Chapter 8
Direct Integrals

Direct integrals are a generalization of direct sums. For a compact group every representation is a direct sum of irreducibles. This property fails in general for non-compact groups. The best one can get for general groups is a direct integral decomposition into factor representations. The latter is a notion more general than irreducibility. For nice groups these notions coincide, and then every unitary representation is a direct integral of irreducible representations.

8.1 Von Neumann Algebras

Let H be a Hilbert space. For a subset M of the space of bounded operators $\mathcal{B}(H)$ on H, define the *commutant* to be

$$M^\circ \stackrel{\text{def}}{=} \{T \in \mathcal{B}(H) : Tm = mT \ \forall m \in M\}.$$

So the commutant is the centralizer of M in $\mathcal{B}(H)$. If $M \subset N \subset \mathcal{B}(H)$, then $N^\circ \subset M^\circ$. We write $M^{\circ\circ}$ for the *bi-commutant*, i.e., the commutant of M°. For a subset M of $\mathcal{B}(H)$ we define its *adjoint set* to be the set M^* of all adjoints m^* where m is in M. The set M is called a *self-adjoint set* if $M = M^*$.

We define a *von Neumann algebra* to be a sub-*-algebra $\mathcal{A}$ of $\mathcal{B}(H)$ that satisfies $\mathcal{A}^{\circ\circ} = \mathcal{A}$. A von Neumann algebra is closed in the operator norm, and so every von Neumann algebra is a C^*-algebra. The converse does not hold (See Exercise 8.6).

For a subset $M \subset \mathcal{B}(H)$, one has $M \subset M^{\circ\circ}$ and hence $M^{\circ\circ\circ} \subset M^\circ$. Since, on the other hand, also $M^\circ \subset (M^\circ)^{\circ\circ} = M^{\circ\circ\circ}$, it follows $M^\circ = M^{\circ\circ\circ}$, so M° is a von Neumann algebra if M is a self-adjoint set. In particular, for a self-adjoint set M the algebra $M^{\circ\circ}$ is the smallest von Neumann algebra containing M, called the *von Neumann algebra generated by M*.

Let $\mathcal{A} \subset \mathcal{B}(H)$ be a von Neumann algebra. Then $Z(\mathcal{A}) = \mathcal{A} \cap \mathcal{A}^\circ$ is the *center* of $\mathcal{A}$, i.e., the set of elements a of $\mathcal{A}$ that commute with every other element of $\mathcal{A}$. A von Neumann algebra $\mathcal{A}$ is called a *factor* if the center is trivial, i.e., if $Z(\mathcal{A}) = \mathbb{C}\,\mathrm{Id}$.

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Examples 8.1.1

- $\mathcal{A} = \mathcal{B}(H)$ is a factor, this is called a *type-I factor*.
- $\mathcal{A} = \mathbb{C}\,\mathrm{Id}$ is a factor.
- The algebra of diagonal matrices in $\mathrm{M}_2(\mathbb{C}) \cong \mathcal{B}(\mathbb{C}^2)$ is a von Neumann algebra, which is not a factor.
- Let V, W be two Hilbert spaces. The algebra $\mathcal{B}(V) \otimes \mathcal{B}(W)$ acts on the Hilbert tensor product $V\hat{\otimes}W$ via $A \otimes B(v \otimes w) = A(v) \otimes B(w)$. Then the von Neumann algebra generated by the image of $\mathcal{B}(V) \otimes \mathcal{B}(W)$ is the entire $\mathcal{B}(V\hat{\otimes}W)$ (See Exercise 8.2).

8.2 Weak and Strong Topologies

Let H be a Hilbert space. On $\mathcal{B}(H)$ one has the topology induced by the operator norm, called the *norm topology*. There are other topologies as well. For instance, every $v \in H$ induces a seminorm on $\mathcal{B}(H)$ through $T \mapsto \|Tv\|$. The topology given by this family of seminorms is called the *strong topology* on $\mathcal{B}(H)$. Likewise, any two $v, w \in H$ induce a seminorm by $T \mapsto |\langle Tv, w\rangle|$. The topology thus induced is called the *weak topology*. It is clear that norm convergence implies strong convergence and that strong convergence implies weak convergence. Therefore, for a set $\mathcal{A} \subset \mathcal{B}(H)$ one has

$$\mathcal{A} \subset \overline{\mathcal{A}}^n \subset \overline{\mathcal{A}}^s \subset \overline{\mathcal{A}}^w,$$

where $\overline{\mathcal{A}}^n$ denotes the closure of $\mathcal{A}$ in the norm topology, or norm closure, $\overline{\mathcal{A}}^s$ the strong closure, and $\overline{\mathcal{A}}^w$ the weak closure. In general, these closures will differ from each other. It is easy to see that $\overline{\mathcal{A}}^s, \overline{\mathcal{A}}^w \subset \mathcal{A}^{\circ\circ}$ since multiplication in $\mathcal{B}(H)$ is easily seen to be separately continuous with respect to the weak topology. Hence every von Neumann algebra is strongly and weakly closed.

