Chapter 7
Compact Groups

In this chapter we will show that every unitary representation of a compact group is a direct sum of irreducibles, and that every irreducible unitary representation is finite dimensional. We further prove the Peter-Weyl theorem, which gives an explicit decomposition of the regular representation of the compact group K on $L^2(K)$.

The term *compact group* will always mean a compact topological group, which is a Hausdorff space.

7.1 Finite Dimensional Representations

Let K be a compact group, and let (τ, V_τ) be a finite dimensional representation, i.e., the complex vector space V_τ is finite dimensional.

Lemma 7.1.1 *On the space V_τ, there exists an inner product, such that τ becomes a unitary representation. If τ is irreducible, this inner product is uniquely determined up to multiplication by a positive constant.*

Proof Let $(\cdot, \cdot)$ be any inner product on V_τ. We define a new inner product $\langle v, w\rangle$ for $v, w \in V_\tau$ to be equal to $\int_K (\tau(k)v, \tau(k)w)\, dk$, where we have used the normalized Haar measure that gives K the measure 1. We have to show that this constitutes an inner product. Linearity in the first argument and anti-symmetry are clear. For the positive definiteness let $v \in V_\tau$ with $\langle v, v\rangle = 0$, i.e.,

$$0 = \langle v, v\rangle = \int_K (\tau(k)v, \tau(k)v)\, dk.$$

The function $k \mapsto (\tau(k)v, \tau(k)v)$ is continuous and positive, hence, by Corollary 1.3.6, the function vanishes identically, so in particular, $(v, v) = 0$, which implies $v = 0$ and $\langle \cdot, \cdot\rangle$ is an inner product. With respect to this inner product the representation τ is unitary, as for $x \in K$ one has

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$$\begin{aligned}\langle \tau(x)v, \tau(x)w \rangle &= \int_K (\tau(k)\tau(x)v, \tau(k)\tau(x)w)\, dk \\ &= \int_K (\tau(kx)v, \tau(kx)w)\, dk \\ &= \int_K (\tau(k)v, \tau(k)w)\, dk = \langle v, w \rangle,\end{aligned}$$

as K is unimodular.

Finally, assume that τ is irreducible, let $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ be two inner products that make τ unitary. Let (τ_1, V_1) and (τ_2, V_2) denote the representation (τ, V_τ) when equipped with the inner products $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$, respectively. Since V_τ is finite dimensional, the identity $\mathrm{Id} : V_1 \to V_2$ is a bounded non-zero intertwining operator for τ_1 and τ_2. By Corollary 6.1.9 there exists a number $c > 0$ such that $c \cdot \mathrm{Id}$ is unitary. But this implies that $c^2 \langle v, w \rangle_2 = \langle v, w \rangle_1$ for all $v, w \in V_\tau$. □

Proposition 7.1.2 *A finite dimensional representation of a compact group is a direct sum of irreducible representations.*

Proof Let (τ, V) be a finite dimensional representation of the compact group K. We want to show that τ is a direct sum of irreducibles. We proceed by induction on the dimension of V. If this dimension is zero or one, there is nothing to show. So assume the claim proven for all spaces of dimension smaller than $\dim V$. By the last lemma, we can assume that τ is a unitary representation. If τ is irreducible itself, we are done. Otherwise, there is an invariant subspace $U \subset V$ with $0 \neq U \neq V$. Let $W = U^\perp$ be the orthogonal complement to U in V, so that $V = U \oplus W$. We claim that W is invariant as well. For this let $k \in K$ and $w \in W$. Then for every $u \in U$,

$$\langle \tau(k)w, u \rangle = \langle w, \underbrace{\tau\left(k^{-1}\right)u}_{\in U} \rangle = 0.$$

This implies that $\tau(k)w \in U^\perp = W$, so W is indeed invariant. We conclude that τ is the direct sum of the subrepresentations on U and W. As both spaces have dimensions smaller than the one of V, the induction hypothesis shows that both are direct sums of irreducibles, and so is V. □

Definition Let (τ, V_τ) be a finite dimensional representation of a compact group K. The dual space

$$V_\tau^* = \mathrm{Hom}(V_\tau, \mathbb{C})$$

of all linear functionals $\alpha : V_\tau \to \mathbb{C}$ carries a natural representation of K, the *dual representation* τ^* defined by

$$\tau^*(x)\alpha(v) = \alpha\left(\tau(x^{-1})v\right).$$

Suppose that V_τ is a Hilbert space. By the Riesz Representation Theorem for every $\alpha \in V_\tau^*$ there exists a unique vector v_α such that

$$\alpha(w) = \langle w, v_\alpha \rangle$$

holds for every $w \in V_\tau$. One instals a Hilbert space structure on the dual V_τ^* by setting

$$\langle \alpha, \beta \rangle = \left\langle v_\beta, v_\alpha \right\rangle.$$

Lemma 7.1.3 *If the representation τ is irreducible, then so is the dual representation τ^*. The same holds for the property of being unitary. For $x \in K$ and $\alpha \in V_\tau^*$ one gets the intertwining relation*

$$v_{\tau^*(x)\alpha} = \tau(x) v_\alpha,$$

so the map $\alpha \mapsto v_\alpha$ is an anti-linear intertwining operator between V_τ^* and V_τ.

Proof Suppose that $W^* \subset V_\tau^*$ is a subrepresentation. Then the space $(W^*)^\perp$ of all $v \in V_\tau$ with $\alpha(v) = 0$ for every $\alpha \in W^*$ is a subrepresentation of V_τ. If τ is irreducible the latter space is trivial and so then is W^*.

For the remaining assertions, we first show the claimed intertwining relation. For $w \in V_\tau$ we use unitarity of τ to get

$$\begin{aligned} \left\langle w, v_{\tau^*(x)\alpha} \right\rangle &= \tau^*(x)\alpha(w) = \alpha(\tau(x^{-1})w) \\ &= \left\langle \tau(x^{-1})w, v_\alpha \right\rangle = \langle w, \tau(x) v_\alpha \rangle. \end{aligned}$$

Varying w, the relation follows. Now the unitarity of τ^* follows by transport of structure,

$$\begin{aligned} \left\langle \tau^*(x)\alpha, \tau^*(x)\beta \right\rangle &= \left\langle v_{\tau^*(x)\beta}, v_{\tau^*(x)\alpha} \right\rangle = \left\langle \tau(x) v_\beta, \tau(x) v_\alpha \right\rangle \\ &= \left\langle v_\beta, v_\alpha \right\rangle = \langle \alpha, \beta \rangle \end{aligned}$$

The Lemma is proven. □

7.2 The Peter-Weyl Theorem

Let K be a compact group, and let $\widehat{K}$ be the set of all equivalence classes of irreducible unitary representations of K. Let $\widehat{K}_{\text{fin}}$ be the subset of all finite dimensional irreducible representations. We want to show that $\widehat{K} = \widehat{K}_{\text{fin}}$.

A *matrix coefficient* for a unitary representation τ of K on V_τ is a function of the form $k \mapsto \langle \tau(k)v, w \rangle$ for some $v, w \in V_\tau$. The matrix coefficients are continuous functions, so they lie in the Hilbert space $L^2(K)$. We need to know that the set of matrix coefficients, where τ runs through all finite dimensional representations is closed under taking complex conjugates. To see this we use Lemma 7.1.3 for a finite dimensional unitary representation (τ, V_τ). So let $v, v' \in V_\tau$ and let $\alpha, \beta \in V_\tau^*$ be their Riesz duals, i.e., $v = v_\alpha$ and $v' = v_\beta$ in the notation of the last section. Then

$$\overline{\langle \tau(x)v, v'\rangle} = \langle v_\beta, \tau(x)v_\alpha\rangle = \langle v_\beta, v_{\tau^*(x)\alpha}\rangle = \langle \tau^*(x)\alpha, \beta\rangle$$

shows that the complex conjugate of a matrix coefficient is indeed a matrix coefficient.

Now, for every class in $\widehat{K}_{\mathrm{fin}}$ choose a representative (τ, V_τ). Choose an orthonormal basis $e_1, \dots, e_n$ of V_τ and write $\tau_{ij}(k) \stackrel{\mathrm{def}}{=} \langle \tau(k)e_i, e_j\rangle$ for the corresponding matrix coefficient. It is easy to see that for every $v, w \in V_\tau$ the function $k \mapsto \langle \tau(k)v, w\rangle$ is a linear combination of the τ_{ij}, $1 \le i, j \le \dim V_\tau$. In what follows we shall write $\dim(\tau)$ for $\dim V_\tau$.

