Chapter 9
Basic Concepts on the Calculus of Variations

9.1 Introduction to the Calculus of Variations

We emphasize the main references for this chapter are [37, 38, 68].

Here we recall that a functional is a function whose co-domain is the real set. We denote such functionals by $F : U \to \mathbb{R}$, where U is a Banach space. In our work format, we consider the special cases:

1. $F(u) = \int_\Omega f(x,u,\nabla u)\,dx$, where $\Omega \subset \mathbb{R}^n$ is an open, bounded, and connected set.
2. $F(u) = \int_\Omega f(x,u,\nabla u, D^2 u)\,dx$, here

$$Du = \nabla u = \left\{ \frac{\partial u_i}{\partial x_j} \right\}$$

and

$$D^2 u = \{D^2 u_i\} = \left\{ \frac{\partial^2 u_i}{\partial x_k \partial x_l} \right\},$$

for $i \in \{1,\ldots,N\}$ and $j,k,l \in \{1,\ldots,n\}$.

Also, $f : \overline{\Omega} \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \to \mathbb{R}$ is denoted by $f(x,s,\xi)$ and we assume

1.
$$\frac{\partial f(x,s,\xi)}{\partial s}$$

and

2.
$$\frac{\partial f(x,s,\xi)}{\partial \xi}$$

are continuous $\forall (x,s,\xi) \in \overline{\Omega} \times \mathbb{R}^N \times \mathbb{R}^{N \times n}$.

Remark 9.1.1. We also recall that the notation $\nabla u = Du$ may be used.

Now we define our general problem, namely problem $\mathscr{P}$ where

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$$\text{Problem } \mathscr{P}: \text{ minimize } F(u) \text{ on } U,$$

that is, to find $u_0 \in U$ such that

$$F(u_0) = \min_{u \in U}\{F(u)\}.$$

At this point, we introduce some essential definitions.

Definition 9.1.2 (Space of Admissible Variations). Given $F : U \to \mathbb{R}$ we define the space of admissible variations for F, denoted by $\mathscr{V}$ as

$$\mathscr{V} = \{\varphi \mid u + \varphi \in U, \forall u \in U\}.$$

For example, for $F : U \to \mathbb{R}$ given by

$$F(u) = \frac{1}{2}\int_\Omega \nabla u \cdot \nabla u \, dx - \langle u, f \rangle_U,$$

where $\Omega \subset \mathbb{R}^3$ and

$$U = \{u \in W^{1,2}(\Omega) \mid u = \hat{u} \text{ on } \partial\Omega\}$$

we have

$$\mathscr{V} = W_0^{1,2}(\Omega).$$

Observe that in this example U is a subset of a Banach space.

Definition 9.1.3 (Local Minimum). Given $F : U \to \mathbb{R}$, we say that $u_0 \in U$ is a local minimum for F if there exists $\delta > 0$ such that

$$F(u) \geq F(u_0), \forall u \in U, \text{ such that } \|u - u_0\|_U < \delta,$$

or equivalently

$$F(u_0 + \varphi) \geq F(u_0), \forall \varphi \in \mathscr{V}, \text{ such that } \|\varphi\|_U < \delta.$$

Definition 9.1.4 (Gâteaux Variation). Given $F : U \to \mathbb{R}$ we define the Gâteaux variation of F at $u \in U$ on the direction $\varphi \in \mathscr{V}$, denoted by $\delta F(u, \varphi)$ as

$$\delta F(u, \varphi) = \lim_{\varepsilon \to 0} \frac{F(u + \varepsilon\varphi) - F(u)}{\varepsilon},$$

if such a limit is well defined. Furthermore, if there exists $u^* \in U^*$ such that

$$\delta F(u, \varphi) = \langle \varphi, u^* \rangle_U, \forall \varphi \in U,$$

we say that F is Gâteaux differentiable at $u \in U$, and $u^* \in U^*$ is said to be the Gâteaux derivative of F at u. Finally we denote

$$u^* = \delta F(u) \text{ or } u^* = \frac{\partial F(u)}{\partial u}.$$

9.2 Evaluating the Gâteaux Variations

Consider $F : U \to \mathbb{R}$ such that

$$F(u) = \int_\Omega f(x, u, \nabla u)\, dx$$

where the hypothesis indicated in the last section is assumed. Consider $u \in C^1(\bar{\Omega}; \mathbb{R}^N)$ and $\varphi \in C_c^1(\bar{\Omega}; \mathbb{R}^N)$ and let us evaluate $\delta F(u, \varphi)$:

From Definition 9.1.4,

$$\delta F(u, \varphi) = \lim_{\varepsilon \to 0} \frac{F(u + \varepsilon \varphi) - F(u)}{\varepsilon}.$$

Observe that

$$\lim_{\varepsilon \to 0} \frac{f(x, u + \varepsilon \varphi, \nabla u + \varepsilon \nabla \varphi) - f(x, u, \nabla u)}{\varepsilon} = \frac{\partial f(x, u, \nabla u)}{\partial s} \cdot \varphi + \frac{\partial f(x, u, \nabla u)}{\partial \xi} \cdot \nabla \varphi.$$

Define

$$G(x, u, \varphi, \varepsilon) = \frac{f(x, u + \varepsilon \varphi, \nabla u + \varepsilon \nabla \varphi) - f(x, u, \nabla u)}{\varepsilon},$$

and

$$\tilde{G}(x, u, \varphi) = \frac{\partial f(x, u, \nabla u)}{\partial s} \cdot \varphi + \frac{\partial f(x, u, \nabla u)}{\partial \xi} \cdot \nabla \varphi.$$

Thus we have

$$\lim_{\varepsilon \to 0} G(x, u, \varphi, \varepsilon) = \tilde{G}(x, u, \varphi).$$

Now we will show that

$$\lim_{\varepsilon \to 0} \int_\Omega G(x, u, \varphi, \varepsilon)\, dx = \int_\Omega \tilde{G}(x, u, \varphi)\, dx.$$

Suppose to obtain contradiction that we do not have

$$\lim_{\varepsilon \to 0} \int_\Omega G(x, u, \varphi, \varepsilon)\, dx = \int_\Omega \tilde{G}(x, u, \varphi)\, dx.$$

Hence, there exists $\varepsilon_0 > 0$ such that for each $n \in \mathbb{N}$ there exists $0 < \varepsilon_n < 1/n$ such that

$$\left| \int_\Omega G(x, u, \varphi, \varepsilon_n)\, dx - \int_\Omega \tilde{G}(x, u, \varphi)\, dx \right| \geq \varepsilon_0. \tag{9.1}$$

Define

$$c_n = \max_{x \in \overline{\Omega}} \{|G(x, u(x), \varphi(x), \varepsilon_n) - \tilde{G}(x, u(x), \varphi(x))|\}.$$

Since the function in question is continuous on the compact set $\overline{\Omega}$, $\{x_n\}$ is well defined. Also from the fact that $\overline{\Omega}$ is compact, there exists a subsequence $\{x_{n_j}\}$ and $x_0 \in \bar{\Omega}$ such that

$$\lim_{j \to +\infty} x_{n_j} = x_0.$$

Thus

$$\begin{aligned} \lim_{j \to +\infty} c_{n_j} &= c_0 \\ &= \lim_{j \to +\infty} \{|G(x_{n_j}, u(x_{n_j}), \varphi(x_{n_j}), \varepsilon_{n_j}) - \tilde{G}(x_0, u(x_0), \varphi(x_0))|\} = 0. \end{aligned}$$

