Chapter 3
Topics on Linear Operators

The main references for this chapter are Reed and Simon [52] and Bachman and Narici [6].

3.1 Topologies for Bounded Operators

First we recall that the set of all bounded linear operators, denoted by $\mathscr{L}(U,Y)$, is a Banach space with the norm

$$\|A\| = \sup\{\|Au\|_Y \mid \|u\|_U \leq 1\}.$$

The topology related to the metric induced by this norm is called the uniform operator topology.

Let us introduce now the strong operator topology, which is defined as the weakest topology for which the functions

$$E_u : \mathscr{L}(U,Y) \to Y$$

are continuous where

$$E_u(A) = Au, \forall A \in \mathscr{L}(U,Y).$$

For such a topology a base at origin is given by sets of the form

$$\{A \mid A \in \mathscr{L}(U,Y),\ \|Au_i\|_Y < \varepsilon, \forall i \in \{1,\ldots,n\}\},$$

where $u_1,\ldots,u_n \in U$ and $\varepsilon > 0$.

Observe that a sequence $\{A_n\} \subset \mathscr{L}(U,Y)$ converges to A concerning this last topology if

$$\|A_n u - Au\|_Y \to 0,\ \text{as } n \to \infty, \forall u \in U.$$

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In the next lines we describe the weak operator topology in $\mathscr{L}(U,Y)$. Such a topology is weakest one such that the functions

$$E_{u,v} : \mathscr{L}(U,Y) \to \mathbb{C}$$

are continuous, where

$$E_{u,v}(A) = \langle Au, v\rangle_Y, \forall A \in \mathscr{L}(U,Y), u \in U,\ v \in Y^*.$$

For such a topology, a base at origin is given by sets of the form

$$\{A \in \mathscr{L}(U,Y) \mid |\langle Au_i, v_j\rangle_Y| < \varepsilon, \forall i \in \{1,\ldots,n\},\ j \in \{1,\ldots,m\}\},$$

where $\varepsilon > 0$, $u_1,\ldots,u_n \in U$, $v_1,\ldots,v_m \in Y^*$.

A sequence $\{A_n\} \subset \mathscr{L}(U,Y)$ converges to $A \in \mathscr{L}(U,Y)$ if

$$|\langle A_n u, v\rangle_Y - \langle Au, v\rangle_Y| \to 0,$$

as $n \to \infty$, $\forall u \in U,\ v \in Y^*$.

3.2 Adjoint Operators

We start this section recalling the definition of adjoint operator.

Definition 3.2.1. Let U,Y be Banach spaces. Given a bounded linear operator $A : U \to Y$ and $v^* \in Y^*$, we have that $T(u) = \langle Au, v^*\rangle_Y$ is such that

$$|T(u)| \leq \|Au\|_Y \cdot \|v^*\| \leq \|A\|\|v^*\|_{Y^*}\|u\|_U.$$

Hence $T(u)$ is a continuous linear functional on U and considering our fundamental representation hypothesis, there exists $u^* \in U^*$ such that

$$T(u) = \langle u, u^*\rangle_U, \forall u \in U.$$

We define A^* by setting $u^* = A^*v^*$, so that

$$T(u) = \langle u, u^*\rangle_U = \langle u, A^*v^*\rangle_U$$

that is,

$$\langle u, A^*v^*\rangle_U = \langle Au, v^*\rangle_Y, \forall u \in U,\ v^* \in Y^*.$$

We call $A^* : Y^* \to U^*$ the adjoint operator relating $A : U \to Y$.

Theorem 3.2.2. *Let U,Y be Banach spaces and let $A : U \to Y$ be a bounded linear operator. Then*

$$\|A\| = \|A^*\|.$$

Proof. Observe that

$$\begin{aligned}
\|A\| &= \sup_{u\in U}\{\|Au\| \mid \|u\|_U = 1\} \\
&= \sup_{u\in U}\left\{\sup_{v^*\in Y^*}\{\langle Au, v^*\rangle_Y \mid \|v^*\|_{Y^*} = 1\}, \|u\|_U = 1\right\} \\
&= \sup_{(u,v^*)\in U\times Y^*}\{\langle Au, v^*\rangle_Y \mid \|v^*\|_{Y^*} = 1, \|u\|_U = 1\} \\
&= \sup_{(u,v^*)\in U\times Y^*}\{\langle u, A^*v^*\rangle_U \mid \|v^*\|_{Y^*} = 1, \|u\|_U = 1\} \\
&= \sup_{v^*\in Y^*}\left\{\sup_{u\in U}\{\langle u, A^*v^*\rangle_U \mid \|u\|_U = 1\}, \|v^*\|_{Y^*} = 1\right\} \\
&= \sup_{v^*\in Y^*}\{\|A^*v^*\|, \|v^*\|_{Y^*} = 1\} \\
&= \|A^*\|. \qquad (3.1)
\end{aligned}$$

In particular, if $U = Y = H$ where H is Hilbert space, we have

Theorem 3.2.3. *Given the bounded linear operators $A, B : H \to H$ we have*

1. $(AB)^* = B^*A^*$,
2. $(A^*)^* = A$,
3. *if A has a bounded inverse A^{-1}, then A^* has a bounded inverse and*

$$(A^*)^{-1} = (A^{-1})^*.$$

4. $\|AA^*\| = \|A\|^2$.

Proof.

1. Observe that

$$(ABu, v)_H = (Bu, A^*v)_H = (u, B^*A^*v)_H, \forall u, v \in H.$$

2. Observe that

$$(u, Av)_H = (A^*u, v)_H = (u, A^{**}v)_H, \forall u, v \in H.$$

3. We have that

$$I = AA^{-1} = A^{-1}A,$$

so that

$$I = I^* = (AA^{-1})^* = (A^{-1})^*A^* = (A^{-1}A)^* = A^*(A^{-1})^*.$$

4. Observe that

$$\|A^*A\| \leq \|A\|\|A^*\| = \|A\|^2,$$

and

$$\|A^*A\| \geq \sup_{u\in U}\{(u, A^*Au)_H \mid \|u\|_U = 1\}$$

$$= \sup_{u \in U}\{(Au, Au)_H \mid \|u\|_U = 1\}$$
$$= \sup_{u \in U}\{\|Au\|_H^2 \mid \|u\|_U = 1\} = \|A\|^2, \tag{3.2}$$

and hence

$$\|A^*A\| = \|A\|^2.$$

Definition 3.2.4. Given $A \in \mathscr{L}(H)$ we say that A is self-adjoint if

$$A = A^*.$$

Theorem 3.2.5. *Let U and Y be Banach spaces and let $A : U \to Y$ be a bounded linear operator. Then*

$$[R(A)]^\perp = N(A^*),$$

where

$$[R(A)]^\perp = \{v^* \in Y^* \mid \langle Au, v^* \rangle_Y = 0, \ \forall u \in U\}.$$

Proof. Let $v^* \in N(A^*)$. Choose $v \in R(A)$. Thus there exists u in U such that $Au = v$ so that

$$\langle v, v^* \rangle_Y = \langle Au, v^* \rangle_Y = \langle u, A^*v^* \rangle_U = 0.$$

Since $v \in R(A)$ is arbitrary we have obtained

$$N(A^*) \subset [R(A)]^\perp.$$

Suppose $v^* \in [R(A)]^\perp$. Choose $u \in U$. Thus,

$$\langle Au, v^* \rangle_Y = 0,$$

so that

$$\langle u, A^*v^* \rangle_U, \forall u \in U.$$

Therefore $A^*v^* = \theta$, that is, $v^* \in N(A^*)$. Since $v^* \in [R(A)]^\perp$ is arbitrary, we get

$$[R(A)]^\perp \subset N(A^*).$$

This completes the proof.

The next result is relevant for subsequent developments.

Lemma 3.1. *Let U, Y be Banach spaces and let $A : U \to Y$ be a bounded linear operator. Suppose also that $R(A) = \{A(u) \ : \ u \in U\}$ is closed. Under such hypotheses, there exists $K > 0$ such that for each $v \in R(A)$ there exists $u_0 \in U$ such that*

$$A(u_0) = v$$

and

$$\|u_0\|_U \leq K\|v\|_Y.$$

Proof. Define $L = N(A) = \{u \in U \,:\, A(u) = \theta\}$ (the null space of A). Consider the space U/L, where

$$U/L = \{\overline{u} \,:\, u \in U\},$$

where

$$\overline{u} = \{u + w \,:\, w \in L\}.$$

Define $\overline{A} : U/L \to R(A)$, by

$$\overline{A}(\overline{u}) = A(u).$$

Observe that $\overline{A}$ is one-to-one, linear, onto, and bounded. Moreover $R(A)$ is closed so that it is a Banach space. Hence by the inverse mapping theorem we have that $\overline{A}$ has a continuous inverse. Thus, for any $v \in R(A)$, there exists $\overline{u} \in U/L$ such that

$$\overline{A}(\overline{u}) = v$$

so that

$$\overline{u} = \overline{A}^{-1}(v),$$

and therefore

$$\|\overline{u}\| \leq \|\overline{A}^{-1}\| \|v\|_Y.$$

Recalling that

$$\|\overline{u}\| = \inf_{w \in L}\{\|u + w\|_U\},$$

we may find $u_0 \in \overline{u}$ such that

$$\|u_0\|_U \leq 2\|\overline{u}\| \leq 2\|\overline{A}^{-1}\| \|v\|_Y,$$

and so that

$$A(u_0) = \overline{A}(\overline{u_0}) = \overline{A}(\overline{u}) = v.$$

Taking $K = 2\|\overline{A}^{-1}\|$ we have completed the proof.