Theorem 7.2.1 (Peter-Weyl Theorem).

(a) *For $\tau \ne \gamma$ in $\widehat{K}_{\mathrm{fin}}$ one has*

$$\langle \tau_{ij}, \gamma_{rs}\rangle = \int_K \tau_{ij}(k)\overline{\gamma_{rs}(k)}\, dk = 0.$$

So the matrix coefficients of non-equivalent representations are orthogonal.

(b) *For $\tau \in \widehat{K}_{\mathrm{fin}}$ one has $\langle \tau_{ij}, \tau_{rs}\rangle = 0$, except for the case when $i = r$ and $j = s$. In the latter case the products are $\langle \tau_{ij}, \tau_{ij}\rangle = \frac{1}{\dim(\tau)}$. One can summarize this by saying that the family*

$$\left(\sqrt{\dim(\tau)}\, \tau_{ij}\right)_{\tau,i,j}$$

is an orthonormal system in $L^2(K)$.

(c) *It even is complete, i.e., an orthonormal basis.*

(d) *The translation-representations $(L, L^2(K))$ and $(R, L^2(K))$ decompose into direct sums of finite-dimensional irreducible representations.*

Proof For (a) let $\tau \ne \gamma$ in $\widehat{K}_{\mathrm{fin}}$. Let $T : V_\tau \to V_\gamma$ be linear and set $S = S_T = \int_K \gamma(k^{-1})T\tau(k)\, dk$. Then one has $S\tau(k) = \gamma(k)S$, hence $S = 0$ by Corollary 6.1.9. Let (e_j) and (f_s) be orthonormal bases of V_τ and V_γ, respectively, and choose $T_{js} : V_\tau \to V_\gamma$ given by $T_{js}(v) = \langle v, e_j\rangle f_s$. Let $S_{js} = S_{T_{js}}$ as above. One gets

$$\begin{aligned}
0 = \langle S_{js}e_i, f_r\rangle &= \int_K \langle \gamma(k^{-1})T_{js}\tau(k)e_i, f_r\rangle\, dk \\
&= \int_K \langle \gamma(k^{-1})\langle \tau(k)e_i, e_j\rangle f_s, f_r\rangle\, dk \\
&= \int_K \langle \tau(k)e_i, e_j\rangle \underbrace{\langle \gamma(k^{-1})f_s, f_r\rangle}_{=\langle f_s, \gamma(k)f_r\rangle = \overline{\langle \gamma(k)f_r, f_s\rangle}}\, dk \\
&= \int_K \tau_{ij}(k)\overline{\gamma_{rs}(k)}\, dk = \langle \tau_{ij}, \gamma_{rs}\rangle.
\end{aligned}$$

To prove (b), we perform the same computation for $\gamma = \tau$ to get

$$\langle S_{js} e_i, e_r \rangle = \langle \tau_{ij}, \tau_{rs} \rangle.$$

In this case the matrix S_{js} is a multiple of the identity $S_{js} = \lambda \mathrm{Id}$ for some $\lambda \in \mathbb{C}$, so if $i \neq r$ we infer $\langle S_{js} e_i, e_r \rangle = 0$, hence $\langle \tau_{ij}, \tau_{rs} \rangle = 0$. Assume $j \neq s$. We claim that $S_{js} = 0$, which implies the same conclusion, so in total we get the first assertion of (b). To show $S_{js} = 0$ recall that $S = S_{js} = \lambda\, \mathrm{Id}$, so the trace equals

$$\begin{aligned} \lambda \dim V_\tau &= \mathrm{tr}\,(S) = \mathrm{tr}\, \left(\int_K \tau(k)^{-1} T \tau(k)\, dk \right) \\ &= \int_K \mathrm{tr}\, \left(\tau(k)^{-1} T \tau(k) \right) dk = \int_K \mathrm{tr}\,(T)\, dk = \mathrm{tr}\,(T), \end{aligned}$$

but as $j \neq s$, the trace of T is zero, hence S is zero and so is $\langle S e_i, e_i \rangle = \langle \tau_{ij}, \tau_{i,s} \rangle$. Finally, we consider the case $j = s$ and $i = r$. Then $S_{jj} = \lambda_j \mathrm{Id}$ for some $\lambda_j \in \mathbb{C}$. Our computation shows $\lambda_j = \langle \tau_{ij}, \tau_{ij} \rangle$, independent of i. But $\tau_{ij}(k) = \overline{\tau_{ji}(k^{-1})}$ and therefore, as K is unimodular we get

$$\begin{aligned} \langle \tau_{ij}, \tau_{ij} \rangle &= \int_K \tau_{ij}(k) \overline{\tau_{ij}(k)}\, dk = \int_K \overline{\tau_{ji}(k^{-1})} \tau_{ji}(k^{-1})\, dk \\ &= \int_K \overline{\tau_{ji}(k)} \tau_{ji}(k)\, dk = \langle \tau_{ji}, \tau_{ji} \rangle. \end{aligned}$$

We conclude $\lambda_j = \langle \tau_{ij}, \tau_{ij} \rangle = \langle \tau_{ji}, \tau_{ji} \rangle = \lambda_i$. We call this common value λ and we have to show that $\lambda = \frac{1}{\dim(\tau)}$. Write $n = \dim V_\tau$ and note that $\mathrm{Id} = \sum_{j=1}^n T_{jj}$. Therefore $(n\lambda)\mathrm{Id} = \sum_{j=1}^n S_{jj} = \int_K \tau(k^{-1}) \mathrm{Id} \tau(k)\, dk = \mathrm{Id}$ and the claim follows.

Finally, to show (c), let $\tau \in \widehat{K}_{\mathrm{fin}}$, and let M_τ be the subspace of $L^2(K)$ spanned by all matrix coeficients of the representation τ. If $h(k) = \langle \tau(k)v, w \rangle$, then one has

$$\begin{aligned} h^*(k) &= \overline{h(k^{-1})} = \langle \tau(k)w, v \rangle \in M_\tau, \\ L_{k_0} h(k) &= h(k_0^{-1}k) = \langle \tau(k)v, \tau(k_0)w \rangle \in M_\tau, \\ R_{k_0} h(k) &= h(kk_0) = \langle \tau(k)\tau(k_0)v, w \rangle \in M_\tau. \end{aligned}$$

This means that the finite-dimensional space M_τ is closed under adjoints, and left and right translations. Let M be the closure in $L^2(K)$ of the span of all M_τ, where $\tau \in \widehat{K}_{\mathrm{fin}}$. Then M decomposes into a direct sum of irreducible representations under the left or the right translation. By the discussion preceding the theorem, M is also closed under complex conjugation. We want to show that $L^2(K) = M$, or, equivalently, $M^\perp = 0$. So assume $M^\perp$ is not trivial. Our first claim is that $M^\perp$ contains a non-zero continuous function. Let $H \neq 0$ in $M^\perp$. Let $(\phi_U)_U$ be a Dirac net. Then the net $\phi_U * H$ converges to H in the L^2-norm. Since $M^\perp$ is closed under translation it follows that $\phi_U * H \in M^\perp$ for every U. As there must exist some U

with $\phi_U * H \neq 0$, the first claim follows. So let $F_1 \in M^\perp$ be continuous. After applying a translation and a multiplication by a scalar, we can assume $F_1(e) > 0$. Set $F_2(x) = \int_K F_1\left(y^{-1}xy\right) dy$. Then $F_2 \in M^\perp$ is invariant under conjugation and $F_2(e) > 0$. Finally put $F(x) = F_2(x) + \overline{F_2\left(x^{-1}\right)}$. Then the function F is continuous, $F \in M^\perp$, $F(e) > 0$, and $F = F^*$. Consider the operator $T(f) = f * F = R(F)f$ for $f \in L^2(K)$. Since $R : L^1(K) \to \mathcal{B}\left(L^2(K)\right)$ is a $*$-representation, T is self-adjoint. Further, as $Tf(x) = \int_K f(y)F\left(y^{-1}x\right) dy$, the operator T is an integral operator with continuous kernel $k(x, y) = F\left(y^{-1}x\right)$. By Proposition 5.3.3, $T = T^* \neq 0$ is a Hilbert-Schmidt operator, hence compact, and thus it follows that T has a real eigenvalue $\lambda \neq 0$ with finite dimensional eigenspace V_λ. We claim that V_λ is stable under left-translations. For this let $f \in V_\lambda$, so $f * F = \lambda f$. Then, for $k \in K$ one has $(L_k f) * F = L_k(f * F) = \lambda L_k f$. This implies that V_λ with the left translation gives a finite dimensional unitary representation of K, hence it contains an irreducible subrepresentation $W \subset V_\lambda \subset M^\perp$. Let $f, g \in W$, and let $h(k) = \overline{\langle L_k f, g\rangle}$ be the corresponding matrix coefficient. One has $h(k) = \int_K f(k^{-1}x)\overline{g(x)}\, dx$, so $h = \overline{g * f^*} \in M^\perp$. On the other hand, $h \in M$, and so $\langle h, h\rangle = 0$, which is a contradiction. It follows that the assumption is wrong, so $M = L^2(K)$.

