Chapter 6
Representations

In this chapter we introduce the basic concepts of representation theory of locally compact groups. Classically, a representation of a group G is an injective group homomorphism from G to some $\mathrm{GL}_n(\mathbb{C})$, the idea being that the "abstract" group G is "represented" as a matrix group.

In order to understand a locally compact group, it is necessary to consider its actions on possibly infinite dimensional spaces like $L^2(G)$. For this reason, one considers infinite dimensional representations as well.

6.1 Schur's Lemma

For a Banach space V, let $\mathrm{GL}_{\mathrm{cont}}(V)$ be the set of bijective bounded linear operators T on V. It follows from the Open Mapping Theorem C.1.5 that the inverses of such operators are bounded as well, so that $\mathrm{GL}_{\mathrm{cont}}(V)$ is a group. Let G be a topological group. A *representation* of G on a Banach space V is a group homomorphism of G to the group $\mathrm{GL}_{\mathrm{cont}}(V)$, such that the resulting map $G \times V \to V$, given by $(g, v) \mapsto \pi(g)v$, is continuous.

Lemma 6.1.1 *Let π be a group homomorphism of the topological group G to $\mathrm{GL}_{\mathrm{cont}}(V)$ for a Banach space V. Then π is a representation if and only if*

(a) *the map $g \mapsto \pi(g)v$ is continuous at $g = 1$ for every $v \in V$, and*

(b) *the map $g \mapsto \|\pi(g)\|_{\mathrm{op}}$ is bounded in a neighborhood of the unit in G.*

Proof Suppose π is a representation. Then (a) is obvious. For (b) note that for every neighborhood Z of zero in V there exists a neighborhood Y of zero in V and a neighborhood U of the unit in G such that $\pi(U)Y \subset Z$. This proves (b). For the converse direction write

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$$\begin{aligned}\|\pi(g)v - \pi(g_0)v_0\| &\le \|\pi(g_0)\|_{\text{op}}\|\pi(g_0^{-1}g)v - v_0\| \\ &\le \|\pi(g_0)\|_{\text{op}}\|\pi(g_0^{-1}g)(v - v_0)\| \\ &\quad + \|\pi(g_0)\|_{\text{op}}\|\pi(g_0^{-1}g)v_0 - v_0\| \\ &\le \|\pi(g_0)\|_{\text{op}}\|\pi(g_0^{-1}g)\|_{\text{op}}\|v - v_0\| \\ &\quad + \|\pi(g_0)\|_{\text{op}}\|\pi(g_0^{-1}g)v_0 - v_0\|.\end{aligned}$$

Under the assumptions given, both terms on the right are small if g is close to g_0, and v is close to v_0. □

Examples 6.1.2

- For a continuous group homomorphism $\chi : G \to \mathbb{C}^\times$ define a representation π_χ on $V = \mathbb{C}$ by $\pi_\chi(g)v = \chi(g) \cdot v$.
- Let $G = \mathrm{SL}_2(\mathbb{R})$ be the group of real 2×2 matrices of determinant one. This group has a natural representation on $\mathbb{C}^2$ given by matrix multiplication.

Definition Let V be a Hilbert space. A representation π on V is called a *unitary representation* if $\pi(g)$ is unitary for every $g \in G$. That means π is unitary if $\langle \pi(g)v, \pi(g)w \rangle = \langle v, w \rangle$ holds for every $g \in G$ and all $v, w \in V$.

