Chapter 12
Duality Applied to Elasticity

12.1 Introduction

The first part of the present work develops a new duality principle applicable to nonlinear elasticity. The proof of existence of solutions for the model in question has been obtained in Ciarlet [21]. In earlier results (see [65] for details) the concept of complementary energy is equivalently developed under the hypothesis of positive definiteness of the stress tensor at a critical point. In more recent works, Gao [33, 34, 36] applied his triality theory to similar models obtaining duality principles for more general situations, including the case of negative definite optimal stress tensor.

We emphasize our main objective is to establish a new and different duality principle which allows the local optimal stress tensor to not be either positive or negative definite. Such a result is a kind of extension of a more basic one obtained in Toland [67]. Despite the fact we do not apply it directly, we follow a similar idea. The optimality conditions are also new. We highlight the basic tools on convex analysis here used may be found in [25, 54, 67] for example. For related results about the plate model presented in Ciarlet [22], see Botelho [11, 13].

In a second step, we present other two duality principles which qualitatively agree with the triality theory proposed by Gao (see again [33, 34], for details).

However, our proofs again are obtained through more traditional tools of convex analysis. Finally, in the last section, we provide a numerical example in which the optimal stress field is neither positive nor negative definite.

At this point we start to describe the primal formulation.

Consider $\Omega \subset \mathbb{R}^3$ an open, bounded, connected set, which represents the reference volume of an elastic solid under the loads $f \in L^2(\Omega;\mathbb{R}^3)$ and the boundary loads $\hat{f} \in L^2(\Gamma;\mathbb{R}^3)$, where Γ denotes the boundary of Ω. The field of displacements resulting from the actions of f and $\hat{f}$ is denoted by $u \equiv (u_1,u_2,u_3) \in U$, where u_1, u_2, and u_3 denote the displacements relating the directions x, y, and z, respectively, in the Cartesian system (x,y,z).

Here U is defined by

$$U = \{u = (u_1,u_2,u_3) \in W^{1,4}(\Omega;\mathbb{R}^3) \mid u = (0,0,0) \equiv \theta \text{ on } \Gamma_0\} \quad (12.1)$$

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and $\Gamma = \Gamma_0 \cup \Gamma_1$, $\Gamma_0 \cap \Gamma_1 = \emptyset$ (for details about the Sobolev space U see [2]). We assume $|\Gamma_0| > 0$ where $|\Gamma_0|$ denotes the Lebesgue measure of Γ_0.

The stress tensor is denoted by $\{\sigma_{ij}\}$, where

$$\sigma_{ij} = H_{ijkl}\left(\frac{1}{2}(u_{k,l} + u_{l,k} + u_{m,k}u_{m,l})\right), \qquad (12.2)$$
$$\{H_{ijkl}\} = \{\lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\},$$

$\{\delta_{ij}\}$ is the Kronecker delta and $\lambda, \mu > 0$ are the Lamé constants (we assume they are such that $\{H_{ijkl}\}$ is a symmetric constant positive definite fourth-order tensor).

The boundary value form of the nonlinear elasticity model is given by

$$\begin{cases} \sigma_{ij,j} + (\sigma_{mj}u_{i,m})_{,j} + f_i = 0, & \text{in } \Omega, \\ u = \theta, & \text{on } \Gamma_0, \\ \sigma_{ij}n_j + \sigma_{mj}u_{i,m}\mathbf{n}_j = \hat{f}_i, & \text{on } \Gamma_1, \end{cases} \qquad (12.3)$$

where $\mathbf{n}$ denotes the outward normal to the surface Γ.

The corresponding primal variational formulation is represented by $J : U \to \mathbb{R}$, where

$$J(u) = \frac{1}{2}\int_\Omega H_{ijkl}\left(\frac{1}{2}(u_{i,j} + u_{j,i} + u_{m,i}u_{m,j})\right)\left(\frac{1}{2}(u_{k,l} + u_{l,k} + u_{m,k}u_{m,l})\right)dx$$
$$-\langle u, f\rangle_{L^2(\Omega;\mathbb{R}^3)} - \int_{\Gamma_1} \hat{f}_i u_i \, d\Gamma \qquad (12.4)$$

where

$$\langle u, f\rangle_{L^2(\Omega;\mathbb{R}^3)} = \int_\Omega f_i u_i \, dx.$$

Remark 12.1.1. Derivatives must be always understood in the distributional sense, whereas boundary conditions are in the sense of traces. Moreover, from now on by a regular boundary Γ of Ω, we mean regularity enough so that the standard Gauss–Green formulas of integrations by parts and the well-known Sobolev imbedding and trace theorems hold. Finally, we denote by θ the zero vector in appropriate function spaces, the standard norm for $L^2(\Omega)$ by $\|\cdot\|_2$, and $L^2(\Omega;\mathbb{R}^{3\times 3})$ simply by L^2.

12.2 The Main Duality Principle

Now we prove the main result.

Theorem 12.2.1. *Assume the statements of last section. In particular, let $\Omega \subset \mathbb{R}^3$ be an open, bounded, connected set with a regular boundary denoted by $\Gamma = \Gamma_0 \cup \Gamma_1$, where $\Gamma_0 \cap \Gamma_1 = \emptyset$ and $|\Gamma_0| > 0$. Consider the functional $(G \circ \Lambda) : U \to \mathbb{R}$ expressed by*

$$(G \circ \Lambda)(u)$$
$$= \frac{1}{2}\int_\Omega H_{ijkl}\left(\frac{u_{i,j} + u_{j,i}}{2} + \frac{u_{m,i}u_{m,j}}{2}\right)\left(\frac{u_{k,l} + u_{l,k}}{2} + \frac{u_{m,k}u_{m,l}}{2}\right)dx,$$

where $\Lambda : U \to Y \times Y$ *is given by*

$$\Lambda u = \{\Lambda_1 u, \Lambda_2 u\},$$

$$\Lambda_1 u = \left\{ \frac{u_{i,j} + u_{j,i}}{2} \right\}$$

and

$$\Lambda_2 u = \{u_{m,i}\}.$$

Here

$$U = \{u \in W^{1,4}(\Omega; \mathbb{R}^3) \mid u = (u_1, u_2, u_3) = \theta \text{ on } \Gamma_0\}.$$

Define $(F \circ \Lambda_2) : U \to \mathbb{R}$, $(G_K \circ \Lambda) : U \to \mathbb{R}$, *and* $(G_1 \circ \Lambda_2) : U \to \mathbb{R}$ *by*

$$(F \circ \Lambda_2)(u) = \frac{K}{2} \langle u_{m,i}, u_{m,i} \rangle_{L^2(\Omega)},$$

$$G_K(\Lambda u) = G_K(\Lambda_1 u, \Lambda_2 u) = G(\Lambda u) + \frac{K}{4} \langle u_{m,i}, u_{m,i} \rangle_{L^2(\Omega)},$$

and

$$(G_1 \circ \Lambda_2)(u) = \frac{K}{4} \langle u_{m,i}, u_{m,i} \rangle_{L^2(\Omega)},$$

respectively.