Theorem 8.2.1 (von Neumann's Bicommutant Theorem). *Let H be a Hilbert space, and let $\mathcal{A}$ be a unital *-subalgebra of $\mathcal{B}(H)$. Then $\overline{\mathcal{A}}^s = \overline{\mathcal{A}}^w = \mathcal{A}^{\circ\circ}$.*

Proof It suffices to show that $\mathcal{A}^{\circ\circ} \subset \overline{\mathcal{A}}^s$. Let $T \in \mathcal{A}^{\circ\circ}$. We want to show that T lies in the strong closure of $\mathcal{A}$. A neighborhood base of zero in the strong topology is given by the system of all sets of the form $\{S \in \mathcal{B}(H) : \|Sv_j\| < \varepsilon,\ j = 1, \dots, n\}$ where $v_1, \dots, v_n$ are arbitrary vectors in H and $\varepsilon > 0$. So it suffices to show that for given $v_1, \dots, v_n \in H$ and $\varepsilon > 0$ there is $a \in \mathcal{A}$ with $\|Tv_j - av_j\| < \varepsilon$ for $j = 1, \dots n$. For this let $\mathcal{B}(H)$ act diagonally on H^n. The commutant of $\mathcal{A}$ in $\mathcal{B}(H^n)$ is the algebra of all $n \times n$ matrices with entries in $\mathcal{A}^\circ$, and the bicommutant of $\mathcal{A}$ in $\mathcal{B}(H^n)$ is the algebra $\mathcal{A}^{\circ\circ}I$, where $I = I_n$ denotes the $n \times n$ unit matrix. Consider the vector $v = (v_1, \dots, v_n)^t$ in H^n. The closure of $\mathcal{A}v$ in H^n is a closed, $\mathcal{A}$-stable subspace of H^n. As $\mathcal{A}$ is a *-algebra, the orthogonal complement $(\mathcal{A}v)^\perp$ is $\mathcal{A}$-stable as well;

therefore the orthogonal projection P onto the closure of $\mathcal{A}v$ is in the commutant of $\mathcal{A}$ in $\mathcal{B}(H^n)$. It follows that $T \in \mathcal{A}^{\circ\circ} I$ commutes with P and leaves $\overline{\mathcal{A}v}$ stable. One concludes $Tv \in \overline{\mathcal{A}v}$, and so there is, to given $\varepsilon > 0$, an element a of $\mathcal{A}$ such that $\|Tv - av\| < \varepsilon$, which implies the desired $\|Tv_j - av_j\| < \varepsilon$ for $j = 1, \dots, n$. □

The Bicommutant Theorem says that for a *-subalgebra $\mathcal{A}$ of $\mathcal{B}(H)$ the von Neumann algebra generated by $\mathcal{A}$ equals the weak or strong closure of $\mathcal{A}$.

Lemma 8.2.2 *A von Neumann algebra $\mathcal{A}$ is generated by its unitary elements.*

Proof Let $\mathcal{A}$ be a von Neumann algebra in $\mathcal{B}(H)$. Let $\mathcal{A}_{\mathbb{R}}$ be the real vector space of self-adjoint elements, then $\mathcal{A} = \mathcal{A}_{\mathbb{R}} + i\mathcal{A}_{\mathbb{R}}$. Let $T \in \mathcal{A}_{\mathbb{R}}$, and let $f \in \mathcal{S}(\mathbb{R})$ be such that $f(x) = x$ for x in the (bounded) spectrum of T (see Exercise 8.1). By Proposition 5.1.2,

$$T = f(T) = \int_{\mathbb{R}} \hat{f}(y) e^{2\pi i y T}\, dy.$$

The unitary elements $e^{2\pi i y T} \in \mathcal{B}(H)$ are power series in T, so belong to the von Neumann algebra $\mathcal{A}$, and every operator that commutes with the $e^{2\pi i y T}$ will commute with T, so T belongs to the von Neumann algebra generated by the unitaries $e^{2\pi i y T}$.

Let B_1 be the unit ball in $\mathcal{B}(H)$, i.e., the set of all $T \in \mathcal{B}(H)$ with $\|T\|_{\text{op}} \leq 1$.

