

Chapter 2

Voronoi and Delaunay Diagrams

The goal of this book is the introduction of the Voronoi diagram and the Delaunay triangulation of a finite set of points in the plane, and an elucidation of their dual relationship. We begin with convex polygons, which can be constructed as the intersection of a finite number of half-planes or as the convex hull of a finite set of points.

2.1 Convex Polygons

A set X is *convex* if two points $x, y \in X$ imply that every point on the line segment connecting x and y belongs to X . For example, a disk in \mathbb{R}^2 is convex but a circle is not. Importantly, convexity is closed under taking intersections.

Lemma A *The intersection of convex sets is convex.*

Indeed, if points x and y belong to the intersection then every point of the line segment belongs to the intersection. A simple convex set is a closed *half-plane*, which consists of all points on or on one side of a line in \mathbb{R}^2 . Assuming the line does not pass through the origin, we write the half-plane that contains the origin as the set of points $x = (x_1, x_2)$ that satisfy

$$\langle x, u \rangle = x_1 u_1 + x_2 u_2 \leq 1, \tag{2.1}$$

where $u = (u_1, u_2)$ is a non-zero vector. It is a normal vector of the line and its length is one over the distance of the origin from the line. We will use points and vectors interchangeably, so u is also a point in the plane, namely the endpoint of the vector. A *convex polygon* is the intersection of finitely many half-planes. By Lemma A, this intersection is indeed convex. Assuming all these half-planes can be written as in (2.1), the polygon contains the origin in its interior, as in Fig. 2.1.

Every bounded convex polygon can alternatively be constructed by taking the *convex hull* of its vertices. This is the intersection of all convex sets that contain all

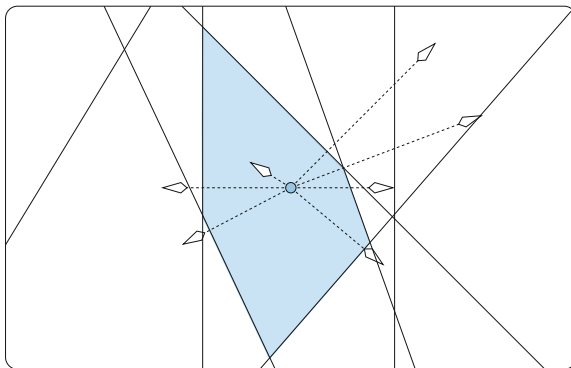


Fig. 2.1 The intersection of finitely many half-planes, each defined by its outward directed normal vector

vertices; in the plane you can visualize it as the region bounded by the rubber band that stretches around the given points.

2.2 Voronoi Diagrams

Let now S be a finite set of points in \mathbb{R}^2 . We call the elements of S *sites* in order to distinguish them from other points in the plane. Recall that the *Euclidean distance* between a point $x = (x_1, x_2)$ and a site $s = (s_1, s_2)$ is $\|x - s\| = \langle x - s, x - s \rangle^{1/2}$, which is equal to $[(x_1 - s_1)^2 + (x_2 - s_2)^2]^{1/2}$. For each site, we are interested in the region of points that are at least as close as to any other site:

$$V_s = \{x \in \mathbb{R}^2 \mid \|x - s\| \leq \|x - t\|, \forall t \in S\}, \tag{2.2}$$

calling V_s the *Voronoi region* of s . It is named after the Ukrainian mathematician Geogy Voronoi [1, 2]. The set of points x that satisfy $\|x - s\| \leq \|x - t\|$ is a closed half-plane, so V_s is the intersection of finitely many half-planes and therefore a convex polygon. Any two Voronoi regions intersect at most along their boundaries, and together, the Voronoi regions cover the entire plane. The *Voronoi diagram* of S is the set of Voronoi regions, one for each site in S ; see Fig. 2.2.

Suppose we pick a point x in the interior of the region V_s and draw a circle with center x and radius $\|x - s\|$. Then s lies on the circle and all other sites lie outside the circle. On the other hand, if x lies on the boundary of V_s then it also belongs to at least one other region, say to V_t . In this case, s and t both lie on the circle. Indeed, if we move x along the edge $V_s \cap V_t$, we get a pencil of circles, all passing through s and t . If x is an endpoint of that edge, then there is typically a third region, V_u , that contains x . In this case, s, t, u all lie on the circle around x and other other sites lie outside this circle.

