Chapter 8
Topological Spaces

In geometry, we can decide which points are near and which are far by computing distance. In topology, no such notion is available and distance is replaced by the weaker concept of neighborhoods. A full treatment of this idea is beyond the scope of this course. Indeed, we are motivated to side-step the technical difficulties by restricting ourselves to spaces in which connectivity can be defined by elementary means. The primary tool to achieve this goal are simplicial complexes.

8.1 Topology and Topology Equivalence

Let $\mathbb{X}$ be a set of points. A *topology* of $\mathbb{X}$ is a collection of subsets, called *open sets*, such that

(i) $\mathbb{X}$ is open and the empty set is open;
(ii) the intersection of any two open sets is open;
(iii) the union of any family of open sets is open.

The set $\mathbb{X}$ together with the topology is called a *topological space*. By a *neighborhood* of a point we mean an open set that contains that point. For example, the plane together with the topology generated by the Euclidean metric is a topological space. To construct it, we call the set of points at distance less than $r > 0$ from a point $x \in \mathbb{R}^2$ an *open disk*. Taking finite intersections and arbitrary unions of open disks, we get a collection of open sets that satisfies the above three conditions. It is usually referred to as the *Euclidean topology* of the plane. Similarly, we can construct topologies for subsets of the plane. Consider for example the unit disk, $\mathbb{B}^2 = \{x \in \mathbb{R}^2 \mid \|x\| \leq 1\}$. The topology *inherited* from the topology of the plane consists of all intersections of open sets with $\mathbb{B}^2$. Thus, $\mathbb{B}^2$ together with the topology inherited from $\mathbb{R}^2$ is a topological space.

A function from one topological space to another is *continuous* if the preimage of every open set is open. This is derived from the familiar notion of continuity in calculus. A *homeomorphism* between two topological spaces is a bijective function

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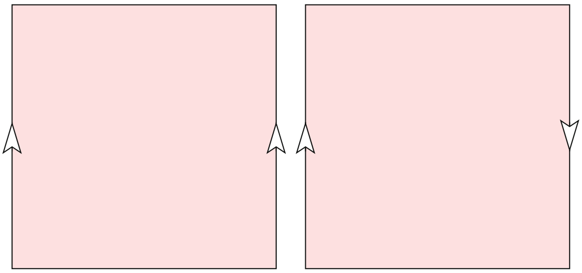

Fig. 8.1 *Left* the cylinder. *Right* the Möbius strip

$f : \mathbb{X} \to \mathbb{Y}$ such that f and f^{-1} are both continuous. If such a function exists, then $\mathbb{X}$ and $\mathbb{Y}$ are said to be *homeomorphic*, or *topologically equivalent*, or they have the same *topology type*, and this is denoted by writing $\mathbb{X} \approx \mathbb{Y}$. An *embedding* is a function $g : \mathbb{X} \to \mathbb{Y}$ whose restriction to the image, $g(\mathbb{X}) \subseteq \mathbb{Y}$, is a homeomorphism. For example, the open disk is homeomorphic to the plane. To prove this, we introduce $\mathbb{S}^1 = \{x \in \mathbb{R}^2 \mid \|x\| = 1\}$ and note that $\mathbb{B}^2 - \mathbb{S}^1$ is the prototypical open disk. The function $f : \mathbb{B}^2 - \mathbb{S}^1 \to \mathbb{R}^2$ defined by $f(x) = \frac{x}{1-\|x\|}$ is bijective and bicontinuous implying that $\mathbb{B}^2 - \mathbb{S}^1 \approx \mathbb{R}^2$. While f is a homeomorphism between the open disk and the plane, the identity defined by $g(x) = x$ is an embedding of the open disk in the plane.

The basic question in topology is to classify spaces up to topology type. For example, most coffee cups have the same type as the solid torus, which is a ball with a handle. However, this is a difficult undertaking in general, with known answers restricted to very limited situations.