Theorem 3.1. *Let U, Y be Banach spaces and let $A : U \to Y$ be a bound linear operator. Assume $R(A)$ is closed. Under such hypotheses*

$$R(A^*) = [N(A)]^{\perp}.$$

Proof. Let $u^* \in R(A^*)$. Thus there exists $v^* \in Y^*$ such that

$$u^* = A^*(v^*).$$

Let $u \in N(A)$. Hence,

$$\langle u, u^* \rangle_U = \langle u, A^*(v^*) \rangle_U = \langle A(u), v^* \rangle_Y = 0.$$

Since $u \in N(A)$ is arbitrary, we get $u^* \in [N(A)]^{\perp}$, so that

$$R(A^*) \subset [N(A)]^{\perp}.$$

Now suppose $u^* \in [N(A)]^\perp$. Thus

$$\langle u, u^* \rangle_U = 0, \ \forall u \in N(A).$$

Fix $v \in R(A)$. From the Lemma 3.1, there exists $K > 0$ (which does not depend on v) and $u_v \in U$ such that

$$A(u_v) = v$$

and

$$\|u_v\|_U \leq K\|v\|_Y.$$

Define $f : R(A) \to \mathbb{R}$ by

$$f(v) = \langle u_v, u^* \rangle_U.$$

Observe that

$$|f(v)| \leq \|u_v\|_U \|u^*\|_{U^*} \leq K\|v\|_Y \|u^*\|_{U^*},$$

so that f is a bounded linear functional. Hence by a Hahn–Banach theorem corollary there exists $v^* \in Y^*$ such that

$$f(v) = \langle v, v^* \rangle_Y \equiv F(v), \ \forall v \in R(A),$$

that is, F is an extension of f from $R(A)$ to Y.

In particular

$$f(v) = \langle u_v, u^* \rangle_U = \langle v, v^* \rangle_Y = \langle A(u_v), v^* \rangle_Y \ \forall v \in R(A),$$

where $A(u_v) = v$, so that

$$\langle u_v, u^* \rangle_U = \langle A(u_v), v^* \rangle_Y \ \forall v \in R(A).$$

Now let $u \in U$ and define $A(u) = v_0$. Observe that

$$u = (u - u_{v_0}) + u_{v_0},$$

and

$$A(u - u_{v_0}) = A(u) - A(u_{v_0}) = v_0 - v_0 = \theta.$$

Since $u^* \in [N(A)]^\perp$ we get

$$\langle u - u_{v_0}, u^* \rangle_U = 0$$

so that

$$\begin{aligned}
\langle u, u^* \rangle_U &= \langle (u - u_{v_0}) + u_{v_0}, u^* \rangle_U \\
&= \langle u_{v_0}, u^* \rangle_U \\
&= \langle A(u_{v_0}), v^* \rangle_Y \\
&= \langle A(u - u_{v_0}) + A(u_{v_0}), v^* \rangle_Y \\
&= \langle A(u), v^* \rangle_Y. \qquad (3.3)
\end{aligned}$$

Hence,

$$\langle u, u^* \rangle_U = \langle A(u), v^* \rangle_Y, \ \forall u \in U.$$

We may conclude that $u^* = A^*(v^*) \in R(A^*)$. Since $u^* \in [N(A)]^\perp$ is arbitrary we obtain

$$[N(A)]^\perp \subset R(A^*).$$

The proof is complete.

We finish this section with the following result.

Definition 3.2.6. Let U be a Banach space and $S \subset U$. We define the positive conjugate cone of S, denoted by $S^\oplus$ by

$$S^\oplus = \{u^* \in U^* \ : \ \langle u, u^* \rangle_U \geq 0, \ \forall u \in S\}.$$

Similarly, we define the negative cone of S, denoted by $S^\ominus$ by

$$S^\ominus = \{u^* \in U^* \ : \ \langle u, u^* \rangle_U \leq 0, \ \forall u \in S\}.$$

Theorem 3.2.7. *Let U, Y be Banach spaces and $A : U \to Y$ be a bounded linear operator. Let $S \subset U$. Then*

$$[A(S)]^\oplus = (A^*)^{-1}(S^\oplus),$$

where

$$(A^*)^{-1} = \{v^* \in Y^* \ : \ A^* v^* \in S^\oplus\}.$$

Proof. Let $v^* \in [A(S)]^\oplus$ and $u \in S$. Thus,

$$\langle A(u), v^* \rangle_Y \geq 0,$$

so that

$$\langle u, A^*(v^*) \rangle_U \geq 0.$$

Since $u \in S$ is arbitrary, we get

$$v^* \in (A^*)^{-1}(S^\oplus).$$

From this

$$[A(S)]^\oplus \subset (A^*)^{-1}(S^\oplus).$$

Reciprocally, let $v^* \in (A^*)^{-1}(S^\oplus)$. Hence $A^*(v^*) \in S^\oplus$ so that for $u \in S$ we obtain

$$\langle u, A^*(v^*) \rangle_U \geq 0,$$

and therefore

$$\langle A(u), v^* \rangle_Y \geq 0.$$

Since $u \in S$ is arbitrary, we get $v^* \in [A(S)]^{\oplus}$, that is,

$$(A^*)^{-1}(S^{\oplus}) \subset [A(S)]^{\oplus}.$$

The proof is complete.

3.3 Compact Operators

We start this section defining compact operators.

Definition 3.3.1. Let U and Y be Banach spaces. An operator $A \in \mathscr{L}(U,Y)$ (linear and bounded) is said to compact if A takes bounded sets into pre-compact sets. Summarizing, A is compact if for each bounded sequence $\{u_n\} \subset U$, $\{Au_n\}$ has a convergent subsequence in Y.

Theorem 3.3.2. *A compact operator maps weakly convergent sequences into norm convergent sequences.*

Proof. Let $A : U \to Y$ be a compact operator. Suppose

$$u_n \rightharpoonup u \text{ weakly in U.}$$

By the uniform boundedness theorem, $\{\|u_n\|\}$ is bounded. Thus, given $v^* \in Y^*$ we have

$$\begin{aligned} \langle v^*, Au_n \rangle_Y &= \langle A^*v^*, u_n \rangle_U \\ &\to \langle A^*v^*, u \rangle_U \\ &= \langle v^*, Au \rangle_Y. \end{aligned} \tag{3.4}$$

Being $v^* \in Y^*$ arbitrary, we get that

$$Au_n \rightharpoonup Au \text{ weakly in } Y. \tag{3.5}$$

Suppose Au_n does not converge in norm to Au. Thus there exists $\varepsilon > 0$ and a subsequence $\{Au_{n_k}\}$ such that

$$\|Au_{n_k} - Au\|_Y \geq \varepsilon, \forall k \in \mathbb{N}.$$

As $\{u_{n_k}\}$ is bounded and A is compact, $\{Au_{n_k}\}$ has a subsequence converging para $\tilde{v} \neq Au$. But then such a sequence converges weakly to $\tilde{v} \neq Au$, which contradicts (3.5). The proof is complete.

Theorem 3.3.3. *Let H be a separable Hilbert space. Thus each compact operator in $\mathscr{L}(H)$ is the limit in norm of a sequence of finite rank operators.*

Proof. Let A be a compact operator in H. Let $\{\phi_j\}$ an orthonormal basis in H. For each $n \in \mathbb{N}$ define

$$\lambda_n = \sup\{\|A\psi\|_H \mid \psi \in [\phi_1, \ldots, \phi_n]^{\perp} \text{ and } \|\psi\|_H = 1\}.$$

It is clear that $\{\lambda_n\}$ is a nonincreasing sequence that converges to a limit $\lambda \geq 0$. We will show that $\lambda = 0$. Choose a sequence $\{\psi_n\}$ such that

$$\psi_n \in [\phi_1, \ldots, \phi_n]^{\perp},$$

$\|\psi_n\|_H = 1$, and $\|A\psi_n\|_H \geq \lambda/2$. Now we will show that

$$\psi_n \rightharpoonup \theta, \text{ weakly in } H.$$

Let $\psi^* \in H^* = H$,; thus there exists a sequence $\{a_j\} \subset \mathbb{C}$ such that

$$\psi^* = \sum_{j=1}^{\infty} a_j \phi_j.$$

Suppose given $\varepsilon > 0$. We may find $n_0 \in \mathbb{N}$ such that

$$\sum_{j=n_0}^{\infty} |a_j|^2 < \varepsilon.$$

Choose $n > n_0$. Hence there exists $\{b_j\}_{j>n}$ such that

$$\psi_n = \sum_{j=n+1}^{\infty} b_j \phi_j,$$

and

$$\sum_{j=n+1}^{\infty} |b_j|^2 = 1.$$

Therefore

$$\begin{aligned} |(\psi_n, \psi^*)_H| &= \left| \sum_{j=n+1}^{\infty} (\phi_j, \phi_j)_H a_j \cdot b_j \right| \\ &= \left| \sum_{j=n+1}^{\infty} a_j \cdot b_j \right| \\ &\leq \sqrt{\sum_{j=n+1}^{\infty} |a_j|^2} \sqrt{\sum_{j=n+1}^{\infty} |b_j|^2} \\ &\leq \sqrt{\varepsilon}, \end{aligned} \tag{3.6}$$

if $n > n_0$. Since $\varepsilon > 0$ is arbitrary,

$$(\psi_n, \psi^*)_H \to 0, \text{ as } n \to \infty.$$

Since $\psi^* \in H$ is arbitrary, we get

$$\psi_n \rightharpoonup \theta, \text{ weakly in } H.$$

Hence, as A is compact, we have

$$A\psi_n \to \theta \text{ in norm },$$

so that $\lambda = 0$. Finally, we may define $\{A_n\}$ by

$$A_n(u) = A\left(\sum_{j=1}^{n}(u,\phi_j)_H\phi_j\right) = \sum_{j=1}^{n}(u,\phi_j)_H A\phi_j,$$

for each $u \in H$. Thus

$$\|A - A_n\| = \lambda_n \to 0, \text{ as } n \to \infty.$$

The proof is complete.

3.4 The Square Root of a Positive Operator

Definition 3.4.1. Let H be a Hilbert space. A mapping $E : H \to H$ is said to be a projection on $M \subset H$ if for each $z \in H$ we have

$$Ez = x,$$

where $z = x + y$, $x \in M$, and $y \in M^{\perp}$.

Observe that

1. E is linear,
2. E is idempotent, that is, $E^2 = E$,
3. $R(E) = M$,
4. $N(E) = M^{\perp}$.

Also observe that from

$$Ez = x$$

we have

$$\|Ez\|_H^2 = \|x\|_H^2 \leq \|x\|_H^2 + \|y\|_H^2 = \|z\|_H^2,$$

so that

$$\|E\| \leq 1.$$

Definition 3.4.2. Let $A, B \in \mathscr{L}(H)$. We write

$$A \geq \theta$$

if

$$(Au,u)_H \geq 0, \forall u \in H,$$

and in this case we say that A is positive. Finally, we denote

$$A \geq B$$

if

$$A - B \geq \theta.$$

Theorem 3.4.3. *Let A and B be bounded self-adjoint operators such that $A \geq \theta$ and $B \geq \theta$. If $AB = BA$, then*

$$AB \geq \theta.$$

Proof. If $A = \theta$, the result is obvious. Assume $A \neq \theta$ and define the sequence

$$A_1 = \frac{A}{\|A\|}, \;\; A_{n+1} = A_n - A_n^2, \forall n \in \mathbb{N}.$$

We claim that

$$\theta \leq A_n \leq I, \forall n \in \mathbb{N}.$$

We prove the claim by induction.