Therefore there exists $j_0 \in \mathbb{N}$ such that if $j > j_0$, then

$$c_{n_j} < \varepsilon_0 / |\Omega|.$$

Thus, if $j > j_0$, we have

$$\begin{aligned} &\left| \int_\Omega G(x, u, \varphi, \varepsilon_{n_j}) \, dx - \int_\Omega \tilde{G}(x, u, \varphi) \, dx \right| \\ &\qquad \leq \int_\Omega |G(x, u, \varphi, \varepsilon_{n_j}) - \tilde{G}(x, u, \varphi)| \, dx \leq c_{n_j} |\Omega| < \varepsilon_0, \end{aligned} \tag{9.2}$$

which contradicts (9.1). Hence, we may write

$$\lim_{\varepsilon \to 0} \int_\Omega G(x, u, \varphi, \varepsilon) \, dx = \int_\Omega \tilde{G}(x, u, \varphi) \, dx,$$

that is,

$$\delta F(u, \varphi) = \int_\Omega \left\{ \frac{\partial f(x, u, \nabla u)}{\partial s} \cdot \varphi + \frac{\partial f(x, u, \nabla u)}{\partial \xi} \cdot \nabla \varphi \right\} dx.$$

Theorem 9.2.1 (Fundamental Lemma of Calculus of Variations). *Consider an open set $\Omega \subset \mathbb{R}^n$ and $u \in L^1_{loc}(\Omega)$ such that*

$$\int_\Omega u\varphi \, dx = 0, \forall \varphi \in C_c^\infty(\Omega).$$

Then $u = 0$, a.e. in Ω.

Remark 9.2.2. Of course a similar result is valid for the vectorial case. A proof of such a result was given in Chap. 8.

Theorem 9.2.3 (Necessary Conditions for a Local Minimum). *Suppose $u \in U$ is a local minimum for a Gâteaux differentiable $F : U \to \mathbb{R}$. Then*

$$\delta F(u,\varphi) = 0, \forall \varphi \in \mathcal{V}.$$

Proof. Fix $\varphi \in \mathcal{V}$. Define $\phi(\varepsilon) = F(u+\varepsilon\varphi)$. Since by hypothesis ϕ is differentiable and attains a minimum at $\varepsilon = 0$, from the standard necessary condition $\phi'(0) = 0$, we obtain $\phi'(0) = \delta F(u,\varphi) = 0$.

Theorem 9.2.4. *Consider the hypotheses stated in Section 9.1 on $F : U \to \mathbb{R}$. Suppose F attains a local minimum at $u \in C^2(\bar{\Omega};\mathbb{R}^N)$ and additionally assume that $f \in C^2(\overline{\Omega},\mathbb{R}^N,\mathbb{R}^{N\times n})$. Then the necessary conditions for a local minimum for F are given by the Euler–Lagrange equations:*

$$\frac{\partial f(x,u,\nabla u)}{\partial s} - div\left(\frac{\partial f(x,u,\nabla u)}{\partial \xi}\right) = \theta, \text{ in } \Omega.$$

Proof. From Theorem 9.2.3, the necessary condition stands for $\delta F(u,\varphi) = 0, \forall \varphi \in \mathcal{V}$. From the above this implies, after integration by parts

$$\int_\Omega \left(\frac{\partial f(x,u,\nabla u)}{\partial s} - div\left(\frac{\partial f(x,u,\nabla u)}{\partial \xi}\right)\right)\cdot \varphi \, dx = 0,$$

$$\forall \varphi \in C_c^\infty(\Omega,\mathbb{R}^N).$$

The result then follows from the fundamental lemma of calculus of variations.

9.3 The Gâteaux Variation: A More General Case

Theorem 9.3.1. *Consider the functional $F : U \to \mathbb{R}$, where*

$$U = \{u \in W^{1,2}(\Omega,\mathbb{R}^N) \mid u = u_0 \text{ in } \partial\Omega\}.$$

Suppose

$$F(u) = \int_\Omega f(x,u,\nabla u)\, dx,$$

where $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n}$ is such that for each $K > 0$ there exists $K_1 > 0$ which does not depend on x such that

$$|f(x,s_1,\xi_1) - f(x,s_2,\xi_2)| < K_1(|s_1 - s_2| + |\xi_1 - \xi_2|)$$
$$\forall s_1,s_2 \in \mathbb{R}^N, \xi_1,\xi_2 \in \mathbb{R}^{N\times n}, \text{ such that } |s_1| < K, |s_2| < K,$$
$$|\xi_1| < K, |\xi_2| < K.$$

Also assume the hypotheses of Section 9.1 except for the continuity of derivatives of f. Under such assumptions, for each $u \in C^1(\overline{\Omega};\mathbb{R}^N)$ and $\varphi \in C_c^\infty(\Omega;\mathbb{R}^N)$, we have

$$\delta F(u,\varphi) = \int_\Omega \left\{\frac{\partial f(x,u,\nabla u)}{\partial s}\cdot\varphi + \frac{\partial f(x,u,\nabla u)}{\partial \xi}\cdot\nabla\varphi\right\} dx.$$

Proof. From Definition 9.1.4,

$$\delta F(u,\varphi) = \lim_{\varepsilon\to 0} \frac{F(u+\varepsilon\varphi) - F(u)}{\varepsilon}.$$

Observe that

$$\lim_{\varepsilon\to 0} \frac{f(x,u+\varepsilon\varphi,\nabla u+\varepsilon\nabla\varphi) - f(x,u,\nabla u)}{\varepsilon} = \frac{\partial f(x,u,\nabla u)}{\partial s}\cdot\varphi + \frac{\partial f(x,u,\nabla u)}{\partial \xi}\cdot\nabla\varphi, \text{ a.e in } \Omega.$$

Define

$$G(x,u,\varphi,\varepsilon) = \frac{f(x,u+\varepsilon\varphi,\nabla u+\varepsilon\nabla\varphi) - f(x,u,\nabla u)}{\varepsilon},$$

and

$$\tilde{G}(x,u,\varphi) = \frac{\partial f(x,u,\nabla u)}{\partial s}\cdot\varphi + \frac{\partial f(x,u,\nabla u)}{\partial \xi}\cdot\nabla\varphi.$$

Thus we have

$$\lim_{\varepsilon\to 0} G(x,u,\varphi,\varepsilon) = \tilde{G}(x,u,\varphi), \text{ a.e in } \Omega.$$

Now we will show that

$$\lim_{\varepsilon\to 0}\int_\Omega G(x,u,\varphi,\varepsilon)\,dx = \int_\Omega \tilde{G}(x,u,\varphi)\,dx.$$

It suffices to show that (we do not provide details here)

$$\lim_{n\to\infty}\int_\Omega G(x,u,\varphi,1/n)\,dx = \int_\Omega \tilde{G}(x,u,\varphi)\,dx.$$

Observe that for an appropriate $K > 0$, we have

$$|G(x,u,\varphi,1/n)| \leq K(|\varphi| + |\nabla\varphi|), \text{ a.e. in } \Omega. \tag{9.3}$$

By the Lebesgue dominated convergence theorem, we obtain

$$\lim_{n\to+\infty}\int_\Omega G(x,u,\varphi,1/(n))\,dx = \int_\Omega \tilde{G}(x,u,\varphi)\,dx,$$

that is,

$$\delta F(u,\varphi) = \int_\Omega \left\{\frac{\partial f(x,u,\nabla u)}{\partial s}\cdot\varphi + \frac{\partial f(x,u,\nabla u)}{\partial \xi}\cdot\nabla\varphi\right\}dx.$$

9.4 Fréchet Differentiability

In this section we introduce a very important definition, namely, Fréchet differentiability.

Definition 9.4.1. Let U, Y be Banach spaces and consider a transformation $T : U \to Y$. We say that T is Fréchet differentiable at $u \in U$ if there exists a bounded linear transformation $T'(u) : U \to Y$ such that

$$\lim_{v \to \theta} \frac{\|T(u+v) - T(u) - T'(u)(v)\|_Y}{\|v\|_U} = 0,\ v \neq \theta.$$

In such a case $T'(u)$ is called the Fréchet derivative of T at $u \in U$.

