even be considered as generalized conjunction operations—some generalized biimplication could also serve as a starting point.

This was done by V. Novák as he introduced the notion of an EQ-algebra, cf. Novák and De Baets (2009).

The main operation of an EQ-algebra is a generalized biimplication operation $\sim$, also called fuzzy equality operation or fuzzy equivalence operation, accompanied by the binary operations of meet and multiplication ($\otimes$).

The operation $\sim$ offers a natural interpretation of the main connective in Novák's fuzzy type theory FTT, cf. Sect. 3.7.3.

The essential difference between residuated lattices and EQ-algebras lies in the definition of implication operation. Unlike residuated lattices, where the adjointness property is the essential link between the strong, i.e. monoidal conjunction and the (residual) implication, in EQ-algebras the implication operation is defined directly from the fuzzy equality $\sim$. So the adjointness property might be relaxed. This has as consequence that the strong conjunction operation can be non-commutative without

[8] The t-conorms are binary operations in [0, 1] which make this unit interval into an ordered abelien monoid with 0 as unit element of the monoid. They are in 1-1 correspondance with t-norms.

forcing the consideration of two kinds of implication, as is the case for the usual non-commutative generalizations of t-norms.

The relation between EQ-algebras and residuated lattices is quite intricate and it seems that the former open the door to another look at the latter. When considering implication only, it can be shown that the corresponding reducts of EQ-algebras are BCK-algebras, and so, residuated lattices are "hidden" inside. On the other hand, EQ-algebras form a variety and they are not equivalent with residuated lattices; in fact, EQ-algebras generalize residuated lattices because they relax the tie between multiplication and residuation, i.e. between conjunction and implication in the corresponding logics.

Such logics, called EQ-logics, are studied in Dyba and Novák (2011) via their algebraic semantics. Completeness theorems are proved, and relationships to the t-norm based logics discussed.

3.7 Basing fuzzy Set Theory on t-norm Logics

With the previously discussed t-norm based fuzzy logics a toolbox is given to develop fuzzy set theory. Of course, there are still quite different ways to approach this problem depending, e.g., on whether one is interested to have some more model-based approach, or whether one prefers a primarily axiomatic one.

In both respects Hájek has offered ideas how to attack this problem. They are special cases in a much wider spectrum of approaches as explained in Gottwald (2006a, b). Nevertheless they deserve to be mentioned here.

3.7.1 ZF-Style Approaches

Two (slightly different) model-based approaches of a ZF-like fuzzy set theory have been presented by Petr Hájek and Zuzana Haniková in Hájek and Haniková (2001) and in Hájek and Haniková (2003). They are based upon the first-order *basic t-norm logic* $\mathsf{BL}\forall\Delta$, enriched with the $\triangle$-operator of Baaz (1996).[9]

In a language with primitive predicates $\in, \subseteqq, =$ the axioms chosen in Hájek and Haniková (2001, 2003) are suitable versions of (i) extensionality, (ii) pairing, (iii) union, (iv) powerset, (v) $\in$-induction (i.e. foundation), (vi) separation, (vii) collection (a form of replacement), (viii) infinity, together with (ix) an axiom stating the existence of the *support* of each fuzzy set.[10]

A kind of "standard" model for these theories is formed w.r.t. some complete BL-chain $\mathbf{L} = \langle L, \wedge, \vee, *, \rightarrow, 0, 1\rangle$ and designed in the style of the Boolean-valued

[9] This logic is in detail explained, e.g., in Hájek (1998).

[10] In Hájek and Haniková (2001) the axiom of $\in$-induction is missing.

models for standard ZF set theory, see e.g. Bell (1985). This model is based upon the hierarchy

$$V_0^{\mathbf{L}} = \emptyset\,, \qquad V_{\alpha+1}^{\mathbf{L}} = \left\{ f \in {}^{\mathsf{dom}\,(u)}L \mid \mathsf{dom}\,(u) \subseteq V_\alpha^{\mathbf{L}} \right\}$$

with unions at limit stages. In Hájek and Haniková (2001), the primitive predicates $\in, \subseteqq, =$ are interpreted as

$$\begin{aligned}
[\![x \in y]\!] &= \bigvee_{u \in \mathsf{dom}\,(y)} ([\![u = x]\!] * y(u)),\\
[\![x \subseteqq y]\!] &= \bigwedge_{u \in \mathsf{dom}\,(x)} (x(u) \Rightarrow [\![u \in y]\!]),\\
[\![x = y]\!] &= \Delta[\![x \subseteqq y]\!] * \Delta[\![y \subseteqq x]\!].
\end{aligned}$$

The last condition forces the equality to be crisp, and makes the authors' standard form of the axiom of extensionality trivially true in the model.

In Hájek and Haniková (2003), however, these primitive predicates are in a simpler way determined by

$$[\![x \in y]\!] \;=\; y(x),$$

together with

$$[\![x = y]\!] \;=\; \begin{cases} 1 & \text{if } x = y,\\ 0 & \text{otherwise.} \end{cases}$$

The main results are that the structure $V^{\mathbf{L}} = \bigcup_{\alpha \in \mathsf{On}} V_\alpha^{\mathbf{L}}$ together with the different interpretations of the primitive predicates gives in both cases a model of all the (respective) axioms chosen by the authors.

It is interesting to see that the modification in the interpretations of the primitive predicates which distinguishes Hájek and Haniková (2001, 2003) essentially mirrors a similar difference between Klaua (1965) and Klaua (1966b).

3.7.2 A Cantor-Style Approach

Another, primarily axiomatic approach by Hájek (2005) toward a fuzzy set theory, in the sense of a set theory based upon a many-valued logic, is going back to an older approach and has the form of a *Cantorian set theory* over L_∞.

That older approach toward a consistency proof of naive set theory, i.e. set theory with *comprehension* and *extensionality* only, in the realm of Łukasiewicz logic was initiated by Skolem (1957) and resulted in a series of intermediate results, mentioned in Gottwald (2001), which show consistency with respect to more and more extended versions of comprehension. In 1979 White (1979) claimed to have (in the realm of

L_∞) a proof theoretic consistency proof for naive set theory with full *comprehension*.[11]

Two equality predicates come into consideration here—Leibniz equality $=_l$ and extensional equality $=_e$ with definitions

$$x =_l y =_{def} \forall z(x \in z \leftrightarrow y \in z),$$
$$x =_e y =_{def} \forall z(z \in x \leftrightarrow z \in y).$$

Leibniz equality is shown to be a *crisp* predicate, but extensional equality is *not*.

The whole system becomes *inconsistent* by the coincidence assumption

$$x =_l y \quad \leftrightarrow \quad x =_e y.$$

A set of natural numbers can be added. This yields an essentially undecidable and essentially incomplete system, see Hájek (2013).

3.7.3 Fuzzified Mathematical Theories and Fuzzy Type Theories

Having in mind that large parts of modern mathematics got their set theoretic foundation in the 20th century, it has to be recognized too that this set theoretic basis in the beginning often was provided only by naive set theoretic ideas.

Accordingly one might use such a "more naive" approach toward a development of seriously[12] "fuzzified" parts of mathematics, and not necessarily rely upon an axiomatized set theory for such a setting.

An interesting approach in such a direction has been initiated by work of two of Hájek's disciples, Běhounek and Cintula, on fuzzy class theory Běhounek and Cintula (2005). In this paper the authors introduce an axiomatic presentation of Zadeh's notion of fuzzy set, i.e. an elementary fuzzy set theory, cast as two-sorted first-order theory over the first-order fuzzy logic $Ł\Pi\forall$. They offer a reduction of this elementary fuzzy set theory to fuzzy propositional logics and a general method of fuzzification of classical mathematical theories within their formalism. The focus is on set relations and operations that are definable without any structure on the universe of discourse.

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[11] There are, however, still doubts whether this proof is correct or has essential gaps. So, around 2010, Kazushige Terui circulated a note which explains an error in the proof of one of White's crucial theorems (cf. also footnote 78 on p. 92 of Běhounek et al. 2011).

[12] In the early days of (naive) fuzzy set theory, mainly in the 1970s, a lot of papers had been written which offered, independent of any reasonable intended application, nothing but some quite trivial generalizations of standard mathematical notions, usually together with some adaptations of well known elementary results—generalizations which, essentially, did nothing but substituting fuzzy sets for crisp ones which play a role in the understanding of such classical mathematical notions. Such *l'art pour l'art* generalizations I consider here as non-serious ones.

These considerations on fuzzy relations are continued in Běhounek et al. (2008), with their main focus on preordering and equivalence relations, now with $\mathsf{MTL}_\triangle$ as basic logic, and also in Běhounek and Daňková (2009), now with the focus of notions related to relation composition, like different composition operations, images and preimages. In a natural way, properties of fuzzy relations come as graded ones in this context. Almost all basic facts of classical relation theory can be generalized in an essentially canonical way.

Such graded versions of standard properties can also be considered in other contexts, e.g. in elementary set theoretic topology, as done in Běhounek and Kroupa (2007), or also with respect to properties of the t-norm based connectives in propositional fuzzy logics, as done in Běhounek (2012).

It is interesting, in the present context, that the approach from Běhounek and Cintula (2005) toward a theory for fuzzy sets of level one[13] can be extended, e.g. by using many-sorted languages together with the first-order fuzzy logic $\mathsf{MTL}_\triangle$, to theories which allow quantification over fuzzy sets of level two, three etc., and so also to a kind of fuzzy type theory. This extension is also already given in Běhounek and Cintula (2005).

Another Church-style version FTT of a *fuzzy type theory* was offered by Novák (2004), and slightly modified in Novák (2011). FTT differs from the classical type theory essentially by extending the structure of truth values. This structure is assumed to be a residuated lattice with prelinearity and double negation extended by the Monteiro-Baaz Δ-operation, or to be an EQ-algebra. This delta connective offers a natural way in which problems of fuzzy equalities can be avoided in making them crisp ones if necessary.

In Novák (2004), various properties of fuzzy type theory are proved including its completeness. Later papers like Novák (2008); Murinová and Novák (2012) apply it e.g. to model natural language phenomena with fuzzy logic tools.

3.8 Conclusion

Actually, approximately five decades after the introduction of fuzzy sets into knowledge engineering and mathematics, the scientific community owes to Petr Hájek's work, particularly to his system BL and its extensions and generalizations, convincing systems of logics for fuzzy sets. These t-norm based systems seem to offer a family of "canonical" logics for fuzzy sets—at least as long as the choice of the class of t-norms as suitable candidates for non-idempotent, i.e. "interactive" versions of intersections for fuzzy sets remains favored in the fuzzy sets community.

But independently of this situation, the class of t-norm based fuzzy logics, pioneered by the inception of the basic fuzzy logic BL, has become an interesting research area for logicians. And this topic of mathematical fuzzy logics is not only

[13] The notion of level of a fuzzy set is used here as e.g. in Gottwald (1979), and corresponds directly to the notion of rank in classical set theory.

related to fuzzy set theory, as was the main focus of this chapter, it has its independent interest as a field of logic in which one studies logics of comparable truth degrees. And additionally these logics can be understood as particular cases of substructural logics, see Kowalski and Ono (2010), because essentially all of them lack the contraction property.

A series of recent surveys, as well as the current research activities in the field, indicate that Hájek's monograph Hájek (1998) opened a kind of gold mine for investigations in the wider field of non-classical logics.

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Chapter 4
Set Theory and Arithmetic in Fuzzy Logic

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4.1 Introduction

One of Petr Hájek's great endeavours in logic was the development of first-order fuzzy logic BL∀ (1998): this work unified some earlier conceptions of many-valued semantics and their calculi, but it also technically prepared the ground for a natural next step, that being an attempt at employing BL∀ or its extensions as background logics for non-classical axiomatic theories of fuzzy mathematics. Hájek initiated this study in the late nineties, in parallel with a continued investigation of the properties of BL∀ itself. Considering his previous engagements in set theory and arithmetic, and also the key rôles these disciplines play in logic, it seems natural that he focused primarily on these theories, from both mathematical and metamathematical points of view. With time passing, other authors have contributed to the area; other parts of axiomatic fuzzy mathematics based on fuzzy logic have been explored; and the work of several predecessors turned out to be important. Nevertheless, Hájek's (and his co-authors') elegant results stand out as fundamental contributions to the aforementioned axiomatic theories of fuzzy mathematics, and for a large part coincide with the state of the art in these fields of research.

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In this chapter we survey Hájek's contributions to arithmetic and set theory over fuzzy logic, in some cases slightly generalizing the results. Our generalizations always concern the underlying fuzzy logic: Hájek, as the designer of the logic BL∀, naturally worked in this logic or in one of its three prominent extensions—Łukasiewicz, Gödel, or product logic. However, Esteva and Godo's similar, but weaker fuzzy logic MTL of left-continuous t-norms can be, from many points of view, seen as an even more fundamental fuzzy logic; therefore, where meaningful and easy enough, we discuss or present the generalization of Hájek's results to MTL.

The chapter is organized as follows: after the necessary preliminaries given in Sect. 4.2, we address three areas of axiomatic fuzzy mathematics—a ZF-style fuzzy set theory (Sect. 4.3), arithmetic with a fuzzy truth predicate (Sect. 4.4), and naïve Cantor-style fuzzy set theory (Sect. 4.5). The motivation and historical background are presented at the beginning of each section. Owing to the survey character of this chapter, for details and proofs (except for those which are new) we refer the readers to the original works indicated within the text.

4.2 Preliminaries

This chapter deals with some formal theories axiomatized in several first-order fuzzy logics: MTL∀, BL∀, and its three salient extensions—Łukasiewicz logic (Ł∀), Gödel logic (G∀), and product fuzzy logic (Π∀), with or without the connective $\triangle$. We assume the reader's familiarity with the basic apparatus of these fuzzy logics; all standard definitions can be found in the introductory chapter by Běhounek, Cintula, and Hájek (2011), which is freely available online. In this section we only focus on the definitions and theorems needed further on which cannot be found in the chapter.

Of the first-order variants of a fuzzy logic L (see Běhounek et al. 2011, Def. 5.1.2), throughout the chapter we employ exclusively that first-order variant L∀ which includes the axiom $(\forall x)(\chi \vee \varphi) \to \chi \vee (\forall x)\varphi$ (for x not free in χ) ensuring strong completeness with respect to (safe) models over linearly ordered L-algebras.

Convention 4.1 *Let us fix the following notational conventions:*

- *The conjunction $\varphi \,\&\, \dots \,\&\, \varphi$ of n identical conjuncts φ will be denoted by φ^n.*
- *The exponents φ^n take the highest precedence in formulae, followed by prefix unary connectives. The connectives $\to$ and $\leftrightarrow$ take the lowest precedence.*
- *The chain of implications $\varphi_1 \to \varphi_2, \varphi_2 \to \varphi_3, \dots, \varphi_{n-1} \to \varphi_n$ can be written as $\varphi_1 \longrightarrow \varphi_2 \longrightarrow \dots \longrightarrow \varphi_n$, and similarly for $\longleftrightarrow$.*
- *We use the abbreviations $(\forall x\, P t)\varphi$ and $(\exists x\, P t)\varphi$, respectively, for $(\forall x)(x P t \to \varphi)$ and $(\exists x)(x P t \,\&\, \varphi)$, for any infix binary predicate P, term t, formula φ, and variable x.*
- *Negation of an atomic formula can alternatively be expressed by crossing its (usually infix) predicate: $x \notin y =_{\mathrm{df}} \neg(x \in y)$, and similarly for $\neq$, $\not\subseteq$, $\not\approx$, etc.*

As usual, by an *extension* of a logic L we mean a logic which is at least as strong as L and has the same logical symbols as L. (Thus, e.g., BL is an extension of MTL, but $\mathrm{BL}_\triangle$ is not.)

Definition 4.1 Let L be a logic extending MTL∀ or $\mathrm{MTL}\forall_\triangle$. Let T be a theory over L, M a model of T, and φ a formula in the language of T.

We say that φ is *crisp in M* if $M \models \varphi \vee \neg\varphi$, and that φ is *crisp in T* if it is crisp in all models of T.

Taking into account the semantics of L, one can observe that φ is crisp in M iff it only takes the values 0 and 1 in M; the linear completeness theorem for L yields that φ is crisp in T iff $T \vdash_{\mathrm{L}} \varphi \vee \neg\varphi$. By convention we will also say that an n-ary predicate P is crisp in M or T if the formula $P(x_1, \ldots, x_n)$ is crisp in M or T.

Definition 4.2 Let L extend MTL∀ or $\mathrm{MTL}\forall_\triangle$. By $\mathrm{L}_=$ we shall denote the logic L with the identity predicate $=$ that satisfies the reflexivity axiom $x = x$ and the intersubstitutivity schema $x = y \to (\varphi(x) \leftrightarrow \varphi(y))$.

Remark 4.1 It can be observed that the identity predicate $=$ is symmetric and transitive, using suitable intersubstitutivity axioms. The crispness of $=$ can be enforced by the additional axiom $x = y \vee x \neq y$. However, the latter axiom is superfluous in all extensions of $\mathrm{MTL}\forall_{\triangle=}$, and also in those extensions of $\mathrm{MTL}\forall_=$ that validate the schema $(\varphi \to \varphi^2) \to (\varphi \vee \neg\varphi)$, e.g., in $\text{Ł}\forall_=$and $\Pi\forall_=$, since over all these logics the predicate $=$ comes out crisp anyway (the proof is analogous to that due to Hájek 2005, Cor. 1).

Later on we will need the following lemmata, formulated here just for the variants of MTL, but valid as well for any stronger logic (as they only assert some provability claims).

Lemma 4.1 *The following are theorems of propositional* MTL*:*

1. $(\varphi \to \varphi \,\&\, \varphi) \,\&\, (\varphi \to \psi) \to (\varphi \wedge \psi \to \varphi \,\&\, \psi)$
2. $(\varphi \to \varphi \,\&\, \varphi) \,\&\, (\psi \to \psi \,\&\, \psi) \to (\varphi \wedge \psi \to \varphi \,\&\, \psi)$

Proof 1. $\varphi \wedge \psi \longrightarrow \varphi \longrightarrow \varphi \,\&\, \varphi \longrightarrow \varphi \,\&\, \psi$ (the antecedents of the theorem are used in the second and third implication).

2. By prelinearity, we can take the cases $\varphi \to \psi$ and $\psi \to \varphi$. The former case follows by weakening from Lemma 4.1(1); the latter is proved analogously: $\varphi \wedge \psi \longrightarrow \psi \longrightarrow \psi \,\&\, \psi \longrightarrow \varphi \,\&\, \psi$. □

Lemma 4.2 (cf. Haniková 2004) $\mathrm{MTL}\forall_\triangle$ *proves:*

1. $(\exists x)\triangle\varphi \to \triangle(\exists x)\varphi$
2. $(\forall x)\triangle\varphi \leftrightarrow \triangle(\forall x)\varphi$
3. $(\forall x)\triangle(\varphi \,\&\, \psi) \to (\forall x)\triangle\varphi \,\&\, (\forall x)\triangle\psi$
4. $\triangle(\varphi \vee \neg\varphi) \leftrightarrow \triangle(\varphi \to \triangle\varphi)$

Proof By inspection of the $\mathrm{BL}\forall_\triangle$-proofs (Haniková 2004) we can observe that the theorems are valid in $\mathrm{MTL}\forall_\triangle$, too. □

Lemma 4.3 *Let $\varphi(x, y, \dots)$ be a formula of* MTL∀ *and $\psi(x, \dots)$ a formula of* $\text{MTL}\forall_=$*, and t be a term substitutable for both x and y in φ and for x in ψ. Then:*

1. MTL∀ *proves:* $\varphi(t, t) \to (\exists x)\varphi(x, t)$
2. $\text{MTL}\forall_=$ *proves:* $(\forall x = t)(\psi(x)) \leftrightarrow \psi(t)$
3. $\text{MTL}\forall_=$ *proves:* $(\exists x = t)(\psi(x)) \leftrightarrow \psi(t)$

Proof 1. Immediate by the MTL∀-axiom of dual specification.

2. Left to right: $(\forall x)(x = t \to \psi(x)) \longrightarrow (t = t \to \psi(t)) \longleftrightarrow \psi(t)$, by specification and the reflexivity of $=$. Right to left: $\psi(t) \to (x = t \to \psi(x))$ by the intersubstitutivity of equals; generalize on x and shift the quantifier to the consequent.

3. Left to right: $x = t \mathbin{\&} \psi(x) \to \psi(t)$ by the intersubstitutivity of equals; generalize on x and shift the quantifier (as $\exists$) to the antecedent. Right to left: $\psi(t) \longrightarrow (t = t \mathbin{\&} \psi(t)) \longrightarrow (\exists x)(x = t \mathbin{\&} \psi(t))$, by the reflexivity of $=$, dual specification, and Lemma 4.3(1). □

Lemma 4.4 *In* $\text{MTL}\forall_=$*, any formula is equivalent to a formula in which function symbols are applied only to variables and occur only in atomic subformulae of the form $y = F(x_1, \dots, x_k)$.*

Proof Using Lemma 4.3, we can inductively decompose nested terms $s(t)$ by $\varphi(s(t)) \leftrightarrow (\exists x = t)\varphi(s(x))$ and finally by

$$\varphi(F(x_1, \dots, x_k)) \leftrightarrow (\exists y = F(x_1, \dots, x_k))\varphi(y)$$

for all function symbols F. □

We now give a few results on the conservativity of introducing predicate and function symbols.

Definition 4.3 For L a logic, T_1 a theory in a language Γ_1 and $T_2 \supseteq T_1$ a theory in a language $\Gamma_2 \supseteq \Gamma_1$, we say that T_2 is a *conservative extension* of T_1 if $T_2 \vdash_{\text{L}} \varphi$ implies $T_1 \vdash_{\text{L}} \varphi$ for each Γ_1-formula φ.

The proofs of the following theorems are easy adaptations of the proofs due to Hájek (2000). Note that Theorem 4.3 covers introducing constants, too, for $n = 0$ (in which case the congruence axiom becomes trivially provable and need not be explicitly added to the theory).

Theorem 4.2 (Adding predicate symbols; cf. Hájek 2000) *Let* L *extend* MTL∀ *or* $\text{MTL}\forall_\triangle$ *and T be a theory over* L *in a language Γ. Let $P \notin \Gamma$ be an n-ary predicate symbol and $\varphi(x_1, \dots, x_n)$ a Γ-formula. If T' results from T by adding P and the axiom*

$$P(x_1, \dots, x_n) \leftrightarrow \varphi(x_1, \dots, x_n)$$

then T' is a conservative extension of T.

Theorem 4.3 (Adding function symbols; cf. Hájek 2000) *Let* L *extend* $\mathrm{MTL}\forall_{=}$ *or* $\mathrm{MTL}\forall_{\triangle=}$ *and* T *be a theory over* L *in a language* Γ. *Let* $F \notin \Gamma$ *be an* n*-ary function symbol and* φ *a* Γ*-formula with* $n+1$ *free variables. Let* T' *result from* T *by adding the axiom* $\varphi(x_1,\ldots,x_n,F(x_1,\ldots,x_n))$ *and the congruence axiom* $x_1 = z_1 \,\&\, \ldots \,\&\, x_n = z_n \to F(x_1,\ldots,x_n) = F(z_1,\ldots,z_n)$.

1. *If* L *extends* $\mathrm{MTL}\forall_{=}$ *and* $T \vdash_{\mathrm{L}} (\exists y)\varphi(x_1,\ldots,x_n,y)$, *then* T' *is a conservative extension of* T.
2. *If* L *extends* $\mathrm{MTL}\forall_{\triangle=}$ *and* $T \vdash_{\mathrm{L}} (\exists y)\triangle\varphi(x_1,\ldots,x_n,y)$, *then* T' *is a conservative extension of* T.

If, in addition, $T \vdash_{\mathrm{L}} (\exists y)(\varphi(x_1,\ldots,x_n,y) \,\&\, (\forall y')(\varphi(x_1,\ldots,x_n,y') \to y = y'))$, *then each* T'*-formula is* T'*-equivalent to a* T*-formula.*

4.3 ZF-Style Set Theories in Fuzzy Logic

This section intends to give an overview of results on axiomatic set theory developed in fuzzy logic in the style of classical Zermelo–Fraenkel set theory. It draws primarily on Hájek and Haniková's paper (2003), where a ZF-like set theory is developed over $\mathrm{BL}\forall_{\triangle}$. The theory introduced by Hájek and Haniková was called 'fuzzy set theory' for simplicity, and the acronym FST was used; this was not meant to suggest that FST was *the* set theory in fuzzy logic, since clearly there are many possible ways to develop a set theory in fuzzy logic. It was shown that FST theory admitted many-valued models, and that at the same time it faithfully interpreted classical Zermelo–Fraenkel set theory ZF. Moreover, some of its mathematics was developed.

Here, for the sake of precision, we shall use $\mathrm{FST}_{\mathrm{BL}}$ for the above theory of Hájek and Haniková (2003) over $\mathrm{BL}\forall_{\triangle}$, and alongside, we shall consider a theory $\mathrm{FST}_{\mathrm{MTL}}$ developed over $\mathrm{MTL}\forall_{\triangle}$. The focus will be on the theory $\mathrm{FST}_{\mathrm{BL}}$.

We start with a short overview of related ZF-style set theories in non-classical logics. A more comprehensive treatment of the history of the subject can be found in Gottwald's survey (2006); see also Haniková (2004); these take into account also the interesting story of the full comprehension schema (discussed in Sect. 4.5).

An early attempt is presented in the works of Klaua (1965, 1966, 1967), who does not develop axiomatic theory but constructs cumulative hierarchies of sets, defining many-valued truth functions of $=$, $\subseteq$, and $\in$ over a set of truth values that is an MV-algebra. Interestingly, Klaua (1967) constructs a cumulative universe similar to ours in definition of its elements and the value of the membership function, but with a non-crisp equality; his universe then validates extensionality and comprehension, but fails to validate the congruence axioms. Klaua's works have been continued and made more accessible in the works of Gottwald (1976a, b, 1977).

It is instructive to study a selection of chapters on ZF-style set theory in the intuitionistic logic. Powell (1975) defines a ZF-like theory with an additional axiom of double complement (similar in effect to our support), develops some technical means, such as ordinals and ranks, and defines a class of stabilized sets, which it proves to be an inner model of classical ZF. Grayson (1979) omits double complement but uses collection instead of replacement, and constructs, within the theory, a Heyting-valued universe over a complete Heyting algebra. Using a particular Boolean algebra which it constructs, it shows relative consistency with respect to ZF. This paper also offers examples of how (variants of) axioms of classical ZF can strengthen the underlying logic to the classical one. For example, the axiom of foundation, together with a very weak fragment of ZF, implies the law of the excluded middle, which yields the full classical logic (both in intuitionistic logic and in the logics we use here), and thus the theory becomes classical. It also shows—by using $\in$-induction instead of foundation—that some classically equivalent principles are no longer equivalent in a weaker logical setting.

Inspired by the intuitionistic set theory results, Takeuti and Titani (1984) wrote a paper on ZF-style set theory over Gödel logic, giving an axiomatization and presenting some nice mathematics. Later (1992), the authors enhanced their approach to a logical system that combines Łukasiewicz connectives with the product conjunction, the strict negation and a constant denoting $\frac{1}{2}$ on [0, 1] (thus defining the well-known *logic of Takeuti and Titani*, a predecessor of the logics ŁΠ and ŁΠ$\frac{1}{2}$—see Hájek 1998, Sect. 9.1). This logic contains Gödel logic, and it is Gödel logic that is used in the set-theoretic axioms. Equality in this system is many-valued. Within their set-theoretic universe, Takeuti and Titani are then able to reconstruct the algebra of truth values determining the logic, and they also prove a completeness theorem. In her paper (1999), Titani gives analogous constructions, including completeness, for a set theory in lattice-valued logic. This theory was interpreted in $\mathrm{FST_{BL}}$ by Hájek and Haniková (2013).

We will now start developing our theories $\mathrm{FST_{BL}}$ and $\mathrm{FST_{MTL}}$. We will not give proofs for statements that were proved elsewhere, for $\mathrm{FST_{BL}}$; as for a possible generalization for $\mathrm{FST_{MTL}}$, proofs can be obtained by inspection of the $\mathrm{FST_{BL}}$ case. For both theories, we assume the logic contains a (crisp) equality. The only non-logical symbol in the language is a binary predicate symbol $\in$.

Definition 4.4 In both $\mathrm{FST_{BL}}$ and $\mathrm{FST_{MTL}}$ we define:

- *Crispness:* $\mathrm{Cr}(x) \equiv_{\mathrm{df}} (\forall u)\triangle(u \in x \vee u \notin x)$
- *Inclusion:* $x \subseteq y \equiv_{\mathrm{df}} (\forall z \in x)(z \in y)$

Semantically, crisp sets only take the classical membership values. Using Lemma 4.2 one gets:

$$\begin{aligned}\mathrm{Cr}(x) &\longleftrightarrow (\forall u)\triangle(u \in x \to \triangle(u \in x)) \longleftrightarrow \\ &\triangle(\forall u)(u \in x \to \triangle(u \in x)) \longleftrightarrow \triangle\triangle(\forall u)(u \in x \to \triangle(u \in x)),\end{aligned}$$

so crispness itself is a crisp property: one has $\vdash_{\mathrm{MTL}\forall_\triangle} \mathrm{Cr}(x) \leftrightarrow \triangle\mathrm{Cr}(x)$. Thus also $\mathrm{Cr}(x) \longleftrightarrow \triangle\mathrm{Cr}(x) \longleftrightarrow (\triangle\mathrm{Cr}(x))^2 \longleftrightarrow (\mathrm{Cr}(x))^2$.

Definition 4.5 FST_{BL} is a theory over $BL\forall_{\triangle=}$, with a basic predicate symbol $\in$. (FST_{MTL} is defined analogously over $MTL\forall_{\triangle=}$.) The axioms of the theory are as follows:

1. Extensionality: $x = y \leftrightarrow \triangle(x \subseteq y) \,\&\, \triangle(y \subseteq x)$; the condition on the right is $MTL\forall_{\triangle}$-equivalent to $(\forall z)\triangle(z \in x \leftrightarrow z \in y)$
2. Empty set: $(\exists x)\triangle(\forall y)(y \notin x)$; we introduce[1] a new constant $\emptyset$
3. Pair: $(\exists z)\triangle(\forall u)(u \in z \leftrightarrow (u = x \vee u = y))$; we introduce the pairing $\{x, y\}$ and singleton $\{x\}$ function symbols
4. Union: $(\exists z)\triangle(\forall u)(u \in z \leftrightarrow (\exists y)(u \in y \,\&\, y \in x))$; we introduce a unary function symbol $\bigcup x$, and we use $x \cup y$ for $\bigcup\{x, y\}$
5. Weak power: $(\exists z)\triangle(\forall u)(u \in z \leftrightarrow \triangle(u \subseteq x))$; we introduce a unary function symbol $\mathrm{WP}(x)$
6. Infinity: $(\exists z)\triangle(\emptyset \in z \,\&\, (\forall x \in z)(x \cup \{x\} \in z))$
7. Separation: $(\exists z)\triangle(\forall u)(u \in z \leftrightarrow (u \in x \,\&\, \varphi(u, x)))$, if z is not free in φ; we introduce a function symbol $\{u \in z \mid \varphi(u, x)\}$, and we use $x \cap y$ for $\{u \in x \mid u \in y\}$
8. Collection: $(\exists z)\triangle((\forall u \in x)(\exists v)\varphi(u, v) \rightarrow (\forall u \in x)(\exists v \in z)\varphi(u, v))$, if z is not free in φ
9. $\in$-Induction: $\triangle(\forall x)(\triangle(\forall y \in x)\varphi(y) \rightarrow \varphi(x)) \rightarrow \triangle(\forall x)\varphi(x)$
10. Support: $(\exists z)(\mathrm{Cr}(z) \,\&\, \triangle(x \subseteq z))$

Let us remark that making $=$ a crisp predicate is not an altogether arbitrary decision. Indeed, in particular logics, such as Łukasiewicz logic or product logic,[2] even much weaker assumptions on equality than those of Definition 4.2 entail its crispness; this was pointed out by Petr Hájek in an unpublished note. This, together with the fact that a crisp equality is much easier to handle (while it does not prevent a development of a very rich fuzzy set theory), makes the crispness of $=$ a universal choice in our theory.

We consistently use $\triangle$ after existential quantifiers[3] in axioms in order to be able to define some of the standard set-theoretic operations like the empty set, a pair, a union, the set ω, etc., as the Skolem functions of these axioms (i.e., by Theorem 4.3). Notice that if FST_{BL} and FST_{MTL} were defined with the function symbols for these set-theoretic operations in the primitive language, the corresponding Skolem axioms (i.e., $y \notin \emptyset$, $u \in \{x, y\} \leftrightarrow u = x \vee u = y$, etc.) would not contain these $\triangle$'s.

In the weak power set axiom, the second $\triangle$ weakens the statement.

Further, similarly as in set theory over the intuitionistic logic (Grayson 1979), the axiom of foundation in a very weak setting implies the law of excluded middle for all formulae. Therefore, $\in$-induction is used instead. For a reader familiar with Hájek and Haniková's paper (2003), we point out that here we employ a different spelling

[1] At the same time, we add the axiom $y \notin \emptyset$ to the theory; see Theorem 4.3. Henceforth, whenever we add new constants and function symbols, we also add the corresponding axioms implicitly.

[2] In fact, in any logic that proves the schema $(\varphi \rightarrow \varphi^2) \rightarrow (\varphi \vee \neg\varphi)$; cf. Remark 4.1.

[3] Note the semantics of the existential quantifier: mere validity of a formula $(\exists x)\varphi(x)$ in a model **M** does *not* guarantee that there is an object m for which $\|\varphi(m)\|_{\mathbf{M}} = 1$.

of the $\in$-induction schema: originally, the schema read $\triangle(\forall x)((\forall y \in x)\varphi(y) \rightarrow \varphi(x)) \rightarrow \triangle(\forall x)\varphi(x)$. The current form of induction axiom was inspired by Titani's paper (1999). As pointed out by Hájek and Haniková (2013), it is an open problem whether the original $\in$-induction implies the current one (the converse is obviously the case).

Given the above sample of possible problems, the first thing one might like to vouchsafe is that the presented theory really *is fuzzy*, i.e., that it admits many-valued models. Hájek and Haniková (2003) showed this for $\mathrm{FST_{BL}}$, in the following manner.

Take a complete $\mathrm{BL}\forall_\triangle$-chain $\boldsymbol{A} = \langle A, *^{\boldsymbol{A}}, \rightarrow^{\boldsymbol{A}}, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, 0^{\boldsymbol{A}}, 1^{\boldsymbol{A}}, \triangle^{\boldsymbol{A}}\rangle$ and define a universe $V^{\boldsymbol{A}}$ by transfinite induction. Take $\mathrm{Fnc}(x)$ for a unary predicate stating that x is a function, and $\mathrm{Dom}(x)$ and $\mathrm{Rng}(x)$ for unary functions assigning to x its domain and range, respectively. Set:

$$
\begin{aligned}
V_0^{\boldsymbol{A}} &= \{\emptyset\} \\
V_{\alpha+1}^{\boldsymbol{A}} &= \{f\colon \mathrm{Fnc}(f)\ \&\ \mathrm{Dom}(f) = V_\alpha^{\boldsymbol{A}}\ \&\ \mathrm{Rng}(f) \subseteq A\} \text{ for any ordinal } \alpha \\
V_\lambda^{\boldsymbol{A}} &= \bigcup_{\alpha<\lambda} V_\alpha^{\boldsymbol{A}} \text{for a limit ordinal } \lambda \\
V^{\boldsymbol{A}} &= \bigcup_{\alpha\in\mathrm{Ord}} V_\alpha^{\boldsymbol{A}}
\end{aligned}
$$

Observe that $\alpha \leq \beta \in \mathrm{Ord}$ implies $V_\alpha^{\boldsymbol{A}} \subseteq V_\beta^{\boldsymbol{A}}$. Define two binary functions from $V^{\boldsymbol{A}}$ into A, assigning to any $u, v \in V^{\boldsymbol{A}}$ the values $\|u \in v\|$ and $\|u = v\|$ in A:

$$
\begin{aligned}
\|u \in v\| &= v(u) \text{ if } u \in \mathrm{Dom}(v),\ \text{ otherwise } 0^{\boldsymbol{A}} \\
\|u = v\| &= 1^{\boldsymbol{A}} \text{ if } u = v,\ \text{ otherwise } 0^{\boldsymbol{A}}
\end{aligned}
$$

and use induction on the complexity of formulae to define for any formula $\varphi(x_1, \ldots, x_n)$ a corresponding n-ary function from $(V^{\boldsymbol{A}})^n$ into A, assigning to an n-tuple $u_1, \ldots, u_n$ the value $\|\varphi(u_1, \ldots, u_n)\|$:

$$
\begin{aligned}
\|0\| &= 0^{\boldsymbol{A}} \\
\|\psi\ \&\ \chi\| &= \|\psi\| *^{\boldsymbol{A}} \|\chi\|,\ \text{ and similarly for } \rightarrow, \wedge \text{ and } \vee \\
\|\triangle\psi\| &= \triangle^{\boldsymbol{A}}\|\psi\| \\
\|(\forall x)\psi\| &= \textstyle\bigwedge_{u\in V^{\boldsymbol{A}}} \|\psi(x/u)\| \\
\|(\exists x)\psi\| &= \textstyle\bigvee_{u\in V^{\boldsymbol{A}}} \|\psi(x/u)\|
\end{aligned}
$$

For a sentence φ, one says that φ is valid in $V^{\boldsymbol{A}}$ iff $\|\varphi\| = 1^{\boldsymbol{A}}$ is provable in ZF. We are able to demonstrate the following soundness result:

Theorem 4.4 *Let φ be a closed formula provable in* $\mathrm{FST_{BL}}$. *Let $\boldsymbol{A}$ be a complete* $\mathrm{BL}\forall_\triangle$*-chain. Then φ is valid in $V^{\boldsymbol{A}}$.*

We remark that an analogous construction of an $\boldsymbol{A}$-valued universe can be performed for a complete MTL$\forall_{\triangle}$-algebra; based on that, the above result can be stated for $\mathrm{FST}_{\mathrm{MTL}}$ w.r.t. the universe defined over such algebra. In either case, the given construction provides an interpretation of the fuzzy set theory in classical ZF. Currently, there is no completeness theorem available.

Within $\mathrm{FST}_{\mathrm{BL}}$, one can define a class of hereditarily crisp sets and prove it to be an inner model of ZF in $\mathrm{FST}_{\mathrm{BL}}$.

Definition 4.6 In $\mathrm{FST}_{\mathrm{BL}}$ we define the following predicates:

- $\mathrm{HCT}(x) \equiv_{\mathrm{df}} \mathrm{Cr}(x)\ \&\ (\forall u \in x)(\mathrm{Cr}(u)\ \&\ u \subseteq x)$; we write $x \in \mathrm{HCT}$ for $\mathrm{HCT}(x)$
- $\mathrm{H}(x) \equiv_{\mathrm{df}} \mathrm{Cr}(x)\ \&\ (\exists x' \in \mathrm{HCT})(x \subseteq x')$; we write $x \in \mathrm{H}$ for $\mathrm{H}(x)$

Lemma 4.5 $\mathrm{FST}_{\mathrm{BL}}$ *proves that* HCT *and* H *are crisp classes, and moreover, that* H *is transitive.*

It was further shown (Hájek and Haniková 2003) that $\mathrm{FST}_{\mathrm{BL}}$ proves H to be an inner model of ZF. In more detail, for φ a formula in the language of ZF (where the language of classical logic is considered with connectives &, →, 0, and the universal quantifier ∀) one defines a translation φ^{H} inductively as follows:

$$\begin{aligned} \varphi^{\mathrm{H}} &= \varphi \text{ for } \varphi \text{ atomic} \\ 0^{\mathrm{H}} &= 0 \\ (\psi\ \&\ \chi)^{\mathrm{H}} &= \psi^{\mathrm{H}}\ \&\ \chi^{\mathrm{H}} \\ (\psi \to \chi)^{\mathrm{H}} &= \psi^{\mathrm{H}} \to \chi^{\mathrm{H}} \\ ((\forall x)\psi)^{\mathrm{H}} &= (\forall x \in \mathrm{H})(\psi^{\mathrm{H}}) \end{aligned}$$

(Then also $(\neg\psi)^{\mathrm{H}} = \neg(\psi^{\mathrm{H}})$, $(\psi \vee \chi)^{\mathrm{H}} = \psi^{\mathrm{H}} \vee \chi^{\mathrm{H}}$, and $((\exists x)\psi)^{\mathrm{H}} = (\exists x \in \mathrm{H})(\psi^{\mathrm{H}})$).

One can show that the law of the excluded middle holds in H:

Lemma 4.6 *Let $\varphi(x_1, \ldots, x_n)$ be a ZF-formula whose free variables are among $x_1, \ldots, x_n$. Then* $\mathrm{FST}_{\mathrm{BL}}$ *proves*

$$(\forall x_1 \in \mathrm{H}) \ldots (\forall x_n \in \mathrm{H})(\varphi^{\mathrm{H}}(x_1, \ldots, x_n) \vee \neg\varphi^{\mathrm{H}}(x_1, \ldots, x_n)).$$

Considering classical ZF with the axioms of empty set, pair, union, power set, infinity, separation, collection, extensionality, and ∈-induction, one can prove their translations in $\mathrm{FST}_{\mathrm{BL}}$:

Lemma 4.7 *For φ being the universal closure of any of the abovementioned axioms of* ZF, $\mathrm{FST}_{\mathrm{BL}}$ *proves* φ^{H}.

This provides an interpretation of ZF in $\mathrm{FST}_{\mathrm{BL}}$ (in particular, H is an inner model of ZF in $\mathrm{FST}_{\mathrm{BL}}$):

Theorem 4.5 *Let a closed formula* φ *be a theorem of* ZF. *Then* $\mathrm{FST_{BL}} \vdash \varphi^{\mathrm{H}}$.

Moreover, the interpretation is faithful: if $\mathrm{FST_{BL}} \vdash \varphi^{\mathrm{H}}$, then $\mathrm{ZF} \vdash \varphi^{\mathrm{H}}$ (since it is formally stronger), but then $\mathrm{ZF} \vdash \varphi$.

Again, by inspection of the proof, one arrives at the conclusion that exactly the same result can be obtained for $\mathrm{FST_{MTL}}$. This poses the question of a formal difference between $\mathrm{FST_{BL}}$ and $\mathrm{FST_{MTL}}$: it would be interesting to determine to what degree the two theories, built in one fashion over two distinct logics, differ.

We now discuss ordinal numbers in $\mathrm{FST_{BL}}$ (Hájek and Haniková 2013). In order to obtain a suitable definition of ordinal numbers in $\mathrm{FST_{BL}}$, we rely on Theorem 4.5. Recall the classical definition of an ordinal number by a predicate symbol Ord_0:

$$\begin{aligned}\mathrm{Ord}_0(x) \equiv_{\mathrm{df}} & (\forall y \in x)(y \subseteq x)\ \&\\ & (\forall y, z \in x)(y \in z \vee y = z \vee z \in y)\ \&\\ & (\forall q \subseteq x)(q \neq \emptyset \to (\exists y \in q)(y \cap q = \emptyset))\end{aligned}$$

If $x \in \mathrm{H}$, then $\mathrm{Ord}_0(x) \leftrightarrow \mathrm{Ord}_0^{\mathrm{H}}(x)$, and $\mathrm{Ord}_0(x)$ is crisp. We define ordinal numbers to be those sets in H for which $\mathrm{Ord}_0^{\mathrm{H}}$ is satisfied:

Definition 4.7 In $\mathrm{FST_{BL}}$ we define: $\mathrm{Ord}(x) \equiv_{\mathrm{df}} x \in \mathrm{H}\ \&\ \mathrm{Ord}_0(x)$.

Furthermore, we define in $\mathrm{FST_{BL}}$:

$$\mathrm{CrispFn}(f) \equiv_{\mathrm{df}} \mathrm{Rel}(f) \& \mathrm{Cr}(f) \& (\forall x \in \mathrm{Dom}(f))(\langle x, y\rangle \in f\ \&\ \langle x, z\rangle \in f \to y = z)$$

where the property of being a relation, and the operations of ordered pair, domain, and range are defined as in classical ZF.

The *iterated weak power* property is as follows:

$$\begin{aligned}\mathrm{ItWP}(f) \equiv_{\mathrm{df}} & \ \mathrm{CrispFn}(f)\ \&\ \mathrm{Dom}(f) \in \mathrm{Ord}\ \&\ f(\emptyset) = \emptyset\ \&\\ & (\forall \alpha \in \mathrm{Ord})(\alpha \neq \emptyset\ \&\ \alpha \in \mathrm{Dom}(f) \to f(\alpha) = \bigcup_{\beta \in \alpha} \mathrm{WP}(f(\beta)))\end{aligned}$$

The notion is crisp: $\mathrm{ItWP}(f) \leftrightarrow \triangle\mathrm{ItWP}(f)$. Moreover, $\mathrm{ItWP}(f)\ \&\ \mathrm{ItWP}(g)\ \&\ \mathrm{Dom}(f) \leq \mathrm{Dom}(g) \to \triangle(f \subseteq g)$.

Lemma 4.8 $\mathrm{FST_{BL}}$ *proves:* $(\forall \alpha \in \mathrm{Ord})(\exists f)(\mathrm{ItWP}(f)\ \&\ \mathrm{Dom}(f) = \alpha)$.

Definition 4.8 For each $\alpha \in \mathrm{Ord}$, let $\hat{V}_\alpha$ be the unique (crisp) set z such that:

$$(\exists f)(\mathrm{ItWP}(f)\ \&\ \alpha \in \mathrm{Dom}(f)\ \&\ f(\alpha) = z)$$

Then one can show some classical results about ordinal induction and ranks, as:

Theorem 4.6 $\mathrm{FST_{BL}}$ *proves:* $(\forall x)(\exists \alpha \in \mathrm{Ord})(x \in \hat{V}_\alpha)$.

4.4 Arithmetic and the Truth Predicate

In this section we focus on theories of arithmetic over fuzzy logic. We recall the results obtained by Hájek, Paris, and Shepherdson (2000), taking into account also Restall's results (1995); these papers muse on the degree to which considering a logical system formally weaker than the classical one eradicates the paradoxes one obtains when adding a truth predicate to a theory of arithmetic. Then we briefly visit the method which Petr Hájek used in order to show that the first-order satisfiability problem in a standard product algebra is non-arithmetical (Hájek 2001). Interestingly, in all these works, the theory of arithmetic is a crisp one—enriched, in the respective cases, by new language elements that admit a many-valued interpretation.

4.4.1 Classical Arithmetic and the Truth Predicate

We start with a tiny review of theories of arithmetic in classical first-order logic. The language of arithmetic has a unary function symbol s for successors, binary function symbols $+$ for addition and $\cdot$ for multiplication, an object constant 0, and its predicate symbols are $=$ for equality and $\leq$ for ordering.[4] An *arithmetical formula (sentence)* is a formula (sentence) in this language.

We assume $=$ is a logical symbol and the usual axioms for it are implicitly present. *Robinson arithmetic* Q has the following axioms:

$$
\begin{array}{ll}
\text{(Q1)} & s(x) = s(y) \rightarrow x = y \\
\text{(Q2)} & s(x) \neq 0 \\
\text{(Q3)} & x \neq 0 \rightarrow (\exists y)(x = s(y)) \\
\text{(Q4)} & x + 0 = x \\
\text{(Q5)} & x + s(y) = s(x + y) \\
\text{(Q6)} & x \cdot 0 = 0 \\
\text{(Q7)} & x \cdot s(y) = x \cdot y + x \\
\text{(Q8)} & x \leq y \leftrightarrow (\exists z)(z + x = y)
\end{array}
$$

Peano arithmetic PA adds induction, usually as an axiom schema. Here we will need a (classically equivalent) rule: for each arithmetical formula φ, from $\varphi(0)$ and $(\forall x)(\varphi(x) \rightarrow \varphi(s(x)))$ derive $(\forall x)\varphi(x)$.

The *standard model of arithmetic* is the structure $\mathscr{N} = \langle N, 0, s, +, \cdot, \leq \rangle$, where N is the set of natural numbers and $0, s, +, \cdot, \leq$ are the familiar operations and ordering of natural numbers (by an abuse that is quite common, the same notation is maintained for the symbols of the language and for their interpretations on N).

An arithmetization of syntax, first introduced by Gödel, is feasible in theories of arithmetic such as Q or PA; thereby, in particular, each arithmetical formula φ is assigned a Gödel number, denoted $\overline{\varphi}$. Then one obtains a classical diagonal result: for T a theory

[4] One can also take $\leq$ to be a defined symbol, relying on axiom (Q8).

containing PA,[5] and for each formula ψ in the language of T with exactly one free variable, there is a sentence φ in the language of T such that $T \vdash \varphi \leftrightarrow \psi(\overline{\varphi})$.

A theory T such as above (i.e., with a Gödel encoding of formulae), has a truth predicate iff its language contains a unary predicate symbol Tr such that $T \vdash \varphi \leftrightarrow \mathrm{Tr}(\overline{\varphi})$ for each sentence φ of the language. This is what Petr Hájek likes to call the *(full) dequotation scheme,* with the following example for its import: the sentence 'It's snowing' is true if and only if it's snowing. Hence another term in usage 'It's snowing–"It's snowing" lemma'. On the margin, we remark that a per-partes dequotation is native to PA (or indeed, $\mathrm{I}\Sigma_1$): one can define partial truth predicates for fixed levels of the arithmetical hierarchy and fixed number of free variables (Hájek and Pudlák 1993). However, here it is required of Tr that it do the same job uniformly for all formulae.

The juxtaposition of the diagonal result with the requirements posed on a truth predicate reveals that consistent arithmetical theories (over classical logic) cannot define their own truth (a result due to Tarski): taking $\neg\mathrm{Tr}(x)$ for $\psi(x)$, diagonalization yields a sentence φ such that $T \vdash \varphi \leftrightarrow \neg\mathrm{Tr}(\overline{\varphi})$, so $T \vdash \varphi \leftrightarrow \neg\varphi$, a contradiction.

4.4.2 Arithmetic with a Fuzzy Truth Predicate

Hájek et al. (2000) noted that a (crisp) Peano arithmetic might be combined with a (many-valued) truth predicate over Łukasiewicz logic (where the existence of a φ such that $\varphi \leftrightarrow \neg\varphi$ is not contradictory); it then proceeds to develop the theory. We shall reproduce its main results, in combination with those by Restall (1995).

Definition 4.9 PAŁ stands for a Peano arithmetic in Łukasiewicz logic, i.e., a theory with the axioms and rules of first-order Łukasiewicz logic Ł∀, the congruence axioms of equality w.r.t. the primitive symbols of the language of arithmetic, the above axioms (Q1)–(Q8), and the induction rule.

Making PAŁ crisp is easy: one postulates a crispness axiom for the predicate symbol $=$ as the only basic predicate symbol of the theory ($\leq$ is definable). In other words, $x = y \vee x \neq y$ is adopted as a new axiom. Then one can prove crispness for all arithmetical formulae, propagating it over connectives and quantifiers.

However, Restall (1995, actually earlier than Hájek et al. 2000) shows that PAŁ is provably crisp even without a crispness axiom.[6] The proof is a neat example of weakening operating hand in hand with the induction rule, showing that:

1. PAŁ $\vdash x = 0 \vee x \neq 0$
2. If PAŁ $\vdash \varphi(0, y)$ and PAŁ $\vdash \varphi(x, 0)$ and PAŁ $\vdash \varphi(x, y) \to \varphi(s(x), s(y))$, then PAŁ $\vdash \varphi(x, y)$.
3. PAŁ $\vdash (\exists x)(x = 0 \leftrightarrow y = z)$
4. PAŁ $\vdash y = z \vee y \neq z$

and consequently:

[5] An analogous statement can be formed for weaker theories, including Q.

[6] In fact, Restall does not prove the crispness axiom in PAŁ but rather verifies it as a semantic consequence of the theory PAŁ in the standard MV-algebra; note that this is a weaker statement since Ł∀ is not complete w.r.t. the standard MV-algebra. Still, each of the steps can be reconstructed syntactically in PAŁ.

Theorem 4.7 (Restall 1995) *Let* $\varphi(x_1, \ldots, x_n)$ *be an arithmetical formula. Then*

$$\text{PAŁ} \vdash \varphi(x_1, \ldots, x_n) \vee \neg\varphi(x_1, \ldots, x_n).$$

Crispness pertaining to PAŁ as the theory of numbers, as Restall goes on to remark, need not concern *additional* concepts that one may wish to add to it, such as the truth predicate; these may be governed by the laws of Łukasiewicz logic Ł∀.

Definition 4.10 (Hájek et al. 2000) PAŁTr is the theory obtained from PAŁ by expanding its language with a new unary predicate symbol Tr (extending the congruence axioms of = to include Tr, while only arithmetical formulae are considered in the induction rule) and adding the axiom schema $\varphi \leftrightarrow \text{Tr}(\overline{\varphi})$ for each formula φ of the expanded language.

Theorem 4.8 (Hájek et al. 2000) PAŁTr *is consistent.*[7]

Hence any theory obtained by replacing Ł∀ with a weaker logic is consistent too. In choosing a weaker logic, one might want to retain weakening in order to be able to prove crispness of the arithmetical part.

The paper then proceeds to show that one cannot go further and demand that Tr as formalized truth commute with the connectives: such a theory is contradictory.

Theorem 4.9 (Hájek et al. 2000) *The standard model* $\mathscr{N}$ *cannot be expanded to a model of* PAŁTr. *Thus* PAŁTr *has no standard model.*

Actually, Restall (1995) shows that PAŁ as such is ω-inconsistent over the standard MV-algebra $[0, 1]_Ł$. It is yet to be investigated whether Peano arithmetic with a truth predicate developed in a suitable weaker logic than Ł∀ might have standard models.

4.4.3 Non-arithmeticity of Product Logic

Now we turn to a different topic, though with the same arithmetic flavour. We recall a result of Hájek (2001), where a particular expansion of a crisp, finitely axiomatizable arithmetic over first-order product logic Π∀ is considered, in order to show that first-order satisfiability in standard product algebra $[0, 1]_\Pi$ is non-arithmetical.

Definition 4.11 (Hájek 2001)

1. QΠ stands for a crisp theory extending Robinson arithmetic in product logic with finitely many axioms (such as the theory PA^- of Kaye 1991).
2. QΠU expands QΠ with a new unary predicate U and adds the following axioms:

$$\neg(\forall x)Ux$$
$$\neg(\exists x)\neg Ux$$
$$y = s(x) \to (Uy \leftrightarrow (Ux)^2)$$
$$x \leq y \to (Uy \to Ux)$$

[7] In fact, Hájek et al. (2000) proved a stronger statement, for a variant of PAŁTr allowing the predicate symbol Tr to occur in formulae the induction rule is applied to.

Informally speaking, the axioms enforce the truth value of Ux to decrease monotonically (and exponentially) towards 0, but never reaching it, as x is iteratively incremented by the successor function s. Hájek has shown that, among all (classical) structures for the language of arithmetic, *exactly those that are isomorphic to the standard model of arithmetic* ($\mathcal{N}$) *can be expanded to a* $[0, 1]_\Pi$*-model of* QΠU. Hence, one can decide truth in the standard model of arithmetic in the manner indicated in the next theorem. Take $\bigwedge$ QΠU to be the $\wedge$-conjunction of all axioms of QΠU.

Theorem 4.10 (Hájek 2001) *An arithmetical sentence* φ *is true in* $\mathcal{N}$ *iff the formula*

$$\bigwedge \mathrm{Q\Pi U} \wedge \varphi$$

is satisfiable in $[0, 1]_\Pi$.

Hence, first-order satisfiability in $[0, 1]_\Pi$ is a non-arithmetical decision problem. This technique inspired Franco Montagna to prove that also first-order tautologousness in the standard product algebra $[0, 1]_\Pi$, as well as in all standard BL-algebras, are non-arithmetical; these results are to be found in Montagna's paper (2001), actually in the volume containing also Hájek's paper (2001).

4.5 Cantor–Łukasiewicz Set Theory

Another first-order mathematical theory to which Hájek has significantly contributed is naïve set theory over Łukasiewicz logic. As is well known, the rule of contraction (or equivalently the validity of $\varphi \to \varphi \,\&\, \varphi$ in sufficiently strong logics) is needed to obtain a contradiction from the existence of Russell's set by the usual proof. Indeed, the consistency of the unrestricted comprehension schema has been established over several contraction-free logics, including the logic BCK (Petersen 2000) and variants of linear logic (Grishin 1982; Terui 2004). Łukasiewicz logic, which is closely related to the latter logics and like them disvalidates the contraction rule, is thus a natural candidate for the investigation of whether or not it can support a consistent and viable naïve set theory.

The consistency of the unrestricted comprehension schema over Łukasiewicz logic was first conjectured by Skolem (1957). In the 1960s, Skolem (1960), Chang (1963), and Fenstad (1964) obtained various partial consistency results for the comprehension schema restricted to certain syntactic classes of formulae. A proof of the full consistency theorem was eventually published by White (1979). Unlike its predecessors, White's proof was based strictly on proof-theoretical methods and did not attempt at constructing a model for the theory.

White's proof of the consistency of unrestricted comprehension over Łukasiewicz logic prompted Hájek to elaborate the theory, for which he coined the name Cantor–Łukasiewicz set theory. With the consistency of Cantor–Łukasiewicz set theory supposedly established, its non-triviality was questioned: i.e., whether the theory is strong enough to reconstruct reasonably large parts of mathematics (as conjectured already by Skolem). Hájek's contributions (2005, 2013a, 2013b), dealing mainly with arithmetic and decidability in Cantor–Łukasiewicz set theory, gave a partially negative answer to this question. Naïve comprehension over (standard) Łukasiewicz logic has also been developed by Restall (1995), some of whose earlier results Hájek independently rediscovered (2005), and by Yatabe (2007, 2009) who extended some of Hájek's results. We survey the results on Cantor–Łukasiewicz set theory in Sects. 4.5.1–4.5.2.

In 2010 Terui (pers. comm.) found what appears to be a serious gap in White's consistency proof. Consequently, the consistency status of Cantor–Łukasiewicz set theory remains unknown. It is therefore worth asking which of Hájek's and Yatabe's results survive in weaker fuzzy logics, such as IMTL or MTL.[8] This problem is addressed in Sect. 4.5.3 below, giving some initial positive results and indicating the main problems that such enterprize has to face.

4.5.1 Basic Notions of Cantor–Łukasiewicz Set Theory

Definition 4.12 (Hájek 2005) *Cantor–Łukasiewicz set theory,* denoted here by $\mathrm{C}_{\text{Ł}}$,[9] is a theory in first-order Łukasiewicz logic. The language of $\mathrm{C}_{\text{Ł}}$ is the smallest language $\mathscr{L}$ such that it contains the binary *membership predicate* $\in$ and for each formula φ of $\mathscr{L}$ and each variable x contains the *comprehension term* $\{x \mid \varphi\}$. (Thus, comprehension terms in $\mathrm{C}_{\text{Ł}}$ can be nested.) The theory $\mathrm{C}_{\text{Ł}}$ is axiomatized by the unrestricted *comprehension schema:*

$$y \in \{x \mid \varphi(x)\} \leftrightarrow \varphi(y),$$

for each formula φ of $\mathrm{C}_{\text{Ł}}$ and any variables x, y.

Remark 4.2 An alternative way of axiomatizing naïve set theory is to use the comprehension schema of the form:

$$(\exists z)(\forall x)(x \in z \leftrightarrow \varphi) \tag{4.1}$$

for any formula φ in the language containing just the binary membership predicate $\in$ and not containing free occurrences of the variable z. The latter restriction is partly alleviated by the fixed-point theorem (see Theorem 4.13), which makes it possible to introduce sets by self-referential formulae (though not uniquely). The comprehension terms of Definition 4.12 are then the Skolem functions of the comprehension axioms (4.1), conservatively introduceable, eliminable, and nestable by Theorems 4.11 and 4.3 and Lemma 4.4.

Remark 4.3 Clearly, no bivalent or even finitely-valued propositional operator can be admitted in the propositional language of naïve set theories over fuzzy logics on pain of

[8] The consistency status of naïve comprehension over these logics is not known, either. Still, being weaker, they have better odds of consistency even if naïve comprehension turns out to be inconsistent over Łukasiewicz logic.

[9] Hájek (2005 and subsequent papers) denoted the theory by $\mathrm{CŁ}_0$, whereas by CŁ he denoted an inconsistent extension of $\mathrm{CŁ}_0$. In this paper we shall use a systematic symbol C_L for naïve set theory over the logic L. The corresponding theory over *standard* $[0, 1]_{\text{Ł}}$-valued Łukasiewicz logic is called H by White (1979) and Yatabe (2007).

contradiction, as Russell's paradox could easily be reconstructed by means of such an operator. Unrestricted comprehension is thus inconsistent in any fuzzy logic with $\triangle$ (incl. $Ł_\triangle$) as well as in any fuzzy logic with strict negation (e.g., Gödel logic, product logic, and the logics SBL and SMTL). For further restrictions on the fuzzy logic underlying naïve comprehension see Corollary 4.4.

Cantor–Łukasiewicz set theory is in many respects similar to other naïve set theories over various logics, esp. substructural. In particular, the shared features include the distinction between intensional and extensional equality, the fixed-point theorem, the existence of the universal and Russell's set, non-well-foundedness of the universe, etc. The reason for these resemblances is the fact that the proofs of these theorems are mainly based on instances of the comprehension schema and involve just a few logical steps, all of which are available in most usual non-classical logics. Moreover, the comprehension schema ensures the availability of the constructions provided by the axioms of ZF-style set theories, such as pairing, unions, power sets, and infinity. We shall give a brief account of these features of $C_Ł$. Unless a reference is given, the proofs are easy or can be found in papers by Hájek (2005) and Cantini (2003).

First observe that by the comprehension schema, the usual elementary fuzzy set operations are available in $C_Ł$:

Definition 4.13 In $C_Ł$, we define:[10]

$$\emptyset =_{\mathrm{df}} \{q \mid \bot\} \qquad \setminus x =_{\mathrm{df}} \{q \mid q \notin x\}$$
$$x \cap y =_{\mathrm{df}} \{q \mid q \in x \mathbin{\&} q \in y\} \qquad x \cup y =_{\mathrm{df}} \{q \mid q \in x \oplus q \in y\}$$
$$x \sqcap y =_{\mathrm{df}} \{q \mid q \in x \wedge q \in y\} \qquad x \sqcup y =_{\mathrm{df}} \{q \mid q \in x \vee q \in y\}$$

The usual properties of these fuzzy set operations are provable in $C_Ł$.[11] Notice, however, that the notions of kernel and support of a fuzzy set are undefinable in $C_Ł$, as they would make the connective $\triangle$ definable (by setting $\triangle\varphi(y) \equiv y \in \mathrm{Ker}\{x \mid \varphi(x)\}$). Thus unlike ZF-style fuzzy set theories (such as FST of Sect. 4.3), naïve fuzzy set theories can hardly serve as axiomatizations of Zadeh's fuzzy sets, as some of the basic concepts of fuzzy set theory cannot be defined in theories with unrestricted comprehension.[12]

---

[10] See Theorems 4.2–4.3 for the conservativeness of these (and subsequent similar) definitions in $C_Ł$. The symbol $\oplus$ denotes the 'strong' disjunction of Łukasiewicz logic, defined in Ł as $\varphi \oplus \psi \equiv_{\mathrm{df}} \neg(\neg\varphi \mathbin{\&} \neg\psi)$.

[11] The schematic translation of propositional tautologies into theorems of elementary fuzzy set theory (Běhounek and Cintula, 2005) only relies on certain distributions laws for quantifiers, and so works for $C_Ł$ (as well as C_{MTL} introduced in Sect. 4.5.3). The converse direction (disproving theorems not supported by propositional tautologies), however, cannot be demonstrated as in elementary fuzzy set theory (namely, by constructing a model from the counterexample propositional evaluation), since no method of constructing models of $C_Ł$ or C_{MTL} is known. In fact, it is well possible (esp. for C_{MTL}) that the comprehension schema does strengthen the logic of the theory (as it does exclude some algebras of semantic truth values, see comments following Theorem 4.21 and preceding Corollary 4.4 in Sect. 4.5.3).

[12] In order to become a full-fledged theory of fuzzy sets, some kind of (preferably, conservative) extension of naïve fuzzy set theories would be needed (cf. Běhounek 2010; Hájek 2013b, Sect. 3). Such extensions, however, make the comprehension axioms restricted to the formulae in the original

Definition 4.14 In $C_Ł$, we define the following binary predicates:

- *Inclusion:* $x \subseteq y \equiv_{df} (\forall u)(u \in x \to u \in y)$.
- *Extensional equality* (or *co-extensionality*):[13] $x \approx y \equiv_{df} (\forall u)(u \in x \leftrightarrow u \in y)$.
- *Leibniz equality:* $x = y \equiv_{df} (\forall u)(x \in u \leftrightarrow y \in u)$.

We will use $x \neq y$, $x \not\approx y$, $x \notin y$, etc., respectively for $\neg(x = y)$, $\neg(x \approx y)$, $\neg(x \in y)$, etc.

As there is a direct correspondence between sets and properties in $C_Ł$, the definition of Leibniz equality effectively says that the sets which have the same properties (expressible in the language of $C_Ł$) are equal (cf. Leibniz's principle of identity of indiscernibles). Since moreover a concept's intension is often identified with the set of its properties, Leibniz equality can also be understood as *co-intensionality,* or *intensional equality.* Unlike in first-order fuzzy logics with identity (see Sect. 4.2), the predicates $=$ and $\approx$ are defined predicates of $C_Ł$. It turns out that the properties required of the identity predicate (in particular, the intersubstitutivity of identicals) are satisfied by Leibniz equality, but not by extensional equality. Since moreover Leibniz equality turns out to be crisp in $C_Ł$, it can be understood as the crisp identity of the objects of $C_Ł$ (i.e., each model of $C_Ł$ can be factorized by $=$ *salva veritate* of all formulae).

The following theorem lists basic provable properties of both equalities.

Theorem 4.11 $C_Ł$ *proves:*

1. *Both $=$ and $\approx$ are fuzzy equivalence relations; i.e.:*

$$x = x, \quad x = y \to y = x, \quad x = y \,\&\, y = z \to x = z,$$

and analogously for $\approx$. Moreover, $\subseteq$ is a fuzzy preorder whose min-symmetrization is $\approx$:

$$x \subseteq x, \quad x \subseteq y \,\&\, y \subseteq z \to x \subseteq z, \quad x \approx y \leftrightarrow x \subseteq y \wedge y \subseteq x.$$

2. *Leibniz equality is crisp, i.e., $x = y \vee x \neq y$.*
3. *Leibniz equality ensures intersubstitutivity: $x = y \to (\varphi(x) \leftrightarrow \varphi(y))$, for any* $C_Ł$-*formula φ.*
4. *Leibniz equality implies co-extensionality: $x = y \to x \approx y$. The converse (i.e., the extensionality of* $C_Ł$*-sets), however, is inconsistent with* $C_Ł$ (Hájek 2005)*.*[14]

By means of the crisp identity, (crisp) singletons, pairs, and ordered pairs can be defined in $C_Ł$:

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language, and so lose the intuitive appeal of the unrestricted comprehension schema. Cf. Remark 4.4 below.

[13] Cantini (2003) as well as Hájek (2005 and subsequent papers) denote extensional equality by the symbol $=_e$.

[14] In fact, as proved by Hájek (2013a), if $C_Ł \vdash t \notin t$ for a term t, then there is a term t' such that $C_Ł \vdash t \approx t' \,\&\, t \neq t'$. Moreover, he also proved that if $C_Ł \vdash (\forall u)(u \approx t \to u \notin t)$ for a term t, then there are infinitely many terms t_i such that $C_Ł$ proves $t \approx t_i$ and $t_i \neq t_j$, for each $i, j \in \mathbb{N}$. (Thus, for instance, there are infinitely many Leibniz-different empty sets.) The above terms t', t_i are defined by the fixed-point theorem (i.e., Theorem 4.13).

Definition 4.15 In $\mathrm{C}_Ł$, we define (for all $k \geq 1$):

$$\{x\} =_{\mathrm{df}} \{q \mid q = x\} \qquad \{x, y\} =_{\mathrm{df}} \{q \mid q = x \vee q = y\}$$
$$\langle x, y\rangle =_{\mathrm{df}} \{\{x\}, \{x, y\}\} \qquad \langle x_1, \dots, x_k, x_{k+1}\rangle =_{\mathrm{df}} \langle\langle x_1, \dots, x_k\rangle, x_{k+1}\rangle$$

The behavior of these crisp sets is as expected (cf. Theorem 4.20 below). In particular, $\mathrm{C}_Ł$ proves $\langle x, y\rangle = \langle u, v\rangle \leftrightarrow x = u \wedge y = v$. This makes it possible to employ the following notation:

Convention 4.12 *By $\{\langle x, y\rangle \mid \varphi\}$ we abbreviate the comprehension term $\{q \mid (\exists x)(\exists y)(q = \langle x, y\rangle \wedge \varphi)\}$, and similarly for tuples of higher arities.*

Like many other naïve set theories, $\mathrm{C}_Ł$ enjoys the fixed-point theorem that makes self-referential definitions possible:

Theorem 4.13 (The Fixed-Point Theorem) *For each formula $\varphi(x, \dots, z)$ of $\mathrm{C}_Ł$ there is a comprehension term ζ_φ such that $\mathrm{C}_Ł$ proves $\zeta_\varphi \approx \{x \mid \varphi(x, \dots, \zeta_\varphi)\}$.*

Hájek's proof of the Fixed Point Theorem (2005) is just a reformulation of Cantini's proof (2003), which works well in $\mathrm{C}_Ł$. The proof is constructive, i.e., yields effectively and explicitly a particular fixed-point comprehension term ζ_φ for each formula φ.

Convention 4.14 *Let us denote the particular fixed-point comprehension term ζ constructed in the proof of Theorem 4.13 by $\mathrm{FP}_z\{x \mid \varphi(x, \dots, z)\}$. In definitions using the fixed-point theorem, instead of $u =_{\mathrm{df}} \mathrm{FP}_z\{x \mid \varphi(x, \dots, z)\}$ we shall write just $u \approx_{\mathrm{df}} \{x \mid \varphi(x, \dots, u)\}$.*

Thus if we define a fixed point $u \approx_{\mathrm{df}} \{x \mid \varphi(x, \dots, u)\}$, then by Theorem 4.13, $\mathrm{C}_Ł$ proves $q \in u \leftrightarrow \varphi(q, \dots, u)$. The fixed-point theorem thus ensures that the "equation" $\mathrm{C}_Ł \vdash q \in z \leftrightarrow \varphi(q, \dots, z)$ has a solution in z for any formula $\varphi(q, \dots, z)$. Consequently, as usual in non-classical naïve set theories enjoying the fixed-point theorem, $\mathrm{C}_Ł$ proves the (non-unique) existence of a "Quine atom" $u \approx \{u\}$, a set comprised of its own properties $u \approx \{p \mid u \in p\}$, etc.

4.5.2 Arithmetic in Cantor–Łukasiewicz Set Theory

In naïve set theories that enjoy the fixed-point theorem, the set ω of natural numbers can be defined in a more elegant way than in ZF-like set theories, straightforwardly applying the idea that a natural number is either 0 or the successor of another natural number. Identifying 0 with the empty set $\emptyset$ and the successor $s(x)$ of x with $\{x\}$, we define by the fixed-point theorem:

$$\omega \approx_{\mathrm{df}} \{n \mid n = 0 \vee (\exists m \in \omega)(n = s(m))\}. \tag{4.2}$$

The definition is not unique w.r.t. Leibniz identity: Hájek (2013a) showed that there are infinitely many terms ω_i such that $\omega \approx \omega_i$ (so ω_i satisfies the co-extensionality (4.2) as well), but $\omega_i \neq \omega_j$, for each (metamathematical) natural numbers $i, j \in \mathbb{N}$.[15]

$C_Ł$ expanded by the constant ω satisfying (4.2) proves some basic arithmetical properties of ω (cf. Sect. 4.4.1), e.g.:

Theorem 4.15 (Hájek 2005) $C_Ł$ *proves:*

1. $s(x) \neq 0$
2. $s(x) = s(y) \to x = y$
3. $x \in \omega \leftrightarrow s(x) \in \omega$

With suitable definitions of addition and multiplication (as given in Hájek (2013a), namely as ternary predicates, adapting the usual inductive definitions to Łukasiewicz logic by means of min-conjunction $\wedge$), further arithmetical properties, amounting in effect to a $C_Ł$-analogue of Grzegorczyk's weakening Q^- of Robinson arithmetic, can be proved. The proof of essential undecidability of the latter weak classical arithmetic can then be adapted for $C_Ł$, yielding its essential undecidability and incompleteness. The proof proceeds along the usual lines of Gödel numbering and self-reference (Hájek 2013a).

Theorem 4.16 (Hájek 2013a) *The theory* $C_Ł$ *is essentially undecidable and essentially incomplete; i.e., each consistent recursively axiomatizable extension of* $C_Ł$ *is undecidable and incomplete.*

Recall, though, that a theory T over first-order Łukasiewicz logic is considered complete if for each pair φ, ψ of sentences in the language of T, either $\varphi \to \psi$ or $\psi \to \varphi$ is provable in T (Hájek 1998); such theories are also called linear (e.g., Hájek and Cintula 2006). Incompleteness thus means that for some pair φ, ψ of sentences, neither $\varphi \to \psi$ nor $\psi \to \varphi$ is provable in T. The self-referential lemma thus refers to pairs of formulae as well:

Lemma 4.9 (Hájek 2013a) *For each pair* $\psi_1(x_1, x_2), \psi_2(x_1, x_2)$ *of* $C_Ł$*-formulae there is a pair* φ_1, φ_2 *of* $C_Ł$*-sentences such that* $C_Ł$ *proves* $\varphi_1 \leftrightarrow \psi_1(\overline{\varphi}_1, \overline{\varphi}_2)$ *and* $\varphi_2 \leftrightarrow \psi_2(\overline{\varphi}_1, \overline{\varphi}_2)$.

Regarding induction, the situation is tricky:

Theorem 4.17 (Hájek 2005) *If* $C_Ł$ *is consistent, then* $C_Ł$ *extended by the rule*

$$\frac{\varphi(0),\ (\forall x)(\varphi(x) \leftrightarrow \varphi(s(x)))}{(\forall x \in \omega)\varphi(x)},$$

for any φ not containing ω, is consistent as well. However, $C_Ł$ *extended by the same rule for any φ (including those containing the constant ω), is inconsistent.*

Hájek (2005) demonstrated the latter inconsistency claim by developing arithmetic in the extended theory, constructing a truth predicate (cf. Sect. 4.4.2), and showing that it commutes with connectives, which yields inconsistency (Hájek et al. 2000).

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[15] This is a corollary of the theorem given in footnote 14, as ω satisfies its conditions.

In the variant of $C_Ł$ over standard [0, 1]-valued Łukasiewicz logic (called H, see footnote 9), the arithmetic of ω can be shown to be ω-inconsistent (Yatabe 2007; cf. Restall 1995); i.e., $H \vdash \varphi(\overline{n})$ for each numeral $\overline{n}$, but also $H \vdash (\exists n \in \omega)\neg\varphi(n)$ for some formula φ. It is unclear, though, whether the result can be extended to $C_Ł$ (Hájek 2013b).

It can be shown that in every model of $C_Ł$, the set ω contains a crisp initial segment isomorphic to the standard model of natural numbers (Hájek 2013a). However, this segment need not represent a set of the model (cf. the ω-inconsistency of H).

Remark 4.4 In order to be able to handle such collections of elements that need not be sets, but are nevertheless present in models of $C_Ł$, extending $C_Ł$ with classes (which cannot enter the comprehension schema) has been proposed (Hájek 2013b; Běhounek 2010). Although this move may be technically advantageous and can possibly yield an interesting theory, admittedly it destroys the appeal of unrestricted comprehension by restricting it to class-free formulae. It should be kept in mind, though, that the tentative consistency of unrestricted comprehension in $C_Ł$ itself is only admitted by a restriction of its language (see Remark 4.3), and therefore does not apply the comprehension principle unrestrictedly anyway. As this is a common feature of substructural naïve set theories, it suggests that the consistency of naïve comprehension in certain contraction-free substructural logics (and so the necessity of contraction for Russell's paradox) is in a sense "accidental", and that a truly unrestricted comprehension principle would require other logical frameworks (such as paraconsistent or inconsistency-adaptive ones).

4.5.3 *Naïve Comprehension over* MTL

In this section we shall discuss which of Hájek's results in $C_Ł$ can survive the weakening of the underlying logic to the logic MTL. We will only give an initial study, hinting at the main problems of this transition.

Naïve set theory over the first-order logic MTL axiomatized in the same way as in Definition 4.12 will be denoted by C_{MTL}. The basic set operations as well as inclusion and the two equalities can be conservatively introduced in C_{MTL} in the same way as in Definitions 4.13–4.14. Cantini's proof (2003) of the fixed-point theorem (Theorem 4.13; cf. Hájek 2005) works well in C_{MTL}; consequently, the set ω of natural numbers can be introduced in C_{MTL} in the same self-referential way as in $C_Ł$ (see Sect. 4.5.2).

It can be easily observed that similarly as in $C_Ł$ (cf. Theorem 4.11), both equalities $=, \approx$ are fuzzy equivalence relations, inclusion $\subseteq$ is a fuzzy preorder whose min-symmetrization is $\approx$, and Leibniz equality implies intersubstitutivity (and therefore also co-extensionality). It will also be seen later that $\approx$ is provably fuzzy and differs from $=$ (so the extensionality of all sets is inconsistent with C_{MTL}, too), although these facts need be proved in a manner different from that of Hájek (2005).

In Hájek's paper (2005), the crispness of $=$, or the provability of $(x = y) \vee (x \neq y)$, is inferred from the fact that $C_Ł$ proves contraction (or &-idempotence) for the Leibniz equality, i.e., $(x = y) \to (x = y)^2$. Hájek's proof of the latter fact works well in C_{MTL}, too. However, since MTL-algebras (unlike MV-algebras for Łukasiewicz logic) can have non-trivial &-idempotents, crispness in MTL does not generally follow from

&-idempotence. Consequently, in C_{MTL} Hájek's proof only ensures the &-idempotence of the Leibniz identity.

Whether the crispness of $=$ can be proved in C_{MTL} by some additional arguments appears to be an open problem. Below we give some partial results which further restrict the possible truth values of Leibniz identity; the complete solution is, however, as yet unknown. The question is especially pressing since so many proofs of Hájek's advanced results (2005, 2013a) utilize the crispness of $=$ in $C_Ł$. In some cases, the results can be reconstructed in C_{MTL} by more cautious proofs; examples of such theorems (though mostly simple ones) are given below. However, it is currently unclear which part of Hájek's results on $C_Ł$ described in Sects. 4.5.1–4.5.2 can still be recovered in C_{MTL}.

For reference in further proofs, let us first summarize the properties of $\subseteq$, $=$, and $\approx$ that translate readily into C_{MTL}:

Theorem 4.18 (cf. Hájek 2005) C_{MTL} *proves:*

1. $x = x$, $x = y \to y = x$, $x = y \mathbin{\&} y = z \to x = z$, *and analogously for* $\approx$
2. $x \subseteq x$, $x \subseteq y \mathbin{\&} y \subseteq z \to x \subseteq z$, $x \approx y \leftrightarrow x \subseteq y \wedge y \subseteq x$
3. $x = y \to (\varphi(x) \leftrightarrow \varphi(y))$, *for any* C_{MTL}*-formula* φ.
4. $x = y \to x \approx y$
5. $x = y \to (x = y)^2$

Now let us reconstruct in C_{MTL} some basic theorems of $C_Ł$, without relying on the crispness of Leibniz equality. First it can be observed that the &-idempotence of $=$ makes it irrelevant which of the two conjunctions is used between equalities. Consequently, $=$ is not only &-transitive (see Theorem 4.18(1)), but also $\wedge$-transitive, so the notation $x = y = z$ can be used without ambiguity.

Theorem 4.19 C_{MTL} *proves:*

1. $a = b \wedge c = d \leftrightarrow a = b \mathbin{\&} c = d$
2. $x = y \wedge y = z \to x = z$

Proof The claims follow directly from Theorem 4.18(5) and Lemma 4.1. □

Even without assuming the crispness of $=$, singletons and pairs (defined as in Definition 4.15) behave as expected. Unlike $C_Ł$, where crisp cases can be taken due to the crispness of $=$ and the proofs are thus essentially classical, C_{MTL} requires more laborious proofs of these facts.

Theorem 4.20 C_{MTL} *proves:*

1. $\{a\} = \{b\} \leftrightarrow a = b$
2. $\{a, b\} = \{c, d\} \leftrightarrow (a = c \wedge b = d) \vee (a = d \wedge b = c)$
3. $\{a, b\} \subseteq \{c\} \leftrightarrow a = b = c$; *in particular,* $\{a, b\} \approx \{a\} \leftrightarrow a = b$
4. $\langle a, b\rangle = \langle c, d\rangle \leftrightarrow a = c \wedge b = d$
5. $\langle x', y'\rangle \in \{\langle x, y\rangle \mid \varphi(x, y, \dots)\} \leftrightarrow \varphi(x', y', \dots)$
6. $y \approx y \cup \{x\} \leftrightarrow x \in y$

Proof 1. Right to left: by intersubstitutivity. Left to right: $\{a\} = \{b\} \longrightarrow \{a\} \approx \{b\} \longleftrightarrow (\forall x)(x \in \{a\} \leftrightarrow x \in \{b\}) \longleftrightarrow (\forall x)(x = a \leftrightarrow x = b) \longrightarrow a = a \leftrightarrow a = b \longleftrightarrow a = b$.

2. Right to left: Both disjuncts imply the consequent by intersubstitutivity. Left to right:

$$\{a, b\} = \{c, d\} \longrightarrow \{a, b\} \approx \{c, d\} \longleftrightarrow (\forall x)(x = a \vee x = b \leftrightarrow x = c \vee x = d) \longleftrightarrow$$
$$(\forall x)(x = a \vee x = b \rightarrow x = c \vee x = d) \wedge (\forall x)(x = c \vee x = d \rightarrow x = a \vee x = b) \longleftrightarrow$$
$$(\forall x)(x = a \rightarrow x = c \vee x = d) \wedge (\forall x)(x = b \rightarrow x = c \vee x = d) \wedge$$
$$(\forall x)(x = c \rightarrow x = a \vee x = b) \wedge (\forall x)(x = d \rightarrow x = a \vee x = b) \longrightarrow$$
$$(a = c \vee a = d) \wedge (b = c \vee b = d) \wedge (c = a \vee c = b) \wedge (d = a \vee d = b)$$

Distributivity then yields max-disjunction of 16 min-conjunctions, of which 14 are equivalent to $a = b = c = d$, one to $a = c \wedge b = d$, and one to $a = d \wedge b = c$.

3. Right to left: $x \in \{a, b\} \longrightarrow x = a \vee x = b \longleftrightarrow x = c \vee x = c \longleftrightarrow x = c \longleftrightarrow x \in \{c\}$; intersubstitutivity is used in the second step. Left to right:

$$\{a, b\} \subseteq \{c\} \longleftrightarrow (\forall x)(x = a \vee x = b \rightarrow x = c) \longleftrightarrow$$
$$(\forall x)(x = a \rightarrow x = c) \wedge (\forall x)(x = b \rightarrow x = c) \longrightarrow a = c \wedge b = c.$$

4. Right to left: by Theorems 4.20(1)–(2). Left to right: By Theorem 4.20(2),

$$\langle a, b\rangle = \langle c, d\rangle \leftrightarrow (\{a\} = \{c\} \wedge \{a, b\} = \{c, d\}) \vee (\{a\} = \{c, d\} \wedge \{a, b\} = \{c\}).$$

Thus it is sufficient to show the following two implications:

$\{a\} = \{c\} \wedge \{a, b\} = \{c, d\} \longleftrightarrow$	by Theorem 4.20(1)–(2)
$a = c \wedge ((a = c \wedge b = d) \vee (a = d \wedge b = c)) \longleftrightarrow$	by distributivity
$(a = c \wedge a = c \wedge b = d) \vee (a = d \wedge b = c \wedge a = c) \longrightarrow$	by $\wedge$-transitivity of $=$
$a = c \wedge b = d$, and	
$\{a\} = \{c, d\} \wedge \{a, b\} = \{c\} \longrightarrow$	by Theorem 4.18(2)
$\{c, d\} \subseteq \{a\} \wedge \{a, b\} \subseteq \{c\} \longrightarrow$	by Theorem 4.20(3)
$a = b = c = d \longrightarrow a = c \wedge b = d$.	

5. The claim is proved by the following chain of equivalences:

$(\exists x)(\exists y)(\langle x', y'\rangle = \langle x, y\rangle \mathbin{\&} \varphi(x, y, \dots)) \longleftrightarrow$	by Theorems 4.20(4) and 4.19
$(\exists x)(\exists y)(x = x' \mathbin{\&} y = y' \mathbin{\&} \varphi(x, y, \dots)) \longleftrightarrow$	in first-order MTL
$(\exists x = x')(\exists y = y')(\varphi(x, y, \dots)) \longleftrightarrow$	by Lemma 4.3(3)
$\varphi(x', y', \dots)$	

6. The claim is proved by the following chain of equivalences (where the last one follows from Lemma 4.3(2)):

$y \approx y \cup \{x\} \longleftrightarrow (\forall q)(q \in y \leftrightarrow q \in y \vee q = x) \longleftrightarrow$
$(\forall q)(q \in y \rightarrow q \in y \vee q = x) \wedge (\forall q)(q \in y \rightarrow q \in y) \wedge (\forall q)(q = x \rightarrow q \in y) \longleftrightarrow$
$(\forall q)(q = x \rightarrow q \in y) \longleftrightarrow x \in y.$

□

Several useful facts about the Leibniz equality can be derived from considering Russell's set, $\mathrm{r} =_{\mathrm{df}} \{x \mid x \notin x\}$. The following observation is instrumental for these considerations:

Theorem 4.21 $\mathrm{C}_{\mathrm{MTL}}$ *proves:* $(\mathrm{r} \in \mathrm{r})^2 \leftrightarrow \bot$.