Lemma 6.1.3 *A representation π of the group G on a Hilbert space V is unitary if and only if $\pi(g^{-1}) = \pi(g)^*$ holds for every $g \in G$.*

Proof An operator T is unitary if and only if it is invertible and $T^* = T^{-1}$. For a representation π and $g \in G$ the operator $\pi(g)$ is invertible and satisfies $\pi(g^{-1}) = \pi(g)^{-1}$. So π is unitary if and only if for every $g \in G$ one has $\pi(g^{-1}) = \pi(g)^{-1} = \pi(g)^*$. □

Examples 6.1.4

- The representation π_χ defined by a continuous group homomorphism $\chi : G \to \mathbb{C}^\times$ is unitary if and only if χ maps into the compact torus $\mathbb{T}$.
- Let G be a locally compact group. On the Hilbert space $L^2(G)$ consider the *left regular representation* $x \mapsto L_x$ with

$$L_x\phi(y) = \phi(x^{-1}y), \qquad \phi \in L^2(G).$$

This representation is unitary, as by the left invariance of the Haar measure,

$$\begin{aligned}\langle L_x\phi, L_x, \psi\rangle &= \int_G L_x\phi(y)\overline{L_x\psi(y)}\,dy\\ &= \int_G \phi(x^{-1}y)\overline{\psi(x^{-1}y)}\,dy\\ &= \int_G \phi(y)\overline{\psi(y)}\,dy = \langle\phi,\psi\rangle.\end{aligned}$$

Definition Let (π_1, V_1) and (π_2, V_2) be two unitary representations. On the direct sum $V = V_1 \oplus V_2$ one has the *direct sum representation* $\pi = \pi_1 \oplus \pi_2$. More generally, if $\{\pi_i : i \in I\}$ is a family of unitary representations acting on the Hilbert spaces V_i, we write $\bigoplus_{i\in I}\pi_i$ for the direct sum of the representations π_i, $i \in I$ on the Hilbert space $\widehat{\bigoplus}_{i\in I} V_i$. See also Exercise 6.3 and appendix C.3.

Example 6.1.5 Let $G = \mathbb{R}/\mathbb{Z}$, and let $V = L^2(\mathbb{R}/\mathbb{Z})$. Let π be the left regular representation. By the Plancherel Theorem, the elements of the dual group $\widehat{G} = \{e_k : k \in \mathbb{Z}\}$ with $e_k([x]) = e^{2\pi i k x}$ form an orthonormal basis of $L^2(\mathbb{R}/\mathbb{Z})$ so that π is a direct sum representation on $V = \widehat{\bigoplus}_{k\in\mathbb{Z}}\mathbb{C}e_k$, where $e_k(x) = e^{2\pi i k x}$ and G acts on $\mathbb{C}e_k$ through the character $\bar{e}_k$.

Definition A representation (π, V_π) is called a *subrepresentation* of a representation (η, V_η) if V_π is a closed subspace of V_η and π equals η restricted to V_π. So every closed subspace $U \subset V_\eta$ that is stable under η, i.e., $\eta(G)U \subset U$, gives rise to a subrepresentation.

A representation is called *irreducible* if it does not possess any proper subrepresentation, i.e., if for every closed subspace $U \subset V_\pi$ that is stable under π, one has $U = 0$ or $U = V_\pi$.

Example 6.1.6. Let $\mathrm{U}(n)$ denote the group of unitary $n \times n$ matrices, so the group of all $u \in \mathrm{M}_n(\mathbb{C})$ such that $uu^* = I$ (unit matrix), where $u^* = \bar{u}^t$. The natural representation of $\mathrm{U}(n)$ on $\mathbb{C}^n$ is irreducible (See Exercise 6.5).

Definition Let (π, V_π) be a representation of G. A vector $v \in V_\pi$ is called a *cyclic vector* if the linear span of the set $\{\pi(x)v : x \in G\}$ is dense in V_π. In other words, v is cyclic if the only subrepresentation containing v is the whole of π. It follows that a representation is irreducible if and only if every nonzero vector is cyclic.