Above, we showed in particular that $L^2(K)$ decomposes as the closure of the direct sum $\bigoplus_{\tau \in \widehat{K}_{\text{fin}}} M_\tau$, where the the linear span M_τ of all matrix coefficients of τ has dimension $\dim(\tau)^2$. Since each M_τ is stable under left and right translations, this implies that $\left(L^2(K), L\right)$ and $\left(L^2(K), R\right)$ decompose as direct sums of finite dimensional representations. Hence (d) follows from Proposition 7.1.2 and the Peter-Weyl Theorem is proven. □

Definition Let π be a finite dimensional representation of the compact group K. The function $\chi_\pi : K \to \mathbb{C}$ defined by $\chi_\pi(k) = \operatorname{tr} \pi(k)$ is called the *character* of the representation π.

Corollary 7.2.2 *Let π, η be two finite-dimensional irreducible unitary representations of the compact group K. For their characters we have*

$$\left\langle \chi_\pi, \chi_\eta \right\rangle = \begin{cases} 1 & \text{if } \pi = \eta, \\ 0 & \textit{otherwise}. \end{cases}$$

Here the inner product is the one of $L^2(K)$.

Proof The proof follows immediately from the Peter-Weyl Theorem. Note that it is shown in Exercise 7.10 that $\{\chi_\pi : \pi \in \widehat{K}\}$ even forms an orthonormal base of the space $L^2(K/\text{conj})$ of conjugacy invariant L^2-functions on K. □

Let (π, V_π) be a representation of a locally compact group G. An *irreducible subspace* is a closed subspace $U \subset V_\pi$ which is stable under $\pi(G)$ such that the representation (π, U), obtained by restricting each $\pi(k)$ to U, is irreducible.

Theorem 7.2.3

(a) *Let K be a compact group. Then $\widehat{K} = \widehat{K}_{\text{fin}}$, so every irreducible unitary representation of K is finite dimensional.*

(b) *Every unitary representation of the compact group K is an orthogonal sum of irreducible representations.*

Proof Let (π, V_π) be a unitary representation of K. We show that V_π can be written as a direct sum $V_\pi = \overline{\bigoplus_{i\in I} V_i}$, where each V_i is a finite dimensional irreducible subspace of V_π. This proves (b) and if we apply this to a given irreducible representation V_π it also implies (a).

So let (π, V_π) be a given unitary representation of K. Consider the set S of all families $(V_i)_{i\in I}$, where each V_i is a finite dimensional irreducible subrepresentation of V_π and for $i \neq j$ in I we insist that V_i and V_j are orthogonal. We introduce a partial order on S given by $(V_i)_{i\in I} \le (W_\alpha)_{\alpha\in A}$ if and only if $I \subset A$ and for each $i \in I$ we have $V_i = W_i$. The Lemma of Zorn yields the existence of a maximal element $(V_i)_{i\in I}$. We claim that the orthogonal sum $\bigoplus_{i\in I} V_i$ is dense in V_π. This is equivalent to the orthogonal space $W = \left(\bigoplus_{i\in I} V_i\right)^\perp$ being the zero space. Now assume that inside W we find a finite-dimensional irreducible subspace U, then we can extend I by one element i_0 and we set $V_{i_0} = U$ which contradicts the maximality of I. Therefore, it suffices to show that any given non-zero unitary representation (η, W_η) contains a finite-dimensional irreducible subspace. For this let $v, w \in W_\eta$, and let $\psi_{v,w}(x) = \langle \eta(x)v, w\rangle$ be the corresponding matrix coefficient. Then $\psi_{v,w} \in C(K) \subset L^2(K)$ and $\psi_{\eta(y)v,w}(x) = \langle \eta(xy)v, w\rangle = \psi_{v,w}(xy) = R_y\psi_{v,w}(x)$. In other words, for fixed w, the map $v \mapsto \psi_{v,w}$ is a K-homomorphism from V_η to $(R, L^2(K))$. We assume $\langle v, w\rangle \neq 0$. Then this map is non-zero. Since $\left(R, L^2(K)\right)$ is a direct sum of finite dimensional irreducible representations, there exists an orthogonal projection $P : L^2(K) \to F$ to a finite dimensional irreducible subrepresentation, such that $P\left(\psi_{v,w}\right) \neq 0$. So there exists a non-zero K-homomorphism $T : V_\eta \to F$, which is surjective, hence induces an isomorphism from $U = (\ker(T))^\perp \subset V_\eta$ to F. Therefore U is the desired finite-dimensional irreducible subspace. □

We now give a reformulation of the Peter-Weyl Theorem. The group K acts on the space $L^2(K)$ by left and right translations, and these two actions commute, that is to say, we have a unitary representation η of the group $K \times K$ on the Hilbert space $L^2(K)$, given by

$$\eta(k_1, k_2) f(x) = f\left(k_1^{-1} x k_2\right).$$

On the other hand, for $(\tau, V_\tau) \in \widehat{K}$ the group $K \times K$ acts on the finite dimensional vector space End(V_τ)= $\mathrm{Hom}_K(V_\tau, V_\tau)$ by

$$\eta_\tau(k_1, k_2)(T) = \tau(k_1)T\tau\left(k_2^{-1}\right).$$

On End (V_τ) we have a natural inner product

$$\langle S,T\rangle = \dim(V_\tau)\,\mathrm{tr}\,(ST^*)$$

making the representation of $K\times K$ unitary (Compare with Exercise 5.8).

Theorem 7.2.4 (Peter-Weyl Theorem, second version). *There is a natural unitary isomorphism*

$$L^2(K) \;\cong\; \widehat{\bigoplus_{\tau\in\widehat{K}}}\mathrm{End}(V_\tau),$$

which intertwines the conjugation representation η of $K\times K$ on $L^2(K)$ with $\bigoplus_{\tau\in\widehat{K}}\eta_\tau$. This isomorphism maps a given $f\in L^2(K)\subset L^1(K)$ onto $\sum_{\tau\in\widehat{K}}\tau(f)$, where

$$\tau(f)=\int_K f(x)\tau(x)\,dx.$$

In particular, if for a given $f\in L^2(K)$ we define the map $\hat{f}:\widehat{K}\to\widehat{\bigoplus}_{\tau\in\widehat{K}}\mathrm{End}(V_\tau)$ by $\hat{f}(\tau)=\tau(f)$, then we get

$$\|f\|=\|\hat{f}\|$$

for every $f\in L^2(K)$. In this way the Peter-Weyl Theorem presents itself as a generalization of the Plancherel Formula.