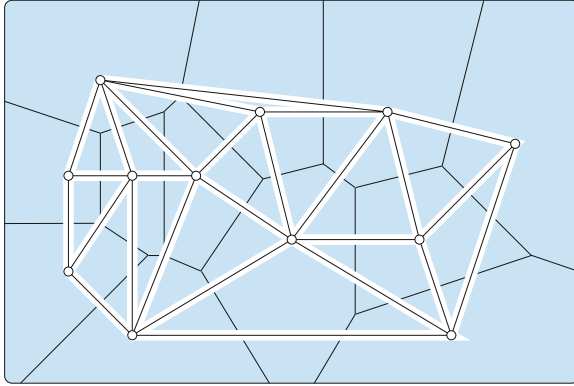


Fig. 2.2 The Voronoi diagram of a finite set of points in \mathbb{R}^2 , and the corresponding Delaunay triangulation superimposed

2.3 Delaunay Triangulations

Given the Voronoi diagram of $S \subseteq \mathbb{R}^2$, we get the *Delaunay triangulation* by connecting two sites by a straight edge whenever the corresponding two Voronoi regions share an edge [3]. Generically, the intersection of any four or more Voronoi regions is empty. Three Voronoi regions may intersect in a point, which is then the endpoint of the three edges formed by taking pairwise intersections. Correspondingly, the three sites form a triangle in the Delaunay triangulation; see Fig. 2.2. We say the points in S are in *general position* if no four points lie on a common circle. It implies that no four Voronoi regions have a non-empty common intersection. A necessary condition for any notion of general position is that violations happen with probability zero, which is the case here. However, computers are finite and work with finite precision, so violations of such conditions become likely, in particular for large data sets. Still, arbitrary small perturbations can remove the violation, and such perturbations can often be simulated without being explicitly constructed.¹

Delaunay triangulations have a number of interesting properties, some of which we now list. To explain them, we define the *star* of a site s in the Delaunay triangulation as the collection of edges and triangles that share s . The *link* of s is the collection of sites and edges in the boundary of the star that do not contain s .

Lemma B *Let S be a finite set of sites in general position in \mathbb{R}^2 .*

- (i) *Sites $s, t, u \in S$ form a triangle in the Delaunay triangulation iff all other sites lie outside the unique circle that passes through s, t, u .*
- (ii) *The triangles in the Delaunay triangulation decompose the convex hull of S .*
- (iii) *The cyclic list of sites in the link of a site s is the same as the cyclic list of edges of V_s .*

¹ Sect. 1.4 in [4]

A standard mistake is to assume that the link of every site bounds a convex polygon. This is not true.

2.4 Planar Graphs

Let n be the number of sites in S . To count the edges and vertices in the Voronoi diagram—as well as the edges and triangles in the Delaunay triangulation—we recall some basic facts about graphs that can be drawn without crossings in the plane. A *graph* consists of finitely many vertices and edges, each edge being a pair of vertices. We draw each vertex as a point and each edge as a curve connecting the points that represent its two vertices. The graph is *planar* if there exists such a drawing in which no two curves cross each other; that is, no two curves intersect other than possibly at shared endpoints. This drawing is called an *embedding* of the graph in the plane. An example is the 1-skeleton of the Delaunay triangulation, which has the extra property that each edge is drawn as a straight line segment. It is not entirely obvious but true that no two of these line segments cross.

Planar graphs cannot have substantially more edges than vertices. To make this precise, let v be the number of vertices, e the number of edges, and f the number of faces of an embedding. Similar to the Euler-Poincaré Formula for convex polyhedra, we have a linear relation provided the graph is connected.

Euler relation *A connected planar graph satisfies $v - e + f = 2$.*

Proof Draw the vertices as points in \mathbb{R}^2 . During the first phase of the construction, we add edges, one by one, always making sure that the two endpoints belong to separate components before we add the edge. After drawing $v - 1$ such edges, we reach a connected graph and therefore the end of the first phase. At this stage, we have $e = v - 1$ edges, and because there are no cycles, we have $f = 1$ face. Plugging these values into the alternating sum gives $v - (v - 1) + 1 = 2$, so the relation holds.

During the second phase, we draw the remaining edges, again one by one. Each edge decomposes a face of the current graph into two. Adding an edge thus increases e and f by 1 each. The two changes cancel, so the claimed relation is maintained. \square

2.5 Maximally Planar Graphs

A planar graph is *maximal* if adding any new edge would render the graph non-planar. It is plausible that in an embedding of a maximal planar graph, every face is bounded by exactly three edges. This is also true, and because of this property, we have 3 edges per face and 2 faces per edge. Assuming $v \geq 3$, this implies $3f = 2e$. Plugging this equation into the Euler Relation, we get