For $n = 1$, it is clear that $A_1 \geq \theta$. And since $\|A_1\| = 1$, we get

$$(A_1 u, u)_H \leq \|A_1\| \|u\|_H \|u\|_H = (Iu, u)_H, \forall u \in H,$$

so that

$$A_1 \leq I.$$

Thus

$$\theta \leq A_1 \leq I.$$

Now suppose $\theta \leq A_n \leq I$. Since A_n is self-adjoint, we have

$$\begin{aligned} (A_n^2(I - A_n)u, u)_H &= ((I - A_n)A_n u, A_n u)_H \\ &= ((I - A_n)v, v)_H \geq 0, \forall u \in H, \end{aligned} \tag{3.7}$$

where $v = A_n u$. Therefore

$$A_n^2(I - A_n) \geq \theta.$$

Similarly, we may obtain

$$A_n(I - A_n)^2 \geq \theta,$$

so that

$$\theta \leq A_n^2(I - A_n) + A_n(I - A_n)^2 = A_n - A_n^2 = A_{n+1}.$$

So, also we have

$$\theta \leq I - A_n + A_n^2 = I - A_{n+1},$$

that is,

$$\theta \leq A_{n+1} \leq I,$$

so that

$$\theta \leq A_n \leq I, \ \forall n \in \mathbb{N}.$$

Observe that

$$\begin{aligned} A_1 &= A_1^2 + A_2 \\ &= A_1^2 + A_2^2 + A_3 \\ &\dots \dots\dots\dots\dots\dots\dots \\ &= A_1^2 + \ldots + A_n^2 + A_{n+1}. \end{aligned} \tag{3.8}$$

Since $A_{n+1} \geq \theta$, we obtain

$$A_1^2 + A_2^2 + \ldots + A_n^2 = A_1 - A_{n+1} \leq A_1. \tag{3.9}$$

From this, for a fixed $u \in H$, we have

$$\begin{aligned} \sum_{j=1}^{n} \|A_j u\|^2 &= \sum_{j=1}^{n} (A_j u, A_j u)_H \\ &= \sum_{j=1}^{n} (A_j^2 u, u)_H \\ &\leq (A_1 u, u)_H. \end{aligned} \tag{3.10}$$

Since $n \in \mathbb{N}$ is arbitrary, we get

$$\sum_{j=1}^{\infty} \|A_j u\|^2$$

is a converging series, so that

$$\|A_n u\| \to 0,$$

that is,

$$A_n u \to \theta, \text{ as } n \to \infty.$$

From this and (3.9), we get

$$\sum_{j=1}^{n} A_j^2 u = (A_1 - A_{n+1})u \to A_1 u, \text{ as } n \to \infty.$$

Finally, we may write

$$\begin{aligned} (ABu, u)_H &= \|A\|(A_1 Bu, u)_H \\ &= \|A\|(BA_1 u, u)_H \\ &= \|A\|(B \lim_{n..} \sum_j = 1^n A_j^2 u, u)_H \\ &= \|A\| \lim_{n\ldots} \sum_j = 1^n (BA_j^2 u, u)_H \\ &= \|A\| \lim_{n\ldots} \sum_j = 1^n (BA_j u, BA_j u)_H \\ &\geq 0. \end{aligned} \tag{3.11}$$

Hence
$$(ABu,u)_H \geq 0, \forall u \in H.$$
The proof is complete.

Theorem 3.4.4. *Let $\{A_n\}$ be a sequence of self-adjoint commuting operators in $\mathscr{L}(H)$. Let $B \in \mathscr{L}(H)$ be a self-adjoint operator such that*

$$A_iB = BA_i, \forall i \in \mathbb{N}.$$

Suppose also that
$$A_1 \leq A_2 \leq A_3 \leq \ldots \leq A_n \leq \ldots \leq B.$$
Under such hypotheses there exists a self-adjoint, bounded, linear operator A such that
$$A_n \to A \text{ in norm },$$
and
$$A \leq B.$$

Proof. Consider the sequence $\{C_n\}$ where

$$C_n = B - A_n \geq 0, \forall n \in \mathbb{N}.$$

Fix $u \in H$. First, we show that $\{C_n u\}$ converges. Observe that

$$C_iC_j = C_jC_i, \forall i, j \in \mathbb{N}.$$

Also, if $n > m$, then
$$A_n - A_m \geq \theta$$
so that
$$C_m = B - A_m \geq B - A_n = C_n.$$
Therefore, from $C_m \geq \theta$ and $C_m - C_n \geq \theta$, we obtain

$$(C_m - C_n)C_m \geq \theta, \text{ if } n > m$$

and also
$$C_n(C_m - C_n) \geq \theta.$$
Thus,
$$(C_m^2 u,u)_H \geq (C_nC_m u,u)_H \geq (C_n^2 u,u)_H,$$
and we may conclude that
$$(C_n^2 u,u)_H$$
is a monotone nonincreasing sequence of real numbers, bounded below by 0, so that there exists $\alpha \in \mathbb{R}$ such that

$$\lim_{n\to\infty} (C_n^2 u,u)_H = \alpha.$$

Since each C_n is self-adjoint we obtain

$$\begin{aligned} \|(C_n - C_m)u\|_H^2 &= ((C_n - C_m)u, (C_n - C_m)u)_H \\ &= ((C_n - C_m)(C_n - C_m)u, u)_H \\ &= (C_n^2 u, u)_H - 2(C_n C_m u, u) + (C_m^2 u, u)_H \\ &\to \alpha - 2\alpha + \alpha = 0, \end{aligned} \tag{3.12}$$

as

$$m, n \to \infty.$$

Therefore $\{C_n u\}$ is a Cauchy sequence in norm, so that there exists the limit

$$\lim_{n\to\infty} C_n u = \lim_{n\to\infty} (B - A_n)u,$$

and hence there exists

$$\lim_{n\to\infty} A_n u, \forall u \in H.$$

Now define A by

$$Au = \lim_{n\to\infty} A_n u.$$

Since the limit

$$\lim_{n\to\infty} A_n u, \forall u \in H$$

exists we have that

$$\sup_{n\in\mathbb{N}} \{\|A_n u\|_H\}$$

is finite for all $u \in H$. By the principle of uniform boundedness

$$\sup_{n\in\mathbb{N}} \{\|A_n\|\} < \infty$$

so that there exists $K > 0$ such that

$$\|A_n\| \le K, \forall n \in \mathbb{N}.$$

Therefore

$$\|A_n u\|_H \le K\|u\|_H,$$

so that

$$\|Au\| = \lim_{n\to\infty} \{\|A_n u\|_H\} \le K\|u\|_H, \forall u \in H$$

which means that A is bounded. Fixing $u, v \in H$, we have

$$(Au, v)_H = \lim_{n\to\infty} (A_n u, v)_H = \lim_{n\to\infty} (u, A_n v)_H = (u, Av)_H,$$

and thus A is self-adjoint. Finally

$$(A_n u, u)_H \le (Bu, u)_H, \forall n \in \mathbb{N},$$

so that

$$(Au,u) = \lim_{n\to\infty}(A_n u,u)_H \leq (Bu,u)_H, \forall u \in H.$$

Hence $A \leq B$.

The proof is complete.

Definition 3.4.5. Let $A \in \mathscr{L}(A)$ be a positive operator. The self-adjoint operator $B \in \mathscr{L}(H)$ such that

$$B^2 = A$$

is called the square root of A. If $B \geq \theta$, we denote

$$B = \sqrt{A}.$$

Theorem 3.4.6. *Suppose $A \in \mathscr{L}(H)$ is positive. Then there exists $B \geq \theta$ such that*

$$B^2 = A.$$

Furthermore B commutes with any $C \in \mathscr{L}(H)$ such that commutes with A.

Proof. There is no loss of generality in considering

$$\|A\| \leq 1,$$

which means $\theta \leq A \leq I$, because we may replace A by

$$\frac{A}{\|A\|}$$

so that if

$$C^2 = \frac{A}{\|A\|}$$

then

$$B = \|A\|^{1/2}C.$$

Let

$$B_0 = \theta,$$

and consider the sequence of operators given by

$$B_{n+1} = B_n + \frac{1}{2}(A - B_n^2), \forall n \in \mathbb{N} \cup \{0\}.$$

Since each B_n is polynomial in A, we have that B_n is self-adjoint and commute with any operator with commutes with A. In particular

$$B_i B_j = B_j B_i, \forall i, j \in \mathbb{N}.$$

First we show that

$$B_n \leq I, \forall n \in \mathbb{N} \cup \{0\}.$$

Since $B_0 = \theta$, and $B_1 = \frac{1}{2}A$, the statement holds for $n = 1$. Suppose $B_n \leq I$. Thus

$$\begin{aligned} I - B_{n+1} &= I - B_n - \frac{1}{2}A + \frac{1}{2}B_n^2 \\ &= \frac{1}{2}(I - B_n)^2 + \frac{1}{2}(I - A) \geq \theta \end{aligned} \tag{3.13}$$

so that

$$B_{n+1} \leq I.$$

The induction is complete, that is,

$$B_n \leq I, \forall n \in \mathbb{N}.$$

Now we prove the monotonicity also by induction. Observe that

$$B_0 \leq B_1,$$

and supposing

$$B_{n-1} \leq B_n,$$

we have

$$\begin{aligned} B_{n+1} - B_n &= B_n + \frac{1}{2}(A - B_n^2) - B_{n-1} - \frac{1}{2}(A - B_{n-1}^2) \\ &= B_n - B_{n-1} - \frac{1}{2}(B_n^2 - B_{n-1}^2) \\ &= B_n - B_{n-1} - \frac{1}{2}(B_n + B_{n-1})(B_n - B_{n-1}) \\ &= (I - \frac{1}{2}(B_n + B_{n-1}))(B_n - B_{n-1}) \\ &= \frac{1}{2}((I - B_{n-1}) + (I - B_n))(B_n - B_{n-1}) \geq \theta. \end{aligned}$$

The induction is complete, that is,

$$\theta = B_0 \leq B_1 \leq B_2 \leq \ldots \leq B_n \leq \ldots \leq I.$$

By the last theorem there exists a self-adjoint operator B such that

$$B_n \to B \text{ in norm.}$$

Fixing $u \in H$ we have

$$B_{n+1}u = B_n u + \frac{1}{2}(A - B_n^2)u,$$

so that taking the limit in norm as $n \to \infty$, we get

$$\theta = (A - B^2)u.$$

Being $u \in H$ arbitrary we obtain

$$A = B^2.$$

It is also clear that

$$B \geq \theta$$

The proof is complete.