8.2 2-Manifolds

Perhaps the best known family for which a complete classification into topological types is known are the compact surfaces. We define a 2-*manifold* (without boundary) as a topological space $\mathbb{X}$ for which every point $x \in \mathbb{X}$ has an open neighborhood homeomorphic to $\mathbb{R}^2$. As mentioned earlier, this is equivalent to saying that x has an open disk neighborhood. If the 2-manifold is connected and compact, then we sometimes call it a *compact surface*. Similarly, a 2-*manifold with* (possibly empty) *boundary* is a topological space $\mathbb{Y}$ for which every point $y \in \mathbb{Y}$ has a neighborhood homeomorphic to $\mathbb{R}^2$ or to $\mathbb{H}^2 = \{x = (x_1, x_2) \mid x_1 \geq 0\}$. The boundary of a 2-manifold with boundary is necessarily a 1-manifold, that is: a collection of closed curves. For example, the *cylinder* is a 2-manifold with boundary, and its boundary consists of two closed curves. As illustrated in Fig. 8.1, it can be constructed from a square by gluing the left edge to the right edge.

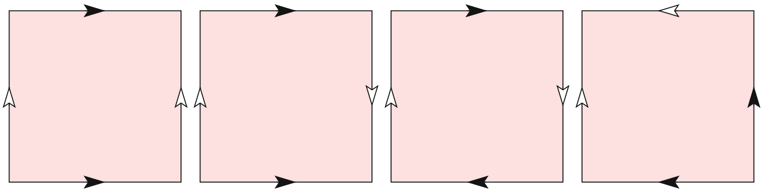

Fig. 8.2 From *left* to *right*: the torus, the Klein bottle, the projective plane, and the sphere

If we glue the same two edges with reversed orientation, as shown in Fig. 8.1 on the right, then we get the *Möbius strip*. In contrast to the cylinder, it is *non-orientable*, by which we mean that it is not possible to assign an unambiguous orientation (cw or ccw) to every point. Indeed, we can walk from a point x—standing on one side of the Möbius strip—to the same point x—standing on the other side of surface. A cw orientation observed at the first visit appears as a ccw orientation at the second visit. This implies that the surface has really only one side, and the two sides make sense only locally.

We can also glue the top to the bottom edge so that all boundary is removed and we get a compact surface. Doing this the obvious way, as shown in Fig. 8.2 on the left, we get the *torus*, and with one reversed orientation, as in the second picture, we get the *Klein bottle*. If we reverse the direction for both pairs of glued edges, as in the third picture, we get the *projective plane*. Finally, if we glue the top edge to the left, and the bottom edge to the right, as in Fig. 8.2 on the right, we get the *sphere*. The torus and the sphere can be embedded in $\mathbb{R}^3$, so we are quite familiar with their curved appearance, but the Klein bottle and the projective plane cannot, which is perhaps the reason why they are more difficult to imagine. The projective plane seems most difficult to imagine of all, so we offer an alternative construction. Starting with the sphere, $\mathbb{S}^2 = \{x \in \mathbb{R}^3 \mid \|x\| = 1\}$, we call points x and $-x$ *antipodal*. Gluing the antipodal points in pairs gives the projective plane. This recipe glues the northern hemisphere to the southern hemisphere, and it glues the equator to itself, like wrapping a hair-band twice around a pony-tail.

8.3 Classification of Compact Surfaces

We use the Euler characteristic and the orientability of a surface for classification. To define the former, we decompose the surface into triangles, making sure that any two meet in a shared edge, or a shared vertex, or not at all. Then we compute the *Euler characteristic* as the alternating sum of simplices: number of vertices minus number of edges plus number of triangles. The four triangulations shown in Fig. 8.3 all have 27 edges and 18 triangles, but they differ in the number of vertices, which from left to right is 9, 9, 10, 11. It follows that $\chi = 0, 0, 1, 2$ for the torus, the Klein bottle, the projective plane, and the sphere. Remembering that the torus and the sphere are orientable, and the Klein bottle and the projective plane are not, we use these two