Proof Since $\tau\mapsto\tau^*$ is a bijection from $\widehat{K}$ onto itself, the Peter-Weyl Theorem yields the orthonormal basis $\sqrt{\dim(\tau)}\tau^*_{kl}$, where the indices are taken with respect to the dual basis of a given orthonormal basis $\{e_1,\dots,e_{\dim(\tau)}\}$ of V_τ. For $f\in L^2(K)$ and indices i,j one has $\langle\tau(f)e_i,e_j\rangle=\int_K f(x)\tau_{ij}(x)\,dx=\langle f,\overline{\tau_{ij}}\rangle$. If we apply this formula to $f=\sigma^*_{kl}=\overline{\sigma_{kl}}$ for some $\sigma\in\widehat{K}$, we see that $\widehat{\sigma^*_{kl}}(\tau)=\tau(\sigma_{kl})=0$ for $\sigma\neq\tau$ and

$$\left\langle\widehat{\tau^*_{kl}}(\tau)e_i,e_j\right\rangle=\begin{cases}\dim(\tau) & \text{if } k=i \text{ and } l=j\\ 0 & \text{otherwise.}\end{cases}$$

Thus it follows that τ^*_{kl} is mapped to the operator $\frac{1}{\dim(\tau)}E^\tau_{kl}\in\mathrm{End}(V_\tau)$, where E^τ_{kl} denotes the endomorphism which sends e_k to e_l and all other basis elements to 0. Hence, the basis element $\sqrt{\dim(\tau)}\tau^*_{kl}\in M_{\tau^*}$ is mapped to $\sqrt{\dim(\tau)}E^\tau_{kl}$. It is trivial to check that these elements form an orthonormal basis of $\mathrm{End}(V_\tau)$ with respect to the given inner product. $\square$

Definition Let (τ,V_τ) and (γ,V_γ) be finite dimensional representations of the compact group K. There is a natural representation $\tau\otimes\gamma$ of the group $K\times K$ on the tensor product space $V_\tau\otimes V_\gamma$ given by

$$(\tau\otimes\gamma)\,(k_1,k_2)=\tau(k_1)\otimes\gamma(k_2).$$

Lemma 7.2.5 *For given $\tau \in \widehat{K}$, there is a natural unitary isomorphism*

$$\Psi : V_\tau \otimes V_{\tau^*} \to \mathrm{End}(V_\tau),$$

which intertwines $\tau \otimes \tau^*$ with η_τ.

Show that the direct summand $\mathrm{End}(V_\pi)$ of $L^2(K)$ equipped with the conjugation action η of $K \times K$ as in the second version of the Peter-Weyl Theorem is equivalent to the irreducible representation $\pi^* \otimes \pi$ of $K \times K$. (Compare with Exercise 5.8.)

Proof The map $\psi : V_\tau \otimes V_{\tau^*} \to \mathrm{End}(V_\tau)$ given by

$$\psi(v \otimes \alpha) = [w \mapsto \alpha(w)v]$$

is linear and sends the simple tensors to the operators of rank one. Every operator of rank one is in the image, so the map is surjective as $\mathrm{End}(V_\tau)$ is linearly generated by the operators of rank one. As the dimensions of the spaces agree, the map is bijective. It further is intertwining, as for $k, l \in K$ one has

$$\begin{aligned} \psi\left(\tau \otimes \tau^*(k,l)(v \otimes \alpha)\right)(w) &= \psi\left(\tau(k)v \otimes \tau^*(k)\alpha\right)(w) \\ &= \alpha(\tau(l^{-1})w)\tau(k)v \\ &= \tau(k)\psi(w \otimes \alpha)\tau(l^{-1})(w) \\ &= [\eta_\tau(k,l)\psi(w \otimes \alpha)](w). \end{aligned}$$

By Corollary 6.1.9 it follows that, modulo a scalar, ψ is unitary. Plugging in test vectors, one sees that $\Psi = \sqrt{\dim(V_\tau)}^{-1}\psi$ satisfies the lemma. $\square$

Corollary 7.2.6 *There is a natural unitary isomorphism*

$$L^2(K) \cong \widehat{\bigoplus_{\tau \in \widehat{K}}} V_\tau \otimes V_{\tau^*},$$

where each finite dimensional space $V_\tau \otimes V_{\tau^*}$ carries the tensor product Hilbert-space structure. This isomorphism intertwines the $K \times K$ representation η with the sum of the representations $\tau \otimes \tau^*$, where τ^* is the representation dual to τ. In particular, we get direct sum decompositions

$$L \cong \widehat{\bigoplus_{\tau \in \widehat{K}}} 1_{V_\tau} \otimes \tau^* \quad \text{and} \quad R \cong \widehat{\bigoplus_{\tau \in \widehat{K}}} \tau \otimes 1_{V_{\tau^*}}$$

for the left and right regular representations of K.

Proof The corollary is immediate from the theorem and the lemma. The assertion about the left and right translation operations follows from restricting to one factor of the group $K \times K$. $\square$

7.3 Isotypes

Let (π, V_π) be a unitary representation of the compact group K. For $(\tau, V_\tau) \in \widehat{K}$ we define the *isotype of* τ or the *isotypical component* of τ in π as the subspace

$$V_\pi(\tau) \stackrel{\text{def}}{=} \sum_{\substack{U \subset V_\pi \\ U \cong V_\tau}} U.$$

This is the sum of all invariant subspaces U, which are K-isomorphic to V_τ. Another description of the isotype is this: There is a canonical map

$$\begin{aligned} T_\tau : \operatorname{Hom}_K(V_\tau, V_\pi) \otimes V_\tau &\to V_\pi \\ \alpha \otimes v &\mapsto \alpha(v). \end{aligned}$$

This map intertwines the action $\mathrm{Id} \otimes \tau$ on $\operatorname{Hom}_K(V_\tau, V_\pi) \otimes V_\tau$ with π, from which it follows that the image of T_τ lies in $V_\pi(\tau)$. Indeed, the image is all of $V_\pi(\tau)$, since if $U \subset V_\pi$ is a closed subspace with $\pi|_U \cong \tau$ via $\alpha : V_\tau \to U$, then $U = T_\tau(\alpha \otimes V_\tau)$ by construction of T_τ. Note that if (τ, V_τ) and (σ, V_σ) are two non-equivalent irreducible representations, then $V_\pi(\tau) \perp V_\pi(\sigma)$, which follows from the fact that if $U, U' \subseteq V_\pi$ are subspaces with $U \cong V_\tau, U' \cong V_\sigma$, then the orthogonal projection $P : V_\pi \to U'$ restricts to a K-homomorphism $P|_U : U \to U'$, which therefore must be 0.

Lemma 7.3.1 *On the vector space* $\operatorname{Hom}_K(V_\tau, V_\pi)$ *there is an inner product, making it a Hilbert space, such that* T_τ *is an isometry.*

Proof Let $v_0 \in V_\tau$ be of norm one. For $\alpha, \beta \in H = \operatorname{Hom}_K(V_\tau, V_\pi)$ set $\langle \alpha, \beta \rangle \stackrel{\text{def}}{=} \langle \alpha(v_0), \beta(v_0) \rangle$. As by Corollary 6.1.9, any element of $\operatorname{Hom}_K(V_\tau, V_\pi)$ is either zero or injective, it follows that $\langle \cdot, \cdot \rangle$ is indeed an inner product on H. We show that H is complete. For this let α_n be a Cauchy-sequence in H. Then $\alpha_n(v_0)$ is a Cauchy-sequence in V_π, so there exists $w_0 \in V_\pi$ such that $\alpha_n(v_0)$ converges to w_0. For $k \in K$ the sequence $\alpha_n(\tau(k)v_0) = \pi(k)\alpha_n(v_0)$ converges to $\pi(k)w_0$. Likewise, for $f \in L^1(K)$ the sequence $\alpha_n(\tau(f)v_0) = \pi(f)\alpha_n(v_0)$ converges to $\pi(f)w_0$. Let $I \subset L^1(K)$ be the annihilator of v_0, i.e., I is the set of all $f \in L^1(K)$ with $\tau(f)v_0 = 0$. It follows that every $f \in I$ also annihilates w_0. Therefore the map $\alpha : V_\tau \cong L^1(K)/I \to V_\pi$ mapping $\tau(f)v_0$ to $\pi(f)w_0$ is well-defined and a K-homomorphism. It follows that α is the limit of the sequence α_n, so H is complete. We now show that $T = T_\tau$ is an isometry. For fixed α the inner product on V_τ given by $(v, w) = \langle \alpha(v), \alpha(w) \rangle$ is K-invariant. Therefore, by Lemma 7.1.1, there is $c(\alpha) > 0$ such that $(v, w) = c(\alpha)\langle v, w \rangle$ for all $v, w \in V_\tau$. So we get $\langle T(\alpha \otimes v), T(\alpha \otimes v) \rangle = (v, v) = c(\alpha)\langle v, v \rangle$. Setting $v = v_0$, we conclude that $c(\alpha) = \langle \alpha, \alpha \rangle$, which proves that T_τ is indeed an isometry. $\square$

It follows from the above lemma that $V_\pi(\tau)$ is isometrically isomorphic to the Hilbert space tensor product $\operatorname{Hom}_K(V_\tau, V_\pi) \hat{\otimes} V_\tau$ and that $\pi|_{V_\pi(\tau)}$ is unitarily equivalent to the representation $\mathrm{Id} \otimes \tau$ on this tensor product. If we choose an orthonormal base $\{\alpha_i : i \in I\}$ of $\mathrm{Hom}_K(V_\tau, V_\pi)$, then we get a canonical isomorphism

$$\mathrm{Hom}_K(V_\tau, V_\pi) \hat{\otimes} V_\tau \cong \widehat{\bigoplus_{i \in I}} V_\tau$$

given by sending an elementary tensor $\alpha \otimes v$ to $\sum_{i \in I} \langle \alpha, \alpha_i \rangle v$. Thus we see that $V_\pi(\tau)$ is unitarily equivalent to a direct sum of V_τ's with multiplicity $I = \dim \mathrm{Hom}_K(V_\tau, V_\pi)$.

