Chapter 10
Complex Construction

There are not many ways to automatically construct interesting topological spaces, and by 'interesting' we mean spaces that go beyond the designed ones, such as the ball, and the sphere. Taking the nerve of a collection of sets is one such method, and we have seen examples: the Delaunay triangulation and the alpha complex. There, we did not have to worry about the embedding in Euclidean space, but in general we do.

10.1 Abstract Simplicial Complexes

It is often convenient to talk about a simplicial complex abstractly, without reference to its realization in a Euclidean space. Suppose we have a finite collection of elements, which can be anything but we call them *vertices*. An *abstract simplicial complex* is a system of subcollection, A, such that $\alpha \in A$ and $\beta \subseteq \alpha$ imply $\beta \in A$. This is the abstract equivalent of Condition (i) for simplicial complexes. There is no abstract equivalent of Condition (ii). The sets α are called *abstract simplices*. The *dimension* of an abstract simplex is one less than its cardinality, and the *dimension* of A is the maximum dimension of any of its abstract simplices. We can draw A in Euclidean space by mapping each vertex to a point in $\mathbb{R}^d$, and mapping each abstract k-simplex to the convex hull of the $k+1$ corresponding points. If this drawing satisfies Conditions (i) and (ii) of a simplicial complex, then we call it a *geometric realization* of A. For example, a 1-dimensional abstract simplicial complex is a graph, and a geometric realization is a straight-line embedding of the graph. Given an abstract simplicial complex, we ask whether it has a geometric realization in $\mathbb{R}^d$. The answer is in the affirmative if d is large enough.

Geometric Realization Theorem *Any abstract simplicial complex of dimension k has a geometric realization in $\mathbb{R}^{2k+1}$.*

For $k = 1$, the theorem says that every graph can be geometrically realized in $\mathbb{R}^3$. Indeed, if we place the vertices in general position, with no four points lie on a common plane, then no two edges can cross. Sometimes, $2k + 1$ is best possible.

H. Edelsbrunner, *A Short Course in Computational Geometry and Topology*,
SpringerBriefs in Mathematical Methods, DOI: 10.1007/978-3-319-05957-0_10,

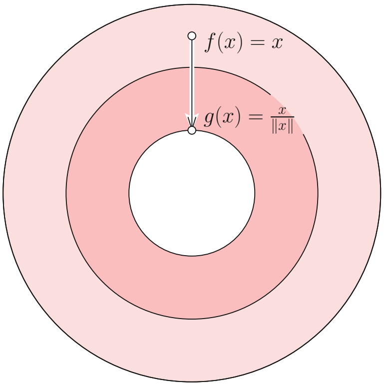

Fig. 10.1 With time, the homotopy moves $f(x)$ to $g(x)$. The half-way function, $\frac{1}{2}f + \frac{1}{2}g$, shrinks the annulus to half its width

For example, the graph with 5 vertices and all 10 edges cannot be drawn without crossing in $\mathbb{R}^2$. Hence, $2k + 1 = 3$ dimensions are necessary to geometrically realize this graph. A generalization of this example to a k-dimensional abstract simplicial complex that has no geometric realization in $\mathbb{R}^{2k}$ can be found in [1].

10.2 Homotopy

We may compare two spaces by deforming one to the other, without gluing or cutting. We begin with deforming maps. Let $f, g : \mathbb{X} \to \mathbb{Y}$ be continuous maps from one topological spaces to another. A *homotopy* between f and g is a continuous map $H : \mathbb{X} \times [0, 1] \to \mathbb{Y}$ for which $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in \mathbb{X}$. The two functions are *homotopic* if a homotopy exists. It is reasonable to think of the second variable as time, going from 0 to 1, and to consider the 1-parameter family of functions $f_t : \mathbb{X} \to \mathbb{Y}$ defined by $f_t(x) = H(x, t)$, which interpolates between $f_0 = f$ and $f_1 = g$.

As example, we consider two functions from the annulus to itself. Let $\mathbb{X} = \{x \in \mathbb{R}^2 \mid 1 \leq \|x\| \leq 3\}$, and define $f, g : \mathbb{X} \to \mathbb{X}$ by setting $f(x) = x$ and $g(x) = x/\|x\|$; see Fig. 10.1. To show that f and g are homotopic, we construct $H : \mathbb{X} \times [0, 1] \to \mathbb{X}$ by setting

$$H(x, t) = (1 - t)f(x) + tg(x). \tag{10.1}$$

We have $f_0 = f$ and $f_1 = g$ by construction. It is also clear that H is well defined, and that it is continuous.