Lemma 8.2.3 *B_1 is weakly compact.*

Proof For $r \geq 0$ and $z \in \mathbb{C}$ let $\bar{B}_r(z)$ be the closed ball around z of radius r. For $T \in B_1$ and $v, w \in H$, one has $|\langle Tv, w\rangle| \leq \|v\|\|w\|$, so the map

$$\psi : B_1 \to \prod_{v,w \in H} \bar{B}_{\|v\|\|w\|}(0)$$

with $\psi(T)_{v,w} = \langle Tv, w\rangle$ embeds B_1 into the Hausdorff space on the right, which is compact by Tychonov's Theorem A.7.1. The weak topology is induced by ψ, so B_1 is weakly compact if we can show that the image of ψ is closed. We claim that this image equals the set A of all elements x of the product such that $(v, w) \mapsto x_{v,w}$ is linear in v and conjugate linear in w. Since convergence in the product space is component-wise, this set is closed. Given $x \in A$ and $w \in H$, the map $\alpha_v : w \mapsto \overline{x_{v,w}}$ is a linear functional on H with $\|\alpha_v\| \leq \|v\|$ and hence there exists an element $Tv \in H$ such that $x_{v,w} = \langle Tv, w\rangle$ for all $w \in H$. One then checks that $v \mapsto Tv$ defines an element in B_1 such that $\psi(T) = x$. □

8.3 Representations

A unitary representation (π, V_π) of a locally compact group G is called a *factor representation* if the von Neumann algebra $\mathrm{VN}(\pi)$ generated by $\pi(G) \subset \mathcal{B}(V_\pi)$ is a factor. So π is a factor representation if and only if $\pi(G)^\circ \cap \pi(G)^{\circ\circ} = \mathbb{C}\,\mathrm{Id}$.

Lemma 8.3.1 *Every irreducible representation is a factor representation.*

Proof It follows from the Lemma of Schur 6.1.7 that $\mathrm{VN}(\pi) = \mathcal{B}(V_\pi)$ for every irreducible representation π. □

Definition Two unitary representations π_1, π_2 of G are called *quasi-equivalent* if there is an isomorphism of *-algebras

$$\phi : \mathrm{VN}(\pi_1) \to \mathrm{VN}(\pi_2)$$

satisfying $\phi(\pi_1(x)) = \pi_2(x)$ for every $x \in G$.

Example 8.3.2 A given unitary representation π is quasi-equivalent to the direct sum representation $\pi \oplus \pi$. This follows from the general fact that any von-Neumann algebra $\mathcal{A} \subset \mathcal{B}(H)$ is isomorphic to $\mathcal{A}\begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \subseteq \mathcal{B}\left(H^2\right)$. (Compare with the proof of von Neumann's Bicommutant Theorem.)

Lemma 8.3.3 *Two irreducible unitary representations of a locally compact group are quasi-equivalent if and only if they are unitarily equivalent.*

Proof Let the unitary representations (π, V_π) and (η, V_η) be unitarily equivalent, i.e., there is a unitary intertwining operator $T : V_\pi \to V_\eta$. Then T induces an isomorphism $\mathrm{VN}(\pi) \to \mathrm{VN}(\eta)$ by mapping S to TST^{-1}. This shows that π and η are also quasi-equivalent. Conversely, let (π, V_π) and (η, V_η) be two irreducible unitary representations of G, and let $\phi : \mathrm{VN}(\pi) \to \mathrm{VN}(\eta)$ be an isomorphism of C^*-algebras such that $\phi(\pi(x)) = \eta(x)$ for all $x \in G$. For $u, v \in V_\pi$ let $T_{u,v} : V_\pi \to V_\pi$ be given by $T_{u,v}(x) \stackrel{\text{def}}{=} \langle x, u\rangle v$. Then $T_{u,v}T_{w,z} = \langle z, u\rangle T_{w,v}$, and $T_{u,v}^* = T_{v,u}$. Let $(e_j)_{j\in I}$ be an orthonormal basis of V_π. For each $j \in I$ the map $P_j = T_{e_j,e_j}$ is the orthogonal projection onto the one dimensional space $\mathbb{C}e_j$ and T_{e_j,e_k} is an isometry from $\mathbb{C}e_j$ to $\mathbb{C}e_k$ and is zero on $\mathbb{C}e_i$ for $i \neq j$. The P_j are pairwise orthogonal projections that add up to the identity in the strong topology. The same holds for the images $\phi(P_j)$. Let $V_{\eta,j} = \phi(P_j)V_\eta$. Then V_η is the direct orthogonal sum of the $V_{\eta,j}$. We claim that $\phi(T_{e_j,e_k})$ is an isometry from $V_{\eta,j}$ to $V_{\eta,k}$ and zero on $V_{\eta,i}$ for $i \neq j$. For this let $x, y \in V_{\eta,j}$, then

$$\begin{aligned}\left\langle \phi(T_{e_j,e_k})x, \phi(T_{e_j,e_k})y\right\rangle &= \left\langle \phi(T_{e_k,e_j}T_{e_j,e_k})x, y\right\rangle \\ &= \left\langle \phi(T_{e_j,e_j})x, y\right\rangle = \langle x, y\rangle.\end{aligned}$$