Also define

$$C = \{u \in U \mid G_K^{**}(\Lambda u) = G_K(\Lambda u)\},$$

where $K > 0$ *is an appropriate constant to be specified.*

For $f \in L^2(\Omega; \mathbb{R}^3)$, $\hat{f} \in L^2(\Gamma; \mathbb{R}^3)$, *let* $J : U \to \mathbb{R}$ *be expressed by*

$$J(u) = G(\Lambda u) - \langle u, f \rangle_{L^2(\Omega; \mathbb{R}^3)} - \int_{\Gamma_1} \hat{f}_i u_i \, d\Gamma. \tag{12.5}$$

Under such hypotheses, we have

$$\inf_{u \in C_1} \{J(u)\}$$
$$\geq \sup_{(\tilde{\sigma}, \sigma, v) \in \tilde{Y}} \left\{ \inf_{z^* \in Y^*} \{F^*(z^*) - \tilde{G}_K^*(\sigma, z^*, v) - \tilde{G}_1^*(\tilde{\sigma}, \sigma, z^*, v)\} \right\},$$

where $\tilde{Y} = A^* \times Y^* \times \hat{Y}^*$, $Y = Y^* = L^2(\Omega; \mathbb{R}^{3 \times 3}) \equiv L^2$,

$$\hat{Y}^* = \{v \in Y^* \text{ such that } W^*(z^*) \text{ is positive definite in } \Omega\}, \tag{12.6}$$

and

$$W^*(z^*) = \frac{z_{mi}^* z_{mi}^*}{K} - \overline{H}_{ijkl} z_{ij}^* z_{kl}^* - \sum_{m,i=1}^{3} \frac{(z_{ij}^* v_{mj})^2}{K/2}. \tag{12.7}$$

Here $C_1 = C_2 \cap C$, where

$$C_2 = \{u \in U \mid \{u_{i,j}\} \in \hat{Y}^*\}.$$

Furthermore,

$$A^* = \{\tilde{\sigma} \in Y^* \mid \tilde{\sigma}_{ij,j} + f_i = 0 \text{ in } \Omega \text{ and } \tilde{\sigma}_{ij}\boldsymbol{n}_j = \hat{f}_i \text{ on } \Gamma_1\}.$$

Also

$$\begin{aligned} F^*(z^*) &= \sup_{v_2 \in Y}\{\langle v_2, z^*\rangle_Y - F(v_2)\} \\ &= \frac{1}{2K}\langle z^*_{mi}, z^*_{mi}\rangle_{L^2(\Omega)}, \end{aligned} \quad (12.8)$$

where we recall that $z^*_{ij} = z^*_{ji}$. Through the relations

$$Q_{mi} = (\sigma_{ij} + z^*_{ij})v_{mj} + (K/2)v_{mi},$$

we define

$$\begin{aligned} \tilde{G}^*_K(\sigma, z^*, v) &= G^*_K(\sigma + z^*, Q) \\ &= \sup_{(v_1,v_2)\in Y\times Y}\{\langle v_1, \sigma + z^*\rangle_Y + \langle v_2, Q\rangle_Y - G_K(v_1, v_2)\}, \end{aligned} \quad (12.9)$$

so that in particular,

$$\begin{aligned} \tilde{G}^*_K(\sigma, z^*, v) &= G^*_K(\sigma + z^*, Q) \\ &= \frac{1}{2}\int_\Omega \overline{H}_{ijkl}(\sigma_{ij} + z^*_{ij})(\sigma_{kl} + z^*_{kl})\,dx \\ &\quad + \frac{1}{2}\int_\Omega (\sigma_{ij} + z^*_{ij})v_{mi}v_{mj}\,dx + \frac{K}{4}\langle v_{mi}, v_{mi}\rangle_{L^2(\Omega)} \end{aligned}$$

if $(\tilde{\sigma}, \sigma, v, z^*) \in B^*$. We emphasize to denote

$$B^* = \{(\tilde{\sigma}, \sigma, v, z^*) \in [Y^*]^4 \mid \sigma_K(\sigma, z^*) \text{ is positive definite in } \Omega\},$$

$$\sigma_K(\sigma, z^*) = \left\{\begin{matrix} \sigma_{11} + z^*_{11} + K/2 & \sigma_{12} + z^*_{12} & \sigma_{13} + z^*_{13} \\ \sigma_{21} + z^*_{21} & \sigma_{22} + z^*_{22} + K/2 & \sigma_{23} + z^*_{23} \\ \sigma_{31} + z^*_{31} & \sigma_{32} + z^*_{32} & \sigma_{33} + z^*_{33} + K/2 \end{matrix}\right\}, \quad (12.10)$$

and

$$\{\overline{H}_{ijkl}\} = \{H_{ijkl}\}^{-1}.$$

Moreover,

$$\begin{aligned}\tilde{G}_1^*(\tilde{\sigma},\sigma,z^*,v) &= G_1^*(\tilde{\sigma},-\sigma,-Q)\\ &= \sup_{v_2\in Y}\{\langle v_2,\tilde{\sigma}-\sigma-Q\rangle_Y - G_1(v_2)\}\\ &= \frac{1}{K}\sum_{m,i=1}^{3}\|\tilde{\sigma}_{mi}-\sigma_{mi}-Q_{mi}\|_2^2\\ &= \frac{1}{K}\sum_{m,i=1}^{3}\|\tilde{\sigma}_{mi}-\sigma_{mi}-(\sigma_{ij}+z^*_{ij})v_{mj}-(K/2)v_{mi}\|_2^2.\end{aligned}$$

Finally, if there exists a point $(u_0,\tilde{\sigma}_0,\sigma_0,v_0,z_0^*)\in C_1\times((\tilde{Y}\times Y^*)\cap B^*)$, *such that*

$$\delta\left\{\langle u_{0^i},-\tilde{\sigma}_{0^{ij},j}-f_i\rangle_{L^2(\Omega)}-\int_{\Gamma_1}u_{0^i}(\hat{f}_i-\tilde{\sigma}_{0^{ij}}\boldsymbol{n}_j)\,d\Gamma \right.$$
$$\left. +F^*(z_0^*)-\tilde{G}_K^*(\sigma_0,z_0^*,v_0)-\tilde{G}_1^*(\tilde{\sigma}_0,\sigma_0,z_0^*,v_0)\right\}=\theta, \tag{12.11}$$

we have

$$\begin{aligned}J(u_0) &= \min_{u\in C_1}\{J(u)\}\\ &= \sup_{(\tilde{\sigma},\sigma,v)\in\tilde{Y}}\left\{\inf_{z^*\in Y^*}\left\{F^*(z^*)-\tilde{G}_K^*(\sigma,z^*,v)-\tilde{G}_1^*(\tilde{\sigma},\sigma,z^*,v)\right\}\right\}\\ &= F^*(z_0^*)-\tilde{G}_K^*(\sigma_0,z_0^*,v_0)-\tilde{G}_1^*(\tilde{\sigma}_0,\sigma_0,z_0^*,v_0).\end{aligned} \tag{12.12}$$

Proof. We start by proving that $G_K^*(\sigma+z^*,Q)=G_{K_L}^*(\sigma+z^*,Q)$ if $\sigma_K(\sigma,z^*)$ is positive definite in Ω, where

$$G_{K_L}^*(\sigma,Q)=\int_\Omega g_{K_L}^*(\sigma,Q)\,dx$$

is the Legendre transform of $G_K: Y\times Y\to\mathbb{R}$. To simplify the notation we denote $(\sigma,Q)=y^*=(y_1^*,y_2^*)$. We first formally calculate $g_{K_L}^*(y^*)$, the Legendre transform of $g_K(y)$, where

$$g_K(y)=H_{ijkl}\left(y_{1^{ij}}+\frac{1}{2}y_{2^{mi}}y_{2^{mj}}\right)\left(y_{1^{kl}}+\frac{1}{2}y_{2^{mk}}y_{2^{ml}}\right)$$
$$+\frac{K}{4}y_{2^{mi}}y_{2^{mi}}. \tag{12.13}$$