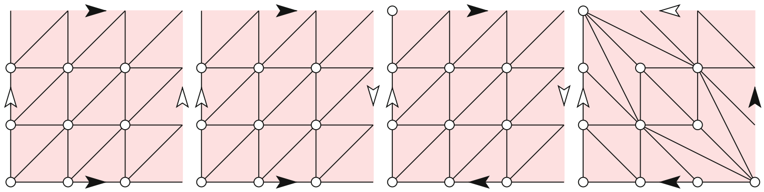

Fig. 8.3 From *left* to *right*: triangulations of the torus, the Klein bottle, the projective plane, and the sphere. Only one copy of each vertex and one copy of each edge is shown

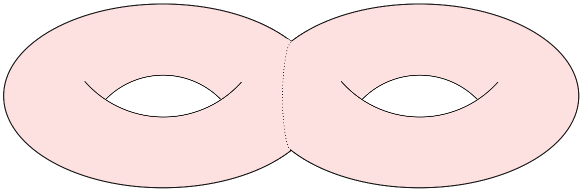

Fig. 8.4 The double torus obtained by forming the connected sum of two tori

pieces of information to tell the four surfaces apart. As observed about a century ago in [1, 2], the method extends to all compact surfaces. It is worth mentioning that such a classification is out of reach for 4-manifolds since it is undecidable whether two triangulated 4-manifolds are homeomorphic [3]. To explain this extension, we write $\mathbb{T}^2$ for the torus, $\mathbb{K}^2$ for the Klein bottle, $\mathbb{P}^2$ for the projective plane, and $\mathbb{S}^2$ for the sphere. Assuming triangulations, we can form the *connected sum* of two by removing a triangle in each and gluing their boundaries to each other; see Fig. 8.4.

Denoting this operation by #, we observe that forming the connected sum with $\mathbb{S}^2$ does not change the topological type. However, forming the connected sum with any of the other three surfaces does change the type. In all three cases, it changes the Euler characteristic, and if we start with an orientable surface and form the connected sum with the projective plane or the Klein bottle, it changes the surface from orientable to non-orientable. Define

$$T_n = \mathbb{T}^2 \# \mathbb{T}^2 \# \ldots \# \mathbb{T}^2, \tag{8.1}$$

$$P_n = \mathbb{P}^2 \# \mathbb{P}^2 \# \ldots \# \mathbb{P}^2 \tag{8.2}$$

by iterating the connected sum $n - 1 \geq 0$ times each. The Euler characteristic of the resulting spaces is $\chi(T_n) = 2 - 2n$ and $\chi(P_n) = 2 - n$. Note that all T_n are orientable and all P_n are non-orientable.

Classification Theorem *Two connected, compact 2-manifolds (without boundary) are homeomorphic iff they have the same Euler characteristic and they are both orientable or both non-orientable.*

This result amounts to saying that the sphere together with the two infinite families defined in (8.1) and (8.2) exhaust all compact surfaces. Note that it also says that $\mathbb{K}^2 \approx \mathbb{P}^2 \# \mathbb{P}^2$.

8.4 Simplicial Complexes

We are now more formal about the decompositions of surfaces into triangles. This is also a good moment to drop the restriction to two dimensions. A set of $k+1$ points, $u_0, u_1, \ldots, u_k$, is *affinely independent* if the k vectors, $u_1 - u_0, u_2 - u_0, \ldots, u_k - u_0$, are linearly independent. A *k-simplex* is the convex hull of $k+1$ affinely independent points. Writing σ for the k-simplex, we call $k = \dim \sigma$ its *dimension*, and u_0 to u_k its *vertices*. Simplices of dimension 0, 1, 2, 3 are usually referred to as *vertices*, *edges*, *triangles*, *tetrahedra*. A *face* of σ is a simplex spanned by a subset of the vertices of σ. Since a set of $k+1$ elements has $\binom{k+1}{\ell+1}$ subsets of size $\ell+1$, σ has this number of ℓ-faces, for $0 \leq \ell \leq k$. The total number of faces is therefore