Now fix some $j_0 \in I$ and choose $f_{j_0} \in V_{\eta,j_0}$ of norm one. For $j \neq j_0$ set $f_j = \phi(T_{e_{j_0},e_j})f_{j_0}$. Consider the isometry $S : V_\pi \to V_\eta$ given by $S(e_j) = f_j$. It then follows that $ST_{e_j,e_k} = \phi(T_{e_j,e_k})S$. The C^*-algebra $\mathrm{VN}(\pi) = \mathcal{B}(V_\pi)$ is generated by the T_{e_j,e_k}, so S is an intertwining operator onto a closed subspace of V_η. As η is irreducible, S must be surjective, i.e., unitary. □

Definition A factor representation π is called a *type-I representation* if π is quasi-equivalent to a representation π_1 whose von Neumann algebra $\mathrm{VN}(\pi_1)$ is a type-I factor. Then π is of type I if and only if π is quasi-equivalent to an irreducible representation.

Example 8.3.4 We here give an example of a factor representation, which is not of type I. Let Γ be a non-trivial group with the property that every conjugacy class in Γ is infinite or trivial. So the only finite conjugacy class in Γ is $\{1\}$. An example of this instance is the free group F_2 generated by two elements. Another example is the group $\mathrm{SL}_2(\mathbb{Z})/\pm 1$.

Consider the regular right representation R of Γ on the Hilbert space $H = \ell^2(\Gamma)$. Let $\mathrm{VN}(R)$ be the von Neumann algebra generated by $R(\Gamma) \subset \mathcal{B}(\ell^2(\Gamma))$.

Proposition 8.3.5 $\mathrm{VN}(R)$ *is a factor, which is not of type I.*

Proof We show that the commutant $\mathrm{VN}(R)^\circ$ is the von Neumann algebra generated by the regular left representation L of Γ. For this consider the natural orthonormal basis $(\delta_\gamma)_{\gamma\in\Gamma}$, which is defined by $\delta_\gamma(\tau) = 1$ if $\gamma = \tau$ and zero otherwise. One has $R_\gamma \delta_{\gamma_0} = \delta_{\gamma_0\gamma^{-1}}$ and $L_\gamma \delta_{\gamma_0} = \delta_{\gamma\gamma_0}$. Let $T \in \mathrm{VN}(R)^\circ$, so $TR_\gamma = R_\gamma T$ for every $\gamma \in \Gamma$. Then $T(\delta_1) = \sum_\gamma c_\gamma \delta_\gamma$ for some coefficients $c_\gamma \in \mathbb{C}$ satisfying $\sum_\gamma |c_\gamma|^2 < \infty$. For $\gamma_0 \in \Gamma$ arbitrary one gets

$$\begin{aligned} T(\delta_{\gamma_0}) &= T\left(R_{\gamma_0^{-1}}\delta_1\right) = R_{\gamma_0^{-1}} T(\delta_1) \\ &= R_{\gamma_0^{-1}} \sum_\gamma c_\gamma \delta_\gamma = \sum_\gamma c_\gamma \delta_{\gamma\gamma_0} \\ &= \sum_\gamma c_\gamma L_\gamma(\delta_{\gamma_0}), \end{aligned}$$

so $T = \sum_\gamma c_\gamma L_\gamma$, where the sum converges in the strong topology. Hence $T \in \mathrm{VN}(L)$. As trivially $\mathrm{VN}(L) \subset \mathrm{VN}(R)^\circ$ we get $\mathrm{VN}(R)^\circ = \mathrm{VN}(L)$. This means that $\mathrm{VN}(L)$ and $\mathrm{VN}(R)$ are each other's commutants. In particular, it follows that each element of $\mathrm{VN}(L)$ can be written as a point-wise convergent sum of the form $\sum_\gamma c_\gamma L_\gamma$, and likewise each element of $\mathrm{VN}(R)$ can be written as a sum of the form $\sum_\gamma d_\gamma R_\gamma$. We show that $\mathrm{VN}(R)$ is a factor. For this we have to show that the intersection of $\mathrm{VN}(R)$ and $\mathrm{VN}(L)$ is trivial. So let $T \in \mathrm{VN}(L) \cap \mathrm{VN}(R)$. Then we have two representations