Proof By comprehension, $\mathrm{r} \in \mathrm{r} \leftrightarrow \mathrm{r} \notin \mathrm{r}$; thus $\mathrm{r} \in \mathrm{r} \,\&\, \mathrm{r} \in \mathrm{r} \longleftrightarrow \mathrm{r} \in \mathrm{r} \,\&\, \mathrm{r} \notin \mathrm{r} \longleftrightarrow \bot$. □

Since $\mathrm{r} \in \mathrm{r} \leftrightarrow \mathrm{r} \notin \mathrm{r}$, the truth value of the formula $\mathrm{r} \in \mathrm{r}$ is the fixed point ρ of negation in the MTL-algebra of semantic truth values in any model of $\mathrm{C}_{\mathrm{MTL}}$. Consequently, $\mathrm{C}_{\mathrm{MTL}}$ has models only over MTL-algebras possessing the fixed point (e.g., there is no model of $\mathrm{C}_{\mathrm{MTL}}$ over Chang's MV-algebra). Moreover, Theorem 4.21 makes it possible to establish the inconsistency of extensionality in $\mathrm{C}_{\mathrm{MTL}}$ without the assumption of the crispness of Leibniz equality:

Corollary 4.1 $\mathrm{C}_{\mathrm{MTL}}$ *plus the extensionality axiom* $x \approx y \rightarrow x = y$ *is inconsistent.*

Proof Since $x = y \rightarrow x \approx y$ is a theorem (Theorem 4.18(4)), under extensionality the equality relations $=$ and $\approx$ would coincide. Thus by Theorems 4.18(5) and 4.20(6), the relation $\in$ would have to yield idempotent values. However, by Theorem 4.21, $\mathrm{r} \in \mathrm{r}$ is not idempotent. □

Theorem 4.21 shows that the fixed point ρ of negation is nilpotent; consequently, there are no non-trivial idempotents smaller than ρ. As a corollary, the truth value of Leibniz identity cannot lie between 0 and ρ:

Corollary 4.2 $\mathrm{C}_{\mathrm{MTL}}$ *proves:* $x \neq y \vee (\mathrm{r} \in \mathrm{r} \rightarrow x = y)$.

Proof By Theorems 4.18(5) and 4.21, and the strong linear completeness of MTL.

A direct proof in $\mathrm{C}_{\mathrm{MTL}}$ can easily be given as well: By prelinearity we can prove that

$$(x = y \rightarrow \mathrm{r} \in \mathrm{r})^2 \vee (\mathrm{r} \in \mathrm{r} \rightarrow x = y).$$

Thus to prove Cor. 4.2 it is sufficient to prove $(x = y \rightarrow \mathrm{r} \in \mathrm{r})^2 \rightarrow (x = y \rightarrow \bot)$. Now, $x = y \longleftrightarrow (x = y)^2 \longrightarrow (\mathrm{r} \in \mathrm{r})^2 \longleftrightarrow \bot$, respectively by Theorem 4.18(5), the assumption $(x = y \rightarrow \mathrm{r} \in \mathrm{r})^2$, and Theorem 4.21. □

Thus, only sufficiently large truth values (namely, those larger than the truth value ρ of $\mathrm{r} \in \mathrm{r}$) can be non-trivial idempotents in any model of $\mathrm{C}_{\mathrm{MTL}}$. This result can be extended by considering the following sets:

Definition 4.16 For each $n \geq 1$, we define $\mathrm{r}_n =_{\mathrm{df}} \{x \mid (x \notin x)^n\}$

By definition, $\mathrm{r}_n \in \mathrm{r}_n \leftrightarrow (\mathrm{r}_n \notin \mathrm{r}_n)^n$. Consequently, the semantic truth value ρ_n of $\mathrm{r}_n \in \mathrm{r}_n$ satisfies $\rho_n = (\neg\rho_n)^n$. Clearly, $\rho_n > 0$ for each n, since otherwise $0 = \rho_n = (\neg\rho_n)^n = (\neg 0)^n = 1^n = 1 \neq 0$, a contradiction. The values ρ_n form a non-increasing chain:

Theorem 4.22 *For each* $n \geq 1$, $\mathrm{C}_{\mathrm{MTL}}$ *proves:* $\mathrm{r}_{n+1} \in \mathrm{r}_{n+1} \to \mathrm{r}_n \in \mathrm{r}_n$.

Proof We shall prove that $(\mathrm{r}_n \in \mathrm{r}_n \to \mathrm{r}_{n+1} \in \mathrm{r}_{n+1})^n \to (\mathrm{r}_{n+1} \in \mathrm{r}_{n+1} \to \mathrm{r}_n \in \mathrm{r}_n)$, whence the theorem follows by prelinearity.

First, by $(\mathrm{r}_n \in \mathrm{r}_n \to \mathrm{r}_{n+1} \in \mathrm{r}_{n+1})^n$ we have $(\mathrm{r}_{n+1} \notin \mathrm{r}_{n+1} \to \mathrm{r}_n \notin \mathrm{r}_n)^n$. Then we obtain:

$$\begin{array}{rlll} \mathrm{r}_{n+1} \in \mathrm{r}_{n+1} & \longleftrightarrow & (\mathrm{r}_{n+1} \notin \mathrm{r}_{n+1})^{n+1} & \text{by definition} \\ & \longrightarrow & (\mathrm{r}_{n+1} \notin \mathrm{r}_{n+1})^{n} & \text{by weakening} \\ & \longrightarrow & (\mathrm{r}_{n} \notin \mathrm{r}_{n})^{n} & \text{by } (\mathrm{r}_{n+1} \notin \mathrm{r}_{n+1} \to \mathrm{r}_n \notin \mathrm{r}_n)^n \\ & \longleftrightarrow & \mathrm{r}_n \in \mathrm{r}_n & \text{by definition.} \end{array}$$

□

As a corollary to Theorems 4.21 and 4.22, the truth values ρ_n are nilpotent for each n:

Corollary 4.3 $(\mathrm{r}_n \in \mathrm{r}_n)^2 \leftrightarrow \bot$

Proof By Theorems 4.21 and 4.22, $(\mathrm{r}_n \in \mathrm{r}_n)^2 \longrightarrow (\mathrm{r}_1 \in \mathrm{r}_1)^2 \longleftrightarrow \bot$. □

The sequence of truth values ρ_n is in fact strictly decreasing, and the sequence of $\neg\rho_n$ strictly increasing:

Theorem 4.23 *In any model of* $\mathrm{C}_{\mathrm{MTL}}$*, the truth values* ρ_n *of* $\mathrm{r}_n \in \mathrm{r}_n$ *form a strictly decreasing chain and the truth values* $\neg\rho_n$ *of* $\mathrm{r}_n \notin \mathrm{r}_n$ *form a strictly increasing chain.*

Proof By Theorem 4.22 we know that $\rho_{n+1} \leq \rho_n$, so $\neg\rho_n \leq \neg\rho_{n+1}$. Suppose $\neg\rho_n = \neg\rho_{n+1}$. Then $\rho_{n+1} = (\neg\rho_{n+1})^{n+1} = (\neg\rho_n)^{n+1} = ((\neg\rho_n)^n \mathbin{\&} \neg\rho_n) = (\rho_n \mathbin{\&} \neg\rho_n) = 0$, but we have already observed that $\rho_{n+1} > 0$ for all n—a contradiction. Thus $\neg\rho_{n+1} \neq \neg\rho_n$, so $\neg\rho_{n+1} > \neg\rho_n$ and $\rho_{n+1} < \rho_n$. □

As a corollary we obtain that the theory $\mathrm{C}_{\mathrm{MTL}}$ is *infinite-valued,* as each model's MTL-algebra contains an infinite decreasing chain of truth values below the fixed point of ¬ and an infinite increasing chain of truth values above the fixed point of ¬. Moreover, since $(\neg\rho_n)^n = \rho_n$, which by Corollary 4.3 is not idempotent, $\neg\rho_n$ is not n-contractive.[16] Consequently, there are no models of $\mathrm{C}_{\mathrm{MTL}}$ over n-contractive MTL-algebras, for any $n \geq 1$:

[16] Recall that an element x of an MTL-algebra is called *n-contractive* if $x^{n-1} = x^n$. Equivalently, x is n-contractive if x^{n-1} is idempotent. An MTL-algebra is called n-contractive if all its elements are n-contractive.

Corollary 4.4 *Naïve comprehension is inconsistent in all logics* $\mathrm{C}_n\mathrm{MTL}$ *of* n*-contractive* MTL*-algebras (i.e., in* MTL *plus the axiom* $\varphi^{n-1} \to \varphi^n$*), for any* $n \geq 1$*. Consequently, it is also inconsistent in any extension of any* $\mathrm{C}_n\mathrm{MTL}$*, which class includes all logics* $\mathrm{S}_n\mathrm{MTL}$ *of* n*-nilpotent* MTL*-algebras (i.e.,* MTL *plus the axiom* $\varphi^{n-1} \vee \neg\varphi$*) as well as the logics* NM *and* WNM *of (weak) nilpotent minima.*[17]

By Theorem 4.23, the truth values $\neg\rho_n$ of $\mathrm{r}_n \notin \mathrm{r}_n$ form an increasing sequence. By Corollary 4.3, each $\neg\rho_n$ is nilpotent, since $(\neg\rho_n)^{2n} = ((\neg\rho_n)^n)^2 = \rho_n^2 = 0$. Non-trivial idempotents can thus only occur among truth values larger than all $\neg\rho_n$:

Corollary 4.5 *In any model of* $\mathrm{C}_{\mathrm{MTL}}$*, all non-trivial idempotents are larger than all truth values* $\neg\rho_n$ *of* $\mathrm{r}_n \notin \mathrm{r}_n$*. (In particular, they are larger than the fixed point* ρ_1 *of negation).*

This fact is internalized in the theory by the following strengthening of Corollary 4.2:

Corollary 4.6 *For all* $n \geq 1$*,* $\mathrm{C}_{\mathrm{MTL}}$ *proves:* $x \neq y \vee (\mathrm{r}_n \notin \mathrm{r}_n \to x = y)$*.*

Proof The proof is analogous to that of Corollary 4.2: by prelinearity, it is sufficient to prove $(x = y \to \mathrm{r}_n \notin \mathrm{r}_n)^{2n} \to x \neq y$, which obtains by $x = y \longleftrightarrow (x = y)^{2n} \longrightarrow (\mathrm{r}_n \notin \mathrm{r}_n)^{2n} \longleftrightarrow (\mathrm{r}_n \in \mathrm{r}_n)^2 \longleftrightarrow \bot$, using the previous observations. □

By Corollary 4.5, the truth values of the Leibniz equality can only be 0 or sufficiently large (namely, larger than all ρ_n). At present it is, however, unclear whether they have to be crisp or not. As we have seen in Theorems 4.18–4.20, some basic properties of Leibniz equality known from $\mathrm{C}_{Ł}$ can be proved in $\mathrm{C}_{\mathrm{MTL}}$ by more laborious proofs even without the assumption of the crispness of $=$. However, since most of Hájek's results on arithmetic in $\mathrm{C}_{Ł}$ rely heavily on the crispness of identity, it is unclear whether they can be reconstructed in $\mathrm{C}_{\mathrm{MTL}}$ or not.

4.6 Conclusions

In this chapter we have surveyed (and on a few occasions slightly generalized) the work in axiomatic fuzzy mathematics connected with Petr Hájek. A recurring pattern could be observed in Hájek's work in this area: even in a non-classical setting of mathematical fuzzy logic, he made a point of employing the knowledge and methods he mastered during earlier stages of his career, for example, in comparing axiomatic theories using syntactic interpretations, or in relying on strong independence results in arithmetic.

Even though Hájek's results remain a landmark of these investigations, it could also be seen from our exposition of them that the theories in question (as well as their metamathematics) are still at initial stages of their development, and many interesting questions remain still open. Hájek's investigation into these theories opened the way for interesting research and demonstrated that some intriguing results can be achieved. One of the aims

[17] Owing to the existence of a fixed point ρ_1 of negation, naïve comprehension is furthermore inconsistent in logics with strict negation, i.e., in SMTL and any of its extensions, which include ΠMTL, SBL, Π, and G.

of this chapter was to gather the results in this field of research scattered in several papers and present them in a synoptic perspective, in order to promote further research in this area of axiomatic non-classical mathematics. We therefore conclude it with a list of open problems mentioned or alluded to in this chapter:

- Can a completeness theorem be proved for the ZF-style fuzzy set theory FST over MTL?
- What is the difference between $\mathrm{FST}_{\mathrm{BL}}$ and $\mathrm{FST}_{\mathrm{MTL}}$?
- Can Peano arithmetic with a truth predicate over MTL (or some intermediate logic between MTL and Ł) have standard models?
- Is $\mathrm{C}_{\text{Ł}}$ (or $\mathrm{C}_{\mathrm{MTL}}$) consistent (relative to a well-established classical theory)?
- Is the Leibniz equality $=$ crisp in $\mathrm{C}_{\mathrm{MTL}}$?
- Is ω crisp in $\mathrm{C}_{\text{Ł}}$ ($\mathrm{C}_{\mathrm{MTL}}$)?
- Is $\mathrm{C}_{\mathrm{MTL}}$ (essentially) undecidable and incomplete?
- Is there a method of constructing models of $\mathrm{C}_{\text{Ł}}$ or $\mathrm{C}_{\mathrm{MTL}}$, so that the models would satisfy some required properties?

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Chapter 5
Bridges Between Contextual Linguistic Models of Vagueness and T-Norm Based Fuzzy Logic

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5.1 Introduction

With the benefit of hindsight, one can ascertain that Petr Hájek's *Metamathematics of Fuzzy Logic* (Hájek 2001), published in 1998, has been a real breakthrough in the study of mathematical fuzzy logic.[1] At the end of Chap. 1 (Preliminaries of Hájek 2001) Hájek summarizes his introduction to the topic by expressing the hope that his book validates the following four statements (repeated here in abbreviated form):

- Fuzzy logic is neither a poor man's logic nor poor man's probability.
- Fuzzy logic is a logic.
- There are various systems of fuzzy logic, not just one.
- Fuzzy logic in the narrow sense is a beautiful logic, but is also important for applications: it offers foundations.

[1] The term 'mathematical fuzzy logic' has been successfully propagated by students and colleagues of Petr Hájek only well after the appearance of Hájek (2001). Hájek, like others at that time, referred to 'fuzzy logic in the narrow sense', following Zadeh's distinction between a wider and a narrow sense of fuzzy logic, where the latter meant the study of deductive systems of logics that are based on the real unit interval as set of truth values.

Dedicated to Petr Hájek

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The still increasing stream of work, more recently documented in the handbook Cintula et al. (2011), witnesses that Hájek has succeeded admirably and to the benefit of a by now fairly large and lively community of logicians, mathematicians, and computer scientists, who regularly publish their sophisticated results about (mathematical) fuzzy logic in the best logic journals, which had hardly been the case before the appearance of Hájek (2001).

As Hájek made clear already in the preface to Hájek (2001), fuzzy logic is intended as "a logic of imprecise (vague) propositions". Indeed, vagueness is a significant and ubiquitous phenomenon of human communication. Consequently, adequate models of reasoning with vague information are not only of considerable interest to logicians and computer scientists, but also to philosophers (see, e.g., Keefe and Smith 1999; Keefe 2000; Williamson 1994; Fermüller 2003; Shapiro 2006; Smith 2008; Dietz and Moruzzi 2010 and references there) and to linguists. Of particular interest from a logical point of view are approaches to formal semantics of natural language that can be traced back to Richard Montague's ground breaking work, firmly connecting formal logic and linguistics in the generative grammar tradition (see, e.g., Partee 1997; Heim and Kratzer 1998).

This chapter is motivated by the fact that the most widely studied contemporary linguistic models of vagueness appear to be *incompatible* with the degree based approach offered by fuzzy logic, at least at a first glimpse. To model the behavior of competent speakers and hearers in face of vagueness, linguists—often only implicitly—insist on the following principles (cf. Pinkal 1995; Bosch 1983; Barker 2002; Kennedy 2007; Kyburg and Morreau 2000; Fernando and Kamp 1996):

- Like all declarative sentences, utterances of vague propositions are either (preliminarily) accepted or rejected by competent hearers. Matters of degree typically appear not at the level of truth, but on deeper levels, like processing gradable adverbs, adjectives, and predicate modifiers.
- The central feature of vague language is its specific form of context dependency. Contexts of admissible precisifications are not only needed to sort out ambiguities, but rather are systematically to be taken into account whenever vague expressions are processed, even if the modeled scenario eliminates all ambiguities and strictly epistemic uncertainties from the discourse in question.
- Any linguistically adequate model of vagueness should strive to capture subtle facts about grammaticality. For example, a comprehensive linguistic model should respect that not only *tall* and *very tall* are vague predicates, but also that *clearly tall* and *definitely tall* can be seen as vague expressions. Moreover, the models should, e.g., reflect that *definitely very tall* is an ordinary English expression, while *very definitely tall* presumably sounds much less natural to most native speakers.
- The formal semantics of vague expressions should fit the wider realm of natural language semantics as developed in the above mentioned tradition. The models should not introduce ad hoc features (like 'degrees of truth') that do not already play a role in the context of formal semantics of natural language.

Faced with such a list, fuzzy logicians may shrug their shoulders and go on to explain that the different methodological principles underlying *their* approach to reasoning under vagueness is guided by quite different aims and intended applications.[2] In contrast, the purpose of this contribution is to show that even those models of vague language preferred by linguists, that seem to be very distant from fuzzy logic at first, may be fruitfully analyzed from a fuzzy logic point of view. More precisely, our aim is to bridge the seemingly wide gap between context based linguistic models and fuzzy logic by explicating how fuzzy sets can be extracted systematically from the meaning of predicates in a given context and how one can reconstruct a corresponding degree based semantics of logical connectives in various ways. To make this concrete, we will refer to a specific linguistic framework—dynamic context semantics—as used by Chris Barker (2002) for the analysis of vagueness. While Barker's model certainly exhibits a number of original features, it is nevertheless a fairly characteristic and important example of contemporary linguistic approaches to vagueness (not only due to superficial attributes, like its heavy reliance on lambda notation). Building on a straightforward connection between contexts and fuzzy sets we will compare the information content coded in contextual models and in fuzzy sets, respectively. At this point Hájek's emphasis on logics based on continuous t-norms will receive further vindication: the three fundamental t-norms—Łukasiewicz t-norm, minimum, and product—naturally appear in different ways as limits of degrees extracted from contexts. Motivated by this coincidence, we will also discuss some approaches to the problem of justifying truth functional (fuzzy) semantics of logical connectives, with the aim to relate standard fuzzy logic interpretations to Barker's semantic framework.

We do not pretend to provide a systematic overview of connections between linguistic research and fuzzy logic. Just in passing, we refer to the extensive work of Vilem Novák and his collaborators (see, e.g., Novák 1992, 2008) for an approach that aims at models of natural language expressions, in particular by using so-called fuzzy type theory. Certainly, further examples of bridges between these seemingly quite distant paradigms of dealing with vague language can be found. Moreover, we emphasize that our aim is to address conceptual challenges, not to provide new mathematical results. Our hope is that our remarks amounts to an (admittedly rather indirect) further appraisal of Petr Hájek's great work on t-norm based fuzzy logic from a presumably rather unexpected angle.

[2] Some fuzzy logicians seem tempted to argue that models like that of Barker, that we will take as starting point here, compare unfavorably with fuzzy logic, even if the aim is to model the semantics of natural language. But, as our brief review of Barker's model in Sect. 5.2 will indicate, context based models are usually much more fine grained than those offered by fuzzy logic. They indicate that the mentioned methodological principles successfully support contemporary linguistic research in various ways. We thus take the idea that linguists should replace their own approach to formal semantics of vague language by that of fuzzy logic as a quixotic move, that hardly deserves serious debate. On the other hand, the claim that there is no relation between natural language semantics and fuzzy logic at all seems dubious. After all, both fields attempt to model the successful processing of vaguely stated information. In this endeavor they frequently refer to the same natural language examples and moreover rely both on tools from mathematical logic.

5.2 A Contextual Linguistic Approach to Vagueness

Linguists, like logicians, often focus on *predicates* and *predicate modifiers* in modeling the semantics of vague language. It is impossible to provide a survey of the relevant literature that does justice to all linguistic approaches to vagueness in short space.[3] For our purpose it suffices to note that there seems to be wide agreement that adequate truth conditions for vague sentences have to refer not only to fixed lexical entries, but also to *contexts of utterance* that may be identified with sets of contextually relevant permissible *precisifications*. Moreover, many authors take it for granted that a realistic and complete formal semantics of natural languages has to take into account the context dependence of truth conditions, anyway, e.g., to be able to resolve ambiguities and to handle anaphora. However, some care has to be taken in this respect, since 'context' can mean different things here, that may operate on different levels. For example, in applying the adjective *tall* it is obviously relevant to know whether the reference is to trees in a forest, to basketball players, to school kids, or to *a tall story*. On the other hand, consider a situation where it is clear that the *general context* of *asserting Jana is tall* is a discussion about students in my class and not about basketball players. Even there, something like Lewis's *conversational score* (Lewis 1970) (cf. also Shapiro 2006) is needed to model the intended meaning of *Jana is tall* unambiguously. To see this, imagine the following two options. Either (1) the speaker wants to communicate information about Jana's height to someone who does not know her or (2) both speaker and hearer have precise common knowledge about Jana's height, but the speaker intends to establish a standard of tallness by making this utterance. Reference to such conversational contexts of possible precisifications is convincingly argued to be an essential ingredient of adequate models of communication with vague notions and propositions (see, e.g., Pinkal 1995; Bosch 1983; Barker 2002; Kennedy 2007; Shapiro 2006).

Instead of detailing the mentioned arguments for using contexts, we will illustrate the versatile use of contexts in formal semantics by outlining just one particular, rather recent and prominently published approach, due to Chris Barker (2002). This will serve as motivation and bridgehead—to stick with the metaphor in the title of this contribution—for exploring connections to fuzzy logic in the following sections. Barker casts his analysis of various linguistic features of vagueness in terms of so-called *dynamic semantics* (Heim 2002; Groenendijk and Stokhof 1991), that has been successfully employed to handle, e.g., anaphora. In this approach the meaning $[\![\varphi]\!]$ of a statement (declarative sentence, propositional expression) φ is given by an *update function* operating on the set of contexts, which in turn are modeled as sets of possible worlds. As already indicated above, semantic theories differ in their intended meaning and formal manifestation of the notion of contexts. Barker (2002), following Stalnaker (1998), identifies a context with a set of 'worlds', where in each world the extension of all relevant predicates with respect to the actual universe of discourse is *completely precisified*; i.e., each (relevant) atomic proposition is either

[3] For this we refer to the handbook article Rooij (2011), the collections (van Rooij et al. 2011; Egré and Klinedinst 2011), but also to the classic monograph Pinkal (1995).

true or false in a given world. For gradable adjectives these precisifications are specified by a *delineation* δ: for each world, δ maps every gradable adjective—or more precisely: every reference to the meaning of a gradable adjective—into a particular value or degree of a corresponding scale. These values represent local standards of acceptance. For instance, if $\delta(w)$ is the delineation function associated with the world w, then $d = \delta(w)(^{\uparrow}tall)$ yields the standard of tallness in w expressed, say, in cm; i.e. every individual that is at least d cm tall in w will be accepted as *tall* in w.

In fact, only a simple form of update functions is needed; namely *filters*,[4] where $[\![\varphi]\!](C) \subseteq C$ for all contexts C—the result $[\![\varphi]\!](C)$ being the set of worlds in C that survive the update of C with the assertion that φ. This observation entails that dynamic semantics is just a notational variant of a more traditional specification of 'truth at a world': φ is true (accepted) at w if $w \in [\![\varphi]\!](C)$ and φ is false (rejected) at w if $w \notin [\![\varphi]\!](C)$. Moreover, we assume that all worlds of a context refer to the *same universe of discourse* U.

Gradable predicates, like *tall*, relate individuals with degrees on some fixed scale. The denotation of *tall* is modeled by a function tall, such that $\mathsf{tall}(w)(\mathsf{a})$ returns the degree of tallness, i.e. the height (again measured, say, in cm) for individual a in the possible world w. Note that different degrees of tallness for the same individual a in different possible worlds are not attributed to the vagueness of *tall*, but to the hearer's uncertainty about a's height. Barker's approach thus demonstrates how epistemic and vagueness related uncertainty interact with each other.

Accordingly, Barker presents the (dynamic) *meaning* of *tall* by

$$[\![tall]\!] =_{df} \lambda x \lambda C.\{w \in C : \delta(w)(^{\uparrow}tall) \leq \mathsf{tall}(w)(x)\}.$$

Note that here we slightly deviate from Barker's notation presented in Barker (2002) in two ways: First, Barker lets $\mathsf{tall}(d, \mathsf{a})$ denote the set of worlds in which the individual a is at least d cm tall. As argued by Kennedy (2007), such a formalism is more flexible and better suited for non-linear scale structures. Here however, we focus only on linearly ordered arithmetic scales. Therefore $\mathsf{tall}(w)(\mathsf{a})$ directly denotes a degree of tallness (on the relevant scale for heights of persons), as described above. Moreover, we will use addition and subtraction on degrees in order to simplify some definitions below. Secondly, Barker does not distinguish between $[\![tall]\!]$ and the purely referential use of it. Our notation $^{\uparrow}tall$ is meant to indicate that the circularity is of a harmless type.

Among other features, this semantic setup allows Barker to capture the intuitive difference in the meaning of the modifiers *very, definitely*, and *clearly*. They are implemented as *predicate modifiers*, i.e., the first argument of $[\![very]\!]$, $[\![definitely]\!]$, or $[\![clearly]\!]$ is the predicate (e.g. $[\![tall]\!]$) that is to be modified.

To define $[\![very]\!]$ Barker uses an underlying ternary relation very over degrees, such that $\mathsf{very}(s, d, d')$ holds if and only if the difference between d and d' is larger than the vague (world dependent) standard s. With these notational simplifications

[4] By a *filter* we (here) just mean a function f which maps any set to one of its subsets, i.e. $f : \mathscr{P}(S) \rightarrow \mathscr{P}(S)$, with $f(X) \subseteq X$ for all $X \in \mathscr{P}(S)$.

Barker's definition of $[\![very]\!]$ reads as follows:

$$[\![very]\!] =_{df} \lambda\alpha\lambda x\lambda C.\ \{w \in \alpha(x)(C) : \exists d(w[d/\alpha] \in \alpha(x)(C) \wedge \\ \delta(w)(^{\uparrow}\alpha) + \delta(w)(^{\uparrow}very) \leq d)\},$$

where the first argument α denotes the predicate to be modified by $[\![very]\!]$ and $w[d/\alpha]$ denotes a world that is like w, except for setting $\delta(w)(^{\uparrow}\alpha) = d$. E.g., in $w[185\,\text{cm}/[\![tall]\!]]$ the standard of tallness is 185 cm. Therefore, $[\![Ann\ is\ very\ tall]\!] = ([\![very]\!]([\![tall]\!]))(\mathsf{Ann})$ is a filter (update) that is survived by exactly those worlds of a given context, where *Ann* exceeds the standard of tallness by at least some margin s. This margin s not only depends on the meaning of *tall* and *very*, but also on the world itself. Thus the vagueness of *very* is modeled by a twofold context dependence: (1) the meaning of *very* may obviously vary from context to context, but (2) even within a fixed context different worlds may show different standards of accepting that an individual is *very tall*, granted that it is accepted as *tall*.

Barker's presentation of $[\![very]\!]$, and also of other predicate modifiers, however has two subtle problems regarding the type of $\delta(w)(^{\uparrow}very)$ and regarding the composition of predicate modifiers. In the following we show how to enhance his definitions in order to circumvent these issues while maintaining his original intentions regarding the semantics of *very*, *definitely*, and *clearly*.

An interesting feature of Barker's representation of $[\![very]\!]$ is that the value $\delta(w)(^{\uparrow}very)$ does not depend on the vague predicate in question. This implies that by stating *Jana is very tall* one communicates also the intended use of the word *very*, thus possibly affecting how the sentence *Jana is very clever* will be evaluated in a subsequent context. However this presupposes that *tall*, *clever*, and *very*, all refer to the same scale. Moreover, even on the same scale it is hard to see why the (absolute) margin involved by uttering *very huge* should be the same as for uttering *very tiny*. Therefore, we will stipulate that this margin for each world may differ for different predicates, denoted by e.g. $\delta(w)(^{\uparrow}very, ^{\uparrow}tall)$.[5] Secondly, in Barker's original setup it is not possible to iterate predicate modifiers. For example *very very tall* cannot be represented as $[\![very]\!]([\![very]\!]([\![tall]\!]))$, although this is clearly intended by Barker. The reason for this is that the definition of $[\![very]\!]$ uses the local threshold value of the modified predicate α and adjusts it to a new degree d as expressed by $w[d/\alpha]$. This does not work if α is a composite predicate such as *very tall*, because there simply is no local threshold value for *very tall* registered by the delineation δ. Instead for each world both threshold values $\delta(w)(^{\uparrow}very, ^{\uparrow}tall)$ and $\delta(w)(^{\uparrow}tall)$ are needed to decide whether the world survives the context update. (The situation gets even more involved when turning to other complex predicates such as *very clearly tall*). We solve this problem by introducing the function $\Delta(w)(\alpha, x)$ denoting the difference between the threshold value for α and the actual degree to which α applies to the individual x in the world w. For a simple (atomic) predicate such as *tall* we have

[5] Note that $\delta(w)$ is polymorphic: for simple predicates such as *tall* it has only one argument. However, if the first argument is a reference to a modifier like $[\![very]\!]$ or $[\![clearly]\!]$ then a reference to a predicate is expected as second argument.

$$\Delta(w)(^{\uparrow}tall, x) =_{df} \mathsf{tall}(w)(x) - \delta(w)(^{\uparrow}tall).$$

Based on this function we can define the predicate modifier $[\![very]\!]$ as

$$[\![very]\!] =_{df} \lambda\alpha\lambda x\lambda C.\{w \in C : \Delta(w)(\alpha, x) \geq \delta(w)(^{\uparrow}very, {}^{\uparrow}\alpha)\}.$$

By defining the behavior of Δ on predicates modified by *very* as follows, $[\![very]\!]$ becomes fully iterable:

$$\Delta(w)(^{\uparrow}very(\alpha), x) =_{df} \Delta(w)(\alpha, x) - \delta(w)(^{\uparrow}very).$$

Note that, on the level of an individual world w, the update function for *very* refers only to information pertaining to w. In contrast, Barker suggests to model *definitely* as a kind of modal operator:

$$[\![definitely]\!] =_{df} \lambda\alpha\lambda x\lambda C.\{w \in \alpha(x)(C) : \forall d(w[d/\alpha] \in C \rightarrow w[d/\alpha] \in \alpha(x)(C))\}.$$

This means that a world $w \in C$ survives the update with $[\![$*Jane is definitely tall*$]\!]$ if and only if all worlds in C in which Jane has the same height as in w judge Jane as tall according to their local standard. Note that the hearer of the utterance may be uncertain about Jane's actual height. This uncertainty is reflected in the model if Jane has different heights (degrees of tallness) in different worlds of the context. Consequently, in general, *definitely tall* is not just equivalent to '*tall* in all worlds of the context'. However, if there is no uncertainty about Jane's height, i.e. if Jane has the same height in all worlds, then $[\![$*definitely tall*$]\!]$ does not filter out any world ($[\![$*definitely tall*$]\!](C) = C$) in case Jane's height is above the local standard for tallness and filters out all worlds ($[\![$*definitely tall*$]\!](C) = \emptyset$) in case Jane's height is below the local standard for tallness.

Again this definition, as given by Barker, poses an obstacle when iterating predicate modifiers such as in *definitely very tall*: the use of $w[d/\alpha]$ does not (yet) scale up to composite predicates. However defining $w[d/\alpha(\beta)] =_{df} w[d/\beta]$ for composite predicates yields a robust notion of substitution in a world, i.e. we discard all predicate modifiers and only change the threshold value of the underlying atomic predicate in the respective world. Evaluated at a particular world $w \in C$ the sentence *Jane is definitely very tall* can then be understood as intended, namely as *Jane is very tall* in all worlds in C in which she has the same height as in w. Note that there is no direct analogon for defining $\Delta(w)(^{\uparrow}definitely, x)$, since, unlike for *very*, there is no world dependent margin $\delta(w)(^{\uparrow}definitely, \alpha)$ for *definitely*. This matches the intuition that it is (at least somewhat) odd to apply the modifier *very* to the predicate *definitely tall*, in contrast to applying *definitely* to *very tall*, which seems quite appropriate. Barker's model captures this intuition by insisting that *definitely*, in contrast to *very*, is not gradable.[6]

[6] Note that nevertheless both, *very* and *definitely*, are understood as vague adjectives, in the sense of being systematically context dependent.

Barker recognizes that alternative models, where $[\![definitely]\!]$ is gradable and thus may be meaningfully iterated to convey emphasis, might be more realistic. However, he prefers to explore such an alternative by attributing it to the modifier $[\![clearly]\!]$, instead. In fact, the following presentation of the meaning of *clearly* combines essential elements of $[\![very]\!]$ as well as of $[\![definitely]\!]$ (See also Table 5.1):

$$[\![clearly]\!] =_{df} \lambda\alpha\lambda x\lambda(C).\{w \in C : \forall d(w[d/\alpha] \in C \\ \rightarrow \Delta(w[d/\alpha])(\alpha, x) \geq \delta(w)(^{\uparrow}clearly, {}^{\uparrow}\alpha))\}.$$

The reference $^{\uparrow}clearly$ as an argument of $\delta(w)$ indicates that *clearly* itself is vague: $\delta(w)(^{\uparrow}clearly, {}^{\uparrow}\alpha)$ returns a world dependent margin for α analogously to $\delta(w)(^{\uparrow}very, {}^{\uparrow}\alpha)$. However there is an essential difference between $[\![very]\!]$ and $[\![clearly]\!]$: while for *very tall* one compares the local standard of tallness with the local value for an individual x's height in each world w, *clearly tall* checks whether for all worlds where x has the same height as in w the individual x is *tall* even by the margin $\delta(w)(^{\uparrow}clearly)$. This comparison of all worlds in the context that share the same height is completely analogous to the definition of $[\![definitely]\!]$. Moreover, defining $\Delta(w)(^{\uparrow}clearly(\alpha), x)$ accordingly as follows enables iterating *clearly* to obtain e.g. *very clearly*

$$\Delta(w)(^{\uparrow}clearly(\alpha), x) =_{df} \min_{\{d: w[d/\alpha] \in C\}} \{\Delta(w[d/\alpha]\})(\alpha, x) - \delta(w)(^{\uparrow}clearly, {}^{\uparrow}\alpha)\}.$$

5.3 Extracting Fuzzy Sets from Contexts

Our main pillar in building a bridge between linguistics and fuzzy logics consists in connecting the dynamic, context based meaning of predicates like *tall* with fuzzy sets. We define logical operators *and*, *or*, and *not* directly on predicates[7] in a straightforward manner and explore how they relate to the corresponding operations on fuzzy sets. Note that linguists may seek to preserve a subtle difference in the meaning of statements like *Jana is tall and clever* and *Jana is tall and Jana is clever*, respectively. In any case, it is straightforward to lift our analysis of predicate operators to the propositional level.

We introduce the notion of an *element filter*. These are filters parameterized by an element of the universe.[8] Element filters that we have already encountered are, e.g., $[\![tall]\!]$ but also $[\![very]\!]([\![tall]\!])$, where for a given element a both $[\![tall]\!](\mathsf{a})$ and $([\![very]\!]([\![tall]\!]))(\mathsf{a})$ are filters.

[7] For brevity we focus on monadic predicates, but the concepts can easily be extended to relations of higher arity.

[8] As already implicitly assumed above (following Barker), we stipulate that the relevant element is in the universe of the context to which the filter is applied. (Otherwise the result simply remains undefined.)

Table 5.1 Example of a context C with j denoting *Jane*, valuating the sentences *Jane is tall* (φ_{tall}), *Jane is very tall* (φ_{vry}), *Jane is definitely tall* (φ_{def}), and *Jane is clearly tall* (φ_{cle})

w	$\mathsf{tall}(w)(\mathsf{j})$	$\delta(w)(^{\uparrow}tall)$	$\delta(w)(^{\uparrow}very, ^{\uparrow}tall)$	$\delta(w)(^{\uparrow}clearly, ^{\uparrow}tall)$	φ_{tall}	φ_{vry}	φ_{def}	φ_{cle}
w_1	180	185	5	10				
w_2	185	190	10	5				
w_3	185	180	5	5	✓	✓		
w_4	190	185	10	10	✓		✓	
w_5	190	185	5	10	✓	✓	✓	
w_6	190	185	5	5	✓	✓	✓	✓
w_7	190	185	10	5	✓		✓	✓

Given a context C we extract a fuzzy set from the meaning $\alpha = [\![P]\!]$ of a predicate P by applying for each element a the filter $\alpha(\mathsf{a})$ to C and measuring the amount of surviving worlds of C.

In the following we consider only finite sets of worlds as contexts and moreover stipulate that all considered contexts share the same universe U. Although adjectives like *tall* or *heavy* at the first glance refer to continuous scales, we argue that the scales of *perceived* heights or weights are discrete by imposing some level of granularity that is due to our perception and to cognitive limitations. This allows one to straightforwardly determine the membership degree of an individual a in the fuzzy set $[\alpha]_C$ by counting the worlds in C before and after applying the filter α.[9]

We identify fuzzy sets with their membership functions to obtain:

Definition 5.1 Let C be a context over a universe U and α an element filter. Then the fuzzy set $[\alpha]_C$ is given by

$$[\alpha]_C : U \to [0, 1] : \quad x \mapsto \frac{|\alpha(x)(C)|}{|C|}$$

Note that the collection of fuzzy sets $[\alpha]_C$ for all relevant element filters α carries less information than C itself. This will get apparent when we compare logical operators defined on predicates with corresponding operations on fuzzy sets. Extending the framework of Barker, we model compound predicates (like *tall and clever*), built up from logically simpler predicates (*tall*, *clever*), as follows.

Definition 5.2

- $[\![and]\!] =_{df} \lambda\alpha\lambda\beta\lambda x\lambda C.\alpha(x)(C) \cap \beta(x)(C)$
- $[\![or]\!] =_{df} \lambda\alpha\lambda\beta\lambda x\lambda C.\alpha(x)(C) \cup \beta(x)(C)$[10]
- $[\![not]\!] =_{df} \lambda\alpha\lambda x\lambda C.C\backslash(\alpha(x)(C))$

[9] Of course, the approach can be generalized to infinite contexts by imposing suitable probability measures on possible worlds. We will implicitly use such a model in Sect. 5.4, below. In any case, we do not claim any originality, but rather follow a well established concept here.

[10] In natural language one can also find *exclusive* disjunction, e.g. *Jana is either tall or clever* (*but not both*), but note that exclusive disjunction can be modeled as well in the obvious way.

Note that in the above definition $\alpha = [\![A]\!]$ and $\beta = [\![B]\!]$ are element filters representing the meaning of the predicates A and B, respectively. Using infix notation, $[\![A \textit{ and } B]\!]$ is an element filter as well. In general, applying $[\![A \textit{ and } B]\!]$ is not equivalent to applying the element filters $[\![A]\!]$ and $[\![B]\!]$ consecutively. We may additionally define

- $[\![\textit{and}^{>}]\!] =_{df} \lambda\alpha\lambda\beta\lambda x\lambda C.\beta(x)(\alpha(x)(C))$.

Then $[\![A \textit{ and}^{>} B]\!]$ is not only different from $[\![A \textit{ and } B]\!]$, but also from $[\![B \textit{ and}^{>} A]\!]$. One might argue that this form of conjunction corresponds to the natural language expression *and moreover* or to certain uses of *but*. It is interesting to note that in this model the non-commutativity of $[\![A \textit{ and}^{>} B]\!]$ arises only if one of the vague predicates A and B is built up using modalities like *definitely* or *clearly*. Otherwise, both forms of conjunction coincide. If no modalities are involved, all worlds are tested individually and independently of the context in which they are appearing. Consequently exactly those worlds survive the update where both A and B hold. However, consider the predicates *tall and definitely tall* and *tall and*$^{>}$ *definitely tall*. Let the context C consist of the two worlds w_1 and w_2, where there is no uncertainty about *Ann's* height, but where in w_1 Ann is judged tall and in w_2 she is not. Then *tall and definitely tall* filters out both worlds, whereas *tall and*$^{>}$ *definitely tall* filters out w_1 in the first step and therefore w_2 survives the update.

Material implication[11] is expressed by composing $[\![\textit{not}]\!]$ and $[\![\textit{or}]\!]$, as usual:

$$[\![\textit{if}]\!] =_{df} \lambda\alpha\lambda\beta\lambda C.(C \backslash \alpha(x)(C)) \cup \beta(x)(C).$$

The membership degree of x in the fuzzy set $[A \textit{ and } B]_C$[12] is determined by applying the filter $[\![A \textit{ and } B]\!](x)$ to the context C and calculating the fraction of worlds in C that survive this update. Proceeding a step further on our bridge from linguistics to fuzzy logics, the question arises whether we can determine $[A \textit{ and } B]_C(x)$ from the membership degrees $[A]_C(x)$ and $[B]_C(x)$ alone. This, of course, would give us a fully truth functional semantics for *and*, *or*, and *not*. However, fuzzy sets abstract away from the internal structure of contexts that may show various possible dependencies of worlds. We illustrate this by the following example.

Let C be a context consisting of the five possible worlds w_1 to w_5 as in Table 5.2. Furthermore, let $[\![\textit{Jana}]\!] = \mathsf{j}$ be in the universe and let tall, clever, and heavy be the denotations of the unary predicates *tall*, *clever*, and *heavy*, respectively, just as demonstrated for *tall* in Sect. 5.2.

Then $[\![\textit{heavy}]\!]$ is an element filter where $[\![\textit{heavy}]\!](\mathsf{j})(C) = \{w_3\}$. Accordingly, $[\textit{heavy}]_C(\mathsf{j}) = 1/5$. Likewise we have $[\textit{clever}]_C(\mathsf{j}) = [\textit{tall}]_C(\mathsf{j}) = 3/5$. Since these latter are equal, also the membership degrees of j in the fuzzy sets $[\textit{tall and heavy}]_C$ and $[\textit{clever and heavy}]_C$, respectively, had to be equal if the (context update) meaning of *and* were truth functional. But we obtain $[\![\textit{tall and heavy}]\!](\mathsf{j})(C) = \{w_3\}$, thus

[11] As is well known, it is questionable whether material implication has a natural language equivalent. We include this logical connective here mainly for the purpose of comparison.

[12] For the sake of readability we write $[X]_C$ instead of $[[\![X]\!]]_C$.

Table 5.2 Example context C

w	$\delta(w)(\uparrow tall)$	$\mathsf{tall}(w)(\mathsf{j})$	$\delta(w)(\uparrow clever)$	$\mathsf{clever}(w)(\mathsf{j})$	$\delta(w)(\uparrow heavy)$	$\mathsf{heavy}(w)(\mathsf{j})$
w_1	170	175	100	105	80	75
w_2	160	170	120	125	75	70
w_3	170	180	100	95	90	100
w_4	180	175	105	100	85	75
w_5	170	165	110	115	70	65

$[tall\ and\ heavy]_C(\mathsf{j}) = 1/5$, while, on the other hand, $[clever\ and\ heavy]_C(\mathsf{j}) = 0$. Note that by extracting the three fuzzy sets from the corresponding element filters we lose the information about the specific overlap of the corresponding updates in the given context.

The following bounds encode optimal knowledge about membership degrees for fuzzy sets extracted from logically compound predicates with respect to membership degrees referring to the corresponding components.

Proposition 5.1 *Let C be a context, $d \in U$, and let $\alpha = [\![A]\!]$ and $\beta = [\![B]\!]$ be two element filters. Then the following bounds are tight:*

- $\max\{0, [\alpha]_C(d) + [\beta]_C(d) - 1\} \leq [A\ and\ B]_C(d) \leq \min\{[\alpha]_C(d), [\beta]_C(d)\}$,
- $\max\{[\alpha]_C(d), [\beta]_C(d)\} \leq [A\ or\ B]_C(d) \leq \min\{1, [\alpha]_C(d) + [\beta]_C(d)\}$,
- $[not\ A]_C(d) = 1 - [\alpha]_C(d)$.

Proof The value $1 - [\alpha]_C(d)$ for negation follows directly from the relevant definitions.

For conjunction and disjunction note that the membership degree $[\alpha]_C(u)$ can—according to Definition 5.1—be identified with the probability that a randomly chosen possible world w survives the corresponding update $[\![\alpha]\!](u)$. The operators *and* and *or* then calculate the conjunction and disjunction, respectively of these events. The given bounds arise in the extremal cases where the two sets $\alpha(d)(C)$ and $\beta(d)(C)$ are maximally disjoint or maximally overlapping and thus directly follow from the Fréchet inequalities (Fréchet 1935). □

Note that $*_G = \min$ and $\bar{*}_G = \max$ are the Gödel t-norm and co-t-norm, respectively. Moreover, $*_Ł = \lambda x, y. \max\{0, x + y - 1\}$ and $\bar{*}_Ł = \lambda x, y. \min\{1, x + y\}$ are the Łukasiewicz t-norm and co-t-norm, respectively. In other words, Proposition 5.1 shows that the truth functions of (strong) conjunction and (strong) disjunction in Gödel and Łukasiewicz logic (see Hájek 2001) correspond to opposite extremal cases of context based evaluations of conjunction and disjunction.

The bounds for the material implication $[\![if]\!]$, as defined above, can be derived easily as well:

$$\max\{1 - [\alpha]_C(d), [\beta]_C(d)\} \leq [if\ A\ then\ B]_C(d) \leq \min\{1, 1 - [\alpha]_C(d) + [\beta]_C(d)\}.$$

Note the emergence of the residual Łukasiewicz implication (R-implication) as upper bound and the so-called S-implication with respect to the Gödel co-t-norm as lower bound. (See Klement and Navara (1999) for more information on these two different forms of implication.)

Remark Although motivated in a different vein, Paris (2000) obtains essentially the same bounds for truth functions that seek to approximate probabilities. Moreover he suggests that a reasonable determinate truth value for a compound statement could be obtained by taking the arithmetic mean value of the corresponding lower and upper bounds, computed as above for the outermost logical connective of the statement. However, such a truth function is non-associative.

The above analysis of logical predicate operators can be easily lifted to the propositional level. For a sentence like *Jana is tall* its meaning [[*Jana is tall*]] is a filter, rather than an element filter. Logical connectives on propositions can be defined in analogy to Definition 5.2:

Definition 5.3
- $[\![\varphi \wedge \psi]\!] =_{df} \lambda C.[\![\varphi]\!](C) \cap [\![\psi]\!](C)$
- $[\![\varphi \vee \psi]\!] =_{df} \lambda C.[\![\varphi]\!](C) \cup [\![\psi]\!](C)$
- $[\![\neg\varphi]\!] =_{df} \lambda C.C \backslash [\![\varphi]\!](C)$

Likewise we may augment:

- $[\![\varphi \to \psi]\!] =_{df} \lambda C.(C \backslash [\![\varphi]\!])(C) \cup [\![\psi]\!](C)$ and
- $[\![\varphi \wedge^{>} \psi]\!] =_{df} \lambda C.([\![\psi]\!]([\![\varphi]\!]))(C)$.

In the following the set of all propositions formed in this way is called Prop. Similarly to the predicate level we can associate a 'degree of truth' $\|\varphi\|_C$ for every $\varphi \in$ Prop by applying the filter $[\![\varphi]\!]$ to the context C:

$$\|\varphi\|_C =_{df} \frac{|[\![\varphi]\!](C)|}{|C|}.$$

In other words, we identify the degree of truth of φ in a context C with the fraction of worlds in C that survive the update with the filter $[\![\varphi]\!]$. Returning to the context C specified in Table 5.2, *Jana is tall* is true to degree $3/5$ in C since three out of five worlds in C classify *Jana*'s height as above the relevant local standard of tallness.

Once more we note that contexts allow to model specific constraints on the worlds (i.e. contextually relevant possible precisifications) of which they consist. Therefore, in general, there are no truth functions that determine $\|\varphi \wedge \psi\|_C$ and $\|\varphi \vee \psi\|_C$ in terms of $\|\varphi\|_C$ and $\|\psi\|_C$ alone. However the optimal bounds of Proposition 5.1 also apply at the level of sentences. In particular:

- $*_Ł(\|\varphi\|_C, \|\psi\|_C) \leq \|\varphi \wedge \psi\|_C \leq *_G(\|\varphi\|_C, \|\psi\|_C)$, and
- $\bar{*}_G(\|\varphi\|_C, \|\psi\|_C) \leq \|\varphi \vee \psi\|_C \leq \bar{*}_Ł(\|\varphi\|_C, \|\psi\|_C)$,

where $*_G(\bar{*}_G)$ and $*_Ł(\bar{*}_Ł)$ are the Gödel and Łukasiewicz t-norms (co-t-norms), respectively. (Analogously for material implication.)

5.4 Saturated Contexts

Having determined bounds for truth functions applied to arbitrary contexts, we now turn to a special class of contexts, called *saturated contexts*. In a saturated context the degrees to which predicates apply as well as all relevant thresholds values are defined by intervals. All values (up to a certain level of granularity) in the given interval are assumed to occur in that context with equal frequency. Moreover, the intervals for different attributes and corresponding threshold values are assumed to be independent of each other. This means that, e.g., an adequate saturated context for uttering *Jane is tall* can be completely defined by giving lower and upper bounds for Jane's height (denoted by h_{j}^l and h_{j}^u) and for possible threshold values for tallness (denoted by tall^l and tall^u). Saturated contexts thus naturally arise when modeling situations, where only those bounds are known, but no further information, e.g., about dependencies between the values or about varying likelihood for the individual possible values, is available to the hearer of an utterance. As we will see below, this lack of specific information is crucial, when one seeks to extract not only *bounds* for truth functions of logical connectives (as in Sect. 5.3), but a *specific* truth-functional semantics.

In the last section we stipulated contexts to be *finite* sets of worlds and argued why this is a natural assumption in linguistics, due to the granularity imposed by limits of distinguishabilty. Here however we will be interested only in contexts with arbitrary fine granularity; in fact we will analyze the limit case, where we can treat the set of possible values for a particular magnitude as an intervals of real numbers. To motivate this move, consider a hearer of *Jane is tall* who only knows about *Jane*'s height that it is between $h_{\mathsf{j}}^l = 179\,\text{cm}$ and $h_{\mathsf{j}}^u = 181\,\text{cm}$. Moreover, for sake of simplicity, let the hearer be certain that it is adequate (in the given context C) to call a person *tall* if and only it its height is at least $\mathsf{tall}^l = \mathsf{tall}^u = 180\,\text{cm}$. If the granularity is too coarse and the interval only includes the three values 179, 180, and 181 cm as possible values for *Jane*'s height, then in two out of these three possible worlds Jane is judged to be tall, hence $[\mathsf{tall}]_C(\mathsf{j}) = 2/3$. This value however is just an artifact imposed by the very low level of granularity, as for higher levels of granularity the value $[\mathsf{tall}]_C(\mathsf{j})$ approaches $1/2$. (In other words, the fraction of worlds in the given context where *Jane*'s height is above the threshold of tallness, is arbitrarily close to $1/2$ for sufficient high levels of granularities.) From now on we will only be interested in the limit case where we can interpret $[h_{\mathsf{j}}^l, h_{\mathsf{j}}^u]$ as a real interval and calculate $[\mathsf{tall}]_C(\mathsf{j}) = 1/2$, corresponding to the intuition that exactly half of this interval of possible values for *Jane*'s height is cut off by the given threshold value for tallness. Note that Definition 1 only applies to finite contexts. However, the value $[\mathsf{tall}]_C(\mathsf{j})$ can also be interpreted as the probability that a randomly chosen possible world w survives the corresponding update $[\![tall]\!](\mathsf{j})$, assuming a uniform distribution over all worlds of the original context. This point of view will enable us to analyze the relevant limit cases directly.

As pointed out above, saturated contexts abstract away from information about dependencies or varying likelihood of possible values. This abstraction allows one

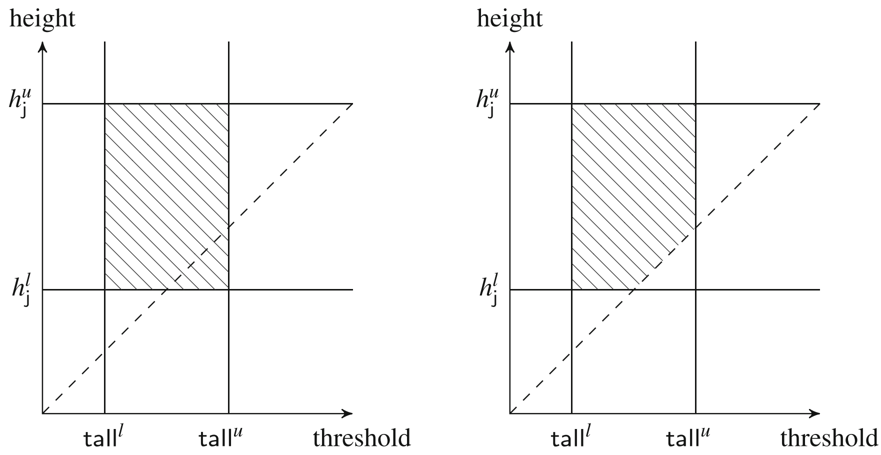

Fig. 5.1 Illustration of a saturated context before and after an update with *Jane is tall*

to compute the fuzzy membership value $[\mathsf{tall}]_C(\mathsf{j})$ from given values h^l_j, h^u_j, tall^l, and tall^u alone. For the actual computation, one has to distinguish between six cases, depending on the relative position of the two intervals $[h^l_\mathsf{j}, h^u_\mathsf{j}]$ and $[\mathsf{tall}^l, \mathsf{tall}^u]$: either they are completely disjoint with (1) $h^l_\mathsf{j} > \mathsf{tall}^u$ or with (2) $h^u_\mathsf{j} < \mathsf{tall}^l$; or one of them is contained in the other one with (3) $h^u_\mathsf{j} \geq \mathsf{tall}^u, h^l_\mathsf{j} \leq \mathsf{tall}^l$ or with (4) $h^u_\mathsf{j} \leq \mathsf{tall}^u, h^l_\mathsf{j} \geq \mathsf{tall}^l$; or they are properly overlapping with (5) $h^u_\mathsf{j} < \mathsf{tall}^u, h^u_\mathsf{j} \geq \mathsf{tall}^l \geq h^l_\mathsf{j}$ or with (6) $h^l_\mathsf{j} > \mathsf{tall}^l, h^u_\mathsf{j} \geq \mathsf{tall}^u \geq h^l_\mathsf{j}$.

For cases (1) and (2) it is easy to see that the fuzzy membership degree in question is 1 (or 0, respectively). Case (6) is depicted by Fig. 5.1. The hatched area of size $A = (h^u_\mathsf{j} - h^l_\mathsf{j})(\mathsf{tall}^u - \mathsf{tall}^l)$ on the left hand side represents the possible worlds in a saturated contexts determined by the boundary values of the two intervals before applying the update. The hatched area on the right hand side represents the worlds after applying the update that corresponds to accepting *Jane is tall*: all worlds under the diagonal are eliminated by the element filter $[\![tall]\!](\mathsf{j})$. Its size is $A - (\mathsf{tall}^u - h^l_\mathsf{j})^2/2$. Putting these observations together, the probability that a randomly chosen world of the context survives the update—and thus the membership degree for j in $[\mathsf{tall}]_C$ for this case—is readily computed as

$$\text{case (5): } [\mathsf{tall}]_C(\mathsf{j}) = \frac{A - \frac{1}{2}(\mathsf{tall}^u - h^l_\mathsf{j})^2}{A} \quad \text{with } A = (h^u_\mathsf{j} - h^l_\mathsf{j})(\mathsf{tall}^u - \mathsf{tall}^l).$$

For the remaining three cases the fuzzy membership degree can be computed analogously, leading to

$$[\mathsf{tall}]_C(\mathsf{j}) = \begin{cases} \frac{\frac{1}{2}(h_{\mathsf{j}}^u - \mathsf{tall}^l)^2}{A} & \text{in case (6)} \\ \frac{(\mathsf{tall}^u - \mathsf{tall}^l)(h_{\mathsf{j}}^u - \mathsf{tall}^u) + \frac{1}{2}(\mathsf{tall}^u - \mathsf{tall}^l)^2}{A} & \text{in case (3)} \\ \frac{(h_{\mathsf{j}}^l - \mathsf{tall}^l)(h_{\mathsf{j}}^u - h_{\mathsf{j}}^l) + \frac{1}{2}(h_{\mathsf{j}}^u - h_{\mathsf{j}}^l)^2}{A} & \text{in case (4).} \end{cases}$$

For two *independent* predicates, say *tall* and *clever*, saturated contexts induce a fully compositional semantics for logical connectives such as *and*. In other words, $[\textit{tall and clever}]_C$ is determined by the values $[\textit{tall}]_C$ and $[\textit{clever}]_C$. The independence of the two predicates is crucial here: every possible combination of a degree of tallness and a degree of cleverness for, say, *Jane* and corresponding threshold values for tallness and cleverness is assumed to occur with equal probability as (part of) a world in C. The probability that *Jane* is tall in a randomly selected world $w \in C$ is $[\textit{tall}(\mathsf{j})]_C$, while the probability that she is clever in w is $[\textit{clever}(\mathsf{j})]_C$. The independence implies that the probability of *Jane* being *tall and clever* at w is modeled as the joint probability, i.e., by the *product t-norm*:

$$[\textit{tall and clever}]_C(\mathsf{j}) = [\textit{tall}]_C(\mathsf{j}) \cdot [\textit{clever}]_C(\mathsf{j}).$$

As in Sect. 5.2 this analysis can be lifted with an analogous argument to the sentential level in order to model e.g. *Jane is tall and Ann is clever* as

$$\|\textit{Jane is tall} \wedge \textit{Ann is clever}\|_C = \|\textit{Jane is tall}\|_C \cdot \|\textit{Ann is clever}\|_C.$$

In contrast, the sentence *Jane is tall and Ann is tall* is not modeled in a truth-functional way by saturated contexts, in general. While saturation reflects the assumption that the *heights* of Jane and Ann are independent of each other, the respective *judgments of tallness* are not independent, since they refer to the same threshold value $\delta(w)(^{\uparrow}\textit{tall})$ in each world w of the context. In other words, the probability that *Jane is tall and Ann is tall* holds in a randomly selected world is not just a function of the probability that *Jane is tall* and the probability that *Ann is tall*, respectively. Rather, for arbitrary saturated contexts, one has to take into account the particular intervals of tallness for both *Ann* and *Jane* in relation to the possible thresholds to obtain the value for the compound statement. However, let us consider an interesting special case of saturated contexts, where there is *perfect knowledge* about the height of the individuals, but still vagueness in the meaning of *tall*. This means that all possible worlds agree on their values for the heights of *Jane*, $\mathsf{tall}(w)(\mathsf{j})$, and of *Ann*, $\mathsf{tall}(w)(\mathsf{a})$, while differing in their threshold value $\delta(w)(^{\uparrow}\textit{tall})$ for tallness. It is easy to see that in this case the membership degree of a conjunction (or disjunction) amounts to the minimum (or maximum, respectively) of the components' fuzzy membership degrees:

$$\|\textit{Jane is tall} \wedge \textit{Ann is tall}\|_C = \min(\|\textit{Jane is tall}\|_C, \|\textit{Ann is tall}\|_C),$$
$$\|\textit{Jane is tall} \vee \textit{Ann is tall}\|_C = \max(\|\textit{Jane is tall}\|_C, \|\textit{Ann is tall}\|_C).$$

In other words, the Gödel t-norm and co-t-norm appear as truth-functions for conjunction and disjunction in this specific case.

5.5 Dialogue Semantics

Giles's game (Giles 1970, 1974) is a combination of a *dialogue game* and a *betting scheme*, originally proposed by Robin Giles for reasoning in physical theories. Arguments about logically complex statements are reduced to arguments about atomic statements governed by dialogue rules that are intended to capture the meaning of logical connectives. In the final state of the dialogue game the players place bets on the results of dispersive experiments that decide about 'truth' and 'falsity' of occurrences of corresponding atomic statements. Below we present a re-interpretation of Giles's game in terms of Barker's contexts of precisifications instead of physical experiments. We also show how to account for predicate modifiers like *very* or *definitely* in this approach by extending the betting part of the game.

The dialogue part of Giles's game is a two-player zero-sum game with perfect information. The players are called *you* and *me*, with *me* initially asserting a logically complex statement. The game can be considered an evaluation game, since the players devise their strategies with respect to a payoff function that is determined by a given context C (in our case) or by given success probabilities of experiments associated with atomic assertions (in Giles's original setup).

At any point in the game each player asserts a multi-set of propositions, which we will call her *tenet*. Accordingly a game state is denoted as $[\psi_1, \ldots, \psi_n \mid \varphi_1, \ldots, \varphi_m]$ where $[\psi_1, \ldots, \psi_n]$ is your tenet and $[\varphi_1, \ldots, \varphi_m]$ is mine, respectively. Initial game states take the form $[\ \mid \varphi]$; i.e., I assert a single statement φ, while your tenet is empty. In each move of the game one of the players picks one of the statements asserted by her opponent and either challenges it or grants it explicitly. In both cases the picked statement is deleted from the state and therefore cannot be challenged again. The other player has to respond to the challenge in accordance with the following rules, that can actually be traced back to Lorenzen (1960).

Rule 1 (Implication). A player asserting *If* φ *then* ψ is obliged to assert ψ if her opponent challenges by asserting φ.
Rule 2 (Disjunction). A player asserting φ *or* ψ is obliged to assert either φ *or* ψ at her own choice.
Rule 3 (Conjunction). A player asserting φ *and* ψ is obliged to assert φ or ψ at her opponent's choice.

Negation is considered equivalent to the implication of a statement $\bot$ that is always evaluated as 'false'. Thus we obtain:
Rule 4 (Negation). A player asserting *not* φ is obliged to assert $\bot$ if his opponent challenges by asserting φ.

As already indicated, Giles stipulated that at the final state of the game the players have to pay a fixed amount of money, say 1€, for each atomic statement in their tenet that is evaluated as 'false' according to an associated experiment. These experiments may show dispersion, i.e., they may yield different answers upon repetition. However a fixed *risk value* $\langle p \rangle$ specifies the probability that the experiment associated with the atomic statement p results in a negative answer. My total risk, i.e., the expected amount of money[13] that I have to pay to you for an atomic state $[q_1, \ldots, q_n \mid p_1, \ldots, p_m]$ therefore is

$$\langle q_1, \ldots, q_n \mid p_1, \ldots, p_m \rangle = \sum_{i=1}^{m} \langle p_i \rangle - \sum_{j=1}^{n} \langle q_j \rangle .$$

Giles proved the following:

Theorem 5.1 (Giles 1970, 1974) *For all assignments of risk values to atomic statements I have a strategy to avoid positive risk in the game starting with my assertion that φ if and only if φ corresponds to a valid formula of Łukasiewicz logic.*

As has been demonstrated in Fermüller and Metcalfe (2009) an alternative rule for conjunction, that corresponds to the 'strong conjunction' interpreted by the Łukasiewicz t-norm $*_Ł = \lambda x, y. \max\{0, x + y - 1\}$, can be specified as follows.

Rule 5 (Strong conjunction). A player asserting φ *and** ψ is obliged to assert either both, φ as well as ψ, or to assert $\bot$.

The optional assertion of $\bot$ in this conjunction rule corresponds to a principle of limited liability that limits the amount of money to be paid for false statements to 1€, also for logically complex statements. An extended discussion of this principle can be found in Fermüller (2010). Further variants of Giles's game for other fuzzy logics have been presented in (Ciabattoni et al. 2005; Fermüller 2009). Here, we adapt the betting part of the game in order to relate the game to evaluations with respect to Barker's context model.

In our intended application the dialogue game ends in a state where you and me assert (in general) vague statements, like *John is tall*, that are logically atomic, i.e., they do not contain logical connectives. Instead of referring to dispersive experiments in physics, we now evaluate such atomic statements with respect to a given context C, consisting of a finite number of relevant precisifications (see Sect. 5.2). We stipulate that for each occurrence of an atom p in the final state a world $w \in C$ is randomly picked. The player that asserts the relevant occurrence of p has to pay 1€ to the opponent player if w does not survive the update of C with $[\![p]\!]$. Like in Giles's original scenario, we may speak of a *risk value* $\langle p \rangle$ associated with p in context C. Assuming a uniform distribution over C, we obtain

$$\langle p \rangle = 1 - \frac{|[\![p]\!](C)|}{|C|}.$$

[13] Note that risk, here, refers to *expected* payments and not to guaranteed bounds. If I am unlucky then, for a final state $[p \mid p]$, the experiment associated with p might yield a negative answer for *my* assertion that p, but might nevertheless yield a positive answer for *your* assertion that p. Accordingly I have to pay 1€ to you, although my corresponding total *risk* remains 0, independently of $\langle p \rangle$.

Let us illustrate this setup with a concrete example. Suppose I state

If Peter is heavy then John is tall

in a given context C. (Remember that we here stipulate the meaning '*if*' to correspond to material implication and not to refer to any causal or conceptual connection.) According to the dialogue rule for implication you may grant my statement, in which case the game ends in the empty state [|], where no risks or payments result. However, if you find more worlds v in C where *Peter* satisfies the standard $\delta(v)(^{\uparrow}heavy)$ of accepting heaviness than worlds w where *John* satisfies the standard $\delta(w)(^{\uparrow}tall)$ of tallness then it is rational for you (in the sense of game theory) to assert that *Peter is heavy*, thereby obliging me to assert that *John is tall*. The resulting state $\big[\textit{Peter is heavy} \mid \textit{John is tall}\big]$ carries my risk (i.e., expected amount of money in €, that I have to pay to you)

$$\langle \textit{John is tall} \rangle - \langle \textit{Peter is heavy} \rangle,$$

where

$$\langle \textit{John is tall} \rangle = 1 - \frac{|\{w \in C : \delta(w)(^{\uparrow}tall) > \mathsf{tall}(w)(\mathsf{j})\}|}{|C|}$$

and

$$\langle \textit{Peter is tall} \rangle = 1 - \frac{|\{w \in C : \delta(w)(^{\uparrow}heavy) > \mathsf{heavy}(w)(\mathsf{p})\}|}{|C|}$$

(j and p denote *John* and *Peter*, respectively).

By analyzing the proof of Theorem 1 (see Fermüller and Metcalfe 2009; Fermüller 2010) we obtain a direct connection between the dialogue rules and the t-norm based truth functions of Łukasiewicz logic. For this purpose, risk values for atomic statements are generalized inductively to risk values for complex statements taking into account that whenever I can choose I will *minimize* my risk, whereas a choice by *you* amounts to *maximizing* my risk over corresponding alternatives.

Proposition 5.2 *My risk involved in the assertion of a logically complex statement arises from the risks $\langle\varphi\rangle$ and $\langle\psi\rangle$ of its immediate sub-statements, as specified in the following table:*

my statement	*my risk*
φ *and* ψ	$\max\{\langle\varphi\rangle, \langle\psi\rangle\}$
φ *or* ψ	$\min\{\langle\varphi\rangle, \langle\psi\rangle\}$
If φ *then* ψ	$\max\{0, \langle\psi\rangle - \langle\varphi\rangle\}$
φ *and** ψ	$\min\{1, \langle\psi\rangle + \langle\varphi\rangle\}$

Note that the functions in Proposition 5.2 turn into the corresponding truth functions of Łukasiewicz logic by stipulating that the truth value $v_{Ł}(\varphi)$ of φ is obtained from its risk value by $v_{Ł}(\varphi) = 1 - \langle\varphi\rangle$.

In addition to modeling the evaluation of atomic propositions like *John is tall* we may specify game rules for the predicate modifiers *very*, *definitely*, *and clearly*, as well. For *very* the evaluation scheme does not have to be modified substantially: as above, we randomly pick a possible world $w \in C$ and test whether the proposition is locally true at w. Thus, e.g., the risk value $\langle \textit{John is very tall} \rangle$ is calculated as

$$\langle \textit{John is very tall} \rangle = 1 - \frac{|\{w \in C : \delta(w)(^{\uparrow}\textit{tall}) + \delta(w)(^{\uparrow}\textit{very}) > \mathsf{tall}(w)(\mathsf{j})\}|}{|C|}$$

with j denoting *John*.

However, for other predicate modifiers like *definitely* or *clearly* we cannot decide if the proposition holds at w without taking into account also the other worlds in C. We have to change the evaluation scheme accordingly.

Assume that I assert, e.g., *John is definitely tall* in a context C. Reflecting Barker's definition of *definitely* discussed in Sect. 5.2, this assertion is evaluated as follows. First, a world $w \in C$ is picked randomly. Then *you* choose a world $v \in C$ where *John* is just as *tall* as in w (i.e., where $\mathsf{tall}(w)(\mathsf{j}) = \mathsf{tall}(v)(\mathsf{j})$). Finally, we stipulate that I have to pay 1€ to you if *John* is not *tall* at v, i.e. if $\delta(v)(^{\uparrow}\textit{tall}) > \mathsf{tall}(v)(\mathsf{j})$. The pay off scheme for *your* assertions of an atomic statement involving *definitely* is completely symmetric.

As defined by Barker, *clearly* acts like a combination of *definitely* and of *very*, where the vague standard $\delta(w)(^{\uparrow}\textit{clearly})$ is used instead of $\delta(w)(^{\uparrow}\textit{very})$. Therefore the proposition John is clearly tall is evaluated analogously to John is definitely tall. The only difference is that I now owe you 1€ if $\delta(v)(^{\uparrow}\textit{tall}) + \delta(v)(^{\uparrow}\textit{clearly}) > \mathsf{tall}(v)(\mathsf{j})$ holds.

5.6 Contexts and Similarity Based Reasoning

Remember that the intended use of contexts in linguistic models of vagueness is to specify sets of plausible alternatives of precisified interpretations (classical worlds), given the current information of the hearer of an utterance. While Barker's dynamic semantics filters out those worlds of a context that become implausible upon accepting the relevant utterance, one may alternatively be interested in evaluating the degree of plausibility or 'truth' of a sentence with respect to the information coded in the given context as a whole. In this endeavor it seems natural to start with the observation that the individual worlds that form a concrete context are to higher or lesser degree similar to each other. After all, vagueness in this model amounts to the fact that, while hearers don't have access to precise criteria for judging a statement as definitely true or false, they are nevertheless supposed to evaluate with respect to a *given set* of such precise criteria that is *constrained in a specific manner* reflecting the context of discourse. Taking the degrees of similarity between the worlds as a basis of an evaluation that is *graded* accordingly provides a further bridge between contextual models and fuzzy logic.

A (fuzzy) similarity relation on a set A is a function $S : A \times A \to [0, 1]$ that is reflexive, symmetric, transitive with respect to some t-norm $*$:

- $S(x, x) = 1$ for all $x \in A$,
- $S(x, y) = S(y, x)$ for all $x, y \in A$, and
- $S(x, y) * S(y, z) \leq S(x, z)$ for all $x, y, z \in A$.

Similarity relations are well investigated for concrete underlying t-norms. For the Product t-norm the concept goes back to Menger (1951) and has been studied by Ovchinnikov (1991); for the Gödel t-norm (min) see Zadeh (1971); for the Łukasiewicz t-norm see Ruspini (1977) and Bezdek and James (1978). Similarity relations and fuzzy sets are closely related. Given a similarity relation S on A and a (crisp) subset B of A one obtains a normalized fuzzy subset B^* of A—the fuzzy set of elements close to B—by defining the membership degree for every $x \in A$ as follows:

$$\mu_{B^*}(x) = \sup_{v \in B} S(x, v).$$

Conversely, similarity relations are induced by fuzzy sets, according to Valverde's representation theorem Valverde (1985). For every similarity relation S there is a fuzzy set F such that

$$S(x, y) = \min(\mu_F(x) \Rightarrow_* \mu_F(y), \mu_F(y) \Rightarrow_* \mu_F(x)),$$

where $\Rightarrow_*$ is the residuum of some (left continuous) t-norm.

Based on the principle that a proposition can be identified with the set of worlds in which it holds, various different formal models of similarity based reasoning have been defined in the literature (Dubois et al. 1997; Esteva et al. 2000; Godo and Rodrıguez 2008) provide relevant overviews). In particular Dubois, Esteva and Godo, with various collaborators, have studied different entailment relations arising from similarity relations over sets of worlds, i.e., of *contexts* in our current terminology. These entailment relations generalize Ruspini's notion of graded implication Ruspini (1991) given by

$$I_S(p \mid q) = \inf_{v \models q} \sup_{w \models p} S(v, w),$$

where S is a similarity relation over a set of classical worlds and p, q are atomic propositions. Intuitively $I_S(p \mid q)$ measures the extent to which all worlds in which q holds are close to some world in which p holds. In the following we will illustrate just one out of many options that arise for linguistic models of vagueness following this approach.

Once a particular (fuzzy) similarity relation S is declared on a context C, the machinery of (Esteva et al. 2000; Godo and Rodrıguez 2008) can be directly applied to define logics that refer to graded entailment relations derived from S on C. But the question arises *how* one obtains a similarity relation that adequately reflects the semantic information represented by C. As to be expected, there is no unique canonical way of doing so.[14] However an interesting possibility emerges if one follows

[14] Note that the mentioned literature does not address this problem. There, the similarity relation over worlds is assumed as given and remains independent of the structure of the worlds themselves

linguists in assuming that the context contains a *comparison class of paradigmatic cases* as reference of judgment (see Kennedy (2007), Rooij (2011), and further references there). For example, restricting attention to a single individual, say *Jana*, and a particular gradable adjective, say *tall*, we assume that a subset P (paradigmatic worlds) of the context C singles out those worlds in which Jana satisfies the respective standard of tallness. Since we let the comparison class consist just of Jana in accepting a person as tall, we take all worlds in P to be maximally similar to each other; i.e., we assign $S(v, w) = 1$ for $v, w \in P$. For each world $u \in C - P$ we again consider the height (degree of tallness) of *Jana* in u and define

$$S(u, w) = 1 - \frac{\delta(u)(^{\uparrow}tall) - \mathsf{tall}(u)(\mathsf{j})}{\max_{v \in C}(\delta(v)(^{\uparrow}tall) - \mathsf{tall}(v)(\mathsf{j}))}$$

if $w \in P$, where, like in Sect. 5.2, $\delta(v)(^{\uparrow}tall)$ denotes the standard of tallness in world v and $\mathsf{tall}(v)(\mathsf{j})$ the height of *Jana* in v. In other words: the closer *Jana*'s height in a world gets to the standard of accepting tallness there, the more similar this world is to a paradigmatic world. Therefore $S(u, w)$ in this case does not directly depend on w, but on the whole class of paradigmatic worlds. (For $u \in P$ and $w \in C - P$ the similarity $S(u, w)$ is defined analogously.) If both worlds u and w are in $C - P$ then we define analogously[15]

$$S(u, w) = 1 - \frac{|((\delta(u)(^{\uparrow}tall) - \mathsf{tall}(u)(\mathsf{j})) - (\delta(w)(^{\uparrow}tall) - \mathsf{tall}(w)(\mathsf{j}))|}{\max_{v \in C}(\delta(v)(^{\uparrow}tall) - \mathsf{tall}(v)(\mathsf{j}))}.$$

It is straightforward to check that S, thus defined, is reflexive, symmetric, and transitive with respect to the Łukasiewicz t-norm. (To obtain fuzzy similarity relations with respect to other t-norms one has to use alternative definitions of S or to impose specific constraints on the contexts.) This opens the way to systematically assign degrees of acceptability of statements like *If Jana is tall then Peter is tall* in contexts C where not only *tall* is vague, but where there might also be uncertainty about the respective heights of Jana and Peter. We only need to apply Ruspini's measure I_S to the arguments *If Jana is tall* and *Peter is tall*, where S is extracted from C as indicated. Of course it is more problematic to extract suitable similarity relations from contexts, where we look at different gradable adjectives simultaneously and where more than one individual is designated as paradigmatic. While there is no technical obstacle in doing so, criteria for evaluating the adequateness of the resulting formal model are much less clear. In any case, as already mentioned, once a similarity relation is fixed one may employ the results of Esteva, Godo et al., to generalize

(at least in principle). However, for our current purpose, we have to take into account that similarities typically depend on the particular valuations of atomic formulas that characterize the individual worlds.

[15] In the case where there is no uncertainty about Jane's height, this definition can be simplified by changing the numerator to $|\delta(u)(^{\uparrow}tall) - \delta(w)(^{\uparrow}tall)|$.

to logically complex assertions and more general entailment relations; thus—so to speak—having crossed yet another bridge between contextual linguistic models and mathematical fuzzy logic.

5.7 Summary and Outlook

We commenced by observing that linguists prefer to analyze the semantics of vague expressions by reference to contexts of utterance that register relevant possible precisifications from the hearer's perspective. This seems to be at variance with the degree based approach to vagueness suggested by fuzzy logic. However, taking Barker's version of dynamic (update) semantics (Barker 2002) as a point of reference, we have demonstrated that fuzzy sets can be associated in a systematic manner with contexts and corresponding filters as used in Barker's model. While the structure of context filters used to specify the different meanings of modifiers like *very*, *definitely*, and *clearly* allows to take into account information that is abstracted away in corresponding fuzzy sets, standard t-norm based operators faithfully register the extremal cases that may result from applying logical connectives to vague predicates and sentences.

It is rather straightforward to identify intermediate truth values with the fraction of worlds in a given context that survive certain updates codifying the meaning of vague expressions. But it is much less clear how to derive specific truth functions in such a setting (beyond providing the indicated bounds). This problem, of course, is just a particular instance of a well known challenge for deductive fuzzy logic: how to justify truth functions with respect to more fundamental semantic notions like, e.g., votes or arguments for and against accepting a vague assertion. In (Paris 2000), the author provides a useful overview over semantic frameworks for fuzzy logics that support truth functionality. Here we have selected two examples of such frameworks—dialogue games and similarity based reasoning—to illustrate how one may connect context based update semantics with t-norm based fuzzy logics.

We emphasize that both, Barker's specific update functions over contexts and the indicated fuzzy semantics, should be understood as just particular spots on either side of the river separating formal semantics of natural language as pursued by linguists from fuzzy logic. Other sites for building bridges crossing that troubled water should be explored as well. On the linguistic side, context and precisification based approaches suggested, e.g., by Kennedy (2007), Kyburg and Morreau (2000), and already earlier by Pinkal (1995) and Bosch (1983) are certainly worth investigating from this perspective. On the fuzzy logic side, e.g., voting semantics Lawry (1998), acceptability semantics Paris (1997), re-randomising semantics (Hisdal 1988; Hájek 2001), and approximation semantics (Bennett et al. 2000; Paris 2000) are alternative candidates for constructing corresponding bridgeheads. We plan to explore at least some of these options in future work. In any case, we hope to have shown already here that the attempt to bridge the gap between linguistic views on vagueness and the machinery offered by fuzzy logic is neither a futile nor a completely trivial matter.

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Part III
Semantics and Consequence Relation in Many-Valued Logic

This part deals with the semantics of many-valued logics, and contains three chapters. The first two chapters introduce new kinds of semantics, one based on the principle that all truth degrees are to be preserved (not only degree 1), and the other based on the behavior of the formula and of its derivatives in the neighbourhood of a point. The third chapter is about a classification of the most important many-valued logics in terms of general semantic principles.

Semantics is a fundamental concept in many-valued logic. Hájek basically proposed three kinds of semantics, one based on the whole variety of the algebras for the logic (consisting of prelinear, commutative, integral, and bounded residuated lattices), the second based on the chains of the variety, and the third called *the standard semantics*, based on algebras having [0, 1] as lattice reduct. But in all cases, consequence relation is defined in terms of valuations, as preservation of degree 1. Hence, the first two chapters of this part, which introduce two alternative semantics, constitute an important and original contribution to Hájek's research. As regards the third chapter, the idea of classifying logics according to semantic principles is new and opens an interesting line of research.

In more detail, in the chapter *Consequence and degrees of truth in many-valued logic*, by Josep Maria Font, the author discusses an alternative notion of consequence relation. Instead of requiring, like in Hájek's book, preservation of degree 1, the new interpretation requires preservation of all truth degrees. Interestingly, the two semantics provide the same set of theorems, but consequence relations are quite different. It is argued that the new definition is a better rendering of Bolzano's idea of consequence in terms of preservation of truth when truth comes in degrees. In the chapter, the author extends some results previously obtained for Łukasiewicz logic to the broader framework of substructural logics.

The chapter, *The differential semantics of Łukasiewicz syntactic consequence*, by Daniele Mundici, investigates the problem of strong standard completeness for Łukasiewicz logic. As we said, Petr Hájek devoted much effort in the standard semantics. Among other things, he proved that the most important fuzzy logics are complete, but not strongly complete (with the exception of Gödel logic), with respect to the standard semantics. Now Mundici replaces the traditional consequence relation

$$\Gamma \models \varphi \text{ iff all valuations validating } \Gamma \text{ validate } \varphi,$$

by another, more geometric notion, still based on the standard semantics, but taking into account not only the behavior of a set of formulas at a point (valuation) but also in a neighborhood of the point, and considering also the derivatives of the corresponding truth functions. Somewhat surprisingly, strong standard completeness is completely restored. This chapter not only proposes a new notion of standard semantics for which strong standard completeness holds, but opens the possibility of proving strong standard completeness (with respect to the new interpretation) for other fuzzy logics, or possibly, for Łukasiewicz first-order logic.

In the chapter *Two principles in many-valued logic*, by Stefano Aguzzoli and Vincenzo Marra, the authors discuss two basic principles, which are both valid in classical logic. The principle (P1) says that two formulas are equivalent if they receive truth value 1 for the same valuations. The principle (P2) says that given two different valuations v and w, there is a formula φ such that $v(\phi) = 0$ and $\mathrm{w}(\varphi) \neq 0$. The three main logics of continuous t-norms are characterized in terms of the above-mentioned principles. That is, among all logics of continuous t-norms, Łukasiewicz logic is the unique one that satisfies (P2), Gödel logic is the unique one that satisfies (P1), and product logic is the unique one such that each of its extensions, with the exception of classical logic, which fails both (P1) and (P2).

Chapter 6
Consequence and Degrees of Truth in Many-Valued Logic

Josep Maria Font

6.1 Introduction

Let me begin by calling your attention to one of the main points made by Petr Hájek in the introductory, vindicating section of his influential book (Hájek 1998) (the italics are his):

> *Logic studies the notion(s) of consequence*. It deals with propositions (sentences), sets of propositions and the relation of consequence among them. [page 1]
> […]
> *Fuzzy logic is a logic*. It has its syntax and semantics and notion of consequence. It is a study of consequence. [page 5]

Petr's book contains no discussion on how consequence in mathematical fuzzy logic should be defined, or why. He simply defines his consequences either by a Hilbert-style axiomatization or semantically by the *truth-preserving* paradigm, which takes 1 as the only designated truth value in the real interval [0, 1] or in other algebraic structures which are ordered and have a maximum value 1. That is, if Γ is a set of formulas and φ is a formula, then[1]

$$\Gamma \vdash \varphi \iff \begin{array}{l} e(\varphi) = 1 \text{ whenever } e(\alpha) = 1 \text{ for all } \alpha \in \Gamma, \\ \text{for any evaluation } e \text{ in the model.} \end{array} \tag{6.1}$$

I would also like to call your attention to a result about propositional Gödel-Dummett logic G, whose consequence $\vdash_{\mathrm{G}}$ is defined axiomatically on p. 97 of Hájek

[1] In this chapter I will represent logics as consequences by the symbol $\vdash$, independently of the way they are defined, be it of semantical or syntactical origin, and will add sub- or superscripts when needed. The symbol $\vDash$ will only be used for satisfaction of equations in (classes of) algebras.

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(1998). G is proved in Theorem 4.2.17 to be strongly complete with respect to the standard Gödel algebra over [0, 1] taking the minimum as the t-norm whose residuum interprets the implication. Then Theorem 4.2.18 reads:

Theorem 6.1 *For each theory T over* G*, each formula φ and each rational r such that $0 < r \leqslant 1$, $T \vdash_{\mathrm{G}} \varphi$ iff each evaluation e such that $e(\alpha) \geqslant r$ for each axiom α of T satisfies $e(\varphi) \geqslant r$.*

The same holds if we take all reals in [0, 1] instead of the rationals (by the density of $\mathbb{Q}$ inside $\mathbb{R}$), but Petr establishes this only to give a relation with "partial truth", which has been previously discussed in the book in the framework of the Rational Pavelka Logic.