We recall that

$$g_{K_L}^*(y^*)=\langle y,y^*\rangle_{\mathbb{R}^{18}}-g_K(y) \tag{12.14}$$

where $y \in \mathbb{R}^{18}$ is the solution of equation

$$y^* = \frac{\partial g_K(y)}{\partial y}. \tag{12.15}$$

Thus

$$y^*_{1ij} = \sigma_{ij} = H_{ijkl}\left(y_{1kl} + \frac{1}{2}y_{2mk}y_{2ml}\right) \tag{12.16}$$

and

$$y^*_{2mi} = Q_{mi} = H_{ijkl}\left(y_{1kl} + \frac{1}{2}y_{2ok}y_{2ol}\right)y_{2mj} + (K/2)y_{2mi} \tag{12.17}$$

so that

$$Q_{mi} = \sigma_{ij}y_{2mj} + (K/2)y_{2mi}. \tag{12.18}$$

Inverting these last equations, we have

$$y_{2mi} = \overline{\sigma}^K_{ij}Q_{mj} \tag{12.19}$$

where $\{\overline{\sigma}^K_{ij}\} = \sigma_K^{-1}(\sigma)$,

$$\sigma_K(\sigma) = \left\{\begin{array}{lll} \sigma_{11} + K/2 & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} + K/2 & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} + K/2 \end{array}\right\} \tag{12.20}$$

and also

$$y_{1ij} = \overline{H}_{ijkl}\sigma_{kl} - \frac{1}{2}y_{2mi}y_{2mj}. \tag{12.21}$$

Finally

$$g^*_{K_L}(\sigma, Q) = \frac{1}{2}\overline{H}_{ijkl}\sigma_{ij}\sigma_{kl} + \frac{1}{2}\bar{\sigma}^K_{ij}Q_{mi}Q_{mj}. \tag{12.22}$$

Now we will prove that $g^*_{K_L}(y^*) = g^*_K(y^*)$ if $\sigma_K(y^*_1) = \sigma_K(\sigma)$ is positive definite. First observe that

$$\begin{aligned} g^*_K(y^*) &= \sup_{y \in \mathbb{R}^{18}} \{\langle y_1, \sigma\rangle_{\mathbb{R}^9} + \langle y_2, Q\rangle_{\mathbb{R}^9} - g_K(y)\} \\ &= \sup_{y \in \mathbb{R}^{18}} \Bigg\{ \langle y_1, \sigma\rangle_{\mathbb{R}^9} + \langle y_2, Q\rangle_{\mathbb{R}^9} \\ &\quad -\frac{1}{2}H_{ijkl}\left(y_{1ij} + \frac{1}{4}y_{2mi}y_{2mj}\right)\left(y_{1kl} + \frac{1}{2}y_{2mk}y_{2ml}\right) \\ &\quad -\frac{K}{4}y_{2mi}y_{2mi}\Bigg\} \end{aligned}$$

$$= \sup_{(\bar{y}_1, y_2) \in \mathbb{R}^9 \times \mathbb{R}^9} \left\{ \langle \bar{y}_{1ij} - \frac{1}{2} y_{2mi} y_{2mj}, \sigma_{ij} \rangle_{\mathbb{R}} + \langle y_2, Q \rangle_{\mathbb{R}^9} \right.$$
$$\left. - \frac{1}{2} H_{ijkl} [\bar{y}_{1ij}][\bar{y}_{1kl}] - \frac{K}{4} y_{2mi} y_{2mi} \right\}.$$

The result follows just observing that

$$\sup_{\bar{y}_1 \in \mathbb{R}^9} \left\{ \langle \bar{y}_{1ij}, \sigma_{ij} \rangle_{\mathbb{R}} - \frac{1}{2} H_{ijkl} [\bar{y}_{1ij}][\bar{y}_{1kl}] \right\} = \frac{1}{2} \overline{H}_{ijkl} \sigma_{ij} \sigma_{kl} \tag{12.23}$$

and

$$\sup_{y_2 \in \mathbb{R}^9} \left\{ \langle -\frac{1}{2} y_{2mi} y_{2mj}, \sigma_{ij} \rangle_{\mathbb{R}} + \langle y_2, Q \rangle_{\mathbb{R}^9} - \frac{K}{4} y_{2mi} y_{2mi} \right\}$$
$$= \frac{1}{2} \overline{\sigma}^K_{ij} Q_{mi} Q_{mj} \tag{12.24}$$

if $\sigma_K(y_1^*) = \sigma_K(\sigma)$ is positive definite.

Now observe that using the relation

$$Q_{mi} = (\sigma_{ij} + z^*_{ij}) v_{mj} + (K/2) v_{mi},$$

we have

$$\begin{aligned} \tilde{G}^*_K(\sigma, z^*, v) &= G^*_K(\sigma + z^*, Q) \\ &= \int_\Omega g^*_{K_L}(\sigma + z^*, Q)\, dx, \end{aligned} \tag{12.25}$$

if $\sigma_K(\sigma + z^*)$ is positive definite.

Also, considering the concerned symmetries, we may write

$$\begin{aligned} \tilde{G}^*_K(\sigma, z^*, v) + \tilde{G}^*_1(\tilde{\sigma}, \sigma, z^*, v) &= G^*_K(\sigma + z^*, Q) + G^*_1(\tilde{\sigma}, -\sigma, -Q) \\ &\geq \langle \Lambda_1 u, \sigma \rangle_{L^2} + \langle \Lambda_2 u, z^* + Q \rangle_{L^2} \\ &\quad + \langle \Lambda_1 u, \tilde{\sigma} - \sigma \rangle_{L^2} - \langle \Lambda_2 u, Q \rangle_{L^2} \\ &\quad - G^{**}_K(\Lambda u) - G_1(\Lambda_2 u), \end{aligned} \tag{12.26}$$

$\forall u \in U,\ z^* \in Y^*,\ (\tilde{\sigma}, \sigma, v) \in \tilde{Y}$, so that

$$\begin{aligned} &\tilde{G}^*_K(\sigma, z^*, v) + \tilde{G}^*_1(\tilde{\sigma}, \sigma, z^*, v) \\ &\quad \geq \langle \Lambda_2 u, z^* \rangle_{L^2} + \langle \Lambda_1 u, \tilde{\sigma} \rangle_{L^2} \\ &\qquad - G_K(\Lambda u) - G_1(\Lambda_2 u) \\ &\quad = \langle \Lambda_2 u, z^* \rangle_{L^2} + \langle u, f \rangle_{L^2(\Omega;\mathbb{R}^3)} \\ &\qquad + \int_{\Gamma_1} \hat{f}_i u_i \, d\Gamma - G_K(\Lambda u) - G_1(\Lambda_2 u), \end{aligned} \tag{12.27}$$

$\forall u \in C_1,\ z^* \in Y^*,\ (\tilde{\sigma}, \sigma, v) \in \tilde{Y}$. Hence

$$\begin{aligned} &-F^*(z^*) + \tilde{G}^*_K(\sigma, z^*, v) + \tilde{G}^*_1(\tilde{\sigma}, \sigma, z^*, v) \\ &\geq -F^*(z^*) + \langle \Lambda_2 u, z^* \rangle_{L^2} + \langle u, f \rangle_{L^2(\Omega;\mathbb{R}^3)} \\ &\quad + \int_{\Gamma_1} \hat{f}_i u_i \, d\Gamma - G_K(\Lambda u) - G_1(\Lambda_2 u), \end{aligned} \tag{12.28}$$

$\forall u \in C_1,\ z^* \in Y^*,\ (\tilde{\sigma}, \sigma, v) \in \tilde{Y}$, and thus

$$\begin{aligned} &\sup_{z^* \in Y^*} \{-F^*(z^*) + \tilde{G}^*_K(\sigma, z^*, v) + \tilde{G}^*_1(\tilde{\sigma}, \sigma, z^*, v)\} \\ &\geq \sup_{z^* \in Y^*} \{-F^*(z^*) + \langle \Lambda_2 u, z^* \rangle_{L^2} + \langle u, f \rangle_{L^2(\Omega;\mathbb{R}^3)} \\ &\quad + \int_{\Gamma_1} \hat{f}_i u_i \, d\Gamma - G_K(\Lambda u) - G_1(\Lambda_2 u)\}, \end{aligned} \tag{12.29}$$

$\forall u \in C_1, (\tilde{\sigma}, \sigma, v) \in \tilde{Y}$.