$$\sum_\gamma c_\gamma L_\gamma = T = \sum_\gamma d_\gamma R_\gamma.$$

In particular, $\sum_\gamma c_\gamma \delta_\gamma = T(\delta_1) = \sum_\gamma d_\gamma \delta_{\gamma^{-1}}$, which implies $d_\gamma = c_{\gamma^{-1}}$, so for $\alpha \in \Gamma$, on the one hand,

$$T(\delta_\alpha) = \sum_\gamma c_\gamma L_\gamma \delta_\alpha = \sum_\gamma c_\gamma \delta_{\gamma\alpha} = \sum_\gamma c_{\gamma\alpha^{-1}} \delta_\gamma$$

and on the other,

$$T(\delta_\alpha) = \sum_\gamma c_\gamma R_{\gamma^{-1}} \delta_\alpha = \sum_\gamma c_\gamma \delta_{\alpha\gamma} = \sum_\gamma c_{\alpha^{-1}\gamma} \delta_\gamma.$$

This means that the function $\gamma \mapsto c_\gamma$ is constant on conjugacy classes. Since the sums must converge, this function can only be supported on finite conjugacy classes. As there is only one of them, it follows that $c_\gamma = 0$ except for $\gamma = 1$, so $T \in \mathbb{C}\,\mathrm{Id}$.

Finally we show that VN(R) is not of type I. For this consider the map $\sigma : \mathrm{VN}(R) \to \mathbb{C}$; $T \mapsto \langle T\delta_1, \delta_1\rangle$. This map is evidently continuous with respect to the strong and weak topologies. We show $\sigma(ST) = \sigma(TS)$ for all $S, T \in \mathrm{VN}(R)$. By continuity it suffices to show this for $S = R_\gamma$ and $T = R_\tau$, where $\gamma, \tau \in \Gamma$. Then we have

$$\sigma(ST) = \sigma(R_\gamma R_\tau) = \sigma(R_{\gamma\tau}) = \left\langle \delta_{\gamma\tau}, \delta_1 \right\rangle = \begin{cases} 1 & \text{if } \gamma\tau = 1, \\ 0 & \text{otherwise.} \end{cases}$$

The last condition is symmetric in γ and τ, since in the group Γ we have $\gamma\tau = 1 \Leftrightarrow \tau\gamma = 1$, so the same calculation gives $\sigma(ST) = \sigma(TS)$ as claimed.

We now show that for every selfadjoint projection $P \neq 0$ in VN(R) one has $0 < \sigma(P) \leq 1$. We first observe that for $T = \sum_{\gamma\in\Gamma} c_\gamma R_\gamma \in \mathrm{VN}(R)$ one has $\sigma(T) = c_1$. Next let P be a selfadjoint projection, which is the same as an orthogonal projection. So it satisfies $P^* = P = P^2$. We write $P = \sum_{\gamma\in\Gamma} c_\gamma R_\gamma$ and we get

$$\sum_\gamma c_\gamma R_\gamma = P = P^2 = \sum_\gamma \left(\sum_\delta c_\delta c_{\delta^{-1}\gamma} \right) R_\gamma.$$

So in particular $c_1 = \sum_\delta c_\delta c_{\delta^{-1}}$. The condition $P = P^* = \sum_\gamma \overline{c_{\gamma^{-1}}} R_\gamma$ implies $c_{\gamma^{-1}} = \overline{c_\gamma}$ and therefore $\sigma(P) = c_1 = \sum_\gamma |c_\gamma|^2$. This implies $c_1 > 0$ and $c_1 \geq c_1^2$, so $1 \geq c_1$.

Now assume there is a *-isomorphism $\phi : \mathcal{B}(H) \to \mathrm{VN}(R)$ for some Hilbert space H. Since VN(R) is infinite-dimensional, the space H is infinite-dimensional. So let $(e_j)_{j\in\mathbb{N}}$ be an orthogonal sequence in H. Let Q_j be the orthogonal projection with image $\mathbb{C}e_j$ and let $P_j = \phi(Q_j)$. Then P_j is a selfadjoint projection. Further Q_j is conjugate to Q_k in $\mathcal{B}(H)$, since there are unitary operators interchanging e_j and e_k. Then P_j and P_k are conjugate in VN(R) and therefore $\sigma(P_j) = \sigma(P_k)$ is a fixed number $c > 0$. Now $Q_1 + \cdots + Q_n$ again is a selfadjoint projection, so the same holds for $P_1 + \cdots + P_n$. So we have

$$1 \geq \sigma(P_1 + \cdots + P_n) = \sigma(P_1) + \cdots + \sigma(P_n) = nc,$$

Since this holds foe every n, we get $c = 0$, a contradiction! Hence ϕ does not exist and VN(R) is not of type I. □

8.4 Hilbert Integrals

A family of vectors $(\xi_i)_{i\in I}$ in a Hilbert space H is called a *quasi-orthonormal basis* if the non-zero members of the family form an orthonormal basis of H.