9.5 Elementary Convexity

In this section we develop some proprieties concerning elementary convexity.

Definition 9.5.1. A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be convex if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y), \forall x, y \in \mathbb{R}^n, \lambda \in [0,1].$$

Proposition 9.5.2. *If $f : \mathbb{R}^n \to \mathbb{R}$ is convex and differentiable, then*

$$f(y) - f(x) \geq \langle f'(x), y - x \rangle_{\mathbb{R}^n}, \forall x, y \in \mathbb{R}^n.$$

Proof. Pick $x, y \in \mathbb{R}^n$. By hypothesis

$$f((1-\lambda)x + \lambda y) \leq (1-\lambda) f(x) + \lambda f(y), \forall \lambda \in [0,1].$$

Thus

$$\frac{f(x + \lambda(y-x)) - f(x)}{\lambda} \leq f(y) - f(x), \forall \lambda \in (0,1].$$

Letting $\lambda \to 0^+$ we obtain

$$f(y) - f(x) \geq \langle f'(x), y - x \rangle_{\mathbb{R}^n}.$$

Since $x, y \in \mathbb{R}^n$ are arbitrary, the proof is complete.

Proposition 9.5.3. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function. If*

$$f(y) - f(x) \geq \langle f'(x), y - x \rangle_{\mathbb{R}^n}, \forall x, y \in \mathbb{R}^n,$$

then f is convex.

Proof. Define $f^*(x^*)$ by

$$f(x^*) = \sup_{x \in \mathbb{R}^n} \{\langle x, x^* \rangle_{\mathbb{R}^n} - f(x)\}.$$

Such a function f^* is called the Fenchel conjugate of f. Observe that by hypothesis,

$$f^*(f'(x)) = \sup_{y \in \mathbb{R}^n} \{\langle y, f'(x) \rangle_{\mathbb{R}^n} - f(y)\} = \langle x, f'(x) \rangle_{\mathbb{R}^n} - f(x). \tag{9.4}$$

On the other hand

$$f^*(x^*) \geq \langle x, x^* \rangle_{\mathbb{R}^n} - f(x), \forall x, x^* \in \mathbb{R}^n,$$

that is,

$$f(x) \geq \langle x, x^* \rangle_{\mathbb{R}^n} - f^*(x^*), \forall x, x^* \in \mathbb{R}^n.$$

Observe that from (9.4)

$$f(x) = \langle x, f'(x) \rangle_{\mathbb{R}^n} - f^*(f'(x))$$

and thus

$$f(x) = \sup_{x^* \in \mathbb{R}^n} \{\langle x, x^* \rangle_{\mathbb{R}^n} - f(x^*)\}, \forall x \in \mathbb{R}^n.$$

Pick $x, y \in \mathbb{R}^n$ and $\lambda \in [0,1]$. Thus, we may write

$$\begin{aligned} f(\lambda x + (1-\lambda)y) &= \sup_{x^* \in \mathbb{R}^n} \{\langle \lambda x + (1-\lambda)y, x^* \rangle_{\mathbb{R}^n} - f^*(x^*)\} \\ &= \sup_{x^* \in \mathbb{R}^n} \{\lambda \langle x, x^* \rangle_{\mathbb{R}^n} + (1-\lambda)\langle y, x^* \rangle_{\mathbb{R}^n} - \lambda f^*(x^*) \\ &\quad - (1-\lambda) f^*(x^*)\} \\ &\leq \lambda \{ \sup_{x^* \in \mathbb{R}^n} \{\langle x, x^* \rangle_{\mathbb{R}^n} - f^*(x^*)\}\} \\ &\quad + (1-\lambda)\{ \sup_{x^* \in \mathbb{R}^n} \{\langle y, x^* \rangle_{\mathbb{R}^n} - f^*(x^*)\}\} \\ &= \lambda f(x) + (1-\lambda) f(y). \end{aligned} \tag{9.5}$$

Since $x, y \in \mathbb{R}^n$ and $\lambda \in [0,1]$ are arbitrary, we have that f is convex.

Corollary 9.5.4. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be twice differentiable and*

$$\left\{ \frac{\partial^2 f(x)}{\partial x_i \partial x_j} \right\},$$

positive definite, for all $x \in \mathbb{R}^n$. Then f is convex.

Proof. Pick $x, y \in \mathbb{R}^n$. Using Taylor's expansion we obtain

$$f(y) = f(x) + \langle f'(x), y - x \rangle_{\mathbb{R}^n} + \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 f(\bar{x})}{\partial x_i \partial x_j}(y_i - x_i)(y_j - x_j),$$

for $\bar{x} = \lambda x + (1-\lambda)y$ (for some $\lambda \in [0,1]$). From the hypothesis we obtain

$$f(y) - f(x) - \langle f'(x), y - x\rangle_{\mathbb{R}^n} \geq 0.$$

Since $x, y \in \mathbb{R}^n$ are arbitrary, the proof is complete.

Similarly we may obtain the following result.

Corollary 9.5.5. *Let U be a Banach space. Consider $F : U \to \mathbb{R}$ Gâteaux differentiable. Then F is convex if and only if*

$$F(v) - F(u) \geq \langle F'(u), v - u\rangle_U, \forall u, v \in U.$$

Definition 9.5.6 (The Second Variation). Let U be a Banach space. Suppose $F : U \to \mathbb{R}$ is a Gâteaux differentiable functional. Given $\varphi,\ \eta \in \mathscr{V}$, we define the second variation of F at u, relating the directions $\varphi,\ \eta$, denoted by

$$\delta^2 F(u, \varphi, \eta),$$

by

$$\delta^2 F(u, \varphi, \eta) = \lim_{\varepsilon \to 0} \frac{\delta F(u + \varepsilon\eta, \varphi) - \delta F(u, \varphi)}{\varepsilon}.$$

If such a limit exists $\forall \varphi,\ \eta \in \mathscr{V}$, we say that F is twice Gâteaux differentiable at u. Finally, if $\eta = \varphi$, we denote $\delta^2 F(u, \varphi, \eta) = \delta^2 F(u, \varphi)$.

Corollary 9.5.7. *Let U be a Banach space. Suppose $F : U \to \mathbb{R}$ is a twice Gâteaux differentiable functional and that*

$$\delta^2 F(u, \varphi) \geq 0, \forall u \in U, \varphi \in \mathscr{V}.$$

Then, F is convex.

Proof. Pick $u, v \in U$. Define $\phi(\varepsilon) = F(u + \varepsilon(v - u))$. By hypothesis, ϕ is twice differentiable, so that

$$\phi(1) = \phi(0) + \phi'(0) + \phi''(\tilde{\varepsilon})/2,$$

where $|\tilde{\varepsilon}| \leq 1$. Thus

$$F(v) = F(u) + \delta F(u, v - u) + \delta^2 F(u + \tilde{\varepsilon}(v - u), v - u)/2.$$

Therefore, by hypothesis,

$$F(v) \geq F(u) + \delta F(u, v - u).$$

Since F is Gâteaux differentiable, we obtain

$$F(v) \geq F(u) + \langle F'(u), v - u\rangle_U.$$

Being $u, v \in U$ arbitrary, the proof is complete.

Corollary 9.5.8. *Let U be a Banach space. Let $F : U \to \mathbb{R}$ be a convex Gâteaux differentiable functional. If $F'(u) = \theta$, then*

$$F(v) \geq F(u), \forall v \in U,$$

that is, $u \in U$ is a global minimizer for F.