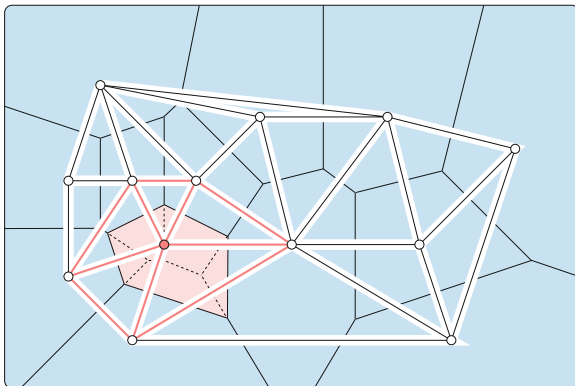


Fig. 2.3 Adding a new site to the Voronoi diagram and the corresponding Delaunay triangulation

$$e = 3v - 6, \quad (2.3)$$

$$f = 2v - 4. \quad (2.4)$$

The Delaunay triangulation can only have fewer edges and faces than the maximal planar graph for the same set of points. Writing n for the number of sites, we therefore get fewer than $3n$ edges and fewer than $2n$ triangles. Correspondingly, the Voronoi diagram of S has fewer than $3n$ edges and fewer than $2n$ vertices.

The *degree* of a vertex is the number of incident edges. Since every edge is incident to only two vertices, the sum of degrees, over all vertices, is less than $6n$. This implies that the average degree in the Delaunay triangulation is less than 6, and that there is at least one vertex of degree at most 5.

2.6 Incremental Construction

A popular algorithm for constructing the Voronoi diagram adds one site at a time. To add a site, s , we first find the Voronoi region that contains s , and second construct the Voronoi region of s by stealing pieces of the surrounding regions. The new Voronoi region is of course convex, which implies that its boundary is connected and can be computed by stepping from one neighboring region to the next, as illustrated in Fig. 2.3. The corresponding algorithm for the Delaunay triangulation removes and adds triangles. The triangles that are removed correspond to the Voronoi vertices that lie in the stolen region, which implies that their circumcircles enclose the new site. The new triangles are all incident to the new site, and they cover the same area in the plane. To implement this algorithm, it is convenient to add three dummy sites that span a triangle containing the entire set S . Starting with this triangle as the initial Delaunay triangulation, we add one site at a time:

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Let  $\mathcal{D}_0$  consist of the triangle  $xyz$ ;
for  $i = 1$  to  $n$  do
1   find the subset of triangles,  $A \subseteq \mathcal{D}_{i-1}$ ,
    whose circumcircles enclose the  $i$ -th site in  $S$ ;
2   let  $B$  contain the triangles formed by joining
    the  $i$ -th site to the boundary edges of  $\bigcup A$ ;
3    $\mathcal{D}_i = (\mathcal{D}_{i-1} - A) \cup B$ 
endfor.

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To implement this algorithm, we need a test that decides whether a point lies inside or outside the circle that passes through three other points. Such a test will be discussed during the next lecture.

2.7 Expectations

Instead of giving a detailed analysis of the run time, we just count the triangles that are constructed during the course of the algorithm. It is possible to choose the sites, and their ordering, such that this number is a constant times n^2 . However, this is not typical, which is a number that is linear in n . To substantiate this claim, we compute the average over the $n!$ orderings of the n sites. Note that there is no assumption on the configuration of the sites.

To understand this average, we count the triangles while we go backward in the ordering. In other words, we ask for the expected number of triangles constructed when we add the last, or n -th site. Every site is equally likely to be the last, so this is the average number of triangles in the star of a vertex, now averaged over all sites, which we know is less than 6. Let t_i be the expected number of triangles constructed (and possibly already destructed) after adding the first i sites. Since expectations are additive, we have

$$t_n < t_{n-1} + 6. \quad (2.5)$$

We have $t_3 = 1$ and therefore $t_n < 6n$. In words, if we add the sites in a random order, then the expected number of triangles constructed during the course of the algorithm is less than $6n$. Less than $2n$ appear in the final triangulation, which implies that roughly two thirds of the triangles are wasted. Similarly, the expected number of edges constructed by the algorithm is less than $6n$, and about half of them are wasted.

References

1. Voronoi G (1907) Nouvelles applications des paramètres continus à la théorie des formes quadratiques. *J Reine Angew Math* 133:97–178
2. Voronoi G (1908) Nouvelles applications des paramètres continus à la théorie des formes quadratiques. *J Reine Angew Math* 134:198–287

3. Delaunay B (1934) Sur la sphère vide. *Izv Akad Nauk SSSR, Otdelenie Matematicheskikh i Estestvennykh Nauk* 7:793–800
4. Edelsbrunner H (2001) *Geometry and topology for mesh generation*. Cambridge University Press, Cambridge, England