Let X be a set and $\mathcal{D}$ a σ-algebra of subsets of X. A *Hilbert bundle* over X is a family of Hilbert spaces $(H_x)_{x\in X}$ and a family of maps $\xi_i : X \to \bigcup_{x\in X} H_x$ (disjoint union) with $\xi_i(x) \in H_x$, such that for each $x \in X$ the family $(\xi_i(x))$ is a quasi-orthonormal basis of H_x, and for each $i \in I$ the set of all $x \in X$ with $\xi_i(x) = 0$ is measurable.

A *section* is a map $s : X \to \bigcup_{x\in X} H_x$ with $s(x) \in H_x$ for every $x \in X$. A section is called *measurable section* if for every $j \in I$ the function $x \mapsto \langle s(x), \xi_j(x)\rangle$ is measurable on X, and there exists a countable set $I_s \subset I$, such that the function $x \mapsto \langle s(x), \xi_i(x)\rangle$ vanishes identically for every $i \notin I_s$.

Let μ be a measure on $\mathcal{D}$. A measurable section s is called a *nullsection* if it vanishes outside a set of measure zero. The *direct Hilbert integral* is the vector space of all measurable sections s, which satisfy

$$\|s\|^2 \stackrel{\text{def}}{=} \int_X \|s(x)\|^2 \, d\mu(x) \; < \; \infty$$

modulo the space of nullsections.

This space, written as $H = \int_X H_x \, d\mu(x)$, is a Hilbert space with the inner product $\langle s, t\rangle = \int_X \langle s(x), t(x)\rangle \, d\mu(x)$. To show the completeness, for $i \in I$ let X_i be the set of all $x \in X$ with $\xi_i(x) \neq 0$. We get a map $P_i : H \to L^2(X_i)$ given by $P_i(s)(x) = \langle s(x), \xi_i(x)\rangle$. These maps combine to give a unitary isomorphism,

$$H = \int_X H_x \, d\mu(x) \xrightarrow{\cong} \widehat{\bigoplus_{i\in I}} L^2(X_i).$$

Example 8.4.1 Direct sums are special cases of direct integrals. Let $H = \bigoplus_{j\in I} H_j$ be a direct sum of separable Hilbert spaces. This space equals the direct integral $\int_X H_x \, d\mu(x)$ with $X = I$ and μ the counting measure on X.

Let (H_x, ξ_j) be a Hilbert bundle and μ a measure on X. Let G be a locally compact group, and for every $x \in X$ let η_x be a unitary representation of G on H_x, such that for every $g \in G$ and all $i, j \in I$ the map $x \mapsto \langle \eta_x(g)\xi_i(x), \xi_j(x)\rangle$ is measurable. Then $(\eta(g)s)(x) \stackrel{\text{def}}{=} \eta_x(g)s(x)$ defines a unitary representation of G on $H = \int_X H_x \, d\mu(x)$.

Example 8.4.2 Let A be a locally compact abelian group with dual group $\widehat{A}$ equipped with the Plancherel measure. Each character $\chi : A \to \mathbb{T} = U(\mathbb{C})$ determines a one-dimensional representation of A on $H_\chi = \mathbb{C}$. Consider the constant section

$\xi_1(\chi) = 1 \in \mathbb{C} = H_\chi$. Let $\eta_\chi(y) = \chi(y)$. Then the direct integral satisfies

$$\int_{\widehat{A}} H_\chi \, d\chi \cong L^2(\widehat{A})$$

with $(\eta(y)\xi)(\chi) = \chi(y)\xi(\chi)$. It follows then from the Plancherel Theorem 3.4.8 that $(\eta, L^2(\widehat{A}))$ is unitarily equivalent to the left regular representation $(L, L^2(A))$ of A via the Fourier transform.

8.5 The Plancherel Theorem

A locally compact group G is called a *type-I group* if every factor representation of G is of type I, i.e., is quasi-equivalent to an irreducible one.

Examples 8.5.1

- Abelian groups are of type I. For an abelian group A and a unitary representation π of A, the von Neumann algebra $\mathrm{VN}(\pi)$ is commutative. So, if $\mathrm{VN}(\pi)$ is a factor, it must be isomorphic to $\mathbb{C}$, which means that π is quasi-equivalent to a one-dimensional representation.
- Compact groups are of type I. For a compact group any unitary representation is a direct sum of irreducible representations.
- Nilpotent Lie groups are of type I. See [BCD+72] Chapter VI.
- Semisimple Lie groups are of type I. See [HC76].
- A discrete group is of type I if and only if it contains a normal abelian subgroup of finite index. See [Tho68].