3.5 About the Spectrum of a Linear Operator

Definition 3.5.1. Let U be a Banach space and let $A \in \mathscr{L}(U)$. A complex number λ is said to be in the resolvent set $\rho(A)$ of A, if

$$\lambda I - A$$

is a bijection with a bounded inverse. We call

$$R_\lambda(A) = (\lambda I - A)^{-1}$$

the resolvent of A in λ.

If $\lambda \notin \rho(A)$, we write

$$\lambda \in \sigma(A) = \mathbb{C} - \rho(A),$$

where $\sigma(A)$ is said to be the spectrum of A.

Definition 3.5.2. Let $A \in \mathscr{L}(U)$.

1. If $u \neq \theta$ and $Au = \lambda u$ for some $\lambda \in \mathbb{C}$, then u is said to be an eigenvector of A and λ the corresponding eigenvalue. If λ is an eigenvalue, then $(\lambda I - A)$ is not injective and therefore $\lambda \in \sigma(A)$.
   The set of eigenvalues is said to be the point spectrum of A.
2. If λ is not an eigenvalue but
   $$R(\lambda I - A)$$
   is not dense in U and therefore $\lambda I - A$ is not a bijection, we have that $\lambda \in \sigma(A)$. In this case we say that λ is in the residual spectrum of A, or briefly $\lambda \in Res[\sigma(A)]$.

Theorem 3.5.3. *Let U be a Banach space and suppose that $A \in \mathscr{L}(U)$. Then $\rho(A)$ is an open subset of $\mathbb{C}$ and*

$$F(\lambda) = R_\lambda(A)$$

is an analytic function with values in $\mathscr{L}(U)$ on each connected component of $\rho(A)$. For $\lambda, \mu \in \sigma(A)$, $R_\lambda(A)$, and $R_\mu(A)$ commute and

$$R_\lambda(A) - R_\mu(A) = (\mu - \lambda)R_\mu(A)R_\lambda(A).$$

Proof. Let $\lambda_0 \in \rho(A)$. We will show that λ_0 is an interior point of $\rho(A)$.

Observe that symbolically we may write

$$\begin{aligned}\frac{1}{\lambda - A} &= \frac{1}{\lambda - \lambda_0 + (\lambda_0 - A)} \\ &= \frac{1}{\lambda_0 - A}\left[\frac{1}{1 - \left(\frac{\lambda_0 - \lambda}{\lambda_0 - A}\right)}\right] \\ &= \frac{1}{\lambda_0 - A}\left(1 + \sum_{n=1}^{\infty}\left(\frac{\lambda_0 - \lambda}{\lambda_0 - A}\right)^n\right). \end{aligned} \tag{3.14}$$

Define

$$\hat{R}_\lambda(A) = R_{\lambda_0}(A)\left\{I + \sum_{n=1}^{\infty}(\lambda - \lambda_0)^n (R_{\lambda_0})^n\right\}. \tag{3.15}$$

Observe that

$$\|(R_{\lambda_0})^n\| \leq \|R_{\lambda_0}\|^n.$$

Thus, the series indicated in (3.15) will converge in norm if

$$|\lambda - \lambda_0| < \|R_{\lambda_0}\|^{-1}. \tag{3.16}$$

Hence, for λ satisfying (3.16), $\hat{R}(A)$ is well defined and we can easily check that

$$(\lambda I - A)\hat{R}_\lambda(A) = I = \hat{R}_\lambda(A)(\lambda I - A).$$

Therefore

$$\hat{R}_\lambda(A) = R_\lambda(A), \text{ if } |\lambda - \lambda_0| < \|R_{\lambda_0}\|^{-1},$$

so that λ_0 is an interior point. Since $\lambda_0 \in \rho(A)$ is arbitrary, we have that $\rho(A)$ is open. Finally, observe that

$$\begin{aligned} R_\lambda(A) - R_\mu(A) &= R_\lambda(A)(\mu I - A)R_\mu(A) - R_\lambda(A)(\lambda I - A)R_\mu(A) \\ &= R_\lambda(A)(\mu I)R_\mu(A) - R_\lambda(A)(\lambda I)R_\mu(A) \\ &= (\mu - \lambda)R_\lambda(A)R_\mu(A). \end{aligned} \tag{3.17}$$

Interchanging the roles of λ and μ we may conclude that R_λ and R_μ commute.

Corollary 3.5.4. *Let U be a Banach space and $A \in \mathscr{L}(U)$. Then the spectrum of A is nonempty.*

Proof. Observe that if

$$\frac{\|A\|}{|\lambda|} < 1$$

we have

$$\begin{aligned}(\lambda I - A)^{-1} &= [\lambda(I - A/\lambda)]^{-1} \\ &= \lambda^{-1}(I - A/\lambda)^{-1}\end{aligned}$$

$$= \lambda^{-1}\left(I + \sum_{n=1}^{\infty}\left(\frac{A}{\lambda}\right)^n\right). \tag{3.18}$$

Therefore we may obtain

$$R_\lambda(A) = \lambda^{-1}\left(I + \sum_{n=1}^{\infty}\left(\frac{A}{\lambda}\right)^n\right).$$

In particular

$$\|R_\lambda(A)\| \to 0, \text{ as } |\lambda| \to \infty. \tag{3.19}$$

Suppose, to obtain contradiction, that

$$\sigma(A) = \emptyset.$$

In such a case $R_\lambda(A)$ would be an entire bounded analytic function. From Liouville's theorem, $R_\lambda(A)$ would be constant, so that from (3.19) we would have

$$R_\lambda(A) = \theta, \forall \lambda \in \mathbb{C},$$

which is a contradiction.

Proposition 3.5.5. *Let H be a Hilbert space and $A \in \mathscr{L}(H)$.*

1. *If $\lambda \in Res[\sigma(A)]$, then $\overline{\lambda} \in P\sigma(A^*)$.*
2. *If $\lambda \in P\sigma(A)$, then $\overline{\lambda} \in P\sigma(A^*) \cup Res[\sigma(A^*)]$.*

Proof.

1. If $\lambda \in Res[\sigma(A)]$, then
$$R(A - \lambda I) \neq H.$$
Therefore there exists $v \in (R(A - \lambda I))^\perp$, $v \neq \theta$ such that
$$(v, (A - \lambda I)u)_H = 0, \forall u \in H$$
that is,
$$((A^* - \overline{\lambda} I)v, u)_H = 0, \forall u \in H$$
so that
$$(A^* - \overline{\lambda} I)v = \theta,$$
which means that $\overline{\lambda} \in P\sigma(A^*)$.
2. Suppose there exists $v \neq \theta$ such that
$$(A - \lambda I)v = \theta,$$
and
$$\overline{\lambda} \notin P\sigma(A^*).$$

Thus

$$(u,(A-\lambda I)v))_H = 0, \forall u \in H,$$

so that

$$((A^* - \overline{\lambda} I)u, v)_H, \forall u \in H.$$

Since

$$(A^* - \overline{\lambda} I)u \neq \theta, \forall u \in H, u \neq \theta,$$

we get $v \in (R(A^* - \overline{\lambda} I))^{\perp}$, so that $R(A^* - \overline{\lambda} I) \neq H$.
Hence $\overline{\lambda} \in Res[\sigma(A^*)]$.

Theorem 3.5.6. *Let $A \in \mathscr{L}(H)$ be a self-adjoint operator, then*

1. $\sigma(A) \subset \mathbb{R}$.
2. *Eigenvectors corresponding to distinct eigenvalues of A are orthogonal.*

Proof. Let $\mu, \lambda \in \mathbb{R}$. Thus, given $u \in H$ we have

$$\|(A-(\lambda+\mu i))u\|^2 = \|(A-\lambda)u\|^2 + \mu^2\|u\|^2,$$

so that

$$\|(A-(\lambda+\mu i))u\|^2 \geq \mu^2\|u\|^2.$$

Therefore if $\mu \neq 0$, $A-(\lambda+\mu i)$ has a bounded inverse on its range, which is closed. If $R(A-(\lambda+\mu i)) \neq H$, then by the last result $(\lambda-\mu i)$ would be in the point spectrum of A, which contradicts the last inequality. Hence, if $\mu \neq 0$, then $\lambda+\mu i \in \rho(A)$.
To complete the proof, suppose

$$Au_1 = \lambda_1 u_1,$$

and

$$Au_2 = \lambda_2 u_2,$$

where

$$\lambda_1, \lambda_2 \in \mathbb{R},\ \lambda_1 \neq \lambda_2, \text{ and } u_1, u_2 \neq \theta.$$

Thus

$$\begin{aligned}(\lambda_1-\lambda_2)(u_1,u_2)_H &= \lambda_1(u_1,u_2)_H - \lambda_2(u_1,u_2)_H \\ &= (\lambda_1 u_1, u_2)_H - (u_1, \lambda_2 u_2)_H \\ &= (Au_1,u_2)_H - (u_1,Au_2)_H \\ &= (u_1,Au_2)_H - (u_1,Au_2)_H \\ &= 0. \end{aligned} \tag{3.20}$$

Since $\lambda_1 - \lambda_2 \neq 0$ we get

$$(u_1,u_2)_H = 0.$$

3.6 The Spectral Theorem for Bounded Self-Adjoint Operators

Let H be a complex Hilbert space. Consider $A : H \to H$ a linear bounded operator, that is, $A \in \mathscr{L}(H)$, and suppose also that such an operator is self-adjoint. Define

$$m = \inf_{u \in H}\{(Au,u)_H \mid \|u\|_H = 1\},$$

and

$$M = \sup_{u \in H}\{(Au,u)_H \mid \|u\|_H = 1\}.$$

Remark 3.6.1. It is possible to prove that for a linear self-adjoint operator $A : H \to H$ we have

$$\|A\| = \sup\{|(Au,u)_H| \mid u \in H,\ \|u\|_H = 1\}.$$

This propriety, which prove in the next lines, is crucial for the subsequent results, since, for example, for A, B linear and self-adjoint and $\varepsilon > 0$, we have

$$-\varepsilon I \leq A - B \leq \varepsilon I,$$

we also would have

$$\|A - B\| < \varepsilon.$$

So, we present the following basic result.