10.3 Homotopy Equivalence

To compare two spaces, we construct two homotopies, both involving the identity map. Specifically, $\mathbb{X}$ and $\mathbb{Y}$ are *homotopy equivalent* if there exist continuous functions $a : \mathbb{X} \to \mathbb{Y}$ and $b : \mathbb{Y} \to \mathbb{X}$ such that $b \circ a$ is homotopic to $\mathrm{id}_{\mathbb{X}} : \mathbb{X} \to \mathbb{X}$ and $a \circ b$ is homotopic to $\mathrm{id}_{\mathbb{Y}} : \mathbb{Y} \to \mathbb{Y}$. In this case, we also say that $\mathbb{X}$ and $\mathbb{Y}$ have the same *homotopy type*, and we write $\mathbb{X} \simeq \mathbb{Y}$. We already did most of the work to show that the annulus, $\mathbb{X}$, has the same homotopy type as the circle, $\mathbb{Y} = \mathbb{S}^1$. Let a map the point $x \in \mathbb{X}$ to the point $a(x) = x/\|x\|$ in $\mathbb{Y}$, and let b be the canonical embedding of the circle in the annulus, that is: $b(y) = y \in \mathbb{X}$. We proved earlier that $g = b \circ a$ and $\mathrm{id}_{\mathbb{X}}$ are homotopic. To see the other direction, we note that $f = a \circ b$ is the identity on the circle: $f = \mathrm{id}_{\mathbb{Y}}$, which is stronger than f and $\mathrm{id}_{\mathbb{Y}}$ being homotopic. It follows that the annulus has the homotopy type of the circle. Even simpler than the circle is the point, and a space that has the homotopy type of the point is said to be *contractible*. For example, the disk is contractible, and so is every tree.

We note that being homotopy equivalent is weaker than being homeomorphic. Indeed, the annulus and the circle have the same homotopy type but not the same topology type. The two notions are different because the former permits a local change of dimension while the latter does not. On the other hand, having isomorphic homology groups is yet weaker than having the same homotopy type:

$$\mathbb{X} \approx \mathbb{Y} \Longrightarrow \mathbb{X} \simeq \mathbb{Y} \quad \Longrightarrow \quad \mathsf{H}_p(\mathbb{X}) \simeq \mathsf{H}_p(\mathbb{Y}), \tag{10.2}$$

for all dimensions p. To compute the Betti numbers of $\mathbb{X}$, we may therefore find a homotopy equivalent space $\mathbb{Y}$ and compute its Betti numbers.

10.4 Nerves

We are now ready to introduce the announced construction of simplicial complexes from arbitrary collections of sets. We prefer finite, so let X be a finite collection of sets. The *nerve* of X is the system of subcollections of X whose sets have a non-empty common intersection:

$$\mathrm{Nrv}\, X = \left\{ \emptyset \neq V \subseteq X \mid \bigcap V \neq \emptyset \right\}. \tag{10.3}$$

Here, we use a short-form for taking the intersection of all sets in a collection: $\bigcap V = \bigcap_{v \in V} v$. We note that the nerve is an abstract simplicial complex because $V \in \mathrm{Nrv}\, X$ and $U \subseteq V$ implies $U \in \mathrm{Nrv}\, X$. To make this more clear, we call the singleton sets in the nerve abstract vertices, the pairs abstract edges, etc.

As mentioned earlier, the Delaunay triangulation of a point set is the nerve of the collection of Voronoi regions. More precisely, the Delaunay triangulation is the geometric realization of the nerve obtained by mapping each Voronoi region to the point

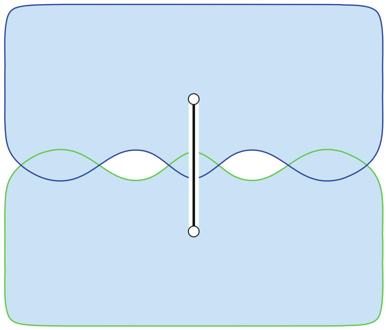

Fig. 10.2 Two contractible regions whose union is a disk with two holes. The nerve consists of two vertices connected by an edge, which has a different homotopy type

that generates the region. Here it is important that the points are in general position, else the dimension of the nerve can be higher than that of the ambient Euclidean space. A geometric realization in this space would then be impossible. However, we can always go to higher-dimensional Euclidean spaces to construct geometric realizations. Independent of the realization, the nerve has the same homotopy type as the union.

Nerve Theorem *If all sets in* X *are closed and triangulable, and all non-empty common intersections of the sets are contractible, then* $\text{Nrv}\, X$ *and* $\bigcup X$ *have the same homotopy type.*

If all sets in X are convex, then their non-empty common intersections are convex and therefore contractible. This is the situation for restricted Voronoi regions, so the Nerve Theorem implies that the alpha complex has the same homotopy type as the union of balls it represents. It follows that it has isomorphic homology groups and therefore the same Betti numbers. If the sets are not convex then be aware that the contractibility of the sets themselves is not sufficient to conclude homotopy equivalence. To get an example, it suffices to have two contractible regions overlap in multiple locations, as in Fig. 10.2. We note that the example is similar to Fig. 3.2, which shows two regions in an Apollonius diagram.