$$\sum_{\ell=0}^{k} \binom{k+1}{\ell+1} = 2^{k+1} - 1, \tag{8.3}$$

the number of subsets minus 1 because we do not count the empty set. Sometimes it is convenient to call $\emptyset$ a face of σ, namely the unique (-1)-face, but we will refrain from this practice. When we triangulate a surface, we choose the triangles such that they have only proper intersections. Similarly, we define a *simplicial complex* as a finite collection of simplices, K, such that

(i) for every simplex $\sigma \in K$, every face of σ is in K;
(ii) for every two simplices $\sigma, \tau \in K$, the intersection, $\sigma \cap \tau$, is either empty or a face of both simplices.

If the intersection of two simplices is a common face, then (i) implies that it is a simplex in K. The *dimension* of K is the largest dimension of any simplex in K. A *subcomplex* is a subset of the simplices that is itself a simplicial complex. The *Euler characteristic* of K is the alternating sum of simplex numbers: $\chi(K) = s_0 - s_1 + s_2 - \ldots \pm s_k$, where k is the dimension of K and s_i is the number of i-simplices, for $0 \leq i \leq k$.

8.5 Triangulations

Until now, we have avoided any mention of the space in which the simplicial complex lives. Clearly, $k+1$ points in $\mathbb{R}^d$ can be affinely independent only if $k \leq d$. Suppose K is a simplicial complex in $\mathbb{R}^d$. The *underlying space* of K is the union of the simplices,

$$|K| = \bigcup_{\sigma \in K} \sigma, \tag{8.4}$$

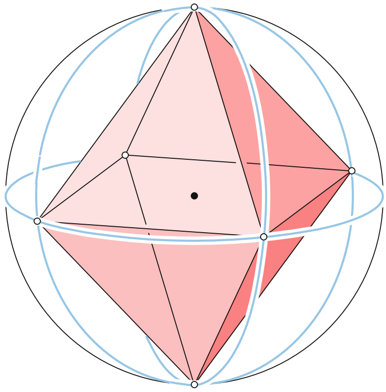

Fig. 8.5 Triangulation of the sphere. Each triangle of the octahedron projects centrally to a spherical triangle that covers an eighth of the sphere

together with the topology inherited from the Euclidean topology of $\mathbb{R}^d$. We recall that this means that the open sets in $| K |$ are the intersections of the open sets in $\mathbb{R}^d$ with the underlying space. Note that the underlying space is a set of points, while the simplicial complex is a set of simplices. Sometimes, this difference is important but more often than not, it is convenient to ignore it. A *triangulation* of a space $\mathbb{X}$ is a simplicial complex, K, whose underlying space is homeomorphic to $\mathbb{X}$.

Many times we talk about a topological space, we do so in reference to a triangulation of the space. It therefore pays off to elaborate on the concept by showing a few examples of spaces and triangulations. The simplicial complexes consisting of 18 triangles and their edges and vertices depicted in Fig. 8.3 are triangulations of the torus, the Klein bottle, the projective plane, and the sphere. We note a few pitfalls that need to be avoided when constructing a triangulation. For example, the triangulation of the square used for the torus on the left does not work for the sphere, because the identification of boundary edges would render some of the diagonal edges invalid, namely the ones for which the endpoints are the same. Similarly, the identification would render some edges the same, which is not allowed since the simplicial complex is a set and not a multi-set. Another popular method for constructing a triangulation is by projection. Take for example any convex polytope in $\mathbb{R}^3$ whose faces are all triangles, such as the octahedron. Fixing a point in the interior of the polytope as the origin, the central projection from the boundary of the polytope to the sphere is a homeomorphism; see Fig. 8.5.

References

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