Theorem 3.6.2. *Let $A : H \to H$ be a bounded linear self-adjoint operator. Define*

$$\alpha = \max\{|m|, |M|\},$$

where

$$m = \inf_{u \in H}\{(Au,u)_H \mid \|u\|_H = 1\},$$

and

$$M = \sup_{u \in H}\{(Au,u)_H \mid \|u\|_H = 1\}.$$

Then

$$\|A\| = \alpha.$$

Proof. Observe that

$$(A(u+v), u+v)_H = (Au,u)_H + (Av,v)_H + 2(Au,v)_H,$$

and

$$(A(u-v), u-v)_H = (Au,u)_H + (Av,v)_H - 2(Au,v)_H.$$

Thus,

$$4(Au,v) = (A(u+v), u+v)_H - (A(u-v), u-v)_H \leq M\|u+v\|_U^2 - m\|u-v\|_U^2,$$

so that
$$4(Au,v)_H \leq \alpha(\|u+v\|_U^2 + \|u-v\|_U^2).$$
Hence, replacing v by $-v$, we obtain
$$-4(Au,v)_H \leq \alpha(\|u+v\|_U^2 + \|u-v\|_U^2),$$
and therefore
$$4|(Au,v)_H| \leq \alpha(\|u+v\|_U^2 + \|u-v\|_U^2).$$
Replacing v by βv, we get
$$4|(A(u),v)_H| \leq 2\alpha(\|u\|_U^2/\beta + \beta\|v\|_U^2).$$
Minimizing the last expression in $\beta > 0$, for the optimal
$$\beta = \|u\|_U/\|v\|_U,$$
we obtain
$$|(Au,v)_H| \leq \alpha\|u\|_U\|v\|_U, \forall u,v \in U.$$
Thus
$$\|A\| \leq \alpha.$$
On the other hand,
$$|(Au,u)_H| \leq \|A\|\|u\|_U^2,$$
so that
$$|M| \leq \|A\|$$
and
$$|m| \leq \|A\|,$$
so that
$$\alpha \leq \|A\|.$$
The proof is complete.

At this point we start to develop the spectral theory. Define by P the set of all real polynomials defined in $\mathbb{R}$. Define
$$\Phi_1 : P \to \mathscr{L}(H),$$
by
$$\Phi_1(p(\lambda)) = p(A), \forall p \in P.$$
Thus we have

1. $\Phi_1(p_1 + p_2) = p_1(A) + p_2(A),$
2. $\Phi_1(p_1 \cdot p_2) = p_1(A)p_2(A),$
3. $\Phi_1(\alpha p) = \alpha p(A), \forall \alpha \in \mathbb{R},\ p \in P,$
4. if $p(\lambda) \geq 0,$ on $[m,M],$ then $p(A) \geq \theta.$

We will prove (4):

Consider $p \in P$. Denote the real roots of $p(\lambda)$ less or equal to m by $\alpha_1, \alpha_2, \ldots, \alpha_n$ and denote those that are greater or equal to M by $\beta_1, \beta_2, \ldots, \beta_l$. Finally denote all the remaining roots, real or complex, by

$$\nu_1 + i\mu_1, \ldots, \nu_k + i\mu_k.$$

Observe that if $\mu_i = 0$, then $\nu_i \in (m, M)$. The assumption that $p(\lambda) \geq 0$ on $[m, M]$ implies that any real root in (m, M) must be of even multiplicity.

Since complex roots must occur in conjugate pairs, we have the following representation for $p(\lambda)$:

$$p(\lambda) = a \prod_{i=1}^{n} (\lambda - \alpha_i) \prod_{i=1}^{l} (\beta_i - \lambda) \prod_{i=1}^{k} ((\lambda - \nu_i)^2 + \mu_i^2),$$

where $a \geq 0$. Observe that

$$A - \alpha_i I \geq \theta,$$

since

$$(Au, u)_H \geq m(u, u)_H \geq \alpha_i (u, u)_H, \forall u \in H,$$

and by analogy

$$\beta_i I - A \geq \theta.$$

On the other hand, since $A - \nu_k I$ is self-adjoint, its square is positive, and hence since the sum of positive operators is positive, we obtain

$$(A - \nu_k I)^2 + \mu_k^2 I \geq \theta.$$

Therefore,

$$p(A) \geq \theta.$$

The idea is now to extend the domain of Φ_1 to the set of upper semicontinuous functions, and such set we will denote by C^{up}.

Observe that if $f \in C^{up}$, there exists a sequence of continuous functions $\{g_n\}$ such that

$$g_n \downarrow f, \text{ pointwise },$$

that is,

$$g_n(\lambda) \downarrow f(\lambda), \forall \lambda \in \mathbb{R}.$$

Considering the Weierstrass Theorem, since $g_n \in C([m, M])$, we may obtain a sequence of polynomials $\{p_n\}$ such that

$$\left\| \left(g_n + \frac{1}{2^n} \right) - p_n \right\|_\infty < \frac{1}{2^n},$$

where the norm $\|\cdot\|_\infty$ refers to $[m,M]$. Thus

$$p_n(\lambda) \downarrow f(\lambda), \text{ on } [m,M].$$

Therefore

$$p_1(A) \geq p_2(A) \geq p_3(A) \geq \ldots \geq p_n(A) \geq \ldots$$

Since $p_n(A)$ is self-adjoint for all $n \in \mathbb{N}$, we have

$$p_j(A)p_k(A) = p_k(A)p_j(A), \forall j,k \in \mathbb{N}.$$

Then the $\lim_{n\to\infty} p_n(A)$ (in norm) exists, and we denote

$$\lim_{n\to\infty} p_n(A) = f(A).$$

Now recall the Dini's theorem.

Theorem 3.6.3 (Dini). *Let $\{g_n\}$ be a sequence of continuous functions defined on a compact set $K \subset \mathbb{R}$. Suppose $g_n \to g$ point-wise and monotonically on K. Under such assumptions the convergence in question is also uniform.*

Now suppose that $\{p_n\}$ and $\{q_n\}$ are sequences of polynomial such that

$$p_n \downarrow f, \text{ and } q_n \downarrow f,$$

we will show that

$$\lim_{n\to\infty} p_n(A) = \lim_{n\to\infty} q_n(A).$$

First observe that being $\{p_n\}$ and $\{q_n\}$ sequences of continuous functions we have that

$$\hat{h}_{nk}(\lambda) = \max\{p_n(\lambda), q_k(\lambda)\}, \forall \lambda \in [m,M]$$

is also continuous, $\forall n,k \in \mathbb{N}$. Now fix $n \in \mathbb{N}$ and define

$$h_k(\lambda) = \max\{p_k(\lambda), q_n(\lambda)\}.$$

Observe that

$$h_k(\lambda) \downarrow q_n(\lambda), \forall \lambda \in \mathbb{R},$$

so that by Dini's theorem

$$h_k \to q_n, \text{ uniformly on } [m,M].$$

It follows that for each $n \in \mathbb{N}$ there exists $k_n \in \mathbb{N}$ such that if $k > k_n$ then

$$h_k(\lambda) - q_n(\lambda) \leq \frac{1}{n}, \forall \lambda \in [m,M].$$

Since

$$p_k(\lambda) \leq h_k(\lambda), \forall \lambda \in [m,M],$$

we obtain

$$p_k(\lambda) - q_n(\lambda) \leq \frac{1}{n}, \forall \lambda \in [m, M].$$

By analogy, we may show that for each $n \in \mathbb{N}$ there exists $\hat{k}_n \in \mathbb{N}$ such that if $k > \hat{k}_n$, then

$$q_k(\lambda) - p_n(\lambda) \leq \frac{1}{n}.$$

From above we obtain

$$\lim_{k\to\infty} p_k(A) \leq q_n(A) + \frac{1}{n}.$$

Since the self-adjoint $q_n(A) + 1/n$ commutes with the

$$\lim_{k\to\infty} p_k(A)$$

we obtain

$$\begin{aligned}\lim_{k\to\infty} p_k(A) &\leq \lim_{n\to\infty}\left(q_n(A) + \frac{1}{n}\right) \\ &\leq \lim_{n\to\infty} q_n(A). \end{aligned} \tag{3.21}$$

Similarly we may obtain

$$\lim_{k\to\infty} q_k(A) \leq \lim_{n\to\infty} p_n(A),$$

so that

$$\lim_{n\to\infty} q_n(A) = \lim_{n\to\infty} p_n(A) = f(A).$$

Hence, we may extend $\Phi_1 : P \to \mathscr{L}(H)$ to $\Phi_2 : C^{up} \to \mathscr{L}(H)$, where C^{up}, as earlier indicated, denotes the set of upper semicontinuous functions, where

$$\Phi_2(f) = f(A).$$

Observe that Φ_2 has the following properties:

1. $\Phi_2(f_1 + f_2) = \Phi_2(f_1) + \Phi_2(f_2)$,
2. $\Phi_2(f_1 \cdot f_2) = f_1(A) f_2(A)$,
3. $\Phi_2(\alpha f) = \alpha \Phi_2(f), \forall \alpha \in \mathbb{R},\ \alpha \geq 0$,
4. if $f_1(\lambda) \geq f_2(\lambda), \forall \lambda \in [m, M]$, then

$$f_1(A) \geq f_2(A).$$

The next step is to extend Φ_2 to $\Phi_3 : C^{up}_- \to \mathscr{L}(H)$, where

$$C^{up}_- = \{f - g \mid f, g \in C^{up}\}.$$

For $h = f - g \in C^{up}_-$ we define

$$\Phi_3(h) = f(A) - g(A).$$

Now we will show that Φ_3 is well defined. Suppose that $h \in C_-^{up}$ and

$$h = f_1 - g_1 \text{ and } h = f_2 - g_2.$$

Thus

$$f_1 - g_1 = f_2 - g_2,$$

that is

$$f_1 + g_2 = f_2 + g_1,$$

so that from the definition of Φ_2 we obtain

$$f_1(A) + g_2(A) = f_2(A) + g_1(A),$$

that is,

$$f_1(A) - g_1(A) = f_2(A) - g_2(A).$$

Therefore Φ_3 is well defined. Finally observe that for $\alpha < 0$

$$\alpha(f - g) = -\alpha g - (-\alpha)f,$$

where $-\alpha g \in C^{up}$ and $-\alpha f \in C^{up}$. Thus

$$\Phi_3(\alpha f) = \alpha f(A) = \alpha \Phi_3(f), \forall \alpha \in \mathbb{R}.$$