Theorem 7.3.2

(a) $V_\pi(\tau)$ *is a closed invariant subspace of* V_π.

(b) $V_\pi(\tau)$ *is* K*-isomorphic to a direct Hilbert sum of copies of* V_τ.

(c) V_π *is the direct Hilbert sum of the isotypes* $V_\pi(\tau)$ *where* τ *ranges over* $\widehat{K}$.

Proof As $V_\pi(\tau)$ is an isometric image of a complete space, it is complete, hence closed. The space $V_\pi(\tau)$ is a sum of invariant spaces, hence invariant, so (a) follows. Now let $V_\pi = \widehat{\bigoplus}_i V_i$ be any decomposition into irreducibles. Set $\tilde{V}_\pi(\tau) = \widehat{\bigoplus}_{i: V_i \cong V_\tau} V_i$. Then it follows that $\tilde{V}_\pi(\tau) \subset V_\pi(\tau)$ as the latter contains the direct sum and is closed. Now clearly V_π is the direct Hilbert sum of the spaces $\tilde{V}_\pi(\tau)$, and hence it is also the direct Hilbert sum of the $V_\pi(\tau)$, as the latter are pairwise orthogonal. This implies (c) and a fortiori $\tilde{V}_\pi(\tau) = V_\pi(\tau)$ and thus (b). □

Proposition 7.3.3 *Let* (π, V_π) *be a unitary representation of the compact group* K. *For* $\tau \in \widehat{K}$ *the orthogonal projection* $P : V_\pi \to V_\pi(\tau)$ *is given by*

$$P(v) = \dim(\tau) \int_K \overline{\chi_\tau(x)} \pi(x) v \, dx.$$

Proof We have to show that for any two vectors $v, w \in V_\pi$ one has $\langle Pv, w \rangle = \dim(\tau) \int_K \overline{\chi_\tau(x)} \langle \pi(x) v, w \rangle \, dx$. Let (v, w) denote the right hand side of this identity. Write $v = v_0 + v_1$, where $v_0 \in V_\pi(\tau)$ and $v_1 \in V_\pi(\tau)^\perp$. Likewise decompose w as $w_0 + w_1$. Then $\langle Pv, w \rangle = \langle v_0, w_0 \rangle$. The Peter-Weyl theorem implies that $(v_0, w_0) = \langle v_0, w_0 \rangle$. To see this, we decompose $V_\pi(\tau)$ into a direct sum of irreducibles, each equivalent to V_τ. It then suffices to assume that v_0, w_0 lie in the same summand, since otherwise we have $\langle Pv, w \rangle = 0 = \langle v_0, w_0 \rangle$. The result then follows from expressing v_0, w_0 in terms of an orthonormal basis of V_τ. The spaces $V_\pi(\tau)$ and its orthocomplement are invariant under π, therefore $(v_0, w_1) = 0 = (v_1, w_0)$. Finally, as $V_\pi(\tau)^\perp$ is a direct sum of isotypes different from τ, the Peter-Weyl theorem also implies that $(v_1, w_1) = 0$. As the map $(\cdot, \cdot)$ is additive in both components, we get

$$\begin{aligned}(v, w) &= (v_0, w_0) + (v_0, w_1) + (v_1, w_0) + (v_1 + w_1)\\ &= (v_0, w_0) = \langle v_0, w_0 \rangle = \langle Pv, w \rangle,\end{aligned}$$

as claimed. □

Example 7.3.4 It follows from the Peter-Weyl Theorem that the isotype $L^2(K)_R(\tau)$ of the right regular representation $(R, L^2(K))$ for the irreducible representation τ of the compact group K is the linear span of the functions $\tau_{ij}(x) = \left\langle \tau(x)e_i, e_j \right\rangle$. In particular, it follows that all functions in $L^2(K)_R(\tau)$ are continuous. Similarly, the isotype $L^2(K)_L(\tau)$ of the left regular representation $(L, L^2(K))$ is given by the linear span of the functions $\overline{\tau}_{ij}$, the complex conjugates of the τ_{ij}.

7.4 Induced Representations

Let K be a compact group, and let $M \subset K$ be a closed subgroup. Let (σ, V_σ) be a finite dimensional unitary representation of M. We now define the *induced representation* $\pi_\sigma = \mathrm{Ind}_M^K(\sigma)$ as follows. First define the Hilbert-space $L^2(K, V_\sigma)$ of all measurable functions $f : K \to V_\sigma$ satisfying $\int_K \|f(x)\|_\sigma^2 \, dk < \infty$ modulo nullfunctions, where $\|\cdot\|_\sigma$ is the norm in the space V_σ. This is a Hilbert-space with inner product $\langle f, g \rangle = \int_K \langle f(k), g(k) \rangle_\sigma \, dk$. Choosing an orthonormal basis of V_σ gives an isomorphism $L^2(K, V_\sigma) \cong L^2(K)^{\dim(V_\sigma)}$, which shows completeness of $L^2(K, V_\sigma)$.

The space of the representation π_σ is the space $\mathrm{Ind}_M^K(V_\sigma)$ of all $f \in L^2(K, V_\sigma)$ such that for every $m \in M$ the identity $f(mk) = \sigma(m)f(k)$ holds almost everywhere in $k \in K$. This is a closed subspace of $L^2(K, V_\sigma)$ as we have

$$\mathrm{Ind}_M^K(V_\sigma) = \bigcap_{m \in M} \ker T_m,$$

where for given $m \in M$ the continuous operator $f \mapsto L_{m^{-1}} f - \sigma(m) f$ is denoted by T_m. The representation π_σ is now defined by

$$\pi_\sigma(y)f(x) = f(xy).$$

The representation π_σ is clearly unitary.

It suffices to consider finite dimensional, indeed irreducible representations σ here, since an arbitrary representation σ of M decomposes as a direct sum $\sigma = \bigoplus_{i \in I} \sigma_i$ of irreducibles and there is a canonical isomorphism

$$\mathrm{Ind}_M^K \left(\bigoplus_{i \in I} \sigma_i \right) \cong \bigoplus_{i \in I} \mathrm{Ind}_M^K(\sigma_i).$$

So suppose that σ is irreducible. As K is compact, π_σ decomposes as a direct sum of irreducible representations $\tau \in \widehat{K}$, each occurring with some multiplicity $[\pi_\sigma : \tau] \stackrel{\text{def}}{=} \dim \mathrm{Hom}_K \left(V_\tau, V_{\pi_\sigma} \right)$.

Theorem 7.4.1 (Frobenius reciprocity). If σ *is irreducible, the multiplicities* $[\pi_\sigma : \tau]$ *are all finite and can be given as*

$$[\pi_\sigma : \tau] = [\tau|_M : \sigma].$$

More precisely, for every irreducible representation (τ, U) *there is a canonical isomorphism* $\mathrm{Hom}_K\left(U, \mathrm{Ind}_M^K(V_\sigma)\right) \to \mathrm{Hom}_M\left(U|_M, V_\sigma\right)$.