Therefore,

$$\begin{aligned} &\sup_{z^* \in Y^*} \{-F^*(z^*) + \tilde{G}^*_K(\sigma, z^*, v) + \tilde{G}^*_1(\tilde{\sigma}, \sigma, z^*, v)\} \\ &\geq F(\Lambda_2 u) + \langle u, f \rangle_{L^2(\Omega;\mathbb{R}^3)} + \int_{\Gamma_1} \hat{f}_i u_i \, d\Gamma \\ &\quad - G_K(\Lambda u) - G_1(\Lambda_2 u), \end{aligned} \tag{12.30}$$

$\forall u \in C_1, (\tilde{\sigma}, \sigma, v) \in \tilde{Y}$, that is,

$$\begin{aligned} &\sup_{z^* \in Y^*} \{-F^*(z^*) + \tilde{G}^*_K(\sigma, z^*, v) + \tilde{G}^*_1(\tilde{\sigma}, \sigma, z^*, v)\} \\ &\geq -J(u), \end{aligned} \tag{12.31}$$

$\forall u \in C_1, (\tilde{\sigma}, \sigma, v) \in \tilde{Y}$. Finally,

$$\begin{aligned} &\inf_{u \in C_1} \{J(u)\} \\ &\geq \sup_{(\tilde{\sigma}, \sigma, v) \in \tilde{Y}} \left\{ \inf_{z^* \in Y^*} \left\{ F^*(z^*) - \tilde{G}^*_K(\sigma, z^*, v) - \tilde{G}^*_1(\tilde{\sigma}, \sigma, z^*, v) \right\} \right\}. \end{aligned} \tag{12.32}$$

Now suppose there exists a point $(u_0, \tilde{\sigma}_0, \sigma_0, z^*_0, v_0) \in C_1 \times ((\tilde{Y} \times Y^*) \cap B^*)$, such that

$$\begin{aligned} &\delta \left\{ \langle u_{0^i}, -\tilde{\sigma}_{0^{ij},j} - f_i \rangle_{L^2(\Omega)} - \int_{\Gamma_1} u_{0^i} (\hat{f}_i - \tilde{\sigma}_{0^{ij}} \mathbf{n}_j) \, d\Gamma \right. \\ &\left. + F^*(z^*_0) - \tilde{G}^*_K(\sigma_0, z^*_0, v_0) - \tilde{G}^*_1(\tilde{\sigma}_0, \sigma_0, z^*_0, v_0) \right\} = \theta, \end{aligned} \tag{12.33}$$

that is,

$$\delta\left\{\langle u_{0^i}, -\tilde{\sigma}_{0^{ij},j} - f_i\rangle_{L^2(\Omega)} - \int_{\Gamma_1} u_{0^i}(\hat{f}_i - \tilde{\sigma}_{0^{ij}}\mathbf{n}_j)\,d\Gamma \right.$$
$$+F^*(z_0^*) - \frac{1}{2}\int_\Omega \bar{H}_{ijkl}(\sigma_{0^{ij}} + z^*_{0^{ij}})(\sigma_{0^{kl}} + z^*_{0^{kl}})\,dx$$
$$-\frac{1}{2}\int_\Omega(\sigma_{0^{ij}} + z^*_{0^{ij}})v_{0^{mi}}v_{0^{mj}}\,dx - \frac{K}{4}\langle v_{0^{mi}}, v_{0^{mi}}\rangle_{L^2(\Omega)}$$
$$\left. - \sum_{m,i=1}^{3}\frac{1}{K}\|\tilde{\sigma}_{0^{mi}} - \sigma_{0^{mi}} - (\sigma_{0^{ij}} + z^*_{0^{ij}})v_{0^{mj}} - K/2v_{0^{mi}}\|_2^2\right\} = \theta.$$

Observe that the variation in $\tilde{\sigma}$ gives us

$$\tilde{\sigma}_{0^{mi}} - \sigma_{0^{mi}} - (\sigma_{0^{ij}} + z^*_{0^{ij}})v_{0^{mj}} - (K/2)v_{0^{mi}} = (K/2)u_{0^{m,i}} \text{ in } \Omega. \tag{12.34}$$

From this and recalling that $\tilde{\sigma}_{ij} = \tilde{\sigma}_{ji}$, so that we may use the replacement

$$\tilde{\sigma}_{ij} = \frac{\tilde{\sigma}_{ij} + \tilde{\sigma}_{ji}}{2} = \tilde{\sigma}_{ji}$$

(observe that a similar remark is valid for $\sigma_{0^{ij}} + z^*_{0^{ij}}$), the variation in σ gives us

$$-\bar{H}_{ijkl}(\sigma_{0^{kl}} + z^*_{0^{kl}}) - v_{0^{mi}}v_{0^{mj}}/2$$
$$+\frac{u_{0^{i,j}} + u_{0^{j,i}}}{2} + u_{0^{m,i}}v_{0^{mj}} = 0, \tag{12.35}$$

in Ω. From (12.34) and the variation in v we get

$$-(\sigma_{0^{ij}} + z^*_{0^{ij}})v_{mj} - (K/2)v_{0^{mi}}$$
$$+(\sigma_{0^{ij}} + z^*_{0_{ij}})u_{0^{m,j}} + (K/2)u_{0^{m,i}} = 0, \tag{12.36}$$

so that

$$\{v_{0^{ij}}\} = \{u_{0^{i,j}}\}, \text{ in } \Omega. \tag{12.37}$$

From this and (12.35) we get

$$\sigma_{0^{ij}} + z^*_{0^{ij}} = H_{ijkl}\left(\frac{u_{0^{k,l}} + u_{0^{l,k}}}{2} + \frac{u_{0^{m,k}}u_{0^{m,l}}}{2}\right). \tag{12.38}$$

Through such relations the variation in z^* gives us

$$z^*_{0^{ij}} = \frac{K}{2}(u_{0^{i,j}} + u_{0^{j,i}}) \text{ in } \Omega. \tag{12.39}$$

Finally, from the variation in u, we get

$$\tilde{\sigma}_{0^{ij},j} + f_i = 0, \text{ in } \Omega, \tag{12.40}$$

$$u_0 = \theta \text{ on } \Gamma_0,$$

and

$$\tilde{\sigma}_{0ij}\mathbf{n}_j = \hat{f}_i \text{ on } \Gamma_1,$$

where from (12.34), (12.37), and (12.39), we have

$$\tilde{\sigma}_{0ij} = H_{ijkl}\left(\frac{u_{0k,l}+u_{0l,k}}{2}+\frac{u_{0m,k}u_{0m,l}}{2}\right) + H_{mjkl}\left(\frac{u_{0k,l}+u_{0l,k}}{2}+\frac{u_{0p,k}u_{0p,l}}{2}\right)u_{0i,m}. \tag{12.41}$$

Replacing such results in the dual formulation we obtain

$$J(u_0) = F^*(z_0^*) - \tilde{G}_K^*(\sigma_0, z_0^*, v_0) - \tilde{G}_1^*(\tilde{\sigma}_0, \sigma_0, z_0^*, v_0). \tag{12.42}$$

From the hypothesis indicated in (12.6), the extremal relation through which z_0^* is obtained is in fact a global one.