This result, which in Hájek (1998) appears to be an anecdotal result on the standard semantics of G, has an alternative view when it is reformulated as the coincidence of two consequences: if we define

$$\Gamma \vdash^{\leqslant} \varphi \iff \begin{array}{l} e(\varphi) \geqslant r \text{ whenever } e(\alpha) \geqslant r \text{ for all } \alpha \in \Gamma, \\ \text{for any evaluation } e \text{ and any value } r \text{ in the model,} \end{array} \tag{6.2}$$

then Theorem 6.1 says that $\vdash_{\mathrm{G}}$ and $\vdash^{\leqslant}$ coincide when the model at hand is the Gödel algebra of rationals, or equivalently of the reals, in [0, 1]. This is a more interesting perspective, and it is then natural to wonder whether it holds for other many-valued logics, and why, and whether it is just a technical result or whether it hides some deeper insights.

For future reference let me say now that when considering the definition (6.2) in general, if the model has a complete lattice structure, then it can be equivalently put in the form

$$\Gamma \vdash^{\leqslant} \varphi \iff e(\varphi) \geqslant \bigwedge\{e(\alpha) : \alpha \in \Gamma\} \text{ for all evaluations } e. \tag{6.3}$$

We will see that this setting has also been popular. When 1 is the maximum of the ordered model set, either (6.1), (6.2) or (6.3) yield the same set of *theorems*:

$$\emptyset \vdash \varphi \iff \emptyset \vdash^{\leqslant} \varphi \iff e(\varphi) = 1 \text{ for all evaluations } e. \tag{6.4}$$

Note that, while this is clearly included in (6.1) and (6.2), for this to follow from (6.3) the implicit assumption that the infimum of an empty set is the maximum of the order is needed.

Investigating these and similar issues we discover a connection with the area of *logics preserving degrees of truth*, which has been gaining momentum recently; see Bou (2008, 2012), Bou et al. (2009), Font (2007, 2009), Font et al. (2006). So I will begin by discussing this idea in general (Sect. 6.2); then I will review the results obtained so far in the literature for the case of Łukasiewicz's infinite-valued logic (Sect. 6.3) and for a larger family ofsubstructural logics (Sect. 6.4). The resulting logics are particularly interesting for *abstract algebraic logic* (Sect. 6.5). I will briefly review some results on the Deduction Theorem (Sect. 6.6) and on their axiomatization (Sect. 6.7). The chapter ends with some research proposals.

6.2 Some Motivation and Some History

Speaking generally, logics defined like (6.1) are called *truth-preserving*, while logics defined like (6.2) or (6.3) are called logics that *preserve degrees of truth*. Just a few words to argue why I think that the latter reflects the semantical idea of many-valued logic better than the former; for a lengthier discussion in a wider context, see Font (2009).

The idea of logical consequence as a truth-preserving one, firmly established from Bolzano to Tarski and beyond, is reasonably unproblematic when there is a single notion of truth in the models, and even more when there is a single model. However, it is at least surprising that it has not raised any significant debate in the context of many-valued logic.

Phrases such as "*Truth comes in degrees*" (Cintula et al. 2011, p. v) or "*Truth of a fuzzy proposition is a matter of degree*" (Hájek 1998, p. 2) appear as a starting justification in many papers and books on fuzzy logic or many-valued logic. One may discuss the meaning of these degrees of truth, their philosophical significance, whether they adequately reflect the phenomenon of *vagueness*, and so on, and for those wanting to do this Smith (2008) is a very enlightening exposition. But I think that for the (mathematical) logician the important thing is not to discuss what they *are* or *should be*, but how they are *used* (to define a logic).

Now, if logic dealt only with *tautologies*, then it would be natural to define them as those propositions that are always true, that is, their truth value always attains the maximum degree, as in (6.4). However, if it is *consequence* that matters, then it seems more natural to demand that consequence preserves truth not only in its maximum degree, but in all the available degrees. Thus, the usage of (6.1) in many-valued contexts raises some dissatisfaction: it seems as if, while all points in the model are considered as truth values when the task is to determine the truth value of a complex formula from the truth values of the atomic formulas,[2] only 1 is really treated as a truth bearer when the task is to establish consequence. Under this view, the other points in the model seem to be treated rather as expressing *degrees of falsity*.[3]

Scheme (6.2) can even be considered as an alternative rendering of the same idea of preservation of truth: not of absolute truth, but of that truth that comes in degrees and characterizes the many-valued landscape. While individual points in a model V may still be regarded as *truth values* in that they are the values assigned to propositions by each of the evaluations, (6.2) suggests identifying *degrees of truth* with the sets $\uparrow r = \{s \in V : r \leqslant s\}$, and then implements the idea that consequence

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[2] In whatever mechanism; one need not assume truth-functionality for this discussion to make sense.

[3] Scott in Scott (1974, p. 421) calls them "degrees of error", see below. Gottwald (2001, Sect. 3.1) seems to be sympathetic with this idea as well.

is the relation that *preserves* all these sets; it is in this sense that it is called "preserving degrees of truth". Since the two schemes produce the same set of tautologies (6.4), separate consideration of the logics obtained by the two paradigms is only of interest when assigning the central rôle in logic to consequence. The second paradigm is potentially as general as the first one; it can be applied to any semantics where truth values are ordered and there is a maximum one, which is indeed a very reasonable and common assumption.[4] It may also be taken to justify interpreting generalized matrices as the most general *structures of degrees of truth*, but this is another issue, discussed in Font (2009).

Logics of the form $\vdash^{\leqslant}$ appeared in the literature much earlier than Hájek (1998), but were only thoroughly studied much more recently. The idea seems to have sprung up independently, but sporadically, in several circles in the early 1970s.

The best motivated precedent is found in the well-known papers by Scott (1973, 1974). These papers contain a critical view on the interpretation of many-valued logic in general, and particularly on the usage of schemes similar to (6.1), perhaps with more than one element playing the rôle of 1, the "designated elements" in the theory of logical matrices (the italics are his):

> One quirk of many-valued logic that always puzzled me was the distribution of *designated elements*. They were somehow "truer" than the others. [...] On the one hand we were denying bivalence by contemplating multivalued systems; but on the other, a return to bivalence was provided by the scheme of designation. Scott (1973, p. 266)

Scott wants to find an interpretation of the non-classical truth values that justifies both the truth tables and the rules of Łukasiewicz logic, and eventually proves completeness. He first interprets the truth values as "types of propositions" and later on as "degrees of error in deviation from the truth", see Scott (1973, p. 271, 1974, p. 421). He then makes a proposal, summarized in the phrase "to replace many values by many valuations", which actually amounts to considering not just a single matrix but a set of n matrices for each n-valued logic, the designated sets being the principal filters of the n-element Łukasiewicz chain; therefore, this proposal turns out to be essentially scheme (6.2). That such an idea leads to a definitely different "conditional assertion" (i.e., consequence relation) is already observed by realizing that *Modus Ponens* in its usual form would fail, but would still hold in the restricted form

$$\text{if } \vdash^{\leqslant} \alpha \to \beta \text{ then } \alpha \vdash^{\leqslant} \beta, \tag{6.5}$$

which will re-appear in Theorems 6.12 and 6.13. What Scott does explicitly for this consequence in Scott (1974) is to define a set of Gentzen-style rules (of the "multiple conclusion" kind) and to prove its completeness, in the sense that the derivable sequents of this calculus coincide with the entailments of the consequence $\vdash^{\leqslant}$ (extended to be of the "multiple conclusion" kind as well). Surprisingly, this calculus does not contain the fusion connective, nor any rules expressing its residuated

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[4] That the truth degrees can be compared (i.e., ordered) seems to be another essential ingredient motivating fuzzy logic: "*We shall understand [fuzzy logic in the narrow sense] as a logic with a comparative notion of truth*" (Hájek 1998, p. 2).

character with respect to implication. In any case, only the completeness part of Scott (1974) seems to have had some impact on the evolution of research on many-valued logic in the following years; the proposal of a different consequence relation seems to have passed unnoticed.

Another mathematically clear though philosophically less motivated precedent of the idea is found in the contemporary Cleave (1974), where the author studies Körner's reinterpretation of Kleene's strong 3-valued logic as a logic (and an algebra) of "inexact predicates". He defines a first-order logic, in a language without implication, as a consequence relation, and chooses to do so by explicitly using (6.3) from Kleene's truth tables, justifying this move only in that it is a generalization of the classical case.[5] The associated relation of logical equivalence, which here coincides with interderivability, turns out to be the identity of truth functions, and so the corresponding Lindenbaum-Tarski construction can be easily performed. Algebraic structures related to this logic are just presented as the De Morgan algebras, but this is wrong; actually they should be the Kleene algebras (i.e., the De Morgan algebras satisfying the inequality $x \wedge \neg x \preccurlyeq y \vee \neg y$), see Balbes and Dwinger (1974, Sect. XI.3) and Font (1997, Sect. 5.1). The main goal of the paper, though, is to present a Gentzen-style axiomatization and to prove its completeness by Schütte-style methods.

At the end of the seventies Pavelka (1979) incorporates degrees of truth to the landscape of many-valued logic in a novel way, but not in the sense of preservation of degrees of truth as we are considering. Inspired by Goguen (1969), he introduces fuzzy logics as fuzzy consequence relations between fuzzy sets of formulas and formulas, with membership degrees coinciding with truth degrees. Moreover, he represents each truth degree $r \in [0, 1]$ as a constant $\overline{r}$ of the language.[6] He considers an axiomatic system where each inference rule is coupled with a rule to calculate provability degrees (degrees of truth of statements saying that something follows from something), and proves what has since been termed "Pavelka-style completeness", which is the coincidence of the degree of membership of a formula φ to the consequences of a fuzzy set of formulas $\widetilde{\Gamma}$ with the degree of provability of φ from $\widetilde{\Gamma}$. Later on this proposal was reformulated in Hájek (1995) by taking only constants for the rationals in [0, 1] (hence the name "Rational Pavelka Logic") and considering "graded formulas", i.e., pairs (φ, r) intended to mean "proposition φ has truth degree at least r", so that the syntax is a calculus of these graded formulas. In Hájek (1998, Sect. 3.3) these pairs are finally taken to be aliases for the formulas $\overline{r} \to \varphi$, because in an evaluation e in the unit interval, $e(\overline{r} \to \varphi) = 1$ if and only if $r \leqslant e(\varphi)$. In this way, Pavelka's idea can be studied in an expansion of the ordinary truth-preserving logic of Łukasiewicz; however, while degrees of truth *seem* to play a more proper rôle in it and in other more recent works in the same line (see Esteva et al. (2007)

[5] As we now know, coincidence of this way of expressing semantical consequence with the truth-preserving one also holds in other, non-classical cases, see Theorems 6.1, 6.2, 6.3 and 6.5.

[6] Pavelka develops his proposal for **L**-valued fuzzy sets, where **L** is an arbitrary complete residuated lattice, but proves his completeness result for the cases where **L** is [**0**, **1**] and all its finite subalgebras.

and references therein), the intended semantics is still truth-preserving, as there is no quantification over all truth degrees when considering consequence.[7]

At the end of the eighties the first major paper where the expression "logics preserving degrees of truth" was coined as having a technical, semantical meaning was published; this was Nowak (1990), preceded by the shorter Nowak (1987). There three schemes implementing the same idea are compared in an abstract algebraic context; the one which amounts to (6.3) is given the same name, and two other variants are called "weakly preserving degrees of truth" and "strongly preserving degrees of truth". The "weakly preserving" and the standard cases are characterized in Nowak (1990, Theorems 3.2 and 4.6) in terms of the abstract properties of selfextensionality (see Sect. 6.5) and projective generation. In this paper the expression "structures of degrees of truth" is proposed to denote any algebraic structure with an ordering relation; finally Theorem 7.6 is obtained:

Theorem 6.2 *Let* $\vdash$ *and* $\vdash^{\leqslant}$ *be the logics defined according to schemes* (6.1) *and* (6.3) *with respect to some algebra with a complete lattice reduct. If the algebra is a complete linear Heyting algebra then* $\vdash \, = \, \vdash^{\leqslant}$.

This does not apply directly to Theorem 6.1, for $[0, 1] \cap \mathbb{Q}$ is not a complete lattice; however, the version of Theorem 6.1 for real numbers, which is also true, is clearly equivalent to the particular case of Theorem 6.2 for the Heyting algebra structure of [0, 1].

Some years later the implication of Theorem 6.2 was refined and shown to be an equivalence: see Theorem 6.5 below.

Another, independent appearance of essentially the same property is found in the conference paper Baaz and Zach (1998), contemporary to Hájek (1998), in a study of Gödel-Dummett logic in the modern framework of fuzzy logics. Here the two logics defined from the two schemes (6.1) and (6.3) are considered when evaluations are restricted to a subset $V \subseteq [0, 1]$, with Gödel's operations; let us denote them as $\vdash_V$ and $\vdash_V^{\leqslant}$ respectively. Then Proposition 2.2 of Baaz and Zach (1998) reads:

Theorem 6.3 *For each closed* $V \subseteq [0, 1]$, $\vdash_V \, = \, \vdash_V^{\leqslant}$.

The same setting and result appear again in Baaz et al. (2007, Proposition 2.15), and after it the authors remark that this is "a unique feature of Gödel logics", but support this claim only with an example showing that it does not hold in Łukasiewicz logic, namely, the failure of *Modus Ponens* for the logic $\vdash_V^{\leqslant}$ when V is the Łukasiewicz algebra $[\mathbf{0}, \mathbf{1}]$ instead of the Gödel algebra on [0, 1]. This statement of uniqueness can be considered correct if understood as referring to the basic fuzzy logics, as has been extended and made precise later on: see Theorem 6.5 and property 5 after Theorem 6.7.

While the discussion in Scott (1973, 1974) is obviously centred on the issue of how to define entailment or consequence in many-valued logic, the discussion in Nowak (1987, 1990) is related to a more general problem, considered by the first

[7] For other approaches to graded consequence, even farther removed from preservation of degrees of truth, see (Chakraborty and Dutta 2010; Gerla 2001).

time in Suszko (1961, Sects. 10, 11) and dealt with in more depth in Wójcicki (1984, Chap. III) and in Wójcicki (1988, Sects. 1.6, 2.10). The central, non-trivial issue is *the relation between logical truth and logical consequence*, technically formulated as the problem of whether and when there is a logic (i.e., a consequence relation) $\vdash_L$ having a given set of formulas L as the set of its theorems, and, if so, how it should be defined in a natural way. Wójcicki's proposal, slightly different in his two studies, amounts to assuming that two binary connectives $\wedge$ and $\to$ exist so that one can define

$$\begin{aligned} \alpha_1, \dots, \alpha_n \vdash_L \varphi &\iff \alpha_1 \wedge \dots \wedge \alpha_n \to \varphi \in L, \\ \emptyset \vdash_L \varphi &\iff \varphi \in L. \end{aligned} \tag{6.6}$$

Moreover, the consequence $\vdash_L$ is assumed to be finitary. Of course these stipulations will define a consequence only when L satisfies certain conditions relative to $\wedge$ and $\to$, which Wójcicki determines. His motivation is clearly twofold, for he is aware of the general problem, which he discusses at length, but also of the fact that in the case of Łukasiewicz many-valued logics it is perfectly natural to consider several consequences; actually in 1973 he already published one of the few early papers on this topic, Wójcicki (1973), devoted to comparing the truth-preserving and the *Modus Ponens*-based consequences[8] based on several subalgebras of Łukasiewicz's algebra $[\mathbf{0}, \mathbf{1}]$. This paper, however, does still not consider the consequences preserving degrees of truth; these appear first in the lecture notes Wójcicki (1984), and the quotations in Nowak (1987) make it clear that it was this particular case what inspired Nowak's general study.

6.3 The Łukasiewicz Case

Thus, in Wójcicki (1984, 1988) several consequences defined from each of the sets of tautologies of Łukasiewicz ξ-valued logics are studied, for $\xi \leqslant \aleph_1$[9]; the two we are interested in now are an axiomatically defined one (which I denote here by $\vdash_\xi$) where the axioms are all these tautologies and the only rule is *Modus Ponens*, and one defined by scheme (6.6), denoted in Wójcicki (1988) by $\mathscr{L}_\xi^{(\leqslant\models}$. Concerning the latter, it is only shown that for each finite n the logic $\mathscr{L}_n^{(\leqslant\models}$ coincides with the consequence $\vdash_n^{\leqslant}$ that preserves degrees of truth from the n-element subalgebra of $[\mathbf{0}, \mathbf{1}]$ in the sense of (6.3), and that $\vdash_n^{\leqslant}$ is strictly weaker than $\vdash_n$.

[8] Among the main results, he proved that the two consequences coincide for the finite subalgebras but not for the denumerable one or for the whole interval, in which cases the truth-preserving consequences are not finitary. However, they do coincide on finite sets of assumptions; thus, if one considers only the associated finitary consequences, then the two fully coincide.

[9] For $\xi \leqslant \aleph_0$, ξ is the cardinality of the subalgebra of $[\mathbf{0}, \mathbf{1}]$ taken as the model; in the $\aleph_0$ case, it is the rational subalgebra. $\aleph_1$ is used to refer to the whole algebra $[\mathbf{0}, \mathbf{1}]$.

Explicitly continuing the work in Wójcicki (1988), in the mid-1990s an algebraic study of the logics $\vdash_n^{\leqslant}$ was begun[10] in Gil (1996, Sect. 5.5). The logics, which appear here only marginally in the study of n-sided Gentzen systems, are shown not to be algebraizable but to be finitely equivalential, and to satisfy a Deduction Theorem for the formula $(x \to y)^n \vee y$; the (unpublished) work does not go any further.

The totally general case of logics preserving degrees of truth from arbitrary subalgebras of $[\mathbf{0}, \mathbf{1}]$ appears only in the twenty-first century, namely in Font and Jansana (2001, Sect. C), where two consequences $\vdash_{\mathbf{S}}$ and $\vdash_{\mathbf{S}}^{\leqslant}$, defined by (6.1) and (6.2) respectively, are associated with each subalgebra $\mathbf{S}$ of $[\mathbf{0}, \mathbf{1}]$. These logics, again, appear only marginally in this paper, as examples for some points of abstract algebraic logic, and only the following basic properties are of interest here:

1. For each $\mathbf{S}$ except the 2-element algebra, $\vdash_{\mathbf{S}}$ is a proper extension of $\vdash_{\mathbf{S}}^{\leqslant}$, but the two logics have the same theorems.
2. For all infinite $\mathbf{S}$, all the logics $\vdash_{\mathbf{S}}$ and all the logics $\vdash_{\mathbf{S}}^{\leqslant}$ have the same theorems: the tautologies of $\vdash_{[\mathbf{0},\mathbf{1}]}$, i.e., of ordinary Łukasiewicz logic.
3. If $\mathbf{S}_1 \neq \mathbf{S}_2$ then $\vdash_{\mathbf{S}_1} \neq \vdash_{\mathbf{S}_2}$ and $\vdash_{\mathbf{S}_1}^{\leqslant} \neq \vdash_{\mathbf{S}_2}^{\leqslant}$.
4. The logics $\vdash_{\mathbf{S}}$ and $\vdash_{\mathbf{S}}^{\leqslant}$ are finitary if and only if the algebra $\mathbf{S}$ is finite.

Point 1 extends the already mentioned remark of Baaz et al. (2007) about the failure of Theorems 6.1 and 6.3: they fail in the Łukasiewicz case, not only for the whole algebra on [0, 1] but for any subalgebra. Point 2 reinforces the idea that the logics preserving degrees of truth are only interesting when considering the consequence relation, as even for different (infinite) subalgebras of $[\mathbf{0}, \mathbf{1}]$ they yield the same tautologies.

The result in point 4, which extends a result from Wójcicki (1973),[11] suggests that one move to create a more uniform setting admitting a smoother treatment inside the framework of abstract algebraic logic might be to force all logics under discussion to be finitary.[12] This is done in Font (2003), where the schemes (6.1) and (6.2) are used to define the consequences only of finite Γ; the same symbols $\vdash$ and $\vdash^{\leqslant}$ will be used from now on. Moreover, since all models under consideration have a lattice structure,[13] (6.2) can be replaced by the conjunction of the two conditions

$$\begin{aligned} \alpha_1, \ldots, \alpha_n \vdash^{\leqslant} \varphi &\iff e(\varphi) \geqslant e(\alpha_1) \wedge \cdots \wedge e(\alpha_n) \text{ for all } e, \\ \emptyset \vdash^{\leqslant} \varphi &\iff e(\varphi) = 1 \text{ for all } e, \end{aligned} \tag{6.7}$$

[10] Later results, see Theorem 6.4 and the comments before Theorem 6.5, will make it clear that the logics presented in Gil et al. (1993) also coincide with $\vdash_n^{\leqslant}$, but this was not explicit at the time of its publication.

[11] In Wójcicki (1973) only the "if" part is proved, and only for $\vdash_{\mathbf{S}}$.

[12] As a matter of fact, finitarity is part of the definition of a logic in most studies outside abstract algebraic logic, and also in some inside it.

[13] Clearly a meet-semilattice structure is enough; some recent, purely abstract studies of logics defined by (6.7) in a context beyond fuzzy, many-valued or substructural logics, such as Font (2011) or Jansana (2012), build on this fact.

where e ranges over all evaluations in the universe of the algebra taken as the truth structure defining the logic. Since order is equationally definable through the lattice operations, it is clear that this definition only depends on the equations satisfied by the model algebra. This is used in Font (2003) to observe that all the finitary logics so defined from infinite subalgebras of $[\mathbf{0}, \mathbf{1}]$ will coincide, that is, that *there is only one finitary logic preserving an infinity of degrees of truth from Łukasiewicz algebra*; it will be denoted by $\vdash^{\leqslant}_{\infty}$. By contrast, the finitary logics defined from the same subalgebras by (6.1) depend on the quasi-equations that hold in the subalgebra, and using a result on quasi-varieties of MV-algebras from Gispert and Torrens (1998) it was proved in Font (2003, Theorem 21) that these quasi-equations depend only on the rationals contained in the subalgebra.

The paper Font et al. (2006) is devoted to a more systematic and complete study of the unique finitary logic $\vdash^{\leqslant}_{\infty}$ that preserves an infinity of degrees of truth from $[\mathbf{0}, \mathbf{1}]$. The main results are several characterizations of its algebraic counterparts and its full generalized models, its classification in the hierarchies of abstract algebraic logic, the presentation of a Gentzen system adequate for it, which is also related to the ordinary truth-preserving logic of Łukasiewicz, and its characterization through Tarski-style conditions (i.e., abstract conditions on its consequence operator). However, most of the results in Font et al. (2006) were extended considerably in Bou et al. (2009), so it is better to review this paper here.

6.4 Widening the Scope: Fuzzy and Substructural Logics

In the last two decades the study of mathematical fuzzy logic, and particularly its algebraic study, has enormously widened its scope thanks to the work of many people around the world, above all Petr and his collaborators. Hájek (1998) draws a framework where all extensions of his basic logical system BL are encompassed. This logic was later characterized as the logic of all continuous t-norms on [0, 1] and their residua, and it was soon superseded as a ground foundation for the universe of fuzzy logics by MTL, the logic of all left-continuous t-norms and their residua; in turn, MTL was soon identified to be an axiomatic extension of FL_{ew}, the canonical contractionless substructural logic, associated with the class of residuated lattices. Thus, the algebraic study of many-valued logics found its natural environment in the realm of substructural logics; this was to be expected, because the residuation property had been recognized very early as one of the key properties characterizing the behaviour of implication, as in Goguen (1969). It should be noted, however, that when moving from extensions of MTL to substructural logics in general we drop what is considered by some to be an essential ingredient of fuzzy logics, namely their *linearity*. In any case, the dominant paradigm is still truth preservation; good, encyclopaedic overviews of these trends are Chapters I and II of Cintula et al. (2011) for fuzzy logics *stricto sensu* and Galatos et al. (2007) for the wider panorama of substructural logics.

The *residuated lattices* relevant to this discussion are always assumed to be *commutative* and *integral*[14]; the latter property means that the unit 1 of the monoidal structure is also the maximum of the order structure. Therefore each variety K of residuated lattices gives rise to what can be considered a truth-preserving logic $\vdash_{\mathsf{K}}$ defined by (6.1) applied to all algebras in K. In each case, the lattice structure naturally induces a companion logic $\vdash_{\mathsf{K}}^{\leqslant}$ defined by applying (6.7) to all algebras in K. As already observed, this definition depends only on the equations that hold in the models, so in this case a convenient way of highlighting this is to define $\vdash_{\mathsf{K}}^{\leqslant}$ as the finitary logic satisfying

$$\begin{aligned} \alpha_1, \ldots, \alpha_n \vdash_{\mathsf{K}}^{\leqslant} \varphi &\iff \mathsf{K} \vDash \alpha_1 \wedge \cdots \wedge \alpha_n \preccurlyeq \varphi\,, \\ \emptyset \vdash_{\mathsf{K}}^{\leqslant} \varphi &\iff \mathsf{K} \vDash \varphi \approx 1\,, \end{aligned} \tag{6.8}$$

where $\preccurlyeq$ and $\approx$ are formal symbols for the ordering[15] and the identity relations. When the variety K is generated by a single algebra, then (6.8) can be stated equivalently with this algebra as a unique model, which approaches it to (6.2), (6.3) and (6.4), thus making the interpretation of $\vdash_{\mathsf{K}}^{\leqslant}$ as a logic that preserves degrees of truth from a single model more natural.

The logics $\vdash_{\mathsf{K}}^{\leqslant}$ have been collectively studied in Bou (2008, 2012); and in Bou et al. (2009)[16] in some depth, touching on all aspects already listed at the end of Sect. 6.3, and particularly considering the relations with their companion logics $\vdash_{\mathsf{K}}$; the previous results concerning the latter are systematized in Galatos et al. (2007). It is not possible to summarize the contents of those papers in full, so I will just highlight some points and especially those with some relation with previous work. First, the basic properties and relations match those already found in the Łukasiewicz case:

Theorem 6.4 *For each variety K of residuated lattices the following hold:*

1. *The logic $\vdash_{\mathsf{K}}$ is the extension of $\vdash_{\mathsf{K}}^{\leqslant}$ with either the rule of Modus Ponens or the rule of $\star$-Adjunction (i.e., from φ and ψ to infer $\varphi \star \psi$).*
2. $\alpha_1, \ldots, \alpha_n \vdash_{\mathsf{K}}^{\leqslant} \varphi \iff \emptyset \vdash_{\mathsf{K}} \alpha_1 \wedge \cdots \wedge \alpha_n \to \varphi$, *and* $\emptyset \vdash_{\mathsf{K}}^{\leqslant} \varphi \iff \emptyset \vdash_{\mathsf{K}} \varphi$, *for all* $\alpha_1, \ldots, \alpha_n, \varphi$.

---

[14] In accordance with most of the literature starting with Ward and Dilworth (1939), here *residuated lattices* are algebras of similarity type $(\wedge, \vee, \star, \to, 1, 0)$ such that $\wedge$, $\vee$ are lattice operations, $\star$ is a commutative monoidal operation (usually called "fusion", "intensional conjunction" or "multiplicative conjunction") with unit 1 also being the maximum of the lattice, and $\to$ is its residuum. A constant 0 is included in the type but in the general case there is no need to postulate anything about it; so these residuated lattices coincide with the FL_{ei}-algebras of Galatos et al. (2007), where the term "residuated lattice" denotes in turn a much larger class. The smaller class of FL_{ew}-algebras is found when postulating that 0 is the minimum of the order, and includes the algebras of most well-known substructural logics such as MTL, BL, $Ł_\infty$, G, Π, etc.

[15] Observe that, in a lattice, an order relation $\alpha \preccurlyeq \beta$ holds if and only if the equation $\alpha \wedge \beta \approx \alpha$ holds; thus, using $\preccurlyeq$ is just a more intuitive way of writing identities of that particular form.

[16] An important error in the proof of Theorem 4.4 in Bou et al. (2009) has been corrected in Bou and Font (2012).

3. *The algebraic counterpart of both logics is the variety* K*, and for each* $\mathbf{A} \in \mathsf{K}$ *the filters of* $\vdash^{\leqslant}_{\mathsf{K}}$ *on* $\mathbf{A}$ *are its lattice filters, while the filters of* $\vdash_{\mathsf{K}}$ *on* $\mathbf{A}$ *are its implicative filters, which are the lattice filters closed under* $\star$*.*

Point 2 tells us that the relation found by Wójcicki in (6.6) for the finite Łukasiewicz logics extends to all varieties of residuated lattices; in terms of Wójcicki (1984), this says that $\vdash^{\leqslant}_{\mathsf{K}}$ is "the well-determined logic" associated with the theorems (tautologies) of $\vdash_{\mathsf{K}}$. This property is often viewed as justifying that it is not necessary to consider the logics $\vdash^{\leqslant}_{\mathsf{K}}$; it would say that the implication connective of $\vdash_{\mathsf{K}}$ already reflects the notion of a consequence preserving degrees of truth. Admittedly, this is a serious objection, but I think it actually rests on a more basic issue, that of whether the implication connective adequately represents consequence or entailment. This is usually made to depend on the kind of Deduction Theorem the logic satisfies, and it will be shown in Theorem 6.9 that either of the two logics satisfies the ordinary Deduction Theorem for the connective $\to$ if and only if they actually coincide. Thus, when put in this context, the objection appears to be much weaker.

For the first statement of point 3 to make sense, the notion of *algebraic counterpart* mentioned there has necessarily to be defined in a non-*ad hoc* way, in the context of some general theory of the algebraization of logic, and this is indeed the case, as will be explained after Theorem 6.6. In any event, a technical consequence of the second part of point 3 is that the logic $\vdash^{\leqslant}_{\mathsf{K}}$ coincides with the logic defined by the class of matrices $\{\langle \mathbf{A}, F\rangle : \mathbf{A} \in \mathsf{K},\ F \text{ a lattice filter of } \mathbf{A}\}$. In the particular case of the class MV of MV-algebras, this shows that the logic $\vdash^{\leqslant}_{\mathsf{MV}}$ coincides with the logic studied in Gil et al. (1993), called there "lattice-like Łukasiewicz logic"; moreover, since MV is the variety generated by the Łukasiewicz algebra $[\mathbf{0}, \mathbf{1}]$, the logic $\vdash^{\leqslant}_{\mathsf{MV}}$ is actually the finitary logic $\vdash^{\leqslant}_{\infty}$ preserving an infinity of degrees of truth from the Łukasiewicz algebra mentioned before, a fact probably known to the authors of Gil et al. (1993) but not mentioned there. Thus, some of the results stated (without proof) in this paper anticipate for this particular case the more general ones obtained in Bou et al. (2009); some will be mentioned later on.

The issue of the precise formulation and the scope of Theorems 6.1, 6.2 and 6.3 is settled in Bou et al. (2009, Theorem 4.12) as follows.

Theorem 6.5 *Let* K *be a variety of residuated lattices. Then the two logics* $\vdash^{\leqslant}_{\mathsf{K}}$ *and* $\vdash_{\mathsf{K}}$ *coincide if and only if* K *is a variety of (generalized) Heyting algebras.*

The qualifier "generalized" appears here to cover the case where 0 is not postulated to be the minimum of the order[17]; when it is, that is, when K is actually a variety of FL_{ew}-algebras, the "generalized" can be deleted. Thus, Theorems 6.1, 6.2 and 6.3 cover the case of all the intermediate logics (the axiomatic extensions of intuitionistic logic). Moreover, the converse of the implication in Theorem 6.2 and the claim in Baaz et al. (2007) that "the coincidence of the two entailment relations [i.e., the

[17] Generalized Heyting algebras can be described informally as "Heyting algebras without minimum"; a residuated lattice is a generalized Heyting algebra if and only if the fusion operation $\star$ coincides with the lattice conjunction $\wedge$.

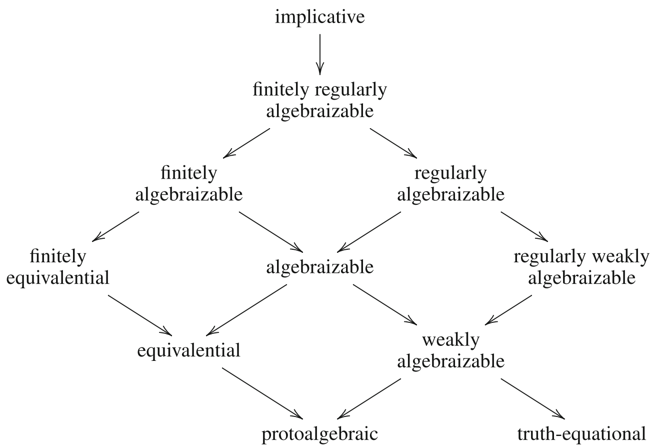

Fig. 6.1 The most important classes of logics in the Leibniz hierarchy; for a finer classification see Cintula and Noguera (2010). "→" means "included in" or "implies"

property of Theorem 6.3] is a unique feature of Gödel's logics" are seen to hold if adequately formulated. The conclusion is that the fuzzy logics preserving degrees of truth may have some interest only for logics that are *not* extensions of Gödel-Dummett logic.

6.5 Abstract algebraic logic classification

In the last two decades, abstract algebraic logic has emerged as an elaborate framework for the study of the algebraic semantics of propositional logics and the relations between metalogical properties of the logics and purely algebraic or model-theoretic properties of their classes of algebraic models; see (Czelakowski 2001; Font and Jansana 2009; Font et al. 2009). The advances in abstract algebraic logic have been partly motivated by advances in the study of many different non-classical logics, and as explained in Cintula et al. (2011, p. 104), they in turn have provided tools for the systematization of the landscape of mathematical fuzzy logic.

One of the main goals of abstract algebraic logic has been to develop methods to classify logics according to some abstract criteria and to study the relations between a logic and its algebraic models in each of the "levels" created by the classification. This has originated two *hierarchies* of a very different character, each with its advantages and its disadvantages.

The *Leibniz hierarchy* (Fig. 6.1) is the more complicated and developed of the two, and can be described in several ways; the one giving it its name is by the behaviour of the so-called Leibniz operator on the theories of the logic, or on the lattices of its filters on arbitrary algebras. As the diagram shows, almost all its members belong to the large class of *protoalgebraic logics*, which can be characterized in several ways: the simplest one is probably by the existence of a set[18] of binary formulas $\Delta(x, y)$ satisfying the basic properties

$$\begin{array}{rl} \text{Reflexivity}: & \emptyset \vdash \delta(x, x) \text{ for every } \delta \in \Delta \\ \textit{Modus Ponens}: & x, \Delta(x, y) \vdash y\,. \end{array} \tag{6.9}$$

The importance of belonging to this hierarchy is that for the logics in these classes the machinery of the theory of matrices can be used in full strength, far beyond the general completeness theorems that hold for all logics whatsoever; many techniques adapted from universal algebra and lattice theory give important, profound results relating a logic with its algebraic models, in particular with the lattices of its filters on arbitrary algebras. The central part of the hierarchy comprises several variants of *algebraizable logics*, all arising from the class introduced by Blok and Pigozzi in their seminal monograph Blok and Pigozzi (1989); these logics enjoy the highest degree of equivalence between a logic and the equational consequence relative to a class of algebras, an equivalence expressible by a pair of mutually inverse definable transformers, whose paradigm is the relation between classical logic and the variety of Boolean algebras, or between intuitionistic logic and Heyting algebras. At the top of this diagram lies the more restricted but still large class of *implicative logics*. These logics slightly generalize those studied by Rasiowa in her highly influential book Rasiowa (1974), and are algebraizable in a very simple and standard way; many of the logics algebraically studied in the past belong to this class.

The *Frege hierarchy* (Fig. 6.2) is less complicated and has also been less studied. Its classifying principle is based on several replacement properties that the logics and some of their generalized models may have. Its weakest, largest level is the class of *selfextensional* logics, originally defined by Wójcicki (1988) as those whose interderivability relation $\dashv\vdash$ is a congruence of the formula algebra.

There are some important theorems connecting the two hierarchies; for instance, every protoalgebraic and Fregean logic with theorems is regularly algebraizable (Font and Jansana 2009, Theorem 3.18). But in general the two hierarchies are orthogonal in the sense that there are logics in the topmost level of each which do not belong even to the lowest level of the other (Theorem 6.6 provides a proper class of examples). As is to be expected, logics belonging to higher levels in both hierarchies enjoy a very nice algebraic behaviour.

The general classification of the two logics associated with each variety K of residuated lattices, determined in Bou et al. (2009, Corollary 4.2) and Galatos et al. (2007, Theorem 2.29), is as follows. The logics $\vdash_{\mathsf{K}}$ have a very good location in the Leibniz hierarchy but in general not in the Frege hierarchy, while the logics $\vdash_{\mathsf{K}}^{\leqslant}$ are in a so-to-speak dual situation.

[18] For a finitary logic this set can be taken finite; if moreover the logic has a conjunction, then the set can be reduced to a single formula.

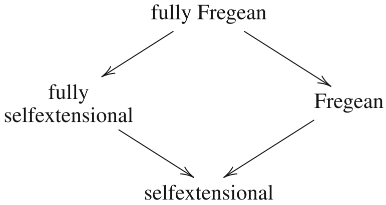

Fig. 6.2 The classes in the Frege hierarchy. "→" means "included in" or "implies"

Theorem 6.6 *Let* K *be any variety of residuated lattices. Then:*

1. *The logic* $\vdash_{\mathsf{K}}$ *is implicative, but need not be even selfextensional.*
2. *The logic* $\vdash_{\mathsf{K}}^{\leqslant}$ *is fully selfextensional, but need not be even protoalgebraic.*

Abstract algebraic logic provides a canonical definition of *the algebraic counterpart*, which applies to an arbitrary logic. The definition uses the notion of a generalized model of a logic, but for restricted classes in the hierarchies the general definition may have a more workable equivalent characterization. In the case of algebraizable logics, this is the notion of *equivalent algebraic semantics*. Thus from point 1 above, point 3 in Theorem 6.4 for $\vdash_{\mathsf{K}}$ means that the equivalent algebraic semantics of $\vdash_{\mathsf{K}}$ is exactly K; for a proof of these facts see Galatos et al. (2007). For selfextensional logics with a conjunction, it is proved in Font and Jansana (2009) that the algebraic counterpart coincides with the notion of the *intrinsic variety* of a logic, which in this case is the variety defined by the set of equations $\{\varphi \approx \psi : \varphi \dashv\vdash \psi\}$. Since the logics $\vdash_{\mathsf{K}}^{\leqslant}$ have a conjunction and by point 2 above they are in particular selfextensional, this can be applied to them; but (6.8) implies that $\varphi \dashv\vdash_{\mathsf{K}}^{\leqslant} \psi \iff \mathsf{K} \vDash \varphi \approx \psi$, and therefore the intrinsic variety of $\vdash_{\mathsf{K}}^{\leqslant}$ is exactly K, which justifies point 3 of Theorem 6.4 regarding $\vdash_{\mathsf{K}}^{\leqslant}$; all this is proved in Bou et al. (2009).

In principle no better classification in the Leibniz hierarchy is possible for the logics $\vdash_{\mathsf{K}}^{\leqslant}$ in general, because many of them fail to be protoalgebraic; for instance $\vdash_{\infty}^{\leqslant}$ has been known not to be protoalgebraic since 1993, see Gil et al. (1993) and Font (2003) for a proof. Those that are protoalgebraic are characterized in several ways in Bou et al. (2009, Theorem 4.6 and Corollary 4.11), and it happens that they are not just protoalgebraic, but automatically finitely equivalential (here the standard notation $x^n = x \star \dots^{(n} \star x$ is used):

Theorem 6.7 *Let* K *be any variety of residuated lattices. Then the following conditions are equivalent:*

(i) *The logic* $\vdash_{\mathsf{K}}^{\leqslant}$ *is protoalgebraic.*
(ii) *The logic* $\vdash_{\mathsf{K}}^{\leqslant}$ *is equivalential.*
(iii) *The logic* $\vdash_{\mathsf{K}}^{\leqslant}$ *is finitely equivalential, with* $(x \to y)^n \star (y \to x)^n$ *as equivalence formula, for some* $n \in \omega$.
(iv) $\mathsf{K} \vDash x \wedge \big((x \to y)^n \star (y \to x)^n\big) \preccurlyeq y$, *for some* $n \in \omega$.

This result somehow generalizes facts already known to hold for Łukasiewicz logics. That the $\vdash_n^{\leqslant}$ are finitely equivalential, with these equivalence formulas, is already proved in Gil (1996); and the equivalence between the first three conditions is already stated (without proof) in Gil et al. (1993) for logics that are extensions of $\vdash_\infty^{\leqslant}$.

The equivalence between (i) and (iv) suggests that the denumerable family of varieties of residuated lattices defined by the equations in point (iv) may have both an algebraic and a logical interest. This family turns out to be related to other denumerable families of varieties (some already known) which are studied in Bou et al. (2009); here are some of the consequences of the relations found there:

1. If $\vdash_{\mathsf{K}}^{\leqslant}$ is protoalgebraic, then there is some $n \in \omega$ such that all algebras in K are n-contractive.[19]
2. It follows from 1. that for the majority of best-known fuzzy logics, their companion preserving degrees of truth is not protoalgebraic. This concerns Łukasiewicz logic, product logic, MTL, BL, FL_{ew}, etc. It is important for the general theory of abstract algebraic logic that natural examples of non-protoalgebraic logics are found, because at the time of their introduction in Blok and Pigozzi (1986) it was believed that only pathological logics could fail to be protoalgebraic.
3. If K is a variety of MTL-algebras, then $\vdash_{\mathsf{K}}^{\leqslant}$ is protoalgebraic if and only if there is some $n \in \omega$ such that all chains in K are ordinal sums of simple n-contractive MTL-chains (Horčík et al. 2007). Observe that not all finite MTL-chains satisfy this condition.
4. When K is a variety of BL-algebras, the implication in 1. is an equivalence. In contrast with the MTL case, this implies that the logic preserving degrees of truth with respect to any finite BL-chain is protoalgebraic. In particular, this confirms that the finite-valued Łukasiewicz logics preserving degrees of truth (the $\vdash_n^{\leqslant}$ of Sect. 6.3) are protoalgebraic, hence finitely equivalential.
5. There is only one variety K generated by a family of continuous t-norms over $[0, 1]$ such that $\vdash_{\mathsf{K}}^{\leqslant}$ is protoalgebraic, namely the variety G of Gödel algebras. Here what is new is the uniqueness, because by Theorems 6.3 and 6.5 we already know that $\vdash_{\mathsf{G}}^{\leqslant} = \vdash_{\mathsf{G}}$, and hence $\vdash_{\mathsf{G}}^{\leqslant}$ is not just protoalgebraic but implicative. This unique feature of Gödel-Dummett logic adds to the already mentioned statement in Baaz et al. (2007) concerning Theorem 6.3.

Comparing with Theorem 6.6, which states the good classifications of the logics $\vdash_{\mathsf{K}}$ and $\vdash_{\mathsf{K}}^{\leqslant}$ in the Leibniz and the Frege hierarchies respectively, it seems it is not possible for each logic in the pair to go further in the hierarchy where the other one is well placed without so-to-speak trivializing the situation, due to Bou et al. (2009, Proposition 4.3 and Theorem 4.12):

[19] A residuated lattice is *n-contractive*, also called "n-potent" in the literature, when it satisfies the equation $x^n \approx x^{n+1}$. The associated logics are also called "n-contractive".

Theorem 6.8 *Let* K *be any variety of residuated lattices. Then the following conditions are equivalent:*

(i) *The logic* $\vdash^{\leqslant}_{\mathsf{K}}$ *is weakly algebraizable.*
(ii) *The logic* $\vdash^{\leqslant}_{\mathsf{K}}$ *is Fregean.*
(iii) *The logic* $\vdash_{\mathsf{K}}$ *is selfextensional.*
(iv) *The logics* $\vdash_{\mathsf{K}}$ *and* $\vdash^{\leqslant}_{\mathsf{K}}$ *coincide.*

Moreover, when these conditions hold, the (unique) logic is both implicative and fully Fregean.

Together with Theorem 6.6, this implies that the logics preserving degrees of truth, when they are properly so (i.e., when they are not truth-preserving) are fully selfextensional but not Fregean.

Also, this gives another view on the possibility of extending Theorems 6.1, 6.2 and 6.3 to other logics: this is only possible for logics placed in the highest levels of both hierarchies.

6.6 The Deduction Theorem

The research on different forms of the *Deduction Theorem* (DDT) and its algebraic counterparts is at the core of abstract algebraic logic, and is one of its best developed and best understood areas. However, its results hold only inside the Leibniz hierarchy (because all logics with the DDT are protoalgebraic), and hence it may happen that its investigation for the generality of the logics $\vdash^{\leqslant}_{\mathsf{K}}$ (some of which are protoalgebraic while some aren't) is more difficult and less standardized than that for the logics $\vdash_{\mathsf{K}}$.

It is well known (Galatos et al. 2007, Corollary 2.15) that all the logics $\vdash_{\mathsf{K}}$ satisfy the *Local Deduction Theorem* (LDDT) for the family $\{x^n \to y : n \in \omega\}$; that is, they satisfy, for all Γ, α, β:

$$\Gamma, \alpha \vdash_{\mathsf{K}} \beta \iff \text{there is some } n \in \omega \text{ such that } \Gamma \vdash_{\mathsf{K}} \alpha^n \to \beta. \tag{6.10}$$

This extends the result for $\vdash_{\infty}$, known at least since 1964, see Pogorzelski (1964, Thesis T3.3) and also Wójcicki (1973, Lemma 2) for a detailed proof. It is shown in Bou (2008), using Theorem 11.2 of Galatos et al. (2007) plus the well-known equivalence (Font et al. 2009, Theorem 3.10) between the DDT for an algebraizable logic and the property of having equationally definable principal congruences for its equivalent algebraic semantics, that $\vdash_{\mathsf{K}}$ has the DDT for some implication[20] $\delta(x, y)$, that is, it satisfies, for all Γ, α, β,

$$\Gamma, \alpha \vdash_{\mathsf{K}} \beta \iff \Gamma \vdash_{\mathsf{K}} \delta\big(\alpha, \beta\big), \tag{6.11}$$

[20] In principle the general theorem concerns an arbitrary set of formulas acting collectively as an implication, but since in the present case all logics are finitary and have a conjunction, one can directly speak about a single formula.

if and only if all algebras in K are n-contractive for some n, and moreover in such a case the implication formula can be taken to be $x^n \to y$. This determines exactly the scope of the Deduction Theorem for the logics preserving truth, and extends the result for the finite-valued logics $\vdash_n$, also known from Pogorzelski (1964) and Wójcicki (1973). In particular it follows from point 1 after Theorem 6.7 that whenever the logic $\vdash_{\mathsf{K}}^{\leqslant}$ is protoalgebraic, the logic $\vdash_{\mathsf{K}}$ has this DDT.

Is the situation for the logics $\vdash_{\mathsf{K}}^{\leqslant}$ comparable? Is there an exact determination of the scope of the DDT, or one of its variants, for this family of logics? The fact that in general they are not even protoalgebraic means there are no general techniques and makes this investigation more difficult. However, some partial results have already been obtained. Concerning the "classical" DDT, that is, when the operation is the "real" implication $x \to y$ itself, the results in Bou et al. (2009, Proposition 2.8 and Theorem 4.12) remove the possibility that it may hold in any other case than those already known and expected:

Theorem 6.9 *Let* K *be any variety of residuated lattices. Then the following conditions are equivalent:*

(i) *The logic* $\vdash_{\mathsf{K}}^{\leqslant}$ *satisfies the* DDT (6.11) *for* $\delta(x, y) = x \to y$.
(ii) *The logic* $\vdash_{\mathsf{K}}$ *satisfies the* DDT (6.11) *for* $\delta(x, y) = x \to y$.
(iii) *The logics* $\vdash_{\mathsf{K}}$ *and* $\vdash_{\mathsf{K}}^{\leqslant}$ *coincide.*

Observe that one half of the DDT is the rule of *Modus Ponens*. Its failure for the logics $\vdash_n^{\leqslant}$ was already observed in Scott (1973, 1974), as recalled in Sect. 6.2; this is also observed in the comments after Lemma 2.17 in Baaz et al. (2007), which establishes necessary and sufficient conditions for a binary function on [0, 1] to be the Gödel conditional; one of them is that the associated binary operation satisfies the DDT. The conjunction of Theorems 6.5 and 6.9 adds some further explanation for this: It is well-known that Gödel's conditional is the residuum of the maximum t-norm, which is the only t-norm turning a residuated lattice structure on [0, 1] into a (generalized) Heyting algebra, and by Theorem 6.5 this is equivalent to point (iii) of Theorem 6.9.

The proceedings paper Bou (2008) determines the cases where the DDT holds for two large classes of cases:

Theorem 6.10 *Let* K *be a variety of MTL-algebras. Then the logic* $\vdash_{\mathsf{K}}^{\leqslant}$ *satisfies the* DDT (6.11) *for some implication* $\delta(x, y)$ *if and only if it is protoalgebraic, that is (see point 3 above) if and only if there is some* $n \in \omega$ *such that all chains in* K *are ordinal sums of simple* n*-contractive MTL-chains. In such a case, the formula* $\delta(x, y) = (x \to y)^n \vee y$ *can be taken as the implication satisfying the* DDT.

The presence of protoalgebraicity in relation with the DDT is expected, because the properties (6.9) follow easily from (6.11), so that every logic with the DDT is protoalgebraic; the interesting part is the converse implication. The more restricted case of the extensions of $\vdash_{\infty}^{\leqslant}$ had already been considered in Gil (1996), Gil et al. (1993); in the first work it is proved that the logics $\vdash_n^{\leqslant}$ satisfy the DDT for the same formula δ, and in the second it is stated (without proof) that one such extension satisfies the DDT for some δ if and only if there is some $n \in \omega$ such that the associated algebras are n-contractive, and that the formula δ can be taken as above.

The other case studied in Bou (2008), with a not so neat but still useful conclusion, is the following.

Theorem 6.11 *Assume that* $\mathbf{A}$ *is a finite residuated lattice satisfying the same equations with at most 3 variables as the variety* K. *Then the logic* $\vdash^{\leqslant}_{\mathsf{K}}$ *satisfies the* DDT (6.11) *for some implication* δ *if and only if the following conditions are satisfied:*

1. *The logic* $\vdash^{\leqslant}_{\mathsf{K}}$ *is protoalgebraic.*
2. $\mathbf{A}$ *is distributive as a lattice.*
3. *The lattice operation* $\wedge$ *is residuated in* $\mathbf{A}$, *and every subalgebra of* $\mathbf{A}$ *is closed under the map* $\langle a, b\rangle \longmapsto \max\{c \in A : a \wedge c \leqslant b\}$.

Observe that the second part of condition 3 is a weakened form of the property that the residuum operation is term-definable; thus, these three conditions are close to (but weaker than) saying that the algebra $\mathbf{A}$ is a (generalized) Heyting algebra.

The interest of this result is that its assumption covers in particular the simpler case where the algebra $\mathbf{A}$ generates the variety K, a situation that may be common in applications where one wants to consider the logic preserving degrees of truth from a single truth structure. Observe also that, due to the general assumption on $\mathbf{A}$, condition 2 implies that all the members of K are distributive as lattices. In contrast with Theorem 6.10, here a general form of the formula δ satisfying the DDT has not been determined, but it is known that it cannot be the formula found in Theorem 6.10. It is also known that neither of the three conditions is superfluous.

6.7 Axiomatizations

6.7.1 In the Gentzen style

As explained in Sect. 6.2, Scott (1974) presented a multiple-conclusion Gentzen-style calculus and used it to prove completeness for what is actually a multiple-conclusion version of the logic $\vdash^{\leqslant}_{[\mathbf{0},\mathbf{1}]}$, which coincides with $\vdash^{\leqslant}_{\mathsf{MV}}$, where MV is the variety of MV-algebras. This calculus leaves little room for generalization to other logics of the form $\vdash^{\leqslant}_{\mathsf{K}}$, and anyway this idea has not been followed in the literature.

Consequence in the logics $\vdash^{\leqslant}_{\mathsf{K}}$ reflects the properties of order in K, and these can be expressed by properties of the closure operator of lattice-filter-generation in the algebras in K. These properties, in turn, can be expressed in an abstract form yielding the so-called *Tarski-style* conditions, and in a syntactic form as Gentzen-style rules. The case where $\mathsf{K} = \mathsf{RL}$, the variety of all residuated lattices, is treated in Theorem 5.9 and Corollary 5.10 of Bou et al. (2009), where the following is proved (we assume we deal with sequents of the form $\Gamma \rhd \varphi$ where Γ is a finite set of formulas):

Theorem 6.12 *Let $\mathfrak{G}$ be the Gentzen calculus that has the structural axiom, all the structural rules, the following logical axioms*

$$\begin{array}{rcl} \emptyset \rhd 1 & \qquad & \varphi, \psi \rhd \psi \\ \emptyset \rhd \varphi \to \varphi & & \varphi \wedge \psi \rhd \varphi \\ \varphi \to (\psi \to \xi) \rhd \psi \to (\varphi \to \xi) & & \varphi \wedge \psi \rhd \psi \end{array}$$

and the following logical rules

$$\frac{\varphi \rhd \xi \qquad \psi \rhd \xi}{\varphi \vee \psi \rhd \xi}\ (\text{r1}) \qquad \frac{\emptyset \rhd \varphi \to \psi}{\varphi \rhd \psi}\ (\text{r2})$$

$$\frac{\varphi \vee \psi \rhd \xi}{\varphi \rhd \xi} \qquad \frac{\varphi \star \psi \rhd \xi}{\varphi \rhd \psi \to \xi}$$

$$\frac{\varphi \vee \psi \rhd \xi}{\psi \rhd \xi} \qquad \frac{\varphi \rhd \psi \to \xi}{\varphi \star \psi \rhd \xi}\,.$$

Then the calculus $\mathfrak{G}$ axiomatizes the logic $\vdash^{\leqslant}_{\mathsf{RL}}$ ψ in the following sense: For any formulas $\varphi_1, \ldots, \varphi_n, \psi$, it holds that $\varphi_1, \ldots, \varphi_n \vdash^{\leqslant}_{\mathsf{RL}} \psi$ if and only if the sequent $\varphi_1, \ldots, \varphi_n \rhd \psi$ is derivable in $\mathfrak{G}$.

If $\rhd$ is read as $\preccurlyeq$, there are few surprises in the formulation of this calculus. Notice rule (r2), which corresponds to the already mentioned weak form of *Modus Ponens* (6.5), and rule (r1), which corresponds to the rule of *Proof by Cases*, but in a weak form where no side assumptions appear; this reflects the fact that the lattices in RL need not be distributive.

This base calculus can be extended to obtain a calculus for the logic $\vdash^{\leqslant}_{\mathsf{K}}$ when an equational presentation of the variety K is known. In such a case, every equation $\varphi \approx \psi$ is re-written as the pair of sequents $\varphi \rhd \psi$ and $\psi \rhd \varphi$, and these are added to the logical axioms of the calculus; it is straightforward that the resulting calculus axiomatizes the logic $\vdash^{\leqslant}_{\mathsf{K}}$ in the same sense as in Theorem 6.12.

However, these Gentzen calculi seem not to have interesting properties from a proof-theoretic point of view.

6.7.2 In the Hilbert style

The property in point 2 of Theorem 6.4 might suggest that any axiomatization of $\vdash_{\mathsf{K}}$ provides one of $\vdash^{\leqslant}_{\mathsf{K}}$ just by looking at the theorems of the former logic having the form $\alpha_1 \wedge \cdots \wedge \alpha_n \to \varphi$; but this is hardly satisfactory as an axiomatic presentation of the real relation of consequence of $\vdash^{\leqslant}_{\mathsf{K}}$, for we cannot recognize neither the axioms nor the rules of inference that it satisfies.

A first result is found in Bou et al. (2009, Theorem 2.12):

Theorem 6.13 *The logic* $\vdash^{\leqslant}_{\mathsf{K}}$ *can be presented by the axiomatic system having the set* $\mathrm{Taut}(\mathsf{K}) = \{\varphi : \mathsf{K} \vDash \varphi \approx 1\}$ *as its set of axioms, and the rules*[21]

$$\begin{aligned} \textit{Adjunction for} \wedge : &\quad \varphi\,,\psi \;\; \rhd \;\varphi \wedge \psi \\ \textit{Restricted Modus Ponens} : &\quad \varphi \;\rhd\; \psi \;\; \textit{provided that}\; \varphi \to \psi \in \mathrm{Taut}(\mathsf{K}) \end{aligned}$$

as its rules of inference. If an axiomatization of $\vdash_{\mathsf{K}}$ *is known, with axioms* $\mathrm{Ax}(\mathsf{K})$ *and Modus Ponens as the only rule, then the set* $\mathrm{Ax}(\mathsf{K})$ *can replace the set* $\mathrm{Taut}(\mathsf{K})$ *in the list of axioms of* $\vdash^{\leqslant}_{\mathsf{K}}$*.*

It is interesting to notice how this restricted form of *Modus Ponens* corresponds to the fact (6.5) already observed in 1973, and was also a derived rule in the axiomatization of Scott (1974, Theorem 3.2). This presentation, however, is not very satisfactory. Both its axioms and one of its rules depend on determination of the set $\mathrm{Taut}(\mathsf{K})$, which is in principle infinite; when it is decidable, then both the axioms and rules of this system will be decidable, so that it can be more properly called "in the Hilbert style". This set is the set of theorems of the logic $\vdash_{\mathsf{K}}$, which due to the LDDT (6.10) can in theory be axiomatized with *Modus Ponens* as its only rule; in the cases where such an axiomatization is known, then the set of axioms can replace the set $\mathrm{Taut}(\mathsf{K})$ in the list of axioms of the above presentation of $\vdash^{\leqslant}_{\mathsf{K}}$, but it is still not possible to do the same in the restricted rule of *Modus Ponens*, so in principle this never gives an axiomatic presentation of $\vdash^{\leqslant}_{\mathsf{K}}$ by a finite set of rule schemes.

This last difficulty is solved in Bou (2012, Corollary 2.4):

Theorem 6.14 *Assume* $\mathrm{Ax}(\mathsf{K})$ *is a set of axioms which, together with the only rule of Modus Ponens, axiomatizes the logic* $\vdash_{\mathsf{K}}$*. Then the logic* $\vdash^{\leqslant}_{\mathsf{K}}$ *can be presented by the axiomatic system having the formula* 1 *as its only axiom, and the following sets of schemes (*$\alpha\,,\varphi\,,\psi$ *are arbitrary formulas) as its inference rules:*

$$\begin{aligned} \mathsf{K}\text{-specific rule} : &\quad \alpha \;\rhd\; \alpha \star \varphi \;\textit{for every}\; \varphi \in \mathrm{Ax}(\mathsf{K}) \\ \text{Adjunction for} \wedge : &\quad \varphi\,,\psi \;\rhd\; \varphi \wedge \psi \\ \textit{Modus Ponens}\text{ for} \star : &\quad \alpha \star \big(\varphi \star (\varphi \to \psi)\big) \;\rhd\; \alpha \star \psi \\ \text{Weakening for} \star : &\quad \varphi \star \psi \;\rhd\; \varphi \\ \text{Associativity for} \star : &\quad (\varphi \star \psi) \star \alpha \;\rhd\; \varphi \star (\psi \star \alpha) \\ \text{Commutativity for} \star : &\quad \varphi \star \psi \;\rhd\; \psi \star \varphi \end{aligned}$$

This is in principle applicable to all the logics $\vdash^{\leqslant}_{\mathsf{K}}$, for we might take $\mathrm{Taut}(\mathsf{K})$ as the set $\mathrm{Ax}(\mathsf{K})$; however this might result in a non-recursive axiomatization (some would even refuse to call such a system a "Hilbert-style axiomatization"). The interesting thing is that if some *finite* axiomatization of $\vdash_{\mathsf{K}}$ is known, then the above procedure

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[21] The symbol $\rhd$ is here just a neutral replacement for other symbols like $\vdash$ or $\Rightarrow$, which might lead to misunderstanding if used to describe sequents or rules in the present context.

turns it into a finite axiomatization of $\vdash_{\mathsf{K}}^{\leqslant}$, because for each axiom schema φ of $\vdash_{\mathsf{K}}$ one can put a variable that does not appear in φ in the place of α in the K-specific rule and one obtains a single rule (schema) for $\vdash_{\mathsf{K}}^{\leqslant}$. The majority of the well-known fuzzy logics fall under this assumption, so this provides finite axiomatizations of the logics preserving degrees of truth with respect to the most common many-valued truth structures.

6.8 Conclusions

Logics preserving degrees of truth, in the technical sense established in the Introduction and in Sect. 6.2, seem to formalize a notion of consequence for many-valued logics that treats all truth values on an equal footing, i.e., by considering that all these values express a certain degree of truth without designating one of them (or a subset) as “the truth”, and giving them the same rôle in a truth-preserving definition of consequence. This possibility has hardly been explored at all in the literature on many-valued logic, save for a short proposal by Scott (1973) and a few other scattered results. The recent systematic study of logics preserving degrees of truth inside the large group of substructural logics has prompted a more technical approach using the tools of abstract algebraic logic. The cases of the two logics $\vdash_{\mathsf{K}}$ and $\vdash_{\mathsf{K}}^{\leqslant}$ associated with each variety K of (commutative, integral) residuated lattices, the first one preserving truth as represented by 1 and the second one preserving degrees of truth, have been reviewed, in particular their classification in the Leibniz and the Frege hierarchies of abstract algebraic logic. It appears that, from the point of view of abstract algebraic logic, the theory of the logics preserving degrees of truth is much richer and diverse than that of the logics preserving truth; in particular some properties of the former seem to depend heavily on those of the associated variety K, while the latter seem to show a more uniform and predictable behaviour.

The survey in this chapter has been limited to published work. While arising from motivations around many-valued and fuzzy logic, the study of logics preserving degrees of truth has been progressively extended, as witnessed by the most recent research reported on in Sects. 6.4–6.7. In hindsight it is now clear that some of the restrictions adopted in the present study (in order to produce a reasonably smooth and powerful development and results) are not essential to its motivations, and may seem *ad hoc* to some readers. Actually, the basic idea (6.2) of a logic preserving degrees of truth requires very little for its application. Thus one can see several natural **directions for future research** in this area. Let me end the chapter by commenting on some of them:

- The results in Sect. 6.5 about the classification of the logics under study in the Leibniz hierarchy of abstract algebraic logic consider only the traditional classes of protoalgebraic, equivalential and algebraizable logics. However, after Raftery (2006) the new class of *truth-equational logics* has been added to this hierarchy (since it is defined by conditions on the Leibniz operator) without being a subclass of protoalgebraic logics. The logics $\vdash_{\mathsf{K}}^{\leqslant}$ are not protoalgebraic in general; therefore investigating whether they are truth-equational or not makes sense, but as far as I know this has not been done.

- As already noted in footnote 13, most of the theoretical background for the study and classification of the logics $\vdash_{\mathsf{K}}^{\leqslant}$ is dependent upon only the connective of lattice conjunction $\wedge$, i.e., it can be developed for *meet-semilattices* rather than lattices. Thus, the investigation of fragments still containing $\wedge$ but perhaps not some of the other connectives may produce interesting results.
- The pairs of companion logics $\left(\vdash_{\mathsf{K}}, \vdash_{\mathsf{K}}^{\leqslant}\right)$ have been studied when K is a variety of residuated lattices. However, it is well known that the natural algebraic counterparts of finitary, finitely algebraizable logics are *quasivarieties*. One of the reasons for the restriction to varieties may be that the logics $\vdash_{\mathsf{K}}$, the algebraizable members of the pairs, have been studied in Galatos et al. (2007) only in this case (corresponding to axiomatic extensions of the basic substructural logic FL). But I think the main reason is the fact that the companion logics $\vdash_{\mathsf{K}}^{\leqslant}$ defined as in (6.8) are determined by varieties, i.e., if defined from an arbitrary class of algebras, the resulting logic coincides with that defined by the variety it generates. However, this is due to the presence of a meet-semilattice conjunction $\wedge$, while the more general definition (6.2) makes sense in quasi-varieties and requires only an ordering relation $\leqslant$. Hence, it would make sense to extend this research to other quasivarieties of algebras corresponding to special substructural logics with a more limited language, such as BCK logic.
- Another restriction has been the assumption of *integrality*, that is, that the unit 1 of the monoidal structure is also the maximum of the lattice structure. Again, there is nothing leading specifically to this choice in the basic idea of a logic preserving degrees of truth. From (6.2) it follows that the theorems of such logics will be the formulas that are always evaluated as the maximum of the order; hence when the algebraic structures need not have a maximum the resulting logic will have no theorems, and in particular will not have the same theorems as the logic preserving truth. A case study of this situation, concerning relevance logic R, is Font and Rodríguez (1994). Removing integrality in general, however, raises several unexpected fundamental questions, both from the motivational side and from the technical one, and has been discussed in Font (2007). To highlight only two: It is not clear that in this case it makes sense to preserve all degrees of truth, and it is not clear that the theoretical support of (Font 2011; Jansana 2012) on selfextensional logics with conjunction can still be used.

The best conclusion is that there is still a great deal of room for research in this area.

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Chapter 7
The Differential Semantics of Łukasiewicz Syntactic Consequence

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7.1 Prelude: Semantics for Hájek Propositional Basic Logic

Basic logic (BL) was invented by Hájek to formalize reasoning about continuous t-norms and their residua. Certain axioms satisfied by any such t-norm were singled out in Hájek (1998, Definition 2.2.4); provability of a formula φ, as well as provability of φ from a set Θ of premises, were defined via Modus Ponens, in the usual way, Hájek (1998, Definition 2.2.17). BL-algebras, BL-evaluations of formulas, and satisfiability, were then defined in Hájek (1998, Definition 2.3.3) and Hájek (1998, Definition 2.3.8), and the following completeness theorem was proved in Hájek (1998, Theorem 2.3.19):

7.1.1 *A formula φ is provable iff every BL-evaluation satisfies φ.*

The following theorem directly follows from Hájek (1998, Theorem 2.4.3):

7.1.2 *For any formula φ and set Θ of formulas, φ is provable from Θ iff every BL-evaluation satisfying all $\theta \in \Theta$ also satisfies φ, in symbols, $\Theta \models_{BL} \varphi$.*

Yet in Hájek (1998, Remark 2.3.23) Hájek champions a different semantics for BL. Let us agree to say that φ is a *t-tautology* if φ is satisfied by every evaluation of φ into

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a BL-algebra arising from a continuous t-norm. The resulting t-tautology semantics is more adherent to the original motivation of BL-logic: for, Hájek's BL-axioms in Hájek (1998, Definition 2.2.4) are the result of his contemplation of continuous t-norms. The question arises: do the BL-axioms prove all t-tautologies? The problem whether BL is *the* logic of continuous t-norms is again posed in a final section (Hájek 1998, Sect. 9.4.6).

In the same pages Hájek (1998, Sect. 9.4.1), it is noted that the traditional semantic consequence relation $\models$ in Ł$_\infty$ fails to be strongly complete. A counterexample is given in Hájek (1998, Remark 3.2.14); stated otherwise, $\models$ is not compact, even though model-sets $\text{Mod}(\psi)$ of Ł$_\infty$-formulas $\psi(X_1 \dots, X_n)$ are compact subsets of the n-cube $[0, 1]^n$, and compactness has a pervasive role in MV-algebra theory, (Cignoli et al. 2000; Mundici 2011).

One is then left with two rather similar problems involving the mutual role of syntax vs. semantics in BL and in Ł$_\infty$:

(A) *Fixed semantics, amendable axioms.* In case BL were not complete for t-tautology semantics, how to strengthen the BL-axioms to obtain a complete logic for continuous t-norms?
(B) *Fixed axioms, amendable semantics.* It being ascertained that $[0, 1]$-valuations fail to yield a strongly complete semantics for Ł$_\infty$, what new notion of "model" of a set of Ł$_\infty$-formulas should be devised to get a strongly complete semantics? (The same question can be asked also for BL, see Sect. 7.6.)

In Hájek (1998) Hájek himself gave the first substantial contribution to Problem (A), by adding to BL two (admittedly not too simple) axioms which, at the time of Hájek (1998, Remark 2.3.23) and Hájek (1998) were not guaranteed to follow from the BL-axioms. The redundancy of these two axioms was finally proved in Cignoli et al. (2000, Theorem 5.2), thus solving Problem (A) in the best possible way: the logic originally invented by Hájek is indeed complete for valuations in t-algebras, the subset of BL-algebras directly given by continuous t-norms.

Since the strong completeness of $[0, 1]$-valuations has been settled in the negative, and the Łukasiewicz axioms are here to stay, in order to solve Problem (B) we are left with no other choice but to modify the traditional semantics of Ł$_\infty$, looking for a novel, genuinely semantical notion of $[0, 1]$-valuation. This is our aim in this chapter.

7.2 Tangents, Differentials and Semantic Consequence Relations in Ł$_\infty$

We refer to Cignoli et al. (2000) and Mundici (2011) for notation and background on MV-algebras and infinite-valued Łukasiewicz propositional logic Ł$_\infty$. The set FORM_n of Ł$_\infty$-formulas in the variables $X_1, \dots, X_n$ has the same definition as its boolean counterpart. The Łukasiewicz connectives $\odot, \oplus$ of conjunction and disjunction are definable in terms of negation $\neg$ and implication $\rightarrow$. Conversely, implication is definable from $\neg$ and $\oplus$.

While in boolean logic formulas take their values in the set $\{0, 1\}$, $Ł_\infty$-formulas are evaluated in the real interval $[0, 1]$. Let $\mathsf{VAL}_n \subseteq [0,1]^{\mathsf{FORM}_n}$ denote the set of valuations (also known as evaluations, assignments, models, interpretations, possible worlds,...). The *truth-functionality* property of $Ł_\infty$ yields the following crucial identification:

7.2.1 *The set* VAL_n *can be identified with the n-cube* $[0, 1]^n \subseteq \mathbb{R}^n$ *via the restriction map* $V \in \mathsf{VAL}_n \mapsto v = V\restriction\{X_1, \ldots, X_n\} \in [0, 1]^{\{X_1,\ldots,X_n\}} = [0, 1]^n$. *For any fixed formula* $\varphi \in \mathsf{FORM}_n$, *the map* $V \in \mathsf{VAL}_n \mapsto V(\varphi) \in [0, 1]$ *defines the function* $\hat{\varphi}\colon [0, 1]^n \to [0, 1]$ by $\hat{\varphi}(v) = V(\varphi)$. *The continuity and piecewise linearity of* $\hat{\varphi}$ *immediately follow by induction on the number of connectives in* φ.

7.2.2 Following Bolzano and Tarski (see Tarski 1956, footnote on p. 417), $Ł_\infty$ is now equipped with the relation $\models$ of *semantic consequence* by stipulating that for all $\Theta \subseteq \mathsf{FORM}_n$ and $\varphi \in \mathsf{FORM}_n$

$$\Theta \models \varphi \text{ iff } \forall v \in [0,1]^n, \ \ (\hat{\theta}(v) = 1 \text{ for all } \theta \in \Theta \Rightarrow \hat{\varphi}(v) = 1).$$

Mutatis mutandis, this notion of consequence is complemented by a completeness theorem in classical logic and in many nonclassical logics having totally disconnected valuation spaces. But for $Ł_\infty$ the situation is different. Indeed from 7.2.1 we have:

7.2.3 *The space* $[0, 1]^n$ *of valuations in* $Ł_\infty$ *is connected. For every* $\varphi \in \mathsf{FORM}_n$, *valuation* $v \in [0, 1]^n$ *and nonzero vector* $u \in \mathbb{R}^n$ *such that the segment* $[v, v + \epsilon u]$ *is contained in* $[0, 1]^n$ *for some* $\epsilon > 0$, *the directional derivative* $\partial\hat{\varphi}(v)/\partial u$ *exists and varies continuously with* u, *once* v *is kept fixed.*

The following simple example involving formulas of one variable already shows that the differential properties of $\hat{\theta}$ for all $\theta \in \Theta$ are ignored by the semantic consequence relation $\models$ of 7.2.2, although they have no less semantical content than the truth-value $\hat{\theta}(v)$:

7.2.4 *Suppose* $\Theta \subseteq \mathsf{FORM}_1$ *is satisfied by a unique valuation* $v \in [0, 1]$. *Assume further* $0 < v < 1, \ v \in \mathbb{Q}$, *and* $\mathrm{d}\hat{\theta}(v)/\mathrm{d}x = 0$ *for all* $\theta \in \Theta$. *Let* $\varphi = \varphi(X_1)$ *be a formula with* $\hat{\varphi}(v) = 1$ *and* $\hat{\varphi}(w) < 1$ *for all* $w \neq v$. *Then* $\Theta \models \varphi$, *although* $\mathrm{d}\hat{\varphi}(v)/\mathrm{d}x^+ < 0$ *and* $\mathrm{d}\hat{\varphi}(v)/\mathrm{d}x^- < 0$.

Intuitively, the hypothesis means that each $\theta \in \Theta$ is not only true at v, but is also true for all w sufficiently close to v; in other words, θ is "stably" true at v, even if the value of v were known up to a certain small error (depending on θ). Although φ misses this (fault-tolerant) stability property of all $\theta \in \Theta$, φ is a semantic consequence of Θ, $\Theta \models \varphi$. It should be noted that $\Theta \nvdash \varphi$.

Similarly, when $n > 1$ and $\Theta \subseteq \mathsf{FORM}_n$, the higher-order stability properties common to all $\theta \in \Theta$ may be missing in some semantic consequence φ of Θ. And again, $\Theta \nvdash \varphi$.

While directional derivatives are trivial in boolean logic, by 7.2.3 they have an important role in $Ł_\infty$. Accordingly, in 7.3.7 we will give a precise definition of "stable" consequence relation $\models_\partial$ which is sensitive to all higher-order differentiability properties of formulas and their associated piecewise linear functions. In Sect. 7.7 this will be generalized to arbitrary sets Θ of formulas. In 7.3.9 we prove that $Ł_\infty$ is "strongly complete" with respect to $\models_\partial$: indeed, $\Theta \models_\partial \varphi$ coincides with the syntactical consequence relation $\Theta \vdash \varphi$.

We then focus on the relative status of $\models_\partial$ with respect to $\models$. As noted in Cignoli et al. (2000, p. 100 and 4.6.6), from the Chang completeness theorem we have:

7.2.5 *Let Θ be an arbitrary (possibly uncountable) set of formulas. Then $\Theta^{\models} = \Theta^{\vdash}$ iff the Lindenbaum algebra* $\mathsf{LIND}(\Theta)$ *is semisimple.*

7.2.6 Following Dubuc and Poveda (2010), we say that an MV-algebra is *strongly semisimple* if all its principal quotients are semisimple.

Let $\Theta \subseteq \mathsf{FORM}_n$. Building on Busaniche and Mundici (2014), in 7.4.3 we observe that $\mathsf{LIND}(\Theta)$ is strongly semisimple iff $(\Theta \cup \{\psi\})^{\models} = (\Theta \cup \{\psi\})^{\models_\partial}$ for all $\psi \in \mathsf{FORM}_n$.

When $\Theta \subseteq \mathsf{FORM}_1$, $\mathsf{LIND}(\Theta)$ is strongly semisimple iff it is semisimple (see 7.4.5).

If $\Theta \subseteq \mathsf{FORM}_2$ and $\mathsf{LIND}(\Theta)$ is semisimple, then $\mathsf{LIND}(\Theta)$ is strongly semisimple iff the set $\mathrm{Mod}(\Theta) \subseteq [0, 1]^{\{X_1, X_2\}} = [0, 1]^2$ of valuations satisfying Θ has no (Bouligand 1930; Severi 1931) outgoing rational tangent vector at any rational point $v \in \mathrm{Mod}(\Theta)$. See 7.5.4.

As shown in 7.5.5, the existence of a Bouligand-Severi rational outgoing tangent at some rational point v of $\mathrm{Mod}(\Theta)$ entails failure of strong semisimplicity in the semisimple MV-algebra $\mathsf{LIND}(\mathrm{Th}(\mathrm{Mod}(\Theta)))$.

In Sect. 7.6, Problems (A) and (B) are retrospectively considered in the light of the results of the previous sections.

7.3 Semantic Consequence $\models$ and Stable Consequence $\models_\partial$

The following corollary of Chang's completeness theorem is proved in Cignoli et al. (2000, Proposition 3.1.4):

7.3.1 *For each $n = 1, 2, \ldots$, the free n-generated MV-algebra $\mathscr{M}([0, 1]^n)$ consists of all functions $f : [0, 1]^n \to [0, 1]$ that are obtainable from the coordinate functions $\pi_i(x_1, \ldots, x_n) = x_i$ by pointwise application of the MV-algebraic operations of negation $\neg x = 1 - x$ and truncated addition $x \oplus y = \min(1, x + y)$. As already noted in 7.2.1, any such function f is continuous and piecewise linear.*

For any nonempty closed set $X \subseteq [0, 1]^n$ we let $\mathscr{M}(X)$ denote the MV-algebra of restrictions to X of the functions in $\mathscr{M}([0, 1]^n)$, in symbols,

$$\mathscr{M}(X) = \{f \restriction X \mid f \in \mathscr{M}([0, 1]^n)\}.$$

McNaughton's characterization Cignoli et al. (2000, Theorem 9.1.5) of the free MV-algebra $\mathscr{M}([0,1]^n)$ will find no use in this chapter.

In Cignoli et al. (2000, Theorem 3.6.7) one can find a proof of the following result, which follows from the proof of Chang's completeness theorem:

7.3.2 *For every nonempty closed set $X \subseteq [0,1]^n$, $\mathscr{M}(X)$ is a semisimple MV-algebra. Conversely, every n-generated semisimple MV-algebra is isomorphic to $\mathscr{M}(Y)$ for some nonempty closed set $Y \subseteq [0,1]^n$.*

To solve Problem (B), we first prepare the necessary elementary material from simplicial geometry.

Fix $n = 1, 2, \ldots$. For any subset S of the Euclidean space $\mathbb{R}^n$ the *convex hull* $\mathrm{conv}(S)$ is the set of all convex combinations of elements of S. Thus $y \in \mathrm{conv}(S)$ iff there are $y_1, \ldots, y_k \in S$ and real numbers $\lambda_1, \ldots, \lambda_k \geq 0$ such that $\lambda_1 + \cdots + \lambda_k = 1$ and $y = \lambda_1 y_1 + \cdots + \lambda_k y_k$. The set S is said to be *convex* if $S = \mathrm{conv}(S)$. For any subset S of $\mathbb{R}^n$, the *affine hull* of S is the set of all *affine combinations* in $\mathbb{R}^n$ of elements of S. Thus z belongs to the affine hull of S iff there are $z_1, \ldots, z_k \in S$ and $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ such that $\lambda_1 + \cdots + \lambda_k = 1$ and $z = \lambda_1 z_1 + \cdots + \lambda_k z_k$. A set $\{y_1, \ldots, y_m\}$ of points in $\mathbb{R}^n$ is said to be *affinely independent* if none of its elements is an affine combination of the remaining elements. The *relative interior* $\mathrm{relint}(C)$ of a convex set $C \subseteq \mathbb{R}^n$ is the interior of C in the affine hull of C. For $0 \leq m \leq n$, an *m-simplex* in $\mathbb{R}^n$ is the convex hull $T = \mathrm{conv}(v_0, \ldots, v_m)$ of $m + 1$ affinely independent points in $\mathbb{R}^n$. The *vertices* $v_0, \ldots, v_m$ are uniquely determined by T.

We next modify the classical notion of valuation as follows:

7.3.3 For $n = 1, 2, \ldots$ and $0 \leq t \leq n$ let $U = (u_0, u_1, \ldots, u_t)$ be a $(t+1)$-tuple of elements of $\mathbb{R}^n$ where $u_1, \ldots, u_t$ are linearly independent vectors. For each $m = 1, 2, \ldots$ let the t-simplex $T_{U,m} \subseteq \mathbb{R}^n$ be defined by

$$T_{U,m} = \mathrm{conv}(u_0, u_0 + u_1/m,\ u_0 + u_1/m + u_2/m^2, \ldots, u_0 + u_1/m + \cdots + u_t/m^t). \tag{7.1}$$

We say that U is a *differential valuation (of order t, in $\mathbb{R}^n$)* if there is an integer $k > 0$ such that for all $m \geq k$ the n-cube $[0,1]^n$ contains $T_{U,m}$. When this is the case, the set $\mathfrak{p}_U \subseteq \mathscr{M}([0,1]^n)$ is defined by

$$\mathfrak{p}_U = \{f \in \mathscr{M}([0,1]^n) \mid f^{-1}(0) \supseteq T_{U,m} \text{ for some } m\}.$$

Traditional valuations coincide with differential valuations of order 0.

7.3.4 *Let $U = (u_0, u_1, \ldots, u_t)$ be a differential valuation in $\mathbb{R}^n$.*

(i) *For all $m = 1, 2, \ldots,$ we have the inclusion $T_{U,m} \supseteq T_{U,m+1}$.*
(ii) *For every $\epsilon_1, \ldots, \epsilon_t > 0$ there is $m = 1, 2, \ldots$ such that the simplex*

$$S = \mathrm{conv}\{u_0, u_0 + \epsilon_1 u_1,\ u_0 + \epsilon_1 u_1 + \epsilon_2 u_2,\ \ldots,\ u_0 + \epsilon_1 u_1 + \cdots + \epsilon_t u_t\}$$

contains $T_{U,m}$.

(iii) $\mathfrak{p}_U$ *is a prime ideal of* $\mathscr{M}([0,1]^n)$.
(iv) *Every prime ideal* $\mathfrak{p}$ *of* $\mathscr{M}([0,1]^n)$ *has the form* $\mathfrak{p} = \mathfrak{p}_V$ *for some differential valuation* V *in* $\mathbb{R}^n$.

Proof (i)–(ii) are easily verified by induction. For (iii)–(iv) use (ii) and see Busaniche and Mundici (2007, Proposition 2.8, Corollary 2.18). □

The prime ideals $\mathfrak{p}_U$ of $\mathscr{M}([0,1]^n)$ are conveniently visualized as follows:
7.3.5 *Let* $U = (u_0, u_1, \ldots, u_t)$ *be a differential valuation in* $\mathbb{R}^n$. *We then have:*

(0) $\mathfrak{p}_{(u_0)}$ *is the maximal ideal of* $\mathscr{M}([0,1]^n)$ *given by all functions of* $\mathscr{M}([0,1]^n)$ *that vanish at* u_0.
(1) $\mathfrak{p}_{(u_0,u_1)}$ *is the prime ideal of* $\mathscr{M}([0,1]^n)$ *given by all functions* $f \in \mathscr{M}([0,1]^n)$ *vanishing on an interval of the form* $\mathrm{conv}(u_0, u_0 + u_1/m)$ *for some integer* $m > 0$. *Equivalently,* $f(u_0) = 0$ *and* $\partial f(u_0)/\partial u_1 = 0$.
(2) $\mathfrak{p}_{(u_0,u_1,u_2)}$ *is the prime ideal of* $\mathscr{M}([0,1]^n)$ *given by those* $f \in \mathscr{M}([0,1]^n)$ *such that for some integer* $m > 0$, f *vanishes on the segment* $\mathrm{conv}(u_0, u_0 + u_1/m)$, *and* $\partial f(y)/\partial u_2 = 0$ *for every* $y \in \mathrm{relint}(\mathrm{conv}(u_0, u_0 + u_1/m))$.
And inductively,
(t) $\mathfrak{p}_{(u_0,u_1,\ldots,u_t)}$ *is the prime ideal of* $\mathscr{M}([0,1]^n)$ *consisting of all* $f \in \mathscr{M}([0,1]^n)$ *such that for some integer* $m > 0$, f *vanishes on the* $(t-1)$*-simplex*

$$S = \mathrm{conv}\Big(u_0, u_0 + u_1/m, u_0 + u_1/m + u_2/m^2, \ldots, u_0 + u_1/m + \cdots + u_{t-1}/m^{t-1}\Big),$$

and $\partial f(y)/\partial u_t = 0$ *for every* $y \in \mathrm{relint}(S)$.

Observe that $\mathfrak{p}_{(u_0)} \supseteq \mathfrak{p}_{(u_0,u_1)} \supseteq \cdots \supseteq \mathfrak{p}_{(u_0,u_1,\ldots,u_{t-1})} \supseteq \mathfrak{p}_{(u_0,u_1,\ldots,u_t)}$.

Generalizing the classical definitions we can now write:

7.3.6 Let $U = (u_0, u_1, \ldots, u_t)$ be a differential valuation in $\mathbb{R}^n$. Let $\psi(X_1, \ldots, X_n)$ be a formula. We then say that U *satisfies* ψ if $1 - \hat{\psi} \in \mathfrak{p}_U$. Thus:

$$\hat{\psi}(u_0) = 1, \quad \frac{\partial \hat{\psi}(u_0)}{\partial u_1} = 0, \ \ldots, \ \text{and } \hat{\psi} \text{ satisfies Conditions (2) through (t) in 7.3.5.}$$

7.3.7 For $\Theta \subseteq \mathsf{FORM}_n$ and $\psi \in \mathsf{FORM}_n$ we say that ψ is a *stable consequence* of Θ, and we write

$$\Theta \models_\partial \psi,$$

if ψ is satisfied by every differential valuation that satisfies each $\theta \in \Theta$.

Recalling the notation of 7.2.2, we have $\Theta \models \psi$ iff ψ is satisfied by every differential valuation of order 0 satisfying (each formula of) Θ. As a consequence:

7.3.8 *Let* $\Theta \subseteq \mathsf{FORM}_n$ *and* $\psi \in \mathsf{FORM}_n$. *If* $\Theta \models_\partial \psi$ *then* $\Theta \models \psi$.

The stable consequence relation $\models_\partial$ is strongly complete:

7.3.9 *For every* $n = 1, 2, \ldots,$ $\Theta \subseteq \mathsf{FORM}_n$ *and* $\psi \in \mathsf{FORM}_n$, $\Theta \models_\partial \psi$ *iff* $\Theta \vdash \psi$.

