Chapter 1
Roots of Geometry and Topology

Geometric questions have been pondered by people for thousands of years. In contrast, the abstraction to topological questions is only a few hundred years old. According to Galileo Galilei (1623), philosophy is written in a grand book—the Universe—which cannot be understood unless one first learns to comprehend its language, which is mathematics, and its characters, which are triangles, circles, and other geometric figures. These characters will figure prominently throughout this course.

1.1 Platonic Solids

A *convex polyhedron* is the intersection of finitely many closed half-spaces. As a practical exercise, we can build one by slicing off pieces of an apple with straight cuts of a knife. If we do this carefully, we can arrange that all faces are regular polygons of the same size and type, and that all vertices are endpoint of the same number of edges. Examples of such polyhedra with non-empty volume are the tetrahedron, the cube, the octahedron, the dodecahedron, and the icosahedron; see Fig. 1.1. There are only these five examples. They are referred to as *regular polytopes* or as *Platonic solids*, named after then Greek philosopher Plato,[1] who theorized that the elements were constructed from them.

Euclid gave a full description of the five Platonic solids in Book XIII of the Elements [1]. It is interesting that the face vector of the octahedron is the reverse of that of the cube. This is a result of the duality between them: we can map the vertices, edges, faces of the octahedron bijectively to the faces, edges, vertices of the cube so that incidences are preserved. Similarly, there is such a mapping from the dodecahedron, with face vector (20, 30, 12), to the icosahedron, with face vector (12, 30, 20). The vector of the tetrahedron is a palindrome, and there is an incidence preserving map that reverses dimensions to itself.

[1] Born in 428 BC, he was a student of Socrates, and became one of the most influential thinkers of all times.

H. Edelsbrunner, *A Short Course in Computational Geometry and Topology*,
SpringerBriefs in Mathematical Methods, DOI: 10.1007/978-3-319-05957-0_1,

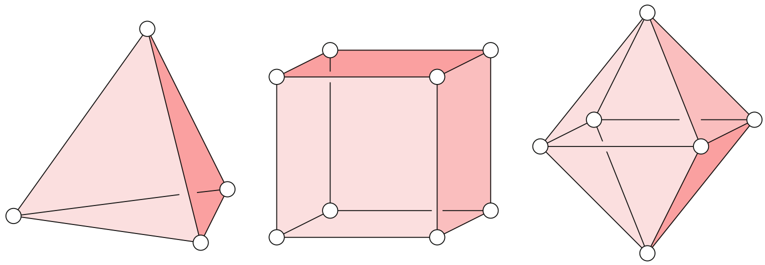

Fig. 1.1 Three of the five Platonic solids together with their face vectors. From *left* to *right*: the tetrahedron, the cube, and the octahedron

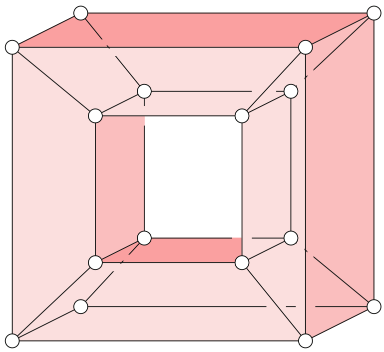

Fig. 1.2 The 'window-frame' polyhedron has a hole through the middle

1.2 Euler Formula

Another pattern we observe for the Platonic solids is that the alternating sum of the face numbers is always the same:

$$\#\text{vertices} - \#\text{edges} + \#\text{faces} = 2. \tag{1.1}$$

This equation hold more generally for bounded convex polyhedra. The relation is originally due to Leonhard Euler and is widely considered the starting point of topology: it is a global statement and it does not depend on the precise geometric shape. Indeed, it does not depend on the convexity of the polyhedron either, but it becomes false if we generalize the class of objects too far. For example, $\#\text{vertices} - \#\text{edges} + \#\text{faces} = 0$ for the 'window-frame' in Fig. 1.2. Indeed, it took more than a century to find a satisfying framework that includes Euler's original observation as a special case and elucidates why and when the relation holds [2]. This generalization is due to Henri Poincaré, which is the reason why the more general result is referred

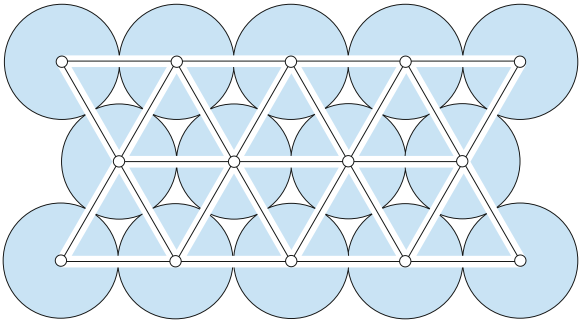

Fig. 1.3 Portion of a densest packing of disks in the plane

to as the Euler-Poincaré formula. It relates the alternating sums of face numbers and Betti numbers:

$$\sum_{i \geq 0} (-1)^i f_i = \sum_{i \geq 0} (-1)^i \beta_i, \tag{1.2}$$

where f_i is the number of i-dimensional faces, and β_i is the ith Betti number. It would be asking too much to explain the meaning of the Betti numbers now, but we will find out in due course. The common value of the two alternating sums is also known as the *Euler* or *Euler-Poincaré characteristic* of the polyhedron.

1.3 Disk Packings

Consider next the question of packing as many disks in a given area as possible. We assume the disks are all the same size, and they can touch each other but not overlap. Arranging pennies on a flat table is a good example. The question is generally difficult because the answer depends on the shape of the area, but it becomes easier when we pack disks in the infinite plane. It is intuitively obvious that the best packing is the arrangement in which the disk centers form the regular hexagonal grid; see Fig. 1.3, and this is also true.