From this, (12.2) and (12.42), the proof is complete.

Remark 12.2.2. About the last theorem, there is no duality gap between the primal and dual problems, if K is big enough so that for the optimal dual point, $\sigma_K(\sigma_0, z_0^*)$ is positive definite in Ω, where

$$\sigma_K(\sigma, z^*) = \left\{\begin{array}{lll} \sigma_{11}+z_{11}^*+K/2 & \sigma_{12}+z_{12}^* & \sigma_{13}+z_{13}^* \\ \sigma_{21}+z_{21}^* & \sigma_{22}+z_{22}^*+K/2 & \sigma_{23}+z_{23}^* \\ \sigma_{31}+z_{31}^* & \sigma_{32}+z_{32}^* & \sigma_{33}+z_{33}^*+K/2 \end{array}\right\}, \tag{12.43}$$

and

$$\sigma_{0ij} + z_{0ij}^* = H_{ijkl}\left(\frac{u_{0k,l}+u_{0l,k}}{2}+\frac{u_{0m,k}u_{0m,l}}{2}\right), \tag{12.44}$$

and, at the same time, K is small enough so that for the fixed point $\{v_{0mj}\} = \{u_{0m,j}\}$ the quadratic form (in z^*) $W^*(z^*)$ is also positive definite in Ω, where

$$W^*(z^*) = \frac{z_{mi}^* z_{mi}^*}{K} - \bar{H}_{ijkl} z_{ij}^* z_{kl}^* - \sum_{m,i=1}^{3} \frac{(z_{ij}^* v_{0mj})^2}{K/2}. \tag{12.45}$$

For $K \approx \mathscr{O}(\min\{H_{1111}/2, H_{2222}/2, H_{1212}/2\})$ there is a large class of external loads for which such a K satisfies the conditions above, including to some extent the large deformation context.

Finally, we have not formally proven, but one may obtain from the relation between the primal and dual variables that

$$\begin{aligned} C &= \{u \in U \mid G_K^{**}(\Lambda u) = G_K(\Lambda u)\} \\ &= \{u \in U \mid \sigma_K(\sigma(u), \theta) \text{ is positive definite in } \Omega\}, \end{aligned} \tag{12.46}$$

where as above indicated

$$\sigma_{ij}(u) = H_{ijkl}\left(\frac{1}{2}(u_{k,l} + u_{l,k} + u_{m,k}u_{m,l})\right). \tag{12.47}$$

12.3 Other Duality Principles

At this point we present another main result, which is summarized by the following theorem.

Theorem 12.3.1. *Let $\Omega \subset \mathbb{R}^3$ be an open, bounded, connected set with a regular boundary denoted by $\Gamma = \Gamma_0 \cup \Gamma_1$, where $\Gamma_0 \cap \Gamma_1 = \emptyset$. Consider the functional $(G \circ \Lambda) : U \to \mathbb{R}$ expressed by*

$$(G \circ \Lambda)(u) = \frac{1}{2}\int_\Omega H_{ijkl}\left(\frac{u_{i,j} + u_{j,i}}{2} + \frac{u_{m,i}u_{m,j}}{2}\right)\left(\frac{u_{k,l} + u_{l,k}}{2} + \frac{u_{m,k}u_{m,l}}{2}\right) dx,$$

where

$$U = \{u = (u_1, u_2, u_3) \in W^{1,4}(\Omega; \mathbb{R}^3) \mid u = (0,0,0) \equiv \theta \text{ on } \Gamma_0\}, \tag{12.48}$$

and $\Lambda : U \to Y = Y^* = L^2(\Omega; \mathbb{R}^{3\times 3}) \equiv L^2$ is given by

$$\Lambda u = \{\Lambda_{ij}(u)\} = \left\{\frac{1}{2}(u_{i,j} + u_{j,i} + u_{m,i}u_{m,j})\right\}.$$

Define $J : U \to \mathbb{R}$ by

$$J(u) = G(\Lambda u) - \langle u, f\rangle_{L^2(\Omega;\mathbb{R}^3)} - \int_{\Gamma_1} \hat{f}_i u_i \, d\Gamma. \tag{12.49}$$

Also define

$$J_K : U \times Y \to \mathbb{R}$$

by

$$J_K(u,p) = G(\Lambda u + p) + K\langle p, p\rangle_{L^2} - \langle u, f\rangle_{L^2(\Omega;\mathbb{R}^3)} - \int_{\Gamma_1} \hat{f}_i u_i \, d\Gamma - \frac{K}{2}\langle p, p\rangle_{L^2},$$

and assume that $K > 0$ is sufficiently big so that $J_K(u,p)$ is bounded below.

Also define

$$J_K^*(\sigma, u) = F_f(\sigma) - G^*(\sigma) + K\left\|\Lambda u - \frac{\partial G^*(\sigma)}{\partial \sigma}\right\|_{L^2}^2 + \frac{1}{2K}\langle \sigma, \sigma\rangle_{L^2}, \tag{12.50}$$

where

$$G^*(\sigma) = \sup_{v\in Y}\{\langle v,\sigma\rangle_{L^2} - G(v)\}$$
$$= \frac{1}{2}\int_\Omega \overline{H}_{ijkl}\sigma_{ij}\sigma_{kl}\,dx, \tag{12.51}$$

$$\{\overline{H}_{ijkl}\} = \{H_{ijkl}\}^{-1}$$

and

$$F_f(\sigma) = \sup_{u\in U}\left\{\langle \Lambda u,\sigma\rangle_{L^2} - \langle u,f\rangle_{L^2(\Omega;\mathbb{R}^3)} - \int_{\Gamma_1}\hat{f}_i u_i\,d\Gamma\right\}.$$

Under such assumptions, we have

$$\inf_{(u,p)\in U}\{J_K(u,p)\} \leq \inf_{(\sigma,u)\in Y^*\times U}\{J_K^*(\sigma,u)\}. \tag{12.52}$$

Finally, assume that Γ_0, $f\in L^2(\Omega;\mathbb{R}^3)$ and $\hat{f}\in L^2(\Gamma;\mathbb{R}^3)$ are such that a local minimum of J_K over $V_0 = B_r(u_0)\times B_r(p_0)$ is attained at some $(u_0,p_0)\in U\times Y$ such that

$$\sigma_0 = \frac{\partial G(\Lambda u_0 + p_0)}{\partial v} \tag{12.53}$$

is negative definite.

Here

$$B_r(u_0) = \{u\in U \mid \|u-u_0\|_U < r\},$$

and

$$B_r(p_0) = \{p\in Y \mid \|p-p_0\|_Y < r\},$$

for some appropriate $r>0$.