Lemma 6.1.7 *(Schur) Let (π, V_π) be a unitary representation of the topological group G. Then the following are equivalent*

(a) (π, V_π) *is irreducible.*

(b) *If T is a bounded operator on V_π such that $T\pi(g) = \pi(g)T$ for every $g \in G$, then $T \in \mathbb{C}\,\mathrm{Id}$.*

Proof Since $\pi(g^{-1}) = \pi(g)^*$, the set $\{\pi(g) : g \in G\}$ is a self-adjoint subset of $\mathcal{B}(V_\pi)$. Thus the result follows from Theorem 5.1.6. □

Let $(\pi, V_\pi), (\eta, V_\eta)$ be representations of G. A continuous linear operator $T : V_\pi \to V_\eta$ is called a *G-homomorphism* or *intertwining operator* if

$$T\pi(g) = \eta(g)T$$

holds for every $g \in G$. We write $\mathrm{Hom}_G(V_\pi, V_\eta)$ for the set of all G-homomorphisms from V_π to V_η. A nice way to rephrase the Lemma of Schur is to say that a unitary representation (π, V_π) is irreducible if and only if $\mathrm{Hom}_G(V_\pi, V_\pi) = \mathbb{C}\,\mathrm{Id}$.

Definition If π, η are unitary, they are called *unitarily equivalent* if there exists a unitary G-homomorphism $T : V_\pi \to V_\eta$.

Example 6.1.8. Let $G = \mathbb{R}$, and let $V_\pi = V_\eta = L^2(\mathbb{R})$. The representation π is given by $\pi(x)\phi(y) = \phi(x+y)$ and η is given by $\eta(x)\phi(y) = e^{2\pi i x y}\phi(y)$. By Theorem 3.3.1 in [Dei05] (see also Exercise 6.4), the Fourier transform $L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is an intertwining operator from π to η.

Corollary 6.1.9 *Let* (π, V_π) *and* (η, V_η) *be two irreducible unitary representations. Then a G-homomorphism T from* V_π to V_η *is either zero or invertible with continuous inverse. In the latter case there exists a scalar* $c > 0$ *such that* cT *is unitary. The space* $\mathrm{Hom}_G(V_\pi, V_\eta)$ *is zero unless* π *and* η *are unitarily equivalent, in which case the space is of dimension* 1.

Proof Let $T : V_\pi \to V_\eta$ be a G-homomorphism. Its adjoint $T^* : V_\eta \to V_\pi$ is also a G-homomorphism as is seen by the following calculation for $v \in V_\pi$, $w \in V_\eta$, and $g \in G$,

$$\begin{aligned}\left\langle v, T^*\eta(g)w\right\rangle &= \langle Tv, \eta(g)w\rangle = \left\langle \eta(g^{-1})Tv, w\right\rangle \\ &= \left\langle T\pi(g^{-1})v, w\right\rangle = \left\langle \pi(g^{-1})v, T^*w\right\rangle \\ &= \left\langle v, \pi(g)T^*w\right\rangle.\end{aligned}$$

This implies that T^*T is a G-homomorphism on V_π, and therefore it is a multiple of the identity $\lambda\,\mathrm{Id}$ by the Lemma of Schur. If T is non-zero, T^*T is non-zero and positive semi-definite, so $\lambda > 0$. Let $c = \sqrt{\lambda^{-1}}$, then $(cT)^*(cT) = \mathrm{Id}$. A similar argument shows that TT^* is bijective, which then implies that cT is bijective, hence unitary. The rest is clear. □

Definition For a locally compact group G we denote by $\widehat{G}$ the set[1] of all equivalence classes of irreducible unitary representations of G. We call $\widehat{G}$ the *unitary dual* of

[1] There is a set-theoretic problem here, since it is not clear why the equivalence classes should form a set. It is, however, not difficult to show that there exists a cardinality α, depending on G, such that

G. It is quite common to make no notational difference between a given irreducible representation π and its unitary equivalence class $[\pi]$, and we will often do so in this book.

Example 6.1.10 If G is a locally compact abelian group, then every irreducible representation is one-dimensional, and therefore the unitary dual $\widehat{G}$ coincides with the Pontryagin dual of G. To see this, let (π, V_π) be any irreducible representation of G. Then $\pi(x)\pi(y) = \pi(xy) = \pi(yx) = \pi(y)\pi(x)$ for all $x, y \in G$, so it follows from Schur's Lemma that $\pi(x) = \lambda(x)\mathrm{Id}_{V_\pi}$ for some $\lambda(x) \in \mathbb{T}$. But this implies that every non-zero closed subspace of V_π is invariant, hence must be equal to V_π. This implies $\dim V_\pi = 1$.