Proof Following Cignoli et al. (2000, Lemma 4.2.7), let $\mathfrak{j}_\Theta = \langle\{1 - \hat{\theta} \mid \theta \in \Theta\}\rangle$ be the ideal of $\mathscr{M}([0,1]^n)$ generated by the functions given by all negations of formulas in Θ. Equivalently, $\mathfrak{j}_\Theta$ is the ideal associated to the congruence $\equiv_\Theta$ of Mundici (2011, Definition 1.11). Then:

$$
\begin{aligned}
\Theta \vdash \psi &\Leftrightarrow 1 - \hat{\psi} \in \mathfrak{j}_\Theta, \text{[7, 4.2.9] or [16, 1.9]}\\
&\Leftrightarrow 1 - \hat{\psi} \text{ belongs to every prime ideal } \mathfrak{p} \supseteq \mathfrak{j}_\Theta, \text{ by subdirect}\\
&\quad\ \text{representation, [7, 1.2.14]}\\
&\Leftrightarrow 1 - \hat{\psi} \text{ belongs to every prime } \mathfrak{p} \text{ such that } 1 - \hat{\theta} \in \mathfrak{p} \text{ for all } \theta \in \Theta,\\
&\quad\ \text{by definition of } \mathfrak{j}_\Theta\\
&\Leftrightarrow \text{ for every differential valuation } U \text{ in } \mathbb{R}^n, \text{ if } 1 - \hat{\theta} \in \mathfrak{p}_U \text{ for all } \theta \in \Theta\\
&\quad\ \text{then } 1 - \hat{\psi} \in \mathfrak{p}_U, \text{ by 7.3.4 (iii)–(iv)}\\
&\Leftrightarrow \psi \text{ is satisfied by all differential valuations } U \text{ satisfying all } \theta \in \Theta,\\
&\quad\ \text{by 7.3.6}\\
&\Leftrightarrow \Theta \models_\partial \psi, \text{ i.e., } \psi \text{ is a stable consequence of } \Theta, \text{ by 7.3.7.} \qquad \square
\end{aligned}
$$

The finitary character of $\models_\partial$, as opposed to the non-compactness of $\models$, is made precise by the following corollary of 7.3.9:

7.3.10 *Let* $\Theta \subseteq \mathsf{FORM}_n$ *and* $\psi \in \mathsf{FORM}_n$. *Then* $\Theta \models_\partial \psi$ *iff* $\{\theta_1, \ldots, \theta_k\} \models_\partial \psi$ *for some finite subset* $\{\theta_1, \ldots, \theta_k\}$ of Θ.

Since $\mathsf{FORM}_n \subseteq \mathsf{FORM}_{n+1}$, one might ask if $\Theta \models_\partial \psi$ depends on n, whence a more accurate notation would be $\Theta \models_{(n,\partial)} \psi$. The following immediate corollary of 7.3.9 shows that such extra notation is unnecessary:

7.3.11 *Let* $\Theta \subseteq \mathsf{FORM}_n$ *and* $\psi \in \mathsf{FORM}_n$. *Then for any* $m \geq n$, $\Theta \models_{(n,\partial)} \psi$ iff $\Theta \models_{(m,\partial)} \psi$.

Remark The linearly independent vectors $u_1, \ldots, u_t$ in 7.3.3 can be replaced by pairwise orthogonal unit vectors $v_1, \ldots, v_t$ in such a way that $\mathfrak{p}_{(u_o,u_1,\ldots,u_t)}$ coincides with $\mathfrak{p}_{(u_o,v_1,\ldots,v_t)}$.

7.4 Strong Semisimplicity and $\models_\partial$

Recall from 7.2.6 the definition of a strongly semisimple MV-algebra. Since $\{0\}$ is a principal ideal of A, every strongly semisimple MV-algebra is semisimple.

7.4.1 *All boolean algebras are strongly semisimple, and so are all simple and all finite MV-algebras.*

Proof Boolean algebras are hyperarchimedean Cignoli et al. (2000, Sect. 6.3). The rest follows from Cignoli et al. (2000, Sect. 3.5 and Proposition 3.6.5). □

For any $\Theta \subseteq \mathsf{FORM}_n$ the notation $\Theta^{\models_\partial}$ is short for $\{\psi \in \mathsf{FORM}_n \mid \Theta \models_\partial \psi\}$.

7.4.2 *Let* $\Theta \subseteq \mathsf{FORM}_n$. *Then* $\mathsf{LIND}(\Theta)$ *is semisimple iff* $\Theta^{\models} = \Theta^{\models_\partial} = \Theta^{\vdash}$. *Thus* $\mathsf{LIND}(\Theta)$ *is not semisimple iff there is* $\psi \in \mathsf{FORM}_n$ *such that every differential valuation* of order 0 *satisfying* Θ *satisfies* ψ, *and there is a differential valuation* U *satisfying* Θ *but not* ψ.

Proof Cignoli et al. (2000, p. 100) and 7.3.9 above. □

7.4.3 *Let* $\Theta \subseteq \mathsf{FORM}_n$. *Then* $\mathsf{LIND}(\Theta)$ *is strongly semisimple iff for all* $\psi \in \mathsf{FORM}_n$, $(\Theta \cup \{\psi\})^{\models} = (\Theta \cup \{\psi\})^{\models_\partial}$.

Proof For any MV-algebra A and ideal $\mathfrak{j}$ of A, the quotient map

$$\mathfrak{i} \mapsto \mathfrak{i}/\mathfrak{j} = \{b/\mathfrak{i} \mid b \in \mathfrak{i}\}$$

determines a 1–1 correspondence between ideals of A containing $\mathfrak{j}$ and ideals of $A/\mathfrak{j}$, Cignoli et al. (2000, Proposition 1.2.10). A well known result in universal algebra, Cohn (1980, Theorem 3.11, p. 61), yields an isomorphism

$$\frac{a}{\mathfrak{i}} \in \frac{A}{\mathfrak{i}} \mapsto \frac{a/\mathfrak{j}}{\mathfrak{i}/\mathfrak{j}} \in \frac{A/\mathfrak{j}}{\mathfrak{i}/\mathfrak{j}}. \tag{7.2}$$

For any $S \subseteq A$, as above we let $\langle S\rangle$ denote the (possibly not proper) ideal of A generated by S. When S is a singleton $\{a\}$ we write $\langle a\rangle$ instead of $\langle\{a\}\rangle$. For $\mathfrak{j}$ an ideal of A we use the self-explanatory notation $S/\mathfrak{j}$ for $\{b/\mathfrak{j} \mid b \in S\}$. For any $a \in A$ we have the trivial identity

$$\frac{\langle a\rangle}{\mathfrak{j}} = \frac{\langle \mathfrak{j} \cup \{a\}\rangle}{\mathfrak{j}}. \tag{7.3}$$

For any element $a/\mathfrak{j} \in A/\mathfrak{i}$, letting $\langle a/\mathfrak{j}\rangle$ be the ideal generated in $A/\mathfrak{j}$ by $a/\mathfrak{j}$, a routine exercise shows

$$\langle a/\mathfrak{j}\rangle = \langle a\rangle/\mathfrak{j} = \{b/\mathfrak{j} \mid b \leq m \centerdot a \text{ for some } m = 1, 2, \ldots\}. \tag{7.4}$$

Here we are using the notation $m \centerdot a$ of Cignoli et al. (2000, p. 33) or Mundici (2011, p. 21) for m-fold truncated addition.

To complete the proof, for any Θ' with $\Theta \subseteq \Theta' \subseteq \Theta^{\vdash}$ we have $\mathsf{LIND}(\Theta) = \mathsf{LIND}(\Theta') = \mathsf{LIND}(\Theta^{\vdash})$, whence it is no loss of generality to assume $\Theta = \Theta^{\vdash}$. The set $\{1 - \hat{\theta} \mid \theta \in \Theta\}$ is automatically an ideal $\mathfrak{j}_\Theta$ of $\mathscr{M}([0, 1]^n)$ and we have the isomorphism

$$\iota\colon \frac{\psi}{\equiv_\Theta} \in \mathsf{LIND}(\Theta) \;\cong\; \frac{1-\hat\psi}{\mathfrak{j}_\Theta} \in \frac{\mathscr{M}([0,1]^n)}{\mathfrak{j}_\Theta}.$$

It follows that the principal ideal $\langle \psi/{\equiv_\Theta}\rangle$ of $\mathsf{LIND}(\Theta)$ generated by the element $\psi/{\equiv_\Theta} \in \mathsf{LIND}(\Theta)$ corresponds via ι to the principal ideal $\langle (1-\hat\psi)/\mathfrak{j}_\Theta\rangle$ generated by the element $\iota(\psi/{\equiv_\Theta}) = (1-\hat\psi)/\mathfrak{j}_\Theta \in \mathscr{M}([0,1]^n)\,/\mathfrak{j}_\Theta$. By (7.3)–(7.4) we have the identities

$$\left\langle \frac{1-\hat\psi}{\mathfrak{j}_\Theta}\right\rangle = \frac{\langle 1-\hat\psi\rangle}{\mathfrak{j}_\Theta} = \frac{\langle \mathfrak{j}_\Theta \cup \{1-\hat\psi\}\rangle}{\mathfrak{j}_\Theta}.$$

Therefore, $\mathsf{LIND}(\Theta)$ is strongly semisimple iff so is $\mathscr{M}([0,1]^n)/\mathfrak{j}_\Theta$ iff for every principal ideal $\langle \mathfrak{j}_\Theta \cup \{1-\hat\psi\}\rangle/\mathfrak{j}_\Theta$ of $\mathscr{M}([0,1]^n)/\mathfrak{j}_\Theta$ the quotient

$$\frac{\mathscr{M}([0,1]^n)/\mathfrak{j}_\Theta}{\langle \mathfrak{j}_\Theta \cup \{1-\hat\psi\}\rangle/\mathfrak{j}_\Theta} \cong \frac{\mathscr{M}([0,1]^n)}{\langle \mathfrak{j}_\Theta \cup \{1-\hat\psi\}\rangle}$$

is semisimple. We are using (7.2). This is the same as saying that $\mathsf{LIND}(\Theta \cup \{\psi\})$ is semisimple for every $\psi \in \mathsf{FORM}_n$. Now apply 7.4.2. □

7.4.4 *For every finite set of* Ł$_\infty$*-formulas* Φ, *the Lindenbaum algebra* $\mathsf{LIND}(\Phi)$ *is strongly semisimple.*

Proof In view of 7.4.3, this is a reformulation of a result by Hay (1963) and Wójcicki (1977) (also see Cignoli et al. (2000, Theorem 4.6.7) and Mundici (2011, Remarks 1.6)), stating that every finitely presented MV-algebra is strongly semisimple. □

By a quirk of fate, when $n = 1$ strong semisimplicity boils down to semisimplicity (see Busaniche and Mundici (2014) for a proof):

7.4.5 *Let* $\Theta \subseteq \mathsf{FORM}_1$. *Then* $\mathsf{LIND}(\Theta)$ *is strongly semisimple iff it is semisimple.*

7.5 Strong Semisimplicity, $\models_\partial$, and Bouligand-Severi Tangents

Let $\Theta \subseteq \mathsf{FORM}_n$, $n = 1, 2, \ldots$. While the strong semisimplicity of $\mathsf{LIND}(\Theta)$ is formulated in purely algebraic terms, a deeper understanding of this property is provided by inspection of the tangent space of $\mathrm{Mod}(\Theta)$ as a compact subset of the Euclidean space $\mathbb{R}^n$. To this purpose we fix some notation and terminology. A point $x \in \mathbb{R}^n$ is said to be *rational* if so are all its coordinates. By a *rational vector* we mean a nonzero vector $w \in \mathbb{R}^n$ such that the line $\mathbb{R}w = \{\lambda w \in \mathbb{R}^n \mid \lambda \in \mathbb{R}\} \subseteq \mathbb{R}^n$ contains a rational point of $\mathbb{R}^n$ other than the origin. Any nonzero scalar multiple of a rational vector is a rational vector. As usual, $||v||$ is the length of the vector $v \in \mathbb{R}^n$.

The following definitions go back to the late twenties and early thirties of the past century, and prove to be very useful to understand the geometry of strong semisimplicity, and its relationship with stable consequence:

7.5.1 (Severi 1927, Sect. 53, p. 59 and p. 392, Severi 1931, Sect. 1, p. 99, Bouligand 1930, p. 32) A half-line $H \subseteq \mathbb{R}^n$ is *tangent* to a set $X \subseteq \mathbb{R}^n$ at an accumulation point x of X if for all $\epsilon, \delta > 0$ there is $y \in X$ other than x such that $||y - x|| < \epsilon$, and the angle between H and the half-line through y originating at x is $< \delta$.

7.5.2 (Bot et al. 2009, p. 16) Let x be an element of a closed subset X of $\mathbb{R}^n$, and u a unit vector in $\mathbb{R}^n$. We then say that u is a *Bouligand-Severi tangent (unit) vector to X at x* if X contains a sequence $x_0, x_1, \ldots$ of elements, all different from x, such that

$$\lim_{i\to\infty} x_i = x \text{ and } \lim_{i\to\infty} \frac{x_i - x}{||x_i - x||} = u.$$

We further say that u is *outgoing* if the open interval $\mathrm{relint}(\mathrm{conv}(x, x + \lambda u))$ is disjoint from X for some $\lambda > 0$.

7.5.3 (Severi 1931, Sect. 5, p.103). *For any nonempty closed subset X of $\mathbb{R}^n$, point $x \in X$, and unit vector $u \in \mathbb{R}^n$ the following conditions are equivalent*:

(i) *For all $m = 1, 2, \ldots$, the cone $\mathscr{C}_{x,u,1/m,1/m^2}$ with apex x, axis parallel to u,* height $1/m$ *and vertex angle* $1/m^2$ *contains infinitely many points of X.*
(ii) *u is a Bouligand-Severi tangent vector to X at x.*
(iii) *The half-line $x + \mathbb{R}_{\geq 0}u$ is tangent to X.*

7.5.4 (Busaniche and Mundici 2014) *Let $\Theta \subseteq \mathsf{FORM}_2$. Suppose $\mathsf{LIND}(\Theta)$ is semisimple. Then $\mathsf{LIND}(\Theta)$ is strongly semisimple iff $\mathrm{Mod}(\Theta)$ does not have any Bouligand-Severi outgoing rational tangent vector at any of its rational points.*

Combining Busaniche and Mundici (2014) with our characterization 7.4.2 we get:

7.5.5 *Let $\Theta \subseteq \mathsf{FORM}_n$. Suppose $\mathsf{LIND}(\Theta)$ is semisimple and $\mathrm{Mod}(\Theta)$ has some Bouligand-Severi outgoing rational tangent vector u at some rational point $v \in \mathrm{Mod}(\Theta)$. Then $\mathsf{LIND}(\Theta)$ is not strongly semisimple.*

There are formulas $\gamma, \lambda \in \mathsf{FORM}_n$ such that $\Theta \cup \{\gamma\} \models \lambda$ but it is not the case that $\Theta \cup \{\gamma\} \models_\partial \lambda$. Specifically, for every stable consequence ψ of $\Theta \cup \{\gamma\}$ we have $\hat{\psi}(v) = 1$ and $\partial\hat{\psi}(v)/\partial u = 0$, but $\hat{\lambda}(v) = 1$ and $\partial\hat{\lambda}(v)/\partial u < 0$.

As in Mundici (2011, Sect. 1.3, 1.4), the operator $\mathrm{Th}\colon X \subseteq [0, 1]^n \mapsto \mathrm{Th}\, X \subseteq \mathsf{FORM}_n$ is defined by

$$\mathrm{Th}\, X = \{\psi \in \mathsf{FORM}_n \mid \hat{\psi}(w) = 1 \text{ for all } w \in X\}.$$

7.5.6 *If there exists a Bouligand-Severi rational outgoing tangent vector at some rational point v of $\mathrm{Mod}(\Theta)$, then $\mathsf{LIND}(\mathrm{Th}(\mathrm{Mod}(\Theta)))$ is semisimple but not strongly semisimple.*

Proof $\mathsf{LIND}(\mathrm{Th}(\mathrm{Mod}(\Theta)))$ is semisimple because $\mathrm{Th}(\mathrm{Mod}(\Theta)) = \Theta^{\models}$. It is not strongly semisimple by Busaniche and Mundici (2014). □

Thus the strong semisimplicity of $\mathsf{LIND}(\mathrm{Th}(\mathrm{Mod}(\Theta)))$, and more generally, of every $\Phi \subseteq \mathsf{FORM}_n$ with $\Phi^{\models} = \Phi^{\models_\partial}$, only depends on the (tangent space of the) set $\mathrm{Mod}(\Theta) \subseteq [0, 1]^n$.

7.6 Concluding Remarks

As shown by the examples of BL and $Ł_\infty$, our starting point is a syntactic consequence relation $\vdash$ based on a set $\mathscr{R}$ of axioms and rules. Then variously defined "semantic" consequence relations are tailored around $\vdash$, until a satisfactory relation is obtained in terms of a certain set of valuations: in the case of BL, a natural complete semantics is provided by t-algebraic valuations; in the case of $Ł_\infty$, strong completeness is achieved by the set of differential valuations, which contains the set of [0, 1]-valuations as the special 0th order case.

Historically, the emergence of semantical notions in first-order logic followed a similar path. Here a long distillation process culminated in a definitive consequence relation $\vdash$. At a later stage, motivation/confirmation of the definitive nature of $\vdash$ would be provided by suitably defined "models" (interpretations, substitutions, evaluations, possible worlds,...). Without them one cannot even speak of the correctness of the set of rules of first-order logic. The completeness problem had a long gestation period. The notions of categoricity and completeness of theories were often confused with the completeness of the set of rules; before the appearance of Tarski models over arbitrary universes, the set of arithmetical models over the *fixed* universe $\mathbb{N}$ was used to evaluate formulas.

Turning retrospectively to Problems (A) and (B), in Sect. 7.1 we didn't mention the following well known fact (Mundici 2011, Proposition 20.7, p. 231):

7.6.1 *For each* $i = 1, 2$ *and any (possibly uncountable) set* Θ *of formulas, let* $\Theta \models_{MVi} \varphi$ *be given by the following stipulation*:

(I) $\Theta \models_{MV1} \varphi$ *iff every A-valuation satisfying every* $\theta \in \Theta$ *also satisfies* φ, *where* A *ranges over arbitrary MV-algebras.*
(II) $\Theta \models_{MV2} \varphi$ *iff every* C*-valuation satisfying every* $\theta \in \Theta$ *also satisfies* φ, *where* C *ranges over arbitrary MV-chains.*

Then $\models_{MV1} = \models_{MV2}$ *=the syntactic consequence relation* $\vdash$ *of* $Ł_\infty$.

Each consequence relation $\models_{MVi}$, while endowing $Ł_\infty$ with a strongly complete semantics, has the same drawbacks as the consequence relation $\models_{BL}$ arising from *all* BL-valuations in 7.1.1–7.1.2: since $\models_{MVi}$ does not directly reflect the intuition behind the original axioms, its applicability is limited.

Consider, for instance, the complexity of the problem whether $\alpha \vdash \beta$, for $\alpha, \beta \in \bigcup_n \mathsf{FORM}_n$. The binary relation

$$\vdash_{\text{fin}} \;=\; \left(\bigcup_n \mathsf{FORM}_n \times \bigcup_n \mathsf{FORM}_n\right) \bigcap \;\vdash$$

turns out to be decidable for BL and for $Ł_\infty$, no less than for boolean logic. However, the proper class of *all* BL- and *all* MV-algebras, which is needed to check $\models_{BL}$ and $\models_{MV}$, has no role in the proof of these decidability results. Actually, the proof depends on subdirect representation and completeness theorems, which, combined with results like the Hay-Wójcicki theorem, yield a dramatic restriction of the set of evaluations needed to check semantic consequence. Suitably small finite chains turn out to be sufficient to decide if β is a consequence of α. In this way we get polytime verifiable certificates for $\alpha \nvdash \beta$ whence the coNP-completeness of $\vdash_{\text{fin}}$ follows. See Baaz et al. (2002) and Mundici (1987). Also see Montagna (2007) for a general discussion of completeness in various logics, including BL and $Ł_\infty$.

The evolving semantical notions of valuation (model, interpretation, possible world,...), strongly impinge on the evolution of the proof theory of $\vdash$. While $\vdash$ is immutable, the recipe $\mathscr{R}$ to check $\alpha \vdash \beta$ is not: we do not even know if "proofs", as we understand them today in boolean logic (let alone $Ł_\infty$ and BL), will one day be superseded by revolutionary polytime decision procedures.

Hájek's intuition of the BL-axioms was confirmed by a completeness result for valuations over t-algebras rather than over arbitrary BL-algebras. Similarly, the Łukasiewicz axioms for $Ł_\infty$, as well as Chang's MV-algebraic axioms, are now endowed by a strongly complete (genuinely semantic) consequence relation $\models_\partial$ that does not resort to valuations over exoteric MV-algebras and their "infinitesimal truth-values". Rather, $\Theta \models_\partial \psi$ depends on (real-valued) differential valuations that check if ψ has the stability properties common to all $\theta \in \Theta$.

Turning to Problem (B), in view of the geometric representation of free BL-algebras in Aguzzoli and Bova (2010), one may reasonably conjecture that some sort of "stable" valuations should also work for BL and yield a strongly complete semantics. The constructions in the present chapter may be of help when taking into account the Łukasiewicz components of the relevant continuous t-norm.

Closing a circle of logic-algebraic-geometric ideas, our results show that the traditional semantic consequence relation $\Theta \models \varphi$ of $Ł_\infty$ fails to be strongly complete because of its total insensitivity to the Bouligand-Severi tangent space of $\mathrm{Mod}(\Theta)$. Strong completeness is retrieved by differential valuations, which take into account the directional derivatives of formulas along the tangent space of $\mathrm{Mod}(\Theta)$.

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7.7 Appendix: Stable Consequence for Arbitrary Sets of Sentences

Up to isomorphism, MV-algebras are the Lindenbaum algebras of sets of formulas on unlimited supplies of variables. So let $\mathscr{X} = \{X_1, X_2, \ldots, X_\alpha, \ldots \mid \alpha < \kappa\}$ be a set of variables of infinite, possibly uncountable cardinality κ, indexed by all ordinals $0 \leq \alpha < \kappa$. Letting $\mathsf{FORM}_{\mathscr{X}}$ be the set of formulas $\psi(X_{\alpha_1}, \ldots, X_{\alpha_t})$ whose variables are contained in $\mathscr{X}$, in 7.7.5 below we will routinely extend Definition 7.3.7 to arbitrary subsets Θ of $\mathsf{FORM}_{\mathscr{X}}$ and formulas $\psi \in \mathsf{FORM}_{\mathscr{X}}$.

Throughout, we will tacitly identify the valuation space $[0, 1]^{\mathscr{X}}$ with the Tychonov cube $[0, 1]^{\kappa}$ equipped with the product topology.

7.7.1 *The free MV-algebra over κ free generators is the MV-algebra $\mathscr{M}([0, 1]^{\kappa})$ of all functions on$[0, 1]^{\kappa}$ obtainable from the coordinate functions $\pi_{\beta}(x) = x_{\beta}, \quad (x \in [0, 1]^{\kappa}, \quad 0 \leq \beta < \kappa)$ by pointwise application of the operations $\neg$ and $\oplus$*, Cignoli et al. (2000, Theorem 9.1.5).

7.7.2 For any finite set $\mathscr{K} = \{X_{\alpha_1}, \ldots, X_{\alpha_d}\} \subseteq \mathscr{X}$ we further identify $[0, 1]^{\mathscr{K}}$ with the set of all $x \in [0, 1]^{\mathscr{X}}$ such that every coordinate x_{β} of x vanishes, with the possible exception of $\beta \in \{\alpha_1, \ldots, \alpha_d\}$. If $\emptyset \neq \mathscr{H} \subseteq \mathscr{K} \subseteq \mathscr{X}$, the MV-algebra $\mathscr{M}([0, 1]^{\mathscr{H}})$ is canonically identified with an MV-subalgebra of $\mathscr{M}([0, 1]^{\mathscr{K}})$ via cylindrification.

7.7.3 Suppose we are given two finite subsets $\mathscr{H} \subseteq \mathscr{K}$ of $\mathscr{X}$ and two differential valuations $U = (u_0, u_1, \ldots, u_d)$ in $\mathbb{R}^{\mathscr{H}}$ and $V = (v_0, v_1, \ldots, v_e) \in \mathbb{R}^{\mathscr{K}}$. We then have two prime ideals $\mathfrak{p}_U$ of $\mathscr{M}([0, 1]^{\mathscr{H}})$ and $\mathfrak{p}_V$ of $\mathscr{M}([0, 1]^{\mathscr{K}}) \supseteq \mathscr{M}([0, 1]^{\mathscr{H}})$. We say that V *dominates* U, in symbols, $V \succeq U$, if $\mathfrak{p}_U = \mathfrak{p}_V \cap \mathscr{M}([0, 1]^{\mathscr{H}})$. When this is the case, the coordinates of v_0 corresponding to the variables of $\mathscr{H}$ agree with those of u_0. Further information on the relationship between U and V can be found in Busaniche and Mundici (2007, Sect. 4).

The following definitions extend 7.3.3, 7.3.6 and 7.3.7 to any set $\mathscr{X}$ of variables of infinite cardinality κ:

7.7.4 By a *differential valuation in* $\mathbb{R}^{\kappa}$ we understand a $\succeq$*-direct system*

$$W = \{U_{\mathscr{H}} \mid \mathscr{H} \subseteq \mathscr{X}, \ \mathscr{H} \text{ finite}\}$$

of differential valuations $U_{\mathscr{H}}$ in $\mathbb{R}^{\mathscr{H}}$, in the sense that for any finite $\mathscr{I}, \mathscr{J} \subseteq \mathscr{X}$, $U_{\mathscr{I} \cup \mathscr{J}}$ dominates both $U_{\mathscr{I}}$ and $U_{\mathscr{J}}$. We say that W *satisfies* a formula $\varphi \in \mathsf{FORM}_{\mathscr{X}}$ if $U_{\mathrm{var}(\varphi)}$ satisfies φ in the sense of 7.3.6, i.e., $1 - \hat{\varphi}$ belongs to the prime ideal $\mathfrak{p}_{U_{\mathrm{var}(\varphi)}}$. (As usual, $\mathrm{var}(\varphi)$ denotes the set of variables occurring in φ.)

7.7.5 For $\Theta \subseteq \mathsf{FORM}_{\mathscr{X}}$ and $\psi \in \mathsf{FORM}_{\mathscr{X}}$ we say that ψ is a *stable consequence* of Θ and we write $\Theta \models_{\partial} \psi$, if ψ is satisfied by every differential valuation W in $\mathbb{R}^{\kappa}$ that satisfies each $\theta \in \Theta$.

The "strong completeness" result 7.3.9 is now extended to arbitrary sets of variables:

7.7.6 *Let $\mathscr{X} \neq \emptyset$ be an arbitrary (finite or infinite) set of variables, $\Theta \subseteq \mathsf{FORM}_{\mathscr{X}}$, and $\psi \in \mathsf{FORM}_{\mathscr{X}}$. Then $\Theta \models_{\partial} \psi$ iff $\Theta \vdash \psi$.*

Proof In the light of 7.3.9 we have only to consider the case when $\mathscr{X}$ has infinite cardinality κ.

By construction, every differential valuation $W = \{U_{\mathscr{I}} \mid \mathscr{I} \subseteq \mathscr{X}, \ \mathscr{I} \text{ finite}\}$ determines the prime ideal

$$\mathfrak{p}_W = \bigcup \left\{\mathfrak{p}_{U_{\mathscr{I}}} \mid \mathscr{I} \subseteq \mathscr{X}, \ \mathscr{I} \text{ finite}\right\}. \tag{7.5}$$

Conversely, suppose $\mathfrak{p}$ is a prime ideal of $\mathscr{M}([0,1]^{\kappa}) = \mathscr{M}([0,1]^{\mathscr{X}})$. Then

$$\mathfrak{p} = \bigcup \left\{ \mathfrak{p} \cap \mathscr{M}([0,1]^{\mathscr{H}}) \mid \mathscr{H} \text{ a finite subset of } \mathscr{X} \right\}.$$

By 7.3.4 (iv), for any finite subset $\mathscr{H}$ of $\mathscr{X}$ there is a differential valuation $U_{\mathscr{H}}$ in $\mathbb{R}^{\mathscr{H}}$ such that $\mathfrak{p} \cap \mathscr{M}([0,1]^{\mathscr{H}}) = \mathfrak{p}_{U_{\mathscr{H}}}$. For any two finite subsets $\mathscr{I}$, $\mathscr{J}$ of $\mathscr{X}$ we have

$$\mathfrak{p}_{U_{\mathscr{I} \cup \mathscr{J}}} \cap \mathscr{M}([0,1]^{\mathscr{I}}) = \mathfrak{p}_{U_{\mathscr{I}}} \quad \text{and} \quad \mathfrak{p}_{U_{\mathscr{I} \cup \mathscr{J}}} \cap \mathscr{M}([0,1]^{\mathscr{J}}) = \mathfrak{p}_{U_{\mathscr{J}}}.$$

We then obtain a differential valuation $W_{\mathfrak{p}} = \{U_{\mathscr{H}} \mid \mathscr{H} \subseteq \mathscr{X},\ \mathscr{H} \text{ finite}\}$ such that $\mathfrak{p} = \mathfrak{p}_{W_{\mathfrak{p}}}$ (notation of 7.5).

Having thus shown that the map $W \mapsto \mathfrak{p}_W$ sends the set of differential valuations in $\mathbb{R}^{\kappa}$ *onto* the set of prime ideals of $\mathscr{M}([0,1]^{\kappa})$, we get the desired result arguing as in the proof of 7.3.9. □

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Chapter 8
Two Principles in Many-Valued Logic

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8.1 Prologue

At the outset of his landmark monograph (Hájek 1998), Petr Hájek writes:

> There are various systems of fuzzy logics, not just one. We have one basic logic (BL) and three of its most important extensions: Łukasiewicz logic, Gödel logic, and the product logic. Hájek (1998, p. 5).

Basic Logic is, of course, the creation of Hájek himself. One of its several virtues is to afford metamathematical comparison of many-valued logics to an unprecedented degree of clarity. Our chapter is intended as a modest contribution to such comparative studies; it will soon transpire that it would have been impossible to write it, in the possible but unfortunate worlds orphan of Hájek (1998).

We assume familiarity with Basic (propositional) Logic, triangular norms (*t-norms*, for short), and BL-algebras; see Hájek (1998), and Sect. 8.2 for an outline. Note that in this chapter 't-norm' means 'continuous t-norm', for the sake of brevity.

Dedicated to Petr Hájek

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We write FORM for the set of formulæ over the countable collection of propositional variables VAR:= $\{X_1, X_2, \ldots\}$, with primitive connectives $\rightarrow$ (implication), & (monoidal conjunction), and $\perp$ (*falsum*). As usual, & is semantically interpreted by a t-norm, $\rightarrow$ by its residuum, and $\perp$ by 0. We adopt the standard abbreviations, $\neg\alpha := \alpha \rightarrow \perp, \alpha\wedge\beta := \alpha\&(\alpha \rightarrow \beta), \alpha\vee\beta := ((\alpha \rightarrow \beta) \rightarrow \beta)\wedge((\beta \rightarrow \alpha) \rightarrow \alpha)$, and $\alpha \leftrightarrow \beta := (\alpha \rightarrow \beta)\&(\beta \rightarrow \alpha)$. We write BL to denote Basic Logic, as axiomatised in (Hájek 1998; Cignoli et al. 2000). An *extension* of BL is a collection of formulæ closed under the (syntactic) consequence relation of BL, and closed under substitutions. If $\mathcal{M}$ is an extension of BL, we always tacitly assume that $\mathcal{M}$ is consistent, we refer to $\mathcal{M}$ as a *many-valued logic*, and we denote by $\vdash_{\mathcal{M}}$ its consequence relation.

Łukasiewicz logic, denoted Ł, is obtained by extending BL with the axiom schema $\neg\neg\varphi \rightarrow \varphi$. *Gödel logic*, denoted G, is obtained by adding to BL the schema $\varphi \rightarrow (\varphi\&\varphi)$. To obtain *Product logic*, written P, one extends BL with $\neg\varphi \vee ((\varphi \rightarrow (\varphi\&\psi)) \rightarrow \psi)$. See Hájek (1998, p. 63, Definitions 4.2.1 and 4.1.1), and Cintula et al. (2011, Chap. I).

Over the real unit interval $[0, 1] \subseteq \mathbb{R}$, consider a BL-algebra $([0, 1], *, \rightarrow_*, 0)$, where $*\colon [0, 1] \times [0, 1] \rightarrow [0, 1]$ is a continuous t-norm with residuum $\rightarrow_*$. By an *algebra of truth values* we shall mean a subalgebra T_* of some such BL-algebra $([0, 1], *, \rightarrow_*, 0)$. Note, in particular, that $\{0, 1\}$ is a subset of any algebra of truth values. We write $T_* \subseteq [0, 1]$ for the underlying set of the algebra of truth values, too, i.e. for the set of truth values itself.

We say that the pair $(\mathcal{L}, T_*)$ is a *real-valued logic* if $\mathcal{L}$ is an extension of BL that is complete with respect to valuations $\mu\colon$ FORM$\rightarrow T_*$ into the given algebra of truth values. Any algebra of truth values T'_* such that $(\mathcal{L}, T'_*)$ is a real-valued logic is said to *induce* $\mathcal{L}$. When T_* =[0,1], we also say that $\mathcal{L}$ is induced by the t-norm $*$. (This makes sense, recalling that $\rightarrow_*$ is uniquely determined by $*$. See Sect. 8.2.) Distinct algebras of truth values may of course induce the same logic $\mathcal{L}$, i.e. the same extension of BL. When we write that $\mathcal{L}$ is a real-valued logic, with no reference to T_*, we mean that there is at least one algebra of truth values T_* that induces $\mathcal{L}$.

With this machinery in place, we consider two principles that a real-valued logic $\mathcal{L}$ may or may not satisfy.

P1. For every algebra T_* of truth values inducing $\mathcal{L}$, the following holds. For each $\alpha, \beta \in$FORM, we have $\vdash_{\mathcal{L}} \alpha \leftrightarrow \beta$ if, and only if,

$$\mu(\alpha) = 1 \iff \mu(\beta) = 1$$

holds for each valuation $\mu\colon$ FORM$\rightarrow T_*$. □

P2. For every algebra T_* of truth values inducing $\mathcal{L}$, the following holds. For each pair of valuations $\mu, \nu\colon$ FORM$\rightarrow T_*$, if $\mu \neq \nu$ then there is a formula $\alpha \in$ FORM such that $\mu(\alpha) > 0$ while $\nu(\alpha) = 0$. □

Our first two results are that **P**1 and **P**2 are characteristic of G and Ł, respectively, to within extensions.

Theorem 8.1 *A real-valued logic* $(\mathcal{L}, T_*)$ *satisfies* **P**1 *if, and only if,* $\mathcal{L}$ *is an extension of Gödel logic.*

Theorem 8.2 *A real-valued logic* $(\mathcal{L}, T_*)$ *satisfies* **P**2 *if, and only if,* $\mathcal{L}$ *is an extension of Łukasiewicz logic.*

Remark 8.1 Observe that the two preceding theorems show that in **P**1 and **P**2 one can safely replace the initial universal quantification by an existential one. In other words, the principles **P**1 and **P**2 display robustness with respect to the specific choice of the algebra of truth values, *salva logica* $\mathcal{L}$. □

We prove Theorem 8.1 in Sect. 8.3, and Theorem 8.2 in Sect. 8.4, after some preliminaries in Sect. 8.2.

The question arises, can one also characterise Product logic by means of general principles such as **P**1 and **P**2. We shall show how to answer this question affirmatively, under one additional assumption. Let us say that the real-valued logic $\mathcal{L}$ is *closed* if there exists an algebra of truth values T^* inducing $\mathcal{L}$ such that the underlying set of T^* is closed in the Euclidean topology of [0,1]. Product logic is the unique *closed* real-valued logic that *fails* both **P**1 and **P**2 hereditarily with respect to real-valued extensions, in the following sense:

Theorem 8.3 *A closed real-valued logic* $\mathcal{L}$ *is Product logic if, and only if,* $\mathcal{L}$ *and all of its non-classical, real-valued extensions fail* **P**1 *and* **P**2.

We prove Theorem 8.3 in Sect. 8.5.

The proofs of Theorems 8.1–8.3 are relatively straightforward applications of known facts about extensions of Basic Logic. The interest of the present contribution, if any, is thus to be sought not so much in the technical depth of the results, as in the significance of the two principles **P**1 and **P**2 in connection with logics of comparative truth. Before turning to the proofs, let us therefore expound on **P**1 and **P**2 a little.

Logics fulfilling **P**1 share with classical logic the feature that each proposition is uniquely determined, up to logical equivalence, by the collection of its true interpretations (that is, models), where 'true' in the latter sentence is to be read as 'true to degree 1'. In the classical case this may be conceived as a consequence of the Principle of Bivalence, along with completeness. (Indeed, if in classical logic α and β evaluate to 1 at exactly the same μ's, then, by bivalence, they evaluate to the same value at each μ; hence $\alpha \leftrightarrow \beta$ is a tautology, and we therefore have $\vdash \alpha \leftrightarrow \beta$ by completeness.) Theorem 8.1 shows that, remarkably, real-valued logics that *fail* the Principle of Bivalence—for instance, Gödel logic—may still satisfy **P**1.

Logics fulfilling **P**2 share with classical logic the feature that distinct models of the logic can be separated by some formula. In more detail, classical logic has the property that if μ and ν are two distinct true interpretations of its axioms, then there is a formula α that can tell apart the two models μ and ν, in the sense that α is not false (i.e. true) in μ but false in ν. A logic failing **P**2, by contrast, must allow two distinct true interpretations μ and ν of its axioms which are indiscernible, in the sense that no proposition is false (i.e. evaluates to degree 0) in ν and not false (i.e. evaluates to degree > 0) in μ.[1] In this precise sense, the given real-valued semantics

[1] It should be emphasised that there is some leeway in formulating the separating conditions $\mu(\alpha) > 0$ and $\nu(\alpha) = 0$ here: see Corollary 8.1 below for equivalent variants.

of such a logic is redundant, as one could identify μ and ν without any logical loss. Theorem 8.2 shows that, remarkably, there is just one [0,1]-valued logic—namely, Łukasiewicz logic—capable of avoiding that redundancy, by actually telling apart any two distinct real numbers in [0,1].

8.2 Preliminary Facts About Real-Valued Logics

We outline the framework of Hájek's Basic Logic. A (*continuous*) *t-norm* is a binary operation $*\colon [0,1]^2 \to [0,1]$, continuous with respect to the Euclidean topology of [0,1], that is associative, commutative, has 1 as neutral element, and is monotonically non-decreasing in each argument:

$$\forall a, b, c \in [0, 1] \ : \ b \leqslant c \implies a * b \leqslant a * c.$$

For $a, b \in [0,1]$, set $a \to_* b := \sup\{c \in [0,1] \mid a * c \leqslant b\}$. It is well known (Hájek 1998, Sect. 2.1.3) that continuity is sufficient to entail $a \to_* b = \max\{c \in [0,1] \mid a * c \leqslant b\}$. The operation $\to_*$ is called the *residuum* of $*$. Recall that the residuum determines the underlying order, that is, $a \leqslant b$ if, and only if, $a \to_* b = 1$. Further recall that the subset of FORM that evaluates to 1 under every valuation $\mu\colon$ FORM$\to ([0,1], *, \to_*, 0)$, is by definition the collection of all tautologies of BL. It is one of the main achievements of Hájek (1998), of course, that this set is recursively axiomatisable by schemata, using *modus ponens* as the only deduction rule; see also Cignoli et al. (2000) for an improved axiomatisation. Moreover, BL is an algebraizable logic, see Hájek (1998, p. 25 and references therein); the algebras in the corresponding variety are called *BL-algebras*, and schematic extensions of BL are in one-one natural correspondence with subvarieties of BL-algebras. Each t-norm $*\colon [0,1]^2 \to [0,1]$ induces a BL-algebra $([0,1], *, \to_*, 0)$, and the variety of BL-algebras is generated by the collection of all t-norms. More generally, each algebra of truth values as defined above is a BL-algebra. We occasionally write 'BL-chain' for 'totally ordered BL-algebra'.

Given algebras of truth-values $T_*, T'_* \subseteq [0,1]$, we say that $\sigma\colon T_* \to T'_*$ is an *isomorphism* if σ is an isomorphism of BL-algebras; equivalently, σ is a bijection, for all $a, b \in T_*$ we have $\sigma(a * b) = \sigma(a) * \sigma(b)$, and $a \leqslant b$ implies $\sigma(a) \leqslant \sigma(b)$.

Recall the three fundamental t-norms.

$$x \odot y := \max\{0, x + y - 1\} \tag{8.1}$$

$$x \min y := \min\{x, y\} \tag{8.2}$$

$$x \times y := xy \tag{8.3}$$

The associated residua evaluate to 1 for each $x, y \in [0,1]$ with $x \leqslant y$; when $x > y$, they are respectively given by:

$$x \to_{\odot} y := 1 - x + y$$
$$x \to_{\min} y := y$$
$$x \to_{\times} y := \frac{y}{x}$$

The algebra of truth values $T_{\odot} := ([0,1], \odot, \to_{\odot}, 0)$ is called the *standard MV-algebra*; the *standard Gödel algebra*, denoted $T_{\min}$, and the *standard Product algebra*, denoted $T_{\times}$, are defined analogously using (8.2)–(8.3) and their residua. The important *completeness theorems* for Ł, G, and P will be tacitly assumed throughout: they state that these logics are complete with respect to evaluations into $T_{\odot}$, $T_{\min}$, and $T_{\times}$, respectively. For proofs and references, consult Hájek (1998, Theorems 3.2.13, 4.2.17, and 4.1.13, and *passim*).

In the remainder of this section we collect technical results needed in the sequel. We begin with a remark that will find frequent application.

Remark 8.2 For any real-valued logic $(\mathcal{L}, T_*)$, let T'_* be an algebra of truth values that is isomorphic to T_*. Then the logic induced by T'_* is again $\mathcal{L}$. This follows immediately from the fact that $\sigma(1) = 1$ and $\sigma^{-1}(1) = 1$ for any isomorphism $\sigma : T_* \to T'_*$. The converse statement is false in general: it is well known that non-isomorphic t-norms may induce the same real-valued logic. However, the following hold.

1. The only t-norm inducing G is the minimum operator, for it is the only idempotent t-norm. See Hájek (1998, Theorem 2.1.16).
2. Each t-norm inducing Ł is isomorphic to $T_{\odot}$. See Hájek (1998, Lemmata 2.1.22.(2) and 2.1.23).
3. Each t-norm inducing P is isomorphic to $T_{\times}$. See Hájek (1998, Lemma 2.1.22.(1)).

□

Lemma 8.1 *For any real-valued logic $(\mathcal{L}, T_*)$, and for any formulæ $\alpha, \beta \in$ FORM, we have:*

$$\begin{aligned} \vdash_{\mathcal{L}} \alpha \leftrightarrow \beta \iff & \vdash_{\mathcal{L}} \alpha \to \beta \text{ and } \vdash_{\mathcal{L}} \beta \to \alpha \\ \iff & \mu(\alpha) = \mu(\beta) \text{ for all valuations } \mu : \mathrm{FORM} \to T_*. \end{aligned}$$

Proof Indeed, $\vdash_{\mathcal{L}} \alpha \leftrightarrow \beta$ iff, by the completeness of $\mathcal{L}$ with respect to T_*, for all valuations μ: FORM$\to T_*$ we have $\mu(\alpha \leftrightarrow \beta) = 1$ iff, since 1 is the neutral element for $*$, $\mu(\alpha \to \beta) = \mu(\beta \to \alpha) = 1$ iff, by the completeness of $\mathcal{L}$ with respect to T_*, $\vdash_{\mathcal{L}} \alpha \to \beta$ and $\vdash_{\mathcal{L}} \beta \to \alpha$ iff, since $\mu(\alpha \to \beta) = 1$ is equivalent to $\mu(\alpha) \leqslant \mu(\beta)$ by the definition of residuum, $\mu(\alpha) = \mu(\beta)$. □

BL-algebras are defined over the signature $(*, \to, \bot)$. *Basic hoops* are the $\bot$-free subreducts of BL-algebras, the latter considered over the extended signature that includes $\top := \bot \to \bot$. Conversely, BL-algebras are *bounded* basic hoops, that is, basic hoops with a minimum element which interprets the new constant $\bot$. Let now $(I, \leqslant)$ be a totally ordered set, and let $\{C_i\}_{i \in I}$ be a family of totally ordered basic hoops, where $C_i := (C_i, *_i, \to_i, 1)$. Assume further that $C_i \cap C_j = \{1\}$ for each $i \neq j \in I$. Then the *ordinal sum* of the family $\{C_i\}_{i \in I}$ is the structure[2]

$$\bigoplus_{i \in I} C_i := \left(\bigcup_{i \in I} C_i, *, \to, 1 \right),$$

where

$$x * y = \begin{cases} x *_i y & \text{if } x, y \in C_i, \\ y & \text{if } x \in C_i,\ y \in C_j \setminus \{1\},\ i > j, \\ x & \text{otherwise,} \end{cases}$$

and

$$x \to y = \begin{cases} x \to_i y & \text{if } x, y \in C_i, \\ y & \text{if } x \in C_i,\ y \in C_j,\ i > j, \\ 1 & \text{otherwise.} \end{cases}$$

Each C_i is called a *summand* of the ordinal sum.

Lemma 8.2 (The Mostert-Shields Structure Theorem) *Each algebra of truth values* $([0, 1], *, \to_*, 0)$ *is isomorphic to an ordinal sum of bounded basic hoops, each of which is isomorphic to one among* $T_\odot$, $T_{\min}$, $T_\times$, *and* $\{0, 1\}$.

Proof This is essentially Mostert and Shields (1957, Theorem B). □

Lemma 8.3 *Let* A *be a subalgebra of an ordinal sum* $\bigoplus_{i \in I} B_i$. *Then there exists* $J \subseteq I$ *and algebras* $\{C_j \mid j \in J\}$ *such that* C_j *is a subalgebra of* B_j *for each* $j \in J$, *and* $A \cong \bigoplus_{j \in J} C_j$.

Proof Direct inspection of the definition of ordinal sum. □

MV-algebras (Cignoli et al. 2000) are (term equivalent to) BL-algebras satisfying the equation $\neg\neg x = x$, where $\neg x$ is short for $x \to \bot$. *Wajsberg hoops* are the $\bot$-free subreducts of MV-algebras; equivalently, MV-algebras are exactly the bounded Wajsberg hoops.

Lemma 8.4 *Each finite BL-chain splits into an ordinal sum of finitely many finite MV-chains.*

Proof This is Aglianò and Montagna (2003, Theorem 3.7), together with the observation that finite Wajsberg hoops are necessarily bounded. □

Lemma 8.5 *Suppose the algebra of truth values* T_* *is not a subalgebra of* $T_\odot$. *Then* T_* *splits into a non-trivial ordinal sum of at least two summands.*

[2] Usage of the symbol $\oplus$ to denote ordinal sums seems fairly standard. It is also standard to use $\oplus$ to denote Łukasiewicz's strong disjunction, see Cignoli et al. (2000). This we will do in Sect. 8.4, where context should prevent confusion.

Proof If T_* is finite, from Lemma 8.4 it follows that T_* is isomorphic to an ordinal sum of finitely many finite MV-chains. Since, by assumption, T_* is not a subalgebra of $T_\odot$, the ordinal sum must contain at least two summands.

If T_* is an infinite subalgebra of [0,1], by Lemmata 8.2 and 8.3 it follows that T_* is isomorphic to an ordinal sum $\bigoplus_{i\in I} C_i$ where each summand C_i is isomorphic to a subalgebra of $T_\odot$, $T_{\min}$, or $T_\times$. If the index set I has at least two elements, we are done; otherwise, by the hypotheses T_* is isomorphic to a subalgebra of $T_{\min}$ or of $T_\times$, and it has more than two elements. Now, by direct inspection, $T_{\min}$ is isomorphic to $\bigoplus_{r\in[0,1)}\{0, 1\}$, while $T_\times$ is isomorphic to $\{0, 1\}\oplus\mathscr{C}$, where $\mathscr{C} = (\,(0, 1], \times, \rightarrow_\times, 1\,)$ is known as the *standard cancellative hoop*. Any subalgebra of $T_{\min}$ with more than two elements is then a non-trivial ordinal sum of copies of $\{0, 1\}$, while any subalgebra of $T_\times$ distinct from $\{0, 1\}$ is of the form $\{0, 1\} \oplus \mathscr{C}'$, for $\mathscr{C}'$ a subhoop of $\mathscr{C}$. In both cases, T_* splits into a non-trivial ordinal sum of at least two summands. □

Lemma 8.6 *Suppose the algebra of truth values T_* splits into a non-trivial ordinal sum of at least two summands, say $\bigoplus_{i\in I} C_i$, where each C_i is a totally ordered basic hoop, and $|I| \geqslant 2$. Then I has a least element, say i_0. Further, let $S \subseteq T_*$ be the support of a summand distinct from C_{i_0}. For any two valuations $\mu, \nu\colon$ FORM$\rightarrow T_*$ such that μ(VAR), ν(VAR)$\subseteq S$, and for any α ∈FORM, we have:*

$$\mu(\alpha) = 0 \quad \Longleftrightarrow \quad \nu(\alpha) = 0.$$

Proof Since T_* is bounded below, the existence of i_0 follows from inspection of the definition of ordinal sum.

We first prove the following *claim* by induction on the structure of formulæ: For any valuation $\mu\colon$ FORM$\rightarrow T_*$ such that $\mu(\text{VAR}) \subseteq S$, and for any α ∈FORM, we have $\mu(\alpha) \in S \cup \{0\}$.

If α is either $\bot$ or $\alpha \in$ VAR, the claim holds trivially. Suppose $\alpha = \beta\&\gamma$. By the induction hypothesis, $\mu(\beta), \mu(\gamma) \in S \cup \{0\}$. If both $\mu(\beta), \mu(\gamma) \in S$ then, by the definition of ordinal sum, $\mu(\beta\&\gamma) \in S$, too. If at least one among β and γ, say β, is such that $\mu(\beta) = 0$, then $\mu(\beta\&\gamma) = 0$. Hence $\mu(\beta\&\gamma) \in S \cup \{0\}$ for all μ such that $\mu(\text{VAR}) \subseteq S$. Next suppose $\alpha = \beta \rightarrow \gamma$. If $\mu(\beta) \leqslant \mu(\gamma)$, then $\mu(\beta \rightarrow \gamma) = 1 \in S$. If $\mu(\beta) > \mu(\gamma) \in S$ then, by the definition of ordinal sum, $\mu(\beta \rightarrow \gamma) \in S$, too. Finally, if $\mu(\beta) \in S$ and $\mu(\gamma) = 0$, then $\mu(\beta \rightarrow \gamma) = 0$. In all cases $\mu(\beta \rightarrow \gamma) \in S \cup \{0\}$. This settles the *claim*.

Consider now $\mu, \nu\colon$ FORM$\rightarrow T_*$ such that $\mu(\text{VAR}), \nu(\text{VAR}) \subseteq S$, and any formula $\alpha \in$ FORM. It suffices to show that $\mu(\alpha) = 0$ implies $\nu(\alpha) = 0$. By the preceding claim, we have $\mu(\alpha), \nu(\alpha) \in S \cup \{0\}$. We proceed again by induction on the structure of formulæ. The base cases $\alpha = \bot$ or $\alpha \in$ VAR hold trivially. Let $\alpha = \beta\&\gamma$. The definition of ordinal sum entails that $\mu(\beta\&\gamma) = 0$ can only occur if at least one of $\mu(\beta)$ and $\mu(\gamma)$, say $\mu(\beta)$, lies in the first summand C_{i_0}. By the preceding claim, $\mu(\beta) = 0$. By induction $\nu(\beta) = 0$, and therefore $\nu(\beta\&\gamma) = 0$. Let $\alpha = \beta \rightarrow \gamma$. Assume $\mu(\beta \rightarrow \gamma) = 0$. The definition of ordinal sum entails either $\mu(\beta) > \mu(\gamma) = 0$, or both $\mu(\beta), \mu(\gamma) \in C_{i_0}$. In the latter case, the preceding claim shows $\mu(\beta) = \mu(\gamma) = 0$, and therefore $\mu(\beta \to \gamma) = 1$, which is a contradiction. In the former case, by induction $\nu(\beta) > \nu(\gamma) = 0$. By the preceding claim, $\nu(\beta) \in S$. By the definition of ordinal sum $\nu(\beta \to \gamma) = 0$. This completes the proof. □

8.3 Logics Satisfying P1

Lemma 8.7 *For any real-valued logic* $(\mathcal{L}, T_*)$*, we have:*

$$\mathcal{L} \textit{ extends } \mathsf{G} \iff T_* \textit{ is a subalgebra of } T_{\min}.$$

Moreover, we have:

$$\mathcal{L} \textit{ extends } \mathsf{G} \textit{ properly } (\text{ i.e. } \mathcal{L} \neq \mathsf{G}) \iff T_* \textit{ is a finite subalgebra of } T_{\min}.$$

Proof $\mathcal{L}$ extends G iff $\vdash_{\mathcal{L}} X_1 \leftrightarrow X_1 \& X_1$ iff, by Lemma 8.1, $\mu(X_1) = \mu(X_1) * \mu(X_1)$ for any valuation $\mu\colon$ VAR$\to T_*$ iff $a = a * a$ for any $a \in T_*$ iff T_* is a subalgebra of $T_{\min}$. (The latter equivalence follows from Remark 8.2.1.) Now, if $\mathcal{L}$ extends G properly, then, by Remark 8.2.1, and the fact that each infinite subalgebra of $T_{\min}$ induces G (Dummett 1959, Theorem 4), the underlying set of T_* cannot be an infinite subset of [0,1], hence T_* is a finite subalgebra of $T_{\min}$. The other direction follows from Hájek (1998, Corollary 4.2.15), stating that any two finite subalgebras of $T_{\min}$ of the same cardinality are isomorphic, and from the axiomatisation of the subvariety of Gödel algebras generated by the n-element chain, essentially given in Gödel (1932). □

Lemma 8.8 *Any real-valued logic that satisfies* **P**1 *is an extension of* G.

Proof We prove the contrapositive: a real-valued logic $\mathcal{L}$ that does not extend G fails **P**1. Indeed, by the hypothesis we have $\nvdash_{\mathcal{L}} X_1 \leftrightarrow X_1 \& X_1$. On the other hand, for any algebra of truth values T_* inducing $\mathcal{L}$, and for any valuation $\mu\colon$ FORM$\to T_*$, we have

$$\mu(X_1) = 1 \;\Rightarrow\; \mu(X_1 \& X_1) = 1, \tag{8.4}$$

$$\mu(X_1 \& X_1) = 1 \;\Rightarrow\; \mu(X_1) = 1. \tag{8.5}$$

Indeed, (8.4) holds by the very definition of t-norm, which includes the condition $1 * 1 = 1$; and (8.5) holds by the fact that t-norms are non-increasing in both arguments, whence $\mu(X_1 \& X_1) \leqslant \mu(X_1)$. Now (8.4)–(8.5) show that $\mathcal{L}$ fails **P**1 for $\alpha = X_1$ and $\beta = X_1 \& X_1$. □

For the proof of the next lemma we recall the notion of semantic consequence with respect to an algebra of truth values T_*. Given a set $\Gamma \subseteq$ FORM and $\alpha \in$ FORM, we say that α is a *semantic consequence* of Γ with respect to T_*, in symbols $\Gamma \vDash_{T_*} \alpha$

if, for any valuation $\mu\colon$ VAR$\to T_*$, the fact that $\mu(\gamma) = 1$ for each $\gamma \in \Gamma$ implies $\mu(\alpha) = 1$.

Lemma 8.9 *Any real-valued logic $\mathcal{L}$ that is an extension of* G *satisfies* **P**1.

Proof Let T_* be an algebra of truth values inducing $\mathcal{L}$. By Lemma 8.7 we know that T_* is a subalgebra of $T_{\min}$. Let $\alpha, \beta \in$FORM be such that $\mu(\alpha) = 1$ iff $\mu(\beta) = 1$, for each valuation $\mu\colon$ FORM$\to T_*$. By the definition of semantic consequence, we have $\alpha \vDash_{T_*} \beta$ and $\beta \vDash_{T_*} \alpha$. Recall that G is strongly complete with respect to $T_{\min}$ (Hájek 1998, Theorem 4.2.17.(2)). By Lemma 8.7, each real-valued extension $\mathcal{L}$ of G distinct from G is induced by a finite subalgebra of $T_{\min}$, and it is moreover strongly complete with respect to any such (essentially unique) subalgebra (Cintula et al. 2009, Proposition 4.18 and Corollary 4.19). In all cases we therefore infer $\alpha \vdash_{\mathcal{L}} \beta$ and $\beta \vdash_{\mathcal{L}} \alpha$. The logic G has the Deduction Theorem by Hájek (1998, Theorem 4.2.10.(1)), and the same proof shows that each extension of G also has the Deduction Theorem. We thereby obtain $\vdash_{\mathcal{L}} \beta \to \alpha$ and $\vdash_{\mathcal{L}} \alpha \to \beta$. Hence, by Lemma 8.1, we conclude $\vdash_{\mathcal{L}} \alpha \leftrightarrow \beta$, as was to be shown. □

Proof of Theorem 8.1 Combine Lemmata 8.8 and 8.9. □

Remark 8.3 Theorem 8.1 holds even if we relax the notion of real-valued logic considerably. Recall that *MTL* (*monoidal t-norm-based logic*) is the logic of all left-continuous t-norms and their residua (Esteva and Godo 2001); write FORM$'$ for the set of well-formed formulæ of MTL. (In contrast to BL, here it is necessary to regard the lattice-theoretic conjunction $\wedge$ as primitive.) The algebraic semantics corresponding to MTL is provided by *MTL-algebras*. By a *standard MTL-algebra* we mean an MTL-algebra induced by a left-continuous t-norm on [0,1] and its residuum. Now replace the definition of real-valued logic by the following. The pair $(\mathcal{L}, T_*)$ is a *real-valued logic* if $\mathcal{L}$ is an extension of MTL that is complete with respect to valuations $\mu\colon$ FORM$' \to T_*$ into an arbitrary MTL-subalgebra T_* of some standard MTL-algebra. It is well known that Remark 8.2.1 holds even if we consider all left-continuous t-norms instead of the continuous ones only. And it is possible to show that Lemmata 8.7, 8.8, and 8.9 continue to hold. Hence Theorem 8.1 holds for real-valued logics in the present sense. □

8.4 Logics Satisfying P2

Lemma 8.10 *For any real-valued logic $(\mathcal{L}, T_*)$, we have:*

$$\mathcal{L} \textit{ extends } \mathsf{L} \iff T_* \textit{ is isomorphic to a subalgebra of } T_{\odot}.$$

Moreover, we have:

$$\mathcal{L} \text{ extends } \text{Ł} \text{ properly } (i.e.\ \mathcal{L} \neq \text{Ł}) \iff$$

$$T_* \text{ is isomorphic to a finite subalgebra of } T_{\odot}.$$

Proof $\mathcal{L}$ extends Ł iff $\vdash_{\mathcal{L}} \neg\neg X_1 \leftrightarrow X_1$ iff (by Lemma 8.1) $\mu(X_1) = \neg\neg\mu(X_1)$ for any valuation $\mu\colon$ VAR$\to T_*$ iff $a = \neg\neg a$ for any $a \in T_*$ iff (by Remark 8.2.2) T_* is an MV-algebra with some underlying set $U \subseteq [0,1]$. Now, if U is finite, say of cardinality n, then T_* is isomorphic to the MV-chain $T_{n-1} = \{\frac{0}{n-1}, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, \frac{n-1}{n-1}\}$, by Cignoli et al. (2000, Proposition 3.6.5), and direct inspection shows that T_{n-1} is a subalgebra of $T_{\odot}$. Assume then that U is infinite. Observe that T_* cannot be a non-trivial ordinal sum of at least two summands: consider such a sum $B \oplus C$, and take $1 \neq c \in C$. Then $\neg\neg c = 1 \neq c$, and hence $B \oplus C$ is not an MV-algebra. By Lemma 8.5 and by Remark 8.2.2, T_* is isomorphic to a subalgebra of $T_{\odot}$. Clearly, if T_* is isomorphic to a subalgebra of $T_{\odot}$ then $\mathcal{L}$ extends Ł. This proves the first statement. Each finite MV-chain generates a proper subvariety of the variety of MV-algebras (see Cignoli et al. 2000, Theorem 8.5.1 for axiomatisations). Thus, if T_* is isomorphic to a finite subalgebra of $T_{\odot}$ then $\mathcal{L}$ extends Ł properly. On the other hand, by Cignoli et al. (2000, Theorem 8.1.1), every infinite subalgebra of $T_{\odot}$ generates the whole variety of MV-algebras. This fact, together with the first assertion of the lemma, suffices to complete the proof. □

Lemma 8.11 *Any real-valued logic $\mathcal{L}$ that satisfies* **P**2 *is an extension of* Ł.

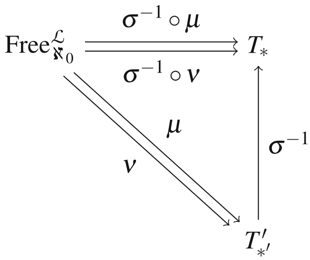
Proof By contraposition, suppose $\mathcal{L}$ is not an extension of Ł. If T_* is an algebra of truth values that induces $\mathcal{L}$, then T_* is not a subalgebra of $T_{\odot}$: for, given that $T_{\odot}$ does induce Ł (cf. Remark 8.2), any such subalgebra clearly induces an extension of Ł. Hence, by Lemma 8.5, T_* splits into a non-trivial ordinal sum of at least two summands. With the notation therein, there exists a summand S of T_* distinct from the first one that is non-trivial, and thus contains two distinct elements $v \neq w$. Let μ_v be the unique valuation that sends each variable to v, and let ν_w be the unique valuation that sends each variable to w. Evidently, we have $\mu_v \neq \nu_w$, so that μ_v and ν_w fail **P**2 by Lemma 8.6. □

Remark 8.4 Let $(\mathcal{L}, T_*)$ be a real-valued logic. In the next lemma we say, somewhat informally, that "$\mathcal{L}$ satisfies **P**2 with respect to T_*", to mean that for any two valuations $\mu \neq \nu\colon$ FORM$\to T_*$ there is $\alpha \in$FORM with $\mu(\alpha) > 0$ and $\nu(\alpha) = 0$. □

Lemma 8.12 *Let $(\mathcal{L}, T_*)$ be a real-valued logic, and let $\sigma : T_* \to T'_{*'}$ be an isomorphism, where $T'_{*'}$ is an algebra of truth values. The logic induced by $T'_{*'}$ is again $\mathcal{L}$, by Remark 8.2. Then $\mathcal{L}$ satisfies* **P**2 *with respect to T_* if, and only if, $\mathcal{L}$ satisfies* **P**2 *with respect to $T'_{*'}$.*

Proof Since $\sigma^{-1}\colon T'_{*'} \to T_*$ is an isomorphism, too, it suffices to show that $\mathcal{L}$ satisfies **P**2 with respect to $T'_{*'}$ if $\mathcal{L}$ satisfies **P**2 with respect to T_*. Proof by contraposition.

Let $\mu \neq \nu\colon$ FORM$\rightarrow T'_{*'}$ be valuations that fail **P**2. Thus, for all formulæ $\alpha \in$ FORM, we have $\mu(\alpha) = 0$ if, and only if, $\nu(\alpha) = 0$. Write $\text{Free}^{\mathcal{L}}_{\aleph_0}$ for the Lindenbam algebra of the logic $\mathcal{L}$. As usual, we may identify formulæ, modulo the logical-equivalence relation induced by $\vdash_{\mathcal{L}}$, with elements of $\text{Free}^{\mathcal{L}}_{\aleph_0}$; and valuations with homomorphisms from $\text{Free}^{\mathcal{L}}_{\aleph_0}$ to T_* (or to $T'_{*'}$, as the case may be). Then the compositions $\sigma^{-1} \circ \mu$ and $\sigma^{-1} \circ \nu$ are valuations into T_*, see the commutative diagram below.



It is not the case that $\sigma^{-1} \circ \mu = \sigma^{-1} \circ \nu$: for else $\mu = \nu$ would follow by pre-composing with σ. Now for any $\alpha \in$ FORM we have:

$$\mu(\alpha) = 0 \quad \text{iff} \quad \nu(\alpha) = 0 \qquad \text{(by assumption)},$$
$$\sigma^{-1}(0) = 0 \qquad \text{(homomorphisms preserve 0)},$$
$$\sigma^{-1}(\mu(\alpha)) = 0 \quad \text{iff} \quad \sigma^{-1}(\nu(\alpha)) = 0 \qquad \text{(by composition)}.$$

Hence $\mathcal{L}$ fails **P**2 with respect to T_*, as was to be shown. □

Lemma 8.13 *Łukasiewicz logic* Ł *satisfies* **P**2.

Remark 8.5 A proof of Lemma 8.13 can be obtained as a consequence of McNaughton's Theorem (Cignoli et al. 2000, 9.1); in fact, the proof can be reduced to the one-variable case (Cignoli et al. 2000, 3.2). Here we give a proof that uses a weaker (and simpler) result from Aguzzoli (1998), thus showing that the full strength of McNaughton's Theorem is not needed to fulfill **P**2. □

Proof In light of Remark 8.2.2 and Lemma 8.12, it suffices to show that Ł satisfies **P**2 with respect to the Łukasiewicz t-norm $\odot$ on [0,1]. For terms s and t over the binary monoidal operation $\odot$ and the unary operation $\neg$, set $s \oplus t := \neg(\neg s \odot \neg t)$. Let us write nt as a shorthand for $t \oplus \cdots \oplus t$ ($n - 1$ occurrences of $\oplus$), and t^n as a shorthand for $t \odot \cdots \odot t$ ($n - 1$ occurrences of $\odot$). We inductively define the set of *basic literals* (in the variables X_i, $i = 1, 2, \ldots$) as follows.

- X_i is a basic literal;
- each term s either of the form $s = nt$ or of the form $s = t^n$, for some integer $n > 0$, is a basic literal, provided that t is a basic literal;
- nothing else is a basic literal.

Given integers $n_1 \geqslant 1$, and $n_2, \ldots, n_u > 1$, we write $(n_1, n_2, \ldots, n_u)X_j$ to denote the basic literal

$$(\cdots((n_i\cdots((n_1X_j)^{n_2}\cdots))^{n_{i+1}})\cdots).$$

In this proof, a *term function* is any function $\lambda_\tau\colon [0,1]^n \to [0,1]$ induced by interpreting over the standard MV-algebra $T_\odot = ([0,1], \odot, \neg, 0)$ a term τ whose variables are contained in $\{X_1, \ldots, X_n\}$. □

Claim 1 *For any integer $n \geqslant 1$, and for any two points $p \neq q \in [0,1]^n$, there is a term τ whose term function $\lambda_\tau\colon [0,1]^n \to [0,1]$ takes value 0 at q, and value > 0 at p.*

Proof Since $p \neq q$ there exists an integer $i \geqslant 1$ such that $p(i) \neq q(i)$, that is, p and q differ at one of their coordinates. If $q(i) < p(i)$ then there are integers $h, k > 0$ such that $q(i) < \frac{h}{k} < p(i)$, with h and k coprime. By Aguzzoli (1998, Corollary 2.8) there is a basic literal $L = (a_1, \ldots, a_u)X_i$ such that $\lambda_L^{-1}(0)$ is the set $[0, \frac{h}{k}]\times[0,1]^{n-1}$, and λ_L is monotone increasing in the variable X_i. Hence $\lambda_L(p) > 0$ and $\lambda_L(q) = 0$. If $p(i) \leqslant q(i)$ for all integers $i \geqslant 1$, then one can choose j such that $p(j) < q(j)$. As before there are integers $h, k > 0$ such that $p(j) < \frac{h}{k} < q(j)$, with h and k coprime, and there is a basic literal $R = (b_1, \ldots, b_w)X_j$ such that $\lambda_R^{-1}(1)$ is the set $[\frac{h}{k}, 1]\times [0,1]^{n-1}$, and λ_R is monotone increasing in the variable X_j. Hence $\lambda_{\neg R}(p) > 0$ and $\lambda_{\neg R}(q) = 0$.

The proof is now completed by a routine translation of Claim 8.4 from terms to formulæ of Ł. □

Remark 8.6 In connection with Claim 8.4, let us observe that term functions in Łukasiewicz logic (even over an arbitrarily large set I of propositional variables) enjoy an even stronger separation property. Recall (see e.g. Engelking 1977, 1.5) that a space is *completely regular* if it is T_1, and points can be separated from closed sets by continuous [0,1]-valued functions. Now, *in each product space* $[0,1]^I$, *points can be separated from closed sets by term functions*. Thus the space of standard models $[0,1]^I$ may be described as *definably completely regular*. The proof is essentially the same as the one above, *mutatis mutandis*; cf. Marra and Spada (2012, Lemma 3.5). The (definable) complete regularity of the space of standard models of Łukasiewicz logic is also proved in Mundici (1986, Lemma 8.1). Also see Remark 8.2 (*ibid.*) □

Proof of Theorem 8.2 In light of Lemmata 8.11 and 8.13, it remains to show that each real-valued extension of Ł that is not Ł itself satisfies **P**2. By Lemmata 8.10 and 8.12, we may safely assume that $\mathcal{L}$ is induced by a finite subalgebra T_* of $T_\odot$. By Cignoli et al. (2000, Proposition 3.6.5), each such subalgebra is isomorphic to $T_m = \left\{\frac{0}{m}, \frac{1}{m}, \ldots, \frac{m-1}{m}, \frac{m}{m}\right\}$, for a uniquely determined integer $m \geqslant 1$. Notice now that if $p \neq q$ are in T_m^n then the term function λ'_τ obtained by restricting to T_m^n the function $\lambda_\tau\colon [0,1]^n \to [0,1]$ provided by Claim 8.4 is such that $\lambda'_\tau(q) = 0$ while $\lambda'_\tau(p) > 0$. Hence $\mathcal{L}$ satisfies **P**2, and the proof is complete. □

To conclude this section, let us discuss two alternative formulations of **P**2. We consider the following conditions, for every algebra T_* of truth values inducing $\mathcal{L}$.

P2$'$. For each pair of valuations $\mu, \nu\colon$ FORM$\to T_*$, if $\mu \neq \nu$ then there is a formula $\alpha \in$ FORM such that $\mu(\alpha) < 1$ while $\nu(\alpha) = 1$. □

P2$''$. For each pair of valuations $\mu, \nu\colon$ FORM$\to T_*$, if $\mu \neq \nu$ then there is a formula $\alpha \in$ FORM such that $\mu(\alpha) = 0$ while $\nu(\alpha) = 1$. □

Corollary 8.1 *A real-valued logic satisfies* **P**2 *if, and only if, it satisfies* **P**2$'$ *if, and only if, it satisfies* **P**2$''$.

Proof Let T_* be an algebra of truth-values inducing the real-valued logic $\mathcal{L}$. It suffices to prove that if $\mathcal{L}$ is an extension of Ł then it satisfies **P**2$'$ and **P**2$''$, and otherwise it fails both.

Assume first that $\mathcal{L}$ is an extension of Ł. Given valuations $\mu \neq \nu$ with values in T_*, by Theorem 8.2 there is a formula α be such that $\mu(\alpha) > 0$ and $\nu(\alpha) = 0$. Then $\mu(\neg\alpha) < 1$ and $\nu(\neg\alpha) = 1$. Hence **P**2$'$ holds. We now show that **P**2$'$ implies **P**2$''$. In light of Remark 8.2.2 and Lemma 8.12, we may safely assume that T_* is a subalgebra of $T_\odot$. Then, if $\mu(\alpha) < 1$ and $\nu(\alpha) = 1$, it is clear by the definition of $\odot$ that there exists an integer $k \geqslant 1$ such that $\mu(\alpha^k) = 0$ and $\nu(\alpha^k) = 1$, where $\alpha^1 = \alpha$ and $\alpha^n = \alpha \odot \alpha^{n-1}$.

Assume now T_* does not induce an extension of Ł. By Theorem 8.2, there are distinct valuations μ and ν such that $\nu(\alpha) = 0$ implies $\mu(\alpha) = 0$ for any formula α. This suffices to show that **P**2$''$ fails. For what concerns **P**2$'$, recall that, by Lemma 8.10 and Lemma 8.5, T_* splits into a non-trivial ordinal sum of at least two summands. Let μ be the valuation assigning 1 to every variable. Then it is easy to check that $\mu(\alpha) \in \{0, 1\}$ for each formula α. Let ν be a valuation such that ν(VAR) is contained in a summand of T_* distinct from the first one. Then, by Lemma 8.6, for each formula α we have $\nu(\alpha) = 0$ iff $\mu(\alpha) = 0$, and hence $\nu(\alpha) = 1$ implies $\mu(\alpha) = 1$, that is, **P**2$'$ fails. □

8.5 Product Logic

Lemma 8.14 *The only many-valued logic that extends* P *properly is classical logic.*

Proof This is essentially Cignoli and Torrens (2000, Corollary 2.10). □

Lemma 8.15 *Product logic* P *fails both* **P**1 *and* **P**2.

Proof (**P**1) Choose the standard product algebra $T_\times$ to induce P. It follows directly from the definition of t-norm that $\mu(X_1) = 1$ if, and only if, $\mu(X_1 \& X_1) = 1$, for any valuation $\mu\colon$ FORM$\to T_\times$. To see that **P**1 fails, it thus suffices to observe that $\nvdash_{\mathsf{P}} X_1 \leftrightarrow X_1 \& X_1$: for else, by soundness and Lemma 8.1, we would have $\mu(X_1 \& X_1) = \mu(X_1)\mu(X_1) = \mu(X_1)$ whatever μ is; this is a contradiction.

(**P**2) By Remark 8.2.3 and Lemma 8.12, it suffices to argue about the product t-norm $T_\times$. By direct inspection, we have the decomposition $T_\times = \{0, 1\} \oplus \mathscr{C}$, where $\mathscr{C}$ is the standard cancellative hoop. The hypotheses of Lemma 8.6 are therefore satisfied, and hence **P**2 fails for any two valuations $\mu \neq \nu\colon \mathrm{FORM} \to T_\times$ such that $\mu(\mathrm{VAR}), \nu(\mathrm{VAR}) \subseteq \mathscr{C}$. □

Lemma 8.16 *Let $\mathcal{L}$ be a closed real-valued logic all of whose non-classical, real-valued extensions fail* **P**1 *and* **P**2*. Then $\mathcal{L} = \mathsf{P}$.*

Proof We know that $\mathcal{L}$ is not an extension of G or L, by Theorems 8.1 and 8.2. Let T_* be any algebra of truth values inducing $\mathcal{L}$. We will show that T_* cannnot be finite, to begin with.

If T_* is finite, by Lemma 8.4 we know that T_* splits into an ordinal sum of finitely many finite MV-chains. If there is just one summand, then $\mathcal{L}$ is an extension of L, and this is a contradiction. If there is more than one summand then, by the definition of ordinal sum, and using the fact that each summand is bounded below by 0, there is an idempotent element $0, 1 \neq e \in T_*$. The subset $G_3 := \{0, e, 1\} \subseteq T_*$ is closed under the BL-algebraic operations, as is checked easily, and all of its elements are idempotent. Hence G_3 is isomorphic to the three-element Gödel algebra. Now consider the collection $\mathcal{E}$ of formulæ that evaluate to 1 under each valuation into G_3. Obviously $\mathcal{E} \supseteq \mathcal{L}$, and $\mathcal{E}$ is closed under substitutions by its very definition. Hence $\mathcal{E}$ is a real-valued extension of $\mathcal{L}$ which by construction is three-valued Gödel logic. Theorem 8.1 implies that $\mathcal{E}$ satisfies **P**1, and we have reached a contradiction.

We may therefore suppose that T_* has an infinite closed subset of [0,1] as its support. By definition, T_* extends to a BL-algebra $([0,1], *', \to_{*'}, 0)$. By Lemmata 8.2 and 8.3, T_* decomposes into an ordinal sum $\bigoplus_{i \in I} C_i$, where each summand C_i is isomorphic to a subalgebra of one amongst $T_\odot$, $T_{\min}$, and $T_\times$. If the index set I has more than one element, then using again the fact that each summand C_i is bounded below by 0, we have an idempotent element $0, 1 \neq e \in T_*$, and hence $\{0, e, 1\}$ is a three-element Gödel subalgebra of T_*. We then reason as above to conclude that $\mathcal{L}$ has three-valued Gödel logic as an extension, reaching a contradiction. Hence I is a singleton, that is, T_* is isomorphic to a subalgebra of $T_\odot$, $T_{\min}$, and $T_\times$. Using Remark 8.2, and Theorems 8.1 and 8.2, T_* cannot be isomorphic to a subalgebra of $T_\odot$—because it fails **P**2—nor can it be isomorphic to a subalgebra of $T_{\min}$—because it fails **P**1. Then T_* is isomorphic to an infinite subalgebra of $T_\times$, and hence $\mathcal{L} = \mathsf{P}$, by Cignoli and Torrens (2000, Corollary 2.9). □

Proof of Theorem 8.3 Lemmata 8.14, 8.15 and 8.16. □

Remark 8.7 Theorem 8.3 fails if we drop the assumption that $\mathcal{L}$ be closed. Indeed, consider the logic $\mathcal{L}$ induced by $\{0, 1\} \oplus \mathscr{C} \oplus \mathscr{C}$, where $\mathscr{C}$ is the standard cancellative hoop (see the proof of Lemma 8.5). Then it can be verified that $\mathcal{L}$ is not closed, that $\mathcal{L}$ is not P, and that all of its non-classical, real-valued extensions fail **P**1 and **P**2. □

8.6 Epilogue

Let us return to Hájek's Programme, as embodied in Hájek (1998). According to Hájek, a real-valued logic may be considered as a "*logic of imprecise (vague) propositions*" Hájek (1998, p. vii), wherein "*truth* [...] *is a matter of degree*" (Hájek 1998,

p. 2). Classical logic may be viewed as a limiting case, where only two degrees of truth, 0 and 1, exist. But as soon as a logic is genuinely real-valued, it must renounce at least one of the familiar features **P**1 and **P**2 of the classical world. We record this fact as a formal statement, by way of conclusion.

Corollary 8.2 *A real-valued logic $\mathcal{L}$ satisfies* **P**1 *and* **P**2 *if, and only if, $\mathcal{L}$ is classical logic if, and only if, $T_* = \{0, 1\}$ is the unique algebra of truth values that induces $\mathcal{L}$.*

Proof That $\mathcal{L}$ is classical logic just in case $\mathcal{L}$ satisfies **P**1 and **P**2 follows from Theorems 8.1–8.2 upon observing that the only common extension of G and L is classical logic, by Hájek (1998, Theorem 4.3.9.(1)). It thus remains to show that $\mathcal{L}$ is classical logic if, and only if, $T_* = \{0, 1\}$ as soon as T_* induces $\mathcal{L}$. By the very definition of t-norm, $T_* = \{0, 1\}$ induces classical logic. On the other hand, if there exists $a \in T_* \setminus \{0, 1\}$ then $\max\{a, a \to_* 0\} < 1$. Indeed, $a \to_* 0 = 1$ would entail $a * 1 = 0$ for $a > 0$, which is impossible. Any valuation $\mu\colon$ FORM$\to T_*$ that sends X_1 to a is therefore such that $\mu(X_1 \vee \neg X_1) < 1$, and the logic induced by T_* cannot be classical. □

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Part IV
Algebra for Many-Valued Logic

Although Hájek is not an algebraist, he offered an important contribution to the theory of residuated lattices, for instance, inventing BL-algebras, which constitute a very interesting variety of residuated lattices, and which were subsequently generalized to GBL-algebras by Nikolaos Galatos and Constantine Tsinakis. Algebras for many-valued logic are related to ℓ-groups, (consider, for instance, Mundici's functor Γ from the category of unital abelian ℓ-groups into the category of MV-algebras), and hence, they constitute a bridge between algebraic logic and a topics, ℓ-groups, belonging to the classical algebra of ordered structures. On the other hand, algebras for Fuzzy Logic can also be presented as residuated lattices (hence, algebras for substructural logics) satisfying the prelinearity property. Therefore, Hájek's research also contributed to clarify the role of prelinearity in residuated lattices.

The two chapters in this part constitute interesting contributions to the above-mentioned aspects of algebra for many-valued logic.

In more detail, the chapter *How do ℓ-groups and p-groups appear in algebraic and quantum structures*? by Anatolij Dvurečenskij, is a summary of the applications of ℓ-groups to many-valued logic and quantum logic. The chapter is interesting because it shows a really impressive number of applications of ℓ-groups to algebraic logic, including algebras for many-valued logic, algebras for quantum logic, and states on MV-algebras or on generalizations of them.

Finally, in the chapter *Semi-linear Varieties of Lattice-Ordered Algebras* by Antonio Ledda, Francesco Paoli, and Constantine Tsinakis, the authors investigate varieties of pointed lattice-ordered algebras satisfying a weak form of distributivity and having a very weak implication. Also, these varieties are very general, and include the reducts of distributive or integral residuated lattices, Boolean algebras with modal operators, and varieties arising from quantum logic. Then the authors focus on prelinearity, a property which is very common in algebra for many-valued logic, and which was investigated in-depth by Hájek. For each such variety

$\mathscr{W}$, the authors investigate its greatest semilinear subvariety $\mathscr{V}$, that is, the variety generated by all totally ordered algebras in $\mathscr{W}$. Moreover, they provide an axiomatization for such subvariety and prove that if $\mathscr{W}$ is finitely based, then so is $\mathscr{V}$. The chapter investigates the prelinearity axiom in a very general context, and includes several interesting results.

Chapter 9
How Do ℓ-Groups and Po-Groups Appear in Algebraic and Quantum Structures?

Anatolij Dvurečenskij

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9.1 Introduction

Lattice-ordered groups (ℓ-groups), as well as partially ordered groups (po-groups), are intimately connected with many algebraic and quantum structures. Such algebraic structures as MV-algebras or BL-algebras are closely tied, for example, with many-valued reasoning and fuzzy logics. Quantum structures model measurements in quantum mechanics and their corresponding algebraic structures are, for example, orthomodular lattices, posets and effect algebras. The aim of this article is to give a survey on the role of ℓ-groups and po-groups in two seemingly unrelated areas: algebras connected with fuzzy logics and quantum structures.

9.1.1 ℓ-Groups and Algebraic Structures

The first important connection between unital Abelian ℓ-groups and MV-algebras, introduced by Chang (1958) for modeling infinitely-valued Łukasziewicz's logic, was established by Mundici (1986). He showed that every MV-algebra is an interval in some unital Abelian ℓ-group with strong unit, and moreover, there is a categorical

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equivalence between the variety of MV-algebras and the category of unital Abelian ℓ-groups. In the last decade, two equivalent non-commutative generalizations of MV-algebras called pseudo MV-algebras (Georgescu and Iorgulescu (2001)), and generalized MV-algebras (see Rachůnek (2002)), have appeared.

A non-commutative generalization of reasoning can be found, for example, in psychological processes, Dvurečenskij (2002): In clinical medicine, an experiment related with transplantation of human organs was performed in which the following two questions were posed to two groups of people : (1) Do you agree to dedicate your organs for medical transplantation after your death? (2) Do you agree to accept organs of a donor if you needed one? When the order of the two questions was changed for the second group, the number of positive answers to the first question was higher than that of the first group.

Today there exists a programming language (see Baudot (2000)) based on a non-commutative logic.

It was also shown by Dvurečenskij (2002), that there is a categorical equivalence between the variety of pseudo MV-algebras and the category of unital ℓ-groups which are not necessarily Abelian.

For BL-algebras which constitute an algebraic semantic of fuzzy logic (see Hájek (1998)), Aglianò and Montagna (2003) showed that every linearly ordered BL-algebra is an ordinal sum of a system consisting of negative cones of Abelian ℓ-groups and negative intervals in Abelian ℓ-groups with strong unit. This result was extended to linearly ordered pseudo BL-algebras in Dvurečenskij (2007). We note that pseudo BL-algebras are a non-commutative extension of BL-algebras introduced in Di Nola et al. (2002a, b). Such algebras have two negations, and it was an open problem whether these two negations commute (cf. Di Nola et al. (2002b, Problem 3.21)). In Dvurečenskij et al. (2010) it was solved in the negative showing that an algebra from Jipsen and Montagna (2006), which was using the ℓ-group of integers $\mathbb{Z}$, and which we now call a kite, provides such a counterexample. This idea was generalized in Dvurečenskij and Kowalski (2014), where a construction of general kites using an arbitrary ℓ-group was studied and the basic properties of such pseudo BL-algebras were established.

9.1.2 Po-Groups and Quantum Structures

When we perform a measurement, we are using the classical probability theory which was axiomatized by Kolmogorov (1933). However when in the beginning of the 20th century physicists started to investigate properties of atoms, it was recognized that measurements in this new physics, which we now call quantum physics, do not satisfy the axioms of Kolmogorov's probability theory. As is well-known, Heisenberg's uncertainty principle (see e.g. (Dvurečenskij and Pulmannová 2000; Varadarajan 1968)) asserts that the position x and the momentum p of an elementary particle cannot be measured simultaneously with arbitrarily prescribed accuracy. If $\Delta_m p$ and $\Delta_m x$ denote the inaccuracies of the measurement of the momentum p and position

x in a state m, then

$$(\Delta_m p)^2 \cdot (\Delta_m x)^2 \geq \frac{1}{4}\hbar^2,$$

where $\hbar = h/2\pi$ and h is Planck's constant. Birkhoff and von Neumann (1936) showed that quantum mechanical events satisfy more general axioms than those of Boolean algebras. These systems of events are called quantum logics or quantum structures. Today we have a whole hierarchy of quantum structures like Boolean algebras, orthomodular lattices and posets, orthoalgebras, which model so-called sharp measurements, or equivalently yes-no experiments, and D-posets and effect algebras which can be used for unsharp measurements because they correspond to many-valued reasoning. For a comprehensive source of information on quantum structures, we recommend the book Dvurečenskij and Pulmannová (2000).