Under such hypotheses, there exists a set $\tilde{V}_0 \subset Y^*\times U$, such that

$$J_K(u_0,p_0) = \inf_{(u,p)\in V_0}\{J_K(u,p)\}$$
$$\leq \inf_{(\sigma,u)\in\tilde{V}_0}\{J_K^*(\sigma,u)\}$$
$$\leq J_K^*(\sigma_0,u_0)$$
$$= J_K(u_0,p_0)$$
$$\approx J(u_0) + \mathscr{O}(1/K). \tag{12.54}$$

Proof. Define

$$G_1(u,p) = G(\Lambda u + p) + K\langle p,p\rangle_{L^2},$$

and

$$G_2(u,p) = \langle u,f\rangle_{L^2(\Omega;\mathbb{R}^3)} + \int_{\Gamma_1}\hat{f}_i u_i\,d\Gamma + \frac{K}{2}\langle p,p\rangle_{L^2}.$$

Observe that $\alpha_K = \inf_{(u,p)\in U\times Y}\{J_K(u,p)\} \in \mathbb{R}$ is such that

$$J_K(u,p) = G_1(u,p) - G_2(u,p) \geq \alpha_K, \forall u \in U, p \in Y.$$

Thus,

$$-G_2(u,p) \geq -G_1(u,p) + \alpha_K, \forall u \in U, p \in Y,$$

so that

$$\langle \Lambda u + p, \sigma \rangle_{L^2} - G_2(u,p) \geq \langle \Lambda u + p, \sigma \rangle_{L^2} - G_1(u,p) + \alpha_K, \forall u \in U, p \in Y.$$

Hence,

$$\sup_{(u,p)\in U\times Y} \{\langle \Lambda u + p, \sigma \rangle_{L^2} - G_2(u,p)\} \geq \langle \Lambda u + p, \sigma \rangle_{L^2} - G_1(u,p) + \alpha_K, \forall u \in U, p \in Y. \tag{12.55}$$

In particular for u, p such that

$$\sigma = \frac{\partial G(\Lambda u + p)}{\partial v},$$

we get

$$p + \Lambda u = \frac{\partial G^*(\sigma)}{\partial \sigma},$$

that is,

$$p = \frac{\partial G^*(\sigma)}{\partial \sigma} - \Lambda u,$$

and

$$G^*(\sigma) = \langle \Lambda u + p, \sigma \rangle_{L^2} - G(\Lambda u + p).$$

Hence

$$\langle \Lambda u + p, \sigma \rangle_{L^2} - G_1(u,p) = G^*(\sigma) - K \left\| \frac{\partial G^*(\sigma)}{\partial \sigma} - \Lambda u \right\|_{L^2}^2.$$

On the other hand,

$$\sup_{(u,p)\in U\times Y} \{\langle \Lambda u + p, \sigma \rangle_{L^2} - G_2(u,p)\} = F_f(\sigma) + \frac{1}{2K}\langle \sigma, \sigma \rangle_{L^2}.$$

Replacing such results in (12.55), we get

$$F_f(\sigma) - G^*(\sigma) + K \left\| \frac{\partial G^*(\sigma)}{\partial \sigma} - \Lambda u \right\|_{L^2}^2 + \frac{1}{2K}\langle \sigma, \sigma \rangle_{L^2} \geq \alpha_K,$$

$\forall \sigma \in Y^*, u \in U$.

Thus,

$$\alpha_K = \inf_{(u,p)\in U\times Y}\{J_K(u,p)\} \leq \inf_{(\sigma,u)\in Y^*\times U}\{J_K^*(\sigma,u)\}. \tag{12.56}$$

Now, let $(u_0, p_0) \in U \times Y$ be such that

$$J(u_0, p_0) = \min_{(u,p) \in V_0} \{J_K(u, p)\}.$$

Defining

$$\sigma_0 = \frac{\partial G(\Lambda u_0 + p_0)}{\partial v}, \tag{12.57}$$

since for the extremal point, we have

$$\delta_u \left\{ G(\Lambda u + p_0) - \langle u, f \rangle_{L^2(\Omega;\mathbb{R}^3)} - \int_{\Gamma_1} \hat{f}_i u_i \, d\Gamma \right\} |_{u=u_0} = \theta,$$

from this and (12.57), we also have

$$\delta_u \left\{ \langle \Lambda u, \sigma_0 \rangle_{L^2} - \langle u, f \rangle_{L^2(\Omega;\mathbb{R}^3)} - \int_{\Gamma_1} \hat{f}_i u_i \, d\Gamma \right\} |_{u=u_0} = \theta,$$

and therefore, since σ_0 is negative definite, we obtain

$$F_f(\sigma_0) = \langle \Lambda u_0, \sigma_0 \rangle_{L^2} - \langle u_0, f \rangle_{L^2(\Omega;\mathbb{R}^3)} - \int_{\Gamma_1} \hat{f}_i u_{0_i} \, d\Gamma. \tag{12.58}$$

From (12.57), we get

$$G^*(\sigma_0) = \langle \Lambda u_0 + p_0, \sigma \rangle_{L^2} - G(\Lambda u_0 + p_0), \tag{12.59}$$

so that, from (12.58) and (12.59), we obtain

$$\begin{aligned} F_f(\sigma_0) - G^*(\sigma_0) + K \left\| \frac{\partial G^*(\sigma_0)}{\partial \sigma} - \Lambda u_0 \right\|_{L^2}^2 + \frac{1}{2K} \langle \sigma_0, \sigma_0 \rangle_{L^2} \\ = G(\Lambda u_0 + p_0) + \frac{K}{2} \langle p_0, p_0 \rangle_{L^2} - \langle u_0, f \rangle_{L^2(\Omega;\mathbb{R}^3)} \\ - \int_{\Gamma_1} \hat{f}_i u_{0_i} \, d\Gamma, \end{aligned} \tag{12.60}$$

that is,

$$J_K^*(\sigma_0, u_0) = J_K(u_0, p_0). \tag{12.61}$$

Observe that, from the hypotheses,

$$J_K(u, p) \geq J_K(u_0, p_0), \forall (u, p) \in V_0.$$

At this point we develop a reasoning similarly to the lines above but now for the specific case of a neighborhood around the local optimal point. We repeat some analogous details for the sake of clarity.

From above,

$$G_1(u,p) = G(\Lambda u + p) + K\langle p,p\rangle_{L^2},$$

and

$$G_2(u,p) = \langle u,f\rangle_{L^2(\Omega;\mathbb{R}^3)} + \int_{\Gamma_1} \hat{f}_i u_i \, d\Gamma + \frac{K}{2}\langle p,p\rangle_{L^2}.$$

Observe that $\alpha = \inf_{(u,p)\in V_0}\{J_K(u,p)\} \in \mathbb{R}$ is such that

$$J_K(u,p) = G_1(u,p) - G_2(u,p) \geq \alpha, \forall (u,p) \in V_0.$$

Thus,

$$-G_2(u,p) \geq -G_1(u,p) + \alpha, \forall (u,p) \in V_0,$$

so that

$$\langle \Lambda u + p, \sigma\rangle_{L^2} - G_2(u,p) \geq \langle \Lambda u + p, \sigma\rangle_{L^2} - G_1(u,p) + \alpha, \forall (u,p) \in V_0.$$

Hence,

$$\begin{aligned}
&\sup_{(u,p)\in U}\{\langle \Lambda u + p, \sigma\rangle_{L^2} - G_2(u,p)\} \\
&\quad \geq \sup_{(u,p)\in V_0}\{\langle \Lambda u + p, \sigma\rangle_{L^2} - G_2(u,p)\} \\
&\quad \geq \langle \Lambda u + p, \sigma\rangle_{L^2} - G_1(u,p) + \alpha, \forall (u,p) \in V_0. \qquad (12.62)
\end{aligned}$$