3.6.1 The Spectral Theorem

Consider the upper semicontinuous function

$$h_\mu(\lambda) = \begin{cases} 1, & \text{if } \lambda \leq \mu, \\ 0, & \text{if } \lambda > \mu. \end{cases} \tag{3.22}$$

Denote

$$E(\mu) = \Phi_3(h_\mu) = h_\mu(A).$$

Observe that

$$h_\mu(\lambda)h_\mu(\lambda) = h_\mu(\lambda), \forall \lambda \in \mathbb{R},$$

so that

$$[E(\mu)]^2 = E(\mu), \forall \mu \in \mathbb{R}.$$

Therefore

$$\{E(\mu) \mid \mu \in \mathbb{R}\}$$

is a family of orthogonal projections. Also observe that if $\nu \geq \mu$, we have

$$h_\nu(\lambda)h_\mu(\lambda) = h_\mu(\lambda)h_\nu(\lambda) = h_\mu(\lambda),$$

so that

$$E(\nu)E(\mu) = E(\mu)E(\nu) = E(\mu), \forall \nu \geq \mu.$$

If $\mu < m$, then $h_\mu(\lambda) = 0$, on $[m,M]$, so that

$$E(\mu) = 0, \text{ if } \mu < m.$$

Similarly, if $\mu \geq M$, then $h_\mu(\lambda) = 1$, on $[m,M]$, so that

$$E(\mu) = I, \text{ if } \mu \geq M.$$

Next we show that the family $\{E(\mu)\}$ is strongly continuous from the right. First we will establish a sequence of polynomials $\{p_n\}$ such that

$$p_n \downarrow h_\mu$$

and

$$p_n(\lambda) \geq h_{\mu+\frac{1}{n}}(\lambda), \text{ on } [m,M].$$

Observe that for any fixed n there exists a sequence of polynomials $\{p_j^n\}$ such that

$$p_j^n \downarrow h_{\mu+1/n}, \text{ point-wise.}$$

Consider the monotone sequence

$$g_n(\lambda) = \min\{p_s^r(\lambda) \mid r,s \in \{1,\ldots,n\}\}.$$

Thus

$$g_n(\lambda) \geq h_{\mu+\frac{1}{n}}(\lambda), \forall \lambda \in \mathbb{R},$$

and we obtain

$$\lim_{n\to\infty} g_n(\lambda) \geq \lim_{n\to\infty} h_{\mu+\frac{1}{n}}(\lambda) = h_\mu(\lambda).$$

On the other hand

$$g_n(\lambda) \leq p_n^r(\lambda), \forall \lambda \in \mathbb{R}, \forall r \in \{1,\ldots,n\},$$

so that

$$\lim_{n\to\infty} g_n(\lambda) \leq \lim_{n\to\infty} p_n^r(\lambda).$$

Therefore

$$\begin{aligned} \lim_{n\to\infty} g_n(\lambda) &\leq \lim_{r\to\infty}\lim_{n\to\infty} p_n^r(\lambda) \\ &= h_\mu(\lambda). \end{aligned} \tag{3.23}$$

Thus

$$\lim_{n\to\infty} g_n(\lambda) = h_\mu(\lambda).$$

Observe that g_n are not necessarily polynomials. To set a sequence of polynomials, observe that we may obtain a sequence $\{p_n\}$ of polynomials such that

$$|g_n(\lambda) + 1/n - p_n(\lambda)| < \frac{1}{2^n}, \forall \lambda \in [m,M],\ n \in \mathbb{N},$$

so that

$$p_n(\lambda) \geq g_n(\lambda) + 1/n - 1/2^n \geq g_n(\lambda) \geq h_{\mu+1/n}(\lambda).$$

Thus

$$p_n(A) \to E(\mu),$$

and

$$p_n(A) \geq h_{\mu+\frac{1}{n}}(A) = E(\mu + 1/n) \geq E(\mu).$$

Therefore we may write

$$E(\mu) = \lim_{n\to\infty} p_n(A) \geq \lim_{n\to\infty} E(\mu + 1/n) \geq E(\mu).$$

Thus

$$\lim_{n\to\infty} E(\mu + 1/n) = E(\mu).$$

From this we may easily obtain the strong continuity from the right.

For $\mu \leq \nu$ we have

$$\begin{aligned} \mu(h_\nu(\lambda) - h_\mu(\lambda)) &\leq \lambda(h_\nu(\lambda) - h_\mu(\lambda)) \\ &\leq \nu(h_\nu(\lambda) - h_\mu(\lambda)). \end{aligned} \tag{3.24}$$

To verify this observe that if $\lambda < \mu$ or $\lambda > \nu$, then all terms involved in the above inequalities are zero. On the other hand if

$$\mu \leq \lambda \leq \nu$$

then

$$h_\nu(\lambda) - h_\mu(\lambda) = 1,$$

so that in any case (3.24) holds. From the monotonicity property we have

$$\begin{aligned} \mu(E(\nu) - E(\mu)) &\leq A(E(\nu) - E(\mu)) \\ &\leq \nu(E(\nu) - E(\mu)). \end{aligned} \tag{3.25}$$

Now choose $a, b \in \mathbb{R}$ such that

$$a < m \text{ and } b \geq M.$$

Suppose given $\varepsilon > 0$. Choose a partition P_0 of $[a,b]$, that is,

$$P_0 = \{a = \lambda_0, \lambda_1, \ldots, \lambda_n = b\},$$

such that

$$\max_{k \in \{1,\ldots,n\}} \{|\lambda_k - \lambda_{k-1}|\} < \varepsilon.$$

Hence

$$\begin{aligned}\lambda_{k-1}(E(\lambda_k) - E(\lambda_{k-1})) &\leq A(E(\lambda_k) - E(\lambda_{k-1})) \\ &\leq \lambda_k(E(\lambda_k) - E(\lambda_{k-1})). \end{aligned} \tag{3.26}$$

Summing up on k and recalling that

$$\sum_{k=1}^{n} E(\lambda_k) - E(\lambda_{k-1}) = I,$$

we obtain

$$\begin{aligned}\sum_{k=1}^{n} \lambda_{k-1}(E(\lambda_k) - E(\lambda_{k-1})) &\leq A \\ &\leq \sum_{k=1}^{n} \lambda_k(E(\lambda_k) - E(\lambda_{k-1})). \end{aligned} \tag{3.27}$$

Let $\lambda_k^0 \in [\lambda_{k-1}, \lambda_k]$. Since $(\lambda_k - \lambda_k^0) \leq (\lambda_k - \lambda_{k-1})$ from (3.26) we obtain

$$\begin{aligned} A - \sum_{k=1}^{n} \lambda_k^0(E(\lambda_k) - E(\lambda_{k-1})) &\leq \varepsilon \sum_{k=1}^{n} (E(\lambda_k) - E(\lambda_{k-1})) \\ &= \varepsilon I. \end{aligned} \tag{3.28}$$

By analogy

$$-\varepsilon I \leq A - \sum_{k=1}^{n} \lambda_k^0(E(\lambda_k) - E(\lambda_{k-1})). \tag{3.29}$$

Since

$$A - \sum_{k=1}^{n} \lambda_k^0(E(\lambda_k) - E(\lambda_{k-1}))$$

is self-adjoint we obtain

$$\|A - \sum_{k=1}^{n} \lambda_k^0(E(\lambda_k) - E(\lambda_{k-1}))\| < \varepsilon.$$

Being $\varepsilon > 0$ arbitrary, we may write

$$A = \int_a^b \lambda \, dE(\lambda),$$

that is,

$$A = \int_{m^-}^{M} \lambda dE(\lambda).$$

3.7 The Spectral Decomposition of Unitary Transformations

Definition 3.7.1. Let H be a Hilbert space. A transformation $U : H \to H$ is said to be unitary if

$$(Uu, Uv)_H = (u, v)_H, \forall u, v \in H.$$

Observe that in this case

$$U^*U = UU^* = I,$$

so that

$$U^{-1} = U^*.$$

Theorem 3.7.2. *Every unitary transformation U has a spectral decomposition*

$$U = \int_{0^-}^{2\pi} e^{i\phi} dE(\phi),$$

where $\{E(\phi)\}$ *is a spectral family on* $[0, 2\pi]$. *Furthermore* $E(\phi)$ *is continuous at* 0 *and it is the limit of polynomials in* U *and* U^{-1}.

We present just a sketch of the proof. For the trigonometric polynomials

$$p(e^{i\phi}) = \sum_{k=-n}^{n} c_k e^{ik\phi},$$

consider the transformation

$$p(U) = \sum_{k=-n}^{n} c_k U^k,$$

where $c_k \in \mathbb{C}, \forall k \in \{-n, \ldots, 0, \ldots, n\}$.