A prototypical example is the system $\mathscr{E}(H)$ of Hermitian operators of a Hilbert space H which are between the zero and the identity operator. They are used for modeling the so called POV-measures. Such measures have been used in mathematics and mathematical physics for many years. For example, a famous result on POV-measures is Naimark's Dilation Theorem, which was proved in 1940 by Naimark (1943).

It is not necessary to go into quantum mechanics for motivation, but there are also less realistic experiments outside quantum mechanics. Such an example is the firefly in a box which is presented in more detail in Sect. 9.4.1 below. This example is due to Foulis and Randall, and a quantum mechanical realization of this example has been done in (Foulis and Randall, 1972, Exam III). It is also mentioned in the books (Cohen 1989; Dvurečenskij and Pulmannová 2000). Another example is the firefly in a three-chamber box, which presented in Wright (1990). This example uses the so called generalized urn models. For more details see Sect. 9.4.1.

Nowadays quantum structures do not describe only events in quantum mechanics but also ones in different areas, for example in computer science, psychiatry, neuroscience (quantum brain—Stern (1994), quantum psychology—Stříženec (2011)), quantum computing, etc.

Quantum structures are algebraic structures where the basic operations are often partial. In the Nineties there appeared two equivalent quantum structures: D-posets by Kôpka and Chovanec (1994), with difference, $-$, of two comparable events as a basic operation, and effect algebras by Foulis and Bennett (1994), with addition, $+$, of two mutually excluding events as a basic operation. Hence, $a + b$ means the disjunction of two mutually excluding events a and b.

The basis quantum structural notion is that of a state, an analogue of a probability measure. Roughly speaking, a state on an effect algebra E (orthomodular poset, orthoalgebra) is a mapping $s : E \to [0, 1]$ which preserves existing sums and $s(1) = 1$. States for MV-algebras were introduced by Mundici in Mundici (1995) as averaging the truth value in Łukasiewicz logic. We have quantum structures which are stateless, which is interesting from the mathematical point of view, but not interesting for quantum measurement. We have situations, for example in BL-algebras, where it is not immediately clear how to define a state for the structure.

States were introduced about 40 years after the appearance of MV-algebras; they do not properly pertain to universal algebra. In the last decade, states have been intensively studied also in the framework of algebraic structures.

The main problem of quantum structures is the fact that not every two events a and b of an effect algebra E need not always be compatible. Two events a and b are said to be *compatible* if there are three events, a_1, b_1, c, such that $a = a_1 + c$, $b = b_1 + c$ and $a_1 + b_1 + c$ is defined. If our effect algebra E is a lattice, then every maximal set of mutually compatible events, called a *block*, is an MV-algebra, and E can be covered by the set of all its blocks, see Riečanová (2000). In such a natural way MV-algebras appeared in the realm of quantum structures.

Ravindran (1996) showed that if an effect algebra satisfies the Riesz Decomposition Property (RDP), i.e. every two decompositions of the unity have a common refinement, then the effect algebra is an interval in an interpolation Abelian po-group with strong unit. When a non-commutative version of effect algebras, pseudo effect algebras, was introduced in Dvurečenskij and Vetterlein (2001a, b), the notion of RDP splits into several variants which coincide in the commutative case. It was shown in Dvurečenskij and Vetterlein (2001a, b), that if a pseudo effect algebra satisfies a certain kind of interpolation, denoted RDP_1, then it is an interval in a unital po-group, not necessarily Abelian, satisfying the analogous type of RDP. Moreover, there is a categorical equivalence between the category of pseudo effect algebras with RDP_1 and the category of unital po-groups with RDP_1.

In general, quantum mechanical measurements are also non-commutative: the result of some experiment may depend on the order of the measurements. Consider, for example, a beam of particles which are prepared in a certain state, and which are sent through a sequence of three polarizing filters F_1, F_2, F_3. It is well-known that the order of the filters makes a difference in general. For example, let the filter be polarizing in planes perpendicular to the particle beam, such that F_1 polarizes vertically, F_2 horizontally and F_3 at a 45° angle. If we place the filters in the order F_1, F_2, F_3, then no particles are detected, but in the order F_1, F_3, F_2, particles are detected; the difference is due to quantum interference.

In the literature, such phenomena are also known as sequential conjunctions or sequentially independent effects or sequential probability models (see Gudder and Nagy (2002) or Foulis (2002), respectively).

In this chapter we survey a number of algebraic and quantum structures which are related to ℓ-groups and more generally to po-groups. Section 9.2 presents MV-algebras and their non-commutative generalizations called pseudo MV-algebras and we present their relation to ℓ-groups. We show a categorical equivalence of the variety of pseudo MV-algebras with the category of unital ℓ-groups. We describe some subvarieties of the variety of pseudo MV-algebras, for example, perfect pseudo MV-algebras, and cover varieties of the variety of MV-algebras. We also introduce states on pseudo MV-algebras and study state-morphism MV-algebras. Section 9.3 gathers some results on BL-algebras and pseudo BL-algebras and their relation to ℓ-groups. We give a construction of pseudo BL-algebras, called kites, starting with an ℓ-group. Finally, Sect. 9.4 introduces quantum structures. We show some simple examples, a firefly in a box and a firefly in a three-chamber box. We present orthomodular lattices and posets, effect algebras and pseudo effect algebras. We show how important are different kinds of the Riesz Decomposition Property for their representation as an interval in the positive cone of po-groups.

9.2 MV-Algebras and Pseudo MV-Algebras

In this section, we review some old and new results on these algebras.

9.2.1 MV-Algebras

MV-algebras were introduced by Chang (1958) in 1958 as an algebraic semantics of many-valued logic. The original list of axioms was rather long. Later it was shown that the following ones suffice. For more information on MV-algebras, see (Cignoli et al. 2000; Mundici 2011).

We notice that an MV-algebra is an algebra $\mathbf{M} = (M; \oplus, ^*, 0, 1)$ of type (2, 1, 0, 0) such that, for all $a, b, c \in M$, we have

(i) $a \oplus b = b \oplus a$;
(ii) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$;
(iii) $a \oplus 0 = a$;
(iv) $a \oplus 1 = 1$;
(v) $(a^*)^* = a$;
(vi) $0^* = 1$;
(vii) $(a^* \oplus b)^* \oplus b = (a \oplus b^*)^* \oplus a$.

If we define $a \le b$ iff there is an element $c \in M$ such that $a \oplus c = b$, then $\le$ is a partial order and $a \vee b = (a^* \oplus b)^* \oplus b$. With respect to this order, M is a distributive lattice. In addition, we can define another two binary operations $a \odot b = (a^* \oplus b^*)^*$, and $a \to b = a^* \oplus b$.[1]

Now let $\mathbf{G} = (G; +, 0)$ be a group written additively. We say that a group $(G; \le)$ is a *partially ordered group* (*po-group*, for short) if $\le$ is a partial order on G such that $a \le b$ implies $c + a + d \le c + b + d$ for all $c, d \in G$. If $(G; \le)$ is a lattice with respect to $\le$, we say that $(G; \le)$ is a *lattice-ordered group* (*ℓ-group*, for short). We denote by $G^+ := \{g \in G : g \ge 0\}$ and $G^- := \{g \in G : g \le 0\}$ the *positive* and *negative cone*, respectively, of G. An element $u \in G^+$ is said to be a *strong unit* (or an *order unit*) if given an element $g \in G$, there is an integer $n \ge 0$ such that $g \le nu$, equivalently, $G = \bigcup_n [-nu, nu]$. The pair (G, u) with a fixed strong unit u is called a *unital po-group*. For more information on po-groups and ℓ-group see, for example (Darnel 1995; Fuchs 1963; Glass 1999; Goodearl 1986).

[1] Notational convention: $\odot$ binds stronger than $\oplus$.

Let (G, u) be an Abelian unital ℓ-group. We define $\Gamma(G, u) := [0, u]$ and we endow $\Gamma(G, u)$ with operations:

(i) $a \oplus b = (a + b) \wedge u$,
(ii) $a^* = u - a$,

then $\boldsymbol{\Gamma}(G, u) := (\Gamma(G, u); \oplus, ^*, 0, u)$ is a prototypical example of an MV-algebra because as shown in Mundici (1986), for every MV-algebra $\mathbf{M}$, there is a unique Abelian unital ℓ-group (up to isomorphism of unital ℓ-groups) (G, u) such that $\mathbf{M} \cong \boldsymbol{\Gamma}(G, u)$.

It is clear that the class $\mathscr{MV}$ of MV-algebras is a variety, but the class of Abelian unital ℓ-groups $\mathscr{AUG}$ is not a variety because it is not closed under infinite direct products. However, (Mundici 1986; Cignoli et al. 2000) proved the following crucial theorem:

Theorem 9.1 *There is a categorical equivalence between the variety of MV-algebras and the category of Abelian unital ℓ-groups which is given by the functor $\Gamma : \mathscr{AUG} \to \mathscr{MV}$ such that $(G, u) \mapsto \Gamma(G, u)$.*

The Mundici Theorem is a basic tool in the investigation of MV-algebras, and it was the first result which showed an intimate connection between algebras related to fuzzy logics and ℓ-groups. This result has many applications.

An *ideal* of an MV-algebra $\mathbf{M}$ is any subset $I \subseteq M$ such that (i) $a \le b \in I$ implies $a \in I$, and (ii) $a \oplus b \in I$ whenever $a, b \in I$. Let $\mathrm{Rad}(M)$ be the intersection of all maximal ideals of $\mathbf{M}$. Then $\mathrm{Rad}(M) := \{a \in M : n \odot a \le a^* \text{ for any } n \ge 1\}$, where $0 \odot a = 0$, $1 \odot a = a$, $(n + 1) \odot a = n \odot a \oplus a$ for $n \ge 1$. In a similar way, we introduce $a^0 = 1$, $a^1 = a$ and $a^{n+1} = a^n \odot a$ for $n \ge 1$.

An MV-algebra $\mathbf{M}$ is said to be *perfect* if given $a \in A$, either $a \in \mathrm{Rad}(M)$ or $a^* \in \mathrm{Rad}(M)$. For example, if G is an arbitrary Abelian ℓ-group (not necessarily unital), then the element $(1, 0)$ is a strong unit in the lexicographic product $\mathbb{Z} \overrightarrow{\times} G$, and $\mathbf{M} := \boldsymbol{\Gamma}(\mathbb{Z} \overrightarrow{\times} G, (1, 0))$ is a perfect MV-algebra. The class of perfect MV-algebras is not a variety, but is a category. Di Nola and Lettieri (1994) characterized perfect MV-algebras as follows: The variety generated by perfect MV-algebras is also generated by $\boldsymbol{\Gamma}(\mathbb{Z} \overrightarrow{\times} \mathbb{Z}, (1, 0))$ and is axiomatized relative to $\mathscr{MV}$ by the equation $2 \odot x^2 = (2 \odot x)^2$.

Theorem 9.2 *There is a categorical equivalence between the category of perfect MV-algebras and the variety $\mathscr{A}$ of Abelian ℓ-groups which is given by $G \in \mathscr{A} \mapsto \Gamma(\mathbb{Z} \overrightarrow{\times} G, (1, 0))$.*

9.2.2 Pseudo MV-Algebras

Many non-commutative generalizations of MV-algebras and BL-algebras were introduced in the last decade. Two equivalent non-commutative versions of MV-algebras appeared independently: pseudo MV-algebras in the terminology of Georgescu and Iorgulescu (2001) and generalized MV-algebras introduced in Rachůnek (2002).

According to Georgescu and Iorgulescu (2001), a *pseudo MV-algebra* is an algebra $\mathbf{M} = (M; \oplus, ^-, ^\sim, 0, 1)$ of type (2, 1, 1, 0, 0) such that the following axioms hold for all $x, y, z \in M$ with an additional binary operation $\odot$ defined via

$$y \odot x = (x^- \oplus y^-)^\sim$$

(A1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
(A2) $x \oplus 0 = 0 \oplus x = x$;
(A3) $x \oplus 1 = 1 \oplus x = 1$;
(A4) $1^\sim = 0$; $1^- = 0$;
(A5) $(x^- \oplus y^-)^\sim = (x^\sim \oplus y^\sim)^-$;
(A6) $x \oplus (x^\sim \odot y) = y \oplus (y^\sim \odot x) = (x \odot y^-) \oplus y = (y \odot x^-) \oplus x$;
(A7) $x \odot (x^- \oplus y) = (x \oplus y^\sim) \odot y$;
(A8) $(x^-)^\sim = x$.

For example, if u is a strong unit of a (not necessarily Abelian) ℓ-group G,

$$\Gamma(G, u) := [0, u]$$

and

$$\begin{aligned} x \oplus y &:= (x + y) \wedge u, \\ x^- &:= u - x, \\ x^\sim &:= -x + u, \\ x \odot y &:= (x - u + y) \vee 0, \end{aligned}$$

then $\boldsymbol{\Gamma}(G, u) = (\Gamma(G, u); \oplus, ^-, ^\sim, 0, u)$ is a pseudo MV-algebra.

For example, if $\mathbb{Z}$ denotes the ℓ-group of integers, then $\Gamma(\mathbb{Z}, 1)$ is a Boolean algebra and it generates the variety of Boolean algebras.

We note that a pseudo MV-algebra $\mathbf{M}$ is an MV-algebra iff $a \oplus b = b \oplus a$ for all $a, b \in M$. If $a^- = a^\sim$ for each $a \in M$, we say that $\mathbf{M}$ is *symmetric*; this is not sufficient for $\mathbf{M}$ to be an MV-algebra. Indeed, if G is a non-Abelian ℓ-group, then $\Gamma(\mathbb{Z} \overrightarrow{\times} G, (1, 0))$ is symmetric but not an MV-algebra. We denote by $\mathscr{SYM}$ the variety of symmetric pseudo MV-algebras.

As in Theorem 9.1, let $\mathscr{PMV}$ be the variety of pseudo MV-algebras and $\mathscr{UG}$ be the variety of unital ℓ-groups. The crucial representation theorem for pseudo MV-algebras was proved in Dvurečenskij (2002):

Theorem 9.3 *For every pseudo MV-algebra* $\mathbf{M}$, *there is a unique unital* ℓ*-group* (G, u) *(up to isomorphism of unital* ℓ*-groups) such that* $\mathbf{M} \cong \boldsymbol{\Gamma}(G, u)$. *There is a categorical equivalence between the variety of pseudo MV-algebras and the category of Abelian unital* ℓ*-groups which is given by the functor* $\Gamma : \mathscr{UG} \to \mathscr{PMV}$ *such that* $(G, u) \mapsto \Gamma(G, u)$.

Thanks to the fundamental paper of Holland (1963), we know that every ℓ-group can be represented as an ℓ-subgroup of the ℓ-group $\mathrm{Aut}(\Omega)$ of all automorphisms preserving the linear order of a linearly ordered set Ω, where the group operation of two automorphisms is their composition. Using this result and Theorem 9.3, every pseudo MV-algebra can be visualized as follows.

Corollary 9.1 *Every pseudo MV-algebra* $\mathbf{M}$ *can be represented as a pseudo MV-algebra of automorphisms of some linearly ordered set* Ω.

Finally, we define a partial operation $+$ on a pseudo MV-algebra $\mathbf{M}$ in such a way that $a + b$ is defined in M iff $a \odot b = 0$, and then $a + b := a \oplus b$. In other words, if $\mathbf{M} = \boldsymbol{\Gamma}(G, u)$, $a + b$ is simply the group addition defined in G. Given an integer n, let na be defined as follows, $0a = 0$, $1a = a$, $(n + 1)a = (na) + a$ if it is defined in M. If na is defined for any integer n, a is said to be *infinitesimal*. Let $\mathrm{Infinit}(M)$ be the set of all infinitesimal elements of $\mathbf{M}$.

A *state* on a pseudo MV-algebra $\mathbf{M}$ is any mapping $s : M \to [0, 1]$ such that (i) $s(1) = 1$, and (ii) $s(a+b) = s(a)+s(b)$ if $a+b$ is defined in M. Every MV-algebra possesses at least one state, however, this is not the case for pseudo MV-algebras, see Dvurečenskij (2001). For example, the pseudo MV-algebra defined in Corollary 9.2 below is stateless.

9.2.3 The Lattice of Subvarieties of Pseudo MV-Algebras

The lattice of subvarieties of the variety of pseudo MV-algebras is much richer than the lattice of subvarieties of the variety of MV-algebras, which is countable according to the theorem of Komori, for example see (Komori 1981; Cignoli et al. 2000). An equational basis relative to $\mathscr{MV}$ for every subvariety of MV-algebras was presented by Di Nola and Lettieri (1999).

Theorem 9.4 *The lattice of subvarieties of pseudo MV-algebras is uncountable.*

This can be shown in two ways: It is well-known that the lattice of subvarieties $\mathscr{L}(\mathscr{LG})$ of the variety $\mathscr{LG}$ of ℓ-groups is uncountable, for example see Darnel (1995). Let $\mathscr{G}$ be a subvariety of ℓ-groups and let $\mathscr{M}(\mathscr{G})$ be the class of pseudo MV-algebras $\{\Gamma(G, u) : G \in \mathscr{G}\}$. Then $\mathscr{M}(\mathscr{G})$ is a subvariety (see (Jakubík 2003; Dvurečenskij and Holland 2007)) and the system of subvarieties $\{\mathscr{M}(\mathscr{G}) : \mathscr{G} \in \mathscr{L}(\mathscr{LG})\}$ is uncountable.

Alternatively, let $\mathscr{P}(\mathscr{G})$ be the variety of pseudo MV-algebras generated by the class of pseudo MV-algebras $\{\Gamma(\mathbb{Z} \overrightarrow{\times} G, (1, 0)) : G \in \mathscr{G}\}$. We again obtain an uncountable system of subvarieties of symmetric pseudo MV-algebras, see Di Nola et al. (2008).

We say that an ideal I of a pseudo MV-algebra $\mathbf{M}$ is *normal* if $x \oplus I = I \oplus x$ for each $x \in M$; we note that $x \oplus I := \{x \oplus a : a \in I\}$ and in a similar way we define $I \oplus x$. Let us denote by $\mathscr{M}$ the class of pseudo MV-algebras $\mathbf{M}$ such that either every maximal ideal of $\mathbf{M}$ is normal or $\mathbf{M}$ is trivial. In (Di Nola et al. 2008, (6.1)), it was shown that $\mathscr{M}$ is a variety which contains many important varieties of pseudo MV-algebras, in particular, the variety of normal-valued pseudo MV-algebras. We note that the variety of normal-valued ℓ-groups is the largest nontrivial subvariety of the variety of ℓ-groups, see Darnel (1995).

It is well-known, for example see Cignoli et al. (2000), that the standard MV-algebra on the real interval $[0, 1] := \Gamma(\mathbb{R}, 1)$, where $\mathbb{R}$ is the ℓ-group of the real numbers, is a generator of the variety of MV-algebras.

We say that a pseudo MV-algebra $\Gamma(G, u)$ is *doubly transitive* if G is a doubly transitive ℓ-group (for unexplained notions of ℓ-groups theory, see for example Glass (1999)). This notion is important because we have the following results (see Dvurečenskij and Holland (2007)):

Theorem 9.5 *Every doubly transitive pseudo MV-algebra generates the variety of pseudo MV-algebras.*

Here are two examples of doubly transitive permutation ℓ-groups:

- The system $\mathrm{Aut}(\mathbb{R})$ of all order preserving automorphisms of the real line $\mathbb{R}$ equipped with the natural order.
- The unital permutation group $(\mathrm{BAut}(\mathbb{R}), u)$, where $u \in \mathrm{Aut}(\mathbb{R})$ is the translation $u(t) = t + 1$, $t \in \mathbb{R}$, and

$$\mathrm{BAut}(\mathbb{R}) = \{g \in \mathrm{Aut}(\mathbb{R}) : \exists\, n \in \mathbb{N},\ \ u^{-n} \leq g \leq u^n\}. \tag{9.2.1}$$

Then the MV-algebra $\boldsymbol{\Gamma}(\mathbb{R}, 1)$ is a subalgebra of the doubly transitive pseudo MV-algebra $\boldsymbol{\Gamma}(\mathrm{BAut}(\mathbb{R}), u)$.

Corollary 9.2 *The doubly transitive pseudo MV-algebra* $\boldsymbol{\Gamma}(\mathrm{BAut}(\mathbb{R}), u)$ *generates the variety of pseudo MV-algebras.*

9.2.4 Perfect Pseudo MV-Algebras

We define the *radical* of a symmetric pseudo MV-algebra $\mathbf{M}$, $\mathrm{Rad}(M)$, as the intersection over all maximal ideals of M, and let $\mathrm{Rad}(M)^* := \{a^- : a \in \mathrm{Rad}(M)\} = \{a^\sim : a \in \mathrm{Rad}(M)\}$. We say that a nontrivial symmetric pseudo MV-algebra $\mathbf{M}$ is *perfect* if $\mathrm{Rad}(M) \cup \mathrm{Rad}(M)^* = M$.

If G is an ℓ-group, then

$$\mathscr{E}(G) := \boldsymbol{\Gamma}(\mathbb{Z} \overrightarrow{\times} G, (1, 0)) \tag{9.2.2}$$

is a perfect pseudo MV-algebra.

Let $\mathscr{PPMV}$ be the category whose objects are perfect pseudo MV-algebras and morphisms are homomorphisms of pseudo MV-algebras. Let $\mathscr{LG}$ be the variety of ℓ-groups. The following result was established in Di Nola et al. (2008).

Theorem 9.6 *There is a categorical equivalence between the category of perfect symmetric pseudo MV-algebras and the variety $\mathscr{LG}$ of ℓ-groups which is given by $G \in \mathscr{LG} \mapsto \Gamma(\mathbb{Z} \overrightarrow{\times} G, (1,0))$.*

Perfect pseudo MV-algebras can be generalized to the so called n-perfect ones, i.e., such that they can be split into $n+1$ comparable slices, where n is a fixed integer $n \geq 1$, Dvurečenskij (2011). Then 1-perfect pseudo MV-algebras are perfect ones.

Let $n \geq 1$ be a fixed integer. A nontrivial pseudo MV-algebra $\mathbf{M}$ is said to be *n-perfect* if there are nonempty subsets $M_0, M_1, \ldots, M_n$ of M such that

(a) $M_i \cap M_j = \emptyset$ and $M_i \leq M_j$ for all $i < j$, that is, if $x \in M_i$ and $y \in M_j$, then $x \leq y$;
(b) $M = M_0 \cup M_1 \cup \cdots \cup M_n$;
(c) $M_i^- = M_{n-i} = M_i^\sim$ for any $i = 0, 1, \ldots, n$;
(d) if $x \in M_i$ and $y \in M_j$, then $x \oplus y \in M_{i \oplus j}$, where $i \oplus j = \min\{i + j, n\}$;

and is written, for short, as $M = (M_0, M_1, \ldots, M_n)$.

For example, if G is an ℓ-group,

$$\mathscr{E}_n(G) := \boldsymbol{\Gamma}(\mathbb{Z} \overrightarrow{\times} G, (n,0)) \tag{9.2.3}$$

is an n-perfect symmetric pseudo MV-algebra.

Theorem 9.7 *Let* $\mathbf{M}$ *with* $M = (M_0, M_1, \ldots, M_n)$ *be an n-perfect pseudo MV-algebra.*

(i) Let $a \in M_i$, $b \in M_j$. If $i+j < n$, then $a+b$ is defined in M and $a+b \in M_{i+j}$; if $a+b$ is defined in M, then $i+j \leq n$.
(ii) $M_i + M_j = M_{i+j}$ whenever $i + j < n$.
(iii) If $a \in M_i$ and $b \in M_j$, and $i + j > n$, then $a + b$ is not defined in M.
(iv) Given $a \in M_1$, there is $a' \in M_1$ such that $a' \leq a$ and na' is defined in M and $na' \in M_n$.
(v) M_0 is a normal and maximal ideal of $\mathbf{M}$ *such that $M_0 + M_0 = M_0$.*
(vi) M_0 is a unique maximal ideal of $\mathbf{M}$, $\mathbf{M} \in \mathscr{M}$, *and* $M_0 = \mathrm{Rad}(M) = \mathrm{Infinit}(M)$.
(vii) $\mathbf{M}$ *admits a unique state, namely $s(M_i) = i/n$ for each $i = 0, 1, \ldots, n$. Then $M_i = s^{-1}(\{i/n\})$ for any $i = 0, 1, \ldots, n$.*
(viii) Let $\mathbf{M} = (M_0', M_1', \ldots, M_n')$ *be another representation of M satisfying* (a)–(d). *Then $M_i = M_i'$ for each $i = 0, 1, \ldots, n$.*

An n-perfect pseudo MV-algebra $\mathbf{M}$ such that $M = (M_0, M_1, \ldots, M_n) = \boldsymbol{\Gamma}(G, u)$ is said to be *strong* if there is $a \in M_1$ such that (i) a belongs to the commutative center of G, and (ii) $na = 1$; this element a is said to be a *strong cyclic element of order n*. For example, $\mathscr{E}_n(G)$ is strong with the element $a = (1, 0)$ as a strong cyclic element of order n. On the other hand, every symmetric 1-perfect pseudo MV-algebra is strong with $a = 1$.

Theorem 9.8 *An n-perfect pseudo MV-algebra* $\mathbf{M}$ *is isomorphic to $\mathscr{E}_n(G)$ if and only if* $\mathbf{M}$ *is strong. In such a case, G is unique up to isomorphism of ℓ-groups.*

Let $\mathscr{SPPMV}_n$ denote the category of strong n-perfect pseudo MV-algebras, where objects are pairs $(\mathbf{M}, a)$ with a strong n-perfect pseudo MV-algebra $\mathbf{M}$ and a fixed strong cyclic element $a \in M$ of order n, and morphisms are homomorphisms of pseudo MV-algebras preserving fixed strong cyclic elements. The mapping $\mathscr{E}_n : \mathscr{LG} \to \mathscr{SPPMV}_n$ defined by (9.2.3) is a functor such that $(\mathscr{E}_n(G), (1, 0))$ is an object of the category $\mathscr{SPPMV}_n$. Therefore, we can formulate a generalization of Theorem 9.6, see Dvurečenskij (2008).

Theorem 9.9 *$\mathscr{E}_n$ defines a categorical equivalence between the variety $\mathscr{LG}$ of ℓ-groups and the category $\mathscr{SPPMV}_n$ of strong n-perfect pseudo MV-algebras.*

For any $n \geq 1$, all categories $\mathscr{SPPMV}_n$ are categorically equivalent.

If $\mathscr{K}$ is a family of algebras, $\mathrm{V}(\mathscr{K})$ denotes the variety generated by $\mathscr{K}$.

Theorem 9.10 *Let G be a doubly transitive ℓ-group. Then* $\mathrm{V}(\mathscr{SPPMV}_n) = \mathrm{V}(\mathscr{E}_n(G))$.

9.2.5 Covers of the Variety of MV-Algebras

We say that a variety $\mathscr{V}$ is a *cover* of a variety $\mathscr{W}$ if (i) $\mathscr{W} \subset \mathscr{V}$ and (ii) if $\mathscr{V}'$ is another variety such that $\mathscr{W} \subseteq \mathscr{V}' \subseteq \mathscr{V}$, then either $\mathscr{V}' = \mathscr{W}$ or $\mathscr{V}' = \mathscr{V}$. The variety $\mathscr{MV}$ of MV-algebras is an important subvariety within the variety $\mathscr{PMV}$ of pseudo MV-algebras. In this subsection we describe some symmetric covers of the variety $\mathscr{MV}$. We will follow the main ideas presented in Dvurečenskij and Holland (2009). Every cover of the Abelian variety of lattice-ordered groups is either representable (generated by a totally ordered group) or generated by a Scrimger group S_p for some prime p, see Darnel (1995). Moreover, by Holland and Medvedev (1994), there are uncountably many covers of the variety of Abelian ℓ-groups. By Dvurečenskij and Holland (2009), $\mathscr{MV}$ has uncountably many covers in $\mathscr{SYM} \cap \mathscr{M}$, where $\mathscr{M}$, as it was already stated, denotes the variety of pseudo MV-algebras where every maximal ideal is normal or a pseudo MV-algebra that is trivial.

Holland (2007) found some non-commutative covers of the variety of Boolean algebras, i.e., the variety generated by $\Gamma(\mathbb{Z}, 1)$. Looking for covers of $\mathscr{MV}$ within the variety of symmetric pseudo MV-algebras, $\mathscr{SYM}$, we can distinguish the following two important subcases of covers (Theorems 9.11–9.12).

For a variety $\mathscr{G}$ of ℓ-groups, we define a variety of pseudo MV-algebras

$$\mathscr{E}(\mathscr{G}) := \mathrm{V}(\{\mathscr{E}(G) : G \in \mathscr{G}\}),$$

where $\mathscr{E}(\mathscr{G}) := \Gamma(\mathbb{Z} \overrightarrow{\times} G, (1, 0))$ was defined in (9.2.2).

Theorem 9.11 (Covers where at least one element has a non-commutative radical). *If $\mathscr{G}$ is a cover of the variety of Abelian ℓ-groups, $\mathscr{A}$, then the variety $\mathscr{MV} \vee \mathscr{E}(\mathscr{G}) \subseteq \mathscr{SYM} \cap \mathscr{M}$ is a cover of the variety of MV-algebras, $\mathscr{MV}$, such that at least one of its elements has a non-commutative radical.*

Conversely, if $\mathscr{V} \subseteq \mathscr{SYM} \cap \mathscr{M}$ *is a cover of the variety* $\mathscr{MV}$ *such that at least one element* $\mathbf{M} \in \mathscr{V}$ *has a non-commutative radical, then there is a unique cover* $\mathscr{G}$ *of the variety* $\mathscr{A}$ *such that* $\mathscr{V} = \mathscr{MV} \vee \mathscr{E}(\mathscr{G})$.

For any prime number p, the *Scrimger group* S_p is defined by

$$\mathbb{Z} \overrightarrow{\rtimes}_{\phi} (\pi_{i=0}^{p-1} \mathbb{Z}_i),$$

where $\mathbb{Z}_i = \mathbb{Z}$ and $\overrightarrow{\rtimes}_{\phi}$ denotes the lexicographically ordered semidirect product. Then if $p \neq q$, then $\mathscr{S}_p \neq \mathscr{S}_q$, and $\mathscr{S}_p$ is a cover of $\mathscr{A}$.

Theorem 9.12 (Covers where each element has a commutative radical). *Let* p *be any prime number,* $n \geq 1$ *an integer,* S_p *be the Scrimger* ℓ*-group with a fixed strong unit* $u_n = (p^n; 0, \ldots, 0)$*, and let* $\Sigma(S_p, n)$ *be the variety of symmetric pseudo MV-algebras in* $\mathscr{M}$ *generated by the* p^n*-perfect pseudo MV-algebra* $\boldsymbol{\Gamma}(S_p, u_n)$*. Then* $\mathscr{MV} \vee \Sigma(S_p, n)$ *is a cover of* $\mathscr{MV}$ *such that every element of* $\Sigma(S_p, n)$ *has a commutative radical.*

Conversely , if $\mathscr{V} \subseteq \mathscr{SYM} \cap \mathscr{M}$ *is a cover of* $\mathscr{MV}$ *such that every element* $\mathbf{M} \in \mathscr{V}$ *has commutative radical, then there is a unique prime* p *and a unique* $n \geq 1$ *such that* $\mathscr{V} = \mathscr{MV} \vee \Sigma(S_p, n)$.

Remark 9.1 (1) If $\mathscr{V}$ is a non-commutative cover of the variety of Boolean algebras, $\mathscr{B}$, then $\mathscr{MV} \cap \mathscr{V} = \mathscr{B}$, and $\mathscr{MV} \vee \mathscr{V}$ is a cover of $\mathscr{MV}$. (The converse is not true)

(2) Holland's examples, see Holland (2007): Let

$$T = \left\{ \sum m_i t^{n_i} \; : \; m_i, n_i \in \mathbb{Z} \right\},$$

$$(r, n)(s, m) = (r + t^n s, n + m),$$

$$S_t = T \overleftarrow{\times} \mathbb{Z}, \quad \mathscr{C}_t = \mathrm{V}(\Gamma(S_t, (1, 0))).$$

The system $\{\mathscr{C}_t : t \in \mathbb{R}^+\}$ is an uncountable system of non-commutative covers of the variety of Boolean algebras, $\mathscr{B}$.

The system $\{\mathscr{C}_t \vee \mathscr{MV} : t \in \mathbb{R}^+\}$, is an uncountable system of covers of $\mathscr{MV}$ which are not symmetric but they are from $\mathscr{M}$.

We finish this subsection with an open question:

Problem 9.1 Are there other covers of $\mathscr{MV}$ outside of $\mathscr{SYM}$ but in $\mathscr{M}$?

9.2.6 State MV-Algebras and State Pseudo MV-Algebras

MV-algebraic states were introduced by Mundici in Mundici (1993) about 40 years after C.C. Chang defined MV-algebras. States have been introduced on D-posets, of which MV-algebras form an important subclass, see Kôpka and Chovanec (1994). An important characterization of states on MV-algebras by regular Borel probability measures was recently done in Kroupa (2006), Panti (2008). States can also be understood as averaging processes for truth-value in Łukasiewicz logic. However, the notion of MV-algebraic state does not properly pertain to universal algebra.

In Flaminio and Montagna (2009) the authors find an algebraizable logic whose equivalent algebraic semantics is the variety of state MV-algebras. In other words, they expanded MV-algebras by a unary operator τ whose properties resemble the properties of a state. It is the so called *internal state* or a *state-operator*. The corresponding subdirectly irreducible algebras are not necessarily linearly ordered. A stronger structure is a state-morphism MV-algebra, where the state-operator τ is an MV-endomorphism such that $\tau^2 = \tau$. Subdirectly irreducible state-morphism MV-algebras were described in Di Nola and Dvurečenskij (2009), Di Nola et al. (2009, 2010). A complete characterization of subdirectly irreducible state MV-algebras together with generators of the variety of state MV-algebras was presented in Dvurečenskij et al. (2011). A generalization of state-morphism MV-algebras for an arbitrary algebra was done in Botur and Dvurečenskij (2013).

We say that an operator τ from an MV-algebra into itself is a *state-operator* or an *internal state* if

(a) $\tau(1) = 1$;
(b) $\tau(x \oplus y) = \tau(x) \oplus \tau(y \ominus (x \odot y))$;
(c) $\tau(x^*) = \tau(x)^*$;
(d) $\tau(\tau(x) \oplus \tau(y)) = \tau(x) \oplus \tau(y)$,

where $x \ominus y := (x^* \oplus y)^*$, and a *state MV-algebra* is a pair (M, τ), where τ is a fixed state-operator.

A *state-morphism* is an endomorphism τ on an MV-algebra M such that $\tau^2 = \tau$, and the pair (M, τ) is said to be a *state-morphism MV-algebra*. This notion can be defined in the same way also for pseudo MV-algebras.

The following facts were proved in (Flaminio and Montagna 2009; Di Nola et al. 2010):

Lemma 9.1 (1) *In a state MV-algebra* (M, τ)*, the following conditions hold:*

(1a) $\tau(0) = 0$.
(1b) *If* $x \odot y = 0$*, then* $\tau(x) \odot \tau(y) = 0$ *and* $\tau(x \oplus y) = \tau(x) \oplus \tau(y)$.
(1c) $\tau(\tau(x)) = \tau(x)$.
(1d) *Let* $\tau(M) := \{\tau(a) : a \in M\}$*. Then* $\tau(M) = (\tau(M), \oplus, ^*, 0, 1)$ *is an MV-subalgebra of* M*, and* τ *is the identity on it.*
(1e) *If* $x \leq y$*, then* $\tau(x) \leq \tau(y)$.
(1f) $\tau(x) \odot \tau(y) \leq \tau(x \odot y)$.
(1g) $\tau(x \to y) = \tau(x) \to \tau(x \wedge y)$.
(1h) *If* (M, τ) *is subdirectly irreducible, then* $\tau(M)$ *is linearly ordered.*

(2) *The following conditions on state-morphism MV-algebras hold:*

(2a) *In a state-morphism MV-algebra* (M, τ)*,* $\tau(M)$ *is a retract of* M*, that is,* τ *is a homomorphism from* M *onto* $\tau(M)$*, the identity map is an embedding from* $\tau(M)$ *into* M*, and the composition* $\tau \circ \mathrm{Id}_{\tau(M)}$*, that is, the restriction of* τ *to* $\tau(M)$ *is the identity on* $\tau(M)$*.*

(2b) *A state MV-algebra* (M, τ) *is a state-morphism MV-algebra iff it satisfies* $\tau(x \vee y) = \tau(x) \vee \tau(y)$ *iff it satisfies* $\tau(x \wedge y) = \tau(x) \wedge \tau(y)$*.*

(2c) *Any linearly ordered state MV-algebra is a state-morphism MV-algebra.*

The relation between state MV-algebras and ℓ-groups can be described as follows, see Di Nola and Dvurečenskij (2009).

Let (G, u) be an Abelian unital ℓ-group. A *state-operator* on (G, u) is a group homomorphism $\tau : G \to G$ such that (i) τ preserves the order $\leq$ on G, (ii) $\tau(u) = u$, (iii) $\tau^2 = \tau$, and (iv) τ on $\tau(G)$ is an ℓ-group homomorphism. A *state-morphism operator* on (G, u) is any ℓ-group homomorphism $\tau : G \to G$ such that $\tau(u) = u$ and $\tau^2 = \tau$.

Theorem 9.13 *Let* $\mathbf{M} = \boldsymbol{\Gamma}(G, u)$*.*

(1) *Every state-operator* τ *on* M *can be uniquely extended to a state-operator* τ_u *on* (G, u)*. Conversely, the restriction of any state-operator of* (G, u) *to* M *gives a state-operator on* M*.*

(2) *Every state-morphism* τ *on* M *can be uniquely extended to a state-morphism operator* τ_u *on* (G, u)*. Conversely, the restriction of any state-morphism operator of* (G, u) *to* M *gives a state-morphism operator on* M*.*

Any filter or ideal F of M such that $\tau(F) \subseteq F$ is said to be a τ-*filter* or a τ-ideal, respectively. There is a one-to-one correspondence between τ-filters (τ-ideals) and congruences on state MV-algebras or state-morphism MV-algebras.

To present the following characterization of subdirectly irreducible state MV-algebras, we need the following notions.

We say that a *hoop* is an algebra $\mathbf{M} = (M; \odot, \to, 1)$ of type (2, 2, 0) such that $(M; \odot)$ is a commutative monoid and, for all $x, y, z \in M$,

(i) $x \odot 1 = x$;
(ii) $(x \odot y) \to z = x \to (y \to z)$;
(iii) $(x \to y) \odot x = (y \to x) \odot y$.

A hoop $\mathbf{M} = (M; \odot, \to, 1)$ is a *Wajsberg hoop*, if for all $x, y \in M$, we have

$$(x \to y) \to y = (y \to x) \to x.$$

In other words, a Wajsberg hoop is a subreduct (subalgebra of a reduct) of an MV-algebra in the language $\{1, \odot, \to\}$, where $x \to y := x^* \oplus y$, $x, y \in M$.

Clearly, $\mathrm{Ker}_1(\tau) := \{a \in M : \tau(a) = 1\}$ is a τ-filter of M, and $(\mathrm{Ker}_1(\tau); \odot, \to, 1)$ is a Wajsberg subhoop of M. We say that two Wajsberg subhoops, $\mathbf{B}$ and $\mathbf{C}$, of an MV-algebra $\mathbf{M}$ have the *disjunction property* if for all $x \in B$ and $y \in C$, if $x \vee y = 1$, then either $x = 1$ or $y = 1$.

Theorem 9.14 *Suppose that (M, τ) is a subdirectly irreducible state MV-algebra. Then:*

1. *If* $\mathrm{Ker}_1(\tau) = \{1\}$*, then* $\tau(M)$ *is subdirectly irreducible.*
2. $\mathrm{Ker}_1(\tau)$ *is either trivial or a subdirectly irreducible hoop.*
3. $\mathrm{Ker}_1(\tau)$ *and* $\tau(M)$ *have the disjunction property.*

Conversely, suppose that (M, τ) is a state MV-algebra satisfying conditions (1), (2) *and* (3) *in the first part of this theorem. Then (M, τ) is subdirectly irreducible, and hence, the above conditions constitute a characterization of subdirectly irreducible state MV-algebras.*

The following characterization of subdirectly irreducible state-morphism MV-algebras was proved in (Di Nola and Dvurečenskij 2009; Di Nola et al. 2009; Dvurečenskij 2011; Botur and Dvurečenskij 2013). We note that $\mathrm{Rad}_1(M)$ is defined as the intersection of all maximal filters of M.

Theorem 9.15 *A state-morphism MV-algebra (M, τ) is subdirectly irreducible if and only if it satisfies at least one of the following pairwise incompatible conditions.*

(i) *M is linearly ordered, τ is the identity on M and the MV-reduct of M is a subdirectly irreducible MV-algebra.*

(ii) *The state morphism operator τ is not faithful, M has no nontrivial Boolean elements and is a local MV-algebra,* $\mathrm{Ker}_1(\tau)$ *is a subdirectly irreducible hoop, and* $\mathrm{Ker}_1(\tau)$ *and* $\tau(M)$ *have the disjunction property.*
Moreover, M is linearly ordered if and only if $\mathrm{Rad}_1(M)$ *is linearly ordered, and in such a case, M is a subdirectly irreducible MV-algebra such that the smallest nontrivial τ-filter of (M, τ), and the smallest nontrivial MV-filter for M coincide.*
If $\mathrm{Rad}_1(M) = \mathrm{Ker}(\tau)$*, then M is linearly ordered.*

(iii) *The state morphism operator τ is not faithful, M has a nontrivial Boolean element. There are a linearly ordered MV-algebra B, a subdirectly irreducible MV-algebra C, and an injective MV-homomorphism $h : B \to C$ such that (M, τ) is isomorphic to $(B \times C, \tau_h)$, where $\tau_h(x, y) = (x, h(x))$ for any $(x, y) \in B \times C$.*

We note that the notion of a state-morphism MV-algebra can be extended also for pseudo MV-algebras as a pair (M, τ), where τ is an idempotent endomorphism of a pseudo MV-algebra M.

If $\mathscr{K}$ is a variety of pseudo MV-algebras, then the class $\mathscr{K}_\tau$ of all state-morphism pseudo MV-algebras (M, τ), where $M \in \mathscr{K}$ and τ is any state-morphism on M, forms a variety, too.

Let $B \in \mathscr{K}$. Then an algebra $D(B) := (B \times B, \tau_B)$, where τ_B is defined by $\tau_B(x, y) = (x, x)$, $x, y \in B$, is a state-morphism pseudo MV-algebra (denoted by $(B \times B, \tau_B) \in \mathscr{K}_\tau$); we call τ_B a *diagonal state-operator*. If a state-morphism pseudo MV-algebra (C, τ) can be embedded into some diagonal state-morphism pseudo MV-algebra, $(B \times B, \tau_B)$, (C, τ) is said to be a *subdiagonal state-morphism pseudo MV-algebra*, or, more precisely, B-subdiagonal.

Theorem 9.16 *For every subdirectly irreducible state-morphism pseudo MV-algebra* (M, τ), *there is a subdirectly irreducible pseudo MV-algebra* B *such that* (M, τ) *is* B*-subdiagonal.*

As usual, given a class $\mathcal{K}$ of algebras, $\mathrm{I}(\mathcal{K})$, $\mathrm{H}(\mathcal{K})$, $\mathrm{S}(\mathcal{K})$, $\mathrm{P}(\mathcal{K})$ and $\mathrm{P_U}(\mathcal{K})$ will denote the class of isomorphic images, of homomorphic images, of subalgebras, of direct products and of ultraproducts of algebras from $\mathcal{K}$, respectively. Moreover, $\mathrm{V}(\mathcal{K})$ will denote the variety generated by $\mathcal{K}$, and we set $\mathsf{D}(\mathcal{K}) = \{D(\mathbf{A}) : \mathbf{A} \in \mathcal{K}\}$.

The following result was proved for MV-algebras in Dvurečenskij et al. (2011) and for pseudo MV-algebras in Botur and Dvurečenskij (2013).

Theorem 9.17 (1) *For every class* $\mathcal{K}$ *of pseudo MV-algebras* $\mathcal{V}(\mathsf{D}(\mathcal{K})) = \mathrm{V}(\mathcal{K})_\tau$.

(2) *Let* $\mathcal{K}_1$ *and* $\mathcal{K}_2$ *be two classes of pseudo MV-algebras. Then* $\mathsf{V}(\mathsf{D}(\mathcal{K}_1)) = \mathrm{V}(\mathsf{D}(\mathcal{K}_2))$ *if and only if* $\mathrm{V}(\mathcal{K}_1) = \mathrm{V}(\mathcal{K}_2)$.

As a corollary we have solved in Dvurečenskij et al. (2011) an open problem formulated in Di Nola and Dvurečenskij (2009) that the diagonal state-morphism MV-algebra of the standard MV-algebra $\boldsymbol{\Gamma}(\mathbb{R}, 1)$ is a generator of the variety of state-morphism MV-algebras. Similarly Botur and Dvurečenskij (2013), the diagonal state-morphism pseudo MV-algebra of the pseudo MV-algebra $\mathrm{BAut}(\mathbb{R})$ defined by (9.2.1) is a generator of the variety of state-morphism pseudo MV-algebras.

We formulate another open problem:

Problem 9.2 Describe some interesting generators of the variety of state MV-algebras.

9.3 BL-Algebras and Pseudo BL-Algebras Versus ℓ-Groups

Hájek in his monograph Hájek (1998) presented the problem of finding a basic fuzzy logic as a common roof for the most important fuzzy logics, namely Łukasiewicz, Gödel and product logic. BL-algebras are the Lindenbaum algebras of Hájek's basic logic. The variety of BL-algebras is generated by all BL-algebras with universe [0, 1] and $\odot$ a continuous t-norm. Di Nola et al. (2002a, b) presented a non-commutative version of BL-algebras which we call *pseudo BL-algebras*.

9.3.1 Pseudo BL-Algebras and BL-Algebras

According to (Di Nola et al. 2002a, b) a pseudo BL-algebra is an algebra $\mathbf{M} = (M; \odot, \vee, \wedge, \to, \rightsquigarrow, 0, 1)$ of type (2, 2, 2, 2, 2, 0, 0) satisfying the following conditions:

(i) $(M; \odot, 1)$ is a monoid (need not be commutative), i.e., $\odot$ is associative with neutral element 1.

(ii) $(M; \vee, \wedge, 0, 1)$ is a bounded lattice;
(iii) $x \odot y \leqslant z$ iff $x \leqslant y \to z$ iff $y \leqslant x \rightsquigarrow z$ $x, y \in M$;
(iv) $(x \to y) \odot x = x \wedge y = y \odot (y \rightsquigarrow x),\ x, y \in M$;
(v) $(x \to y) \vee (y \to x) = 1 = (x \rightsquigarrow y) \vee (y \rightsquigarrow y),\ x, y \in M$.

We note that $\wedge, \vee$ and $\odot$ have higher binding priority than $\to$ or $\rightsquigarrow$, and M is a distributive lattice. We note that a pseudo BL-algebra is a BL-algebra iff $\odot$ is commutative iff $\to = \rightsquigarrow$. We define two unary operations (negations) $^-$ and $^\sim$ on M such that $x^- := x \to 0$ and $x^\sim := x \rightsquigarrow 0$ for any $x \in M$. It is easy to show that

$$x \odot y = 0 \Leftrightarrow y \leq x^\sim \Leftrightarrow x \leq y^-.$$

We say that a *pseudo hoop* is an algebra $\mathbf{M} = (M; \odot, \to, \rightsquigarrow, 1)$ of type $(2, 2, 2, 0)$ such that, for all $x, y, z \in M$,

(i) $x \odot 1 = x = 1 \odot x$;
(ii) $x \to x = 1 = x \rightsquigarrow x$;
(iii) $(x \odot y) \to z = x \to (y \to z)$;
(iv) $(x \odot y) \rightsquigarrow z = y \rightsquigarrow (x \rightsquigarrow z)$;
(v) $(x \to y) \odot x = (y \to x) \odot y = x \odot (x \rightsquigarrow y) = y \odot (y \rightsquigarrow x)$.

If $\odot$ is commutative (equivalently $\to = \rightsquigarrow$), $\mathbf{M}$ is said to be a *hoop*.

A pseudo hoop $\mathbf{M}$ is said to be *Wajsberg* if, for all $x, y \in M$,

(W1) $(x \to y) \rightsquigarrow y = (y \to x) \rightsquigarrow x$;
(W2) $(x \rightsquigarrow y) \to y = (y \rightsquigarrow x) \to x$.

For example, if G is an ℓ-group written multiplicatively, then for the negative cone $G^- = \{g \in G :\ g \leq e\}$ we define $a \odot b = a \cdot b$, $a \to b = (b \cdot a^{-1}) \wedge e$, and $a \rightsquigarrow b = (a^{-1} \cdot b) \wedge e$. Then $(G^-; \odot, \to, \wedge, e)$ is a pseudo Wajsberg hoop.

Let $\{\mathbf{M}_i :\ i \in I\}$ be a system of pseudo hoops with a linearly ordered index set $(I; \leq)$ such that $M_i \cap M_j = \{1\}$ for all $i \neq j$, $i, j \in I$. We set $M = \bigcup_{i \in I} M_i$ and on the universe M we define the operations $\odot$, $\to$ and $\rightsquigarrow$ as follows:

$$x \odot y \ = \begin{cases} x \odot_i y & \text{if} \ \ x, y \in M_i, \\ x & \text{if} \ \ x \in M_i \setminus \{1\},\ y \in M_j,\ i < j, \\ y & \text{if} \ \ x \in M_i,\ y \in M_j \setminus \{1\},\ i > j, \end{cases}$$

$$x \to y = \begin{cases} x \to_i y & \text{if} \ \ x, y \in M_i, \\ y & \text{if} \ \ x \in M_i,\ y \in M_j \setminus \{1\},\ i > j, \\ 1 & \text{if} \ \ x \in M_i \setminus \{1\},\ y \in M_j,\ i < j, \end{cases}$$

and

$$x \rightsquigarrow y = \begin{cases} x \rightsquigarrow_i y & \text{if} \ \ x, y \in M_i, \\ y & \text{if} \ \ x \in M_i,\ y \in M_j \setminus \{1\},\ i > j, \\ 1 & \text{if} \ \ x \in M_i \setminus \{1\},\ y \in M_j,\ i < j. \end{cases}$$

Then $\mathbf{M}$ with M, 1, $\odot$, $\to$ and $\rightsquigarrow$ is a pseudo hoop called the *ordinal sum* of $\{\mathbf{M}_i : i \in I\}$.

Aglianò and Montagna showed in Aglianò and Montagna (2003) that every linearly ordered BL-algebra can be decomposed as an ordinal sum of linearly ordered Wajsberg hoops. This result was generalized in Dvurečenskij (2007):

Theorem 9.18 *Every linearly ordered pseudo hoop (linear pseudo BL-algebra) can be uniquely represented as the ordinal sum of a family of linearly ordered pseudo Wajsberg hoops (whose first component is a linearly ordered pseudo Wajsberg algebra).*

Equivalently, every linearly ordered pseudo hoop is the ordinal sum of a system whose every component is either the negative cone of a linearly ordered ℓ-group or a negative interval in a linearly ordered unital ℓ-group with strong unit.

The RDP can be defined also for pseudo BL-algebras.

We say that a pseudo hoop $\mathbf{M}$ satisfies the *Riesz decomposition property* (RDP for short) if $a \geq b \odot c$ implies that there are two elements $b_1 \geq b$ and $c_1 \geq c$ such that $a = b_1 \odot c_1$. Similarly, as for ℓ-groups, we have the following result, Botur et al. (2012):

Theorem 9.19 *Every pseudo hoop satisfies* RDP.

9.3.2 Kites

We say that a pseudo BL-algebra $\mathbf{M}$ is *good*, if $x^{-\sim} = x^{\sim -}$ for every $x \in M$. Every pseudo MV-algebra and every BL-algebra is good. From Theorem 9.18 we have that every linearly ordered pseudo BL-algebra, hence every representable pseudo BL-algebra, is good. Good pseudo BL-algebras are important for example in the investigation of states on pseudo BL-algebras. There are two notions of a state for pseudo BL-algebras, a *Bosbach state* (Georgescu (2004)), and a *Riečan state* (Riečan (2000)), and in Dvurečenskij and Rachůnek (2006) it was shown that for good pseudo BL-algebras both notions coincide. It was an open problem whether every pseudo BL-algebra is good, see Di Nola et al. (2002b, Problem 3.21). This was answered in the negative in Dvurečenskij et al. (2010), where a special type of pseudo BL-algebras generated by the ℓ-group of integers, $\mathbb{Z}$, defined in Jipsen and Montagna (2006) was used, which we now call a kite. This construction was generalized in Dvurečenskij and Kowalski (2014) for arbitrary ℓ-groups. We note that for kites, instead of arrows we will use divisions, / and \, and instead of $\odot$, we use multiplication $\cdot$.

Let $\mathbf{G}$ be an ℓ-group written multiplicatively and I, J be sets with $|J| \leqslant |I|$. Since only the cardinalities of I and J matter for the construction, it is harmless to think of these sets as ordinals. For injections $\lambda, \rho \colon J \to I$ we define an algebra with the universe $(G^+)^J \uplus (G^-)^I$. We order this universe by keeping the original coordinatewise ordering within $(G^+)^J$ and $(G^-)^I$, and setting $x \leqslant y$ for all $x \in (G^+)^J$, $y \in (G^-)^I$. It is easy to verify that this is a (bounded) lattice ordering of

$(G^+)^J \uplus (G^-)^I$. Notice also that the case $I = J$ is not excluded, so the element e^I may appear twice: at the bottom of $(G^+)^J$ and at the top of $(G^-)^I$. To avoid confusion in the definitions below, we adopt a convention of writing $a_i^{-1}, b_i^{-1}, \ldots$ for elements of $(G^-)^I$ and $f_j, g_j, \ldots$ for elements of $(G^+)^J$. In particular, we will write e^{-1} for e as an element of G^-. We also put 1 for the constant sequence $(e^{-1})^I$ and 0 for the constant sequence e^J. With these conventions in place we are ready to define multiplication, putting:

$$\begin{aligned}
\langle a_i^{-1} : i \in I\rangle \cdot \langle b_i^{-1} : i \in I\rangle &= \langle (b_i a_i)^{-1} : i \in I\rangle \\
\langle a_i^{-1} : i \in I\rangle \cdot \langle f_j : j \in J\rangle &= \langle a_{\lambda(j)}^{-1} f_j \vee e : j \in J\rangle \\
\langle f_j : j \in J\rangle \cdot \langle a_i^{-1} : i \in I\rangle &= \langle f_j a_{\rho(j)}^{-1} \vee e : j \in J\rangle \\
\langle f_j : j \in J\rangle \cdot \langle g_j : j \in J\rangle &= \langle e : j \in J\rangle = 0.
\end{aligned}$$

Definition 9.1 Divisions, / and \, corresponding to multiplication defined as above on $(G^+)^J \uplus (G^-)^I$ are defined by:

$$\begin{aligned}
\langle a_i^{-1} : i \in I\rangle \backslash \langle b_i^{-1} : i \in I\rangle &= \langle a_i b_i^{-1} \wedge e^{-1} : i \in I\rangle \\
\langle b_i^{-1} : i \in I\rangle / \langle a_i^{-1} : i \in I\rangle &= \langle b_i^{-1} a_i \wedge e^{-1} : i \in I\rangle \\
\langle a_i^{-1} : i \in I\rangle \backslash \langle f_j : j \in J\rangle &= \langle a_{\lambda(j)} f_j : j \in J\rangle \\
\langle f_j : j \in J\rangle / \langle a_i^{-1} : i \in I\rangle &= \langle f_j a_{\rho(j)} : j \in J\rangle \\
\langle f_j : j \in J\rangle \backslash \langle g_j : j \in J\rangle &= \langle a_i^{-1} : i \in I\rangle, \\
\text{where } a_i^{-1} &= \begin{cases} f_{\rho^{-1}(i)}^{-1} g_{\rho^{-1}(i)} \wedge e^{-1} & \text{if } \rho^{-1}(i) \text{ is defined} \\ e^{-1} & \text{otherwise} \end{cases} \\
\langle g_j : j \in J\rangle / \langle f_j : j \in J\rangle &= \langle b_i^{-1} : i \in I\rangle, \\
\text{where } b_i^{-1} &= \begin{cases} g_{\lambda^{-1}(i)} f_{\lambda^{-1}(i)}^{-1} \wedge e^{-1} & \text{if } \lambda^{-1}(i) \text{ is defined} \\ e^{-1} & \text{otherwise,} \end{cases} \\
\langle a_i^{-1} : i \in I\rangle / \langle f_j : j \in J\rangle &= (e^{-1})^I = \langle f_j : j \in J\rangle \backslash \langle a_i^{-1} : i \in I\rangle.
\end{aligned}$$

Lemma 9.2 *For any ℓ-group* **G** *and any choice of appropriate sets I, J and maps λ, ρ, the algebra $K_{I,J}^{\lambda,\rho}(\mathbf{G})$ is a pseudo BL-algebra.*

We will call the pseudo BL-algebra we have just defined a *kite* of **G**, and denote it by $K_{I,J}^{\lambda,\rho}(\mathbf{G})$. Observe that if we take $I = J$, then λ and ρ become permutations of the set of coordinates and so the kite construction is reminiscent of wreath product.

We note that the example from Jipsen and Montagna (2006) is $\mathbb{Z}^+ \uplus (\mathbb{Z}^- \times \mathbb{Z}^-) = K_{2,1}^{\lambda,\rho}(\mathbb{Z})$ with $\lambda(0) = 0$, $\rho(0) = 1$ which is a pseudo BL-algebra that is not good.

Theorem 9.20 *Let* $\mathbf{G}$ *be a non-trivial* ℓ*-group.*

(1) *A pseudo BL-algebra* $K_{I,J}^{\lambda,\rho}(\mathbf{G})$ *is good if and only if* $\lambda(J) = \rho(J)$.

(2) *A pseudo BL-algebra* $K_{I,J}^{\lambda,\rho}(\mathbf{G})$ *is a pseudo MV-algebra if and only if* $\lambda(J) = I = \rho(J)$.

(3) *A pseudo BL-algebra* $K_{I,J}^{\lambda,\rho}(\mathbf{G})$ *is subdirectly irreducible if and only if* $\mathbf{G}$ *is subdirectly irreducible, and for all* i, j, *there exists an integer* m *such that* $(\rho \circ \lambda^{-1})^m(i) = j$ *or* $(\lambda \circ \rho^{-1})^m(i) = j$.

A kite $K_{I,J}^{\lambda,\rho}(\mathbf{G})$ is *n-dimensional* if $|I| = n$, and *finite-dimensional* if $|I| < \infty$. We think of the index sets I and J as ordinals (hence $\mathbb{Z} \neq \omega$). By Dvurečenskij and Kowalski (2014), if $K_{I,J}^{\lambda,\rho}(\mathbf{G})$ is finite-dimensional and subdirectly irreducible, then it is isomorphic to one of the following kites:

1. $K_{n,n}^{\lambda,\rho}(\mathbf{G})$, $\lambda(j) = j$, $\rho(j) = j + 1 \pmod n$,
2. $K_{n+1,n}^{\lambda,\rho}(\mathbf{G})$, with $\lambda(j) = j$ and $\rho(j) = j + 1$.

Theorem 9.21 *Let* $K_{I,J}^{\lambda,\rho}(\mathbf{G})$ *be a subdirectly irreducible kite and* $\mathbf{G}$ *non-trivial. Then* $K_{I,J}^{\lambda,\rho}(\mathbf{G})$ *is isomorphic to precisely one of:*

0. $K_{0,0}^{\emptyset,\emptyset}(\mathbf{G})$, $K_{1,1}^{id,id}(\mathbf{G})$,
1. $K_{n,n}^{\lambda,\rho}(\mathbf{G})$, *with* $\lambda(j) = j$ *and* $\rho(j) = j + 1 \pmod n$.
2. $K_{\mathbb{Z},\mathbb{Z}}^{\lambda,\rho}(\mathbf{G})$, *with* $\lambda(j) = j$ *and* $\rho(j) = j + 1$.
3. $K_{\omega,\omega}^{\lambda,\rho}(\mathbf{G})$, *with* $\lambda(j) = j$ *and* $\rho(j) = j + 1$.
4. $K_{\omega,\omega}^{\lambda,\rho}(\mathbf{G})$, *with* $\lambda(j) = j + 1$ *and* $\rho(j) = j$.
5. $K_{n+1,n}^{\lambda,\rho}(\mathbf{G})$, *with* $\lambda(j) = j$ *and* $\rho(j) = j + 1$.

Moreover, types (1) *and* (2) *consist entirely of pseudo MV-algebras, the other types contain no pseudo MV-algebras except the two-element Boolean algebra. A kite of type* (3) *or* (4) *is good if and only if it is a two-element Boolean algebra. A kite of type* (5) *is good if and only if* $J = \emptyset$.

Finite-dimensional kites are important because:

Theorem 9.22 *The variety of pseudo BL-algebras* K *generated by all kites is generated by all finite-dimensional kites, and it is the varietal join of varieties* K_n, *generated by* n*-dimensional kites, that is,*

$$\mathsf{K} = \bigvee_{n=1}^{\infty} \mathsf{K}_n.$$

Kites provide new covers of the variety of Boolean algebras $\mathscr{BA}$.

As an abbreviation, given an integer $n \geq 1$, we set $\mathbb{Z}_n^{\dagger} = K_{I_n,J_n}^{\lambda_n,\rho_n}(\mathbb{Z})$, where $\lambda, \rho : J_n \to I_n$ are given by $\lambda(i) = i$ and $\rho(i) = i + 1$ for each $i \in J_n$. In addition, we define $\mathbb{Z}_0^{\dagger} := K_{I_1,J_1}^{\lambda_1,\rho_1}(O)$, where O is the trivial ℓ-group consisting only of the identity. Then $\mathbb{Z}_0^{\dagger}$ is the two-element Boolean algebra, and therefore, $\mathbb{Z}_0^{\dagger}$ generates the variety of Boolean algebras, $\mathscr{BA}$.

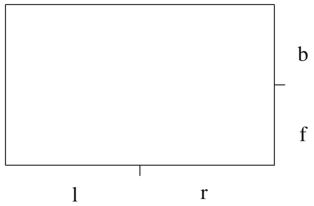

Fig. 9.1 Firefly in a box

Theorem 9.23 *For any integer $n \geq 1$, $\mathrm{V}(\mathbb{Z}_n^\dagger)$ is a cover of the variety of Boolean algebras $\mathscr{BA}$, and $n \neq m$ implies $\mathrm{V}(\mathbb{Z}_n^\dagger) \neq \mathrm{V}(\mathbb{Z}_m^\dagger)$.*

9.4 Quantum Structures

As it was mentioned in the introduction, measurements in quantum mechanics do not satisfy the axioms of Kolmogorov (1933), and quantum mechanical events form more complex structures than Boolean algebras. We call such a structure a *quantum logic*, see Birkhoff and Neumann (1936). At present, there is a whole variety of such structures, called *quantum structures*, important examples of which are orthomodular lattices and posets, orthoalgebras, effect algebras or D-posets, and their non-commutative extensions like pseudo effect algebras.

9.4.1 Examples of Quantum Structures Without Quantum Mechanics

In this subsection we present some simple examples far from quantum physics whose propositional logic does form not a Boolean algebra, but rather an orthomodular lattice and an orthoalgebra, see (Dvurečenskij and Pulmannová, 2000, Chap. 4)

1. *Firefly in a box.* Consider a system consisting of a firefly in a box with a clear plastic window at the front and another one on the side pictured in Fig. 9.1.

Suppose each window has a thin vertical line drawn down the center to divide the window in half. We shall consider two experiments on the system: Experiment A consists of looking at the front window, while experiment B consists of looking at the side window. We assume that we cannot observe simultaneously both windows. The outcomes of A and B are: See a light in the left half l_A and l_B, right half r_A and r_B of window or see no light n_A and n_B, respectively. It is clear that $n_A = n_B =: n$ and we put $l_A =: l$, $r_A =: r$, $l_B =: f$, $r_B =: b$ (f for the front, b for the back).

The associated Hasse diagram of this orthomodular lattice (for definition, see below) is given by Fig. 9.2 which in fact is a pasting in an atom and a co-atom of two three-atom Boolean algebras. We note that, for example, l' denotes the negation of l, etc.

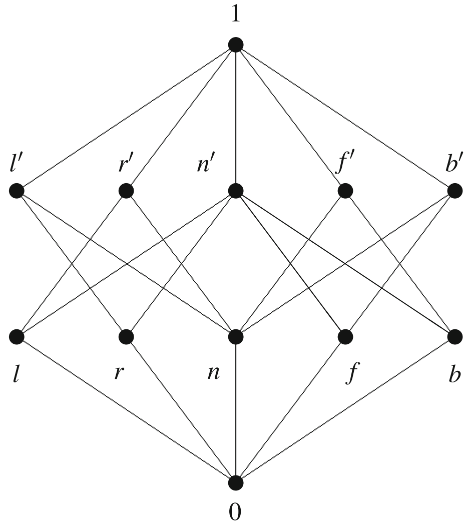

Fig. 9.2 Hasse diagram for Firefly in a box

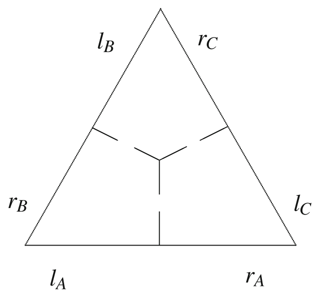

Fig. 9.3 Firefly in a three-chamber box

2. *Firefly in a three-chamber box.* Consider again a firefly, but now in a three-chamber box pictured in Fig. 9.3.

The firefly is free to roam around the three chambers and to light up at will. The sides of the box are windows with vertical lines down their centers. We make three experiments, corresponding to the three windows A, B and C. For each experiment E, we record l_E, r_E, n_E if we see, respectively, a light to the left or right of the center line or no light. It is clear that we can identify $r_A = l_C =: e$, $r_C = l_B =: c$, $r_B = l_A =: a$, but now we cannot identify $f := n_A$, $b := n_B$, $d := n_C$.

The propositional system of this model is an orthoalgebra, for definition see below, whose corresponding Hasse diagram is given by Fig. 9.4. It is again a pasting in a "triangular way" in an atom and a co-atom of three three-atom Boolean algebras.

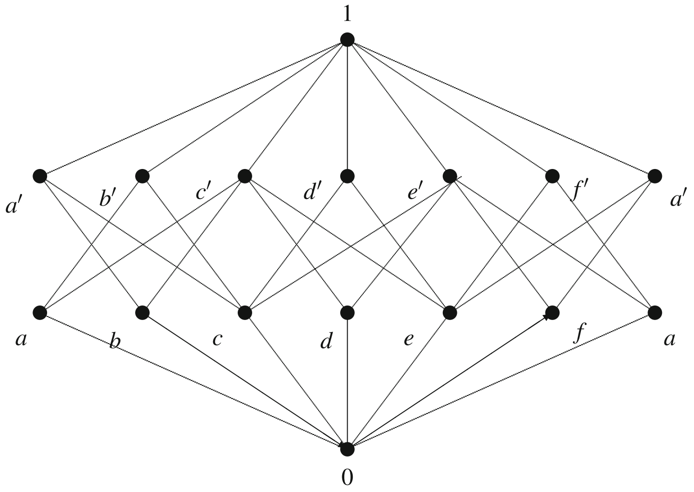

Fig. 9.4 Hasse diagram for Firefly in a three-chamber box

9.4.2 Orthomodular Lattices, Orthomodular Posets, and Orthoalgebras

An orthomodular lattice satisfies a weaker form of distributive law, called *orthomodular law.*

An *orthomodular poset* (OMP, in short) is a bounded poset $\mathbf{L} = (L, \leqslant, ', 0, 1)$ with a unary operation $' : L \to L$ (an *orthocomplementation*) such that the following conditions are satisfied for all $a, b, c \in L$:

(i) $a \leqslant b \Rightarrow b' \leqslant a'$;
(ii) $(a')' = a$;
(iii) $a \in L, a \vee a' = 1$;
(iv) $a, b \in L, a \leqslant b' \Rightarrow a \vee b$ exists in L;
(v) (*orthomodular law*) $a \leqslant b \Rightarrow \exists\, c \in L$ such that $c \leqslant a'$ and $a \vee c = b$.

If an OMP $\mathbf{L}$ is a lattice, we call it an *orthomodular lattice* (OML). For more information about OMLs see, for example, (Varadarajan 1968; Dvurečenskij and Pulmannová 2000).

We say that two elements a, b of an OML $\mathbf{L}$ are *compatible*, and we write $a \leftrightarrow b$, if there are three mutually orthogonal elements $a_1, b_1, c \in L$ such that $a = a_1 \vee c$ and $b = b_1 \vee c$. An OML $\mathbf{L}$ is a Boolean algebra iff the distributivity law holds, equivalently, $a \leftrightarrow b$ for all $a, b \in L$. If M is a maximal set of mutually compatible elements of $\mathbf{L}$, it is a Boolean algebra, and L can be covered by a system of Boolean algebras. In any block, the system has a so called classical character, and in general a quantum character.

For example, let H be a real, complex or quaternionic Hilbert space and let $\mathscr{L}(H)$ be the system of all closed subspaces of H endowed with set-theoretic inclusion, and let $M' := M^{\perp} = \{x \in H : x \perp y, \ \forall\, y \in M\}$. Then $\mathscr{L}(H); \leq,', \{0\}, H)$ is an OML which is not a Boolean algebra.

The OML $\mathscr{L}(H)$ is one of the most important examples of OML's. It is the basis of the so called Hilbert space quantum mechanics.

A little bit more general structure than a Hilbert space is an inner product space S which is not necessarily complete in the metric sense. We can define two systems of closed subspaces of S: $\mathscr{E}(S) := \{M \subseteq S : M + M^{\perp} = S\}$, the system of *splitting subspaces*, and $\mathscr{F}(S) := \{M \in S : M^{\perp\perp} = M\}$, the system of *orthogonally closed subspaces*. Then $\mathscr{E}(S) \subseteq \mathscr{F}(S)$, $\mathscr{E}(S)$ is always an OMP, but not necessarily an OML, and $\mathscr{F}(S)$ is a complete lattice, where the orthomodular law can fail: it is an OML iff S is a Hilbert space, iff the orthomodular law holds for $\mathscr{F}(S)$. Equivalently, S is a Hilbert space iff $\mathscr{E}(S) = \mathscr{F}(S)$. For more information on these spaces, see Dvurečenskij (1993).

Finally, an *orthoalgebra* (OA) is a set A containing two special elements 0, 1 and equipped with a partial binary operation $+$ satisfying, for all $a, b, c \in A$, the following conditions:

(OA1) If $a+b$ is defined, then $b+a$ is defined and $a+b = b+a$ (commutativity);
(OA2) If $a+b$ and $(a+b)+c$ are defined, then $b+c$ and $a+(b+c)$ are defined, and $(a+b)+c = a+(b+c)$ (associativity);
(OA3) For every $a \in A$ there is a unique $b \in A$ such that $a+b$ is defined and $a+b = 1$ (orthocomplementation);
(OA4) If $a+a$ is defined, then $a = 0$ (consistency).

9.4.3 *Effect Algebras*

Kôpka and Chovanec introduced in Kôpka and Chovanec (1994) D-posets, where the primary operation is a subtraction of two comparable events. An equivalent structure is an *effect algebra* where a primary notion is a partial operation of addition of mutually "excluding" events, see Foulis and Bennett (1994).

Following to Foulis and Bennett (1994), we say that an *effect algebra* is a partial algebra $\mathbf{E} = (E; +, 0, 1)$ with a partially defined operation $+$ and two constant elements 0 and 1 such that, for all $a, b, c \in E$,

(i) $a+b$ is defined in E if and only if $b+a$ is defined, and in such a case $a+b = b+a$;
(ii) $a+b$ and $(a+b)+c$ are defined if and only if $b+c$ and $a+(b+c)$ are defined, and in such a case $(a+b)+c = a+(b+c)$;
(iii) for any $a \in E$, there exists a unique element $a' \in E$ such that $a + a' = 1$;
(iv) if $a+1$ is defined in E, then $a = 0$.

If we define $a \leq b$ if and only if there exists an element $c \in E$ such that $a+c = b$, then $\leq$ is a partial ordering on E, and we write $c := b - a$; then $a' = 1 - a$ for any $a \in E$. As a basic source of information about effect algebras we can recommend the monograph Dvurečenskij and Pulmannová (2000). An effect algebra is not necessarily a lattice ordered set.

We show two important classes of effect algebras. (1) If E is a system of fuzzy sets on Ω, that is, $E \subseteq [0, 1]^{\Omega}$, such that (i) $1 \in E$, (ii) $f \in E$ implies $1 - f \in E$, and (iii) if $f, g \in E$ and $f(\omega) \leq 1 - g(\omega)$ for any $\omega \in \Omega$, then $f + g \in E$, then E is an effect algebra of fuzzy sets that need not be a Boolean algebra as well as not a lattice. (2) If G is an Abelian partially ordered group written additively, $u \in G^{+}$, then $\Gamma(G, u) := [0, u] = \{g \in G : 0 \leq g \leq u\}$ is the universe of an effect algebra $\boldsymbol{\Gamma}(G, u) = (\Gamma(G, u); +, 0, 1)$ with $0 = 0$, $1 = u$ and $+$ is the group addition of elements if it exists in $\Gamma(G, u)$. Such effect algebras are said to be *interval effect algebras*.

For example, let $\mathscr{B}(H)$ denote the system of all Hermitian operators of a Hilbert space H. We can define a partial order $\leq$ for two Hermitian operators A and B such that $A \leq B$ iff $(Ax, x) \leq (Bx, x)$ for any $x \in H$. Then $\mathscr{B}(H)$ is a po-group and the identity operator I is a strong unit of $\mathscr{B}(H)$. The effect algebra $\mathscr{E}(H) := \Gamma(\mathscr{B}(H), I)$ is interval, and it is important for measurements in quantum mechanics via the so called POV-measures.

Any Boolean algebra, OML, OMP, orthoalgebra is an effect algebra. For example, if $\mathbf{E}$ is an OMP, we set $a + b = a \vee b$ whenever $a \leq b'$. By Dvurečenskij and Pulmannová (2000, Sect 1.5), we have (i) an orthoalgebra E is an OMP iff $a \vee b$ is defined whenever $a + b$ is defined in E, in which case $a + b = a \vee b$. (ii) An effect algebra is an OMP such that $a + b = a \vee b$ iff the existence of $x + y, x + z, y + z$ entails $x + y + z$ is defined in E. (iii) An effect algebra is an orthoalgebra iff $a \wedge a' = 0$ for every $a \in E$, iff $a + a \in E$ implies $a = 0$.

We say that an effect algebra $\mathbf{E}$ satisfies the Riesz Decomposition Property (RDP for short) if for all $a_1, a_2, b_1, b_2 \in E$ such that $a_1 + a_2 = b_1 + b_2$, there are four elements $c_{11}, c_{12}, c_{21}, c_{22}$ such that $a_1 = c_{11} + c_{12}$, $a_2 = c_{21} + c_{22}$, $b_1 = c_{11} + c_{21}$ and $b_2 = c_{12} + c_{22}$.

We say that a po-group G satisfies the *interpolation property* if whenever $a_1, a_2 \leq b_1, b_2$, there exists an element $c \in G$ such that $a_1, a_2 \leq c \leq b_1, b_2$. We note that RDP fails for $\mathscr{E}(H)$. But if H is a complex separable Hilbert space, $\mathscr{E}(H)$ can be covered by a system of maximal systems of mutually commuting operators, Pulmannová (2002); every such a maximal system is an MV-algebra, so that RDP holds locally in $\mathscr{E}(H)$.

The basic result on effect algebras with RDP is the following result by Ravindran (1996) which says that every effect algebra with RDP is interval.

Theorem 9.24 *If* $\mathbf{E}$ *is an effect algebra with* RDP, *there exists a unique Abelian unital po-group with interpolation* (G, u) *(up to isomorphism of unital po-groups) such that* $\mathbf{E} \cong \boldsymbol{\Gamma}(G, u)$.

If $\mathbf{E}$ is an MV-algebra, we define a partial operation $+$ on E as follows: $a + b$ is defined iff $a \odot b = 0$ (equivalently, $a \leq b^{*}$), in which case $a + b = a \oplus b$. It is possible to show that $\mathbf{E} = (E; +, 0, 1)$ is an effect algebra with RDP. Conversely, any lattice ordered effect algebra $\mathbf{E}$ with RDP is in fact term equivalent to an MV-algebra.

We say that two elements a and b in an effect algebra $\mathbf{E}$ are compatible, if there are three elements $a_1, b_1, c \in E$ such that $a = a_1 + c$, $b = b_1 + c$, and $a_1 + a_2 + c$ is defined in E. For example, if E satisfies RDP, then every two elements of E are compatible. The maximal system of mutually compatible elements of a lattice ordered effect algebra E, called a *block*, is an MV-algebra, and E can be covered by MV-algebras, see Riečanová (2000).

In a similar way as for perfect MV-algebras, we can define perfect effect algebras with RDP, and show that the category of perfect effect algebras is categorically equivalent to the category of directed Abelian po-groups with interpolation, see Dvurečenskij (2007); a typical perfect effect algebra is of the form $\Gamma(\mathbb{Z} \overrightarrow{\times} G, (1,0))$ for some directed Abelian po-group G with interpolation.

9.4.4 Pseudo Effect Algebras

Pseudo effect algebras are a non-commutative generalization of effect algebras and they were introduced in (Dvurečenskij and Vetterlein 2001a, b).

We say that a *pseudo effect algebra* is a partial algebra $\mathbf{E} = (E; +, 0, 1)$, where $+$ is a partial binary operation and 0 and 1 are constants, such that for all $a, b, c \in E$, the following holds

(i) $a + b$ and $(a + b) + c$ exist if and only if $b + c$ and $a + (b + c)$ exist, and in this case $(a + b) + c = a + (b + c)$;
(ii) there is exactly one $d \in E$ and exactly one $e \in E$ such that $a + d = e + a = 1$;
(iii) if $a + b$ exists, there are elements $d, e \in E$ such that $a + b = d + a = b + e$;
(iv) if $1 + a$ or $a + 1$ exists, then $a = 0$.

If we define $a \le b$ if and only if there exists an element $c \in E$ such that $a + c = b$, then $\le$ is a partial ordering on E such that $0 \le a \le 1$ for any $a \in E$. It is possible to show that $a \le b$ if and only if $b = a + c = d + a$ for some $c, d \in E$. We write $c = a \,/\, b$ and $d = b \setminus a$. Then

$$(b \setminus a) + a = a + (a \,/\, b) = b,$$

and we write $a^- = 1 \setminus a$ and $a^\sim = a \,/\, 1$ for any $a \in E$.

For basic properties of pseudo effect algebras see (Dvurečenskij and Vetterlein 2001a, b). A pseudo effect algebra is an effect algebra iff $+$ is commutative. Every pseudo MV-algebra can be viewed as a pseudo effect algebra, see (Dvurečenskij and Vetterlein 2001a, b), in the same way as MV-algebras are viewed as effect algebras.

For example, if (G, u) is a unital (not necessarily Abelian) po-group with strong unit u, and

$$\Gamma(G, u) := \{g \in G : \ 0 \le g \le u\},$$

then $\boldsymbol{\Gamma}(G, u) = (\Gamma(G, u); +, 0, u)$ is a pseudo effect algebra if we restrict the group addition $+$ to $\Gamma(G, u)$. Every pseudo effect algebra $\mathbf{E}$ that is isomorphic to some

$\boldsymbol{\Gamma}(G, u)$ is said to be an *interval pseudo effect algebra*. Every interval effect algebra admits a state. This is not true in general for interval pseudo effect algebras, neither for pseudo MV-algebras.

For pseudo effect algebras we can define three different types of the Riesz Decomposition Properties, denoted RDP, RDP_1, and RDP_2. It turns out that RDP and RDP_1 coincide for effect algebras. According to (Dvurečenskij and Vetterlein 2001a, b), we say that an effect algebra $\mathbf{E}$ satisfies

(a) the Riesz Decomposition Property (RDP for short) if, for all $a_1, a_2, b_1, b_2 \in E$ such that $a_1 + a_2 = b_1 + b_2$, there are four elements $c_{11}, c_{12}, c_{21}, c_{22}$ such that $a_1 = c_{11} + c_{12}$, $a_2 = c_{21} + c_{22}$, $b_1 = c_{11} + c_{21}$ and $b_2 = c_{12} + c_{22}$;
(b) RDP_1 if it satisfies RDP, and $x \le c_{12}$ and $y \le c_{21}$ imply $x + y$, $y + x$ exist and $x + y = y + x$;
(c) RDP_2 if it satisfies RDP, and $c_{12} \wedge c_{21} = 0$.

Then RDP_2 implies RDP_1, and RDP_1 implies RDP, but the converse is not true, in general. If G is now a po-group, not necessarily Abelian, we define the analogous RDP's also for G in the same way as we did for pseudo effect algebras, where now the elements $a_1, a_2, b_1, b_2, c_{11}, c_{12}, c_{21}, c_{22}, x, y$ belong to the positive cone G^+.

Theorem 9.25 *For every pseudo effect algebra* $\mathbf{E}$ *with* RDP_1*, there is a unique unital po-group* (G, u) *not necessarily Abelian (up to isomorphism of unital po-groups) satisfying* RDP_1 *such that* $\mathbf{E} \cong \boldsymbol{\Gamma}(G, u)$.

In addition, there is a categorical equivalence between the category of pseudo effect algebras with RDP_1 *and the category of unital po-groups satisfying* RDP_1.

Theorem 9.26 (1) *Every pseudo effect algebra with* RDP_2 *is a lattice, and it can be transformed into a pseudo MV-algebra using* $a \oplus b := (b^- \setminus (a \wedge b^-))^\sim$.

(2) *A pseudo effect algebra with* RDP_1 *satisfies* RDP_2 *if and only if it is a lattice.*

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Chapter 10
Semi-linear Varieties of Lattice-Ordered Algebras

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10.1 Introduction

A variety $\mathscr{V}$ of lattice-ordered algebras is said to be *semi-linear* in case it is generated by its totally ordered members (in more traditional algebraic parlance, the term 'representable' is often used in place of 'semi-linear'). Due to the congruence distributivity of $\mathscr{V}$, $\mathscr{V}$ is semi-linear if and only if its subdirectly irreducible members are totally ordered (Burris and Sankappanavar, 1981, Theorem 6.8, p. 165). Needless to say, semi-linearity is a welcome property insofar as it often makes a class of algebras very tractable for computation and proof purposes. Many well-understood varieties in algebraic logic are known to be semi-linear: examples include Abelian ℓ-groups and varieties arising from many-valued logic (such as MTL algebras and thus, in particular, BL algebras, MV algebras or Gödel algebras: Cintula et al. 2011). Petr Hájek, besides giving fundamental contributions to the investigation of many such classes of algebras, has repeatedly underscored the central role played by semi-linearity in fuzzy logic:

> Among the logics of residuated lattices, fuzzy logics [...] are distinguished by the property of semilinearity, i.e., completeness w.r.t. a class of linearly ordered residuated lattices. The

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main scope of mathematical fuzzy logic thus can be delimited as the study of *intuitionistic substructural semilinear logics* (Běhounek et al. 1998, p. 58).

On the other hand, one can easily find just as many important varieties that fail to be semi-linear. A prime example is given by the variety of (pointed) *residuated lattices* (Jipsen and Tsinakis 2002; Galatos et al. 2007; Metcalfe et al. 2010) and by several of its subvarieties, most notably ℓ-groups and Heyting algebras; we also mention orthomodular lattices (Bruns and Harding 2000) and interior algebras (Blok 1976). In these cases, it may be useful to be in a position to axiomatize the semi-linear subvariety[1] $\mathscr{W}$ of the variety $\mathscr{V}$ of our interest, relative to a given basis for $\mathscr{V}$—and, in fact, elegant axiomatizations have been devised in many individual cases, for example residuated lattices (Blount and Tsinakis 2003), ℓ-groups (Anderson and Feil 1988), or Heyting algebras (van Dalen 2002). Yet, it is natural to ask the following question: given a variety $\mathscr{V}$ for which an equational basis is known, is it possible to provide a general criterion for axiomatizing its semi-linear subvariety, without having to proceed on a piecemeal fashion?

We address this problem from a fairly general standpoint. In fact, we consider varieties of pointed lattice-ordered algebras obeying a restricted distribution condition and admitting a binary implication term that satisfies a minimal set of reasonable properties. Examples of these varieties are ubiquitous in algebraic logic:

1. integral residuated lattices;
2. distributive residuated lattices;
3. the $\{\cdot\}$-free subreducts of the algebras under (1) or (2);
4. expansions of the algebras under (1) or (2) by any additional signature—hence, in particular, Boolean algebras with operators and modal algebras; and
5. some varieties arising from quantum logic, e.g. Chajda et al.'s *basic algebras* (Chajda et al. 2009).