Proof Let V^c be the subspace of V_{π_σ} consisting of all continuous functions $f : K \to V_\sigma$ with $f(mk) = \sigma(m)f(k)$. The space V^c is stable under the K-action and dense in the Hilbert space V_{π_σ}, which can be seen by approximating any f in V_{π_σ} by $\pi_\sigma(\phi)f$ with Dirac functions ϕ in $C(K)$ of arbitrary small support. Let $\alpha \in \mathrm{Hom}_K\left(U, \mathrm{Ind}_M^K(V_\sigma)\right)$. We show that the image of α lies in V^c. For this recall that by the Peter-Weyl Theorem the space $L^2(K)$ decomposes into a direct sum of isotypes $L^2(K)(\gamma)$ for $\gamma \in \widehat{K}$. Here we consider the K-action by right translations only. Each isotype $L^2(K)(\gamma)$ is finite dimensional and consists of continuous functions. We have isometric K-homomorphisms,

$$\alpha : U \to \mathrm{Ind}_M^K(V_\sigma) \hookrightarrow L^2(K, V_\sigma) \xrightarrow{\cong} L^2(K) \otimes V_\sigma,$$

where K acts trivially on V_σ. This implies that $\alpha(U) \subset L^2(K)(\tau) \otimes V_\sigma$ consists of continuous functions. Let $\delta : V^c \to V_\sigma$ be given by $\delta(f) = f(1)$, and define $\psi : \mathrm{Hom}_K(U, \mathrm{Ind}_M^K(V_\sigma)) \to \mathrm{Hom}_M(U|_M, V_\sigma)$ by $\psi(\alpha)(u) = \delta(\alpha(u)) = \alpha(u)(1)$. We claim the ψ is a bijection. For injectivity assume that $\psi(\alpha) = 0$. Then for every $u \in U$ and $k \in K$ one has $\alpha(u)(k) = \pi_\sigma(k)\alpha(u)(1) = \alpha(\tau(k)u)(1) = \psi(\alpha)(\tau(k)u) = 0$, which means $\alpha = 0$.

For surjectivity let $\beta \in \mathrm{Hom}_M(U, V_\sigma)$ and define an element $\alpha \in \mathrm{Hom}_{\mathbb{C}}\left(U, \mathrm{Ind}_M^K(V_\sigma)\right)$ by $\alpha(u)(k) = \beta\left(\tau(k^{-1})u\right)$. By definition, α is a K-homomorphism and $\beta = \psi(\alpha)$. The theorem is proven. □

Example 7.4.2 Let M be a closed subgroup of the compact group K. Then K/M carries a unique Radon measure μ that is invariant under the left translation action of the group K and is normalized by $\mu(K/M) = 1$. The group K acts on the Hilbert space $L^2(K/M, \mu)$ by left translations and this constitutes a unitary representation. This representation is isomorphic to the induced representation $\mathrm{Ind}_M^K(\mathbb{C})$ induced from the trivial representation. An isomorphism between these representations is given by the map $\Phi : L^2(K/M) \to \mathrm{Ind}_M^K(\mathbb{C})$, which maps $\psi \in L^2(K/M)$ to the function $\Phi(\psi) : K \to \mathbb{C}$ defined by

$$\Phi(\psi)(k) \stackrel{\text{def}}{=} \psi\left(k^{-1}M\right).$$

Now, for any $\tau \in \widehat{K}$ the multiplicity $[\tau|_M : 1]$ equals $\dim V_\tau^M$, where V_τ^M denotes the space of M-invariant vectors in V_τ. Thus by Frobenius we get

$$L^2(K/M) \cong \widehat{\bigoplus_{\tau \in \widehat{K}}} \dim\left(V_\tau^M\right) V_\tau$$

where $\dim\left(V_\tau^M\right) V_\tau$ denotes the $\dim\left(V_\tau^M\right)$-fold direct sum of the V_τ's.

7.5 Representations of SU(2)

In this section we consider the irreducible representations of the compact group SU(2). We use the description of these representations to construct decompositions of the Hilbert spaces $L^2(S^3)$ and $L^2(S^2)$, thus giving a glance into the harmonic analysis of the spheres. For this recall the *n-dimensional sphere*, $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$, where $\|x\| = \sqrt{x_1^2 + \cdots + x_{n+1}^2}$ is the *euclidean norm* on $\mathbb{R}^{n+1}$. The set S^n inherits a topology from $\mathbb{R}^{n+1}$. For a subset $A \subset S^n$ let IA be the set of all ta, where $a \in A$ and $0 \le t \le 1$. The set $IA \subset \mathbb{R}^n$ is Borel measurable if and only if $A \subset S^n$ is, (Exercise 7.13). For a measurable set $A \subset S^n$, define the normalized Lebesgue measure as $\mu(A) \stackrel{\text{def}}{=} \frac{\lambda(IA)}{\lambda(IS^n)}$, where λ denotes the Lebesgue measure on $\mathbb{R}^{n+1}$. As a consequence of the transformation formula on $\mathbb{R}^{n+1}$, the Lebesgue measure λ is invariant under the action of the orthogonal group

$$\mathrm{O}(n+1) \stackrel{\text{def}}{=} \left\{ g \in \mathrm{M}_{n+1}(\mathbb{R}) : g^t g = 1 \right\}.$$

The group $\mathrm{O}(n+1)$ can also be described as the group of all $g \in \mathrm{M}_{n+1}(\mathbb{R})$ such that $\|gv\| = \|v\|$ holds for every $v \in \mathbb{R}^{n+1}$. It follows from this description, that $\mathrm{O}(n+1)$ leaves stable the sphere S^n and that the measure μ is invariant under $\mathrm{O}(n+1)$. We denote by $\mathrm{SO}(n+1)$ the *special orthogonal group*, i.e., the group of all $g \in \mathrm{O}(n+1)$ of determinant one. For the next lemma, we consider $\mathrm{O}(n)$ as a subgroup of $\mathrm{O}(n+1)$ via the embedding $g \mapsto \left(\begin{smallmatrix} 1 & 0 \\ 0 & g \end{smallmatrix}\right)$.

Lemma 7.5.1 *Let $n \in \mathbb{N}$, and let $e_1 = (1, 0, \dots, 0)^t$ be the first standard basis vector of $\mathbb{R}^{n+1}$. The matrix multiplication $g \mapsto ge_1$ gives an identification*

$$S^n \cong \mathrm{O}(n+1)/\mathrm{O}(n) \cong \mathrm{SO}(n+1)/\mathrm{SO}(n).$$

This map is invariant under left translations and the normalized Lebesgue measure on S^n is the unique normalized invariant measure on this quotient space.

Proof The group $\mathrm{O}(n)$ is the subgroup of $\mathrm{O}(n+1)$ of all elements with first column equal to e_1, so it is the stabilizer of e_1 and one indeed gets a map $\mathrm{O}(n+1)/\mathrm{O}(n) \to S^n$. As the invariance of the measure is established by the transformation formula, we only need to show surjectivity. Now let $v \in S^n$. Then there always exists a rotation in $\mathrm{SO}(n+1)$ that transforms e_1 into v. One simply chooses a rotation around an axis that is orthogonal to both e_1 and v. This proves surjectivity. The assertion on the measure is due to the uniqueness of invariant measures. □

Recall that SU(2) is the group of all matrices $g \in \mathrm{M}_2(\mathbb{C})$ that are unitary: $g^*g = gg^* = 1$ and satisfy $\det(g) = 1$. These conditions imply

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} : \begin{pmatrix} a \\ b \end{pmatrix} \in S^3 \right\},$$

where we realize the three sphere S^3 as the set of all $z \in \mathbb{C}^2$ with $|z_1|^2 + |z_2|^2 = 1$. From this description and the fact that the Lebesgue measure on S^3 is invariant under O(4) as well as the uniqueness of invariant measures we get the following lemma.

Lemma 7.5.2 *The map* $\mathrm{SU}(2) \to S^3$, *mapping the matrix* $g \in \mathrm{SU}(2)$ *to its first column, is a homeomorphism. Via this homeomorphism, the normalized Lebesgue measure on* S^3 *coincides with the normalized Haar measure on* SU(2).

We want to obtain a convenient formula for computations with the Haar integral on $\mathrm{SU}(2) \cong S^3$. Recall from Calculus that the gamma function $\Gamma : (0, \infty) \to \mathbb{R}$ is defined by the integral $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$. Note that $\Gamma(1) = 1$ and that $\Gamma(x+1) = x\Gamma(x)$ for every $x > 0$, which implies that $\Gamma(n) = (n-1)!$ for every $n \in \mathbb{N}$. Moreover, via the substitution $t = r^2$ we get the alternative formula $\Gamma(x) = 2\int_0^\infty r^{2x-1} e^{-r^2}\, dr$, which we shall use below.