Let G and H be locally compact groups, and let (π, V_π), (σ, V_σ) be unitary representations of G and H, respectively. On the Hilbert tensor product $V_\pi \hat{\otimes} V_\sigma$ (see Appendix C.3) we define a representation $\pi \otimes \sigma$ of the product group $G \times H$ by linear extension of

$$v \otimes w \ \mapsto \ \pi(x)v \otimes \sigma(y)w$$

for $(x, y) \in G \times H$, $v \in V_\pi$, and $w \in V_\sigma$.

Recall that the unitary dual $\widehat{G}$ consists of all equivalence classes of irreducible unitary representations of G. On $\widehat{G}$ we will install a natural σ-algebra in the case that G has a countable dense subset.

Lemma 8.5.2 *Assume that G has a countable dense subset. Then every irreducible unitary representation (π, V_π) has countable dimension, i.e., the Hilbert space V_π has a countable orthonormal system.*

Proof Let (π, V_π) be an irreducible unitary representation of G. A subset $\mathcal{T} \subset V_\pi$ is called *total*, if the linear span of $\mathcal{T}$ is dense in V_π. By the orthonormalization scheme it suffices to show that there is a countable total set in V_π. Let $0 \neq v \in V_\pi$. Then the set $\pi(G)v$ is total in V_π, as V_π is irreducible. Let $D \subset G$ be a countable dense subset. Then the set $\pi(D)v$ is dense in $\pi(G)v$, hence also total in V_π.

Assume that G has a dense countable subset. For a countable cardinal $n = 1, 2, \dots \aleph_0$, let H_n denote a fixed Hilbert space of dimension n. For each class C in $\widehat{G}$ we fix a representative $\pi \in C$ with representation space H_n, which exists by Lemma 8.5.2. The cardinal n is uniquely determined by $C = [\pi]$ and is called the *dimension* of the representation. Let $\widehat{G}_n$ be the subset of $\widehat{G}$ consisting of all classes $[\pi]$ of dimension n. On $\widehat{G}_n$ we install the smallest σ-algebra making all maps $[\pi] \mapsto \langle \pi(g)v, w\rangle$ measurable, where g ranges in G and v, w range over H_n. On $\widehat{G} = \bigcup_n \widehat{G}_n$ we install the union σ-algebra.

The prescription $\eta(x, y) = L_x R_y$ defines a unitary representation of $G \times G$ on the Hilbert space $L^2(G)$. Note that if G is second countable, then it contains a dense countable subset, i.e., is separable.

Theorem 8.5.3 *Let G be a second countable, unimodular, locally compact group of type I. There is a unique measure μ on $\widehat{G}$ such that for $f \in L^1(G) \cap L^2(G)$ one has*

$$\|f\|_2^2 = \int_{\widehat{G}} \|\pi(f)\|_{\mathrm{HS}}^2 \, d\mu(\pi).$$

The map $f \mapsto (\pi(f))_\pi$ extends to a unitary $G \times G$ equivariant map

$$L^2(G) \cong \int_{\widehat{G}} \mathrm{HS}(V_\pi) \, d\mu(\pi),$$

where the representation of η_π of *$G \times G$ on the space of Hilbert-Schmidt operators $HS(V_\pi)$ is given by $\eta_\pi(x, y)(T) = \pi(x) T \pi(y^{-1})$ for each $\pi \in \widehat{G}$ and $x, y \in G$.*

The proof is in [Dix96], 18.8.1.

This Plancherel Theorem generalizes the Plancherel Theorem in the abelian case, Theorem 3.4.8, as well as the Peter-Weyl Theorem in the compact case, Theorems 7.2.1 and 7.2.4. Concrete examples for groups, which are neither abelian nor compact will be given in Theorem 10.3.1 and Theorem 11.3.1.

8.6 Exercises

Exercise 8.1 For $S > 0$ show that there exists a function $f : \mathbb{R} \to \mathbb{R}$ which is infinitely differentiable, of compact support and satisfies $f(x) = x$ for $|x| \le S$.

(Hint: Let $g(x) = 1$ for $|x| \le S + 1$ and $g(x) = 0$ otherwise. Let $h = \phi * g$ for some smooth Dirac function with support in $[-1, 1]$. Set $f(x) = xh(x)$.)

Exercise 8.2 Let V, W be Hilbert spaces.