To justify this assertion, we compute the *packing density*, which is the percentage of the plane covered by disks. Since the pattern is the same everywhere, we can just compute the covered fraction of an equilateral triangle spanned by the centers of three mutually touching disks. Assuming the distance between any two of the three points is 2, the area of the triangle is $\sqrt{3}$. There are three disks, each covering a

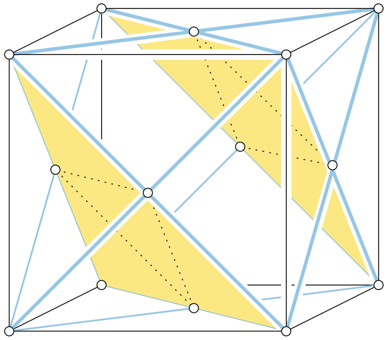

Fig. 1.4 We have a sphere centered at each vertex and at the center of each face of the cube

portion of the triangle with one sixth of its area. It follows the packing density of the hexagonal grid is

$$\varrho_2 = \frac{\pi}{2\sqrt{3}} \quad = \quad 90.68...\%. \tag{1.3}$$

Every other packing of disks in the plane covers at most that fraction of the plane.

1.4 Sphere Packings

The physically more relevant question is how well spheres can be packed in three dimensional space, which is significantly more difficulty than for disks in the plane. The answer has been known since Johannes Kepler, who worked on the problem and stated the conjecture in 1611, but a proof has been completed only recently [3]. The densest packing is perhaps obvious: arrange a layer of spheres in a hexagonal grid, as in Fig. 1.3, and put on top another such layer, resting in the crevices of the first layer. A third layer is added on top of the second layer, and so on. Drawing the layers as diagonal planes of the cube, we notice that one such configuration is obtained by placing the centers of the sphere at the corners and the face centers of a cube tiling; see Fig. 1.4. This gives us a short-cut to computing the packing density of the arrangement, which is the fraction of the cube covered by the spheres (or rather, balls). Assuming the spheres have unit radius, the edges of the cube have length $2\sqrt{2}$. The cube overlaps with one eight of each corner sphere and one half of each face center sphere. The total volume of these pieces is four times the volume of a unit sphere, which is $16\pi/3$. The cube itself has volume $16\sqrt{2}$. The packing density is therefore

$$\varrho_3 = \frac{\pi}{3\sqrt{2}} \quad = \quad 74.04...\%. \tag{1.4}$$

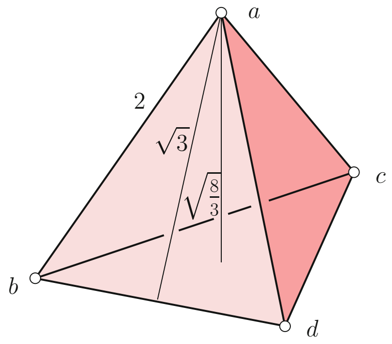

Fig. 1.5 A regular tetrahedron with edges of length 2 has height $\sqrt{8/3}$. Its four triangles have height $\sqrt{3}$ each

1.5 Space Filling

To illustrate the difficulty of the sphere packing question, we note a false belief that goes back to Aristotle,[2] namely that copies of the regular tetrahedron can be used to tile the 3-dimensional Euclidean space. If this were true, then we could compute the best packing density that can be achieved in $\mathbb{R}^3$ by calculating the percentage of the tetrahedron covered by the spheres centered at its four vertices. Take the tetrahedron with vertices $a = (\sqrt{2}, 0, 0, 0)$, $b = (0, \sqrt{2}, 0, 0)$, $c = (0, 0, \sqrt{2}, 0)$, $d = (0, 0, 0, \sqrt{2})$ in $\mathbb{R}^4$, and notice that its six edges all have length 2. The height of each triangular face is the distance between a and $\frac{1}{2}(b + c)$, which is $\sqrt{3}$, and the height of the tetrahedron is the distance between a and $\frac{1}{3}(b+c+d)$, which is $\sqrt{8/3}$; see Fig. 1.5. To compute the volume, we take the area of a triangle, which is $\sqrt{3}$, times the height over 3, which gives $\sqrt{8}/3 = 0.94....$ The dihedral angle between any two faces of the tetrahedron is $2\alpha = 2\arctan(1/\sqrt{2})$, which is about 70.52°. From this, we get the solid angle at any vertex as $6\alpha - \pi = 0.55...$; this is the surface area of the portion of the sphere inside the tetrahedron. The total volume of the four spheres inside the tetrahedron is 4/3 times as much. The fraction of the tetrahedron covered by the four spheres is therefore

$$\frac{4(6\arctan\frac{1}{\sqrt{2}} - \pi)}{\sqrt{8}} = 77.96...\%. \tag{1.5}$$

We see that this percentage is higher than ϱ_3 given in (1.4). It follows that the packing in layers is locally not optimal. This is the deeper reason why proving its global optimality has been so difficult.

---

[2] He was born in 384 BC, studied under Plato, and was the tutor of Alexander the Great.

References

1. Heath TL (eds) (1956) Euclid. The thirteen books of Euclid's elements, 2nd unabr edn. Dover, England
2. Lakatos I (1976) Proofs and Refutations: the logic of mathematical discovery. Cambridge University Press, Cambridge, England
3. Hales T (2005) A proof of the Kepler conjecture. Ann Math 162:1065–1185