Lemma 7.5.3 *Let* $f : S^3 \to \mathbb{C}$ *be any integrable function, and for each* $m \in \mathbb{N}_0$ let $F_m : \mathbb{C}^2 \to \mathbb{C}$ *be defined by* $F_m(rx) = r^m f(x)$ *for every* $x \in S^3$ *and* $r > 0$. *Further let* $c_m \stackrel{\text{def}}{=} \pi^{-2}\Gamma\left(\frac{m}{2}+2\right)^{-1}$. *Then*

$$\int_{S^3} f(x)\, d\mu(x) = c_m \int_{\mathbb{C}^2} F_m(z) e^{-(\|z\|^2)} d\lambda(z),$$

where λ *stands for the Lebesgue measure on* $\mathbb{C}^2 \cong \mathbb{R}^4$, *and* $\|z\|^2 = |z_1|^2 + |z_2|^2$.

Proof Integration in polar coordinates on $\mathbb{C}^2$ implies

$$\begin{aligned}\int_{\mathbb{C}^2} F_m(z) e^{-(\|z\|^2)}\, d\lambda(z) &= c \int_0^\infty r^3 \int_{S^3} F_m(rx) e^{-r^2}\, d\mu(x)\, dr \\ &= c\left(\int_0^\infty r^{3+m} e^{-r^2}\, dr\right) \int_{S^3} f(x)\, d\mu(x) \\ &= \frac{c}{2}\Gamma\left(\frac{m}{2}+2\right) \int_{S^3} f(x)\, d\mu(x).\end{aligned}$$

where c is some positive constant (the non-normalized volume of S^3). To compute the constant c let $f \equiv 1$ and $m = 0$. Since $\Gamma(2) = 1$ one gets

$$c = 2\int_{\mathbb{C}^2} e^{-(\|z\|^2)} d\lambda(z) = 2\left(\int_{\mathbb{C}} e^{-|z_1|^2} dz_1\right)\left(\int_{\mathbb{C}} e^{-|z_2|^2} dz_2\right) = 2\pi^2.$$

The lemma follows. □

For $m \in \mathbb{N}_0$ let $\mathcal{P}_m$ denote the set of homogeneous polynomials on $\mathbb{C}^2$ of degree m. In other words, $\mathcal{P}_m$ is the space of all polynomial functions $p : \mathbb{C}^2 \to \mathbb{C}$ that satisfy $p(tz) = t^m p(z)$ for every $z \in \mathbb{C}^2$ and every $t \in \mathbb{C}$. Every $p \in \mathcal{P}_m$ can uniquely be written as $p(z) = \sum_{k=0}^m c_k\, z_1^k z_2^{m-k}$. For $p, \eta \in \mathcal{P}_m$ define

$$\langle p, \eta \rangle_m \stackrel{\text{def}}{=} \langle p|_{S^3}, \eta|_{S^3} \rangle_{L^2(S^3)} = \int_{S^3} p(x)\overline{\eta(x)}\, d\mu(x).$$

It then follows from Lemma 7.5.3 that

$$\langle p, \eta \rangle_m = c_{2m} \int_{\mathbb{C}^2} p(z)\overline{\eta(z)} e^{-\|z\|^2}\, d\lambda(z).$$

We define a representation π_m of SU(2) on $\mathcal{P}_m$ by

$$(\pi_m(g)p)(z) \stackrel{\text{def}}{=} p\left(g^{-1}(z)\right).$$

Theorem 7.5.4 *For every $m \geq 0$, the representation $(\pi_m, \mathcal{P}_m)$ is irreducible. Every irreducible unitary representation of* SU(2) *is unitarily equivalent to one of the representations $(\mathcal{P}_m, \pi_m)$. Thus*

$$\widehat{\mathrm{SU}(2)} = \{[(\mathcal{P}_m, \pi_m)] : m \in \mathbb{N}_0\},$$

where $[(\mathcal{P}_m, \pi_m)]$ denotes the equivalence class of $(\mathcal{P}_m, \pi_m)$.

Proof This is Theorem 10.2.2 in [Dei05]. □

The next corollary follows from the Peter-Weyl Theorem.

Corollary 7.5.5 *The* SU(2) *representation on $L^2(S^3)$ is isomorphic to the orthogonal sum $\bigoplus_{m \geq 0} (m+1)\mathcal{P}_m$, where $\mathcal{P}_m$ is the space of homogeneous polynomials of degree m and each $\mathcal{P}_m$ occurs with multiplicity $m+1$.*

We want to close this section with a study of the two-sphere S^2. For each $\lambda \in \mathbb{T}$ consider the matrix $g_\lambda \stackrel{\text{def}}{=} \mathrm{diag}(\lambda, \bar{\lambda})$. Then we may regard $\mathbb{T} \cong \{g_\lambda : \lambda \in \mathbb{T}\}$ as a closed subgroup of SU(2). Recall that we can identify $\mathrm{SU}(2) \cong S^3$. The map $p : S^3 \to S^2$ of the following lemma is known as the *Hopf fibration* of S^3.

Lemma 7.5.6 *Let us realize the two-sphere as*

$$S^2 = \left\{(v, x) : v \in \mathbb{C}, x \in \mathbb{R} \text{ and } |v|^2 + x^2 = 1\right\}.$$

Then the map $\eta : S^3 \to S^2$ defined by $\eta(a, b) = \left(2a\bar{b}, |a|^2 - |b|^2\right)$ factors through a homeomorphism $\mathrm{SU}(2)/\mathbb{T} \cong S^2$, which maps the normalized SU(2)*-invariant measure on* $\mathrm{SU}(2)/\mathbb{T}$ *to the normalized Lebesgue measure on S^2.*

Proof For $(a, b) \in S^3$ we compute $|2a\bar{b}|^2 + (|a|^2 - |b|^2)^2 = 4|ab|^2 + |a|^4 - 2|ab|^2 + |b|^4 = (|a|^2 + |b|^2)^2 = 1$, which shows that the image of η lies in S^2. To see that the map is surjective let $(v, x) \in S^2$ be given. Choose $a, b \in \mathbb{C}$ with $|a|^2 + |b|^2 = 1$ and $|a|^2 - |b|^2 = x$, which is possible since $|x| \leq 1$. Then $|v| = 2|ab|$, and there exists a complex number z of modulus one such that $v = 2za\bar{b}$. It then follows that $\eta(za, b) = (v, x)$.

We now claim that $\eta(a,b) = \eta(a',b')$ if and only if there exists $\lambda \in \mathbb{T}$ with $(a,b) = \lambda(a',b')$. The if direction is easy to check, so assume now that $(2a\bar{b}, |a|^2 - |b|^2) = (2a'\bar{b}', |a'|^2 - |b'|^2)$. Since $|a|^2 + |b|^2 = 1 = |a'|^2 + |b'|^2$ we get $1 - 2|b|^2 = |a|^2 - |b|^2 = |a'|^2 - |b'|^2 = 1 - 2|b'|^2$, from which it follows that $|b| = |b'|$. Hence $\lambda = \overline{b'}/\bar{b} \in \mathbb{T}$ with $b = \lambda b'$. Since $a\bar{b} = a'\bar{b}'$ it also follows that $a = \lambda a'$.

Finally, since

$$\begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix} = \begin{pmatrix} \lambda a & -\overline{\lambda b} \\ \lambda b & \overline{\lambda a} \end{pmatrix}$$

it is then clear that η factorizes through a bijection $\mathrm{SU}(2)/\mathbb{T} \cong S^2$. Since η is continuous and all spaces are compact, this bijection is also a homeomorphism.