Given any such variety $\mathscr{V}$, we provide an explicit equational basis (relative to $\mathscr{V}$) for the semi-linear subvariety $\mathscr{W}$ of $\mathscr{V}$. In particular, we show that if $\mathscr{V}$ is finitely based, then so is $\mathscr{W}$. Our proof takes advantage of ideas developed in Blount and Tsinakis (2003) for residuated lattices and in Chajda and Kühr (2013) for basic algebras, generalizing them to a more abstract setting. This is in line with the approach taken by (van van Alten 2013), who, using different techniques, provides a distinct axiomatization of the prelinear subquasivariety of a given quasivariety of lattice-ordered algebras.

To attain this goal, we put to good use some tools from the theory of *quasi-subtractive varieties* (Kowalski et al. 2011), a generalization of Gumm's and Ursini's subtractive varieties (Gumm and Ursini 1984), introduced to account for some known isomorphism theorems between ideal and congruence lattices that are not corollaries of general theorems in the theory of subtractive varieties. The required machinery is

[1] From now on, when we speak of *the* semilinear subvariety of a given variety $\mathscr{V}$, we invariably mean its *largest* semilinear subvariety. This is the variety generated by all totally ordered members of $\mathscr{V}$, equivalently, all totally ordered subdirectly irreducible members of $\mathscr{V}$.

briefly illustrated in Sect. 10.2. The following Sect. 10.3, is devoted to the introduction of the concept of an LI-algebra and to the proof of our main result. A final section discusses some special cases and applications of our criterion.

10.2 Preliminaries on Quasi-Subtractive Varieties

All the results mentioned in this section are stated without a proof; all the relevant proofs can be found in Kowalski et al. (2011).

A variety $\mathscr{V}$, of signature ν, such that there exists an essentially nullary term 1 that is equationally definable in $\mathscr{V}$ over ν, is 1-*subtractive* (or simply *subtractive* when no ambiguity is possible) if there is a binary term of signature ν, denoted by $\rightarrow$ and written in infix notation, such that $\mathscr{V}$ satisfies the following equations:

S1 $x \rightarrow x \approx 1$
S2 $1 \rightarrow x \approx x$

$\mathscr{V}$ is called 1-*permutable* if for any algebra $\mathbf{A} \in \mathscr{V}$ and for any congruences θ, φ of $\mathbf{A}$, $[1^{\mathbf{A}}]_{\theta \circ \varphi} = [1^{\mathbf{A}}]_{\varphi \circ \theta}$, where$[1^{\mathbf{A}}]_{\theta \circ \varphi}$ and $[1^{\mathbf{A}}]_{\varphi \circ \theta}$ denote the equivalence classes of $1^{\mathbf{A}}$ relative to the congruences $\theta \circ \varphi$ and $\varphi \circ \theta$, respectively. In their paper Gumm and Ursini (1984), Gumm and Ursini essentially observe that a variety $\mathscr{V}$ with 1 is 1-permutable iff it is 1-subtractive.

In Kowalski et al. (2011), the next generalization of the preceding concept was suggested:

Definition 10.1 A variety $\mathscr{V}$, of signature ν, such that there exists a nullary term 1 and a unary term $\Box$ of the same signature, equationally definable in $\mathscr{V}$, is called *quasi-subtractive* with respect to 1 and $\Box$ iff there exists a binary term $\rightarrow$ (hereafter written in infix notation) of signature ν such that $\mathscr{V}$ satisfies the following equations:

Q1 $\Box x \rightarrow x \approx 1$
Q2 $1 \rightarrow x \approx \Box x$
Q3 $\Box (x \rightarrow y) \approx x \rightarrow y$
Q4 $\Box (x \rightarrow y) \rightarrow (\Box x \rightarrow \Box y) \approx 1$

Observe that, given Q3, Q4 is equivalent to $(x \rightarrow y) \rightarrow (\Box x \rightarrow \Box y) \approx 1$. Although the latter equation is simpler, Q4 is more reminiscent of the K axiom for modal algebras. On occasion, we will say that "$\rightarrow$ witnesses quasi-subtractivity with respect to 1 and $\Box$ for $\mathscr{V}$", possibly using some stylistical variants of this expression. Members of quasi-subtractive varieties will be called, by extension, quasi-subtractive as well.

In their article on assertionally equivalent quasivarieties (Blok and Raftery 2008), Blok and Raftery introduce a notion of τ-*class* that relativizes the usual notion of congruence class to a given *translation*, namely, to a finite set of equations in a single variable. If $\mathscr{V}$ is a variety of type ν, $\mathbf{A} \in \mathscr{V}, \theta \subseteq A^2$ and $\tau(x) = \{\delta_i(x) \approx \varepsilon_i(x) : i \leq n\}$ is a function from the formula algebra $\mathbf{Fm}$ of type ν to $\wp(\mathbf{Fm} \times \mathbf{Fm})$, the τ-*class* of θ in $\mathbf{A}$—in symbols $[\tau^{\mathbf{A}}]_\theta$—is defined as

$$[\tau^{\mathbf{A}}]_\theta = \left\{a \in \mathbf{A} : \delta_i^{\mathbf{A}}(a)\,\theta\,\varepsilon_i^{\mathbf{A}}(a) \text{ for every } i \leq n\right\}.$$

A variety $\mathscr{V}$ is said to be *τ-regular* if for any congruences θ, φ on any $\mathbf{A} \in \mathscr{V}$, $[\tau^{\mathbf{A}}]_\theta = [\tau^{\mathbf{A}}]_\varphi$ implies $\theta = \varphi$; if $\tau(x) = \{x \approx 1\}$, we get as a special case the standard notion of 1-regularity.

As shown in (Blok and Raftery 1999, Theorem 5.2), τ-regularity is a Mal'cev property: a variety $\mathscr{V}$ is τ-regular for $\tau(x) = \{\delta_i(x) \approx \varepsilon_i(x) : i \leq m\}$ iff there exist binary terms $p_1, \ldots, p_n$ such that:

$$\begin{aligned} &\vDash_{\mathscr{V}} \tau(p_j(x,x)), \text{ for } j \leq n; \text{ and} \\ &\{\tau(p_j(x,y)) : j \leq n\} \vDash_{\mathscr{V}} x \approx y, \end{aligned} \tag{10.1}$$

where $\{t_i \approx s_i : i \leq n\} \vDash_{\mathscr{V}} t \approx s$ means that for all $\mathbf{A} \in \mathscr{V}$ and for all $a \in A$, if $t_i^{\mathbf{A}}(a) = s_i^{\mathbf{A}}(a)$ for every $i \leq n$, then $t^{\mathbf{A}}(a) = s^{\mathbf{A}}(a)$. In case $m = 1$ (and rewriting δ_1 as δ and ε_1 as ε), we are also guaranteed (Barbour and Raftery 1997, Theorem 5.2) that there exist $(2n+2)$-ary terms $t_1, \ldots, t_k$ such that $\mathscr{V}$ satisfies the identities

$$\begin{aligned} &x \approx t_1\left(x, y, \delta\left(\overrightarrow{p(x,y)}\right), \varepsilon\left(\overrightarrow{p(x,y)}\right)\right) \\ &t_j\left(x, y, \varepsilon\left(\overrightarrow{p(x,y)}\right), \delta\left(\overrightarrow{p(x,y)}\right)\right) \approx t_{j+1}\left(x, y, \delta\left(\overrightarrow{p(x,y)}\right), \varepsilon\left(\overrightarrow{p(x,y)}\right)\right), \\ &(1 \leq j < k) \\ &t_k\left(x, y, \varepsilon\left(\overrightarrow{p(x,y)}\right), \delta\left(\overrightarrow{p(x,y)}\right)\right) \approx y, \end{aligned} \tag{10.2}$$

where $\delta\left(\overrightarrow{p(x,y)}\right)$ is an abbreviation for the sequence $\delta(p_1(x,y)), \ldots, \delta(p_n(x,y))$, and similarly for $\varepsilon\left(\overrightarrow{p(x,y)}\right)$. A third equivalent characterization of τ-regularity is as follows: $\mathscr{V}$ is τ-regular in case its *τ-assertional logic*, whose consequence relation $\vdash_{\mathscr{V}}$ is defined by

$$\Gamma \vdash_{\mathscr{V}} t \text{ iff } \{\tau(s) : s \in \Gamma\} \vDash_{\mathscr{V}} \tau(t),$$

is strongly and finitely algebraizable with $\mathscr{V}$ as equivalent variety semantics.

Blok and Raftery also consider a property of τ-permutability appropriately generalizing the notion of 1-permutability to varieties which need not be pointed: a variety $\mathscr{V}$ is *τ-permutable* iff for any congruences θ, φ on any $\mathbf{A} \in \mathscr{V}$, $[\tau^{\mathbf{A}}]_{\theta \circ \varphi} = [\tau^{\mathbf{A}}]_{\varphi \circ \theta}$. Every quasi-subtractive variety is $\{\Box x \approx 1\}$-permutable, while the converse statement need not hold (Kowalski et al. 2011). For the sake of brevity, the notation "$\{\Box x \approx 1\}$" will be streamlined to "$(\Box x, 1)$" in every relevant context.

Every 1-subtractive variety with witness term $\rightarrow$ is automatically quasi-subtractive with witness terms $\rightarrow$, 1, and the identity term as box. The table on the next page lists some other examples of quasi-subtractive varieties. Observe that some of these varieties are indeed subtractive but can be viewed as *properly* quasi-subtractive with a different choice of witness terms.

Variety	Ref.	$x \to y$	$\Box x$	1-Subtr.?
Residuated lattices	Galatos et al. (2007)	$(x\backslash y) \wedge 1$	$x \wedge 1$	Yes
Quasi-MV algebras	Ledda et al. (2006)	$x' \oplus y$	$x \oplus 0$	No
Var. with a comm. TD term	Blok and Pigozzi (1994)	$p(x, p(x, y, x), 1)$	$p(x, 1, 1)$	
Pseudointerior algebras	Blok and Pigozzi (1994)	$x \to y$	x°	No
Interior algebras	Blok (1976)	$\Box(\neg x \vee y)$	$\Box x$	Yes
Integral k-potent res. lattices	Galatos et al. (2007)	$(x\backslash y)^k$	x^k	Yes

The next concept of open filter is as central for the investigation of quasi-subtractive varieties as the Gumm-Ursini concept of ideal is for the investigation of subtractive varieties:

Definition 10.2 Let $\mathscr{V}$ be a variety whose signature ν is as in Definition 10.1. A *$\mathscr{V}$-open filter term* in the variables $\overrightarrow{x}$ is an $n+m$-ary term $p\left(\overrightarrow{x}, \overrightarrow{y}\right)$ of signature ν such that:

$$\{\Box x_i \approx 1 : i \leq n\} \vDash_{\mathscr{V}} \Box p\left(\overrightarrow{x}, \overrightarrow{y}\right) \approx 1.$$

The wording "$\mathscr{V}$-open filter term" will be simplified to " open filter term" whenever this replacement is unambiguous. The same applies to "$\mathscr{V}$-open filter" below.

Definition 10.3 Let $\mathscr{V}$ be as in Definition 10.2. A *$\mathscr{V}$-open filter* of $\mathbf{A} \in \mathscr{V}$ is a subset $F \subseteq A$ with the following properties:

(i) F is closed with respect to all $\mathscr{V}$-open filter terms p: whenever $a_1, ..., a_n \in F, b_1, ..., b_m \in A$, $p\left(\overrightarrow{a}, \overrightarrow{b}\right) \in F$;
(ii) for every $a \in A$, we have that $a \in F$ iff $\Box a \in F$.

Observe that 1 is a member of any open filter since the constant term 1 is an open filter term.

In the theory of subtractive varieties, ideal generation can be nicely described. A similar result holds for open filters. If $\mathbf{A}$ is any algebra in a variety $\mathscr{V}$ of the appropriate signature, and we define for $X \subseteq A$:

$$\uparrow X = X \cup \{a : \Box a \in X\};$$

$$\Gamma X = \left\{p^{\mathbf{A}}\left(\overrightarrow{a}, \overrightarrow{b}\right) : \overrightarrow{a} \in X, \overrightarrow{b} \in A, p \text{ an open filter term}\right\},$$

we get:

Lemma 10.1 *Let $\mathscr{V}$ be a quasi-subtractive variety, $\mathbf{A} \in \mathscr{V}$ and $X \subseteq A$. The $\mathscr{V}$-open filter $[X)$ generated by X is precisely $\uparrow \Gamma X$.*

Among its consequences, the preceding theorem yields a characterization of joins of open filters and the following interesting property:

Lemma 10.2 *Let $\mathscr{V}$ be a quasi-subtractive variety. Then the lattice of open filters of any $\mathbf{A} \in \mathscr{V}$ is modular.*

If $\mathscr{V}$ is a 1-subtractive variety, the ideals of any $\mathbf{A} \in \mathscr{V}$ coincide with the deductive filters[2] on $\mathbf{A}$ of the 1-assertional logic of $\mathscr{V}$ (Ursini 1994); if, moreover, $\mathscr{V}$ is 1-regular, the congruence lattice of any $\mathbf{A} \in \mathscr{V}$ is isomorphic to the lattice of such deductive filters and, therefore, to its ideal lattice (Czelakowski 1981). What happens, instead, if the variety at issue is quasi-subtractive with respect to $\Box$ and 1 and $(\Box x, 1)$-regular? The next result is an analogue of Ursini's result for subtractive varieties:

Lemma 10.3 *If $\mathscr{V}$ is a quasi-subtractive variety and $\mathbf{A} \in \mathscr{V}$, then the $\mathscr{V}$-open filters of $\mathbf{A}$ coincide with the deductive filters on $\mathbf{A}$ of the $(\Box x, 1)$-assertional logic of $\mathscr{V}$.*

With this, we are halfway through our task. For the remaining half, we make a note of a result essentially due to Blok and Pigozzi (1989), although they focus on the more general scenario of an arbitrary translation τ:

Theorem 10.1 *If $\mathscr{V}$ is $(\Box x, 1)$-regular, then the congruence lattice of any $\mathbf{A} \in \mathscr{V}$ is isomorphic to the lattice of deductive filters on $\mathbf{A}$ of the $(\Box x, 1)$-assertional logic of $\mathscr{V}$.*

By Lemma 10.3 and Theorem 10.1, we get:

Corollary 10.1 *If $\mathscr{V}$ is quasi-subtractive and $(\Box x, 1)$-regular, then in any $\mathbf{A} \in \mathscr{V}$ there is a lattice isomorphism between the congruence lattice of $\mathbf{A}$ and the lattice of $\mathscr{V}$-open filters on $\mathbf{A}$.*

Besides generalizing the correspondence theorem for ideal determined varieties, Corollary 10.1 subsumes many lattice isomorphism results that do not follow from the theorem itself, to be found e.g. in the theories of residuated lattices,[3] of pseudointerior algebras, or of quasi-MV algebras.

10.3 Axiomatizing the Semi-Linear Subvariety

As a first step towards our goal, we need an umbrella heading that encompasses the varieties of lattice-ordered algebras of our interest. Therefore, we introduce the concept of *LI-algebra*, a label whose 'L' should be suggestive of 'lattice' and whose 'I' should remind of 'implication'.

Definition 10.4 An *LI-algebra* is an algebra $\mathbf{A}$ that has a term reduct $(A, \wedge, \vee, \rightsquigarrow, 1)$ of signature $(2, 2, 2, 0)$ such that:

[2] If $F \subseteq A$ and $\mathbf{A}$ has the same signature as $\mathbf{Fm}$, F is said to be a deductive filter on $\mathbf{A}$ of the logic $(\mathbf{Fm}, \vdash)$ just in case F is closed with respect to all the $\vdash$-entailments: if $\Gamma \vdash t$ and $s^{\mathbf{A}}\left(\overrightarrow{a}\right) \in F$ for all $s \in \Gamma$, then $t^{\mathbf{A}}\left(\overrightarrow{a}\right) \in F$.

[3] The variety of residuated lattices is actually 1-ideal determined and, in fact, in every residuated lattice the lattice of congruences is isomorphic to the lattice of ideals in the sense of Gumm-Ursini, which in turn coincide with convex normal subalgebras of such. There is a further isomorphism theorem, however (namely, between congruences and *deductive filters* in the sense of Galatos et al. (2007)), which does not instantiate the correspondence theorem for ideal determined varieties, but follows from Corollary 10.1.

- $(A, \wedge, \vee, 1)$ is a pointed lattice satisfying:

$$\text{(D):}\ \ (x \vee y) \wedge 1 \approx (x \wedge 1) \vee (y \wedge 1)$$

- The following conditions concerning $\rightsquigarrow$ are satisfied:

$$\begin{aligned}&(\text{A1})x \rightsquigarrow y \approx 1 \text{ iff } x \leq y\\&(\text{A2})1 \rightsquigarrow x \approx x \wedge 1\\&(\text{A3})x \vee y \rightsquigarrow z \approx (x \rightsquigarrow z) \wedge (y \rightsquigarrow z)\\&(\text{A4})z \rightsquigarrow x \wedge y \approx (z \rightsquigarrow x) \wedge (z \rightsquigarrow y)\end{aligned}$$

We assume that lattice operations bind more strongly than $\rightsquigarrow$. Let us now exemplify the preceding definition.

Example 10.1 (Residuated lattices). Recall that a *residuated lattice* is an algebra $\mathbf{A} = (A, \cdot, \wedge, \vee, \backslash, /, 1)$ such that (i) $(A, \cdot, 1)$ is a monoid, (ii) $(A, \wedge, \vee)$ is a lattice, and (iii) for all $x, y, z \in A$, $xy \leq z \iff x \leq z/y \iff y \leq x\backslash z$. A *pointed residuated lattice* is an algebra $\mathbf{A} = (A, \cdot, \wedge, \vee, \backslash, /, 1, 0)$ such that $(A, \cdot, \wedge, \vee, \backslash, /, 1)$ is a residuated lattice and 0 is a nullary operation. Residuated lattices and hence pointed residuated lattices form finitely based equational classes of algebras (Blount and Tsinakis 2003).

Not all residuated lattices can be viewed as instances of LI-algebras, because they fail, in general, to satisfy (D). However, all distributive (pointed) residuated lattices and all integral (pointed) residuated lattices are LI-algebras with $x \rightsquigarrow y = x\backslash y \wedge 1$. Therefore the class of LI-algebras includes, in particular: ℓ-groups; MTL algebras (thus, BL algebras, MV algebras and product algebras); Heyting algebras; and Sugihara algebras.

Example 10.2 (Subreducts of residuated lattices). Observe that nothing in Definition 10.4 hinges on the presence of a monoidal operation whose residual is $\rightsquigarrow$. Consequently, this definition equally applies to all the $(\wedge, \vee, \backslash, /, 1)$-subreducts of the residuated lattices in Example 10.1 (see van Alten and Raftery (2004) for a detailed study of these and other subreducts in the commutative case).

Example 10.3 (Expansions of residuated lattices). The property of being an LI-algebra is obviously preserved upon arbitrary expansions of the signature. As a result, any expansion of any residuated lattice in Example 10.1 continues to be a LI-algebra. In particular, Boolean algebras with operators and modal algebras make instances of our concept.

Example 10.4 (Basic algebras). Basic algebras were introduced in Chajda et al. (2009) as algebras arising from lattices with sectionally antitone involutions. The theory of basic algebras presents connections with the theories of MV algebras (which can be viewed as associative basic algebras), orthomodular lattices, and lattice-ordered effect algebras. Basic algebras are LI-algebras with $x \rightsquigarrow y = \neg x \oplus y$.

Throughout the rest of this chapter, $\mathscr{V}$ will refer to a generic variety of LI-algebras. In the next lemmas, we list some arithmetical properties of $\mathscr{V}$.

Lemma 10.4 *Let* $\mathbf{A} \in \mathscr{V}$*, and let* $a, b \in A$*. The following equalities hold:*

(i) $a \rightsquigarrow b = (a \rightsquigarrow b) \wedge 1$
(ii) $a \wedge 1 \rightsquigarrow b = a \wedge 1 \rightsquigarrow b \wedge 1$
(iii) $a \leq 1$ *implies* $a \rightsquigarrow b = a \rightsquigarrow b \wedge 1$
(iv) $(a \rightsquigarrow b) \wedge 1 \leq a \wedge 1 \rightsquigarrow b \wedge 1$

Proof (i) By (A1), (A3) and absorption, $(a \rightsquigarrow b) \wedge 1 = (a \rightsquigarrow b) \wedge (a \wedge b \rightsquigarrow b) = a \vee (a \wedge b) \rightsquigarrow b = a \rightsquigarrow b$.

(ii) By (A1), (A4) and (i), $a \wedge 1 \rightsquigarrow b \wedge 1 = (a \wedge 1 \rightsquigarrow b) \wedge (a \wedge 1 \rightsquigarrow 1) = (a \wedge 1 \rightsquigarrow b) \wedge 1 = a \wedge 1 \rightsquigarrow b$.

(iii) From (ii); (iv) By absorption and (A3),

$$a \rightsquigarrow b = a \vee (a \wedge 1) \rightsquigarrow b = (a \rightsquigarrow b) \wedge (a \wedge 1 \rightsquigarrow b),$$

whence by (i) and (ii)

$$(a \rightsquigarrow b) \wedge 1 = a \rightsquigarrow b \leq a \wedge 1 \rightsquigarrow b = a \wedge 1 \rightsquigarrow b \wedge 1. \qquad \square$$

Lemma 10.5 $\mathscr{V}$ *satisfies the quasiequation* $x \vee y \approx 1 \Longrightarrow x \rightsquigarrow y \approx y$.

Proof Let $a, b \in \mathbf{A} \in \mathscr{V}$. Then $a \rightsquigarrow b = (a \rightsquigarrow b) \wedge 1 = (a \rightsquigarrow b) \wedge (b \rightsquigarrow b) = a \vee b \rightsquigarrow b = 1 \rightsquigarrow b = b \wedge 1 = b$, for $b \leq a \vee b = 1$. □

The crucial observation that paves the way for an application of the results in Sect. 10.2 is the fact that $\mathscr{V}$ is quasi-subtractive and $(\Box x, 1)$-regular:

Lemma 10.6 $\mathscr{V}$ *is quasi-subtractive with respect to* 1 *and* $\Box x = x \wedge 1$*, as witnessed by* $x \rightarrow y = x \rightsquigarrow y$*; moreover,* $\mathscr{V}$ *is* $(\Box x, 1)$*-regular with respect to the same constant* 1 *and the same unary term* $\Box x$.

Proof To show that $\mathscr{V}$ is quasi-subtractive, we need to check one by one the four conditions under Definition 10.1. However, (Q1) follows from (A1); (Q2) is exactly (A2); (Q3) amounts to Lemma 10.4.(i); finally, (Q4) follows from Lemma 10.4.(iv) and (A1).

Now, let us consider Eq. (10.1) with $n = 2, m = 1, \delta_1(x) = x \wedge 1, \varepsilon_1(x) = 1, p_1(x, y) = x \rightsquigarrow y$ and $p_2(x, y) = y \rightsquigarrow x$. It is easy to check that this choice of witness terms vouches for the $(\Box x, 1)$-regularity of $\mathscr{V}$, given (A1) and Lemma 10.4 (i). □

Corollary 10.2 *If* $\mathbf{A} \in \mathscr{V}$*, the congruence lattice of* $\mathbf{A}$ *is isomorphic to the lattice of open filters of* $\mathbf{A}$.

Proof By Lemma 10.6 and Corollary 10.1. □

In the following, we consider the equation

$$\text{(S1)} \quad \left(t\left(z \rightsquigarrow x_1, ..., z \rightsquigarrow x_n, \overrightarrow{y}\right) \wedge 1\right) \vee (x_1 \rightsquigarrow z) \vee ... \vee (x_n \rightsquigarrow z) \approx 1$$

which is actually a family of equations, one for each $\mathscr{V}$-open filter term $t\left(\overrightarrow{x}, \overrightarrow{y}\right)$ in the variables $\overrightarrow{x}$. Unwieldy as it may seem, (S1) can however be broken down into a conjunction of two more manageable conditions (cf. van Alten 2013).

Lemma 10.7 *(S1) is equivalent to the conjunction of*

$$(Prel)\ (x \rightsquigarrow y) \vee (y \rightsquigarrow x) \approx 1$$

and

$$(Q)\, x_1 \vee z \geq 1\&...\&x_n \vee z \geq 1 \Longrightarrow t\left(\overrightarrow{x}, \overrightarrow{y}\right) \vee z \geq 1.$$

Proof From left to right, observe that (Prel) is a special case of (S1), because x is an open filter term and $x \rightsquigarrow y = (x \rightsquigarrow y) \wedge 1$. Moreover, let $a_i \vee c \geq 1$ for all $i \leq n$, whence $(a_i \vee c) \wedge 1 = 1$ and, by (D), $(a_i \wedge 1) \vee (c \wedge 1) = 1$. Applying Lemma 10.4 (ii) and Lemma 10.5, $c \wedge 1 \rightsquigarrow a_i = c \wedge 1 \rightsquigarrow a_i \wedge 1 = a_i \wedge 1$. Since open filters are generated by their open elements (Lemma 10.1), we are allowed to pick $a_i \leq 1$ (recall that $\Box a = a \wedge 1$), whence (S1) and Lemma 10.5 again give

$$\begin{aligned}
1 &= \left(t\left(c \wedge 1 \rightsquigarrow a_1, ..., c \wedge 1 \rightsquigarrow a_n, \overrightarrow{b}\right) \wedge 1\right) \vee (a_1 \rightsquigarrow c \wedge 1) \vee ... \vee (a_n \rightsquigarrow c \wedge 1) \\
&= \left(t\left(a_1 \wedge 1, ..., a_n \wedge 1, \overrightarrow{b}\right) \wedge 1\right) \vee (a_1 \rightsquigarrow c \wedge 1) \vee ... \vee (a_n \rightsquigarrow c \wedge 1) \\
&= \left(t\left(a_1, ..., a_n, \overrightarrow{b}\right) \wedge 1\right) \vee (c \wedge 1) \\
&= \left(t\left(a_1, ..., a_n, \overrightarrow{b}\right) \vee c\right) \wedge 1
\end{aligned}$$

Conversely, replacing in (Q) the variables x_i by $z \rightsquigarrow x_i$, and the variable z by $(x_1 \rightsquigarrow z) \vee ... \vee (x_n \rightsquigarrow z)$, its consequent is exactly (S1) by Lemma 10.4 (i) and (D); its antecedent, however, follows from (Prel) for the same reasons. □

We can now state and prove the main result of this chapter:

Theorem 10.2 *The semi-linear subvariety $\mathscr{W}$ of $\mathscr{V}$ is axiomatized by (S1).*

Proof We first show that every totally ordered algebra in $\mathscr{V}$ satisfies (S1). We distinguish two cases. If there is an i such that $a_i \leq c$, then $a_i \rightsquigarrow c = 1$, whence our result follows since, by absorption and Lemma 10.4 (i),

$$\begin{aligned}
&\left(t\left(c \rightsquigarrow a_1, ..., c \rightsquigarrow a_n, \overrightarrow{b}\right) \wedge 1\right) \vee (a_1 \rightsquigarrow c) \vee ... \vee 1 \vee ... \vee (a_n \rightsquigarrow c) \\
&= (a_1 \rightsquigarrow c) \vee ... \vee 1 \vee ... \vee (a_n \rightsquigarrow c) \\
&= ((a_1 \rightsquigarrow c) \wedge 1) \vee ... \vee 1 \vee ... \vee ((a_n \rightsquigarrow c) \wedge 1) = 1
\end{aligned}$$

On the other hand, if for all i, $c \leq a_i$, then, since t is a $\mathscr{V}$-open filter term in $\overrightarrow{x}$,

$$t\left(c \rightsquigarrow a_1, ..., c \rightsquigarrow a_n, \overrightarrow{b}\right) \wedge 1 = t\left(\overrightarrow{1}, \overrightarrow{b}\right) \wedge 1 = 1,$$

whence our result, again, follows along similar lines.

It remains to prove that every subdirectly irreducible algebra in the subvariety axiomatized by (S1) is totally ordered. Let therefore **A** be such an algebra, and let $a, b \in A$ be such that $a \nleq b$ and $b \nleq a$, that is, $a \rightsquigarrow b \neq 1$ and $b \rightsquigarrow a \neq 1$. In particular, by Lemma 10.4 (i), $a \rightsquigarrow b, b \rightsquigarrow a < 1$. Owing to Lemma 10.7, **A** satisfies (Prel), whereby $(a \rightsquigarrow b) \vee (b \rightsquigarrow a) = 1$. For $a \leq 1$ in A, let

$$a^{\perp} = \{b : a \vee b \geq 1\}$$

We now show that $\{a \rightsquigarrow b\}^{\perp\perp}$ and $\{b \rightsquigarrow a\}^{\perp\perp}$ are open filters that intersect to the smallest open filter $\uparrow 1$, and strictly include it. Sets of the form $a^{\perp}$ are open filters: by Lemma 10.7 they are closed with respect to open filter terms, while it is easy to check, using (D), that if $b \wedge 1 \in a^{\perp}$, then also $b \in a^{\perp}$. Consequently, so is $B^{\perp}$, for any nonempty B, because $B^{\perp} = \bigcap \left\{b^{\perp} : b \in B\right\}$. Also $\{a \rightsquigarrow b\}^{\perp\perp}$ and $\{b \rightsquigarrow a\}^{\perp\perp}$ are nonzero, because they respectively contain the elements $a \rightsquigarrow b$ and $b \rightsquigarrow a$, both outside the positive cone. Finally, let $c \in \{a \rightsquigarrow b\}^{\perp\perp}$ and $c \in \{b \rightsquigarrow a\}^{\perp\perp}$. Then for every y, if $y \vee (a \rightsquigarrow b) \geq 1$, it follows that $c \vee y \geq 1$. In particular, for $y = b \rightsquigarrow a$, we obtain that $c \vee (b \rightsquigarrow a) \geq 1$, and similarly, $c \vee (a \rightsquigarrow b) \geq 1$. Letting now $y = c$, we obtain $c = c \vee c \geq 1$. By Corollary 10.2, then, **A** has no monolith, a contradiction. □

Although Theorem 10.2 is not sufficient to ensure that $\mathscr{W}$ is finitely based in case $\mathscr{V}$ is, we can take advantage of the following result, proved in (Kowalski et al. in press), that implies the existence of a *finite* axiomatization of $\mathscr{W}$ relative to $\mathscr{V}$, at least if $\mathscr{V}$ has a finite signature. Recall from Sect. 10.2 that $(\Box x, 1)$-regularity is a Mal'cev property, witnessed by terms $p_1, ..., p_n$. It also implies the existence of terms $t_1, ..., t_k$ abiding by the conditions specified in, respectively, Eqs. (10.1) and (10.2).

Theorem 10.3 *Let $\mathscr{K}$ be a variety of signature ν that is quasi-subtractive with respect to $\Box$ and 1, and $(\Box x, 1)$-regular. Moreover, let the former property be witnessed by the term $x \rightarrow y$ and the latter be witnessed by $p_1(x, y), ..., p_n(x, y)$. Let $t_1, ..., t_k$ be as in Eq. (10.2). Suppose, finally, that $\mathbf{A} \in \mathscr{K}$ and that $F = \uparrow F \subseteq A$ contains 1 and is closed with respect to the terms $\Box x, \Box p_1(1, \Box x), ..., \Box p_n(1, \Box x)$. Then F is closed with respect to* all *the open filters terms (and so is an open filter) iff it closed with respect to the following terms:*

- $(\Box x \rightarrow (\Box y \rightarrow \Box z)) \rightarrow z$;
- *for any $j, l \in \{0, ..., n\}$, $i \in \{1, ..., k\}$, and any m-ary $f \in \nu$:*

$$\Box p_l\left(\Box p_j\left(f(\overrightarrow{x}), f\left(\alpha_1^i, ..., \alpha_m^i\right)\right), \Box p_j\left(f(\overrightarrow{x}), f\left(\beta_1^i, ..., \beta_m^i\right)\right)\right), \text{ and}$$
$$\Box p_l\left(\Box p_j\left(\Box f(\overrightarrow{x}), \Box f\left(\alpha_1^i, ..., \alpha_m^i\right)\right), \Box p_j\left(\Box f(\overrightarrow{x}), \Box f\left(\beta_1^i, ..., \beta_m^i\right)\right)\right),$$

where

$$\alpha_k^i = t_i(x_k, y_k, \Box y_1^k, ..., \Box y_n^k, \overrightarrow{\Box p(x_k, y_k)});$$
$$\beta_k^i = t_{i+1}(x_k, y_k, , \overrightarrow{\Box p(x_k, y_k)}, \Box y_1^k, ..., \Box y_n^k).$$

Corollary 10.3 *If $\mathscr{V}$ is finitely based, so is its semi-linear subvariety $\mathscr{W}$.*

Proof By means of Theorem 10.2, we have exhibited a possibly infinite equational basis for $\mathscr{W}$ relative to $\mathscr{V}$, namely, the family of identities (S1). Now, $\mathscr{V}$ is quasi-subtractive with respect to $x \wedge 1$ and 1, as well as $(x \wedge 1, 1)$-regular. Its open filters contain a whenever they contain $a \wedge 1$, contain 1, and are closed with respect to the terms

$$\begin{aligned} x \wedge 1 = \Box x &= \Box p_1(1, \Box x) = 1 \rightsquigarrow \Box x; \\ \Box p_2(1, \Box x) &= \Box(\Box x \rightsquigarrow 1) \\ &= (x \wedge 1 \rightsquigarrow 1) \wedge 1 \\ &= 1. \end{aligned}$$

Therefore Theorem 10.3 applies, and we can streamline this basis to a finite one. □

10.4 Specializations and Applications

We conclude this chapter by pointing to the reader's attention some special cases and applications of the results in the preceding section.

If $\mathscr{V}$ is such that, for every $\mathbf{A} \in \mathscr{V}$, the pointed lattice term reduct $(A, \wedge, \vee, 1)$ has 1 as its top element, the situation drastically simplifies. In fact, while (D) is clearly redundant in this case, (A1) and (A2) imply that $\mathscr{V}$ is 1-ideal determined and its open filters coincide with its ideals in the sense of Gumm and Ursini.

To demonstrate the strength and applicability of Theorem 10.2, we will first identify $\mathscr{V}$ with the variety of residuated lattices satisfying (D), and $\mathscr{W}$ with its semilinear subvariety, deriving the characterization of $\mathscr{W}$ in Blount and Tsinakis (2003)[4] as a consequence of this theorem. To do so, we will use a known finite basis of open filter terms in order to streamline (S2) to a finite equational basis for $\mathscr{W}$ relative to $\mathscr{V}$. Subsequently, we prove that the basis obtained in this way can be reduced to the one in Blount and Tsinakis (2003). The same strategy will then be applied to the $(\wedge, \vee, \backslash, /, 1)$-subreducts of residuated lattices satisfying (D). The former application yields an alternative proof of a well-known result, whereas the last one is, to the best of our knowledge, new.

[4] It should be noted that a more delicate analysis in Blount and Tsinakis (2003) demonstrates that (D) can be omitted from the hypothesis of the theorem. Such refinements of special instances of a general result are to be expected.

Theorem 10.4 *Let $\mathscr{V}$ be a variety of residuated lattices that satisfies (D). Then its semi-linear subvariety $\mathscr{W}$ is axiomatized by the single equation*

$$\text{(S2) } \lambda_z\left((x \vee y) \backslash x\right) \vee \rho_w\left((x \vee y) \backslash y\right) \approx 1,$$

where $\lambda_y(x) = y \backslash xy \wedge 1$, $\rho_y(x) = yx/y \wedge 1$.

Proof Since $\mathscr{V}$ is quasi-subtractive with respect to 1 and $x \wedge 1$, by Theorem 10.3 its open filters coincide with the deductive filters of its $(x \wedge 1, 1)$-assertional logic, namely, the extension of the substructural logic **FL** by the axiom

$$((\alpha \vee \beta) \wedge 1) \backslash ((\alpha \wedge 1) \vee (\beta \wedge 1)) .$$

By results in (Galatos and Ono 2006, Sect. 4.2), these "deductive filters" (in the sense of abstract algebraic logic) coincide with upsets of convex normal subalgebras, which are likewise called *deductive filters* by residuated lattice practicioners. It follows from the same results that, in order to ensure that a $\mathscr{V}$-open filter is closed under all open filter terms, it suffices to check that it is closed under the following three: xy (in the variables x, y), $\lambda_y(x)$, $\rho_y(x)$ (in the variable x). By Theorem 10.2, therefore, an equational basis for $\mathscr{W}$ is given by:

$$\text{(S3) } 1 \leq (z\backslash x \wedge 1)(z\backslash y \wedge 1) \vee x\backslash z \vee y\backslash z$$
$$\text{(S4) } 1 \leq \lambda_z(y\backslash x \wedge 1) \vee x\backslash y$$
$$\text{(S5) } 1 \leq \rho_z(y\backslash x \wedge 1) \vee x\backslash y$$

What remains to be proved, then, is that (S3)-(S5) are jointly equivalent to (S2).[5] To begin with, observe that (S3) follows from (S4) or (S5) by letting $z = 1$. Note, next, that (S2) is equivalent to the quasi-identity

$$x \vee y \approx 1 \Rightarrow \lambda_z(x) \vee \rho_w(y) \approx 1$$

(see Blount and Tsinakis 2003, Lemma 6.5). This fact also implies that, in the presence of (D), (S2) is equivalent to

$$\text{(S2}') \ \lambda_z(x\backslash y \wedge 1) \vee \rho_w(y\backslash x \wedge 1) \approx 1.$$

Now (S4) and (S5) can be rewritten as

$$\text{(S4}') \ \lambda_z(x\backslash y \wedge 1) \vee (y\backslash x \wedge 1) \approx 1,$$

and

$$\text{(S5}') \ \rho_z(x\backslash y \wedge 1) \vee (y\backslash x \wedge 1) \approx 1.$$

[5] Compare Galatos et al. (2007, p. 426).

It is now clear that (S2′) is equivalent to the conjunction of (S4′) and (S5′). □

We now turn to the $(\wedge, \vee, \backslash, /, 1)$-subreducts of residuated lattices satisfying (D). We observed in Example 10.2 that these algebras form a class $\mathscr{K}$ of LI-algebras. As a consequence, $\mathscr{V} = V(\mathscr{K})$ is quasi-subtractive and $(\Box x, 1)$-regular, with the same witness term as for residuated lattices. It should be noted that the equations below involve all the operation symbols. A purely implicational characterization of the variety of semilinear integral residuated lattices, relative to the variety of integral residuated lattices, was conjectured in van Alten (2002) and proven in Kühr (2007). Theorem 10.5 below presents a multiplication-free characterization of the semilinear subvariety of the variety $\mathscr{V}$ of residuated lattices.

Our first goal will be that of giving a manageable description of $\mathscr{V}$-open filters.

Lemma 10.8 *Let* $\mathbf{A} \in \mathscr{V}$*, and* $F \subseteq A$*. Then* F *is a* $\mathscr{V}$*-open filter of* $\mathbf{A}$ *iff it is upward closed, and it is closed under all interpretations of the following* $\mathscr{V}$*-open filter terms (in the variables* x, y*):* $x \wedge 1$*,* $(x\backslash z)\backslash z$*,* $z/(z/x)$*, and*

$$t(x, y, z) := (\Box x \rightsquigarrow (\Box y \rightsquigarrow \Box z)) \rightsquigarrow z.$$

Proof By (Galatos and Ono 2006, Lemma 4.7), a subset F of the universe of a residuated lattice is a deductive filter (hence an open filter) in case it is upward closed, it is closed under modus ponens (if $a, a\backslash b \in F$, then $b \in F$), and it is closed under all interpretations of the open filter terms $x \wedge 1$, $(x\backslash z)\backslash z$, $z/(z/x)$. Observe that $F \subseteq A$ is a $\mathscr{V}$-open filter iff it obeys the same conditions. In fact, open filters are upward closures of congruence classes of 1, and the monoidal operation does not occur in the previous conditions. However, by Lemma 20 in Kowalski et al. (2011), closure under all interpretations of the term $t(x, y, z)$ suffices to guarantee modus ponens in any quasi-subtractive algebra. □

Observe that the upward closure condition in Lemma 10.8 is equivalent to the provision that if $a \wedge 1 \in F$, then $a \in F$. Therefore, by Lemma 10.1, closure under all interpretations of the open filter terms in Lemma 10.8 guarantees closure under all interpretations of *any* open filter term.

Theorem 10.5 *The semilinear subvariety of* $\mathscr{V}$ *is axiomatized, relative to* $\mathscr{V}$*, by the equations:*

(S6) $1 \leq ((z\backslash x \wedge 1)\backslash((z\backslash y \wedge 1)\backslash v \wedge 1))\backslash v \vee x\backslash z \vee y\backslash z;$

(S7) $1 \leq z\backslash x \vee x\backslash z;$

(S8) $1 \leq ((z\backslash x \wedge 1)\backslash y)\backslash y \vee x\backslash z;$

(S9) $1 \leq y/(y/(z\backslash x \wedge 1)) \vee x\backslash z.$

Proof By Lemma 10.8 and Theorem 10.2, the semilinear subvariety of $\mathscr{V}$ is axiomatized, relative to $\mathscr{V}$, by the equations (S7), (S8), (S9) and

$$(\mathrm{S6}')1 \leq ((z\backslash x \wedge 1)\backslash((z\backslash y \wedge 1)\backslash(v \wedge 1) \wedge 1) \wedge 1)\backslash v \vee x\backslash z \vee y\backslash z$$

However,

$$\begin{aligned}
&((z\backslash x \wedge 1)\backslash((z\backslash y \wedge 1)\backslash(v \wedge 1) \wedge 1) \wedge 1)\backslash v \vee x\backslash z \vee y\backslash z \\
&= ((z\backslash x \wedge 1)\backslash((z\backslash y \wedge 1)\backslash v \wedge (z\backslash y \wedge 1)\backslash 1 \wedge 1))\backslash v \vee x\backslash z \vee y\backslash z \\
&= ((z\backslash x \wedge 1)\backslash((z\backslash y \wedge 1)\backslash v \wedge 1))\backslash v \vee x\backslash z \vee y\backslash z.
\end{aligned}$$ □

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Part V
More Recent Trends

Petr Hájek's book *Metamathematics of Fuzzy Logic* is extremely interesting not only because the author was able to present the status of Fuzzy Logic in a very formal mathematical context, but also because he presented several interesting and original ideas. One of them was the logic BL, which was proposed as a common generalization of the three main fuzzy logics, Łukasiewicz, Gödel and product logics, and hence as the most general fuzzy logic. However, after the book several new and weaker logics appeared. First of all, Esteva and Godo noted that the continuity of the t-norm is not needed to have the residuation property: left continuity is enough. Hence, they invented the Monoidal T-norm-based Logic MTL. Since then, other logics, obtained by deleting either commutativity, or integrality, or by weakening the language, arose. Hájek himself was aware of this trend, and called the new logics *flea logics* after the joke about fleas we mentioned in the introduction. Then Petr Cintula and his collaborators tried to give a general treatment to these new logics, starting from the weakly implicative logics.

Other original and fruitful ideas are contained in the last chapters of the book. There, many lines of research, including probabilistic and possibilistic logics, fuzzy modal logics, fuzzy description logics, generalized quantifiers, etc., have been introduced. These topics are not developed in full detail, but the basic lines are explained clearly and constitute a source of ideas for further research.

The chapter *On Possibilistic Modal Logics Defined Over MTL-Chains* authored by Felix Bou, Francesc Esteva, and Lluis Godo, discusses some modal logics generalizing the so-called possibilistic logic, a graded logic to reason under uncertainty with classical propositions, in a fuzzy context, taking MTL as a background logic. Like in the classical context, the semantics is in terms of possibility distributions over possible worlds, but unlike an early approach over Łukasiewicz logic in a 1994 chapter co-authored by Petr Hájek, Dagmar Harmancová, Francesc Esteva, and Lluis Godo, the expression defining the possibility degree of a sentence is in terms of sup and of the monoid operation, and not in terms of sup and meet. This chapter is interesting because it carries further and generalizes an important research line proposed by Hájek in his book *Metamathematics of Fuzzy Logic*, as mentioned above.

In the chapter *The Quest for the Basic Fuzzy Logic*, by Petr Cintula, Rostislav Hořcik, and Carles Noguera, the authors start from Hájek's seminal work on fuzzy logic, and make a sharp proposal, namely they propose to call Hájek's basic fuzzy logic HL (after Hájek), instead of BL, and to call the BL-algebras HL-algebras. Moreover, the authors take the search of the really basic fuzzy logic as a leitmotiv for their chapter. Historically, this search has led to weaker and weaker fuzzy logics. After Esteva and Godo'sr logic MTL, many generalizations, obtained, for instance, by removing commutativity or weakening, have been presented. In particular, Cintula introduced some very general conditions which are still sufficient to obtain a fuzzy logic. In this chapter, the authors introduce a very interesting (and very weak) system, called SL, and they provide arguments to show that it is the required most basic fuzzy logic.

Chapter 11
On Possibilistic Modal Logics Defined Over MTL-Chains

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11.1 Introduction

In this chapter, as our humble homage to Petr Hájek, our aim is to revisit Hájek et al.'s paper[1] (Hájek et al. 1994) where a modal logic over a finitely-valued Łukasiewicz logic is defined to capture possibilistic reasoning. In this chapter we go further in two aspects: first, by generalizing Hájek's approach in the sense of considering modal logics over an arbitrary finite MTL-chain, and second, by considering a slightly different possibilistic semantics for the necessity and possibility modal operators.

Indeed, in Hájek et al. (1994) the authors defined a modal logic to reason about possibility and necessity degrees[2] of many-valued propositions. This logic was a generalization of the so-called *Possibilistic logic* (see e.g. Dubois et al. 1994; Dubois and Prade 2004), a well-known uncertainty logic to reasoning with graded beliefs on classical propositions by means of necessity and possibility measures. Possibilistic

[1] Actually, Hájek et al. (1994) was for F. Esteva and L. Godo the first joint paper with P. Hájek.
[2] In the sense of Possibility Theory (Dubois and Prade 1988).

Dedicated to Petr Hájek

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logic deals with weighted formulas (φ, r), where φ is a classical proposition and $r \in [0, 1]$ is a weight, interpreted as a lower bound for the necessity degree of φ. The semantics of these degrees is defined in terms of possibility distributions $\pi : \Omega \to [0, 1]$ on the set Ω of classical interpretations of a given propositional language. A possibility distribution π on Ω ranks interpretations according to its plausibility level: $\pi(w) = 0$ means that w is rejected, $\pi(w) = 1$ means that w is fully plausible, while $\pi(w) < \pi(w')$ means that w' is more plausible than w. A possibility distribution $\pi : \Omega \to [0, 1]$ induces a pair of dual possibility and necessity measures on propositions, defined respectively as:

$$\Pi(\varphi) = \sup\{\pi(w) \mid w \in \Omega, w(\varphi) = 1\}$$
$$N(\varphi) = \inf\{1 - \pi(w) \mid w \in \Omega, w(\varphi) = 0\}.$$

They are dual in the sense that $\Pi(\varphi) = 1 - N(\neg\varphi)$ for every proposition φ. From a logical point of view, possibilistic logic can be seen as a sort of graded extension of the non-nested fragment of the well-known modal logic of belief KD45.[3]

When we go beyond the classical framework of Boolean algebras of events to generalized algebras of many-valued events, one has to come up with appropriate extensions of the notion of necessity and possibility measures for many-valued events, as explored in Dellunde et al. (2011). A natural generalization, and indeed the one that we will focus on for the main result in this chapter, is to consider Ω as the set of propositional interpretations of some many-valued calculi defined by a t-norm $\odot$ and its residuum $\Rightarrow$. Then, a possibility distribution $\pi : \Omega \to [0, 1]$ induces the following generalized possibility and necessity measures over many-valued propositions:

$$\Pi(\varphi) = \sup\{\pi(w) \odot w(\varphi) \mid w \in \Omega\}$$
$$N(\varphi) = \inf\{\pi(w) \Rightarrow w(\varphi) \mid w \in \Omega\}.$$

Actually, these definitions agree with the ones commonly used in many-valued modal logics (see for example Bou et al. (2011) and the references therein) in the particular case where the many-valued accessibility relations R in Kripke-style frames (W, R) (i.e., binary $[0, 1]$-valued relations $R : W \times W \to [0, 1]$) are indeed defined by possibility distributions $\pi : W \to [0, 1]$ by putting $R(w, w') = \pi(w')$ for any $w, w' \in W$.

Structure of the chapter After this introduction, Sect. 11.2 contains a rather extensive overview of related work. The main contribution of the chapter is the fuzzy modal system given in Sect. 11.3 that is shown to properly capture the intended possibilistic semantics for the modal operators. The result has a limited scope since it only aplies to modal logics over finite MTL-chains (expanded with truth-constants and the Monteiro-Baaz's $\triangle$ operator) and to a language with finitely many variables. The axioms and completeness proof are natural generalizations of the ones for the system

[3] In fact, as it is explained in Sect. 11.2, two-valued possibility and necessity measures over classical propositions can be taken as an alternative semantics for the modal operators in the system KD45.

MVKD45 in Hájek et al. (1994), where the assumption about finitely-many variables is also adopted. In that paper the semantics of the possibility modal operator is defined in terms of a 'sup–min' combination of possibility values of worlds and truth-values of formulas. Here the semantics of the possibility modal operator is defined as a 'sup–$\odot$' combination, where $\odot$ is the monoidal operation of the MTL-chain. As in Hájek et al. (1994), we make an extensive use of the so-called maximal elementary conjunctions, which are definable in our setting. Admittedly, this makes the resulting axiomatization not very elegant. The chapter ends with Sect. 11.4 stating some conclusions and an open problem.

11.2 Related Work on Modal Approaches to Possibilistic Logics

When reviewing the literature on logical formalizations of different kinds of possibilistic reasoning, one can indentify two classes of systems according to the kind of language used and the possibilistic semantics of modal necessity and possibility operators, namely modal-like two-tiered logics and full modal logics. In this section we provide a brief overview of the most relevant ones for our purposes in each class.

11.2.1 Two-Tiered Logics

By two-tiered logics we refer to systems whose language is defined in a two-level manner: non-modal formulas are formulas from a given propositional logic L_1 (e.g. classical propositional logic) and then modal formulas are propositional combinations (according to a second logic L_2) of atomic modal formulas of the kind $\Box\varphi$ and $\Diamond\varphi$, where $\varphi \in L_1$. In this way, the language of these systems allow neither formulas with nested modalities (e.g. $\Box\Diamond\varphi$ is not allowed) nor formulas mixing both non-modal and modal subformulas (e.g. $\varphi \to \Box\psi$ with $\varphi, \psi \in L_1$ is not allowed). In all these systems, models can be considered under the form of a possibility distribution $\pi : \Omega \to [0, 1]$ on the set Ω of propositional evaluations for the logic L_1 (either classical or many-valued).

Among logics falling in this class we can find the following ones:

(i) The logic MEL (Banerjee and Dubois 2009) corresponds to the case of L_1 and L_2 being both classical propositional logic (CPL) and where models are subsets E of the set Ω of classical interpretations for the language of L_1, i.e. $E \subseteq \Omega$. The two-valued possibility distribution corresponding to a model $E \subseteq \Omega$ is nothing but its characteristic function, i.e. the mapping $\pi_E : \Omega \to \{0, 1\}$ where $\pi_E(w) = 1$ if $w \in E$, and $\pi_E(w) = 0$ otherwise. Atomic modal formulas are evaluated in a model E as follows:

$$E \models \Box\varphi \quad \text{if } w(\varphi) = 1 \text{ for all } w \in E.$$

A complete axiomatization of MEL, which indeed can be seen as a fragment of the modal logic KD, is given by the following set of additional axioms and rule to those of CPL:

$$\begin{array}{c} \text{(K)}\, \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi) \\ \text{(D)}\, \Diamond\top \\ \text{necessitation : if } \varphi \text{ is a CPL tautology, derive } \Box\varphi \end{array}$$

In this logic, one can only express two-valued possibilities and necessities, i.e. that a proposition is certainly true ($\Box\varphi$), certainly false ($\Box\neg\varphi$), possibly true ($\Diamond\varphi$) or possibly false ($\Diamond\neg\varphi$). The epistemic status "unknown" can be represented as $\Diamond\varphi \wedge \Diamond\neg\varphi$, or equivalenty $\neg\Box\varphi \wedge \neg\Box\neg\varphi$.

(ii) While keeping $L_1 = L_2 = CPL$, a natural generalization of MEL is to allow graded possibilities and necessities. This is done in Dubois et al. (2012), where the authors define what they call *Generalized Possibilistic logic*, GPL for short. To deal with graded possibility and necessity they fix a finite scale of uncertainty values $U = \left\{0, \frac{1}{k}, \frac{2}{k}, \ldots, 1\right\}$ and for each value $\lambda \in U \setminus \{0\}$ introduce a pair of modal operators $\Box_\lambda$ and $\Diamond_\lambda$. In this case models (epistemic states) are possibility distributions $\pi : \Omega \to U$ on the set Ω of classical interpretations for the language L_1 with values in U, and the evaluation of the modal formulas is as follows:

$$\pi \models \Box_\lambda\varphi \quad \text{if } \; N_\pi(\varphi) = \min\{1 - \pi(w) \mid w(\varphi) = 0\} \geq \lambda.$$

The dual possibility operators are defined as $\Diamond_\lambda\varphi = \neg\Box_{(1-\lambda)^+}\neg\varphi$, where the superscript $^+$ refers to the successor. The semantics of $\Diamond_\lambda\varphi$ is the natural one, i.e. $\pi \models \Diamond_\lambda\varphi$ whenever the possibility degree of φ induced by π, $\Pi(\varphi) = \max\{\pi(w) \mid w(\varphi) = 1\}$, is at least λ. A complete axiomatization of GPL is given in Dubois et al. (2012), an equivalent and shorter axiomatization is given by the following additional set of axioms and rules to those of CPL:

$$\begin{array}{c} \Box_\lambda(\varphi \to \psi) \to (\Box_\lambda\varphi \to \Box_\lambda\psi) \\ \Diamond_1\top \\ \Box_{\lambda_1}\varphi \to \Box_{\lambda_2}\varphi, \text{ if } \lambda_1 \geq \lambda_2 \\ \text{necessitation : if } \varphi \text{ is a CPL tautology, derive } \Box_1\varphi. \end{array}$$

(iii) Another kind of systems that have been proposed in the literature are the ones by Hájek et al. (1995), Hájek (1998). Here the idea is a bit different since it is based on a formalization where L_1 is still CPL but L_2 is Łukasiewicz infinitely-valued logic. The idea here is interpreting the modality $\Box$ as a fuzzy modality in the sense that a formula $\Box\varphi$ (where φ is a classical, Boolean proposition of L_1) is a fuzzy formula whose degree of truth in a given model is taken as the necessity degree of φ. Then Łukasiewicz logic is used to build compound expressions (out of atomic modal ones) and to reason about the truth-degrees of those fuzzy propositions. The logic can be augmented by the introduction of rational truth constants to allow explicitly reasoning with necessity and possibility degrees. A complete axiomatization is given by axioms of Łukasiewicz logic plus the following ones on modalities, where $\Diamond\varphi$ is defined as $\neg_Ł\Box\neg\varphi$ (we add the subindex Ł to differentiate the connectives of Łukasiewicz logic from the ones of CPL) :

$$\Box(\varphi \to \psi) \to_Ł (\Box\varphi \to_Ł \Box\psi)$$
$$\Diamond\top$$
$$\text{necessitation : if } \varphi \text{ is a CPL tautology, then derive } \Box\varphi.$$

(iv) Finally, we mention that the latter fuzzy logic-based approach has been generalized to reason about the necessity and possibility of *fuzzy events* (Dubois et al. 1994), where a fuzzy event refers to a proposition (modulo logical equivalence) in a given fuzzy logic (Dellunde et al. 2011). In these systems both logics L_1 and L_2 refer to two (possibly different) fuzzy logics and address different generalizations of the notion of necessity and possibility degrees of a fuzzy proposition. In general, if the fuzzy logic L_1 is the logic of a (continuous) t-norm $\odot$ and its residuum $\Rightarrow$, possibilistic models are given by possibility distributions $\pi : W \to [0, 1]$, where now W is the set of L_1 interpretations, that evaluate the necessity and possibility degree of a proposition φ from L_1 as follows:

$$\|\Box\varphi\|_\pi = \inf\{\pi(w) \Rightarrow w(\varphi) \mid w \in W\}$$
$$\|\Diamond\varphi\|_\pi = \sup\{\pi(w) \odot w(\varphi) \mid w \in W\}.$$

Basically, two choices of L_1 and L_2 have been addressed in the literature, namely taking L_1 and L_2 to be some variants of Łukasiewicz logic (Flaminio et al. 2011) or some variants of Gödel logic (Dellunde et al. 2011). For instance, in the former case, when L_1 and L_2 coincide with the $(k+1)$-valued Łukasiewicz logic $Ł_k$ expanded with truth-constants, the following is a complete set of additional axioms and inference rules to those of $Ł_k$ for both modal and non-modal formulas:

$$\Box(\varphi \wedge_Ł \psi) \leftrightarrow_Ł (\Box\varphi \wedge_Ł \Box\psi)$$
$$\Diamond\top$$
$$\Box(\overline{r} \oplus \varphi) \leftrightarrow_Ł (\overline{r} \oplus \Box\varphi), \quad \text{for } r \in \{0, 1/k, \ldots, (k-1)/k, 1\}$$

where $\oplus$ refers to the strong disjunction of Łukasiewicz logics and $\overline{r}$ denotes the truth constant of value r. The interested reader is referred to Flaminio et al. (2011) for a general treatment of this kind of logics.

11.2.2 Full Modal Systems

As it regards to full modal systems capturing possibilistic semantics, either within a classical or fuzzy logic approach, one can find less proposals in the literature. Next we point out two approaches.

The most basic "possibilistic" system is indeed the classical modal logic KD45. As it is well-known, the logic KD45 is axiomatized by the modal axioms

$$\begin{array}{l} \text{(K) } \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi) \\ \text{(D) } \Diamond\top \\ \text{(4) } \Box\varphi \to \Box\Box\varphi \\ \text{(5) } \Diamond\varphi \to \Box\Diamond\varphi \end{array}$$

and the necessitation inference rule for $\Box$, and it is sound and complete for the class of Kripke models $M = (W, e, R)$, where W is a non-empty set of worlds, $e : W \times Var \to \{0, 1\}$ provides a truth-evaluation of variables in each world, and the accessibility relation $R \subseteq W \times W$ is actually of the form $R = W \times E$ with $\emptyset \neq E \subseteq W$, see e.g. Pietruszczak (2009). Hence, R can be equivalently described by a two-valued possibility distribution $\pi_E : W \to \{0, 1\}$ with $\pi_E(w) = 1$ if $w \in E$, and $\pi_E(w) = 0$ otherwise. This yields the following truth-evaluation for modal formulas:

$$(M, w) \models \Box\varphi \qquad \text{if} \qquad (M, w') \models \varphi \text{ for each } w' \in E,$$

which clearly shows that it does not depend on the particular world where it is evaluated but only on the whole model.

The other directly related full modal system that we would like to refer to, and that is in fact the main motivation for this chapter, is 1994 Hájek et al.'s paper (Hájek et al. 1994), where a modal account of a certain notion of necessity and possibility of fuzzy events is provided. In particular the logic MVKD45, that we describe below, is developed over the finitely-valued Łukasiewicz logic $Ł_k$ (with truth-values in the set $S_k = \{0, 1/k, \ldots, (k-1)/k, 1\}$) expanded with some unary operators to deal with truth-constants.

Let us summarize here the main ingredients of the logic and the given axiomatization. The language of MVKD45 is that of $Ł_k$ built from a finite set of propositional variables $Var = \{p_1, \ldots, p_n\}$ and connectives $\to$ and $\neg$, expanded with two modal operators $\Box$ and $\Diamond$. Actually, in finitely-valued Łukasiewicz modal logics, one could consider only one of them since the other is definable by duality: e.g. $\Box\varphi$ is $\neg\Diamond\neg\varphi$. Interestingly enough, for all truth-values $r \in S_k$, the unary connectives (r), such that the value of $(r)\varphi$ is 1 if the value of φ is r and 0 otherwise, are definable in $Ł_k$. We also use expressions $(\leq r)\varphi$ and $(\geq r)\varphi$ to denote the disjunctions $\bigvee_{i \in S_k : i \leq r}(i)\varphi$ and $\bigvee_{i \in S_k : i \geq r}(i)\varphi$, respectively.

The semantics of the modal operators is as follows. Models are possibilistic Kripke structures of the form $M = (W, e, \pi)$, where W is a non-empty set of possible worlds,

$e : W \times Var \rightarrow S_k$ is an evaluation of propositional variables for each possible world and $\pi : W \rightarrow S_k$ is a normalized possibility distribution on W. Truth-evaluation of formulas is defined inductively in the usual way (we omit the reference to the model M):

- if $\varphi \in Var$, $\|\varphi\|_w = e(w, \varphi)$,
- if φ is a propositional combination, $\|\varphi\|_w$ is defined using the corresponding truth functions of the $Ł_k$ connectives,
- $\|\Box\varphi\|_w = \min\{\max(1 - \pi(w'), \|\varphi\|_{w'}) \mid w' \in W\}$,
- $\|\Diamond\varphi\|_w = \max\{\min(\pi(w'), \|\varphi\|_{w'}) \mid w' \in W\}$.

Note that this possibilistic semantics is a bit different from the general one we considered in item (iv) of the previous subsection. Actually this semantics was already proposed by Dubois, Prade et al. (see e.g. Dubois et al. 1994) for generalizing necessity and possibility measures over fuzzy sets using Kleene-Dienes implication and minimum respectively.

The following axiomatization provided in Hájek et al. (1994) to capture this semantics makes heavy use of *maximally elementary conjunctions*. Given the finite set of propositional variables $Var = \{p_1, \ldots, p_n\}$, maximally elementary conjunctions (m.e.c.'s for short) are formulas of the kind $(r_1)p_1 \wedge \cdots \wedge (r_n)p_n$. The set of m.e.c.'s will be denoted **mec**. Axioms of MVKD45 are those of $Ł_k$ plus:

- Axioms of KD45:

  $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$
  $\Box\varphi \leftrightarrow \Box\,\Box\varphi$
  $\Diamond\varphi \leftrightarrow \Box\,\Diamond\varphi$
  $\Diamond\top$

- $(r)\Box\varphi \leftrightarrow \Box(r)\Box\varphi$, $(r)\Diamond\varphi \leftrightarrow \Box(r)\Diamond\varphi$

- Possibilistic axioms:

  $((r)\Diamond\varphi \wedge E) \rightarrow (\leq r)(\varphi \wedge \Diamond E)$, with $E \in \mathbf{mec}$
  $(r)\Diamond\varphi \rightarrow \bigvee_{E \in \mathbf{mec}}(\geq r)\Diamond(E \wedge (r)(\varphi \wedge \Diamond E))$, with $r > 0$

Deductions rules are modus ponens and necessitation for $\Box$ (from φ infer $\Box\varphi$). In Hájek et al. (1994), the authors showed that this axiomatization is sound and complete with respect to the possibilistic semantics introduced above.

11.3 Extending the Logic of a Finite MTL-Chain with Possibilistic Modal Operators

In this section our aim is to generalize the above logic MVKD45 from Hájek et al. (1994). On the one hand we consider more general many-valued propositional logics, namely we consider logics of finite linearly-ordered MTL algebras rather than only finitely-valued Łukasiewicz logics. But on the other hand, we consider a different semantics for the modal necessity and possibility operators than the one used in

Hájek et al. (1994) and recalled in Sect. 11.2.2. Actually, the semantics adopted here is consistent with the one taken in Bou et al. (2011), using the monoidal operation and its residuum to evaluate the possibility and necessity operators respectively rather than using the min operation and Kleene-Dienes implication as in Hájek et al. (1994). Main consequences of these changes are that the necessity and possibility operators are no longer dual with respect to the negation of the logic, and that the necessity operator does not satisfy axiom (K).

In what follows, let $\mathbf{A} = (A, \wedge, \vee, \odot, \Rightarrow, 0, 1)$ denote a *finite* MTL-chain. Our modal logic will be defined on top of $\Lambda(\mathbf{A}^c_\triangle)$, the finitely-valued propositional logic of the finite MTL-chain $\mathbf{A}$ expanded with the Monteiro-Baaz's $\triangle$ operator and with truth constants $\overline{r}$ for each $r \in A$. Thus, the language of $\Lambda(\mathbf{A}^c_\triangle)$ is defined from a set of propositional variables using connectives $\wedge, \&, \rightarrow$ and $\triangle$, and truth constants $\overline{r}$.

For our purposes, we can consider the logic $\Lambda(\mathbf{A}^c_\triangle)$ as the consequence relation specified by: a formula φ logically follows from a set of formulas Γ, written $\Gamma \models_{\mathbf{A}^c_\triangle} \varphi$, whenever for each evaluation v of formulas into A such that $v(\psi) = 1$ for all $\psi \in \Gamma$, then $v(\varphi) = 1$ as well. Here, by evaluation we mean a mapping interpreting the connectives $\wedge, \&, \rightarrow$ into the algebra operations $\wedge, \odot, \Rightarrow$ respectively, the connective $\triangle$ into the function $\triangle : A \rightarrow A$ such that $\triangle(1) = 0$ and $\triangle(a) = 0$ for $a \neq 1$, and interpreting each truth-constant $\overline{r}$ into the value $r \in A$.

Then, we extend the language of $\Lambda(\mathbf{A}^c_\triangle)$ with two modal operators $\Box$ and $\Diamond$. Actually, we assume *our modal language to be generated from a finite set* $Var = \{p_1, \ldots, p_n\}$ *of propositional variables* together with the connectives[4] $\wedge, \&, \rightarrow$, truth-constants $\overline{r}$ (for each $r \in A$) and unary operators $\triangle$, $\Box$ and $\Diamond$.

11.3.1 Semantics

Definition 11.1 An $\mathbf{A}$-*valued possibilistic Kripke frame* is a pair $F = \langle W, \pi \rangle$ where W is a non empty set (whose elements are called *worlds*) and π is a normalized A-valued unary relation (i.e., $\pi : W \longrightarrow A$ and there exists $w \in W$ such that $\pi(w) = 1$) called *possibility distribution*. ⊣

Actually, any $\mathbf{A}$-valued possibilistic Kripke frame $F = (W, \pi)$ can be considered as a usual Kripke frame $F = (W, R_\pi)$, where the A-valued binary accessibility relation $R_\pi : W \times W \rightarrow A$ is defined by $R_\pi(w, w') = \pi(w')$. Moreover, R_π is clearly serial, transitive and Euclidean in the following generalized sense:

Serial: for every $w \in W$, there is $w' \in W$ such that $R_\pi(w, w') = 1$
Transitive: for every $w, w', w'' \in W$, $R_\pi(w, w') \odot R_\pi(w', w'') \leq R_\pi(w, w'')$
Euclidean: for every $w, w', w'' \in W$, $R_\pi(w, w') \odot R_\pi(w, w'') \leq R_\pi(w', w'')$.

[4] Other connectives are defined as usual in MTL, for instance $\neg\varphi$ is $\varphi \rightarrow \overline{0}$, $\varphi \vee \psi$ is $((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi)$, and $\varphi \leftrightarrow \psi$ is $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$.

Definition 11.2 An **A**-*valued possibilistic Kripke model* (or simply a *possibilistic Kripke model*) is a 3-tuple $K = \langle W, e, \pi \rangle$ where $\langle W, \pi \rangle$ is an **A**-valued possibilistic Kripke frame and e is a map, called *valuation*, assigning to each variable in Var and each world in W an element of A (i.e., $e : W \times Var \longrightarrow A$). We will say that K is *finite* when W is so. ⊣

If $K = \langle W, e, \pi \rangle$ is a possibilistic Kripke model, the map e can be uniquely extended to a map $\| \cdot \|_{K,w} : Fm \longrightarrow A$ assigning to each formula and each world $w \in W$ an element of A satisfying:

- $\|p\|_{K,w} = e(w, p)$ for each $p \in Var$,
- $\| \cdot \|_{K,w}$ is an algebraic homomorphism for the connectives $\wedge, \vee, \&, \rightarrow, \triangle$,
- $\|\overline{r}\|_{K,w} = r$, for every $r \in A$,
- and the following rules for evaluating modal formulas

$$\|\Diamond\varphi\|_{K,w} = \max_{u\in W}\{\pi(u) \odot \|\varphi\|_{K,u}\}, \qquad \text{(Sem-}\Diamond\text{)}$$

$$\|\Box\varphi\|_{K,w} = \min_{u\in W}\{\pi(u) \Rightarrow \|\varphi\|_{K,u}\}. \qquad \text{(Sem-}\Box\text{)}$$

Notice that the truth-evaluation of modal formulas starting with $\Diamond$ or $\Box$ does not depend on the particular world w but only on W and π. Also we define $\|\varphi\|_K = \min\{\|\varphi\|_{K,w} \mid w \in W\}$. When $\|\varphi\|_K = 1$ (resp. $\|\varphi\|_{K,w} = 1$) we will also write $K \models \varphi$ (resp. $(K, w) \models \varphi$). Finally, we define the notion of (local) logical consequence as follows: for any set of formulas $\Gamma \cup \{\varphi\}$, φ follows from Γ, denoted $\Gamma \models \varphi$, whenever for any model $K = (W, e, \pi)$ and world $w \in W$, if $(K, w) \models \psi$ for every $\psi \in \Gamma$ then $(K, w) \models \varphi$.

Call *reduced* a possibilistic Kripke model $K = (W, e, \pi)$ such that for any worlds $w, w' \in W$, if $e(w, \cdot) = e(w', \cdot)$ then $w = w'$, and hence $\pi(w) = \pi(w')$.[5] Since we are assuming that both the set of propositional variables Var and the MTL-chain **A** are finite, it holds that there is a finite number of reduced models and all of them have a finite number of worlds as well. Next lemma shows that we can actually restrict ourselves to consider the subclass of reduced possibilistic Kripke models.

Proposition 11.1 *For any possibilistic Kripke model K there is a reduced model K' such that $\|\varphi\|_K = \|\varphi\|_{K'}$ for any formula φ.*

Proof Let $K = (W, e, \pi)$ be a possibilistic Kripke model and define an equivalence relation on W as follows: $w \cong w'$ whenever $e(w, p) = e(w', p)$ for all propositional variables $p \in Var$. We will denote by $[w]$ the equivalence class of w. Let us define the model $K' = (W', e', \pi')$ as follows:

1. $W' = W/\cong$
2. for each $w \in W$, $e'([w], p) = e(w, p)$ for all $p \in Var$
3. $\pi' : W' \rightarrow [0, 1]$ is the mapping defined as $\pi'([w]) = \max\{\pi(w') \mid w' \in [w]\}$.

[5] We use the notation $e(w, \cdot)$ to denote the function $p \in Var \longmapsto e(w, p) \in A$.

Clearly, K' is reduced. Let us check by induction that, for any formula φ and any $w \in W$, $\|\varphi\|_{K,w} = \|\varphi\|_{K',[w]}$. Indeed, this is obvious for φ being a propositional variable. The inductive steps for the propositional connectives are also clear, and the interesting steps are the cases of the modal operators:

- Let $\varphi = \Box\psi$. Then, using the induction hypothesis, we have the following chain of equalities:

$$\begin{aligned}
\|\Box\psi\|_{K,w} &= \min_{w\in W}\{\pi(w) \Rightarrow \|\psi\|_{K,w}\} \\
&= \min_{w\in W}\{\pi(w) \Rightarrow \|\psi\|_{K',[w]}\} \\
&= \min_{w\in W}\{(\max_{w'\in[w]} \pi(w')) \Rightarrow \|\psi\|_{K',[w]}\} \\
&= \min_{w\in W}\{\pi'([w]) \Rightarrow \|\psi\|_{K',[w]}\} \\
&= \min_{[w]\in W'}\{\pi'([w]) \Rightarrow \|\psi\|_{K',[w]}\} \\
&= \|\Box\psi\|_{K',[w]}.
\end{aligned}$$

  We point out that the third equality is an easy consequence of the inclusion $\{\pi(w') \Rightarrow \|\psi\|_{K',[w']} : w' \in W\} \supseteq \{(\max_{w'\in[w]} \pi(w')) \Rightarrow \|\psi\|_{K',[w]} : w \in W\}$.
- Let $\varphi = \Diamond\psi$. Then, using the induction hypothesis, we have the following chain of equalities:

$$\begin{aligned}
\|\Diamond\psi\|_{K,w} &= \max_{w\in W}\{\pi(w) \odot \|\psi\|_{K,w}\} \\
&= \max_{w\in W}\{\pi(w) \odot \|\psi\|_{K',[w]}\} \\
&= \max_{w\in W}\{(\max_{w'\in[w]} \pi(w')) \odot \|\psi\|_{K',[w]}\} \\
&= \max_{w\in W}\{\pi'([w]) \odot \|\psi\|_{K',[w]}\} \\
&= \max_{[w]\in W'}\{\pi'([w]) \odot \|\psi\|_{K',[w]}\} \\
&= \|\Diamond\psi\|_{K',[w]}.
\end{aligned}$$

This ends the proof.

This last result, together with the fact that there are only finitely many reduced models, has the following relevant consequences.

Corollary 11.1 *Modulo semantical equivalence, there are only a finite number of different formulas. Therefore, if Γ is a possibly infinite set of formulas, then there exists a finite subset $\Gamma_0 \subseteq \Gamma$ that is semantically equivalent to Γ in the following sense: for any formula φ, $\Gamma \models \varphi$ iff $\Gamma_0 \models \varphi$.*

Throughout the rest of the chapter, we will make use of the following notation conventions:

- $(1)\varphi$ will denote the formula $\triangle\varphi$, and $(0)\varphi$ will denote the formula $\triangle\neg\varphi$
  $(r)\varphi$ will denote the formula $\triangle(\overline{r} \leftrightarrow \varphi)$ (when $r \notin \{0, 1\}$)
  $(\geq r)\varphi$ will denote the formula $\triangle(\overline{r} \rightarrow \varphi)$
  $(> r)\varphi$ will denote the formula $(\geq r)\varphi \wedge \neg(r)\varphi$
  $(\leq r)\varphi$ will denote the formula $\triangle(\varphi \rightarrow \overline{r})$
  $(< r)\varphi$ will denote the formula $(\leq r)\varphi \wedge \neg(r)\varphi$
- Propositional combinations of formulas of the kind $(r)\varphi$, where φ is an arbitrary formula, will be called *B-formulas* (for Boolean formulas)
- As in MVD45, *maximally elementary conjunctions* (m.e.c.'s) are B-formulas that are conjunctions of the form $\bigwedge_{i=1,\ldots,n}(r_i)p_i$ (remember that $p_1, \ldots, p_n$ are the finitely many fixed propositional variables). We will keep denoting by **mec** the set of all m.e.c.'s.

Note that for each B-formulas φ and ψ, the formulas $\varphi \wedge \psi$ and $\varphi\&\psi$ are equivalent, that is, $\|(\varphi \wedge \psi) \leftrightarrow (\varphi\&\psi)\|_K = 1$ for all possibilistic Kripke models K.

Next lemma shows some useful tautologies of the class of possibilistic Kripke models.

Lemma 11.1 *The following equivalences are tautologies for the class of all Possibilistic Kripke models:*

1. $\varphi \leftrightarrow \bigwedge_{r\in A}((r)\varphi \rightarrow \overline{r})$
2. $\Box\varphi \leftrightarrow \bigwedge_{r\in A}(\Diamond(r)\varphi \rightarrow \overline{r})$
3. $\Box(\varphi \rightarrow \overline{r}) \leftrightarrow (\Diamond\varphi \rightarrow \overline{r})$
4. $(0)\Diamond\varphi \leftrightarrow (1)\Box\neg\varphi$
5. $(r)\Diamond\varphi \leftrightarrow \Big((1)\Box(\varphi \rightarrow \overline{r}) \wedge (< 1)\Box(\varphi \rightarrow \overline{r^-})\Big)$, *if* $r > 0$ *and* r^- *is the predecessor of* r .

Proof 1. Obvious.

2. $\|\Box\varphi\|_K = \bigwedge_{w\in W}\{\pi(w) \Rightarrow \|\varphi\|_{K,w}\} =$
$\bigwedge_{r\in A}(\bigwedge\{\pi(w) \Rightarrow r \mid w \in W, \|\varphi\|_{K,w} = r\}) =$
$\bigwedge_{r\in A}(\bigwedge_{w\in W}\{\pi(w) \Rightarrow (\|(r)\varphi\|_{K,w} \Rightarrow r)\}) =$[6]
$\bigwedge_{r\in A}(\bigwedge_{w\in W}\{(\pi(w) \odot \|(r)\varphi\|_{K,w}) \Rightarrow r\}) =$[7]
$\bigwedge_{r\in A}\{(\bigvee_{w\in W}\{\pi(w) \odot \|(r)\varphi\|_{K,w}\}) \Rightarrow r\} =$
$\bigwedge_{r\in A}\{\|\Diamond(r)\varphi\|_K \Rightarrow r\} =$
$\| \bigwedge_{r\in A}(\Diamond(r)\varphi \rightarrow \overline{r})\|_K$.

3. $\|\Box(\varphi \rightarrow \overline{r})\|_K = \bigwedge_{w\in W}\{\pi(w) \Rightarrow (\|\varphi\|_{K,w} \Rightarrow r)\} =$
$\bigwedge_{w\in W}\{(\pi(w) \odot \|\varphi\|_{K,w}) \Rightarrow r\} =$
$(\bigvee_{w\in W}\{(\pi(w) \odot \|\varphi\|_{K,w})\}) \Rightarrow r =$
$\|\Diamond\varphi \rightarrow \overline{r}\|_K$.

[6] Here we use the fact the equation $x \Rightarrow (y \Rightarrow z) = (x \odot y) \Rightarrow z$ holds in every MTL-chain.

[7] Here we use the fact the equation $(x_1 \Rightarrow y) \wedge (x_2 \Rightarrow y) = (x_1 \vee x_2) \Rightarrow y$ holds in every MTL-chain.

4. Taking $r = 0$ in item 3 we get that $\Box(\neg\varphi) \leftrightarrow \neg\Diamond\varphi$ is a tautology, and hence in particular $(1)\Box\neg\varphi \leftrightarrow (0)\Diamond\varphi$ as well.
5. If $r > 0$ then the claim directly follows from the observation that for any formula ψ, $(r)\psi$ is equivalent to $(1)(\psi \rightarrow \overline{r}) \wedge (< 1)(\psi \rightarrow \overline{r^-})$. Then, by item 3, we have that $(r)\Diamond\varphi$ is equivalent to $(1)(\Box\varphi \rightarrow \overline{r}) \wedge (< 1)(\Box\varphi \rightarrow \overline{r^-})$.

Taking into account that item 1 of Lemma 11.1 gives that

$$\Diamond\varphi \leftrightarrow \bigwedge_{r \in A} ((r)\Diamond\varphi \rightarrow \overline{r})$$

is a tautology, items 2, 4 and 5 of the same lemma tell us that, due to the presence of the truth-constants, the modal operators $\Box$ and $\Diamond$ are indeed inter-definable:

$\Box\varphi$ as $\bigwedge_{r \in A} (\Diamond(r)\varphi \rightarrow \overline{r})$, and
$\Diamond\varphi$ as $(> 0)\Box\neg\varphi \wedge \left(\bigwedge_{r \in A\setminus 0} \left((1)\Box(\varphi \rightarrow \overline{r}) \wedge (< 1)\Box(\varphi \rightarrow \overline{r^-})\right) \rightarrow \overline{r}\right)$.

Indeed, the latter is obtained by noticing that, by the above equivalence, $\Diamond\varphi$ is equivalent to $((0)\Diamond\varphi \rightarrow \overline{0}) \wedge (\bigwedge_{r \in A\setminus 0}((r)\Diamond\varphi \rightarrow \overline{r}))$, and then by applying item 4 to the first conjunct and item 5 to the second conjunct.

11.3.2 Syntax

Assuming a Hilbert style axiomatization (with modus ponens as unique inference rule) of $\Lambda(\mathbf{A})$ (i.e., the propositional logic of the MTL-chain $\mathbf{A}$), one can get an axiomatization of $\Lambda(\mathbf{A}^c_\triangle)$, its expansion with the Baaz-Monteiro $\triangle$ operator and canonical truth-constants, by adding (cf. Bou et al. 2011, Prop. A.12):

- the well-known axioms and necessitation rule for $\triangle$ (see e.g. Hájek 1998),
- the following book-keeping axioms[8]:

$$(\overline{r} \& \overline{s}) \leftrightarrow \overline{r \odot s}$$
$$(\overline{r} \rightarrow \overline{s}) \leftrightarrow \overline{r \Rightarrow s}$$
$$(\overline{r} \wedge \overline{s}) \leftrightarrow \overline{\min(r, s)}$$
$$\triangle\overline{r} \leftrightarrow \overline{\triangle r},$$

[8] Notice that these axioms could also be expressed as the following B-formulas:

$$(r \odot s)(\overline{r} \& \overline{s}),\ (r \Rightarrow s)(\overline{r} \rightarrow \overline{s}),\ (\min(r, s))(\overline{r} \wedge \overline{s}),\ (\triangle r)\triangle\overline{r} \text{ and } \bigvee_{r \in A}(r)\varphi.$$

However, the adopted formulation makes less use of the $\triangle$ connective.

- and the witnessing axiom:

$$\bigvee_{r \in A} (\varphi \leftrightarrow \overline{r}).$$

The last book-keeping axiom guarantees that truth-constants in the logic behave as *canonical* ones, in the sense that each truth-constant $\overline{r}$ is actually interpreted as the value r in $\mathbf{A}$.

Taking this into account, next we define an axiomatic system for the modal expansion of $\Lambda(\mathbf{A}^c_\triangle)$ that will be shown to be sound and complete with respect to the class of possibilistc Kripke models defined above.

Definition 11.3 The logic $Pos(\mathbf{A}^c_\triangle)$ has the following axioms:

- Axioms of $\Lambda(\mathbf{A}^c_\triangle)$
- Axioms from KD45:

(4) $\Box\varphi \leftrightarrow \Box\Box\varphi$
(5) $\Diamond\varphi \leftrightarrow \Box\Diamond\varphi$
(D) $\Diamond\top$
(4') $(r)\Box\varphi \leftrightarrow \Box(r)\Box\varphi$, for each $r \in A$
(5') $(r)\Diamond\varphi \leftrightarrow \Box(r)\Diamond\varphi$, for each $r \in A$

- Possibilistic axioms (for each $r \in A$):

(NΠ) $\Box(\varphi \to \overline{r}) \leftrightarrow (\Diamond\varphi \to \overline{r})$
(Π1) $((r)\Diamond\varphi \wedge E) \to (\leq r)(\varphi \& \Diamond E)$, with $E \in \mathbf{mec}$
(Π2) $(r)\Diamond\varphi \to \bigvee_{E \in \mathbf{mec}} (\geq r)\Diamond(E \wedge (r)(\varphi \& \Diamond E))$, with $r > 0$

Deductions rules of $Pos(\mathbf{A}^c_\triangle)$ are modus ponens, necessitation for $\triangle$ (from φ derive $\triangle\varphi$) and monotonicity for $\Box$: if $\varphi \to \psi$ is a theorem, infer $\Box\varphi \to \Box\psi$.

The notion of proof in $Pos(\mathbf{A}^c_\triangle)$, denoted $\vdash$, is defined from the above axioms and rules (notice that the application of the monotonicity rule for $\Box$ is restricted to theorems, in contrast to the other two rules). $\dashv$

Axioms (Π1) and (Π2) actually capture the semantics of the $\Diamond$ operator defined in (Sem-3) as a maximum of values. If $\Diamond\varphi$ takes value r, (Π1) tells us that each element in the maximum must be less of equal than r, while (Π2) expresses that the maximum is actually attained. Notice also that each m.e.c. E correspond to a possible world w, and hence the value of $\Diamond E$ corresponds to the possibility distribution on w.

To prove soundness of $Pos(\mathbf{A}^c_\triangle)$ with respect to the possibilistic Kripke semantics, we need first to prove some auxiliary results in the next lemma.

Lemma 11.2 *Let $K = (W, e, \pi)$ be a possibilistic Kripke model. Then the following conditions hold:*

1. *For each $w \in W$ and formula φ, there is a unique m.e.c. E and truth-value $r \in A$ such that $(K, w) \models E \wedge (r)\varphi$.*

2. *For each formula φ and every m.e.c. E, if $(K, w) \models E$ for some $w \in W$, then there exists a unique value r such that $K \models E \to (r)\varphi$.*
3. *For any m.e.c. E, formula φ and value r, it holds $K \models (>0)\Diamond((r)\varphi \wedge E) \to (E \to (r)\varphi)$.*

Proof Items 1 and 2 are easy. As for item 3, let $w \in W$, and assume $(K, w) \models (>0)\Diamond((r)\varphi \wedge E)$. Then necessarily $\|\Diamond((r)\varphi \wedge E)\|_w > 0$, i.e. there exists $w' \in W$ such that $\pi(w') \odot \|(r)\varphi \wedge E\|_{w'} > 0$. Since $(r)\varphi \wedge E$ is a B-formula, the latter holds iff $\pi(w') > 0$ and $\|(r)\varphi \wedge E\|_{w'} = 1$. Therefore $(K, w') \models E$, and by item 2, r is actually the unique value such that $K \models E \to (r)\varphi$, and hence in particular, $(K, w) \models E \to (r)\varphi$. So we have proved that for any $w \in W$, $(K, w) \models (>0)\Diamond((r)\varphi \wedge E) \to (E \to (r)\varphi)$.