In particular, if $(\sigma,u) \in \tilde{V}_0$, where such a set is defined by the points (σ,u) such that $u \in B_r(u_0)$ and for the σ in question there exists $p \in B_r(p_0)$ such that

$$\sigma = \frac{\partial G(\Lambda u + p)}{\partial v},$$

that is,

$$p + \Lambda u = \frac{\partial G^*(\sigma)}{\partial \sigma},$$

we get

$$p = \frac{\partial G^*(\sigma)}{\partial \sigma} - \Lambda u,$$

and

$$G^*(\sigma) = \langle \Lambda u + p, \sigma\rangle_{L^2} - G(\Lambda u + p).$$

Hence

$$\langle \Lambda u + p, \sigma\rangle_{L^2} - G_1(u,p) = G^*(\sigma) - K\left\|\frac{\partial G^*(\sigma)}{\partial \sigma} - \Lambda u\right\|_{L^2}^2. \qquad (12.63)$$

On the other hand

$$\begin{aligned}
&\sup_{(u,p)\in V_0}\{\langle \Lambda u+p,\sigma\rangle_{L^2}-G_2(u,p)\}\\
&\quad\leq \sup_{(u,p)\in U\times Y}\{\langle \Lambda u+p,\sigma\rangle_{L^2}-G_2(u,p)\}\\
&\quad= F_f(\sigma)+\frac{1}{2K}\langle\sigma,\sigma\rangle_{L^2}. \qquad (12.64)
\end{aligned}$$

Observe that $\sigma_0 \in \tilde{V}_0$. We do not provide details here, but from the generalized inverse function theorem, also an appropriate neighborhood of σ_0 belongs to $\tilde{V}_0$.

Replacing the last relations (12.63) and (12.64) into (12.62), we get

$$\begin{aligned}
&F_f(\sigma)-G^*(\sigma)+K\left\|\frac{\partial G^*(\sigma)}{\partial\sigma}-\Lambda u\right\|_{L^2}^2\\
&\quad+\frac{1}{2K}\langle\sigma,\sigma\rangle_{L^2}\geq\alpha, \qquad (12.65)
\end{aligned}$$

$\forall(\sigma,u)\in\tilde{V}_0$.

Thus,

$$\alpha=\inf_{(u,p)\in V_0}\{J_K(u,p)\}\leq\inf_{(\sigma,u)\in\tilde{V}_0}\{J_K^*(\sigma,u)\}. \qquad (12.66)$$

Finally, since

$$p_0=-\frac{1}{K}\frac{\partial G(\Lambda u_0+p_0)}{\partial p}, \qquad (12.67)$$

we get

$$\|p_0\|_Y\approx\mathscr{O}\left(\frac{1}{K}\right),$$

so that from this, (12.61), and (12.65), we may finally write

$$\begin{aligned}
\alpha=J_K(u_0,p_0)&=\inf_{(u,p)\in V_0}\{J_K(u,p)\}\\
&\leq\inf_{(\sigma,u)\in\tilde{V}_0}\{J_K^*(\sigma,u)\}\\
&\leq J_K^*(\sigma_0,u_0)\\
&=J_K(u_0,p_0)\\
&\approx J(u_0)+\mathscr{O}(1/K). \qquad (12.68)
\end{aligned}$$

The proof is complete.

Remark 12.3.2. Of particular interest is the model behavior as $K\to+\infty$. From (12.68) it seems to be clear that the duality gap between the original primal and dual formulations goes to zero as K goes to $+\infty$.

Our final result is summarized by the next theorem. It refers to a duality principle for the case of a local maximum for the primal formulation.

Theorem 12.3.3. *Let $\Omega \subset \mathbb{R}^3$ be an open, bounded, connected set with a regular boundary denoted by $\Gamma = \Gamma_0 \cup \Gamma_1$, where $\Gamma_0 \cap \Gamma_1 = \emptyset$. Consider the functional $(G \circ \Lambda) : U \to \mathbb{R}$ expressed by*

$$(G \circ \Lambda)(u) = \frac{1}{2}\int_\Omega H_{ijkl}\left(\frac{u_{i,j}+u_{j,i}}{2}+\frac{u_{m,i}u_{m,j}}{2}\right)\left(\frac{u_{k,l}+u_{l,k}}{2}+\frac{u_{m,k}u_{m,l}}{2}\right)dx,$$

where

$$U = \{u = (u_1,u_2,u_3) \in W^{1,4}(\Omega;\mathbb{R}^3) \mid u = (0,0,0) \equiv \theta \text{ on } \Gamma_0\}, \tag{12.69}$$

and $\Lambda : U \to Y = Y^* = L^2(\Omega;\mathbb{R}^{3\times 3}) \equiv L^2$ is given by

$$\Lambda u = \{\Lambda_{ij}(u)\} = \left\{\frac{1}{2}(u_{i,j}+u_{j,i}+u_{m,i}u_{m,j})\right\}.$$

Define $J : U \to \mathbb{R}$ by

$$J(u) = G(\Lambda u) - \langle u, f\rangle_{L^2(\Omega;\mathbb{R}^3)} - \int_{\Gamma_1}\hat{f}_i u_i \, d\Gamma. \tag{12.70}$$

Assume that Γ_0, $f \in L^2(\Omega;\mathbb{R}^3)$, and $\hat{f} \in L^2(\Gamma;\mathbb{R}^3)$ are such that a local maximum of J over $V_0 = B_r(u_0)$ is attained at some $u_0 \in U$ such that

$$\sigma_0 = \frac{\partial G(\Lambda u_0)}{\partial v} \tag{12.71}$$

is negative definite.

Also define

$$J^*(\sigma) = F_f(\sigma) - G^*(\sigma), \tag{12.72}$$

where

$$G^*(\sigma) = \sup_{v\in Y}\{\langle v,\sigma\rangle_{L^2} - G(v)\} = \frac{1}{2}\int_\Omega \overline{H}_{ijkl}\sigma_{ij}\sigma_{kl}\,dx, \tag{12.73}$$

$$\{\overline{H}_{ijkl}\} = \{H_{ijkl}\}^{-1}$$

and

$$F_f(\sigma) = \sup_{u\in U}\left\{\langle \Lambda u,\sigma\rangle_{L^2} - \langle u,f\rangle_{L^2(\Omega;\mathbb{R}^3)} - \int_{\Gamma_1}\hat{f}_i u_i\,d\Gamma\right\}.$$

Under such assumptions, there exists a set $\tilde{V}_0 \subset Y^*$ *such that*

$$-J^*(\sigma_0) = \max_{\sigma \in \tilde{V}_0}\{-J^*(\sigma)\} = \max_{u \in V_0}\{J(u)\} = J(u_0). \quad (12.74)$$

Proof. Define $\alpha = J(u_0)$.

Thus,

$$J(u) = G(\Lambda u) - \langle u, f\rangle_{L^2(\Omega;\mathbb{R}^3)} - \int_{\Gamma_1} \hat{f}_i u_i \, d\Gamma \leq J(u_0) = \alpha,$$

$\forall u \in V_0$.