Proof. Just observe that

$$F(v) \geq F(u) + \langle F'(u), v - u\rangle_U, \forall u, v \in U.$$

Therefore, from $F'(u) = \theta$, we obtain

$$F(v) \geq F(u), \forall v \in U.$$

Theorem 9.5.9 (Sufficient Condition for a Local Minimum). *Let U be a Banach space. Suppose $F : U \to \mathbb{R}$ is a twice Gâteaux differentiable functional at a neighborhood of u_0, so that*

$$\delta F(u_0) = \theta$$

and

$$\delta^2 F(u, \varphi) \geq 0, \forall u \in B_r(u_0),\ \varphi \in \mathscr{V},$$

for some $r > 0$. Under such hypotheses, we have

$$F(u_0) \leq F(u_0 + \varepsilon\varphi), \forall\ \varepsilon,\ \varphi \text{ such that } |\varepsilon| < \min\{r, 1\},\ \|\varphi\|_U < 1.$$

Proof. Fix $\varphi \in \mathscr{V}$ such that $\|\varphi\|_U < 1$. Define

$$\phi(\varepsilon) = F(u_0 + \varepsilon\varphi).$$

Observe that for $|\varepsilon| < \min\{r, 1\}$, for some $\tilde{\varepsilon}$ such that $|\tilde{\varepsilon}| \leq |\varepsilon|$, we have

$$\begin{aligned}
\phi(\varepsilon) &= \phi(0) + \phi'(0)\varepsilon + \phi''(\tilde{\varepsilon})\varepsilon^2/2 \\
&= F(u_0) + \varepsilon\langle\varphi, \delta F(u_0)\rangle_U + (\varepsilon^2/2)\delta^2 F(u_0 + \tilde{\varepsilon}\varphi, \varphi) \\
&= F(u_0) + (\varepsilon^2/2)\delta^2 F(u_0 + \tilde{\varepsilon}\varphi, \varphi) \geq F(u_0).
\end{aligned}$$

Hence,

$$F(u_0) \leq F(u_0 + \varepsilon\varphi), \forall\ \varepsilon,\ \varphi \text{ such that } |\varepsilon| < r,\ \|\varphi\|_U < 1.$$

The proof is complete.

9.6 The Legendre–Hadamard Condition

Theorem 9.6.1. *If $u \in C^1(\bar{\Omega}; \mathbb{R}^N)$ is such that*

$$\delta^2 F(u, \varphi) \geq 0, \forall \varphi \in C_c^\infty(\Omega, \mathbb{R}^N),$$

then

$$f_{\xi_\alpha^i \xi_\beta^k}(x, u(x), \nabla u(x))\rho^i \rho^k \eta_\alpha \eta_\beta \geq 0, \forall x \in \Omega, \rho \in \mathbb{R}^N, \eta \in \mathbb{R}^n.$$

Such a condition is known as the Legendre-Hadamard condition.

Proof. Suppose

$$\delta^2 F(u, \varphi) \geq 0, \forall \varphi \in C_c^\infty(\Omega; \mathbb{R}^N).$$

We denote $\delta^2 F(u, \varphi)$ by

$$\delta^2 F(u, \varphi) = \int_\Omega a(x) D\varphi(x) \cdot D\varphi(x)\, dx + \int_\Omega b(x)\varphi(x) \cdot D\varphi(x)\, dx + \int_\Omega c(x)\varphi(x) \cdot \varphi(x)\, dx, \tag{9.6}$$

where

$$a(x) = f_{\xi\xi}(x, u(x), Du(x)),$$
$$b(x) = 2f_{s\xi}(x, u(x), Du(x)),$$

and

$$c(x) = f_{ss}(x, u(x), Du(x)).$$

Now consider $v \in C_c^\infty(B_1(0), \mathbb{R}^N)$. Thus given $x_0 \in \Omega$ for λ sufficiently small we have that $\varphi(x) = \lambda v\left(\frac{x - x_0}{\lambda}\right)$ is an admissible direction. Now we introduce the new coordinates $y = (y^1, \ldots, y^n)$ by setting $y = \lambda^{-1}(x - x_0)$ and multiply (9.6) by λ^{-n} to obtain

$$\int_{B_1(0)} \{a(x_0 + \lambda y) Dv(y) \cdot Dv(y) + 2\lambda b(x_0 + \lambda y) v(y) \cdot Dv(y) + \lambda^2 c(x_0 + \lambda y) v(y) \cdot v(y)\}\, dy > 0,$$

where $a = \{a_{ij}^{\alpha\beta}\}, b = \{b_{jk}^\beta\}$ and $c = \{c_{jk}\}$. Since a, b and c are continuous, we have

$$a(x_0 + \lambda y) Dv(y) \cdot Dv(y) \to a(x_0) Dv(y) \cdot Dv(y),$$

$$\lambda b(x_0 + \lambda y) v(y) \cdot Dv(y) \to 0,$$

and

$$\lambda^2 c(x_0 + \lambda y) v(y) \cdot v(y) \to 0,$$

uniformly on $\bar{\Omega}$ as $\lambda \to 0$. Thus this limit gives us

$$\int_{B_1(0)} \tilde{f}_{jk}^{\alpha\beta} D_\alpha v^j D_\beta v^k\, dx \geq 0, \forall v \in C_c^\infty(B_1(0); \mathbb{R}^N), \tag{9.7}$$

where

$$\tilde{f}_{jk}^{\alpha\beta} = a_{jk}^{\alpha\beta}(x_0) = f_{\xi_\alpha^i \xi_\beta^k}(x_0, u(x_0), \nabla u(x_0)).$$

Now define $v = (v^1, \ldots, v^N)$, where

$$v^j = \rho^j cos((\eta \cdot y)t)\zeta(y)$$

$$\rho = (\rho^1, \ldots, \rho^N) \in \mathbb{R}^N$$

and

$$\eta = (\eta_1, \ldots, \eta_n) \in \mathbb{R}^n$$

and $\zeta \in C_c^\infty(B_1(0))$. From (9.7) we obtain

$$0 \leq \tilde{f}_{jk}^{\alpha\beta} \rho^j \rho^k \left\{ \int_{B_1(0)} (\eta_\alpha t(-sin((\eta \cdot y)t)\zeta + cos((\eta \cdot y)t)D_\alpha \zeta) \right.$$
$$\left. \cdot \left(\eta_\beta t(-sin((\eta \cdot y)t)\zeta + cos((\eta \cdot y)t)D_\beta \zeta\right) \, dy \right\} \tag{9.8}$$

By analogy for

$$v^j = \rho^j sin((\eta \cdot y)t)\zeta(y)$$

we obtain

$$0 \leq \tilde{f}_{jk}^{\alpha\beta} \rho^j \rho^k \left\{ \int_{B_1(0)} (\eta_\alpha t(cos((\eta \cdot y)t)\zeta + sin((\eta \cdot y)t)D_\alpha \zeta) \right.$$
$$\left. \cdot \left(\eta_\beta t(cos((\eta \cdot y)t)\zeta + sin((\eta \cdot y)t)D_\beta \zeta\right) \, dy \right\} \tag{9.9}$$

Summing up these last two equations, dividing the result by t^2, and letting $t \to +\infty$ we obtain

$$0 \leq \tilde{f}_{jk}^{\alpha\beta} \rho^j \rho^k \eta_\alpha \eta_\beta \int_{B_1(0)} \zeta^2 \, dy,$$

for all $\zeta \in C_c^\infty(B_1(0))$, which implies

$$0 \leq \tilde{f}_{jk}^{\alpha\beta} \rho^j \rho^k \eta_\alpha \eta_\beta.$$

The proof is complete.

9.7 The Weierstrass Condition for $n = 1$

Here we present the Weierstrass condition for the special case $N \geq 1$ and $n = 1$. We start with a definition.

Definition 9.7.1. We say that $u \in \hat{C}^1([a,b]; \mathbb{R}^N)$ if $u : [a,b] \to \mathbb{R}^N$ is continuous in $[a,b]$ and Du is continuous except on a finite set of points in $[a,b]$.