Theorem 11.1 [Soundness] *The logic $Pos(\mathbf{A}^c_\triangle)$ is sound with respect to the class of possibilistic Kripke models.*

Proof Notice that, as observed before, possibilistic Kripke models can be considered as many-valued Kripke models with a transitive, Euclidean and serial accessibility relation. Therefore, from general results in Radzikowska and Kerre (2005), axioms (4), (5) and (D) are automatically sound. Moreover, the related axioms with truth constants (4') and (5') are also sound as an easy computation shows, and the soundness of axiom (NΠ) is just item 3 of Lemma 11.1. Next we prove soundness of axioms (Π1) and (Π2).

(Π1): Assume there is $w \in W$ such that $(K, w) \models (r)\Diamond\varphi \wedge E$, otherwise the result is trivial. Then $\|\Diamond E\|_w = \max\{\pi(w') \mid w' \models E\} = \pi(w)$ since w can only be the unique world in W such that $w \models E$. Then $\|\varphi \& \Diamond E\|_w = \|\varphi\|_w \odot \|\Diamond E\|_w = \|\varphi\|_w \odot \pi(w) \leq \max\{\|\varphi\|_{w'} \odot \pi(w') \mid w' \in W\} = \|\Diamond\varphi\| = r$.

(Π2): Assume $r > 0$, otherwise it is obvious. If $r = \|\Diamond\varphi\|$, then there is $w_0 \in W$ and a m.e.c. E such that $r = \|\varphi\|_{w_0} \odot \pi(w_0)$, $w_0 \models E$ and $\pi(w_0) = \|\Diamond E\|$. Therefore $r = \|\varphi\|_{w_0} \odot \|\Diamond E\|$. Thus $\|E \wedge (r)(\varphi \& \Diamond E)\|_{w_0} = 1$ and hence $\|E \wedge (r)(\varphi \& \Diamond E)\|_{w_0} \odot \pi(w_0) \geq \|\varphi\|_{w_0} \odot \pi(w_0) = r$. Consequently, we have $\|\Diamond(E \wedge (r)(\varphi \& \Diamond E))\| \geq r$.

Modus ponens and necessitation for $\triangle$ are trivially sound, and finally, it is also easy to show that monotonicity inference rule for $\Box$ is also sound when applied to valid implications.

It is worth mentioning that the well-known axiom (K) is not sound in general (although it is indeed sound for B-formulas), except for the case when the finite MTL-chain $\mathbf{A}$ is a Gödel chain (see Bou et al. 2011, Corollay 3.13 for details). Actually this is the main difference with the modal system studied in Hájek et al. (1994), since there the semantics of the necessity operator $\Box$ (defined in Sect. 11.2.1), using min instead of $\odot$, makes axiom (K) sound for every MTL-chain $\mathbf{A}$.

11.3.3 Completeness

To finish this section our aim is to show that the logic $Pos(\mathbf{A}^c_\triangle)$ also enjoys strong completeness with respect to the semantics defined in Sect. 11.3.1. Let us remind that our axiomatization is already complete with respect to the non-modal semantics given by the chain $\mathbf{A}^c_\triangle$, see (Bou et al. 2011, Prop. A.12) for more details. Moreover, for instance, the $\triangle$-deduction theorem

$$\Gamma \cup \{\varphi\} \vdash \psi \text{ iff } \Gamma \vdash \triangle\varphi \to \psi$$

can be straightforwardly proved to hold by induction on the rules of our axiomatization (take into account the monotonicity rule only applies to theorems).

Definition 11.4 A theory is a set of B-formulas. A theory T is consistent if $T \nvdash \overline{0}$. A theory T is complete if for each B-formula ψ, either $T \vdash \psi$ or $T \vdash \neg\psi$. Moreover, we say that two complete theories T and T' are equivalent, written $T \approx T'$, if for each r and φ, $T \vdash (r)\Diamond\varphi$ iff $T' \vdash (r)\Diamond\varphi$. ⊣

Note that any inconsistent theory is complete, and all inconsistent theories are equivalent. On the other hand, using classical techniques, one can show that any consistent theory T can be always extended to a complete and consistent supertheory $T^\star \supseteq T$.

Next we will prove some lemmas necessary for the completeness proof.

Lemma 11.3 *(a) If φ is a B-formula, $\vdash \varphi \leftrightarrow (1)\varphi$, $\vdash \neg\varphi \leftrightarrow (0)\varphi$ and $\vdash \varphi \vee \neg\varphi$.*
(b) If φ is a B-formula and $0 < r < 1$, then $(r)\varphi \vdash \overline{0}$.
(c) T is complete and consistent iff for every formula φ there exists a unique r such that $T \vdash (r)\varphi$.
*(d) For each complete and consistent theory T there is a unique $E \in$ **mec** such that $T \vdash E$. We will denote such a unique m.e.c. E_T.*

Proof (a) Taking into account that $(1)\varphi$ is $\triangle\varphi$ and $(0)\varphi$ is $\triangle\neg\varphi$, and that B-formulas are propositional combinations of formulas starting with $\triangle$, it turns out that the considered formulas are already tautologies in $\mathrm{MTL}_\triangle$ (considering all the $\Box$-formulas and $\Diamond$-formulas as propositional variables), and hence, by completeness of $\mathrm{MTL}_\triangle$, they are also provable in $\mathrm{MTL}_\triangle$, and thus in $\Lambda(\mathbf{A}^c_\triangle)$ as well.

(b) This is an immediate consequence that our axiomatization is complete with respect to the non-modal semantics over the chain $\mathbf{A}^c_\triangle$.

(c) Suppose T is complete and $T \nvdash (r)\varphi$ for each $r \in A$. Then $T \vdash \neg(r)\varphi$ for each $r \in A$, hence $T \vdash \bigwedge_{r\in A} \neg(r)\varphi$, in other words, $T \vdash \neg \bigvee_{r\in A}(r)\varphi$. But this is in contradiction with the witnessing axiom $\bigvee_{r\in A}(\varphi \leftrightarrow \overline{r})$. Conversely, let ψ be a B-formula and let r the unique value such that $T \vdash (r)\psi$ given by the assumption. By (b), it follows that $r \in \{0, 1\}$, hence either $T \vdash (1)\psi$ or $T \vdash (0)\psi$, i.e. either $T \vdash \psi$ or $T \vdash \neg\psi$.

(d) Since T is complete and consistent, by (c), for each propositional variable p_i there is a unique r_i such that $T \vdash (r_i)p_i$, hence T proves the m.e.c. $\bigwedge_{i=1,\dots,n}(r_i)p_i$.

Lemma 11.4 *The following conditions hold:*

(a) For any B-formula φ and any consistent theory T, if $T \vdash (>0)\Diamond\varphi$ then φ is consistent (i.e., $\varphi \nvdash \overline{0}$).
(b) For any B-formula φ, $\vdash \Diamond\neg\varphi \to \neg\Box\varphi$.
(c) For any B-formulas φ and ψ, $\vdash \Box\varphi \to (\Diamond\psi \to \Diamond(\varphi \wedge \psi))$.

Proof (a) Assume φ is inconsistent. Then by the Δ-deduction theorem, $\vdash \Delta\varphi \to \overline{0}$, i.e. $\vdash \neg\Delta\varphi$, but since φ is a B-formula, $\vdash \varphi \leftrightarrow \Delta\varphi$, we have $\vdash \neg\varphi$. By the rule of necessitation for $\Box$, $\vdash \Box\neg\varphi$, hence (by axiom $(N\Pi)$), we have $\vdash \neg\Diamond\varphi$, i.e. $\vdash (0)\Diamond\varphi$, and hence $T \vdash (0)\Diamond\varphi$ as well. But this is in contradiction with the hypothesis that $T \vdash (>0)\Diamond\varphi$.

(b) Let φ be a B-formula. Taking $r = 0$, axiom $(N\Pi)$ gives $\neg\Diamond\neg\varphi \leftrightarrow \Box\neg\neg\varphi$, but if φ is a B-formula $\neg\neg\varphi$ is equivalent to φ. Hence $\vdash \neg\neg\Diamond\neg\varphi \leftrightarrow \neg\Box\varphi$, and since $\psi \to \neg\neg\psi$ is a theorem of MTL, we thus have $\vdash \Diamond\neg\varphi \to \neg\Box\varphi$.

(c) For B-formulas φ and ψ, by (a) of Lemma 11.3, we have that ψ is equivalent to $(\psi \wedge \neg\varphi) \vee (\psi \wedge \varphi)$. Hence $\vdash \Diamond\psi \leftrightarrow (\Diamond(\psi \wedge \neg\varphi) \vee \Diamond(\psi \wedge \varphi))$, hence $\vdash \Diamond\psi \to (\Diamond(\neg\varphi) \vee \Diamond(\psi \wedge \varphi))$, and by (b), $\vdash \Diamond\psi \to (\neg\Box\varphi \vee \Diamond(\psi \wedge \varphi))$. Now, using that $(\neg\varphi \vee \psi) \to (\varphi \to \psi)$ is a theorem of MTL, we also have $\vdash \Diamond\psi \to (\Box\varphi \to \Diamond(\varphi \wedge \psi))$, and hence $\vdash \Box\varphi \to (\Diamond\psi \to \Diamond(\varphi \wedge \psi))$ as well.

Lemma 11.5 *Let T_0 be a complete and consistent theory and let $T_0 \vdash (r)\Diamond\varphi$. Then, the following conditions hold:*

*(a) For any theory $T \approx T_0$ and for any $E \in$ **mec**, $T \vdash E \to (\leq r)(\varphi \& \Diamond E)$.*
*(b) There is a theory $T \approx T_0$ and $E \in$ **mec** such that $T \vdash E \wedge (r)(\varphi \& \Diamond E)$.*

Proof (a) Using (b) of Lemma 11.3, there is a unique value r such that $T \vdash (r)\Diamond\varphi$. If E is a m.e.c., then both $(r)\Diamond\varphi$ and E are B-formulas and then $(r)\Diamond\varphi \wedge E$ is equivalent to $(r)\Diamond\varphi \,\&\, E$. Then axiom $(\Pi 1)$ can be equivalently expressed as $((r)\Diamond\varphi \,\&\, E) \to (\leq r)(\varphi \& \Diamond E)$, and this equivalent in turn to $(r)\Diamond\varphi \to (E \to (\leq r)(\varphi \& \Diamond E))$. Now by applying modus ponens to the latter and $(r)\Diamond\varphi$, we have $T \vdash E \to (\leq r)(\varphi \& \Diamond E)$.

(b) Assume $T_0 \vdash (r)\Diamond\varphi$ with $r > 0$. Then, by modus ponens with axiom $(\Pi 2)$, we get $T_0 \vdash \bigvee_{E \in \mathbf{mec}} (\geq r)\Diamond(E \wedge (r)(\varphi \& \Diamond E))$. Since T_0 is complete,[9] for some E, $T_0 \vdash (\geq r)\Diamond(E \wedge (r)(\varphi \& \Diamond E))$, and since $r > 0$ we have that $T_0 \vdash (>0)\Diamond(E \wedge (r)(\varphi \& \Diamond E))$.

Let D denote $E \wedge (r)(\varphi \& \Diamond E)$, and let $H = \{(s)\Diamond\psi \mid T_0 \vdash (s)\Diamond\psi\}$ be the set of B-formulas of the kind $(s)\Diamond\psi$ provable from T_0. We are going to prove that D is consistent with H. Let H_f be the conjunction of an arbitrary finite subset of H. Obviously, $T_0 \vdash \Box H_f$. Since both D and H_f are Boolean, by (c) of Lemma 11.4, we have $\vdash \Box H_f \to (\Diamond D \to \Diamond(D \wedge H_f))$, and by modus ponens, $T_0 \vdash \Diamond D \to \Diamond(D \wedge H_f)$, and thus $T_0 \vdash (>0)\Diamond D \to (>0)\Diamond(D \wedge H_f)$ as well. But $T_0 \vdash (>0)\Diamond D$, so again by modus ponens, $T_0 \vdash (>0)\Diamond(D \wedge H_f)$. Hence, by (a) of previous Lemma 11.4, $D \wedge H_f$ is consistent. We have thus proved that D is consistent with any arbitrary finite conjunction H_f of H, therefore $\{D\} \cup H$ is

[9] Recall that a complete theory is prime in the classical sense for B-formulas.

consistent. Finally consider T to be a completion of $\{D\} \cup H$. This theory clearly proves D (i.e. $T \vdash E \wedge (r)(\varphi \& \Diamond E)$) and T proves the same formulas of the kind $(s)\Diamond\psi$ than T_0, that is, $T \approx T_0$.

Finally, assume $T_0 \vdash (0)\Diamond\varphi$. Let T be any theory such that $T \approx T_0$. Then by (c) of Lemma 11.3 we have $T \vdash E_T$, and taking $r = 0$ in (a) above, we have $T \vdash E_T \rightarrow (0)(\varphi \& \Diamond E_T)$. Therefore, $T \vdash (0)(\varphi \& \Diamond E_T)$ and the statement is proved.

Corollary 11.2 *Let T_0 be a complete and consistent theory. Then, for any formula φ, $T_0 \vdash (r)\Diamond\varphi$ iff the following two conditions hold:*

(a) For any theory $T \approx T_0$, $T \vdash (\leq r)(\varphi \& \Diamond E_T)$,
(b) There is a theory $T_\varphi \approx T_0$ such that $T_\varphi \vdash (r)(\varphi \& \Diamond E_{T_\varphi})$.

Proof From left to right, it is a direct consequence of the previous lemma by considering for each complete and consistent theory T the corresponding m.e.c. E_T as defined in (c) of Lemma 11.3. For the other direction we reason as follows. Assume conditions (a) and (b) hold and assume further that $T_0 \vdash (s)\Diamond\varphi$ with $s \neq r$. If $s < r$, then applying the 'left to right part' to $T_0 \vdash (s)\Diamond\varphi$ we would get (a): for every $T \approx T_0$, $T \vdash (\leq s)(\varphi \& \Diamond E_T)$, and this would contradict (b): $T_\varphi \vdash (r)(\varphi \& \Diamond E_T)$. In a similar way, if $s > r$, then the application of the 'left to right part' to $T_0 \vdash (s)\Diamond\varphi$ would give (b): $T' \vdash (s)(\varphi \& \Diamond E_T)$ for some $T' \approx T_0$, which would be in contradiction with (a): $T \vdash (\leq r)(\varphi \& \Diamond E_T)$ for all $T \approx T_0$.

From these lemmas we are ready to prove strong completeness but first we define a sort of canonical model that will be used in the completeness proof.

Definition 11.5 Let T_0 be a complete and consistent theory. For each $r \in A$ and formula φ such that $T_0 \vdash (r)\Diamond\varphi$, let T_φ be the complete theory such that $T_\varphi \approx T_0$ and $T_\varphi \vdash (r)(\varphi \& \Diamond E_{T_\varphi})$ (as guaranteed by (b) of Corollary 11.2). Then we define the following possibilistic Kripke model

$$K_0 = (W_0, e_0, \pi_0)$$

where

- $W_0 = \{T_0\} \cup \{T_\varphi \mid \varphi \text{ formula}\}$ is the set of worlds,
- $e_0 : W_0 \times Var \rightarrow A$ is defined by $e_0(T, p) = s$ whenever $T \vdash (s)p$,[10] and
- $\pi_0 : W_0 \rightarrow A$ is defined by $\pi_0(T) = s$ if $T \vdash (s)\Diamond E_T$. ⊣

Note that, so defined, there is at least some $T \in W_0$ such that $\pi_0(T) = 1$. Indeed, since $T_0 \vdash (1)\Diamond\top$, the theory $T_\top$ is such that $T_\top \vdash (1)(\top \& \Diamond E_\top)$, i.e. $T_\top \vdash (1)\Diamond E_\top$. Then, by definition, $\pi_0(T_\top) = 1$. Therefore, $K_0 = (W_0, e_0, \pi)$ is indeed a possibilistic Kripke model according to Definition 11.2. Moreover, it is a finite model, i.e. W_0 is finite. Indeed, W_0 contains at most as many theories T as m.e.c.s E in **mec**, and it is clear that **mec** is a finite set.

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[10] This definition is sound due to (c) of Lemma 11.3.

The truth-evaluation of a formula φ in a world $T \in W_0$, $\|\varphi\|_{T,K_0}$, is defined as usual (see the paragraph after Definition 11.2). In particular, for any formula ψ we have

$$\|\Diamond\psi\|_{K_0} = \max_{T \in W_0} \{\pi(T) \odot \|\psi\|_{T,K_0}\}.$$

Lemma 11.6 (Truth Lemma) *For each formula ψ, value r and $T \in W_0$,*

$$T \vdash (r)\psi \textit{ iff } \|\psi\|_{T,K_0} = r.$$

Proof The proof is by induction, the interesting induction step being for $\psi = \Diamond\varphi$.

Assume first $T \vdash (r)\Diamond\varphi$, and reason as follows. Since $T \in W_0$, i.e. $T \approx T_0$, by definition $T_0 \vdash (r)\Diamond\varphi$ as well. Then by Corollary 11.2, this happens if and only if: (a) $T' \vdash (\leq r)(\varphi \mathbin{\&} \Diamond E_{T'})$ for any $T' \approx T_0$, and (b) there exists $T_\varphi \in W_0$ such that $T_\varphi \vdash (r)(\varphi \mathbin{\&} \Diamond E_{T_\varphi})$. Then this is in turn equivalent to the following equalities:

$$\begin{aligned} r &= \max\{s \mid T' \vdash (s)(\varphi \mathbin{\&} \Diamond E_{T'}), T' \in W_0\} \\ &= \max\{s_1 \odot s_2 \mid T' \vdash (s_1)\varphi,\ T' \vdash (s_2)\Diamond E_{T'}, T' \in W_0\} \\ &= \max\{s_1 \odot s_2 \mid s_1 = \|\varphi\|_{T',K_0}, s_2 = \pi_0(T'), T' \in W_0\} \\ &= \max\{\|\varphi\|_{T',K_0} \odot \pi_0(T') \mid T' \in W_0\} \\ &= \|\Diamond\varphi\|_{K_0}. \end{aligned}$$

Note that in the third equality we apply the induction hypothesis to $(s_1)\varphi$.

For the right-to-left implication, assume $\|\Diamond\varphi\|_{T,K_0} = r$. Since we have already proved the converse implication, we know that $T \not\vdash (r')\Diamond\varphi$ for every $r' \neq r$. Since $T \in W_0$, it is complete and consistent, and by (c) of Lemma 11.3, we get that $T \vdash (r)\Diamond\varphi$.

Theorem 11.2 (Strong Completeness) *$Pos(\mathbf{A}^c_\Delta)$ is strongly complete with respect to the class of A-valued possibilistic Kripke frames, that is, the following conditions are equivalent for any set of formulas $\Gamma \cup \{\varphi\}$:*

(1) $\Gamma \vdash \varphi$
(2) $\Gamma \models \varphi$
(3) For any reduced *(and thus finite) possibilistic Kripke model $K = (W, e, \pi)$ and $w \in W$, if $\|\psi\|_{w,K} = 1$ for all $\psi \in \Gamma$, then $\|\varphi\|_{w,K} = 1$.*

Proof (1) $\Rightarrow$ (2) is soundness (Theorem 11.1) and (2) $\Rightarrow$ (3) is trivial. As for (3) $\Rightarrow$ (1), assume $\Gamma \not\vdash \varphi$. Then $\{(1)\psi \mid \psi \in \Gamma\} \cup \{(< 1)\varphi\}$ is consistent, hence it can be extended to a complete theory T_0. It is clear that $T_0 \vdash (1)\psi$ for every $\psi \in \Gamma$. Moreover, since T_0 is complete, $T_0 \vdash (r)\varphi$, for some $r < 1$. We then build a possibilistic Kripke model $K_0 = (W_0, e, \pi)$ like in Definition 11.5, hence with W_0 being finite. Then, by Lemma 11.6, $\|\psi\|_{T_0,K_0} = 1$ for all $\psi \in \Gamma$ and $\|\varphi\|_{T_0,K_0} = r$, and hence $\|\varphi\|_{T_0,K_0} = r < 1$.

Completeness with respect to reduced models trivially implies that for every finite number of propositional variables n, the corresponding finitary consequence $\vdash$ relation is *decidable*. To conclude, we would also like to notice that the above strong completeness result could also be obtained from a weak completeness result (i.e. completeness for theorems) taking into account Corollary 11.1 and the $\triangle$-deduction theorem.

11.4 Conclusions and Further Work

In this short chapter we have shown how the approach of Hájek et al. (1994) can be easily adapted to define a many-valued modal system that capture reasoning with natural generalizations of possibility and necessity measures over many-valued formulas in a general finite setting.

As recalled in Sect. 11.2, in the classical framework, when the possibility distributions and the accessibility relations are crisp ({0, 1}-valued), possibilistic systems correspond to the classical modal system KD45, which is sound and complete with respect to the class of Kripke frames with serial, transitive and Euclidean accessibility relations. In other words, in the classical setting the tautologies of KD45-models are the same than the tautologies of possibilistic models.

This result extends without difficulty to the many-valued framework when the accessibility relations and the possibility distributions remain {0, 1}-valued. However it is currently unknown whether it also extends in the general many-valued case, when the accessibility relations and possibility distributions are both many-valued (not necessarily finitely-valued, like in this chapter). So the following problem remains *open*: is every tautology of the class of possibilistic models (as defined here in this chapter) a tautology of the class of Kripke models whose accessibility relations are serial, transitive and Euclidean?

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Chapter 12
The Quest for the Basic Fuzzy Logic

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12.1 Introduction

The present work intends to be both a survey and a position paper, conceived as a homage to Petr Hájek and his absolutely crucial contributions to Mathematical Fuzzy Logic (MFL). Our aim is to present some of the main developments in the area, starting with Hájek's seminal works and continuing with the contributions of many others, and we want to do it by taking the search of the basic fuzzy logic as the leitmotif. Indeed, as it will be apparent in the short historical account given later in this introduction, this search has been one of the main reasons for the development of new weaker systems of fuzzy logics and the necessary mathematical apparatus to deal with them. Hájek started the quest when he proposed his basic fuzzy logic BL, complete with respect to the semantics given by all continuous t-norms. Later weaker systems, such as MTL, UL or psMTL^r, complete with respect to broader (but still meaningful for fuzzy logics) semantics, have been introduced and disputed the throne of the basic fuzzy logic. We survey the development of these systems with

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a stress on how they have yielded systematical approaches to MFL. The chapter is also a position paper because we contribute to the quest with our own proposal of a basic fuzzy logic. Indeed, we put forth a very weak logic called SL^{ℓ}, introduced and studied by Cintula and Noguera (2011), Cintula et al. (2013), and propose it as a base of a new framework which allows to work in a uniform way with both propositional and first-order fuzzy logics. We present a wealth of results to illustrate the power and usefulness of this framework, which support our thesis that, from a well-defined point of view, SL^{ℓ} can indeed be seen as the basic fuzzy logic.[1]

12.1.1 T-Norm Based Fuzzy Logics

Mathematical Fuzzy Logic (MFL) started as the study of logics based on particular continuous t-norms,[2] most prominently Łukasiewicz logic Ł, Gödel–Dummett logic G and Product logic Π. These logics are rendered in a language with the truth-constant $\overline{0}$ (falsum) and binary connectives $\to$ (implication) and & (fusion, residuated/strong conjunction). They are complete with respect to the *standard semantics*, which has the real-unit interval $[0, 1]$ as the set of truth degrees and interprets falsum $\perp$ by 0, fusion & by the corresponding t-norm, and the implication $\to$ by its residuum,[3] which always exists for continuous t-norms. On the other hand, these systems are also complete with respect to an algebraic semantics (MV-algebras, Gödel algebras, and product algebras, respectively) and with respect to the subclass of their linearly ordered members, also known as (MV-/Gödel/product) *chains*. These three algebraic semantics are mutually incomparable superclasses of Boolean algebras, which amounts to say that Ł, G and Π are mutually incomparable subclassical logics. In fact, classical logic can be retrieved as axiomatic extension of any of these three systems obtained by adding the excluded middle axiom.

In this context, Petr Hájek introduced a natural question: is it possible to see Ł, G and Π (and, in general, any fuzzy logic with a continuous t-norm-based semantics) as extensions of the same fuzzy logic? In other words: is there *a basic fuzzy logic* underlying all (by then) known fuzzy logic systems? As an answer to this question, he introduced in his monograph (Hájek 1998b) a system, weaker than Ł, G and Π, which he named BL (for *basic logic*). This logic was given by means of a Hilbert-style calculus in the language $\mathscr{L} = \{\&, \to, \overline{0}\}$ of type $\langle 2, 2, 0\rangle$, with the deduction rule of *modus ponens* (MP)—from φ and $\varphi \to \psi$ infer ψ—and the following axioms (taking $\to$ as the least binding connective):

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[1] The chapter is presented (almost) without proofs because (almost) all the claims follow from results proved in previous works. When necessary, we explain how the particular formulations used in this chapter follow from previous results in the literature.

[2] T-norms are commutative, associative, and monotone binary operations on the real unit interval with 1 as the neutral element; see Klement et al. (2000) for a reference book on t-norm and Běhounek et al. (2011) for a detailed survey and historical account on the role of t-norms in mathematical fuzzy logic.

[3] If $*$ is a t-norm, its residuum is defined as the binary function $x \Rightarrow y = \sup\{z \in [0, 1] \mid x * z \leqslant y\}$.

(A1) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
(A2) $\varphi \mathbin{\&} \psi \to \varphi$
(A3) $\varphi \mathbin{\&} \psi \to \psi \mathbin{\&} \varphi$
(A4) $\varphi \mathbin{\&} (\varphi \to \psi) \to \psi \mathbin{\&} (\psi \to \varphi)$
(A5a) $(\varphi \to (\psi \to \chi)) \to (\varphi \mathbin{\&} \psi \to \chi)$
(A5b) $(\varphi \mathbin{\&} \psi \to \chi) \to (\varphi \to (\psi \to \chi))$
(A6) $((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$
(A7) $\overline{0} \to \varphi$

Other connectives are introduced as follows:

$$\begin{aligned} \varphi \wedge \psi &= \varphi \mathbin{\&} (\varphi \to \psi) & \neg\varphi &= \varphi \to \overline{0} \\ \varphi \vee \psi &= ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi) & \overline{1} &= \neg\overline{0} \\ \varphi \leftrightarrow \psi &= (\varphi \to \psi) \wedge (\psi \to \varphi) \end{aligned}$$

Petr Hájek also introduced the corresponding algebraic semantics for his logic. A BL-*algebra* is an algebra $\boldsymbol{A} = \langle A, \wedge, \vee, \cdot, \to, 0, 1\rangle$ such that

- $\langle A, \wedge, \vee, 0, 1\rangle$ is a bounded lattice
- $\langle A, \cdot, 1\rangle$ is a commutative monoid
- for each $a, b, c \in A$ we have

$$\begin{aligned} &a \cdot b \leqslant c \quad \text{iff} \quad b \leqslant a \to c && \text{(residuation)} \\ &(a \to b) \vee (b \to a) = 1 && \text{(prelinearity)} \\ &a \cdot (a \to b) = b \cdot (b \to a) && \text{(divisibility)} \end{aligned}$$

We say that a BL-algebra is:

- *Linearly ordered* (or BL-chain) if its lattice order is total.
- *Standard* if its lattice reduct is the real unit interval [0, 1] ordered in the usual way.

Note that in a standard BL-algebra & is interpreted by a continuous t-norm and $\to$ by its residuum; and vice versa: each continuous t-norm fully determines its corresponding standard BL-algebra.

Hájek proved completeness of BL with respect to BL-algebras and BL-chains and conjectured that BL should be also complete with respect to the standard BL-algebras (i.e., the semantics given by all continuous t-norms). The conjecture was later proved true: Hájek himself showed the completeness by adding two additional axioms (Hájek 1998a) which later were shown to be derivable in BL (Cignoli et al. 2000). Therefore, BL could really be seen, at that time, as a *basic fuzzy logic*. Indeed, it was a genuine *fuzzy* logic because it retained what was then seen as the defining property of fuzzy logics: completeness with respect to a semantics based on continuous t-norms. And it was also *basic* in the following two senses:

1. *it could not be made weaker without losing essential properties and*
2. *it provided a base for the study of all fuzzy logics.*

The first item followed from the completeness of BL w.r.t. the semantics given by *all* continuous t-norms; thus, in a context of continuous t-norm based logics one could not possibly take a weaker system. The second meaning relied on the fact that the three main fuzzy logics (Ł, G, and Π) are all axiomatic extensions of BL and, in fact, the methods used by Hájek to introduce, algebraize, and study BL could be utilized for any other logic based on continuous t-norms. Actually, already Hájek (1998b) developed a uniform mathematical theory for MFL. He considered all axiomatic extensions of BL (not just the three prominent ones) as fuzzy logics (he called them *schematic extensions*) and systematically studied their first-order extensions (inspired by Rasiowa (1974)), extensions with modalities, complexity issues, etc.

Moreover, mainly thanks to the availability of good mathematical characterizations for continuous t-norms and BL-chains, BL has turned out to be a crucial logical system giving rise to an intense research with lots of nice results obtained by many authors (for an up-to-date survey see Busaniche and Montagna (2011)). For these reasons, we want to take on the occasion of the present tribute volume to Petr Hájek to propose that both BL logic and BL-algebras should rightfully be renamed after their creator as *Hájek logic* and *Hájek algebras* (HL and HL-algebras, for short).

Another strong reason supporting abandoning the name 'Basic Logic' is that the development of MFL has shown that HL was actually *not basic enough*. That is, HL was indeed a good basic logic for the initial framework in which it was formulated, but the active research area that Hájek helped creating with his monograph and his weakest logical system soon expanded its horizons to broader frameworks which demanded a revision of the basic logic. Therefore, Hájek had not settled but only initiated the quest for the basic fuzzy logic.

The first step towards a broader point of view was taken by Esteva and Godo, who noticed that the necessary and sufficient condition for a t-norm to have a residuum is not continuity, but left-continuity. Inspired by this fact they introduced by Esteva and Godo (2001) the logic MTL (shorthand for *Monoidal t-norm based Logic*) as an attempt to axiomatize the standard semantics given by all residuated t-norms. It was introduced by means of a Hilbert-style calculus in the language $\mathscr{L} = \{\&, \to, \wedge, \overline{0}\}$ of type $\langle 2, 2, 2, 0\rangle$, ($\wedge$ is no longer a derived connective and has to be considered as primitive). This calculus is the same as the one for HL only axiom (A4) is replaced by the following three axioms:

(A4a) $\varphi \wedge \psi \to \varphi$
(A4b) $\varphi \wedge \psi \to \psi \wedge \varphi$
(A4c) $\varphi \,\&\, (\varphi \to \psi) \to \varphi \wedge \psi$

Similarly to the previous cases, Esteva and Godo introduced a broader class of algebraic structures, MTL-algebras (defined analogously to HL-algebras but without requiring the fulfilment of the divisibility condition) and proved that MTL is complete both w.r.t. the semantics given by all MTL-algebras and w.r.t. MTL-chains. Moreover, Jenei and Montagna (2002) indeed proved MTL to be complete with respect to the semantics given by all left-continuous (i.e. residuated) t-norms. Thus it was a better candidate than HL for a basic fuzzy logic, which could be retrieved as the axiomatic extension of MTL by axiom (A4).

12.1.2 Core Fuzzy Logics

In the broader framework for MFL promoted by Esteva and Godo, i.e. that of logics based on residuated t-norms, the system MTL indeed fulfilled our requirements for a basic fuzzy logic, as expressed in the previous subsection. It was a genuine fuzzy logic enjoying a standard completeness theorem w.r.t. a semantics based on left-continuous t-norms, it could not be made weaker without losing this property and all known fuzzy logics could be obtained as extensions of MTL, thus providing a good base for a new systematical study of MFL. In fact, Petr Hájek saluted MTL as the new basic fuzzy logic and defined (Hájek and Cintula 2006) a precise general framework taking MTL as the basic system and not restricting to its axiomatic extensions (i.e. logics in the same language as MTL) but rather to its axiomatic expansions (by allowing new additional connectives). In particular they introduced two classes of logics which, though not very broad from the general perspective of the whole logical landscape, are still large enough to cover the most studied fuzzy logics. These two classes have provided a useful framework for a general study of these logics and have been utilized in particular in the study of completeness of (propositional and first-order) fuzzy logics w.r.t. distinguished semantics (Cintula et al. 2009) and the arithmetical complexity of first-order fuzzy logics (Montagna and Noguera 2010). The rough idea was to capture, by simple syntactic means, a class of logics which share many desirable properties with MTL.

Definition 12.1 A logic L in a language $\mathscr{L}$ is a *core fuzzy logic* if:

1. L expands MTL.
2. For all $\mathscr{L}$-formulae φ, ψ, χ the following holds:[4]

$$\varphi \leftrightarrow \psi \vdash_{\mathrm{L}} \chi \leftrightarrow \chi', \tag{Cong}$$

   where χ' is a formula resulting from χ by replacing some occurrences of its subformula φ by the formula ψ.
3. L has an axiomatic system with *modus ponens* as the only deduction rule.

Therefore, core fuzzy logics are essentially well-behaved axiomatic expansions of MTL.[5] Observe, that since MTL is a finitary logic[6] and we are only considering adding axioms, not rules, all core fuzzy logics remain finitary. Table 12.1 collects prominent members of the family of core fuzzy logics together with the axioms one needs to add to MTL to obtain them (see the definition of axioms in Table 12.2). An important logic, which does not fall under the scope of the previous definition,

[4] By $\vdash_{\mathrm{L}}$ we denote the provability relation in L, see Sect. 12.2.1 for the formal definition.

[5] The original definition of core fuzzy logics (Hájek and Cintula 2006, Convention 1) required the validity of a variant of deduction theorem (see Theorem 12.1), but is shown equivalent with our definition (Hájek and Cintula 2006, Proposition 3); analogously for $\triangle$-core fuzzy logics introduced in the next definition.

[6] This means that whenever $\Gamma \vdash_{\mathrm{MTL}} \varphi$, there is a finite $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash_{\mathrm{MTL}} \varphi$.

Table 12.1 Some axiomatic extensions of MTL obtained by adding the corresponding additional axiom schemata

Logic	Additional axiom schemata	References
SMTL	(PC)	Hájek (2002)
ΠMTL	(Can)	Hájek (2002)
WCMTL	(WCan)	Montagna et al. (2006)
IMTL	(Inv)	Esteva and Godo (2001)
WNM	(WNM)	Esteva and Godo (2001)
NM	(Inv) and (WNM)	Esteva and Godo (2001)
C_nMTL	(C_n)	Ciabattoni et al. (2002)
C_nIMTL	(Inv) and (C_n)	Ciabattoni et al. (2002)
HL (BL)	(Div)	Hájek (1998b)
SHL (SBL)	(Div) and (PC)	Esteva et al. (2000)
Ł	(Div) and (Inv)	Hájek (1998b); Łukasiewicz (1920)
Π	(Div) and (Can)	Hájek et al. (1996)
G	(C)	Hájek (1998b); Dummett (1959); Gödel (1932)
CPC	(EM)	

Table 12.2 Some usual axiom schemata in fuzzy logics

Axiom schema	Name
$\neg\neg\varphi \to \varphi$	Involution (Inv)
$\neg\varphi \vee ((\varphi \to \varphi \,\&\, \psi) \to \psi)$	Cancellation (Can)
$\neg(\varphi \,\&\, \psi) \vee ((\psi \to \varphi \,\&\, \psi) \to \varphi)$	Weak Cancellation (WCan)
$\varphi \to \varphi \,\&\, \varphi$	Contraction (C)
$\varphi^{n-1} \to \varphi^n$	n-Contraction (C_n)
$\varphi \wedge \neg\varphi \to \overline{0}$	Pseudocomplementation (PC)
$\varphi \wedge \psi \to \varphi \,\&\, (\varphi \to \psi)$	Divisibility (Div)
$(\varphi \,\&\, \psi \to \overline{0}) \vee (\varphi \wedge \psi \to \varphi \,\&\, \psi)$	Weak Nilpotent Minimum (WNM)
$\varphi \vee \neg\varphi$	Excluded Middle (EM)

is the expansion of MTL with the Monteiro–Baaz projection connective $\triangle$ (Baaz 1996; Monteiro 1980). This logic, denoted as $\mathrm{MTL}_\triangle$, is obtained by adding the unary connective $\triangle$ to the language, the rule of $\triangle$-*Necessitation* ($\mathrm{Nec}_\triangle$)—from φ infer $\triangle\varphi$—and the following axioms:

($\triangle$1) $\triangle\varphi \vee \neg\triangle\varphi$
($\triangle$2) $\triangle(\varphi \vee \psi) \to \triangle\varphi \vee \triangle\psi$
($\triangle$3) $\triangle\varphi \to \varphi$
($\triangle$4) $\triangle\varphi \to \triangle\triangle\varphi$
($\triangle$5) $\triangle(\varphi \to \psi) \to (\triangle\varphi \to \triangle\psi)$

Taking $\mathrm{MTL}_\triangle$ as an alternative basic logic, Hájek and Cintula defined another class of fuzzy logics, now with the $\triangle$ connective:

Definition 12.2 A logic L in a language $\mathscr{L}$ is a $\triangle$-*core fuzzy logic* if:

1. L expands $\mathrm{MTL}_\triangle$.
2. For all $\mathscr{L}$-formulae φ, ψ, χ the following holds:

$$\varphi \leftrightarrow \psi \vdash_{\mathrm{L}} \chi \leftrightarrow \chi', \qquad \text{(Cong)}$$

   where χ' is a formula resulting from χ by replacing some occurrences of its subformula φ by the formula ψ.
3. L has an axiomatic system with *modus ponens* and ($Nec_\triangle$) as the only deduction rules.

Expansions of fuzzy logics with $\triangle$ were already systematically studied by Petr Hájek already in his seminal monograph (Hájek 1998b) and have since then been considered for most fuzzy logics, making the class of $\triangle$-core fuzzy logics another largely populated useful class.

Other well-known fuzzy logics in expanded languages fall under the scope of the two classes we have introduced, such as logics with truth-constants for intermediate truth-values (a Petr Hájek's variant of the Pavelka's extension of Łukasiewicz logic (Hájek 1998b; Pavelka 1979; Novák 1990) later studied in many works by other authors (Savický et al. 2006; Esteva et al. 2009; Esteva et al. 2011, Sect. 2)), logics $\mathrm{L}_\sim$ expanded with an extra involutive negation (again initiated by Petr Hájek and followed by others (Esteva et al. 2000; Cintula et al. 2006; Esteva et al. 2000; Flaminio and Marchioni 2006; Haniková and Savický 2008; Esteva et al. 2011, Sect. 4)), or logics combining conjunctions and implications corresponding to different t-norms (Cintula 2003; Horčík and Cintula 2004; Montagna and Panti 2001; Esteva et al. 2001; Esteva et al. 2011, Sect. 5). On the other hand there are logics expanding MTL studied in the literature which are neither core nor $\triangle$-core because they need some additional deduction rules, the prominent examples being the logic PL' (the extension of Łukasiewicz logic with an additional product-like conjunction which has no zero-divisors (Horčík and Cintula 2004)) or logics with truth-hedges (Esteva et al. 2013).

Core and $\triangle$-core fuzzy logics are all finitary and well-behaved from several points of view. In particular, for every such logic L one can define in a natural way a corresponding class of algebraic structures, L-algebras, which provide a complete semantics as in the case of MTL or the previously mentioned logics and, more importantly, the completeness theorem is preserved if we restrict ourselves to *linearly ordered* L-algebras. Moreover, these classes of algebras are always *varieties*, i.e. they can be presented in terms of equations or, equivalently, are closed under formation of homomorphic images, subalgebras and direct products.

Another interesting property shared by the logics in these classes is the deduction theorem. Petr Hájek (1998b) already proved deduction theorems for several fuzzy logics, including his basic logic, and also for his expansions with $\triangle$. One can analogously obtain deduction theorems for all the logics in the classes just defined (in a local form for core fuzzy logics, global for $\triangle$-core):[7]

Theorem 12.1

1. *Let* L *be a core fuzzy logic in a language* $\mathscr{L}$. *For every* $\Gamma \cup \{\varphi, \psi\} \subseteq Fm_{\mathscr{L}}$, $\Gamma, \varphi \vdash_{\mathrm{L}} \psi$ *iff there is* $n \geqslant 0$ *such that* $\Gamma \vdash_{\mathrm{L}} \varphi^n \to \psi$.
2. *Let* L *be a core* $\triangle$*-fuzzy logic in a language* $\mathscr{L}$. *For every* $\Gamma \cup \{\varphi, \psi\} \subseteq Fm_{\mathscr{L}}$, $\Gamma, \varphi \vdash_{\mathrm{L}} \psi$ *iff* $\Gamma \vdash_{\mathrm{L}} \triangle\varphi \to \psi$.

We will give more precise details both about algebraization of logics and about deduction theorems in Sect. 12.2.

12.1.3 Substructural Logics as a Framework for Fuzzy Logics

The quest for the basic fuzzy logic did not end with MTL (or $\mathrm{MTL}_{\triangle}$). Indeed, MTL has been further weakened in two different directions beyond the framework of core fuzzy logics:

- by dropping commutativity of conjunction Hájek (2003b) obtained the logic psMTL^r which Jenei and Montagna (2003) proved to be complete with respect to the semantics on non-commutative residuated t-norms,
- by removing integrality (i.e. not requiring the neutral element of conjunction to be maximum of the order) Metcalfe and Montagna (2007) proposed the logic UL which is, in turn, complete with respect to left-continuous uninorms.

Petr Hájek liked to describe this process of successive weakening of fuzzy logics by telling the joke of the crazy scientist that studied fleas by removing their legs one by one and checking whether they could still jump (Hájek 2005b).[8] Namely, if HL was the original flea, it lost the ‘divisibility leg’ when it was substituted by MTL, and then psMTL^r and UL respectively lost the ‘commutativity and the integrality leg’ while retaining the ability to ‘jump’ (i.e., the completeness w.r.t. intended semantics based on reals).

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[7] We need to recall the following inductively defined notation: $\varphi^0 = \overline{1}, \varphi^1 = \varphi$, and $\varphi^n = \varphi^{n-1} \,\&\, \varphi$.

[8] A prominent biologist conducted a very important experiment. He trained a flea to jump upon giving her a verbal command (“Jump!”). In a first stage of the experiment he removed a flea’s leg, told her to jump, and the flea jumped. So he wrote in his scientific notebook: “Upon removing one leg all flea organs function properly”. So, he removed the second leg, asked the flea to jump, she obeyed, so he wrote again: “Upon removing the second leg all flea organs function properly”. Thereafter he removed all the legs but one, the flea jumped when ordered, so he wrote again: “Upon removing the one but last leg all flea organs function properly”. Then he removed the last leg. Told flea to jump, and nothing happened. He did not want to take a chance, so he repeated the experiment several times, and the legless flea never jumped. So he wrote the conclusion: “Upon removing the last leg the flea loses sense of hearing”.

These weaker fuzzy logics (and even MTL itself (Esteva et al. 2003)) can be fruitfully studied in the context of substructural logics. Recall the bounded full Lambek logic FL,[9] a basic substructural logic which does not satisfy any of the usual three structural rules: exchange, weakening, and contraction. Although firstly presented by means of a Gentzen calculus, it can be given a Hilbert-style presentation and shown to be an algebraizable logic in the sense of Blok and Pigozzi (1989) whose equivalent algebraic semantics is the variety of bounded pointed lattice-ordered residuated monoids (usually referred to as *bounded pointed residuated lattices* or FL-*algebras*). Intuitionistic logic together with logics $\mathrm{FL_e}$, $\mathrm{FL_w}$, and $\mathrm{FL_{ew}}$ are among the most prominent extensions of FL. These logics are obtained by adding some of the structural rules and correspond to subvarieties of residuated lattices satisfying corresponding extra algebraic properties (Galatos et al. 2007). Actually, many fuzzy logics have been shown to be axiomatic extensions of some of these prominent substructural logics by adding some axioms that enforce completeness with respect to some class of linearly ordered residuated lattices (or *chains*). For instance, Gödel–Dummett logic is the logic of linearly ordered Heyting algebras ($\mathrm{FL_{ewc}}$-chains), MTL is the logic $\mathrm{FL}^{\ell}_{\mathrm{ew}}$ of $\mathrm{FL_{ew}}$-chains,[10] UL is the logic $\mathrm{FL}^{\ell}_{\mathrm{e}}$ of $\mathrm{FL_e}$-chains, and psMTL^{r} is the logic $\mathrm{FL}^{\ell}_{\mathrm{w}}$ of $\mathrm{FL_w}$-chains.

This common feature, completeness with respect to their corresponding linearly ordered algebraic structures, has motivated the methodological paper (Běhounek and Cintula 2006) where the authors postulate that *fuzzy logics are the logics of chains*, in the sense that they are logics complete with respect to a semantics of chains. However, all the fuzzy logics mentioned so far do enjoy a stronger property: the *standard completeness theorem*, i.e. completeness with respect to a semantics of algebras defined on the real unit interval [0, 1] which Petr Hájek and many others have considered to be the intended semantics for fuzzy logics. Following Hájek's flea joke, we could say that those fleas are fuzzy logics that *jump well* provided that they satisfy standard completeness. Actually, many authors implicitly (and sometimes even explicitly, e.g. Metcalfe and Montagna (2007)) regard standard completeness as an essential requirement for fuzzy logics. It is, thus, reasonable to expect any candidate for the basic fuzzy logic to satisfy this stronger requirement. But, although they fulfill that, neither $\mathrm{FL}^{\ell}_{\mathrm{e}}$ nor $\mathrm{FL}^{\ell}_{\mathrm{w}}$ can be taken as basic because they are not comparable and hence do not satisfy our second meaning of *basic*. A reasonable candidate could be the logic FL^{ℓ} of FL-chains (a common generalization of $\mathrm{FL}^{\ell}_{\mathrm{e}}$ and $\mathrm{FL}^{\ell}_{\mathrm{w}}$). But, interestingly enough, this logic does not enjoy the standard completeness (as proved by Wang and Zhao (2009)) and, therefore, we must discard it. Moreover, one can also argue that FL^{ℓ} is still *not basic enough* (in the first meaning) because it satisfies a remaining structural rule: associativity. Hence, in the context of substructural logics, it could still be made weaker by removing associativity.

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[9] We use this notation for simplicity in this introduction, even though in the literature the symbol FL is usually used for the unbounded full Lambek logic whereas the bounded FL is denoted as $\mathrm{FL}_{\bot}$.

[10] As notation convention (later precisely introduced in Definition 12.11) given a logic L, we denote by L^{ℓ} the logic of L-chains.

Table 12.3 Axiomatic system of SL

(R)	$\varphi \to \varphi$	(R′)	$\overline{1} \to (\varphi \to \varphi)$
(MP)	$\varphi, \varphi \to \psi \vdash \psi$	(Push)	$\varphi \to (\overline{1} \to \varphi)$
(Sf)	$\varphi \to \psi \vdash (\psi \to \chi) \to (\varphi \to \chi)$	($\overline{1}$)	$\overline{1}$
(Pf)	$\psi \to \chi \vdash (\varphi \to \psi) \to (\varphi \to \chi)$	(Bot)	$\bot \to \varphi$
(As)	$\varphi \vdash (\varphi \to \psi) \to \psi$	($\wedge$1)	$\varphi \wedge \psi \to \varphi$
(As$_{\ell\ell}$)	$\varphi \to ((\varphi \rightsquigarrow \psi) \to \psi)$	($\wedge$2)	$\varphi \wedge \psi \to \psi$
(Symm$_1$)	$\varphi \rightsquigarrow \psi \vdash \varphi \to \psi$	($\wedge$3)	$(\chi \to \varphi) \wedge (\chi \to \psi) \to (\chi \to \varphi \wedge \psi)$
(E$_{\rightsquigarrow 1}$)	$\varphi \to (\psi \to \chi) \vdash \psi \to (\varphi \rightsquigarrow \chi)$	($\vee$1)	$\varphi \to \varphi \vee \psi$
(Res$_1$)	$\psi \to (\varphi \to \chi) \vdash \varphi \,\&\, \psi \to \chi$	($\vee$2)	$\psi \to \varphi \vee \psi$
(Adj$_\&$)	$\varphi \to (\psi \to \psi \,\&\, \varphi)$	($\vee$3)	$(\varphi \to \chi) \wedge (\psi \to \chi) \to (\varphi \vee \psi \to \chi)$
(Adj)	$\varphi, \psi \vdash \varphi \wedge \psi$	($\vee$3$_\rightsquigarrow$)	$(\varphi \rightsquigarrow \chi) \wedge (\psi \rightsquigarrow \chi) \to (\varphi \vee \psi \rightsquigarrow \chi)$

There have actually been several studies on non-associative substructural logics, starting with the original Lambek non-associative calculus (Lambek 1961) (without lattice connectives), and followed (in the full language) e.g. by Buszkowski and Farulewski (2009). Recently, a general algebraic framework to study fuzzy logics considered as a subfamily of (not necessarily associative) substructural logics has been developed by Cintula and Noguera (2011). It is based on the logic SL, a non-associative version of the bounded Full Lambek calculus, introduced by Galatos and Ono (2010).[11] SL is formulated in the language $\mathscr{L}_{\mathrm{SL}} = \{\wedge, \vee, \&, \to, \rightsquigarrow, \overline{0}, \overline{1}, \bot\}$ (we also make use of the defined connectives $\varphi \leftrightarrow \psi = (\varphi \to \psi) \wedge (\psi \to \varphi)$ and $\top = \bot \to \bot$)[12] and axiomatized by means of the Hilbert-style calculus (Galatos and Ono 2010, Fig. 5) presented in Table 12.3.

Moreover, Galatos and Ono proved that SL is an algebraizable logic whose equivalent algebraic semantics is the variety of bounded lattice-ordered residuated unital groupoids, where the monoidal structure of the previous logics has become just a groupoid on account of the lack of associativity (we will see more details in Sect. 12.2). Therefore, if we are looking for a logic complete with respect to standard chains in the non-associative context, it makes sense to consider, in a similar fashion as with FL and its extensions, the logic SL^ℓ as the logic of bounded linearly ordered residuated unital groupoids.

12.1.4 Goals and Outline of the Chapter

The main goal of this chapter is to propose SL^ℓ as a new basic fuzzy logic. The current stage of development in MFL requires a broader framework than that provided

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[11] Technically speaking, Galatos and Ono introduced an unbounded version of this logic and actually never named it. The name SL, standing for 'substructural logic', was proposed by Cintula and Noguera (2011).

[12] When writing formulae in this language we will assume that the increasing binding order of connectives is: first &, then $\{\wedge, \vee\}$, and finally $\{\to, \rightsquigarrow, \leftrightarrow\}$.

by core and △-core fuzzy logics. This is witnessed by several works (some already mentioned) dealing with fuzzy logics which either need some additional deduction rule or are weaker than MTL, e.g. Botur (2011), Esteva et al. (2013), Gabbay and Metcalfe (2007), Hájek (2003a, b, c), Hájek and Ševčík (2004), Horčík and Cintula (2004), Jenei and Montagna (2003), Marchioni and Montagna (2008), Metcalfe and Montagna (2007), Wang and Zhao (2009). For this reason fuzzy logics have started being systematically studied in the context of (not necessarily associative) substructural logics by Cintula and Noguera (2011). However, this work did not offer a basic fuzzy logic in its framework.

On the other hand, SL^{ℓ} has been introduced and studied as an axiomatic extension of SL in the recent paper (Cintula et al. 2013) where, among others, it has been shown to enjoy standard completeness. Based on these results we will defend here the thesis that SL^{ℓ} can serve as a basic fuzzy logic, good enough for the current needs of MFL. We will argue that is genuinely fuzzy and basic in both senses mentioned earlier in this introduction. To this end, we introduce a new class of logics containing core and △-core fuzzy logics and much more: *core semilinear logics*. The adjective 'semilinear' in the name of this class refers to a notion introduced by Cintula and Noguera (2010) in order to capture the idea of fuzzy logics as logics of chains proposed by Běhounek and Cintula (2006). The idea is the following: if a logic has a reasonable implication $\to$ (which is the case of SL and many of its expansions like core and △-core fuzzy logics) then its corresponding algebraic structures can be ordered in terms of the implication ($a \leqslant b$ iff $a \to b \geqslant 1$); the logic is said to be *semilinear* iff it is complete w.r.t. the class of algebras where the order just defined is total. Moreover, the class of core semilinear logics contains both core and △-core fuzzy logics and is defined in formally analogous way. Actually, the class of core semilinear logics provides a convenient intermediate level of generality, between that of core and △-core fuzzy logics and that of weakly implicative semilinear logics of Cintula and Noguera (2011), by fixing SL^{ℓ} (and, therefore, its language) as a common base and allowing for non-axiomatic extensions.

Outline of the chapter After this introduction that has presented the topic (historically and conceptually), the main logical systems, the classes of (△-)core fuzzy logics, and the motivation for the forthcoming class of core semilinear logics,[13] Section 12.2 presents, in mathematical details, the necessary logical and algebraic framework for our approach, which mainly restricts to substructural logics understood as well-behaved expansions of the non-associative logic SL. Section 12.2.1 gives the basic notions, Sect. 12.2.2 presents the useful syntactical notion of almost (MP)-based logics, and Sect. 12.2.3 is devoted to generation of filters, algebraization, and completeness w.r.t. (finitely) subdirectly irreducible algebras. Section 12.3, as the central part of the chapter, focuses on propositional core semilinear logics. After defining them, Sect. 12.3.1 shows several useful characterizations of semilinear

[13] Although we have tried to make this chapter reasonably self-contained, the obvious space limitations do not allow for an extensive presentation of all mentioned logical systems. For an up-to-date encyclopedical account of Mathematical Fuzzy Logic see Cintula et al. (2011).

logics and their axiomatizations, including a presentation of SL^ℓ as axiomatic extension of SL; Section 12.3.2 is a survey on completeness properties of core semilinear logics w.r.t. significant algebraic semantics, in particular we stress the standard completeness of SL^ℓ. Finally, Sect. 12.4 is devoted to first-order predicate counterparts of core semilinear logics, including $\mathrm{SL}^\ell\forall$, the first-order extension of SL^ℓ. Section 12.4.1 shows the axiomatization of these logics, Sect. 12.4.2 presents their semantics based on general and witnessed models, and Sect. 12.4.3 focuses again on distinguished semantics, in particular stressing that $\mathrm{SL}^\ell\forall$ enjoys standard completeness too.

12.2 Logical Framework

In order to deal with the classes of logics mentioned above, we need some flexibility as regards both propositional languages and logics. Therefore, for the sake of reference and in order to fix terminology in a convenient way for this chapter, we shall start with some standard general definitions and conventions.[14]

12.2.1 Basic Syntax and Semantics

In this chapter we consider logics as given by finitary Hilbert-style proof systems expanding that of SL (see Table 12.3 in the introduction). Following Hájek's methodology, we restrict to finitary systems as he did when proposing schematic extensions of HL as a systematical approach to MFL. This does not undermine the suitability of our proposed basic logic SL^ℓ (or its first-order counterpart) because the infinitary systems of fuzzy logic can still be retrieved as its extensions; we disregard them here for simplicity of presentation only.

A *propositional language* $\mathscr{L}$ is a *countable* type, i.e. a function $ar\colon C_{\mathscr{L}} \to \mathsf{N}$, where $C_{\mathscr{L}}$ is a countable set of symbols called *connectives*, giving for each one its *arity*. Nullary connectives are also called *truth-constants*. We write $\langle c, n\rangle \in \mathscr{L}$ whenever $c \in C_{\mathscr{L}}$ and $ar(c) = n$. The basic language in this chapter is $\mathscr{L}_{\mathrm{SL}}$ with binary connectives $\wedge, \vee, \&, \to, \rightsquigarrow$ and truth-constants $\overline{0}, \overline{1}, \bot$ (we also make use of the defined connectives $\top = \bot \to \bot$ and $\varphi \leftrightarrow \psi = (\varphi \to \psi) \wedge (\psi \to \varphi)$). Let $\textsc{Var}$ be a fixed infinite countable set of symbols called *variables*. The set $Fm_{\mathscr{L}}$ of *formulae* in a propositional language $\mathscr{L}$ is the least set containing $\textsc{Var}$ and closed under connectives of $\mathscr{L}$, i.e. for each $\langle c, n\rangle \in \mathscr{L}$ and every $\varphi_1, \ldots, \varphi_n \in Fm_{\mathscr{L}}$, $c(\varphi_1, \ldots, \varphi_n)$ is a formula. $Fm_{\mathscr{L}}$ can be seen as the domain of the absolutely

[14] The interested reader can complement the upcoming short presentation by consulting reference works on (Abstract) Algebraic Logic such as Blok and Pigozzi (1989), Burris and Sankappanavar (1981), Cintula et al. (2011). We deviate slightly from the standard treatment of some basic notions because we are tailoring them to the particular purposes of the present chapter.

free algebra $\mathit{Fm}_{\mathscr{L}}$ of type $\mathscr{L}$ and generators VAR. An $\mathscr{L}$*-substitution* is an endomorphism on the algebra $\mathit{Fm}_{\mathscr{L}}$, i.e. a mapping $\sigma\colon \mathit{Fm}_{\mathscr{L}} \to \mathit{Fm}_{\mathscr{L}}$, such that $\sigma(c(\varphi_1, \ldots, \varphi_n)) = c(\sigma(\varphi_1), \ldots, \sigma(\varphi_n))$ holds for each $\langle c, n\rangle \in \mathscr{L}$ and every $\varphi_1, \ldots, \varphi_n \in \mathit{Fm}_{\mathscr{L}}$. Since an $\mathscr{L}$-substitution is a mapping whose domain is a free $\mathscr{L}$-algebra, it is fully determined by its values on the generators (propositional variables).

An *axiomatic system* $\mathscr{AS}$ in a propositional language $\mathscr{L}$ is a pair $\langle Ax, R\rangle$ where Ax is set of formulae (the *axioms*) and R is a set of pairs $\langle \Gamma, \varphi\rangle$ (the *rules*) where Γ is a *finite* non-empty set of formulae and φ is a formula.[15] Moreover, both Ax and R are closed under arbitrary substitutions. Given $\Gamma \cup \{\varphi\} \subseteq \mathit{Fm}_{\mathscr{L}}$, we say that φ *is provable from* Γ *in* $\mathscr{AS}$, in symbols $\Gamma \vdash_{\mathscr{AS}} \varphi$, if there exists a finite sequence of formulae $\langle \varphi_0, \ldots, \varphi_n\rangle$ (a *proof*) such that:

- $\varphi_n = \varphi$, and
- for every $i \leqslant n$, either $\varphi_i \in \Gamma \cup Ax$ or there is some rule $\langle \Delta, \varphi_i\rangle \in R$ such that $\Delta \subseteq \{\varphi_0, \ldots, \varphi_{i-1}\}$.

Observe that the provability relation $\vdash_{\mathscr{AS}}$ is finitary, i.e., if $\Gamma \vdash_{\mathscr{AS}} \varphi$, then there is a finite $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash_{\mathscr{AS}} \varphi$.

Let $\mathscr{L}_1 \subseteq \mathscr{L}_2$ be propositional languages and $\mathscr{AS}_i$ an axiomatic system in $\mathscr{L}_i$. We say that $\mathscr{AS}_2$ is an expansion of $\mathscr{AS}_1$ by axioms Ax and rules R if all its axioms (rules) are $\mathscr{L}_2$-substitutional instances of axioms (rules) of $\mathscr{AS}_1$ or formulae from Ax (rules from R).

Now we are ready to give our formal convention restricting logics to finitary expansions of SL with well-behaved connectives.

Convention 12.2 *A* logic L *in a language* $\mathscr{L} \supseteq \mathscr{L}_{\mathrm{SL}}$ *is the provability relation given by an axiomatic system* $\mathscr{AS}$ *in* $\mathscr{L}$ *which is an expansion of that of* SL (*see Table* 12.3) *and for all* $\mathscr{L}$*-formulae* φ, ψ, χ *the following holds:*

$$\varphi \leftrightarrow \psi \vdash_{\mathscr{AS}} \chi \leftrightarrow \chi', \qquad \text{(Cong)}$$

where χ' *is a formula resulting from* χ *by replacing some occurrences of its subformula* φ *by a formula* ψ. *In this case we say that* $\mathscr{AS}$ *is a* presentation *of* L (*or that* L is axiomatized by $\mathscr{AS}$) *and write* $\Gamma \vdash_{\mathrm{L}} \varphi$ *instead of* $\Gamma \vdash_{\mathscr{AS}} \varphi$.

Remark 12.1 One can equivalently replace the condition (Cong) by the following:

$$\varphi \leftrightarrow \psi \vdash_{\mathscr{AS}} c(\chi_1, \ldots, \chi_i, \varphi, \ldots, \chi_n) \leftrightarrow c(\chi_1, \ldots, \chi_i, \psi, \ldots, \chi_n) \qquad (\mathrm{Cong}^i_c)$$

for each $\langle c, n\rangle \in \mathscr{L}$ and each $0 \leqslant i < n$. Therefore, since this condition is already satisfied in SL for all its connectives, in order to check whether a particular expansion

[15] Sometimes, especially when listing rules, we use the denotation $\Gamma \vdash \varphi$ rather than $\langle \Gamma, \varphi\rangle$. Also note that axioms could be seen as nullary rules; while this identification would simplify some of the upcoming formulations we have opted for a more conceptually illuminating separation of these two notions.

Table 12.4 Axioms for structural rules

a_1	$\varphi \mathbin{\&} (\psi \mathbin{\&} \chi) \to (\varphi \mathbin{\&} \psi) \mathbin{\&} \chi$	Re-associate to the left
a_2	$(\varphi \mathbin{\&} \psi) \mathbin{\&} \chi \to \varphi \mathbin{\&} (\psi \mathbin{\&} \chi)$	Re-associate to the right
e	$\varphi \mathbin{\&} \psi \to \psi \mathbin{\&} \varphi$	Exchange
c	$\varphi \to \varphi \mathbin{\&} \varphi$	Contraction
i	$\psi \to (\varphi \to \psi)$	Left weakening
o	$\overline{0} \to \varphi$	Right weakening

of SL is a logic in the sense just defined, it is enough to check (Cong^i_c) for all new connectives (this statement remains true if we replace SL by any other logic).

The notion of expansion can naturally be formulated for logics. Let $\mathscr{L}_1 \subseteq \mathscr{L}_2$ be propositional languages and $\mathrm{L_i}$ a logic in $\mathscr{L}_i$. We say that

- L_2 is the *expansion of* L_1 *by axioms Ax and rules R* if it is axiomatized by expanding some presentation of L_1 with axioms *Ax* and rules *R*.
- L_2 is an (axiomatic) *expansion of* L_1 if it is the expansion of L_1 by some axioms and rules (or just axioms respectively).

If $\mathscr{L}_1 = \mathscr{L}_2$, we use the term 'extension' instead of 'expansion'. Let $\mathscr{S}$ be a collection of extensions of a given logic L. We define the following two axiomatic systems and two logics:

$$\bigcap \mathscr{S} = \{\langle \Gamma, \varphi\rangle \mid \Gamma \vdash_{\mathrm{L}} \varphi \text{ for each } \mathrm{L} \in \mathscr{S}\} \qquad \bigwedge \mathscr{S} = \vdash_{\bigcap \mathscr{S}}$$

$$\bigcup \mathscr{S} = \{\langle \Gamma, \varphi\rangle \mid \Gamma \vdash_{\mathrm{L}} \varphi \text{ for some } \mathrm{L} \in \mathscr{S}\} \qquad \bigvee \mathscr{S} = \vdash_{\bigcup \mathscr{S}}$$

It is clear that $\bigwedge \mathscr{S}$ and $\bigvee \mathscr{S}$ are respectively the infimum and the supremum of $\mathscr{S}$ in the set of extensions of L ordered by inclusion. Therefore, the set of extensions of a given logic L always forms a complete lattice. Note that $\bigvee \mathscr{S}$ can be axiomatized by taking the union of arbitrary axiomatic systems for the logics in $\mathscr{S}$. Thus, in particular, if all logics in $\mathscr{S}$ are axiomatic extensions of L, then so is $\bigvee \mathscr{S}$. Therefore, the class of *axiomatic* extensions of L is a sub-join-semilattice of the lattice of all extensions of L. The axiomatization of meets is not so straightforward; at the end of Sect. 12.3.1 we will see how to deal with this problem in the restricted setting of core semilinear logics.

Some important axiomatic extensions of SL are obtained by adding the axioms a_1, a_2, e, c, i, o corresponding to structural rules (see Table 12.4).

Given any $S \subseteq \{a_1, a_2, \mathrm{e}, \mathrm{c}, \mathrm{i}, \mathrm{o}\}$, by SL_S we denote the axiomatic extension of SL by S. If $\{a_1, a_2\} \subseteq S$, then instead of them we write the symbol 'a'. Analogously if $\{\mathrm{i}, \mathrm{o}\} \subseteq S$, instead of them we write the symbol 'w'. Equivalent ways to formulate these axioms are known (Cintula and Noguera 2011, Theorem 2.5.7.). $\mathrm{SL_a}$ is, in fact, the bounded full Lambek logic. Next, we introduce the basic algebraic notions that will allow to provide a semantics for our logics.

Definition 12.3 A *bounded pointed lattice-ordered residuated unital groupoid*, or shortly just SL-*algebra*, is an algebra $\boldsymbol{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, 0, 1, \bot, \top\rangle$ such that

Table 12.5 Equations defining important classes of SL-algebras

a_1	$x \cdot (y \cdot z) \leqslant (x \cdot y) \cdot z$	Re-associate to the left
a_2	$(x \cdot y) \cdot z \leqslant x \cdot (y \cdot z)$	Re-associate to the right
e	$x \cdot y = y \cdot x$	Commutativity
c	$x \leqslant x \cdot x$	Square-increasingness
i	$x \leqslant 1$	Integrality
o	$0 \leqslant x$	Boundedness

- $\langle A, \wedge, \vee, \bot, \top \rangle$ is a bounded lattice
- 1 is the unit of $\cdot$
- for each $a, b, c \in A$ we have

$$a \cdot b \leqslant c \quad \text{iff} \quad b \leqslant a \backslash c \quad \text{iff} \quad a \leqslant c/b\,.$$

The class of all SL-algebras is a variety and it is denoted as $\mathbb{SL}$. Observe that the residuation condition together with the fact that 1 is a neutral element implies that for every SL-algebra $\boldsymbol{A}$ and each $a, b \in A$ we have

$$a \leqslant b \quad \text{iff} \quad 1 \leqslant a \backslash b \quad \text{iff} \quad 1 \leqslant b/a\,.$$

Given an SL-algebra $\boldsymbol{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, 0, 1, \bot, \top \rangle$, an $\boldsymbol{A}$-*evaluation* is a homomorphism from the algebra of formulae to $\boldsymbol{A}$ such that the connectives $\wedge, \vee, \&, \to, \rightsquigarrow, \overline{0}, \overline{1}, \bot, \top$ are respectively interpreted by the functions $\wedge, \vee, \cdot, \backslash, /,$ $0, 1, \bot, \top$, i.e., a mapping from $Fm_{\mathscr{L}}$ to A such that $e(*) = *$ for $* \in \{0, 1, \bot, \top\}$ and $e(\varphi \circ \psi) = e(\varphi) \circ' e(\psi)$, where $\circ \in \{\wedge, \vee, \&, \to, \rightsquigarrow\}$ and $\circ'$ is the corresponding operation from $\{\wedge, \vee, \cdot, \backslash, /\}$.[16] By means of this notion, we can give, more generally, the following definition for the algebraic counterpart of any logic.

Definition 12.4 Let L be a logic in language $\mathscr{L}$ which is the expansion of SL by axioms Ax and rules R. An $\mathscr{L}$-algebra $\boldsymbol{A}$ is an L-algebra if

- its reduct $\boldsymbol{A}_{\mathrm{SL}} = \langle A, \wedge, \vee, \cdot, \backslash, /, 0, 1, \bot, \top \rangle$ is an SL-algebra,
- for every $\varphi \in Ax$ and every $\boldsymbol{A}$-evaluation e, $e(\varphi) \geqslant 1$,
- for each $\langle \Gamma, \varphi \rangle \in R$ and each $\boldsymbol{A}$-evaluation e, if $e(\psi) \geqslant 1$ for all $\psi \in \Gamma$, then $e(\varphi) \geqslant 1$.

$\boldsymbol{A}$ is a linearly ordered (or L-*chain*) if its lattice order is total. The class of all (linearly ordered) L-algebras is denoted by $\mathbb{L}$ (or $\mathbb{L}_{lin}$ respectively).

Table 12.5 shows what equations have to be added to SL-algebras, to obtain, for arbitrary $S \subseteq \{a_1, a_2, e, c, i, o\}$, the class of SL_S-algebras.

[16] Here we opted for a rather nonstandard (in the context of algebraic logic) notational distinction between logical connectives and algebraic operations. The reason is that, in this case, the notational traditions on both sides, algebraic and logical, are so strong that any unification would not be advisable.

The following completeness theorem follows from more general results (see Sect. 12.2.3 where we show more on the connection between logics and algebras) but can also be directly proved by means of the usual Lindenbaum–Tarski process. It shows how L-algebras really give an algebraic semantics for SL and its expansions.

Theorem 12.3 *Let* L *be a logic. Then for every set of formulae Γ and every formula φ the following are equivalent:*

1. $\Gamma \vdash_{\mathrm{L}} \varphi$,
2. *for every* $\boldsymbol{A} \in \mathbb{L}$ *and every* $\boldsymbol{A}$*-evaluation* e*, if* $e(\psi) \geqslant 1$ *for every* $\psi \in \Gamma$*, then* $e(\varphi) \geqslant 1$.

12.2.2 Almost (MP)*-Based Logics and Deduction Theorems*

In the introduction we have formulated the usual deduction theorems for core and $\triangle$-core fuzzy logics (Theorem 12.1). In this section we show how this can be generalized to all logics in the present framework (expansions of SL) provided that the additional rules they satisfy are of a suitable form. Technically, this corresponds to the notion of almost (MP)-based logic that, as shown by Cintula et al. (2013), essentially allows to repeat Hájek's original proof of deduction theorem now in this wide context. To this end, we introduce a few more syntactical notions. Let $\star$ be a new propositional variable not occurring in *Var*, which acts as placeholder for a special kind of substitutions. The notions of formula and substitution are augmented by the prefix $\star$- whenever they are construed over the set of variables $\mathrm{VAR} \cup \{\star\}$ and are left as they are if construed in the original set of variables VAR. If φ and δ are $\star$-formulae, by $\delta(\varphi)$ we denote the formula obtained from δ when one replaces the occurrences of $\star$ by φ; note that if φ is a formula, then so is $\delta(\varphi)$ (i.e., $\star$ does not occur in $\delta(\varphi)$).

Definition 12.5 Given a set of $\star$-formulae Γ, we define the sets Γ^* and $\Pi(\Gamma)$ of $\star$-formulae:

- Γ^* is the smallest set containing $\star$ and $\delta(\gamma) \in \Gamma^*$ for each $\delta \in \Gamma$ and each $\gamma \in \Gamma^*$.
- $\Pi(\Gamma)$ is the smallest set of $\star$-formulae containing $\Gamma \cup \{\overline{1}\}$ and closed under &.

We are now ready to give the formal definition of almost (MP)-based logic.

Definition 12.6 Let bDT be a set of $\star$-formulae closed under all $\star$-substitutions σ such that $\sigma(\star) = \star$. A logic L is *almost* (MP)*-based w.r.t. the set of basic deduction terms* bDT if:

- L has a presentation where the only deduction rules are *modus ponens* and those of the form $\langle \varphi, \gamma(\varphi) \rangle$ for $\gamma \in \mathrm{bDT}$, and

Table 12.6 New axiomatic system for SL

(Adj$_\&$)	$\varphi \to (\psi \to \psi \,\&\, \varphi)$	(Bot)	$\bot \to \varphi$
(Adj$_{\&\rightsquigarrow}$)	$\varphi \to (\psi \rightsquigarrow \varphi \,\&\, \psi)$	(Push)	$\varphi \to (\overline{1} \to \varphi)$
(Res$'$)	$\psi \,\&\, (\varphi \,\&\, (\varphi \to (\psi \to \chi))) \to \chi$	(Pop)	$(\overline{1} \to \varphi) \to \varphi$
(Res$'_\rightsquigarrow$)	$(\varphi \,\&\, (\varphi \to (\psi \rightsquigarrow \chi))) \,\&\, \psi \to \chi$	(&∧)	$(\varphi \wedge \overline{1}) \& (\psi \wedge \overline{1}) \to \varphi \wedge \psi$
(T$'$)	$(\varphi \to (\varphi \,\&\, (\varphi \to \psi)) \,\&\, (\psi \to \chi)) \to (\varphi \to \chi)$		
(T$'_\rightsquigarrow$)	$(\varphi \rightsquigarrow ((\varphi \rightsquigarrow \psi) \,\&\, \varphi) \,\&\, (\psi \to \chi)) \to (\varphi \rightsquigarrow \chi)$		
(∧1)	$\varphi \wedge \psi \to \varphi$	(∨1)	$\varphi \to \varphi \vee \psi$
(∧2)	$\varphi \wedge \psi \to \psi$	(∨2)	$\psi \to \varphi \vee \psi$
(∧3)	$(\chi \to \varphi) \wedge (\chi \to \psi) \to (\chi \to \varphi \wedge \psi)$	(∨3)	$(\varphi \to \chi) \wedge (\psi \to \chi) \to (\varphi \vee \psi \to \chi)$
(MP)	$\varphi, \varphi \to \psi \vdash \psi$	(Adj$_u$)	$\varphi \vdash \varphi \wedge \overline{1}$
(α)	$\varphi \vdash \delta \,\&\, \varepsilon \to \delta \,\&\, (\varepsilon \,\&\, \varphi)$	(β)	$\varphi \vdash \delta \to (\varepsilon \to (\varepsilon \,\&\, \delta) \,\&\, \varphi)$
(α')	$\varphi \vdash \delta \,\&\, \varepsilon \to (\delta \,\&\, \varphi) \,\&\, \varepsilon$	(β')	$\varphi \vdash \delta \to (\varepsilon \rightsquigarrow (\delta \,\&\, \varepsilon) \,\&\, \varphi)$

- for each $\beta \in$ bDT and each formulae φ, ψ, there exist $\beta_1, \beta_2 \in$ bDT* such that:[17]

$$\vdash_{\mathrm{L}} \beta_1(\varphi \to \psi) \to (\beta_2(\varphi) \to \beta(\psi)).$$

L is called (MP)-*based* if it admits the empty set as a set of basic deduction terms.

SL can be shown to be indeed an almost (MP)-based logic. For this, of course, one needs to endow it with a convenient alternative presentation. Consider the axiomatic system from Table 12.6 and let us introduce a convenient notation for the terms appearing on the right-hand side of the rules (α), (α'), (β), and (β'). Given arbitrary formulae δ, ε, we define the following $\star$-formulae:

$$\alpha_{\delta,\varepsilon} = \delta \,\&\, \varepsilon \to \delta \,\&\, (\varepsilon \,\&\, \star) \qquad \beta_{\delta,\varepsilon} = \delta \to (\varepsilon \to (\varepsilon \,\&\, \delta) \,\&\, \star)$$
$$\alpha'_{\delta,\varepsilon} = \delta \,\&\, \varepsilon \to (\delta \,\&\, \star) \,\&\, \varepsilon \qquad \beta'_{\delta,\varepsilon} = \delta \to (\varepsilon \rightsquigarrow (\delta \,\&\, \varepsilon) \,\&\, \star)$$

Note that the terms in the second line generalize the well-known notions of left and right conjugates used in associative logics:[18]

$$\lambda_\varepsilon = \varepsilon \to \star \,\&\, \varepsilon \qquad \rho_\varepsilon = \varepsilon \rightsquigarrow \varepsilon \,\&\, \star$$

[17] We deviate slightly from the original definition from Cintula and Noguera (2011), where β_1, β_2 were required to be in bDT, and follow that from Cintula et al. (2013) which has some technical advantages.

[18] It is usual in the literature on algebraic study of substructural logics to find these terms defined in a slightly more complicated way: $\lambda_\varepsilon = (\varepsilon \to \star \,\&\, \varepsilon) \wedge \overline{1}$ and $\rho_\varepsilon = (\varepsilon \rightsquigarrow \varepsilon \,\&\, \star) \wedge \overline{1}$, although in the usual Hilbert-style axiomatizations of Full Lambek logic the simplified terms without $\wedge\overline{1}$ are used for the *product normality* rules. The reason for this more complicated form is to give algebraic terms which simultaneously cope with product normality rules and adjunction, whereas our formalism allows for a clearer distinction of their respective rôles.

Table 12.7 bDTs of prominent substructural logics

Logic L	$\mathrm{bDT_L}$
SL	$\{\alpha_{\delta,\varepsilon}, \alpha'_{\delta,\varepsilon}, \beta_{\delta,\varepsilon}, \beta'_{\delta,\varepsilon}, \star \wedge \overline{1} \mid \delta, \varepsilon \text{ formulae}\}$
$\mathrm{SL_w}$	$\{\alpha_{\delta,\varepsilon}, \alpha'_{\delta,\varepsilon}, \beta_{\delta,\varepsilon}, \beta'_{\delta,\varepsilon} \mid \delta, \varepsilon \text{ formulae}\}$
$\mathrm{SL_e}$	$\{\alpha_{\delta,\varepsilon}, \beta_{\delta,\varepsilon}, \star \wedge \overline{1} \mid \delta, \varepsilon \text{ formulae}\}$
$\mathrm{SL_{ew}}$	$\{\alpha_{\delta,\varepsilon}, \beta_{\delta,\varepsilon} \mid \delta, \varepsilon \text{ formulae}\}$
$\mathrm{SL_a}$	$\{\lambda_{\varepsilon}, \rho_{\varepsilon}, \star \wedge \overline{1} \mid \varepsilon \text{ a formula}\}$
$\mathrm{SL_{aw}}$	$\{\lambda_{\varepsilon}, \rho_{\varepsilon} \mid \varepsilon \text{ a formula}\}$
$\mathrm{SL_{ae}}$	$\{\star \wedge \overline{1}\}$
$\mathrm{SL_{aew}}$	$\{\star\}$

Cintula et al. (2013) proved that the axiomatic system from Table 12.6 is indeed a presentation of SL, therefore we can obtain the following result for SL and some of its notable axiomatic extension (it also shows how the sets of basic deduction terms, and so posteriorly the axioms systems, of these extensions can be simplified).

Theorem 12.4 (Cintula et al. 2013, Sect. 3.1) *Let* $S \subseteq \{\mathrm{a}, \mathrm{e}, \mathrm{w}\}$. *Then any axiomatic extension of the logic* SL_S *is almost* (MP)*-based with respect to the corresponding set of basic deduction terms listed in Table* 12.7.

Theorem 12.5 (Local deduction theorem (Cintula et al. 2013, Corollary 3.12)) *Let* L *be an almost* (MP)*-based logic with a set of basic deduction terms* bDT. *Then for each set* Γ *of formulae and each formulae* φ *and* ψ *the following holds:*

$$\Gamma, \varphi \vdash_{\mathrm{L}} \psi \qquad \textit{iff} \qquad \Gamma \vdash_{\mathrm{L}} \gamma(\varphi) \to \psi \textit{ for some } \gamma \in \Pi(\mathrm{bDT}^*).$$

Therefore, we obtain a (parameterized or non-parameterized, depending on the presence of variables other than $\star$ in the set bDT) local deduction theorem for SL and its axiomatic extensions (sometimes with a simplified set bDT; see Table 12.7).

12.2.3 Consequences of Algebraization

Given a logic L in a language $\mathscr{L}$ and an $\mathscr{L}$-algebra $\boldsymbol{A}$, a set $F \subseteq A$ is an L-*filter* if for every set of formulae $\Gamma \cup \{\varphi\}$ such that $\Gamma \vdash_{\mathrm{L}} \varphi$ and every $\boldsymbol{A}$-evaluation e it holds: if $e[\Gamma] \subseteq F$, then $e(\varphi) \in F$. By $\mathscr{F}i_{\mathrm{L}}(\boldsymbol{A})$ we denote the set of all L-filters over $\boldsymbol{A}$. Since $\mathscr{F}i_{\mathrm{L}}(\boldsymbol{A})$ is a closure system (it clearly contains A and is closed under arbitrary intersections), one can define a notion of generated filter. Given $X \subseteq A$, the L-filter generated by X, denoted as $\mathrm{Fi}^{\boldsymbol{A}}_{\mathrm{L}}(X)$ is the least L-filter containing X (we omit the indexes when clear from the context). With this terminology one can also prove a semantical (or transferred) version of (parameterized) local deduction theorem; Theorem 12.5 is the particular case in which $\boldsymbol{A}$ is the algebra of formulae (observe that in this case $\varphi \in \mathrm{Fi}(\Gamma)$ iff $\Gamma \vdash_{\mathrm{L}} \varphi$). First we introduce two technical notions:

Definition 12.7 Given a set of $\star$-formulae Γ, an SL-algebra $\boldsymbol{A}$, and a set $X \subseteq A$, we define

- $\Gamma^{\boldsymbol{A}}$ as the set of unary polynomials built using terms from Γ with coefficients from A and variable $\star$, i.e., $\{\delta(\star, a_1, \ldots, a_n) \mid \delta(\star, p_1, \ldots, p_n) \in \Gamma$ and $a_1, \ldots, a_n \in A\}$.
- $\Gamma^{\boldsymbol{A}}(X)$ as the set $\{\delta^{\boldsymbol{A}}(x) \mid \delta(\star) \in \Gamma^{\boldsymbol{A}}$ and $x \in X\}$.

Theorem 12.6 (Cintula et al. 2013, Theorem 3.11) *Let* L *be an almost* (MP)*-based logic in a language* $\mathscr{L}$ *with a set of basic deduction terms* bDT. *Let* $\boldsymbol{A}$ *be an* $\mathscr{L}$*-algebra and* $X \cup \{x\} \subseteq A$. *Then*

$$y \in \mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{A}}(X, x) \quad \textit{iff} \quad \gamma^{\boldsymbol{A}}(x) \backslash y \in \mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{A}}(X) \textit{ for some } \gamma \in (\Pi(\mathrm{bDT}^*))^{\boldsymbol{A}}.$$

On the other hand, Theorem 12.6 can be used to obtain a general form of the usual algebraic description of the filter generated by a set.

Corollary 12.1 (Cintula et al. 2013, Corollary 3.13) *Let* L *be an almost* (MP)*-based logic with a set of basic deduction terms* bDT. *Let* $\boldsymbol{A}$ *be an* L*-algebra and* $X \subseteq A$. *Then*

$$\mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{A}}(X) = \{a \in A \mid a \geqslant x \textit{ for some } x \in (\Pi(\mathrm{bDT}^*))^{\boldsymbol{A}}(X)\}\,.$$

The algebraic completeness result we have seen above (Theorem 12.3) can be strengthened obtaining that SL is actually an algebraizable logic in the sense of Blok and Pigozzi (1989) and $\mathbb{SL}$ is its equivalent algebraic semantics with translations $\rho(p \approx q) = p \leftrightarrow q$ and $\tau(p) = p \wedge \overline{1} \approx \overline{1}$. Indeed, if we consider formal equations in the language $\mathscr{L}_{\mathrm{SL}}$ as expressions of the form $\varphi \approx \psi$ where $\varphi, \psi \in Fm_{\mathscr{L}_{\mathrm{SL}}}$ and if $\models_{\mathbb{SL}}$ denotes the equational consequence with respect to the class $\mathbb{SL}$, it is easy to prove that:

1. $\Pi \models_{\mathbb{SL}} \varphi \approx \psi$ iff $\rho[\Pi] \vdash_{\mathrm{SL}} \rho(\varphi \approx \psi)$
2. $p \vdash_{\mathrm{SL}} \rho[\tau(p)]$ and $\rho[\tau(p)] \vdash_{\mathrm{SL}} p$

Actually, this result can be extended to every logic L and its corresponding class of algebras $\mathbb{L}$. If L is a logic in a language $\mathscr{L}$ which is the expansion of SL by axioms *Ax* and rules *R*, then L-algebras can also be described as the expansions of SL-algebras satisfying:

- the equation $\tau(\varphi)$ for each $\varphi \in Ax$
- the quasiequation $\tau(\varphi_1)$ and $\ldots$ and $\tau(\varphi_n) \Rightarrow \tau(\varphi)$ for each $\langle\{\varphi_1, \ldots, \varphi_n\}, \varphi\rangle$ from *R*.

Therefore, the class $\mathbb{L}$ is always a quasivariety and it is a variety if $R = \emptyset$, i.e. if L is an axiomatic expansion of SL (note that this condition is not necessary as demonstrated e.g. by the logic $\mathrm{MTL}_{\triangle}$). Conversely, given a quasivariety $\mathbb{L}$ of $\mathscr{L}$-algebras, one can always find a quasiequational base obtained by adding a set of equations *E* and a set of quasiequations *Q* to an equational base of $\mathbb{SL}$. Then $\mathbb{L}$ is the equivalent algebraic semantics of the logic obtained as the expansion of SL by

- the axiom $\rho(\varphi, \psi)$ for each equation $\varphi \approx \psi \in E$

- the rule $\langle\{\rho(\varphi_1, \psi_1), \ldots, \rho(\varphi_n, \psi_n)\}, \rho(\varphi, \psi)\rangle$ for each quasiequation $(\varphi_1 \approx \psi_1)$ and … and $(\varphi_n \approx \psi_n) \Rightarrow \varphi \approx \psi \in Q$.