(a) For $A \in \mathcal{B}(V)$ set $\alpha(A)(v \otimes w) = A(v) \otimes w$. Show that α is a norm-preserving $*$-homomorphism from $\mathcal{B}(V)$ to $\mathcal{B}(V \hat{\otimes} W)$.

(b) Let β be the analogous map on the second factor. Show that $\alpha \otimes \beta$ defines a $*$-homomorphism from $\mathcal{B}(V) \otimes \mathcal{B}(W)$ to $\mathcal{B}(V \hat{\otimes} W)$.

(c) Show that the von Neumann algebra generated by the image of $\alpha \otimes \beta$ equals the entire $\mathcal{B}(V \hat{\otimes} W)$.

Exercise 8.3 Give an example of a set of bounded operators on a Hilbert space, for which the norm-closure differs from the strong closure. Also give an example, for which the strong and weak closures differ.

Exercise 8.4 Let H be a Hilbert space, and let $\mathcal{P}$ be the set of all orthogonal projections on H. Let $\mathcal{T}_s$ and $\mathcal{T}_w$ be the restrictions of the strong and weak topologies to the set $\mathcal{P}$. Show that $\mathcal{T}_s = \mathcal{T}_w$.

Exercise 8.5 Let H be a Hilbert space, and let $\mathcal{U}$ be the set of all unitary operators on H. Let $\mathcal{T}_s'$ and $\mathcal{T}_w'$ be the restrictions of the strong and weak topologies to the set $\mathcal{U}$. Show that $\mathcal{T}_s' = \mathcal{T}_w'$.

Exercise 8.6 Show that not every unital C^*-algebra is isomorphic to a von Neumann algebra.

(Hint: Consider an infinite dimensional Hilbert space H and the space $\mathcal{K}$ of compact operators. The algebra $\mathcal{A} = \mathcal{K} + \mathbb{C}\mathrm{Id}$ is a C^*-algebra.)

Exercise 8.7 Let H be a Hilbert space, and let $M \subset \mathcal{B}(H)$ be self-adjoint and commutative, i.e., for $S, T \in M$ one has $S^*, T^* \in M$ and $ST = TS$. Show that the bicommutant $M^{\circ\circ}$ is commutative.

Exercise 8.8 For a von Neumann algebra $\mathcal{A} \subset \mathcal{B}(H)$ let $\mathcal{A}^+$ be the set of all finite sums of elements of the form aa^* for some $a \in \mathcal{A}$. Show:

(a) $\mathcal{A}^+$ is a proper cone, i.e.:

$$\mathcal{A}^+ + \mathcal{A}^+ \subset \mathcal{A}^+, \quad \mathbb{R}^+\mathcal{A}^+ \subset \mathcal{A}^+, \quad \mathcal{A}^+ \cap (-\mathcal{A}^+) = 0.$$

(b) For $a \in \mathcal{A}$ one has

$$a \in \mathcal{A}^+ \quad \Leftrightarrow \quad \exists b \in \mathcal{A} : a = bb^*, \quad \Leftrightarrow \quad a \geq 0.$$

Exercise 8.9 Let $\mathcal{A} \subset \mathcal{B}(H)$ be a von Neumann algebra. A *finite trace* is a linear map $\tau : \mathcal{A} \to \mathbb{C}$ with $\tau(\mathcal{A}^+) \subset \mathbb{R}^+$ and $\tau(ab) = \tau(ba)$ for all $a, b \in \mathcal{A}$. Show:

(a) Let τ be a finite trace on $\mathcal{A} = \mathrm{M}_n(\mathbb{C})$. Then $\tau(a) = c\,\mathrm{tr}\,(a)$ for some $c \geq 0$.

(b) Let $\mathcal{A} = \mathcal{B}(H)$, where H is an infinite-dimensional Hilbert space. Then there is no finite trace on $\mathcal{A}$.

(c) Let Γ be a discrete group, and let $\mathcal{A} = \mathrm{VN}(R)$. Then $\tau\left(\sum_{\gamma\in\Gamma} c_\gamma R_\gamma\right) = c_1$ is a finite trace on $\mathcal{A}$.

Exercise 8.10 Show that a von Neumann algebra $\mathcal{A}$ is generated by all orthogonal projections it contains.

Exercise 8.11 Let G be a locally compact group. For a unitary representation (π, V_π) let its *matrix coefficients* be all continuous functions on G of the form

$$g \;\mapsto\; \psi_{v,w}(g) \stackrel{\mathrm{def}}{=} \langle \pi(g)v, w\rangle, \qquad v, w \in V_\pi .$$

Let G be of type I. Let π be a unitary representation such that all its matrix coefficients are in $L^2(G)$. Show that π is a direct sum of irreducible representations.