10.5 Čech Complexes

Suppose we simplify the construction of the alpha complex by intersecting the disks directly, without first restricting them to the corresponding Voronoi regions. To describe this more formally, let S be a finite set of points in $\mathbb{R}^2$, let $r \geq 0$ be a real number, and write $D_s(r)$ for the disk with center s and radius r, as before. The *Čech complex* is isomorphic to the nerve of the disks:

$$\check{\text{C}}\text{ech}(r) = \{\emptyset \neq T \subseteq S \mid \bigcap_{s \in T} D_s(r) \neq \emptyset\}. \tag{10.4}$$

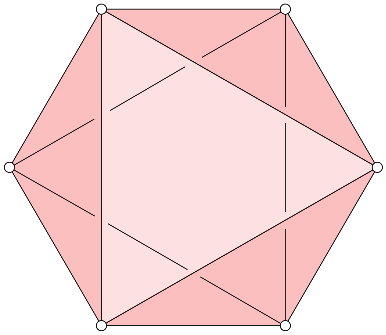

Fig. 10.3 The Vietoris-Rips complex of six points equally spaced on the unit circle and a parameter $\sqrt{3}/2 < r < 1$. It consists of eight triangles connected like the faces of an octahedron

Even if the points are in general position, the radius may be large enough so that the complex has dimension larger than 2. For this to happen, it suffices that four disks have a non-empty common intersection, such as in Fig. 7.2. Nevertheless, the Nerve Theorem implies that the Čech complex has the same homotopy type as the union of disks. Similarly, the Nerve Theorem implies that the alpha complex for radius r, denoted as $A(r)$, has the same homotopy type as the union of disks. Since homotopy equivalence is transitive—every equivalence relation is—this implies $|\,A(r)\,| \simeq |\,\check{\text{C}}\text{ech}(r)\,|$. Note also that $A(r)$ is isomorphic to a subcomplex of $\check{\text{C}}\text{ech}(r)$, simply because the restricted disks are subsets of the unrestricted disks.

It might be interesting to study the structure of the extra simplices in the Čech complex. How does it relate to the substitution method that reduces a long Pie formula to a short Pie formula; see Chap. 7.

10.6 Vietoris-Rips Complexes

To construct the Čech complex, we need to test whether a collection of disks has a non-empty intersection, which can be difficult or, in some metric spaces, impossible. We now define a complex that needs only the distances between the points in S for its construction. Letting $r \geq 0$ be a real number, the *Vietoris-Rips complex* of S and r, denoted as Vietoris-Rips(r), consists of all abstract simplices in 2^S whose vertices are at most a distance $2r$ from one another. In other words, we connect any two vertices at distance at most $2r$ from each other by an edge, and we add a triangle or higher-dimensional simplex to the complex if all its edges are in the complex; see Fig. 10.3.

While the Vietoris-Rips complex is easy to construct, it generally does not have the homotopy type of the union of disks of radius r. Indeed, for the 6 points in Fig. 10.3, the disks with radius r form an annulus, which has $\beta_0 = \beta_1 = 1$ and $\beta_p = 0$ for all $p \neq 0, 1$. In contrast, the Vietoris-Rips complex triangulates a 2-sphere, which has $\beta_0 = \beta_2 = 1$ and $\beta_1 = 0$. This example suggests that Vietoris-Rips complexes can have topological artifacts that do not show up in the data. While this is true, the artifacts are limited.

Vietoris-Rips Lemma *Let S be a finite set in* $\mathbb{R}^2$*. Then* $\hat{C}ech(r) \subseteq Vietoris\text{-}Rips(r)$ $\subseteq$ *Čech*(r'), *where* r' is $\frac{2\sqrt{3}}{3} = 1.154\ldots$ t*imes* r.

Proof The first containment is obvious because Čech(r) and Vietoris-Rips(r) have the same edges. For the second containment, we note that the equilateral triangle with edges of length $2r$ has a circumcircle of radius $r' = \frac{2\sqrt{3}r}{3}$. If three points lin on this circle, then their disks of radius r' have a non-empty common intersection, so the triangle belongs to Čech(r'). The longest of the three edges connecting the points has length at least $2r$, implying that all triangles in Vietoris-Rips(r) have a circumcircle of radius at most r'. This implies the second claimed containment. □

Reference

1. Flores A (1933/34) Über n-dimensionale Komplexe die in $\mathbb{R}_n$ selbstverschlungen sind. Ergeb Math Koll 6:4–7