Theorem 9.7.2 (Weierstrass). *Let $\Omega = (a,b)$ and $f : \bar{\Omega} \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be such that $f_s(x,s,\xi)$ and $f_\xi(x,s,\xi)$ are continuous on $\bar{\Omega} \times \mathbb{R}^N \times \mathbb{R}^N$.*

Define $F : U \to \mathbb{R}$ by

$$F(u) = \int_a^b f(x, u(x), u'(x)) \, dx,$$

where

$$U = \{u \in \hat{C}^1([a,b];\mathbb{R}^N) \mid u(a) = \alpha,\ u(b) = \beta\}.$$

Suppose $u \in U$ minimizes locally F on U, that is, suppose that there exists $\varepsilon_0 > 0$ such that

$$F(u) \leq F(v), \forall v \in U, \text{ such that } \|u - v\|_\infty < \varepsilon_0.$$

Under such hypotheses, we have

$$E(x, u(x), u'(x+), w) \geq 0, \forall x \in [a,b],\ w \in \mathbb{R}^N,$$

and

$$E(x, u(x), u'(x-), w) \geq 0, \forall x \in [a,b],\ w \in \mathbb{R}^N,$$

where

$$u'(x+) = \lim_{h \to 0^+} u'(x+h),$$

$$u'(x-) = \lim_{h \to 0^-} u'(x+h),$$

and

$$E(x, s, \xi, w) = f(x, s, w) - f(x, s, \xi) - f_\xi(x, s, \xi)(w - \xi).$$

Remark 9.7.3. The function E is known as the Weierstrass excess function.

Proof. Fix $x_0 \in (a,b)$ and $w \in \mathbb{R}^N$. Choose $0 < \varepsilon < 1$ and $h > 0$ such that $u + v \in U$ and

$$\|v\|_\infty < \varepsilon_0$$

where $v(x)$ is given by

$$v(x) = \begin{cases} (x - x_0)w, & \text{if } 0 \leq x - x_0 \leq \varepsilon h, \\ \tilde{\varepsilon}(h - x + x_0)w, & \text{if } \varepsilon h \leq x - x_0 \leq h, \\ 0, & \text{otherwise,} \end{cases}$$

where

$$\tilde{\varepsilon} = \frac{\varepsilon}{1 - \varepsilon}.$$

From

$$F(u + v) - F(u) \geq 0$$

we obtain

$$\int_{x_0}^{x_0+h} f(x, u(x) + v(x), u'(x) + v'(x))\, dx - \int_{x_0}^{x_0+h} f(x, u(x), u'(x))\, dx \geq 0. \tag{9.10}$$

Define

$$\tilde{x} = \frac{x - x_0}{h},$$

so that

$$d\tilde{x} = \frac{dx}{h}.$$

From (9.10) we obtain

$$\begin{aligned} h\int_0^1 & f(x_0+\tilde{x}h, u(x_0+\tilde{x}h)+v(x_0+\tilde{x}h), u'(x_0+\tilde{x}h)+v'(x_0+\tilde{x}h)\, d\tilde{x} \\ & -h\int_0^1 f(x_0+\tilde{x}h, u(x_0+\tilde{x}h), u'(x_0+\tilde{x}h))\, d\tilde{x} \geq 0. \end{aligned} \tag{9.11}$$

where the derivatives are related to x.

Therefore

$$\begin{aligned} \int_0^{\varepsilon} & f(x_0+\tilde{x}h, u(x_0+\tilde{x}h)+v(x_0+\tilde{x}h), u'(x_0+\tilde{x}h)+w)\, d\tilde{x} \\ & -\int_0^{\varepsilon} f(x_0+\tilde{x}h, u(x_0+\tilde{x}h), u'(x_0+\tilde{x}h))\, d\tilde{x} \\ & +\int_{\varepsilon}^1 f(x_0+\tilde{x}h, u(x_0+\tilde{x}h)+v(x_0+\tilde{x}h), u'(x_0+\tilde{x}h)-\tilde{\varepsilon}w)\, d\tilde{x} \\ & -\int_{\varepsilon}^1 f(x_0+\tilde{x}h, u(x_0+\tilde{x}h), u'(x_0+\tilde{x}h))\, d\tilde{x} \\ & \geq 0. \end{aligned} \tag{9.12}$$

Letting $h \to 0$ we obtain

$$\begin{aligned} & \varepsilon(f(x_0, u(x_0), u'(x_0+)+w) - f(x_0, u(x_0), u'(x_0+)) \\ & \quad +(1-\varepsilon)(f(x_0, u(x_0), u'(x_0+)-\tilde{\varepsilon}w) - f(x_0, u(x_0), u'(x_0+))) \geq 0. \end{aligned}$$

Hence, by the mean value theorem, we get

$$\begin{aligned} & \varepsilon(f(x_0, u(x_0), u'(x_0+)+w) - f(x_0, u(x_0), u'(x_0+)) \\ & \quad -(1-\varepsilon)\tilde{\varepsilon}(f_{\xi}(x_0, u(x_0), u'(x_0+)+\rho(\tilde{\varepsilon})w)) \cdot w \geq 0. \end{aligned} \tag{9.13}$$

Dividing by ε and letting $\varepsilon \to 0$, so that $\tilde{\varepsilon} \to 0$ and $\rho(\tilde{\varepsilon}) \to 0$, we finally obtain

$$\begin{aligned} & f(x_0, u(x_0), u'(x_0+)+w) - f(x_0, u(x_0), u'(x_0+)) \\ & \qquad -f_{\xi}(x_0, u_(x_0), u'(x_0+)) \cdot w \geq 0. \end{aligned}$$

Similarly we may get

$$\begin{aligned} & f(x_0, u(x_0), u'(x_0-)+w) - f(x_0, u(x_0), u'(x_0-)) \\ & \qquad - f_{\xi}(x_0, u_(x_0), u'(x_0-)) \cdot w \geq 0. \end{aligned}$$

Since $x_0 \in [a,b]$ and $w \in \mathbb{R}^N$ are arbitrary, the proof is complete.

9.8 The Weierstrass Condition: The General Case

In this section we present a proof for the Weierstrass necessary condition for $N \geq 1, n \geq 1$. Such a result may be found in similar form in [37].

Theorem 9.1. *Assume $u \in C^1(\overline{\Omega};\mathbb{R}^N)$ is a point of strong minimum for a Fréchet differentiable functional $F : U \to \mathbb{R}$ that is, in particular, there exists $\varepsilon > 0$ such that*

$$F(u+\varphi) \geq F(u),$$

for all $\varphi \in C_c^\infty(\Omega;\mathbb{R}^n)$ such that

$$\|\varphi\|_\infty < \varepsilon.$$

Here

$$F(u) = \int_\Omega f(x,u,Du)\,dx,$$

where we recall to have denoted

$$Du = \nabla u = \left\{\frac{\partial u_i}{\partial x_j}\right\}.$$

Under such hypotheses, for all $x \in \Omega$ and each rank-one matrix $\eta = \{\rho_i \beta^\alpha\} = \{\rho \otimes \beta\}$, we have that

$$E(x,u(x),Du(x),Du(x)+\rho\otimes\beta) \geq 0,$$

where

$$\begin{aligned} &E(x,u(x),Du(x),Du(x)+\rho\otimes\beta) \\ &= f(x,u(x),Du(x)+\rho\otimes\beta) - f(x,u(x),Du(x)) \\ &\quad -\rho^i\beta_\alpha f_{\xi^i_\alpha}(x,u(x),Du(x)). \end{aligned} \tag{9.14}$$

Proof. Since u is a point of local minimum for F, we have that

$$\delta F(u;\varphi) = 0, \forall \varphi \in C_c^\infty(\Omega;\mathbb{R}^N),$$

that is,

$$\int_\Omega (\varphi\cdot f_s(x,u(x),Du(x)) + D\varphi\cdot f_\xi(x,u(x),Du(x))\,dx = 0,$$

and hence,

$$\begin{aligned} &\int_\Omega (f(x,u(x),Du(x)+D\varphi(x)) - f(x,u(x),Du(x))\,dx \\ &\quad -\int_\Omega (\varphi(x)\cdot f_s(x,u(x),Du(x)) - D\varphi(x)\cdot f_\xi(x,u(x),Du(x))\,dx \\ &\geq 0, \end{aligned} \tag{9.15}$$

$\forall \varphi \in \mathscr{V}$, where

$$\mathscr{V} = \{\varphi \in C_c^\infty(\Omega;\mathbb{R}^N) \ : \ \|\varphi\|_\infty < \varepsilon\}.$$

Choose a unit vector $e \in \mathbb{R}^n$ and write

$$x = (x \cdot e)e + \bar{x},$$

where

$$\bar{x} \cdot e = 0.$$

Denote $D_e v = Dv \cdot e$ and let $\rho = (\rho_1, \ldots, \rho_N) \in \mathbb{R}^N$.