Moreover, if we fix a language $\mathscr{L} \supseteq \mathscr{L}_{\mathrm{SL}}$ and a logic L in $\mathscr{L}$, the translations τ and ρ between formulae and equations give a bijective correspondence between extensions of L and quasivarieties of L-algebras, and a bijective correspondence (its restriction) between axiomatic extensions of L and varieties of L-algebras. These bijections are, actually, dual lattice isomorphisms.

A logic is called *strongly algebraizable* if its corresponding quasivariety is actually a variety. Obviously, strongly algebraizable logics in $\mathscr{L}_{\mathrm{SL}}$ coincide with axiomatic extensions of SL.

Algebraizability also gives a strong correspondence between filters and (relative) congruences in L-algebras, which can be made explicit using the particular forms of the translations. Let $\boldsymbol{Con}_{\mathbb{L}}(\boldsymbol{A})$ denote the lattice of congruences of $\boldsymbol{A}$ relative to $\mathbb{L}$, i.e. those giving a quotient in $\mathbb{L}$. If $\mathbb{L}$ is a variety, then $\boldsymbol{Con}_{\mathbb{L}}(\boldsymbol{A})$ is precisely the lattice of all congruences of $\boldsymbol{A}$. The *Leibniz operator* $\Omega_{\boldsymbol{A}}$ is defined, for any $F \in \mathscr{F}i_{\mathrm{L}}(\boldsymbol{A})$, as $\Omega_{\boldsymbol{A}}(F) = \{\langle a, b\rangle \in A^2 \mid a\backslash b \in F \text{ and } b\backslash a \in F\}$. Now we can state a specific variant of a well-known theorem of abstract algebraic logic (Czelakowski 2001), narrowed down to our setting.

Proposition 12.1 *Let* L *be a logic and* $\boldsymbol{A}$ *an* L*-algebra. The Leibniz operator* $\Omega_{\boldsymbol{A}}$ *is a lattice isomorphism from* $\mathscr{F}i_{\mathrm{L}}(\boldsymbol{A})$ *to* $\boldsymbol{Con}_{\mathbb{L}}(\boldsymbol{A})$. *Its inverse is the function that maps any* $\theta \in \boldsymbol{Con}_{\mathbb{L}}(\boldsymbol{A})$ *to the filter* $\{a \in A \mid \langle a \wedge 1, 1\rangle \in \theta\}$.

Observe that the minimum filter is the one generated by the emptyset, $\mathrm{Fi}(\emptyset)$, and it must correspond to the identity congruence $Id_{\boldsymbol{A}}$. Therefore, using the previous proposition, we obtain that, on any L-algebra $\boldsymbol{A}$, $\mathrm{Fi}(\emptyset) = \{a \in A \mid a \geqslant 1\}$. This set is, of course, contained in any other filter. It is also worth noting that Proposition 12.1 and Corollary 12.1 give a description of the relative principal congruence generated by a pair of elements of a given algebra of an almost (MP)-based logic.

Finally, we focus on a restriction of the completeness theorem (Theorem 12.3) to a couple of subclasses of algebraic models that will play an important rôle when characterizing semilinearity in the next section: relatively (finitely) subdirectly irreducible algebras. Given a class of algebras $\mathbb{K}$ an algebra $\boldsymbol{A}$ is (*finitely*) *subdirectly irreducible relative to* $\mathbb{K}$ if for every (finite non-empty) subdirect representation α of $\boldsymbol{A}$ with a family $\{\boldsymbol{A}_i \mid i \in I\} \subseteq \mathbb{K}$ there is $i \in I$ such that $\pi_i \circ \alpha$ is an isomorphism. The class of all (finitely) subdirectly irreducible algebras relative to $\mathbb{K}$ is denoted as $\mathbb{K}_{\mathrm{R(F)SI}}$. Of course $\mathbb{K}_{\mathrm{RSI}} \subseteq \mathbb{K}_{\mathrm{RFSI}}$. Observe that the trivial algebra is by definition in $\mathbb{K}_{\mathrm{RFSI}}$ but not in $\mathbb{K}_{\mathrm{RSI}}$. Again, the next theorem is a specific variant of a well-known fact of abstract algebraic logic.

Theorem 12.7 *Let* L *be a logic. Then for every set of formulae* Γ *and every formula* φ *the following are equivalent:*

1. $\Gamma \vdash_{\mathrm{L}} \varphi$,
2. *for every countable* $\boldsymbol{A} \in \mathbb{L}_{\mathrm{RSI}}$ *and every* $\boldsymbol{A}$*-evaluation* e, *if* $e(\psi) \geqslant 1$ *for every* $\psi \in \Gamma$, *then* $e(\varphi) \geqslant 1$.

12.3 Core Semilinear Logics

Let us start by recalling two notions mentioned in the introduction: first we give a formal semantical definition of the logic SL^{ℓ} (later in Theorem 12.12 we present some of its natural axiomatizations).

Definition 12.8 The logic SL^{ℓ} is defined as follows for every set Γ of formulae and every formula φ:

1. $\Gamma \vdash_{\mathrm{SL}^{\ell}} \varphi$ if and only if
2. $e(\varphi) \geqslant 1$ for each SL-*chain* $\boldsymbol{A}$ and each $\boldsymbol{A}$-evaluation e such that $e(\psi) \geqslant 1$ for all $\psi \in \Gamma$.

Remark 12.2 Clearly SL^{ℓ} extends SL and so (Cintula and Noguera 2011, Propositions 3.1.15 and 3.1.16) it follows that SL^{ℓ} is a logic in the sense of Convention 12.2 and that the classes of SL^{ℓ}-chains and SL-chains coincide.

The second notion is that of core fuzzy logics formally defined in Definition 12.1. Let us reformulate this definition using the terminology introduced in the previous section (especially Convention 12.2 which stipulates that all the logics considered in this chapter satisfy the condition (Cong)):

Definition 12.9 A logic L is a *core fuzzy logic* if it expands MTL by some set of axioms Ax.

Let us now generalize this class in two aspects: first, we replace MTL by the (much) weaker logic SL^{ℓ} and, second, we include logics axiomatized by using extra rules provided that they satisfy a certain stability condition involving disjunction. As we shall soon see (in Theorem 12.8), these conditions are sufficient and necessary for an expansion of SL^{ℓ} to remain complete w.r.t. chains.

Definition 12.10 A logic L is a *core semilinear logic* if it expands SL^{ℓ} by some sets of axioms Ax and rules R such that for each $\langle \Gamma, \varphi \rangle \in R$ and every formula ψ we have:

$$\Gamma \vee \psi \vdash_{\mathrm{L}} \varphi \vee \psi,$$

where by $\Gamma \vee \psi$ we denote the set $\{\chi \vee \psi \mid \chi \in \Gamma\}$.

Observe that if L is an expansion of a core semilinear logic by axioms Ax and rules R, then L is itself a core semilinear logic iff for each $\langle \Gamma, \varphi \rangle \in R$ we have $\Gamma \vee \psi \vdash_{\mathrm{L}} \varphi \vee \psi$. Thus in particular:

- Any axiomatic expansion of a core semilinear logic is a core semilinear logic.
- Any axiomatic expansion of SL is a core semilinear logic iff it expands SL^{ℓ}.

The first item justifies why Hájek considered all axiomatic extensions (*schematic extensions*) of HL in his framework for fuzzy logics, since they were all complete with respect to chains. Moreover, one can check that MTL is an extension of SL^{ℓ};

therefore MTL and all core fuzzy logics are core semilinear logics. Similarly, it is easy to show that $\triangle$-core fuzzy logics are core semilinear (note that we are adding only one rule $\langle\varphi, \triangle\varphi\rangle$ and we can easily prove that $\varphi \vee \psi \vdash_{\mathrm{L}} \triangle\varphi \vee \psi$ using axioms of $\mathrm{MTL}_{\triangle}$).

By restricting and re-elaborating results from the general theory presented by Cintula and Noguera (2011) (and by using some new results by Cintula et al. (2013) and by Cintula and Noguera (2013)), in this section we present several characterizations of core semilinear logics, some general methods to obtain their Hilbert-style axiomatizations, and a survey of their completeness results.

12.3.1 Characterizations and Properties of Core Semilinear Logics

The first characterization justifies the usage of the adjective 'semilinear'. This terminology comes from the theory of residuated lattices (Olson and Raftery 2007) where it denotes classes of algebras such that in all (relatively) subdirectly irreducible members the lattice order is linear.[19] Such property characterizes core semilinear logics as shown by conditions 3 and 4 of the following theorem. Moreover, as stated in condition 2, this is also equivalent with what we consider the main property of our logics: completeness with respect to the semantics given by chains.

Theorem 12.8 (Semilinearity) *Let* L *be a logic. Then the following are equivalent:*

1. L *is a core semilinear logic.*
2. L *is complete w.r.t.* L*-chains, i.e. the following are equivalent for any set of formulae* $\Gamma \cup \{\varphi\}$*:*
   a. $\Gamma \vdash_{\mathrm{L}} \varphi$
   b. $e(\varphi) \geqslant 1$ *for each* L-chain $\boldsymbol{A}$ *and each* $\boldsymbol{A}$*-evaluation* e *such that* $e(\psi) \geqslant 1$ *for all* $\psi \in \Gamma$.
3. $\mathbb{L}_{\mathrm{RFSI}} = \mathbb{L}_{lin}$.
4. $\mathbb{L}_{\mathrm{RSI}} \subseteq \mathbb{L}_{lin}$.

Proof Logics satisfying condition 2 are, in particular, *weakly implicative semilinear logics* in the sense of Cintula and Noguera (2011); thus we can use a result by Cintula and Noguera (2011, Corollary 3.2.14.) to prove the equivalence of the first two properties (for L_1 being SL^{ℓ} and L_2 being L; we need to check the validity of three premises of that corollary: (a) SL^{ℓ} is a weakly implicative semilinear logic: directly from Definition 12.8 and its following remark, (b) $\vee$ is a protodisjunction: trivially satisfied, and (c) L proves ($\mathrm{MP}_{\vee}$): established by Cintula and Noguera (2011), Proposition 3.2.2.

[19] This follows the tradition of Universal Algebra of calling a class of algebras 'semi*X*' whenever its subdirectly irreducible members have the property *X*; e.g. as in 'semisimple'.

The equivalence of the latter three claims was established by Cintula and Noguera (2011, Theorem 3.1.8). □

Thus, as established in the proof of the theorem above, core semilinear logic are weakly implicative semilinear logics in the sense of Cintula and Noguera (2010, 2011). In fact, in the terminology of those papers, they are exactly algebraically implicative semilinear finitary expansions of SL^{ℓ}.

In order to formulate the syntactic characterization theorem for core semilinear logics (in terms of syntactic properties) we need to make use of special kinds of theories. Recall that a theory is a deductively closed set of formulae, i.e., $T \vdash \varphi$ implies that $\varphi \in T$). We say that a theory T is

- *maximally consistent w.r.t. a formula* φ if $\varphi \notin T$ and for every $\psi \notin T$ we have $T, \psi \vdash \varphi$
- *saturated* if it is maximally consistent w.r.t. some formula φ
- *linear*[20] if for each formulae φ and ψ we have $\varphi \to \psi \in T$ or $\psi \to \varphi \in T$
- *prime* if for each formulae φ and ψ we have $\varphi \in T$ or $\psi \in T$ whenever $\varphi \vee \psi \in T$.

We also need a special formula $(\varphi \to \psi) \vee (\psi \to \varphi)$, called *prelinearity* and usually denoted by $(\mathrm{P}_\vee)$, which could be equivalently replaced in the formulation of the syntactic characterization theorem by any of the following two theorems of SL^{ℓ} (as shown by Cintula and Noguera (2011, Lemma 3.2.8)):

$(\mathrm{lin}_\wedge)$ $(\varphi \wedge \psi \to \chi) \to (\varphi \to \chi) \vee (\psi \to \chi)$
$(\mathrm{lin}_\vee)$ $(\chi \to \varphi \vee \psi) \to (\chi \to \varphi) \vee (\chi \to \psi)$.

Theorem 12.9 (Syntactic characterization theorem) *Let* L *be a logic. Then the following are equivalent:*

1. L *is a core semilinear logic.*
2. L *has the* Semilinearity Property, SLP*, i.e. for every set of formulae* $\Gamma \cup \{\varphi, \psi, \chi\}$ *the following rule holds*

$$\frac{\Gamma, \varphi \to \psi \vdash_{\mathrm{L}} \chi \qquad \Gamma, \psi \to \varphi \vdash_{\mathrm{L}} \chi}{\Gamma \vdash_{\mathrm{L}} \chi}.$$

3. L *has the* Linear Extension Property, LEP*, i.e. for every theory* T *and a formula* φ *such that* $\varphi \notin T$*, there is a* linear *theory* $T' \supseteq T$ *such that* $\varphi \notin T'$.
4. *Saturated theories are linear.*
5. L *proves* $(\mathrm{P}_\vee)$ *and has the* Proof by Cases Property, PCP*, i.e. for every set of formulae* $\Gamma \cup \{\varphi, \psi, \chi\}$ *holds*

$$\frac{\Gamma, \varphi \vdash_{\mathrm{L}} \chi \qquad \Gamma, \psi \vdash_{\mathrm{L}} \chi}{\Gamma, \varphi \vee \psi \vdash_{\mathrm{L}} \chi}.$$

[20] Petr Hájek (1998b) called this kind of theories 'complete'. However, after recent developments (Cintula and Noguera 2011) we prefer the more descriptive terminology used here.

6. L *proves* ($P_\vee$) *and has an axiomatic system* $\langle Ax, R\rangle$ *such that for each* $\langle \Gamma, \varphi\rangle \in R$ *we have:*
$$\Gamma \vee \psi \vdash_{\mathrm{L}} \varphi \vee \psi.$$
7. L *proves* ($P_\vee$) *and has the* Prime Extension Property*,* PEP*, i.e. for every theory* T *and a formula* φ *such that* $\varphi \notin T$*, there is a* prime *theory* $T' \supseteq T$ *such that* $\varphi \notin T'$*.*
8. L *proves* ($P_\vee$) *and its saturated theories are prime.*

Proof The equivalence of the first three claims was proved by Cintula and Noguera (2011, Theorem 3.1.8). To prove 1 implies 4 observe that saturated theories are finitely ∩-irreducible (*ibid.*, Proposition 2.3.7) and so they are linear (*ibid.*, Theorem 3.1.8). Conversely, if saturated theories are linear, then the Abstract Lindenbaum Lemma (*ibid.*, Lemma 2.3.8) clearly implies LEP (i.e. claim 3).

The equivalence of 1 and 5 is also established using results by Cintula and Noguera (2011) (use Theorem 3.2.4 after observing that any L proves ($MP_\vee$) as established in Proposition 3.2.2); the equivalence of 5 and 6 follows from Theorem 2.7.15. We use Theorem 2.7.23 to directly prove that 5 is equivalent with 7 and 7 implies 8. Finally, by using a similar reasoning as in the proof of 4 implies 3, we complete the whole proof by showing that 8 implies 7. □

Remark 12.3 Most of these characterizations are inspired by the original ideas behind the proof of completeness of HL and its schematic extensions by Hájek (1998b): actually in Lemma 2.3.15. he gives a direct proof of transferred PEP (the third line of the following theorem) and in Lemma 2.4.2 he proves LEP by proving SLP first (without giving names to any of these properties).

Observe that while claim 6 is a just minor reformulation of Definition 12.10 of core semilinear logics, it provides an easy way to check whether a logic is core semilinear without having to prove that it extends SL^{ℓ}.

Note that theories are exactly the filters on the term algebra $\mathit{Fm}_{\mathscr{L}}$. Thus it makes sense to generalize the classes of theories we introduced above to filters with '$\Gamma \vdash \varphi$' replaced by '$x \in \mathrm{Fi}(X)$', e.g. a filter F in an L-algebra $\boldsymbol{A}$ is maximally consistent (algebraist would say 'maximal non-trivial') w.r.t. an element $a \in A$ if $a \notin F$ and for every $b \notin F$ we have $a \in \mathrm{Fi}(F \cup \{b\})$. This allows us not only to see the conditions appearing in the syntactic characterization theorem as claims about filters on the term algebra $\mathit{Fm}_{\mathscr{L}}$, but mainly to formulate their *transferred* variants which speak about all L-algebras. We collect these results in the next theorem together with some other useful algebraic properties L-algebras.

Theorem 12.10 *Let* L *be a core semilinear logic and* $\boldsymbol{A}$ *an* L*-algebra. Then:*

1. *For each set* $X \cup \{a, b\} \subseteq A$ *the following holds:*
$$\mathrm{Fi}(X, a) \cap \mathrm{Fi}(X, b) = \mathrm{Fi}(X, a \vee b) \qquad \mathrm{Fi}(X, a \to b) \cap \mathrm{Fi}(X, b \to a) = \mathrm{Fi}(X).$$
2. *Linear and prime filters coincide and contain the set of saturated filters.*

3. *For each filter* $F \in \mathscr{F}i_{\mathrm{L}}(\boldsymbol{A})$ *and each* $a \in A$ *such that* $a \notin F$*, there is a* linear/prime *filter* $F' \supseteq F$ *such that* $a \notin F'$.
4. *The lattice of* L*-filters is distributive.*
5. *The lattice of relative* $\mathbb{L}$*-congruences is distributive.*
6. *The* $\{\vee, \wedge\}$*-reduct of* $\boldsymbol{A}$ *is a distributive lattice.*

Proof We will freely use all equivalent characterizations of core semilinear logics established before.

1. (Cintula and Noguera 2011, Theorems 2.7.18 and 3.1.8).
2. (*ibid.*, Theorems 2.7.23, Theorem 3.1.8, and Proposition 2.3.7).
3. (*ibid.*, Theorem 2.7.23) and claim 2.
4. (*ibid.*, Theorem 2.7.20).
5. Claim 4 and Proposition 12.1.
6. (*ibid.*, Theorem 3.2.12).

The following proposition is, among others, important to establish the soundness of the upcoming crucial definition of L^{ℓ}.

Proposition 12.2 *The intersection of a family of core semilinear logics in the same language is a core semilinear logic.*

Proof First we need to observe that the intersection is a logic in the sense of Convention 12.2. This is established by Cintula and Noguera (2011, Proposition 3.1.16). Then, the fact that it is core semilinear is a simple corollary of the syntactic characterization theorem (e.g. of the Semilinearity Property). □

Definition 12.11 For a logic L we define the logic L^{ℓ} as the least core semilinear logic extending L.

The following two theorems give useful, semantical and syntactical, descriptions of L^{ℓ}. The first one is very general and, besides providing a semantical characterization of L^{ℓ} as the logic of L-chains, it shows how to extend any axiomatization of L into an axiomatization of L^{ℓ}. Roughly speaking, it adds prelinearity and the $\vee$-form of all rules (cf. the syntactic characterization theorem 12.9). Note that Petr Hájek also obtained some logics in these ways: e.g. he showed that G was in fact the logic of linearly ordered Heyting algebras or defined psMTL^r as the logic psMTL-chains.

Theorem 12.11 *Let* L *be a logic. Then:*

- L^{ℓ}*-chains coincide with* L*-chains and the class* $\mathbb{L}^{\ell}$ *of* L^{ℓ}*-algebras is exactly the quasivariety generated by* $\mathbb{L}_{lin}$.
- *If* L *is axiomatized by axioms Ax and rules R, then* L^{ℓ} *is the extension of* L *by the axiom* $(\mathrm{P}_{\vee})$ *and the set of rules* $\{\langle \Gamma \vee \psi, \varphi \vee \psi \rangle \mid \langle \Gamma, \varphi \rangle \in R\}$.
- *If* L *is obtained as the expansion of some core semilinear logic by axioms Ax and rules R, then* L^{ℓ} *is the extension of* L *by the rules* $\{\langle \Gamma \vee \psi, \varphi \vee \psi \rangle \mid \langle \Gamma, \varphi \rangle \in R\}$.

Table 12.8 Axiomatization of L^ℓ for prominent substructural logics

Logic L	Additional axioms needed to axiomatize L^ℓ
SL	$((\varphi \to \psi) \wedge \overline{1}) \vee \gamma((\psi \to \varphi) \wedge \overline{1})$, for every $\gamma \in \{\alpha_{\delta,\varepsilon}, \alpha'_{\delta,\varepsilon}, \beta_{\delta,\varepsilon}, \beta'_{\delta,\varepsilon}\}$
$\mathrm{SL_w}$	$(\varphi \to \psi) \vee \gamma(\psi \to \varphi)$, for every $\gamma \in \{\alpha_{\delta,\varepsilon}, \alpha'_{\delta,\varepsilon}, \beta_{\delta,\varepsilon}, \beta'_{\delta,\varepsilon}\}$
$\mathrm{SL_e}$	$\alpha_{\delta,\varepsilon}((\varphi \to \psi) \wedge \overline{1}) \vee \beta_{\delta',\varepsilon'}((\psi \to \varphi) \wedge \overline{1})$
$\mathrm{SL_{ew}}$	$\alpha_{\delta,\varepsilon}(\varphi \to \psi) \vee \beta_{\delta',\varepsilon'}(\psi \to \varphi)$
$\mathrm{SL_a}$	$(\lambda_\varepsilon(\varphi \to \psi) \wedge \overline{1}) \vee (\rho_{\varepsilon'}(\psi \to \varphi) \wedge \overline{1})$
$\mathrm{SL_{ae}}$	$((\varphi \to \psi) \wedge \overline{1}) \vee ((\psi \to \varphi) \wedge \overline{1})$
$\mathrm{SL_{aew}}$	$(\varphi \to \psi) \vee (\psi \to \varphi)$

Proof Again we use results by Cintula and Noguera (2011). The first claim follows from Proposition 3.1.15; the second follows from Proposition 3.2.9 and Theorem 2.7.27. The third one is an obvious corollary of already established facts. □

The problem of the axiomatizations provided by this theorem is that they require additional new rules. We show that if L is almost (MP)-based we can do better: L^ℓ is then actually an *axiomatic* extension of L by *variations* of the prelinearity axiom. We present two variants, *B* and *C*, of this axiomatization because they generalize two different usual formulations appearing in the literature; for comparison we also add a presentation *A* resulting from the direct application of the previous theorem. Note that this theorem can be used to axiomatize the two logics mentioned above and studied by Petr Hájek: G and psMTL^r.

Theorem 12.12 *Let* L *be an almost* (MP)*-based logic with a set* bDT *of basic deductive terms. Then* L^ℓ *is axiomatized by adding to* L *any of the following:*

A $(\varphi \to \psi) \vee (\psi \to \varphi)$
$(\varphi \to \psi) \vee \chi, \varphi \vee \chi \vdash \psi \vee \chi$
$\varphi \vee \psi \vdash \gamma(\varphi) \vee \psi$*, for every* $\gamma \in \mathrm{bDT}$
B $((\varphi \to \psi) \wedge \overline{1}) \vee \gamma((\psi \to \varphi) \wedge \overline{1})$*, for every* $\gamma \in \mathrm{bDT} \cup \{\star\}$
C $(\varphi \vee \psi \to \psi) \vee \gamma(\varphi \vee \psi \to \varphi)$*, for every* $\gamma \in \mathrm{bDT} \cup \{\star \wedge \overline{1}\}$.

Proof A weaker claim, for *extensions* of SL^ℓ, is proved by Cintula et al. (2013, Theorem 4.29). One can easily see, by inspecting the proof, that the theorem remains valid in our framework of *expansions* of SL^ℓ. □

Table 12.8 collects axiomatizations of important semilinear substructural logics obtained as axiomatization *B* from Theorem 12.2. We present them in the form of axiom schemata, sometimes altered a little (in an equivalent way) for simplicity or to obtain some form known from the literature (Cintula et al. 2013).

As mentioned in Sect. 12.2.2, finding nice Hilbert-style presentations for meets in the lattice of extensions of a given logic (in particular, showing that the meet of axiomatic extensions is itself an axiomatic extension of the base logic) is not straightforward. The following theorem gives a presentation for meets of extensions of a given core semilinear logic by capitalizing on the fact that ∨ enjoys PCP in all core semilinear logics.

Theorem 12.13 *Let* L_1 *and* L_2 *be semilinear extensions of a core semilinear logic* L *defined by the sets of axioms* Ax_i *and rules* R_i. *Then* $\mathrm{L}_1 \cap \mathrm{L}_2$ *is the extension of* L *obtained by adding*

- *the set of axioms* $\{\varphi \vee \psi \mid \varphi \in Ax_1$ *and* $\psi \in Ax_2\}$ *and*
- *the union of the following three sets of rules:*
  - $\langle \Gamma \vee \chi, \varphi \vee \psi \vee \chi \rangle \mid \langle \Gamma, \varphi \rangle \in R_1,\ \psi \in Ax_2,$ *and* χ *a formula*$\}$
  - $\langle \Gamma \vee \chi, \varphi \vee \psi \vee \chi \rangle \mid \langle \Gamma, \varphi \rangle \in R_2,\ \psi \in Ax_1,$ *and* χ *a formula*$\}$
  - $\langle (\Gamma_1 \cup \Gamma_2) \vee \chi, \varphi_1 \vee \varphi_2 \vee \chi \rangle \mid \langle \Gamma_1, \varphi_1 \rangle \in R_1,\ \langle \Gamma_2, \varphi_2 \rangle \in R_2,$ *and* χ *a formula*$\}$

Proof Established by Cintula and Noguera (2013, Theorem 5.10). □

To close the subsection we clarify the position of core semilinear (axiomatic) extensions of given logic in the lattice of all its (axiomatic) extensions.

Corollary 12.2 *Let* L *be a logic. Then the class of core semilinear extensions of* L *is a sublattice of the lattice of extensions of* L^{ℓ}. *Furthermore, the class of core semilinear* axiomatic *extensions of* L *is a principal filter in the lattice of axiomatic extensions of* L^{ℓ}.

12.3.2 Completeness Results

We devote this subsection to completeness theorems for core semilinear logics. As discussed in the introduction, a crucial guideline for Petr Hájek and others when studying new fuzzy logics was to find logical systems complete with respect to a semantics of algebras defined on the real unit interval [0, 1]. This kind of completeness results have been known as *standard completeness theorems*, although this terminology is not univocally defined. Indeed, by *standard* semantics one means the semantics that, due to some design choices, is considered to be the *intended* one for the logic. In some cases it consists of all algebras defined over [0, 1] (e.g. for HL, SHL, MTL, SMTL, or IMTL); in other cases it consists of algebras with a fixed interpretation using particular operations (e.g. for Ł, G or Π where one interprets & as the corresponding t-norm (Hájek 1998b), or for logics with an additional involutive negation $\sim$ where one interprets it as $1 - x$ (Esteva et al. 2000). In all the examples taken from ($\triangle$-)core fuzzy logics, the standard semantics is based on left-continuous t-norms and their residua. Later on, the introduction of weaker systems brought forth an analogous relaxation for the corresponding algebraic structures on [0, 1], such as residuated uninorms (for UL) or residuated non-commutative t-norms (for psMTL^r). Recently, when considering a standard semantics for SL^{ℓ} (Cintula et al. 2013), even associativity has been dropped giving rise to residuated unital groupoids on [0, 1].

Some other works have however focused on other kinds of semantics for fuzzy logics, besides the real-valued one. It is the case of rational-chain semantics, hyperreal-chain semantics or finite-chain semantics (e.g. Cintula et al. 2009; Esteva et al. 2010;

Flaminio 2008; Montagna and Noguera 2010) where instead of $[0, 1]$ one respectively takes the rational unit interval, any hyperreal interval or any finite linearly ordered set as the domain for the intended models. A systematical study of the corresponding completeness properties is better presented in the following general formulation.

Definition 12.12 Let L be a core semilinear logic and $\mathbb{K} \subseteq \mathbb{L}_{lin}$. We say that L has the *Strong* $\mathbb{K}$*-Completeness*, S$\mathbb{K}$C for short, when the following are equivalent for every set of formulae $\Gamma \cup \{\varphi\}$:

1. $\Gamma \vdash_{\mathrm{L}} \varphi$,
2. for every $\boldsymbol{A} \in \mathbb{K}$ and every $\boldsymbol{A}$-evaluation e, if $e(\psi) \geqslant 1$ for every $\psi \in \Gamma$, then $e(\varphi) \geqslant 1$.

If the equivalence above holds for finite Γ (or only for $\Gamma = \emptyset$) we speak about *Finite Strong* $\mathbb{K}$*-Completeness* (or just $\mathbb{K}$*-Completeness*, respectively). The Finite Strong $\mathbb{K}$-Completeness is denoted by FS$\mathbb{K}$C whereas the $\mathbb{K}$-Completeness is denoted by $\mathbb{K}$C.

It is easy to show that the failure of completeness properties is inherited by conservative expansions (recall that a logic L_2 in a language $\mathscr{L}_2$ is a *conservative* expansion of a logic L_1 in a language $\mathscr{L}_1$ if $\mathscr{L}_1 \subseteq \mathscr{L}_2$ and for each set of $\mathscr{L}_1$-formulae $\Gamma \cup \{\varphi\}$ we have that $\Gamma \vdash_{\mathrm{L}_2} \varphi$ iff $\Gamma \vdash_{\mathrm{L}_1} \varphi$).

Proposition 12.3 (Cintula and Noguera 2011, Proposition 3.4.14) *Let* L$'$ *be a conservative expansion of* L*,* $\mathbb{K}'$ *a class of* L$'$*-chains and* $\mathbb{K}$ *the class of their* L*-reducts. If* L$'$ *enjoys the* $\mathbb{K}'$C*, then* L *enjoys the* $\mathbb{K}$C*. The analogous claim holds for* FS$\mathbb{K}'$C *and* S$\mathbb{K}'$C.

We recall now several algebraic characterizations of completeness properties by Cintula et al. (2009) and Cintula and Noguera (2011). In what follows we will use the following operators on classes of algebras of the same type:

- $\mathbf{S}(\mathbb{K})$ is the class of subalgebras of members in $\mathbb{K}$,
- $\mathbf{I}(\mathbb{K})$ is the class of algebras isomorphic to a member in $\mathbb{K}$,
- $\mathbf{H}(\mathbb{K})$ is the class of homomorphic images of members in $\mathbb{K}$,
- $\mathbf{P}(\mathbb{K})$ is the class of direct products of members in $\mathbb{K}$,
- $\mathbf{P}_{fin}(\mathbb{K})$ is the class of finite direct products of members in $\mathbb{K}$,
- $\mathbf{P}_{\mathrm{U}}(\mathbb{K})$ is the class of ultraproducts of members in $\mathbb{K}$,
- $\mathbf{P}_{\sigma\text{-}f}(\mathbb{K})$ is the class of reduced products of members in $\mathbb{K}$ over countably complete filters (i.e. filters closed under countable intersections),
- $\mathbf{V}(\mathbb{K})$ is the variety generated by $\mathbb{K}$, i.e., $\mathbf{V}(\mathbb{K}) = \mathbf{HSP}(\mathbb{K})$,
- $\mathbf{Q}(\mathbb{K})$ is the quasivariety generated by $\mathbb{K}$, i.e., $\mathbf{Q}(\mathbb{K}) = \mathbf{ISPP}_{\mathrm{U}}(\mathbb{K})$.

Let us fix a core semilinear logic L and a class of L-chains $\mathbb{K}$. We present several characterizations of the general completeness properties. The first one relates them respectively with generation of the class of algebras as a variety, a quasivariety and a generalized quasivariety, respectively.

Theorem 12.14 (Cintula and Noguera 2011, Theorem 3.4.3)

1. L *has the* $\mathbb{K}$C *if, and only if,* $\mathbf{H}(\mathbb{L}) = \mathbf{V}(\mathbb{K})$.
2. L *has the* FS$\mathbb{K}$C *if, and only if,* $\mathbb{L} = \mathbf{Q}(\mathbb{K})$.
3. L *has the* S$\mathbb{K}$C *if, and only if,* $\mathbb{L} = \mathbf{ISP}_{\sigma\text{-}f}(\mathbb{K})$.

The completeness properties of L can be also characterized in terms of (finitely) subdirectly irreducible algebras relative to $\mathbb{L}$. Recall that, by Theorem 12.8, finitely subdirectly irreducible L-algebras relative to $\mathbb{L}$ coincide with the class of L-chains, i.e., we have $\mathbb{L}_{\mathrm{RSI}} \subseteq \mathbb{L}_{\mathrm{RFSI}} = \mathbb{L}_{lin}$. Given a class of algebras $\mathbb{M}$, the class of its nontrivial members is denoted $\mathbb{M}^+$. Similarly, $\mathbb{M}^\sigma$ stands for the class of countable members of $\mathbb{M}$.

Theorem 12.15 *We have the following chains of equivalences:*

1. L *has the* $\mathbb{K}$C *iff* $\mathbb{L}_{lin} \subseteq \mathbf{HSP}_{\mathrm{U}}(\mathbb{K})$ *iff* $\mathbb{L}^{\sigma}_{\mathrm{RSI}} \subseteq \mathbf{HSP}_{\mathrm{U}}(\mathbb{K})$.
2. L *has the* FS$\mathbb{K}$C *iff* $\mathbb{L}^{+}_{lin} \subseteq \mathbf{ISP}_{\mathrm{U}}(\mathbb{K})$ *iff* $\mathbb{L}^{\sigma}_{\mathrm{RSI}} \subseteq \mathbf{ISP}_{\mathrm{U}}(\mathbb{K})$.
3. L *has the* S$\mathbb{K}$C *iff* $\mathbb{L}^{\sigma+}_{lin} \subseteq \mathbf{IS}(\mathbb{K})$ *iff* $\mathbb{L}^{\sigma}_{\mathrm{RSI}} \subseteq \mathbf{IS}(\mathbb{K})$.

Proof To prove the first claim we use a result by Dziobiak (1989) showing that for any congruence distributive quasivariety $\mathbb{Q}$ and any subclass of algebras $\mathbb{M} \subseteq \mathbb{Q}$ we have $\mathbf{V}(\mathbb{M}) \cap \mathbb{Q}_{\mathrm{RFSI}} \subseteq \mathbf{HSP}_{\mathrm{U}}(\mathbb{M})$. Indeed from Theorem 12.10 we know that the quasivariety $\mathbb{L}$ is congruence distributive and thus by setting $\mathbb{Q} = \mathbb{L}$ and $\mathbb{M} = \mathbb{K}$ we obtain

$$\mathbf{V}(\mathbb{K}) \cap \mathbb{L}_{lin} = \mathbf{V}(\mathbb{K}) \cap \mathbb{L}_{\mathrm{RFSI}} \subseteq \mathbf{HSP}_{\mathrm{U}}(\mathbb{K}).$$

Now assume that L has the $\mathbb{K}$C. Then by Theorem 12.14 we have $\mathbf{H}(\mathbb{L}) = \mathbf{V}(\mathbb{K})$. Consequently, $\mathbb{L}_{lin} \subseteq \mathbf{HSP}_{\mathrm{U}}(\mathbb{K})$ because $\mathbb{L}_{lin} \subseteq \mathbf{H}(\mathbb{L})$. Further, it is obvious that $\mathbb{L}_{lin} \subseteq \mathbf{HSP}_{\mathrm{U}}(\mathbb{K})$ implies $\mathbb{L}^{\sigma}_{\mathrm{RSI}} \subseteq \mathbf{HSP}_{\mathrm{U}}(\mathbb{K})$ since $\mathbb{L}^{\sigma}_{\mathrm{RSI}} \subseteq \mathbb{L}_{lin}$. Finally, suppose that $\mathbb{L}^{\sigma}_{\mathrm{RSI}} \subseteq \mathbf{HSP}_{\mathrm{U}}(\mathbb{K})$. By Theorem 12.14 it is sufficient to show that $\mathbf{H}(\mathbb{L}) = \mathbf{V}(\mathbb{K})$. Since $\mathbf{V}(\mathbb{L}) = \mathbf{H}(\mathbb{L})$ and $\mathbb{K} \subseteq \mathbb{L}$, we always have $\mathbf{V}(\mathbb{K}) \subseteq \mathbf{H}(\mathbb{L})$. Conversely, L is strongly complete w.r.t. $\mathbb{L}^{\sigma}_{\mathrm{RSI}}$ by Theorem 12.7. Thus by Theorem 12.14 we have $\mathbf{H}(\mathbb{L}) = \mathbf{V}(\mathbb{L}^{\sigma}_{\mathrm{RSI}})$. Consequently, by our assumption we obtain $\mathbf{H}(\mathbb{L}) \subseteq \mathbf{V}(\mathbb{K})$.

The first equivalence of the second claim is proved by Cintula and Noguera (2011, Theorem 3.4.11). In order to prove the remaining one, one can argue similarly as above. Indeed, since $\mathbb{L}^{\sigma}_{\mathrm{RSI}} \subseteq \mathbb{L}^{+}_{lin}$, $\mathbb{L}^{+}_{lin} \subseteq \mathbf{ISP}_{\mathrm{U}}(\mathbb{K})$ implies $\mathbb{L}^{\sigma}_{\mathrm{RSI}} \subseteq \mathbf{ISP}_{\mathrm{U}}(\mathbb{K})$. Conversely, assume that $\mathbb{L}^{\sigma}_{\mathrm{RSI}} \subseteq \mathbf{ISP}_{\mathrm{U}}(\mathbb{K})$. Again using Theorems 12.7 and 12.14, we obtain

$$\mathbb{L} = \mathbf{Q}(\mathbb{L}^{\sigma}_{\mathrm{RSI}}) \subseteq \mathbf{Q}(\mathbf{ISP}_{\mathrm{U}}(\mathbb{K})) = \mathbf{Q}(\mathbb{K}) \subseteq \mathbb{L}\,.$$

Thus L enjoys FS$\mathbb{K}$C by Theorem 12.14.

The last claim is proved by Cintula and Noguera (2011, Theorem 3.4.6). □

Corollary 12.3 *If* L *enjoys* FS$\mathbb{K}$C*, then it enjoys the* $\mathbf{SP}_{\mathrm{U}}(\mathbb{K})$C *as well.*

Alternatively, for logics with finitely many propositional connectives, an algebraic property equivalent to finite strong $\mathbb{K}$-completeness is expressed in terms of partial embeddings. This was, in fact, the property used by Hájek and others to prove standard completeness of HL.

Definition 12.13 Given two algebras $\boldsymbol{A}$ and $\boldsymbol{B}$ of the same language $\mathscr{L}$, we say that a finite subset X of $\boldsymbol{A}$ is partially embeddable into $\boldsymbol{B}$ if there is a one-to-one mapping $f\colon X \to B$ such that for each $\langle c, n\rangle \in \mathscr{L}$ and each $a_1, \ldots, a_n \in X$ satisfying $c^{\boldsymbol{A}}(a_1, \ldots, a_n) \in X, f(c^{\boldsymbol{A}}(a_1, \ldots, a_n)) = c^{\boldsymbol{B}}(f(a_1), \ldots, f(a_n))$.

A class $\mathbb{K}$ of algebras is *partially embeddable into* a class $\mathbb{K}'$ if every finite subset of every member of $\mathbb{K}$ is partially embeddable into a member of $\mathbb{K}'$.

Theorem 12.16 *If the language of* L *is finite, then the following are equivalent:*

1. L *has the* FS$\mathbb{K}$C.
2. *The class* $\mathbb{L}^{\sigma}_{\mathrm{RSI}}$ *is partially embeddable into* $\mathbb{K}$.
3. *The class* $\mathbb{L}^{+}_{lin}$ *is partially embeddable into* $\mathbb{K}$.
4. $\mathbb{L}$ *is partially embeddable into* $\mathbf{P}_{fin}(\mathbb{K})$.

Proof The equivalence of the first three claims is proved by Cintula and Noguera (2011, Theorem 3.4.8).

(3)⇒(4): Let $\boldsymbol{A} \in \mathbb{L}$ and $X \subseteq A$ a finite subset. By a well-known fact from universal algebra, every algebra $\boldsymbol{C}$ in a quasivariety $\mathbb{Q}$ is a subdirect product of subdirectly irreducible algebras relative to $\mathbb{Q}$. Since $\mathbb{L}$ is a quasivariety, it follows that $\boldsymbol{A}$ can be viewed as a subdirect product of a family $\{\boldsymbol{A}_i \in \mathbb{L}_{\mathrm{RSI}} \mid i \in I\}$. Since X is finite, it suffices to consider only finitely many $\boldsymbol{A}_i$'s in order to separate elements of X. Thus X is partially embeddable into a finite direct product of some subdirectly irreducible algebras relative to $\mathbb{L}$. Since $\mathbb{L}_{\mathrm{RSI}} \subseteq \mathbb{L}_{lin}$ by Theorem 12.8, X is partially embeddable into $\mathbf{P}_{fin}(\mathbb{K})$ by (3).

(4)⇒(1): Assume that $\Gamma \nvdash_{\mathrm{L}} \varphi$ for a finite set Γ of formulae. By Theorem 12.7 there is a counter-model $\boldsymbol{A} \in \mathbb{L}_{\mathrm{RSI}}$. By (4) we have also a counter-model $\boldsymbol{B} \in \mathbf{P}_{fin}(\mathbb{K})$. Since $\boldsymbol{B}$ is a direct product of members from $\mathbb{K}$, one of them actually has to be a counter-model as well. □

Remark 12.4 Notice that the implications from 2, 3, or 4 to 1 hold also for infinite languages, whereas the converse ones do not (as shown by Cintula et al. (2009, Example 3.10)).

Let us now deal with particular notable semantics. We consider first the class of all finite L-chains, denoted by $\mathscr{F}$.

Theorem 12.17 (Cintula and Noguera 2011, Theorem 3.4.16.) *The following are equivalent:*

1. L *enjoys the* S$\mathscr{F}$C.
2. *All* L*-chains are finite.*
3. *There exists* $n \in \mathsf{N}$ *such each* L*-chain has at most n elements.*
4. *There exists* $n \in \mathsf{N}$ *such that* $\vdash_{\mathrm{L}} \bigvee_{i<n}(x_i \to x_{i+1})$.

Corollary 12.4 *For any core semilinear logic* L *and a natural number n, the axiomatic extension* $\mathrm{L}_{\leqslant \mathrm{n}}$ *obtained by adding the schema* $\bigvee_{i<n}(x_i \to x_{i+1})$ *is a semilinear logic which is strongly complete with respect the* L*-chains of length less than or equal to n.*

Next we show that the properties of FS$\mathscr{F}$C and $\mathscr{F}$C have purely algebraic characterizations in terms of basic notions studied in universal algebra. We say that a class of algebras $\mathbb{M}$ has:

- the *finite embeddability property* (FEP) if $\mathbb{M}$ is partially embeddable into the class of its finite members,
- the (*strong*) *finite model property* ((S)FMP) if every (quasi-)identity that fails to hold in $\mathbb{M}$ can be refuted in a finite member of $\mathbb{M}$.

The next theorem follows from Theorem 12.16 but can be also seen as an instance of a purely universal-algebraic result by Blok and van Alten (2002) (after replacing the first claim by an equivalent algebraic formulation using Theorem 12.14).

Theorem 12.18 *The following are equivalent:*

1. L *enjoys the* FS$\mathscr{F}$C.
2. $\mathbb{L}$ *enjoys the* SFMP.
3. $\mathbb{L}$ *enjoys the* FEP.

Finally, the algebraic characterization of $\mathscr{F}$C is not much of use because it involves free algebras whose structure is usually quite complex, but we include it for the sake of completeness.

Theorem 12.19 *The following are equivalent:*

1. L *enjoys the* $\mathscr{F}$C.
2. $\mathbb{L}$ *enjoys the* FMP.
3. *The class of finitely generated* $\mathbb{L}$*-free algebras is partially embeddable into the class of finite members of* $\mathbb{L}$.

Proof (1)⇒(2): Since L is algebraizable, the first claim implies the second one.

(2)⇒(3): Assume that $\mathbb{L}$ has the FMP. Let $\boldsymbol{F}$ be a finitely generated $\mathbb{L}$-free algebra and $X \subseteq F$ a finite subset. We will construct a partial embedding

$$f\colon X \to \prod_{\substack{x, y \in X\\ x \neq y}} A_{x,y},$$

where $\boldsymbol{A}_{x,y}$ are going to be finite members of $\mathbb{L}$. Let $x, y \in X$ such that $x \neq y$. Since $\boldsymbol{F}$ is free, x, y can be viewed as equivalence classes of terms. Consider any term t_x belonging to x and similarly any term t_y from y. Then the identity $t_x \approx t_y$ does not hold in $\mathbb{L}$ because $x \neq y$. By FMP there is a finite algebra $\boldsymbol{A}_{x,y} \in \mathbb{L}$ where $t_x \approx t_y$ can be refuted. Since $\boldsymbol{F}$ is free, we have a surjective homomorphism $f_{x,y}\colon F \to A_{x,y}$ such that $f_{x,y}(x) \neq f_{x,y}(y)$. The collection of homomorphisms $f_{x,y}$ induces a homomorphism

$$g\colon F \to \prod_{\substack{x, y \in X\\ x \neq y}} A_{x,y}$$

Table 12.9 Finite strong completeness w.r.t. $\mathscr{F}$ for some core semilinear logics

Logic	S$\mathscr{F}$C	FS$\mathscr{F}$C	$\mathscr{F}$C
$\mathrm{SL}^{\ell}_{\mathrm{S}}$, for each $\mathrm{S} \subseteq \{\mathrm{e, c, i, o}\}$	No	Yes	Yes
$\mathrm{SL}^{\ell}_{\mathrm{aw}}$	No	Yes	Yes
MTL, IMTL, SMTL	No	Yes	Yes
UL, WCMTL, ΠMTL, Π	No	No	No
HL, SHL, Ł	No	Yes	Yes
G, WNM, NM, C_nMTL, C_nIMTL	No	Yes	Yes
CPC	Yes	Yes	Yes

defined by $g(z) = \langle f_{x,y}(z)\rangle_{x,y\in X, x\neq y}$ whose restriction to X gives the desired partial embedding f.

(3)⇒(1): Let φ be a formula which is not a theorem of L, i.e., $\nvdash_{\mathrm{L}} \varphi$. By algebraizability the identity $\overline{1} \approx \overline{1} \wedge \varphi$ does not hold in $\mathbb{L}$. Consequently, $\overline{1} \approx \overline{1} \wedge \varphi$ does not hold in a finitely generated $\mathbb{L}$-free algebra $\boldsymbol{F}$. Since $\boldsymbol{F}$ is partially embeddable into a finite member $\boldsymbol{A} \in \mathbb{L}$, $\overline{1} \approx \overline{1} \wedge \varphi$ does not hold in $\boldsymbol{A}$. By a well-known fact from universal algebra, every algebra $\boldsymbol{C}$ in a quasivariety $\mathbb{Q}$ is a subdirect product of subdirectly irreducible algebras relative to $\mathbb{Q}$ which are homomorphic images of $\boldsymbol{C}$. Since $\mathbb{L}_{\mathrm{RSI}} \subseteq \mathbb{L}_{lin}$, $\boldsymbol{A}$ is a subdirect product of chains $\boldsymbol{B}_i$, $i \in I$, which are homomorphic images of $\boldsymbol{A}$. Thus $\boldsymbol{B}_i$'s have to be finite as well. Consequently, $\overline{1} \approx \overline{1} \wedge \varphi$ does not hold in $\prod_{i\in I} \boldsymbol{B}_i$ and therefore $\overline{1} \approx \overline{1} \wedge \varphi$ can be refuted in one of the $\boldsymbol{B}_i$'s. □

The completeness properties w.r.t. the class $\mathscr{F}$ of finite L-chains are usually used in order to show decidability of theorems and finite consequence of L. More precisely, if L is finitely axiomatizable then $\mathscr{F}$C implies decidability of the set $\{\varphi \mid \vdash_{\mathrm{L}} \varphi\}$ and FS$\mathscr{F}$C implies decidability of $\{\langle \Gamma, \varphi\rangle \mid \Gamma \vdash_{\mathrm{L}} \varphi,\ \Gamma \text{ finite}\}$. Table 12.9 lists several known results on completeness properties w.r.t. $\mathscr{F}$ (see Horčík (2011), Wang (2013) and references thereof).

We now consider the semantics given by chains defined over the rational and the real unit interval. We present both notions together because their completeness properties are much related.[21]

Definition 12.14 The class $\mathscr{R} \subseteq \mathbb{L}_{lin}$ is defined as: $\boldsymbol{A} \in \mathscr{R}$ if the domain of $\boldsymbol{A}$ is the real unit interval [0, 1] and $\leqslant_{\boldsymbol{A}}$ is the usual order on reals. The class $\mathscr{Q} \subseteq \mathbb{L}_{lin}$ is analogously defined requiring the rational unit interval as domain.

Theorem 12.20 (Cintula and Noguera 2011, Theorem 3.4.19)

1. L *has the* FS$\mathscr{Q}$C *iff it has the* S$\mathscr{Q}$C.
2. *If* L *has the* $\mathscr{R}$C, *then it has the* $\mathscr{Q}$C.
3. *If* L *has the* FS$\mathscr{R}$C, *then it has the* S$\mathscr{Q}$C.

[21] Another closely related semantics is that of hyperreal or non-standard reals proposed as a semantics for fuzzy logics by Flaminio (2008). Cintula et al. (2009) provided some results linking hyperreal completeness with real and rational completeness.

Observe that the completeness properties with respect to $\mathcal{Q}$ are, in fact, equivalent to completeness properties with respect to the whole class of densely ordered chains. Indeed, when we have an evaluation over a densely ordered linear model providing a counterexample to some derivation, we can apply the downward Löwenheim–Skolem Theorem to the (countable) subalgebra generated by the image of all formulae by the evaluation and obtain a rational countermodel.

Strong rational completeness also admits a proof-theoretic characterization in terms of the Density Property:[22]

Theorem 12.21 *The following are equivalent:*

1. L *has the* S$\mathcal{Q}$C*.*
2. L *has the* Density Property DP*, i.e. if for any set of formulae* $\Gamma \cup \{\varphi, \psi, \chi\}$ *and any variable* p *not occurring in* $\Gamma \cup \{\varphi, \psi, \chi\}$ *the following meta-rule holds:*

$$\frac{\Gamma \vdash_{\mathrm{L}} (\varphi \to p) \vee (p \to \psi) \vee \chi}{\Gamma \vdash_{\mathrm{L}} (\varphi \to \psi) \vee \chi}.$$

3. L *is the intersection of all its extensions satisfying the* DP.

Proof Again we use results by Cintula and Noguera (2011). The equivalence of 1 and 2 follows Theorem 3.3.8. The equivalence with 3 follows from Theorem 3.3.13. □

The last claim gives some insight into an approach used in the fuzzy logic literature to prove completeness w.r.t. the semantics of densely ordered chains (e.g. by Metcalfe and Montagna (2007) for the logic UL). Indeed, in this approach one starts from a suitable proof-theoretic description of a logic L, which then is extended into a proof-system for the intersection of all extensions of L satisfying the DP just by adding DP as a rule (in the proof-theoretic sense, not as we understand rules here). This rule is then shown to be eliminable (using analogs of the well-known cut-elimination techniques), i.e., the condition 3 is met and hence the original logic is complete w.r.t. its densely ordered chains (of course, our general theory is not helpful in this last step, because here one needs to use specific properties of the logic in question).

Many works in the literature of MFL focus on the study of these completeness properties. Besides the historical papers devoted to particular logical systems, there are more systematic approaches dealing with the study of these properties by Cintula et al. (2009) and Horčík (2011). Table 12.10 collects the results for some prominent core semilinear logics. Unlike FL^{ℓ}, the weakest core semilinear logic SL^{ℓ} does enjoy all these completeness properties, as proved by Cintula et al. (2013). In particular, if one considers residuated groupoids defined over [0, 1] as its intended semantics, then SL^{ℓ} enjoys standard completeness in the strong version, and hence, can be regarded as a genuine fuzzy logic as much as HL, MTL or UL. On the other hand, it can arguably be seen as a basic logic in the meanings described in the introduction.

[22] This property was originally proposed by Takeuti and Titani (1984) in a much more specific context, then was generalized to a wide class of fuzzy logics by Metcalfe and Montagna (2007).

Table 12.10 Real and rational completeness for some core semilinear logics

Logic	$\mathscr{R}$C	FS$\mathscr{R}$C	S$\mathscr{R}$C	$\mathscr{Q}$C	FS$\mathscr{Q}$C = S$\mathscr{Q}$C
SL^{ℓ}_{S}, for each $S \subseteq \{e, c, i, o\}$	Yes	Yes	Yes	Yes	Yes
SL^{ℓ}_{a}, SL^{ℓ}_{ac}	No	No	No	No	No
$UL = SL^{\ell}_{ae}$, SL^{ℓ}_{aw}	Yes	Yes	Yes	Yes	Yes
$MTL = SL^{\ell}_{aew}$, IMTL, SMTL	Yes	Yes	Yes	Yes	Yes
WCMTL, ΠMTL	Yes	Yes	No	Yes	Yes
HL, SHL, Ł, Π	Yes	Yes	No	Yes	Yes
G, WNM, NM, C_nMTL, C_nIMTL	Yes	Yes	Yes	Yes	Yes
CPC	No	No	No	No	No

Indeed, the class of core semilinear logics is based in this logic and provides a useful framework covering virtually all the work done nowadays in MFL; moreover in the context of substructural logics complete w.r.t. chains could not be made weaker. We have therefore defended the role of SL^{ℓ} as basic fuzzy logic in the framework of propositional logics. In the last part of the chapter we argue that this is also the case at the first-order level.

12.4 First-Order Core Semilinear Logics

In this section we present the theory of first-order core semilinear logics. The presentation, definitions, and results of the first two subsections closely follow the work of Cintula and Noguera (2011, Sect. 5) simplified to our setting of core semilinear logics. The third subsection generalizes results of Cintula et al. (2009) (proved there for core *fuzzy* logics) and shortly surveys the undecidability results treated in detail by Hájek et al. (2011).

12.4.1 Syntax

In the following let L be a fixed core semilinear logic in a propositional language $\mathscr{L}$. The language of a first-order extension of L is defined in the same way as in classical first-order logic. In order to fix the notation and terminology we give an explicit definition:

Definition 12.15 A *predicate language* is a triple $\langle Pred_{\mathscr{P}}, Func_{\mathscr{P}}, Ar_{\mathscr{P}} \rangle$, where $Pred_{\mathscr{P}}$ is a non-empty set of *predicate symbols, $Func_{\mathscr{P}}$* is a set (disjoint with $Pred_{\mathscr{P}}$) of *function symbols,* and $Ar_{\mathscr{P}}$ is the *arity function,* assigning to each predicate or function symbol a natural number called the *arity* of the symbol. The function symbols F with $Ar_{\mathscr{P}}(F) = 0$ are called *object* or *individual constants.* The predicates symbols P for which $Ar_{\mathscr{P}}(P) = 0$ are called *truth constants.*[23]

[23] The roles of nullary predicates of $\mathscr{P}$ and nullary connectives of $\mathscr{L}$ are analogous, even though the values of the former are only fixed under a given interpretation of the predicate language, while

$\mathscr{P}$-terms and (atomic) $\mathscr{P}$-formulae of a given predicate language are defined as in classical logic (note that the notion of formula also depends on propositional connectives in $\mathscr{L}$). A $\mathscr{P}$-theory is a set of $\mathscr{P}$-formulae. The notions of free occurrence of a variable, substitutability, open formula, and closed formula (or, synonymously, *sentence*) are defined in the same way as in classical logic. Unlike in classical logic, in fuzzy logics without involutive negation the quantifiers $\forall$ and $\exists$ are not mutually definable, so the primitive language of L$\forall$ has to contain both of them.

There are several variants of the first-order extension of a propositional fuzzy logic L that can be defined. Following Hájek's original approach and his developments (Hájek 1998b, 2007a, b, 2010), here we restrict ourselves to logics of models over linearly ordered algebras (see Sect. 12.4.2) and introduce the first-order logics L$\forall$ and L$\forall^{\mathrm{w}}$ (respectively, complete w.r.t. all models or w.r.t. witnessed models). The axiomatic systems of the logics L$\forall$ and L$\forall^{\mathrm{w}}$ are defined as follows:

Definition 12.16 Let $\mathscr{P}$ be a predicate language. The logic L$\forall$ in language $\mathscr{P}$ has the following axioms:[24]

(P) The axioms of L
($\forall$1) $(\forall x)\varphi(x)\to\varphi(t)$, where the $\mathscr{P}$-term t is substitutable for x in φ
($\exists$1) $\varphi(t)\to(\exists x)\varphi(x)$, where the $\mathscr{P}$-term t is substitutable for x in φ
($\forall$2) $(\forall x)(\chi\to\varphi)\to(\chi\to(\forall x)\varphi)$, where x is not free in χ
($\exists$2) $(\forall x)(\varphi\to\chi)\to((\exists x)\varphi\to\chi)$, where x is not free in χ
($\forall$3) $(\forall x)(\chi\vee\varphi)\to\chi\vee(\forall x)\varphi$, where x is not free in χ.

The deduction rules of L$\forall$ are those of L plus the rule of *generalization*:

(Gen) $\langle\varphi, (\forall x)\varphi\rangle$.

The logic L$\forall^{\mathrm{w}}$ is the extension of L$\forall$ by the axioms:

(C$\forall$) $(\exists x)(\varphi(x)\to(\forall y)\varphi(y))$
(C$\exists$) $(\exists x)((\exists y)\varphi(y)\to\varphi(x))$.

The notions of proof and provability are defined for first-order core semilinear logics in the same way as in first-order classical logic. The fact that the formula φ is provable in L$\forall$ from a theory T will be denoted by $T\vdash_{\mathrm{L}\forall}\varphi$, and analogously for L$\forall^{\mathrm{w}}$; in a fixed context we can write just $T\vdash\varphi$.

Helena Rasiowa (1974) gave a first general theory of first-order non-classical logics based on her notion of propositional implicative logic. The presentation of her first-order logics, which we denote L$\forall^{\mathrm{m}}$, omitted the axiom ($\forall$3).[25] The superscript

(Footnote 23 continued)
the values of the latter are fixed under all such interpretations. The ambiguity of the term *truth constant* is thus a harmless abuse of language.

[24] When we speak about axioms or deduction rules of a propositional logic, we actually consider them with $\mathscr{P}$-formulae substituted for propositional variables.

[25] Actually her axiomatization omitted also the generalization rule, and the axioms ($\forall$2) and ($\exists$2) were replaced by the corresponding rules. However it can be shown that in the context of core semilinear logics her axiomatization and ours (without ($\forall$3)) are equivalent.

'm' stands for 'minimal', because L$\forall^{\mathrm{m}}$ is, in a sense, the weakest first-order extension of L. Indeed, L$\forall^{\mathrm{m}}$ is sound and complete w.r.t. first-order models built over *arbitrary* L-algebras. However, the axioms of L$\forall^{\mathrm{m}}$ are not strong enough to ensure the completeness w.r.t. first-order models (see the next subsection for technical details) over *linearly ordered* L-algebras—i.e., the usual chain completeness theorem, which we have presented as an essential common trait of all core semilinear logics. That is the reason why Hájek needed to add the axiom (∀3) in his presentation of first-order fuzzy logics. This axiom is valid in all models over L-chains (though not generally in models over arbitrary L-algebras) and ensures the chain completeness theorem for the resulting logic L∀.[26] This makes L∀ a natural choice for *the* first-order extension of a given core *semilinear* logic L. Consequently, we denote this first-order logic as L∀ with no superscript (though in some works the more systematic denotation L$\forall^{\ell}$ is used). Finally let us note that in the context of MFL, the logics L$\forall^{\mathrm{m}}$ were rediscovered by Petr Hájek (2000), where he denoted them by L$\forall^{-}$.

Let us list some important theorems that are provable in all logics L∀. Their proofs in MTL or HL are given e.g. by Esteva and Godo (2001) or Hájek (1998b); proofs in a weaker setting are given by Cintula and Noguera (2011).

Theorem 12.22 (Cintula and Noguera 2011, Propositions 4.2.5 and 4.3.2) *Let $\mathscr{P}$ be a predicate language, φ, ψ, χ $\mathscr{P}$-formulae, x a variable not free in χ, and x' a variable not occurring in φ. The following $\mathscr{P}$-formulae are theorems of* L∀:

(T∀1)	$\chi \leftrightarrow (\forall x)\chi$	(T∀11)	$(\exists x)(\chi \to \varphi) \to (\chi \to (\exists x)\varphi)$
(T∀2)	$(\exists x)\chi \leftrightarrow \chi$	(T∀12)	$(\exists x)(\varphi \to \chi) \to ((\forall x)\varphi \to \chi)$
(T∀3)	$(\forall x)\varphi(x) \leftrightarrow (\forall x')\varphi(x')$	(T∀13)	$(\forall x)(\varphi \wedge \psi) \leftrightarrow (\forall x)\varphi \wedge (\forall x)\psi$
(T∀4)	$(\exists x)\varphi(x) \leftrightarrow (\exists x')\varphi(x')$	(T∀14)	$(\exists x)(\varphi \vee \psi) \leftrightarrow (\exists x)\varphi \vee (\exists x)\psi$
(T∀5)	$(\forall x)(\forall y)\varphi \leftrightarrow (\forall y)(\forall x)\varphi$	(T∀15)	$(\forall x)(\varphi \vee \chi) \leftrightarrow (\forall x)\varphi \vee \chi$
(T∀6)	$(\exists x)(\exists y)\varphi \leftrightarrow (\exists y)(\exists x)\varphi$	(T∀16)	$(\exists x)(\varphi \wedge \chi) \leftrightarrow (\exists x)\varphi \wedge \chi$
(T∀7)	$(\forall x)(\varphi \to \psi) \to ((\forall x)\varphi \to (\forall x)\psi)$	(T∀17)	$(\exists x)(\varphi \,\&\, \chi) \leftrightarrow (\exists x)\varphi \,\&\, \chi$
(T∀8)	$(\forall x)(\varphi \to \psi) \to ((\exists x)\varphi \to (\exists x)\psi)$	(T∀18)	$(\exists x)(\varphi^n) \leftrightarrow ((\exists x)\varphi)^n$
(T∀9)	$(\chi \to (\forall x)\varphi) \leftrightarrow (\forall x)(\chi \to \varphi)$	(T∀19)	$(\exists x)\varphi \to \neg(\forall x)\neg\varphi$
(T∀10)	$((\exists x)\varphi \to \chi) \leftrightarrow (\forall x)(\varphi \to \chi)$	(T∀20)	$\neg(\exists x)\varphi \leftrightarrow (\forall x)\neg\varphi$

Remark 12.5 The converse implication of (T∀19) is provable in Ł∀, IMTL∀, or NM∀, i.e., in logics where ¬ is involutive (i.e. proves $\neg\neg\varphi \to \varphi$). Thus in such logics the existential quantifier is definable and the axioms (∃1) and (∃2) become redundant. Actually, for this claim to hold, the presence of an arbitrary unary connective $\sim$ such that $\varphi \to \psi \vdash_{\mathrm{L}} {\sim}\psi \to {\sim}\varphi$ and $\vdash_{\mathrm{L}} \varphi \leftrightarrow {\sim}{\sim}\varphi$ is sufficient (which could be either the 'natural' logical negation given by implication, or a new primitive connective added in logics $\mathrm{L}_{\sim}$).

The provability of the converse implications of (T∀11) or (T∀12) is equivalent to provability of (C∃) or (C∀) resp., i.e., if L∀ proves them, then L∀ = L$\forall^{\mathrm{w}}$. This is the case of Łukasiewicz logic; product logic proves (C∃) (and so the converse of

---

[26] This fact was first observed for Gödel logic by Horn (1969).

(T∀11)) but not (C∀), and Gödel logic proves neither. Finally it is worth noting that the axiom (∀3) is redundant in the axiomatization of Ł∀ and thus $Ł\forall^{\mathrm{m}} = Ł\forall = Ł\forall^{\mathrm{w}}$.

Some syntactic metatheorems valid in propositional core semilinear logics hold analogously for their first-order logics:

Theorem 12.23 (Cintula and Noguera 2011, Theorems 4.2.6, 4.2.8, and 4.3.9 and Corollary 4.2.11) *Let* L *be a core semilinear logic,* $\vdash$ *be either* $\vdash_{\mathrm{L}\forall}$ *or* $\vdash_{\mathrm{L}\forall^{\mathrm{w}}}$*, and* $\mathscr{P}$ *be a predicate language. Then the following holds for each* $\mathscr{P}$*-theory* T*,* $\mathscr{P}$*-sentences* φ*,* ψ*, and* $\mathscr{P}$*-formula* χ*:*

1. *The* intersubstitutability*:*
$$\varphi \leftrightarrow \psi \vdash \chi \leftrightarrow \chi',$$
where χ' *is obtained from* χ *by replacing some occurrences of* φ *by* ψ*.*
2. *The* constants theorem*:*
$$T \vdash \chi(c) \qquad \textit{iff} \qquad T \vdash \chi(x),$$
for any constant c *not occurring in* $T \cup \{\chi\}$*.*
3. *The* proof by cases *property:*
$$\frac{T, \varphi \vdash \chi \qquad T, \psi \vdash \chi}{T, \varphi \vee \psi \vdash \chi}$$
4. *The* semilinearity *property:*
$$\frac{T, \varphi \to \psi \vdash \chi \qquad T, \psi \to \varphi \vdash \chi}{T \vdash \chi}$$
If, furthermore, L *is almost* (MP)*-based with a set of basic deductive terms* bDT*, we can add:*
5. *The* local deduction theorem:
$$T, \varphi \vdash \chi \qquad \textit{iff} \qquad T \vdash \delta(\varphi) \to \chi \quad \textit{for some } \delta \in \Pi(\mathrm{bDT}^*)_{\mathscr{P}},$$
where by $\Pi(\mathrm{bDT}^*)_{\mathscr{P}}$ *we denote the set of formulae resulting from any* $\star$*-formula from* $\Pi(\mathrm{bDT}^*)$ *by replacing all its propositional variables other than* $\star$ *by arbitrary* $\mathscr{P}$*-sentences.*

Petr Hájek (1998b) (or later Hájek and Cintula 2006) used the local deduction theorems for schematic extensions of HL (for (△-)core fuzzy logics resp.) to show the semilinearity property (even though only in the latter it is formulated explicitly), which in turn is a crucial prerequisite for proving the completeness theorem.

12.4.2 General and Witnessed Semantics

In this subsection we shall introduce the (witnessed) semantics of predicate fuzzy logics, corresponding to the axiomatic systems $\mathrm{L}\forall$ and $\mathrm{L}\forall^{\mathrm{w}}$ respectively. To simplify the formulation of upcoming definitions let us fix: a core semilinear logic L in a propositional language $\mathscr{L}$, a predicate language $\mathscr{P} = \langle Pred, Func, Ar\rangle$, and an L-chain $\boldsymbol{B}$.

Definition 12.17 A $\boldsymbol{B}$-*structure* $\mathbf{M}$ for the predicate language $\mathscr{P}$ has the form: $\langle M, (P_{\mathbf{M}})_{P\in Pred}, (F_{\mathbf{M}})_{F\in Func}\rangle$, where M is a non-empty domain; for each n-ary predicate symbol $P \in Pred$, $P_{\mathbf{M}}$ is an n-ary fuzzy relation on M, i.e., a function $M^n \to B$ (identified with an element of B if $n = 0$); for each n-ary function symbol $F \in Func$, $F_{\mathbf{M}}$ is a function $M^n \to M$ (identified with an element of M if $n = 0$).

Let $\mathbf{M}$ be a $\boldsymbol{B}$-structure for $\mathscr{P}$. An $\mathbf{M}$-*evaluation* of the object variables is a mapping v which assigns an element from M to each object variable. Let v be an $\mathbf{M}$-evaluation, x a variable, and $a \in M$. Then by $\mathrm{v}[x\mapsto a]$ we denote the $\mathbf{M}$-evaluation such that $\mathrm{v}[x\mapsto a](x) = a$ and $\mathrm{v}[x\mapsto a](y) = \mathrm{v}(y)$ for each object variable y different from x.

Let $\mathbf{M}$ be a $\boldsymbol{B}$-structure for $\mathscr{P}$ and v an $\mathbf{M}$-evaluation. We define the *values* of terms and the *truth values* of formulae in $\mathbf{M}$ for an evaluation v recursively as follows noting that in the last two clauses, if the infimum or supremum does not exist, then the corresponding value is taken to be undefined, and in all clauses, if one of the arguments is undefined, then the result is undefined:

$$
\begin{aligned}
\|x\|^{\boldsymbol{B}}_{\mathbf{M},\mathrm{v}} &= \mathrm{v}(x) \\
\|F(t_1,\dots,t_n)\|^{\boldsymbol{B}}_{\mathbf{M},\mathrm{v}} &= F_{\mathbf{M}}(\|t_1\|^{\boldsymbol{B}}_{\mathbf{M},\mathrm{v}},\dots,\|t_n\|^{\boldsymbol{B}}_{\mathbf{M},\mathrm{v}}) \quad \text{for } F \in Func \\
\|P(t_1,\dots,t_n)\|^{\boldsymbol{B}}_{\mathbf{M},\mathrm{v}} &= P_{\mathbf{M}}(\|t_1\|^{\boldsymbol{B}}_{\mathbf{M},\mathrm{v}},\dots,\|t_n\|^{\boldsymbol{B}}_{\mathbf{M},\mathrm{v}}) \quad \text{for } P \in Pred \\
\|c(\varphi_1,\dots,\varphi_n)\|^{\boldsymbol{B}}_{\mathbf{M},\mathrm{v}} &= c_{\boldsymbol{B}}(\|\varphi_1\|^{\boldsymbol{B}}_{\mathbf{M},\mathrm{v}},\dots,\|\varphi_n\|^{\boldsymbol{B}}_{\mathbf{M},\mathrm{v}}) \quad \text{for } c \in \mathscr{L} \\
\|(\forall x)\varphi\|^{\boldsymbol{B}}_{\mathbf{M},\mathrm{v}} &= \inf\{\|\varphi\|^{\boldsymbol{B}}_{\mathbf{M},\mathrm{v}[x\mapsto a]} \mid a \in M\} \\
\|(\exists x)\varphi\|^{\boldsymbol{B}}_{\mathbf{M},\mathrm{v}} &= \sup\{\|\varphi\|^{\boldsymbol{B}}_{\mathbf{M},\mathrm{v}[x\mapsto a]} \mid a \in M\}.
\end{aligned}
$$

We say that the $\boldsymbol{B}$-structure $\mathbf{M}$ is

- *safe* if $\|\varphi\|^{\boldsymbol{B}}_{\mathbf{M},\mathrm{v}}$ is defined for each $\mathscr{P}$-formula φ and each $\mathbf{M}$-evaluation v,
- *witnessed* if for each $\mathscr{P}$-formula φ we have:

$$
\begin{aligned}
\|(\forall x)\varphi\|^{\boldsymbol{B}}_{\mathbf{M},\mathrm{v}} &= \min\{\|\varphi\|^{\boldsymbol{B}}_{\mathbf{M},\mathrm{v}[x\mapsto a]} \mid a \in M\} \\
\|(\exists x)\varphi\|^{\boldsymbol{B}}_{\mathbf{M},\mathrm{v}} &= \max\{\|\varphi\|^{\boldsymbol{B}}_{\mathbf{M},\mathrm{v}[x\mapsto a]} \mid a \in M\}.
\end{aligned}
$$

Note that each witnessed structure is safe. To simplify the upcoming definitions and theorems we write $\langle \boldsymbol{B}, \mathbf{M}\rangle \models \varphi$ if $\|\varphi\|^{\boldsymbol{B}}_{\mathbf{M},\mathrm{v}} \geqslant_{\boldsymbol{B}} 1$ for each $\mathbf{M}$-evaluation v.

Definition 12.18 Let $\mathbf{M}$ be a $\boldsymbol{B}$-structure for $\mathscr{P}$ and T a $\mathscr{P}$-theory. Then $\mathbf{M}$ is called a *$\boldsymbol{B}$-model of T* if it is safe and $\langle \boldsymbol{B}, \mathbf{M}\rangle \models \varphi$ for each $\varphi \in T$.

Observe that models are safe structures by definition. Since, obviously, each safe $\boldsymbol{B}$-structure is a $\boldsymbol{B}$-model of the empty theory, we shall use the term *model* for both models and safe structures in the rest of the text.[27] By a slight abuse of language we use the term model also for the pair $\langle \boldsymbol{B}, \mathbf{M}\rangle$.

The following completeness theorems show that the syntactic presentations introduced above succeed in capturing the intended general chain-semantics for first-order fuzzy logics.[28]

Theorem 12.24 (Cintula and Noguera 2011, Theorems 4.3.5 and 4.4.10) *Let* L *be a core semilinear logic, $\mathscr{P}$ a predicate language, T a $\mathscr{P}$-theory, and φ a $\mathscr{P}$-formula. Then the following are equivalent:*

- $T \vdash_{\mathrm{L}\forall} \varphi$.
- $\langle \boldsymbol{B}, \mathbf{M}\rangle \models \varphi$ *for each* L*-chain* $\boldsymbol{B}$ *and each model* $\langle \boldsymbol{B}, \mathbf{M}\rangle$ *of the theory* T.

Theorem 12.25 (Cintula and Noguera 2011, Theorem 4.5.12 and Example 4.5.3) *Let* L *be an axiomatic expansion of* $\mathrm{SL}^{\ell}_{\mathrm{ae}}$, $\mathscr{P}$ *a predicate language, T a $\mathscr{P}$-theory, and φ a $\mathscr{P}$-formula. Then the following are equivalent:*

- $T \vdash_{\mathrm{L}\forall^{\mathrm{w}}} \varphi$.
- $\langle \boldsymbol{B}, \mathbf{M}\rangle \models \varphi$ *for each* L*-chain* $\boldsymbol{B}$ *and each* witnessed *model* $\langle \boldsymbol{B}, \mathbf{M}\rangle$ *of the theory* T.

12.4.3 Standard Semantics

Already in the pioneering works of Petr Hájek, as in the case of propositional fuzzy logics, the general chain completeness we have just seen was not considered sufficient and, in fact, a crucial item in his agenda was again the search for standard completeness theorems with respect to distinguished classes of models. In order to survey the corresponding results in our framework, in this section we restrict ourselves to *countable* predicate languages. We shall say that $\langle \boldsymbol{B}, \mathbf{M}\rangle$ is a $\mathbb{K}$*-model* (of T) for some $\mathbb{K} \subseteq \mathbb{L}_{lin}$ if $\langle \boldsymbol{B}, \mathbf{M}\rangle$ is a model (of T) and $\boldsymbol{B} \in \mathbb{K}$.

Definition 12.19 Let L be a core semilinear logic and $\mathbb{K} \subseteq \mathbb{L}_{lin}$. We say that L$\forall$ enjoys enjoys (*finite*) *strong* $\mathbb{K}$*-completeness* S$\mathbb{K}$C (FS$\mathbb{K}$C *resp.*) if for each countable predicate language $\mathscr{P}$, $\mathscr{P}$-formula φ, and (finite) $\mathscr{P}$-theory T holds:

$$T \vdash_{\mathrm{L}\forall} \varphi \quad \text{iff} \quad \langle \boldsymbol{B}, \mathbf{M}\rangle \models \varphi \text{ for each } \mathbb{K}\text{-model } \langle \boldsymbol{B}, \mathbf{M}\rangle \text{ of the theory } T$$

We say that L$\forall$ enjoys $\mathbb{K}$*-completeness* $\mathbb{K}$C if the equivalence holds for $T = \emptyset$.

[27] In the literature the term 'ℓ-model' is sometimes used instead to stress that $\boldsymbol{B}$ is linearly ordered.

[28] The proofs of Theorems 12.24 and 12.25 for core and $\triangle$-core fuzzy logics are given by Hájek and Cintula (2006). Instances of Theorem 12.24 for various core semilinear logics were originally proved separately; usually for countable predicate languages only (Esteva and Godo 2001; Hájek 1998b).

All these properties are stronger than their corresponding ones for propositional logics:

Theorem 12.26 *Let* L *be a core semilinear logic and* $\mathbb{K}$ *a class of* L*-chains. If* L∀ *has the* S$\mathbb{K}$C (FS$\mathbb{K}$C *or* $\mathbb{K}$C *respectively), then* L *has the* S$\mathbb{K}$C (FS$\mathbb{K}$C *or* $\mathbb{K}$C *respectively*).

As in the propositional case (Theorem 12.15), strong $\mathbb{K}$-completeness is related to an embedding property, although it is a stronger one requiring preservation of existing suprema and infima:

Definition 12.20 Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two algebras of the same type with (defined) lattice operations. We say that an embedding $f : \boldsymbol{A} \to \boldsymbol{B}$ is a *σ-embedding* if $f(\sup C) = \sup f[C]$ (whenever $\sup C$ exists) and $f(\inf D) = \inf f[D]$ (whenever $\inf D$ exists) for each countable $C, D \subseteq A$.

Theorem 12.27 *Let* L *be a core semilinear logic. If every countable* L*-chain* $\boldsymbol{A}$ *can be σ-embedded into some* L*-chain* $\boldsymbol{B} \in \mathbb{K}$*, then* L∀ *enjoys* S$\mathbb{K}$C.

The proof of this theorem is almost straightforward. Unlike in the propositional case, the existence of σ-embeddings is not a necessary condition, as shown by Cintula et al. (2009, Theorem 5.38), but nevertheless it is the usual method for proving these results. It also has an interesting corollary for completeness w.r.t. finite chains (recall the characterizations of S$\mathscr{F}$C in Theorem 12.17).

Corollary 12.5 *Let* L *be a core semilinear logic. Then* L *enjoys the* S$\mathscr{F}$C *iff* L∀ *enjoys the* S$\mathscr{F}$C.

It is obvious that every $\boldsymbol{B}$-structure over a finite L-chain is necessarily witnessed. Thus we have the following proposition which can be used in order to disprove the $\mathscr{F}$C for many logics.

Proposition 12.4 *Let* L *be a core semilinear logic such that* L∀ *enjoys* $\mathscr{F}$C*. Then* L∀ = L$\forall^{\mathrm{w}}$.

For instance, the examples by Hájek (1998b, Lemma 5.3.6) can be used to show that (C∃) is unprovable in G∀ and (C∀) is unprovable both in G∀ and in Π∀. We can also easily show that $\nvdash_{\mathrm{NM}\forall}$ (C∀). Thus these logics do not enjoy $\mathscr{F}$C.

Table 12.11 collects results on real, rational and finite-chain completeness of prominent core semilinear logics. Their proofs are scattered in the literature (e.g. Hájek (1998b); Esteva and Godo (2001)). Cintula et al. (2009) give more information and detailed references. For logics weaker than MTL∀ the negative results are derived from the corresponding failure of completeness at the propositional level, while the positive ones are justified by observing that the embeddings built to prove completeness for the underlying propositional logic are actually σ-embeddings. Observe that the family of logics in the first row of the table include SL$^{\ell}$∀, which enjoys both strong rational and real (i.e. standard) completeness. This fact was not stated before in the literature because the only paper dealing with SL$^{\ell}$ (Cintula et al. 2013) concentrated only on propositional logics, but it is not difficult to check that for SL$^{\ell}$ the obtained embedding is also actually a σ-embedding, i.e., we obtain:

Table 12.11 Completeness properties for some first-order core semilinear logics

Logic	$\mathscr{R}$C	FS$\mathscr{R}$C, S$\mathscr{R}$C	$\mathscr{Q}$C, FS$\mathscr{Q}$C, S$\mathscr{Q}$C	$\mathscr{F}$C, FS$\mathscr{F}$C, S$\mathscr{F}$C
$\mathrm{SL}^{\ell}_{\mathrm{S}}\forall$, for each $\mathrm{S} \subseteq \{\mathrm{e, c, i, o}\}$	Yes	Yes	Yes	No
$\mathrm{SL}^{\ell}_{\mathrm{a}}\forall$	No	No	No	No
$\mathrm{SL}^{\ell}_{\mathrm{aw}}\forall$	Yes	Yes	Yes	No
MTL∀, IMTL∀, SMTL∀	Yes	Yes	Yes	No
WCMTL∀, ΠMTL∀	?	No	?	No
HL∀, SHL∀	No	No	No	No
Ł∀, Π∀	No	No	Yes	No
G∀, WNM∀, NM∀	Yes	Yes	Yes	No
$\mathrm{C_nMTL}\forall$, $\mathrm{C_nIMTL}\forall$	Yes	Yes	Yes	No
CPC∀	No	No	No	Yes

Theorem 12.28 *Let* $\mathrm{S} \subseteq \{\mathrm{e, c, i, o}\}$. *Then for each countable predicate language* $\mathscr{P}$, $\mathscr{P}$*-formula* φ, *and* $\mathscr{P}$*-theory* T *holds:*

$$T \vdash_{\mathrm{SL}^{\ell}_{\mathrm{S}}\forall} \varphi \quad \textit{iff} \quad \langle \boldsymbol{B}, \mathbf{M}\rangle \models \varphi \textit{ for each } \mathscr{R}\textit{-model } \langle \boldsymbol{B}, \mathbf{M}\rangle \textit{ of the theory } T.$$

It is worth adding that for all core fuzzy logics appearing in the table the same results hold for their expansions with $\triangle$; moreover $\mathrm{G}_{\sim}$ behaves like G, while $\mathrm{SHL}_{\sim}$ and ŁΠ behave like HL. Observe the rather surprising behavior of continuous t-norm based logics regarding the rational-chain semantics: while Ł∀, Π∀, and G∀ enjoy the S$\mathscr{Q}$C, the logics HL∀ and SHL∀ do not even have $\mathscr{Q}$C (Cintula et al. 2009). Petr Hájek (1998b) already gave an important hint towards the failure of rational completeness in HL∀ and SHL∀: he found a first-order formula, $(\forall x)(\chi \,\&\, \varphi) \rightarrow (\chi \,\&\, (\forall x)\varphi)$, which holds in every model on a densely ordered HL-chain but, as shown later by Esteva and Godo (2001), it is not a tautology of any of those two logics; therefore it makes sense to extend them with this axiom and, by doing so, one obtains new first-order logics complete with respect to all models over rational HL-chains or, respectively, SHL-chains (Cintula et al. 2009).

Finding particular examples of formulae witnessing failure of $\mathscr{R}$C of a given logic is not an easy task. Even though some examples were found by Petr Hájek (2004b) for some particular cases, a usual method of proving this is to show that the set of standard tautologies (see next definition) is not recursively enumerable, and therefore it cannot coincide with the set of its theorems. Determining the position in the arithmetical hierarchy (e.g. Rogers 1967) of prominent sets of formulae (such as the tautologies of a given logic) is an important field of study in mathematical logic with major contributions done by Petr Hájek. Here we just briefly mention a few results related to fuzzy logics: a full treatment of the arithmetical complexity of first-order ($\triangle$-)fuzzy logics was presented by Hájek et al. (2011). Other references on the topic are Hájek (2001, 2004a, 2005a), Montagna (2001, 2005), Montagna and Noguera (2010).

Table 12.12 Arithmetical complexity of standard semantics

	G∀	Ł∀	Π∀	HL∀
stTAUT	$\mathbf{\Sigma}_1$-c	$\mathbf{\Pi}_2$-c	***NA***	***NA***
stSAT	$\mathbf{\Pi}_1$-c	$\mathbf{\Pi}_1$-c	***NA***	***NA***

First let us introduce some prominent sets of first-order formulae given by a core semilinear logic L:

Definition 12.21 We say that a sentence φ is

- A *general* (resp., *standard*) *tautology* of L∀ if $\langle \boldsymbol{B}, \mathbf{M}\rangle \models \varphi$ for each ($\mathscr{R}$-)model $\langle \boldsymbol{B}, \mathbf{M}\rangle$.
- *Generally* (resp., *standardly*) *satisfiable* in L∀ if $\langle \boldsymbol{B}, \mathbf{M}\rangle \models \varphi$ for some ($\mathscr{R}$-)model $\langle \boldsymbol{B}, \mathbf{M}\rangle$.

The sets of general and standard tautologies and generally and standardly satisfiable sentences are denoted, respectively, by genTAUT, stTAUT, genSAT, and stSAT.

For illustration, let us state the results for four predicate logics: HL∀, Ł∀, G∀, and Π∀. For each of them, the set of general tautologies is $\mathbf{\Sigma}_1$-complete (thus they are recursively axiomatizable, but undecidable) and the set of generally satisfiable formulae is $\mathbf{\Pi}_1$-complete. For the arithmetical complexity of their standard semantics see Table 12.12 (where "-c" stands for "-complete" and "***NA***" for "non-arithmetical"). It can be seen that as far as standard semantics is concerned, the four logics differ drastically with respect to their degree of undecidability.

12.5 Conclusions

In this chapter we have presented a general approach to fuzzy logics based on the logic SL^ℓ. We have introduced a broad class of, both propositional and predicate, core semilinear logics and shown their axiomatizations and completeness with respect to models over chains. Moreover, we have surveyed their completeness results with respect to distinguished semantics and obtained, in particular, that the weakest predicate fuzzy logic of our framework, $\mathrm{SL}^\ell\forall$, enjoys the standard completeness theorem. Therefore, our flea still jumps (and jumps very well, even in the first-order case!) and we can arguably say that the quest for the basic fuzzy logic initiated by Petr Hájek so far seems to culminate with SL^ℓ. Indeed, both for propositional and first-order predicate logics, SL^ℓ provides a good ground level to build broad families of logics containing all the important particular systems of fuzzy logic: propositional fuzzy logics are captured inside the class of core semilinear logics, while first-order fuzzy logics are obtained as extensions of the logic L∀ built over a core semilinear logic L. Moreover, SL^ℓ is the weakest possible logic one could take in the context of substructural logics in a language with lattice connectives, a conjunction which is not required to satisfy any property corresponding to the usual structural rules and its left and right residua. We do not know whether Mathematical Fuzzy Logic will require an even weaker system to serve as the basic fuzzy logic in the future. Only time will tell. What we can say is that, at the moment, we do not see any remaining legs to be pulled.

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