We next need to show that the induced action of $g \in \mathrm{SU}(2)$ on $S^2 \cong \mathrm{SU}(2)/\mathbb{T}$ comes from some linear transformation. Since it maps S^2 to S^2 it is then automatically orthogonal. But if $g = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}$ and $\begin{pmatrix} v \\ x \end{pmatrix} = p \begin{pmatrix} z \\ w \end{pmatrix} \in S^2$ then a short computation shows that

$$\begin{aligned} g \cdot \begin{pmatrix} v \\ x \end{pmatrix} = p \begin{pmatrix} az - \bar{b}w \\ bz + \bar{a}w \end{pmatrix} &= \begin{pmatrix} 2(az - \bar{b}w)(\bar{b}\bar{z} + a\bar{w}) \\ |az - \bar{b}w|^2 - |bz + \bar{a}w|^2 \end{pmatrix} \\ &= \begin{pmatrix} 2a\bar{b}x - a^2 v - \bar{b}^2\bar{v} \\ (|a|^2 - |b|^2)x - abv - \overline{abv} \end{pmatrix}. \end{aligned}$$

This expression is obviously $\mathbb{R}$-linear in v and x. This implies that SU(2) acts on S^2 through orthogonal transformations. As the normalized Lebesgue measure on S^2 is invariant under such transformations, it is invariant under the action of SU(2), so it coincides with the unique invariant measure on $\mathrm{SU}(2)/\mathbb{T}$. □

The left translation on $\mathrm{SU}(2)/\mathbb{T}$ induces a unitary representation π of SU(2) on $L^2(S^2) \cong L^2(\mathrm{SU}(2)/\mathbb{T})$. We will now make use of the Frobenius reciprocity to give an explicit decomposition of this representation.

Proposition 7.5.7 *The representation π of* SU(2) *on $L^2(S^2)$ is isomorphic to the direct sum $\bigoplus_{m\geq 0} \pi_{2m}$.*

Proof As SU(2) is a compact group, Theorem 7.2.3 implies that π is a direct sum of irreducibles. By Theorem 7.5.4, it is a direct sum of copies of the π_m. It remains to show that for $m \geq 0$ the irreducible representation π_m has multiplicity 1 in π if m is even and 0 otherwise. Example 7.4.2 shows that π is equivalent to the induced representation $\mathrm{Ind}_{\mathbb{T}}^{\mathrm{SU}(2)}(1)$, hence the Frobenius reciprocity applies. So by Theorem 7.4.1 we can compute the multiplicity as $[\pi : \pi_m] = [\pi_m|_{\mathbb{T}} : 1]$. Using the basis $z_1^m, z_1^{m-1}z_2, \ldots, z_2^m$ one sees that

$$[\pi_m|_{\mathbb{T}} : 1] = \begin{cases} 1 & m \text{ even}, \\ 0 & \text{otherwise}. \end{cases}$$

The proposition is proven. □

7.6 Exercises

Exercise 7.1 Let $\mathcal{A}$ be an abelian subgroup of $\mathrm{U}(n)$. Show that there is $S \in \mathrm{GL}_2(\mathbb{C})$ such that $S\mathcal{A}S^{-1}$ consists of diagonal matrices only.

Exercise 7.2 Let G be a finite group.

(a) Show that the number of elements of G equals $\sum_{\tau \in \widehat{G}} \dim(V_\tau)^2$.

(b) Show that the space of conjugation-invariant functions has a basis $(\chi_\tau)_{\tau \in \widehat{G}}$. Conclude that the number of irreducible representations of G equals the number of conjugacy classes.

Exercise 7.3 Let (π, V_π) and (η, V_η) be two unitary representations of the compact group K. Suppose there exists a bijective bounded linear operator $T : V_\pi \to V_\eta$ that intertwines π and η, i.e., $T\pi(k) = \eta(k)T$ holds for every $k \in K$. Show that there already exists a unitary intertwining operator $S : V_\pi \to V_\eta$.

Exercise 7.4 For a compact group K write $L^2(\widehat{K}) := \widehat{\bigoplus}_{\tau \in \widehat{K}} \mathrm{End}(V_\tau)$ equipped with the inner product as in the second version of the Peter-Weyl Theorem and let

$$\mathcal{F} : L^2(K) \to L^2(\widehat{K}), \quad \mathcal{F}(f) = \hat{f}$$

be the Plancherell-isomorphism as given in that theorem. Show that the inverse of this isomorphism is given by the inverse Fourier transform

$$\widehat{\mathcal{F}} : L^2(\widehat{K}) \to L^2(K); \widehat{\mathcal{F}}(g)(x) = \sum_{\tau \in \widehat{K}} \dim(\tau) \operatorname{tr}\big(g(\tau)\tau(x^{-1})\big),$$

for $(\tau \mapsto g(\tau)) \in L^2(\widehat{K})$.

Exercise 7.5 Let K and L be compact groups, and let $\tau \in \widehat{K}$ and $\eta \in \widehat{L}$. Show that $\tau \otimes \eta(k, l) = \tau(k) \otimes \tau(l)$ defines an element of $\widehat{K \times L}$ and that the map $(\tau, \eta) \mapsto \tau \otimes \eta$ is a bijection from $\widehat{K} \times \widehat{L}$ to $\widehat{K \times L}$.

Exercise 7.6 Let K be a compact group, let $(\pi, V_\pi) \in \widehat{K}$, and let χ_π be its character. Show that for $x \in K$, $\int_K \pi(kxk^{-1})\, dk = \frac{\chi_\pi(x)}{\dim V_\pi} \mathrm{Id}$. Conclude that $\chi_\pi(x)\chi_\pi(y) = \dim(V_\pi) \int_K \chi_\pi\left(kxk^{-1}y\right) dk$ holds for all $x, y \in K$.

Exercise 7.7 Let (π, V_π) be a finite dimensional representation of the compact group K with character χ_π. Let π^* be the *dual representation* on $V_{\pi^*} = V_\pi^*$. Show that $\chi_{\pi^*} = \overline{\chi_\pi}$.

Exercise 7.8 Keep the notation of the last exercise. The representation π is called a *self-dual representation* if $\pi \cong \pi^*$. Show that the following are equivalent.

(a) π is self-dual,

(b) there exists an antilinear bijective K-homomorphism $C : V_\pi \to V_\pi$,

(c) there exists a real sub vector space $V_\mathbb{R}$ of V_π such that V_π is the orthogonal sum of $V_\mathbb{R}$ and $iV_\mathbb{R}$, and $\pi(K)V_\mathbb{R} = V_\mathbb{R}$.

(d) χ_π takes only real values,

Exercise 7.9 Show that every unitary representation of SU(2) is self dual. (Hint: Use Theorem 7.2.3.)

Exercise 7.10 Let the compact group K act on $L^2(K)$ by conjugation, i.e., $k.f(x) = f(k^{-1}xk)$. Show that the space of K-invariants $L^2(K/\mathrm{conj})$ is closed and that $(\chi_\pi)_{\pi\in\widehat{K}}$ is an orthonormal basis of the Hilbert space $L^2(K/\mathrm{conj})$.

Exercise 7.11 Let (π, V_π) be a representation of a compact group K on a Banach space V_π. Show that $V_\pi = \overline{\bigoplus_{i\in I} V_i}$, where each V_i is a finite-dimensional irreducible subspace.

(Hint: Use matrix coefficients as in the proof of Theorem 7.2.3 to get a map $T : V_\pi \to W$, where W is a finite-dimensional irreducible representation. Then fix a complementary space of $\ker(T)$ inside V_π and apply a projection operator as in Proposition 7.3.3.)

Exercise 7.12 Let K be a compact group. Show that the following are equivalent.

- Every character χ_π for $\pi \in \widehat{K}$ is real valued.
- For every $k \in K$ there exists $l \in K$ such that $lkl^{-1} = k^{-1}$.

Exercise 7.13 For a subset $A \subset S^n$ let IA be defined as in the beginning of Sect. 7.5. Show that A is Borel measurable as a subset of S^n if and only if IA is measurable as a subset of $\mathbb{R}^n$.

Exercise 7.14 Consider the map $\phi : \mathrm{SU}(2) \times \mathbb{T} \to \mathrm{U}(2)$ that sends a pair (g, z) to the matrix zg. Show that ϕ is a surjective homomorphism. Compute $\ker\phi$ and $\widehat{\mathrm{U}(2)}$.

Exercise 7.15 Via Lemma 7.5.6, the group SU(2) acts on $S^2 \cong \mathrm{SU}(2)/\mathbb{T}$. Show that this action determines a surjective homomorphism $\psi : \mathrm{SU}(2) \to \mathrm{SO}(3)$ such that $\ker\psi = \{\pm I\}$. In particular, this gives an isomorphism $\mathrm{SO}(3) \cong \mathrm{SU}(2)/\{\pm I\}$. Use this to compute $\widehat{\mathrm{SO}(3)}$.