Also, let x_0 be any point of Ω. Without loss of generality assume $x_0 = 0$.

Choose $\lambda_0 \in (0,1)$ such that $C_{\lambda_0} \subset \Omega$, where

$$C_{\lambda_0} = \{x \in \mathbb{R}^n \ : \ |x \cdot e| \leq \lambda_0 \text{ and } \|\bar{x}\| \leq \lambda_0\}.$$

Let $\lambda \in (0, \lambda_0)$ and

$$\phi \in C_c((-1,1);\mathbb{R})$$

and choose a sequence

$$\phi_k \in C_c^\infty((-\lambda^2, \lambda);\mathbb{R})$$

which converges uniformly to the Lipschitz function ϕ_λ given by

$$\phi_\lambda = \begin{cases} t + \lambda^2, & \text{if } -\lambda^2 \leq t \leq 0, \\ \lambda(\lambda - t), & \text{if } 0 < t < \lambda \\ 0, & \text{otherwise} \end{cases} \tag{9.16}$$

and such that ϕ_k' converges uniformly to ϕ_λ' on each compact subset of

$$A_\lambda = \{t : -\lambda^2 < t < \lambda, \ t \neq 0\}.$$

We emphasize the choice of $\{\phi_k\}$ may be such that for some $K > 0$ we have $\|\phi\|_\infty < K$, $\|\phi_k\|_\infty < K$ and $\|\phi_k'\|_\infty < K, \forall k \in \mathbb{N}$.

Observe that for any sufficiently small $\lambda > 0$ we have that φ_k defined by

$$\varphi_k(x) = \rho\,\phi_k(x \cdot e)\phi(|\bar{x}|^2/\lambda^2) \in \mathscr{V}, \forall k \in \mathbb{N}$$

so that letting $k \to \infty$ we obtain that

$$\varphi(x) = \rho\,\phi_\lambda(x \cdot e)\phi(|\bar{x}|^2/\lambda^2),$$

is such that (9.15) is satisfied.

Moreover,

$$D_e\varphi(x) = \rho\,\phi_\lambda'(x \cdot e)\phi(|\bar{x}|^2/\lambda^2),$$

and

$$\overline{D}\varphi(x) = \rho\,\phi_\lambda(x \cdot e)\phi'(|\bar{x}|^2/\lambda^2)2\lambda^{-2}\bar{x},$$

where $\overline{D}$ denotes the gradient relating the variable $\bar{x}$.

Note that for such a $\varphi(x)$, the integrand of (9.15) vanishes if $x \notin C_\lambda$, where

$$C_\lambda = \{x \in \mathbb{R}^n \,:\, |x \cdot e| \leq \lambda \text{ and } \|\bar{x}\| \leq \lambda\}.$$

Define C_λ^+ and C_λ^- by

$$C_\lambda^- = \{x \in C_\lambda \,:\, x \cdot e \leq 0\},$$

and

$$C_\lambda^+ = \{x \in C_\lambda \,:\, x \cdot e > 0\}.$$

Hence, denoting

$$\begin{aligned} g_k(x) = &\,(f(x,u(x),Du(x)+D\varphi_k(x)) - f(x,u(x),Du(x)) \\ &-(\varphi_k(x)\cdot f_s(x,u(x),Du(x)+D\varphi_k(x)\cdot f_\xi(x,u(x),Du(x)) \end{aligned} \tag{9.17}$$

and

$$\begin{aligned} g(x) = &\,(f(x,u(x),Du(x)+D\varphi(x)) - f(x,u(x),Du(x)) \\ &-(\varphi(x)\cdot f_s(x,u(x),Du(x)+D\varphi(x)\cdot f_\xi(x,u(x),Du(x)) \end{aligned} \tag{9.18}$$

letting $k \to \infty$, using the Lebesgue dominated converge theorem, we obtain

$$\begin{aligned} &\int_{C_\lambda^-} g_k(x)\,dx + \int_{C_\lambda^+} g_k(x)\,dx \\ \to &\int_{C_\lambda^-} g(x)\,dx + \int_{C_\lambda^+} g(x)\,dx \geq 0, \end{aligned} \tag{9.19}$$

Now define

$$y = y^e e + \bar{y},$$

where

$$y^e = \frac{x \cdot e}{\lambda^2},$$

and

$$\bar{y} = \frac{\bar{x}}{\lambda}.$$

The sets C_λ^- and C_λ^+ correspond, concerning the new variables, to the sets B_λ^- and B_λ^+, where

$$B_\lambda^- = \{y \,:\, \|\bar{y}\| \leq 1, \text{ and } -\lambda^{-1} \leq y^e \leq 0\},$$

$$B_\lambda^+ = \{y \,:\, \|\bar{y}\| \leq 1, \text{ and } 0 < y^e \leq \lambda^{-1}\}.$$

Therefore, since $dx = \lambda^{n+1} dy$, multiplying (9.19) by λ^{-n-1}, we obtain

$$\int_{B_1^-} g(x(y))\,dy + \int_{B_\lambda^- \setminus B_1^-} g(x(y))\,dy + \int_{B_\lambda^+} g(x(y))\,dy \geq 0, \tag{9.20}$$

where

$$x = (x \cdot e)e + \bar{x} = \lambda^2 y^e + \lambda \bar{y} \equiv x(y).$$

Observe that

$$D_e\varphi(x) = \begin{cases} \rho\phi(\|\bar{y}\|^2) & \text{if } -1 \leq y^e \leq 0, \\ \rho\phi(\|\bar{y}\|^2)(-\lambda) & \text{if } 0 \leq y^e \leq \lambda^{-1}, \\ 0, & \text{otherwise.} \end{cases} \tag{9.21}$$

Observe also that

$$|g(x(y))| \leq o(\sqrt{|\varphi(x)|^2 + |D\varphi(x)|^2}),$$

so that from the expression of $\varphi(x)$ and $D\varphi(x)$ we obtain, for

$$y \in B^+_\lambda, \text{ or } y \in B^-_\lambda \setminus B^-_1,$$

that

$$|g(x(y))| \leq o(\lambda), \text{ as } \lambda \to 0.$$

Since the Lebesgue measures of B^-_λ and B^+_λ are bounded by

$$2^{n-1}/\lambda$$

the second and third terms in (9.20) are of $o(1)$ where

$$\lim_{\lambda \to 0^+} o(1)/\lambda = 0,$$

so that letting $\lambda \to 0^+$, considering that

$$x(y) \to 0,$$

and on B^-_1 (up to the limit set B)

$$\begin{aligned} g(x(y)) \to\ & f(0,u(0),Du(0)+\rho\phi(\|\bar{y}\|^2)e) \\ & - f(0,u(0),Du(0)) - \\ & \rho\phi(\|\bar{y}\|^2)ef_\xi(0,u(0),Du(0)) \end{aligned} \tag{9.22}$$

we get

$$\begin{aligned} & \int_B [f(0,u(0),Du(0)+\rho\phi(\|\bar{y}\|^2)e) - f(0,u(0),Du(0)) \\ & -\rho\phi(\|\bar{y}\|^2)ef_\xi(0,u(0),Du(0))]\, d\bar{y}_2 \ldots d\bar{y}_n \\ \geq\ & 0, \end{aligned} \tag{9.23}$$

where B is an appropriate limit set (we do not provide more details here) such that

$$B = \{y \in \mathbb{R}^n \ : \ y^e = 0 \text{ and } \|\bar{y}\| \leq 1\}.$$

Here we have used the fact that on the set in question,

$$D\varphi(x) \to \rho\phi(\|\bar{y}\|^2)e, \text{ as } \lambda \to 0^+.$$

Finally, inequality (9.23) is valid for a sequence $\{\phi_n\}$ (in place of ϕ) such that

$$0 \leq \phi_n \leq 1 \text{ and } \phi_n(t) = 1, \text{ if } |t| < 1 - 1/n,$$

$\forall n \in \mathbb{N}$.