Observe that

$$\overline{p(e^{i\phi})} = \sum_{k=-n}^{n} \bar{c}_k e^{-ik\phi},$$

so that the corresponding operator is

$$p(U)^* = \sum_{k=-n}^{n} \bar{c}_k U^{-k} = \sum_{k=-n}^{n} \bar{c}_k (U^*)^k.$$

Also if

$$p(e^{i\phi}) \geq 0$$

there exists a polynomial q such that

$$p(e^{i\phi}) = |q(e^{i\phi})|^2 = \overline{q(e^{i\phi})}q(e^{i\phi}),$$

so that

$$p(U) = [q(U)]^* q(U).$$

Therefore

$$(p(U)v,v)_H = (q(U)^* q(U)v,v)_H = (q(U)v,q(U)v)_H \geq 0, \forall v \in H,$$

which means

$$p(U) \geq 0.$$

Define the function $h_\mu(\phi)$ by

$$h_\mu(\phi) = \begin{cases} 1, \text{ if } 2k\pi < \phi \leq 2k\pi + \mu, \\ 0, \text{ if } 2k\pi + \mu < \phi \leq 2(k+1)\pi, \end{cases} \tag{3.30}$$

for each $k \in \{0, \pm 1, \pm 2, \pm 3, \dots\}$. Define $E(\mu) = h_\mu(U)$. Observe that the family $\{E(\mu)\}$ are projections and in particular

$$E(0) = 0,$$

$$E(2\pi) = I$$

and if $\mu \leq \nu$, since

$$h_\mu(\phi) \leq h_\nu(\phi),$$

we have

$$E(\mu) \leq E(\nu).$$

Suppose given $\varepsilon > 0$. Let P_0 be a partition of $[0, 2\pi]$, that is,

$$P_0 = \{0 = \phi_0, \phi_1, \dots, \phi_n = 2\pi\}$$

such that

$$\max_{j \in \{1,\dots,n\}} \{|\phi_j - \phi_{j-1}|\} < \varepsilon.$$

For fixed $\phi \in [0, 2\pi]$, let $j \in \{1, \dots, n\}$ be such that

$$\phi \in [\phi_{j-1}, \phi_j].$$

$$\begin{aligned} |e^{i\phi} - \sum_{k=1}^{n} e^{i\phi_k}(h_{\phi_k}(\phi) - h_{\phi_{k-1}}(\phi))| &= |e^{i\phi} - e^{i\phi_j}| \\ &\leq |\phi - \phi_j| < \varepsilon. \end{aligned} \tag{3.31}$$

Thus,

$$0 \leq |e^{i\phi} - \sum_{k=1}^{n} e^{i\phi_k}(h_{\phi_k}(\phi) - h_{\phi_{k-1}}(\phi))|^2 \leq \varepsilon^2$$

so that, for the corresponding operators

$$0 \leq [U - \sum_{k=1}^{n} e^{i\phi_k}(E(\phi_k) - E(\phi_{k-1}))]^*[U - \sum_{k=1}^{n} e^{i\phi_k}(E(\phi_k) - E(\phi_{k-1}))] \leq \varepsilon^2 I$$

and hence

$$\left\| U - \sum_{k=1}^{n} e^{i\phi_k}(E(\phi_k) - E(\phi_{k-1}) \right\| < \varepsilon.$$

Being $\varepsilon > 0$ arbitrary, we may infer that

$$U = \int_0^{2\pi} e^{i\phi}\, dE(\phi).$$

3.8 Unbounded Operators

3.8.1 Introduction

Let H be a Hilbert space. Let $A : D(A) \to H$ be an operator, where unless indicated $D(A)$ is a dense subset of H. We consider in this section the special case where A is unbounded.

Definition 3.8.1. Given $A : D \to H$ we define the graph of A, denoted by $\Gamma(A)$, by

$$\Gamma(A) = \{(u, Au) \mid u \in D\}.$$

Definition 3.8.2. An operator $A : D \to H$ is said to be closed if $\Gamma(A)$ is closed.

Definition 3.8.3. Let $A_1 : D_1 \to H$ and $A_2 : D_2 \to H$ operators. We write $A_2 \supset A_1$ if $D_2 \supset D_1$ and

$$A_2 u = A_1 u, \forall u \in D_1.$$

In this case we say that A_2 is an extension of A_1.

Definition 3.8.4. A linear operator $A : D \to H$ is said to be closable if it has a linear closed extension. The smallest closed extension of A is denoted by $\overline{A}$ and is called the closure of A.

Proposition 3.8.5. *Let $A : D \to H$ be a linear operator. If A is closable, then*

$$\Gamma(\overline{A}) = \overline{\Gamma(A)}.$$

Proof. Suppose B is a closed extension of A. Then

$$\overline{\Gamma(A)} \subset \overline{\Gamma(B)} = \Gamma(B),$$

so that if $(\theta, \phi) \in \overline{\Gamma(A)}$, then $(\theta, \phi) \in \Gamma(B)$, and hence $\phi = \theta$. Define the operator C by

$$D(C) = \{\psi \mid (\psi, \phi) \in \overline{\Gamma(A)} \text{ for some } \phi\},$$

and $C(\psi) = \phi$, where ϕ is the unique point such that $(\psi, \phi) \in \overline{\Gamma(A)}$. Hence

$$\Gamma(C) = \overline{\Gamma(A)} \subset \Gamma(B),$$

so that

$$A \subset C.$$

However $C \subset B$ and since B is an arbitrary closed extension of A we have

$$C = \overline{A}$$

so that

$$\Gamma(C) = \Gamma(\overline{A}) = \overline{\Gamma(A)}.$$

Definition 3.8.6. Let $A : D \to H$ be a linear operator where D is dense in H. Define $D(A^*)$ by

$$D(A^*) = \{\phi \in H \mid (A\psi, \phi)_H = (\psi, \eta)_H, \ \forall \psi \in D \text{ for some } \eta \in H\}.$$

In this case we denote

$$A^*\phi = \eta.$$

A^* defined in this way is called the adjoint operator related to A.

Observe that by the Riesz lemma, $\phi \in D(A^*)$ if and only if there exists $K > 0$ such that

$$|(A\psi, \phi)_H| \leq K\|\psi\|_H, \forall \psi \in D.$$

Also note that if

$$A \subset B \text{ then } B^* \subset A^*.$$

Finally, as D is dense in H, then

$$\eta = A^*(\phi)$$

is uniquely defined. However the domain of A^* may not be dense, and in some situations we may have $D(A^*) = \{\theta\}$.

If $D(A^*)$ is dense, we define

$$A^{**} = (A^*)^*.$$

Theorem 3.8.7. *Let $A : D \to H$ a linear operator, being D dense in H. Then*

1. *A^* is closed,*
2. *A is closable if and only if $D(A^*)$ is dense and in this case*

$$\overline{A} = A^{**},$$

3. *If A is closable, then $(\overline{A})^* = A^*$.*

Proof.

1. We define the operator $V : H \times H \to H \times H$ by

$$V(\phi, \psi) = (-\psi, \phi).$$

Let $E \subset H \times H$ be a subspace. Thus, if $(\phi_1, \psi_1) \in V(E^\perp)$, then there exists $(\phi, \psi) \in E^\perp$ such that

$$V(\phi, \psi) = (-\psi, \phi) = (\phi_1, \psi_1).$$

Hence

$$\psi = -\phi_1 \text{ and } \phi = \psi_1,$$

so that for $(\psi_1, -\phi_1) \in E^\perp$ and $(w_1, w_2) \in E$ we have

$$((\psi_1, -\phi_1), (w_1, w_2))_{H \times H} = 0 = (\psi_1, w_1)_H + (-\phi_1, w_2)_H.$$

Thus

$$(\phi_1, -w_2)_H + (\psi_1, w_1)_H = 0,$$

and therefore

$$((\phi_1, \psi_1), (-w_2, w_1))_{H \times H} = 0,$$

that is,

$$((\phi_1, \psi_1), V(w_1, w_2))_{H \times H} = 0, \forall (w_1, w_2) \in E.$$

This means that

$$(\phi_1, \psi_1) \in (V(E))^\perp,$$

so that

$$V(E^\perp) \subset (V(E))^\perp.$$

It is easily verified that the implications from which the last inclusion results are in fact equivalences, so that

$$V(E^\perp) = (V(E))^\perp.$$

Suppose $(\phi, \eta) \in H \times H$. Thus, $(\phi, \eta) \in V(\Gamma(A))^\perp$ if and only if

$$((\phi, \eta), (-A\psi, \psi))_{H \times H} = 0, \forall \psi \in D,$$

which holds if and only if

$$(\phi, A\psi)_H = (\eta, \psi)_H, \forall \psi \in D,$$

that is, if and only if

$$(\phi, \eta) \in \Gamma(A^*).$$

Thus

$$\Gamma(A^*) = V(\Gamma(A))^\perp.$$

Since $(V(\Gamma(A))^\perp$ is closed, A^* is closed.

2. Observe that $\Gamma(A)$ is a linear subset of $H \times H$ so that

$$\begin{aligned}\overline{\Gamma(A)} &= [\Gamma(A)^\perp]^\perp \\ &= V^2[\Gamma(A)^\perp]^\perp \\ &= [V[V(\Gamma(A))^\perp]]^\perp \\ &= [V(\Gamma(A^*)]^\perp \end{aligned} \tag{3.32}$$

so that from the proof of item 1, if A^* is densely defined, we get

$$\overline{\Gamma(A)} = \Gamma[(A^*)^*].$$

Conversely, suppose $D(A^*)$ is not dense. Thus there exists $\psi \in [D(A^*)]^\perp$ such that $\psi \neq \theta$. Let $(\phi, A^*\phi) \in \Gamma(A^*)$. Hence

$$((\psi, \theta), (\phi, A^*\phi))_{H\times H} = (\psi, \phi)_H = 0,$$

so that

$$(\psi, \theta) \in [\Gamma(A^*)]^\perp.$$

Therefore $V[\Gamma(A^*)]^\perp$ is not the graph of a linear operator. Since $\overline{\Gamma(A)} = V[\Gamma(A^*)]^\perp$ A is not closable.

3. Observe that if A is closable, then

$$A^* = \overline{(A^*)} = A^{***} = (\overline{A})^*.$$

3.9 Symmetric and Self-Adjoint Operators

Definition 3.9.1. Let $A : D \to H$ be a linear operator, where D is dense in H. A is said to be symmetric if $A \subset A^*$, that is, if $D \subset D(A^*)$ and

$$A^*\phi = A\phi, \forall \phi \in D.$$

Equivalently, A is symmetric if and only if

$$(A\phi, \psi)_H = (\phi, A\psi)_H, \forall \phi, \psi \in D.$$

Definition 3.9.2. Let $A : D \to H$ be a linear operator. We say that A is self-adjoint if $A = A^*$, that is, if A is symmetric and $D = D(A^*)$.

Definition 3.9.3. Let $A : D \to H$ be a symmetric operator. We say that A is essentially self-adjoint if its closure $\overline{A}$ is self-adjoint. If A is closed, a subset $E \subset D$ is said to be a core for A if $\overline{A|_E} = A$.

Theorem 3.9.4. *Let $A : D \to H$ be a symmetric operator. Then the following statements are equivalent:*

1. *A is self-adjoint,*
2. *A is closed and $N(A^* \pm iI) = \{\theta\}$,*
3. *$R(A \pm iI) = H$.*

Proof.