Hence,

$$-\langle u, f\rangle_{L^2(\Omega;\mathbb{R}^3)} - \int_{\Gamma_1} \hat{f}_i u_i \, d\Gamma \leq -G(\Lambda u) + \alpha, \forall u \in V_0,$$

so that

$$\langle \Lambda u, \sigma\rangle_{L^2} - \langle u, f\rangle_{L^2(\Omega;\mathbb{R}^3)} - \int_{\Gamma_1} \hat{f}_i u_i \, d\Gamma$$
$$\leq \langle \Lambda u, \sigma\rangle_{L^2} - G(\Lambda u) + \alpha, \forall u \in V_0, \sigma \in Y^*. \quad (12.75)$$

Therefore,

$$\sup_{u \in V_0}\left\{\langle \Lambda u, \sigma\rangle_{L^2} - \langle u, f\rangle_{L^2(\Omega;\mathbb{R}^3)} - \int_{\Gamma_1} \hat{f}_i u_i \, d\Gamma\right\}$$
$$\leq \sup_{v \in Y}\{\langle v, \sigma\rangle_{L^2} - G(v)\} + \alpha, \forall \sigma \in Y^*. \quad (12.76)$$

We define $\tilde{V}_0$ by the points $\sigma \in Y^*$ such that

$$\sup_{u \in V_0}\left\{\langle \Lambda u, \sigma\rangle_{L^2} - \langle u, f\rangle_{L^2(\Omega;\mathbb{R}^3)} - \int_{\Gamma_1} \hat{f}_i u_i \, d\Gamma\right\}$$
$$= \sup_{u \in U}\left\{\langle \Lambda u, \sigma\rangle_{L^2} - \langle u, f\rangle_{L^2(\Omega;\mathbb{R}^3)} - \int_{\Gamma_1} \hat{f}_i u_i \, d\Gamma\right\}$$
$$= F_f(\sigma). \quad (12.77)$$

We highlight that $\sigma_0 \in \tilde{V}_0$, and from the generalized inverse function theorem, any σ in an appropriate neighborhood of σ_0 also belongs to $\tilde{V}_0$ (we do not provide the details here).

From this and (12.76), we get

$$F_f(\sigma) - G^*(\sigma) \leq \alpha = J(u_0), \forall \sigma \in \tilde{V}_0. \quad (12.78)$$

Finally, observe that

$$\begin{aligned}F_f(\sigma_0) - G^*(\sigma_0) &= G(\Lambda u_0) - \langle u_0, f\rangle_{L^2(\Omega;\mathbb{R}^3)} - \int_{\Gamma_1} \hat{f}_i u_{0i}\, d\Gamma \\ &= J(u_0).\end{aligned} \tag{12.79}$$

From this and (12.78), the proof is complete.

12.4 A Numerical Example

Consider the functional $J : U \to \mathbb{R}$ defined by

$$J(u) = \frac{H}{2}\int_0^1 \left(u_x + \frac{1}{2}u_x^2\right)^2 dx - \int_0^1 Pu\, dx,$$

where

$$U = \{u \in W^{1,4}([0,1]) \mid u(0) = u(1) = 0\} = W_0^{1,4}([0,1]),$$

$$H = 10^5$$

$$P = -1000$$

where the units refer to the international system. The condition indicated in (12.45) here stands for $W^*(z^*)$ to be positive definite in a critical point $u_0 \in U$, where

$$W^*(z^*) = \frac{(z^*)^2}{K} - \frac{(z^*)^2}{H} - \frac{(u_0'(x))^2(z^*)^2}{K/2},$$

which is equivalent to

$$\frac{\partial^2 W^*(z^*)}{\partial (z^*)^2} \geq 0,$$

so that, for $K = H/2$, we get

$$(u_0'(x))^2 \leq 0.25, \text{ a.e. in } [0,1],$$

that is,

$$|u_0'(x)| \leq 0.5, \text{ a.e. in } [0,1].$$

We have computed a critical point through the primal formulation, again denoted by $u_0 \in U$. Please see Fig. 12.1. For $u_0'(x)$, see Fig. 12.2.

We may observe that

$$|u_0'(x)| \leq 0.5,$$

in $[0,1]$, so that by the main duality, such a point is a local minimum on the set $C_1 = C \cap C_2$, where

$$\begin{aligned}C &= \{u \in U \mid G_K^{**}(u_x) = G_K(u_x)\} \\ &= \{u \in U \mid H(u_x + u_x^2/2) + K/2 > 0, \text{ in } [0,1]\},\end{aligned} \tag{12.80}$$

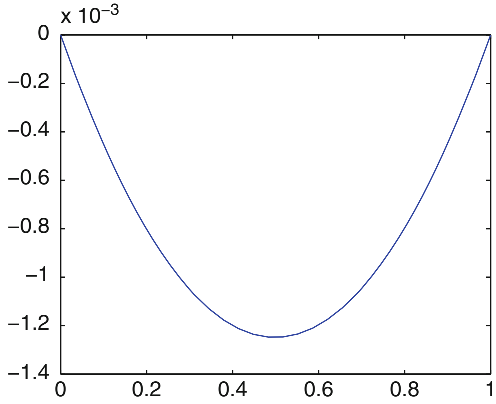

Fig. 12.1 The solution $u_0(x)$ through the primal formulation

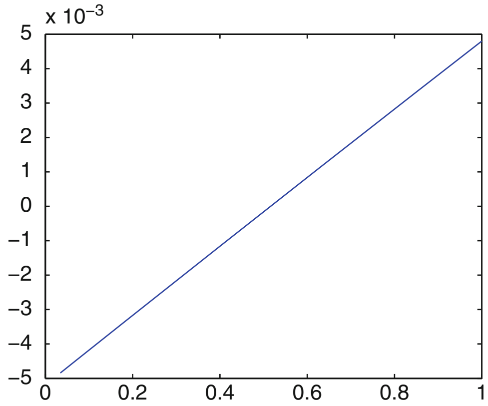

Fig. 12.2 The solution $u_0'(x)$ through the primal formulation

$C_2 = \{u \in U \mid u_x \in \hat{Y}^*\}$, where

$$G_K(u_x) = \frac{H}{2}\int_0^1 (u_x + u_x^2/2)^2\,dx + \frac{K}{4}\int_0^1 u_x^2\,dx,$$

and

$$\hat{Y}^* = \{v \in L^2([0,1]) \mid W^*(z^*) \text{ is positive definite in } [0,1]\}.$$

In fact, plotting the function $F(x) = H(x+x^2/2)^2/2$, we may observe that inside the set $[-0.5, 0.5]$ there is a local minimum, that is, in a close set, the Legendre necessary condition for a local minimum is satisfied. Please see Fig. 12.3.

We emphasize on the concerned sets there is no duality gap between the primal and dual formulations. Also, from the graphic of $u_0'(x)$, it is clear that the stress

$$H(u_0' + 1/2(u_0')^2)$$

is not exclusively positive or negative in $[0,1]$.

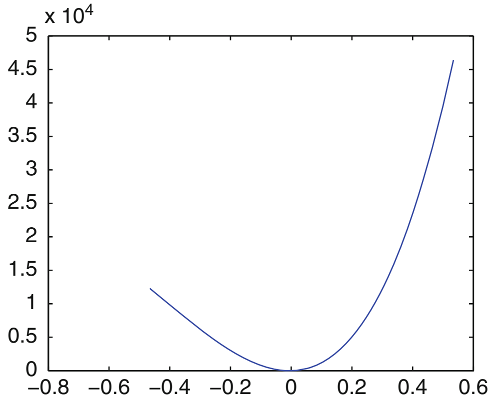

Fig. 12.3 The function $F(x) = H(x + x^2/2)^2/2$

12.5 Conclusion

In this chapter we develop new duality principles applicable to nonlinear finite elasticity. The results are obtained through the basic tools of convex analysis and include sufficient conditions of restricted optimality. It is worth mentioning that the methods developed here may be applied to many other situations, such as nonlinear models of plates and shells. Applications to related areas (specially to the shell model presented in [23]) are planned for future works.