Letting $n \to \infty$, from (9.23), we obtain

$$\begin{aligned} &f(0,u(0),Du(0)+\rho \otimes e) - f(0,u(0),Du(0)) \\ &-\rho \cdot e f_\xi(0,u(0),Du(0)) \geq 0. \end{aligned} \tag{9.24}$$

9.9 The du Bois–Reymond Lemma

We present now a simpler version of the fundamental lemma of calculus of variations. The result is specific for $n = 1$ and is known as the du Bois–Reymond lemma.

Lemma 9.9.1 (du Bois–Reymond). *If $u \in C([a,b])$ and*

$$\int_a^b u\varphi' \, dx = 0, \forall \varphi \in \mathscr{V},$$

where

$$\mathscr{V} = \{\varphi \in C^1[a,b] \mid \varphi(a) = \varphi(b) = 0\},$$

then there exists $c \in \mathbb{R}$ such that

$$u(x) = c, \forall x \in [a,b].$$

Proof. Define

$$c = \frac{1}{b-a}\int_a^b u(t)\, dt,$$

and

$$\varphi(x) = \int_a^x (u(t) - c)\, dt.$$

Thus we have $\varphi(a) = 0$ and

$$\varphi(b) = \int_a^b u(t)\, dt - c(b-a) = 0.$$

Moreover $\varphi \in C^1([a,b])$ so that

$$\varphi \in \mathscr{V}.$$

Therefore

$$\begin{aligned} 0 &\leq \int_a^b (u(x)-c)^2\,dx \\ &= \int_a^b (u(x)-c)\varphi'(x)\,dx \\ &= \int_a^b u(x)\varphi'(x)\,dx - c[\varphi(x)]_a^b = 0. \end{aligned} \tag{9.25}$$

Thus

$$\int_a^b (u(x)-c)^2\,dx = 0,$$

and being $u(x)-c$ continuous, we finally obtain

$$u(x)-c = 0, \forall x \in [a,b].$$

This completes the proof.

Proposition 9.9.2. *If* $u,v \in C([a,b])$ *and*

$$\int_a^b (u(x)\varphi(x)+v(x)\varphi'(x))\,dx = 0,$$

$\forall \varphi \in \mathscr{V}$*, where*

$$\mathscr{V} = \{\varphi \in C^1[a,b] \mid \varphi(a) = \varphi(b) = 0\},$$

then

$$v \in C^1([a,b])$$

and

$$v'(x) = u(x), \forall x \in [a,b].$$

Proof. Define

$$u_1(x) = \int_a^x u(t)\,dt, \forall x \in [a,b].$$

Thus $u_1 \in C^1([a,b])$ and

$$u_1'(x) = u(x), \forall x \in [a,b].$$

Hence, for $\varphi \in \mathscr{V}$, we have

$$\begin{aligned} 0 &= \int_a^b (u(x)\varphi(x)+v(x)\varphi'(x)\,dx \\ &= \int_a^b (-u_1(x)\varphi'(x)+v\varphi'(x))\,dx + [u_1(x)\varphi(x)]_a^b \\ &= \int_a^b (v(x)-u_1(x))\varphi'(x)\,dx. \end{aligned} \tag{9.26}$$

That is,

$$\int_a^b (v(x) - u_1(x))\varphi'(x)\, dx, \forall \varphi \in \mathscr{V}.$$

By the du Bois–Reymond lemma, there exists $c \in \mathbb{R}$ such that

$$v(x) - u_1(x) = c, \forall x \in [a,b].$$

Hence

$$v = u_1 + c \in C^1([a,b]),$$

so that

$$v'(x) = u_1'(x) = u(x), \forall x \in [a,b].$$

The proof is complete.

9.10 The Weierstrass–Erdmann Conditions

We start with a definition.

Definition 9.10.1. Define $I = [a,b]$. A function $u \in \hat{C}^1([a,b];\mathbb{R}^N)$ is said to be a weak extremal of

$$F(u) = \int_a^b f(x, u(x), u'(x))\, dx,$$

if

$$\int_a^b (f_s(x,u(x),u'(x)) \cdot \varphi + f_\xi(x,u(x),u'(x)) \cdot \varphi'(x))\, dx = 0,$$

$\forall \varphi \in C_c^\infty([a,b];\mathbb{R}^N)$.

Proposition 9.10.2. *For any weak extremal of*

$$F(u) = \int_a^b f(x, u(x), u'(x))\, dx$$

there exists a constant $c \in \mathbb{R}^N$ such that

$$f_\xi(x,u(x),u'(x)) = c + \int_a^x f_s(t,u(t),u'(t))\, dt, \forall x \in [a,b]. \tag{9.27}$$

Proof. Fix $\varphi \in C_c^\infty([a,b];\mathbb{R}^N)$. Integration by parts of the extremal condition

$$\delta F(u,\varphi) = 0,$$

implies that

$$\int_a^b f_\xi(x,u(x),u'(x))\cdot\varphi'(x)\,dx$$
$$-\int_a^b\int_a^x f_s(t,u(t),u'(t))\,dt\cdot\varphi'(x)\,dx=0.$$

Since φ is arbitrary, considering the du Bois-Reymond lemma is valid also for $u\in L^1([a,b])$ and the respective N-dimensional version (see [37], page 32 for details), there exists, $c\in\mathbb{R}^N$ such that

$$f_\xi(x,u(x),u'(x))-\int_a^x f_s(t,u(t),u'(t))\,dt=c,\forall x\in[a,b].$$

The proof is complete.

Theorem 9.10.3 (Weierstrass–Erdmann Corner Conditions). *Let $I=[a,b]$. Suppose $u\in\hat{C}^1([a,b];\mathbb{R}^N)$ is such that*

$$F(u)\le F(v),\forall v\in\mathscr{C}_r,$$

for some $r>0$ where

$$\mathscr{C}_r=\{v\in\hat{C}^1([a,b];\mathbb{R}^N)\mid v(a)=u(a),\ v(b)=u(b),$$
$$\text{and } \|u-v\|_\infty<r\}.$$

Let $x_0\in(a,b)$ be a corner point of u. Denoting $u_0=u(x_0)$, $\xi_0^+=u'(x_0+0)$, and $\xi_0^-=u'(x_0-0)$, then the following relations are valid:

1. $f_\xi(x_0,u_0,\xi_0^-)=f_\xi(x_0,u_0,\xi_0^+)$,
2.

$$f(x_0,u_0,\xi_0^-)-\xi_0^-f_\xi(x_0,u_0,\xi_0^-)$$
$$=f(x_0,u_0,\xi_0^+)-\xi_0^+f_\xi(x_0,u_0,\xi_0^+).$$

Remark 9.10.4. The conditions above are known as the Weierstrass–Erdmann corner conditions.