- 1 implies 2:
Suppose A is self-adjoint, let $\phi \in D = D(A^*)$ be such that
$$A\phi = i\phi$$
so that
$$A^*\phi = i\phi.$$
Observe that
$$\begin{aligned} -i(\phi,\phi)_H &= (i\phi,\phi)_H \\ &= (A\phi,\phi)_H \\ &= (\phi,A\phi)_H \\ &= (\phi,i\phi)_H \\ &= i(\phi,\phi)_H, \end{aligned} \tag{3.33}$$
so that $(\phi,\phi)_H = 0$, that is, $\phi = \theta$. Thus
$$N(A - iI) = \{\theta\}.$$
Similarly we prove that $N(A + iI) = \{\theta\}$. Finally, since $\overline{A^*} = A^* = A$, we get that $A = A^*$ is closed.
- 2 implies 3:
Suppose 2 holds. Thus the equation
$$A^*\phi = -i\phi$$
has no nontrivial solution. We will prove that $R(A - iI)$ is dense in H. If $\psi \in R(A - iI)^\perp$, then
$$((A - iI)\phi, \psi)_H = 0, \forall \phi \in D,$$
so that $\psi \in D(A^*)$ and

$$(A - iI)^* \psi = (A^* + iI)\psi = \theta,$$

and hence by above $\psi = \theta$. Now we will prove that $R(A - iI)$ is closed and conclude that

$$R(A - iI) = H.$$

Given $\phi \in D$ we have

$$\|(A - iI)\phi\|_H^2 = \|A\phi\|_H^2 + \|\phi\|_H^2. \tag{3.34}$$

Let $\psi_0 \in H$ be a limit point of $R(A - iI)$. Thus we may find $\{\phi_n\} \subset D$ such that

$$(A - iI)\phi_n \to \psi_0.$$

From (3.34)

$$\|\phi_n - \phi_m\|_H \leq \|(A - iI)(\phi_n - \phi_m)\|_H, \forall m, n \in \mathbb{N}$$

so that $\{\phi_n\}$ is a Cauchy sequence, therefore converging to some $\phi_0 \in H$. Also from (3.34)

$$\|A\phi_n - A\phi_m\|_H \leq \|(A - iI)(\phi_n - \phi_m)\|_H, \forall m, n \in \mathbb{N}$$

so that $\{A\phi_n\}$ is a Cauchy sequence, hence also a converging one. Since A is closed, we get $\phi_0 \in D$ and

$$(A - iI)\phi_0 = \psi_0.$$

Therefore $R(A - iI)$ is closed, so that

$$R(A - iI) = H.$$

Similarly

$$R(A + iI) = H.$$

- 3 implies 1: Let $\phi \in D(A^*)$. Since $R(A - iI) = H$, there is an $\eta \in D$ such that

$$(A - iI)\eta = (A^* - iI)\phi,$$

and since $D \subset D(A^*)$ we obtain $\phi - \eta \in D(A^*)$ and

$$(A^* - iI)(\phi - \eta) = \theta.$$

Since $R(A + iI) = H$ we have $N(A^* - iI) = \{\theta\}$. Therefore $\phi = \eta$, so that $D(A^*) = D$. The proof is complete.

3.9.1 The Spectral Theorem Using Cayley Transform

In this section H is a complex Hilbert space. We suppose A is defined on a dense subspace of H, being A self-adjoint but possibly unbounded. We have shown that $(A+i)$ and $(A-i)$ are onto H and it is possible to prove that

$$U=(A-i)(A+i)^{-1},$$

exists on all H and it is unitary. Furthermore, on the domain of A,

$$A=i(I+U)(I-U)^{-1}.$$

The operator U is called the Cayley transform of A. We have already proven that

$$U=\int_0^{2\pi} e^{i\phi}dF(\phi),$$

where $\{F(\phi)\}$ is a monotone family of orthogonal projections, strongly continuous from the right and we may consider it such that

$$F(\phi)=\begin{cases} 0, \text{ if } \phi\leq 0, \\ I, \text{ if } \phi\geq 2\pi. \end{cases} \tag{3.35}$$

Since $F(\phi)=0$, for all $\phi\leq 0$ and

$$F(0)=F(0^+)$$

we obtain

$$F(0^+)=0=F(0^-),$$

that is, $F(\phi)$ is continuous at $\phi=0$. We claim that F is continuous at $\phi=2\pi$. Observe that $F(2\pi)=F(2\pi^+)$ so that we need only to show that

$$F(2\pi^-)=F(2\pi).$$

Suppose

$$F(2\pi)-F(2\pi^-)\neq\theta.$$

Thus, there exists some $u,v\in H$ such that

$$(F(2\pi)-F(2(\pi^-)))u=v\neq\theta.$$

Therefore

$$F(\phi)v=F(\phi)[(F(2\pi)-F(2\pi^-))u],$$

so that

$$F(\phi)v=\begin{cases} 0, \text{ if } \phi<2\pi, \\ v, \text{ if } \phi\geq 2\pi. \end{cases} \tag{3.36}$$

Observe that

$$U - I = \int_0^{2\pi} (e^{i\phi} - 1)dF(\phi),$$

and

$$U^* - I = \int_0^{2\pi} (e^{-i\phi} - 1)dF(\phi).$$

Let $\{\phi_n\}$ be a partition of $[0,2\pi]$. From the monotonicity of $[0,2\pi]$ and pairwise orthogonality of

$$\{F(\phi_n) - F(\phi_{n-1})\}$$

we can show that (this is not proved in details here)

$$(U^* - I)(U - I) = \int_0^{2\pi} (e^{-i\phi} - 1)(e^{i\phi} - 1)dF(\phi),$$

so that, given $z \in H$, we have

$$((U^* - I)(U - I)z, z)_H = \int_0^{2\pi} |e^{i\phi} - 1|^2 d\|F(\phi)z\|^2,$$

thus, for v defined above

$$\begin{aligned} \|(U - I)v\|^2 &= ((U - I)v, (U - I)v)_H \\ &= ((U - I)^*(U - I)v, v)_H \\ &= \int_0^{2\pi} |e^{i\phi} - 1|^2 d\|F(\phi)v\|^2 \\ &= \int_0^{2\pi^-} |e^{i\phi} - 1|^2 d\|F(\phi)v\|^2 \\ &= 0. \end{aligned} \tag{3.37}$$

The last two equalities result from $e^{2\pi i} - 1 = 0$ and $d\|F(\phi)v\| = \theta$ on $[0,2\pi)$. Since $v \neq \theta$ the last equation implies that $1 \in P\sigma(U)$, which contradicts the existence of

$$(I - U)^{-1}.$$

Thus, F is continuous at $\phi = 2\pi$.

Now choose a sequence of real numbers $\{\phi_n\}$ such that $\phi_n \in (0,2\pi)$, $n = 0, \pm 1, \pm 2, \pm 3, \ldots$ such that

$$-\cot\left(\frac{\phi_n}{2}\right) = n.$$

Now define $T_n = F(\phi_n) - F(\phi_{n-1})$. Since U commutes with $F(\phi)$, U commutes with T_n. Since

$$A = i(I + U)(I - U)^{-1},$$

this implies that the range of T_n is invariant under U and A. Observe that

$$\begin{aligned}\sum_n T_n &= \sum_n (F(\phi_n) - F(\phi_{n-1})) \\ &= \lim_{\phi\to 2\pi} F(\phi) - \lim_{\phi\to 0} F(\phi) \\ &= I - \theta = I. \end{aligned} \tag{3.38}$$

Hence

$$\sum_n R(T_n) = H.$$

Also, for $u \in H$, we have that

$$F(\phi)T_n u = \begin{cases} 0, \text{ if } \phi < \phi_{n-1}, \\ (F(\phi) - F(\phi_{n-1}))u, \text{ if } \phi_{n-1} \leq \phi \leq \phi_n, \\ F(\phi_n) - F(\phi_{n-1}))u, \text{ if } \phi > \phi_n, \end{cases} \tag{3.39}$$

so that

$$\begin{aligned}(I-U)T_n u &= \int_0^{2\pi} (1 - e^{i\phi}) dF(\phi) T_n u \\ &= \int_{\phi_{n-1}}^{\phi_n} (1 - e^{i\phi}) dF(\phi) u. \end{aligned} \tag{3.40}$$

Therefore

$$\begin{aligned}&\int_{\phi_{n-1}}^{\phi_n} (1-e^{i\phi})^{-1} dF(\phi)(I-U)T_n u \\ &= \int_{\phi_{n-1}}^{\phi_n} (1-e^{i\phi})^{-1} dF(\phi) \int_{\phi_{n-1}}^{\phi_n} (1-e^{i\phi}) dF(\phi) u \\ &= \int_{\phi_{n-1}}^{\phi_n} (1-e^{i\phi})^{-1} (1-e^{i\phi}) dF(\phi) u \\ &= \int_{\phi_{n-1}}^{\phi_n} dF(\phi) u \\ &= \int_0^{2\pi} dF(\phi) T_n u = T_n u. \end{aligned} \tag{3.41}$$

Hence

$$\left[(I-U)|_{R(T_n)}\right]^{-1} = \int_{\phi_{n-1}}^{\phi_n} (1-e^{i\phi})^{-1} dF(\phi).$$

From this, from above, and as

$$A = i(I+U)(I-U)^{-1}$$

we obtain

$$AT_n u = \int_{\phi_{n-1}}^{\phi_n} i(1+e^{i\phi})(1-e^{i\phi})^{-1} dF(\phi)u.$$

Therefore defining

$$\lambda = -\cot\left(\frac{\phi}{2}\right),$$

and

$$E(\lambda) = F(-2\cot^{-1}\lambda),$$

we get

$$i(1+e^{i\phi})(1-e^{i\phi})^{-1} = -\cot\left(\frac{\phi}{2}\right) = \lambda.$$

Hence,

$$AT_n u = \int_{n-1}^{n} \lambda dE(\lambda)u.$$

Finally, from

$$u = \sum_{n=-\infty}^{\infty} T_n u,$$

we can obtain

$$\begin{aligned} Au &= A(\sum_{n=-\infty}^{\infty} T_n u) \\ &= \sum_{n=-\infty}^{\infty} AT_n u \\ &= \sum_{n=-\infty}^{\infty} \int_{n-1}^{n} \lambda dE(\lambda)u. \end{aligned} \tag{3.42}$$

Being the convergence in question in norm, we may write

$$Au = \int_{-\infty}^{\infty} \lambda dE(\lambda)u.$$

Since $u \in H$ is arbitrary, we may denote

$$A = \int_{-\infty}^{\infty} \lambda dE(\lambda). \tag{3.43}$$