Proof. Condition (1) is just a consequence of (9.27). For (2), define

$$\tau_\varepsilon(x)=x+\varepsilon\lambda(x),$$

where $\lambda\in C_c^\infty(I)$. Observe that $\tau_\varepsilon(a)=a$ and $\tau_\varepsilon(b)=b$, $\forall\varepsilon>0$. Also $\tau_0(x)=x$. Choose $\varepsilon_0>0$ sufficiently small such that for each ε satisfying $|\varepsilon|<\varepsilon_0$, we have $\tau_\varepsilon'(x)>0$ and

$$\tilde{u}_\varepsilon(x)=(u\circ\tau_\varepsilon^{-1})(x)\in\mathscr{C}_r.$$

Define

$$\phi(\varepsilon) = F(x, \tilde{u}_\varepsilon, \tilde{u}'_\varepsilon(x)).$$

Thus ϕ has a local minimum at 0, so that $\phi'(0) = 0$, that is,

$$\frac{d(F(x, \tilde{u}_\varepsilon, \tilde{u}'_\varepsilon(x)))}{d\varepsilon}|_{\varepsilon=0} = 0.$$

Observe that

$$\frac{d\tilde{u}_\varepsilon}{dx} = u'(\tau_\varepsilon^{-1}(x))\frac{d\tau_\varepsilon^{-1}(x)}{dx},$$

and

$$\frac{d\tau_\varepsilon^{-1}(x)}{dx} = \frac{1}{1+\varepsilon\lambda'(\tau_\varepsilon^{-1}(x))}.$$

Thus,

$$F(\tilde{u}_\varepsilon) = \int_a^b f\left(x, u(\tau_\varepsilon^{-1}(x)), u'(\tau_\varepsilon^{-1}(x))\left(\frac{1}{1+\varepsilon\lambda'(\tau_\varepsilon^{-1}(x))}\right)\right) dx.$$

Defining

$$\bar{x} = \tau_\varepsilon^{-1}(x),$$

we obtain

$$d\bar{x} = \frac{1}{1+\varepsilon\lambda'(\bar{x})}\, dx,$$

that is,

$$dx = (1+\varepsilon\lambda'(\bar{x}))\, d\bar{x}.$$

Dropping the bar for the new variable, we may write

$$F(\tilde{u}_\varepsilon) = \int_a^b f\left(x+\varepsilon\lambda(x), u(x), \frac{u'(x)}{1+\varepsilon\lambda'(x)}\right)\left(1+\varepsilon\lambda'(x)\right)\, dx.$$

From

$$\frac{dF(\tilde{u}_\varepsilon)}{d\varepsilon}|_{\varepsilon=0},$$

we obtain

$$\int_a^b (\lambda f_x(x, u(x), u'(x)) + \lambda'(x)(f(x, u(x), u'(x)) - u'(x) f_\xi(x, u(x), u'(x))))\, dx = 0. \tag{9.28}$$

Since λ is arbitrary, from Proposition 9.9.2, (in fact from its version for $u \in L^1([a,b])$ and respective extension for the N dimensional case, please see [37] for details), we obtain

$$f(x, u(x), u'(x)) - u'(x) f_\xi(x, u(x), u'(x)) - \int_a^x f_x(t, u(t), u'(t))\, dt = c_1$$

for some $c_1 \in \mathbb{R}^N$.

Since $\int_a^x f_x(t,u(t),u'(t))\,dt + c_1$ is a continuous function (in fact absolutely continuous), the proof is complete.

9.11 Natural Boundary Conditions

Consider the functional $f : U \to \mathbb{R}$, where

$$F(u)\int_\Omega f(x,u(x),\nabla u(x))\,dx,$$

$$f(x,s,\xi) \in C^1(\bar{\Omega},\mathbb{R}^N,\mathbb{R}^{N\times n}),$$

and $\Omega \subset \mathbb{R}^n$ is an open bounded connected set.

Proposition 9.11.1. *Assume*

$$U = \{u \in W^{1,2}(\Omega;\mathbb{R}^N); u = u_0 \text{ on } \Gamma_0\},$$

where $\Gamma_0 \subset \partial\Omega$ is closed and $\partial\Omega = \Gamma = \Gamma_0 \cup \Gamma_1$ being Γ_1 open in Γ and $\Gamma_0 \cap \Gamma_1 = \emptyset$. Thus if $\partial\Omega \in C^1$, $f \in C^2(\bar{\Omega},\mathbb{R}^N,\mathbb{R}^{N\times n})$ and $u \in C^2(\bar{\Omega};\mathbb{R}^N)$, and also

$$\delta F(u,\varphi) = 0, \forall \varphi \in C^1(\bar{\Omega};\mathbb{R}^N), \text{ such that } \varphi = 0 \text{ on } \Gamma_0,$$

then u is a extremal of F which satisfies the following natural boundary conditions:

$$n_\alpha f_{\xi^i_\alpha}(x,u(x)\nabla u(x)) = 0, \text{ a.e. on } \Gamma_1, \forall i \in \{1,\ldots,N\}.$$

Proof. Observe that $\delta F(u,\varphi) = 0, \forall \varphi \in C_c^\infty(\Omega;\mathbb{R}^N)$; thus u is an extremal of F and through integration by parts and the fundamental lemma of calculus of variations, we obtain

$$L_f(u) = 0, \text{ in } \Omega,$$

where

$$L_f(u) = f_s(x,u(x),\nabla u(x)) - div(f_\xi(x,u(x),\nabla u(x)).$$

Defining

$$\mathscr{V} = \{\varphi \in C^1(\Omega;\mathbb{R}^N) \mid \varphi = 0 \text{ on } \Gamma_0\},$$

for an arbitrary $\varphi \in \mathscr{V}$, we obtain

$$\begin{aligned}
\delta F(u,\varphi) &= \int_\Omega L_f(u)\cdot\varphi\,dx \\
&\quad + \int_{\Gamma_1} n_\alpha f_{\xi^i_\alpha}(x,u(x),\nabla u(x))\varphi^i(x)\,d\Gamma \\
&= \int_{\Gamma_1} n_\alpha f_{\xi^i_\alpha}(x,u(x),\nabla u(x))\varphi^i(x)\,d\Gamma \\
&= 0, \forall \varphi \in \mathscr{V}.
\end{aligned} \tag{9.29}$$

Suppose, to obtain contradiction, that

$$n_\alpha f_{\xi^i_\alpha}(x_0, u(x_0), \nabla u(x_0)) = \beta > 0,$$

for some $x_0 \in \Gamma_1$ and some $i \in \{1, \ldots, N\}$. Defining

$$G(x) = n_\alpha f_{\xi^i_\alpha}(x, u(x), \nabla u(x)),$$

by the continuity of G, there exists $r > 0$ such that

$$G(x) > \beta/2, \text{ in } B_r(x_0),$$

and in particular

$$G(x) > \beta/2, \text{ in } B_r(x_0) \cap \Gamma_1.$$

Choose $0 < r_1 < r$ such that $B_{r_1}(x_0) \cap \Gamma_0 = \emptyset$. This is possible since Γ_0 is closed and $x_0 \in \Gamma_1$.

Choose $\varphi^i \in C_c^\infty(B_{r_1}(x_0))$ such that $\varphi^i \geq 0$ in $B_{r_1}(x_0)$ and $\varphi^i > 0$ in $B_{r_1/2}(x_0)$. Therefore

$$\int_{\Gamma_1} G(x)\varphi^i(x)\,dx > \frac{\beta}{2}\int_{\Gamma_1} \varphi^i\,dx > 0,$$

and this contradicts (9.29). Thus

$$G(x) \leq 0, \forall x \in \Gamma_1,$$

and by analogy

$$G(x) \geq 0, \forall x \in \Gamma_1,$$

so that

$$G(x) = 0, \forall x \in \Gamma_1.$$

The proof is complete.