6.2 Representations of $L^1(G)$

A unitary representation (π, V_π) of G induces an algebra homomorphism from the convolution algebra $L^1(G)$ to the algebra $\mathcal{B}(V_\pi)$, as the following proposition shows.

Proposition 6.2.1 *Let (π, V_π) be a unitary representation of the locally compact group G. For every $f \in L^1(G)$ there exists a unique bounded operator $\pi(f)$ on V_π such that*

$$\langle \pi(f)v, w\rangle = \int_G f(x)\langle \pi(x)v, w\rangle \, dx$$

holds for any two vectors $v, w \in V_\pi$. The induced map $\pi : L^1(G) \to \mathcal{B}(V_\pi)$ is a continuous homomorphism of Banach--algebras.*

Proof Taking complex conjugates one sees that the claimed equation is equivalent to the equality $\langle w, \pi(f)v\rangle = \int_G \overline{f(x)}\langle w, \pi(x)v\rangle \, dx$. The map $w \mapsto \int_G \overline{f(x)}\langle w, \pi(x)v\rangle \, dx$ is linear. It is also bounded, since

$$\begin{aligned} \left| \int_G \overline{f(x)}\langle w, \pi(x)v\rangle \, dx \right| &\le \int_G |f(x)\langle w, \pi(x)v\rangle| \, dx \\ &\le \int_G |f(x)| \|w\| \|\pi(x)v\| \, dx \\ &= \|f\|_1 \|w\| \|v\|. \end{aligned}$$

every irreducible unitary representation (π, V_π) of G satisfies $\dim V_\pi \le \alpha$. This means that one can fix a Hilbert space H of dimension α and each irreducible unitary representation π can be realized on a subspace of H. Setting the representation equal to 1 on the orthogonal complement one gets a representation on H, i.e., a group homomorphism $G \to \mathrm{GL}(H)$. Indeed, since every irreducible representation has a cyclic vector by Schur's lemma, one can choose α as the cardinality of G. Therefore, each equivalence class has a representative in the set of all maps from G to $\mathrm{GL}(H)$ and so $\hat{G}$ forms a set.

Therefore, by Proposition C.3.1, there exists a unique vector $\pi(f)v \in V_\pi$ such that the equality holds. It is easy to see that the map $v \mapsto \pi(f)v$ is linear. To see that it is bounded, note that the above shows $\|\pi(f)v\|^2 = \langle \pi(f)v, \pi(f)v\rangle \leq \|f\|_1\|v\|\|\pi(f)v\|$, and hence $\|\pi(f)v\| \leq \|f\|_1\|v\|$. A straightforward computation finally shows $\pi(f * g) = \pi(f)\pi(g)$ and $\pi(f)^* = \pi(f^*)$ for $f, g \in L^1(G)$. □

Alternatively, one can define $\pi(f)$ as the Bochner integral $\pi(f) = \int_G f(x)\pi(x)\, dx$ in the Banach space $\mathcal{B}(V_\pi)$. By the uniqueness in the above proposition, these two definitions agree.

The above proposition has a converse, as we shall see in Proposition 6.2.3 below.

Lemma 6.2.2 *Let (π, V_π) be a representation of G. Then for every $v \in V_\pi$ and every $\varepsilon > 0$ there exists a unit-neighborhood U such that for every Dirac function ϕ_U with support in U one has $\|\pi(\phi_U)v - v\| < \varepsilon$. In particular, for every Dirac net $(\phi_U)_U$ the net $(\pi(\phi_U)v)$ converges to v in the norm topology.*

Proof The norm $\|\pi(\phi_U)v - v\|$ equals $\|\int_G \phi_U(x)(\pi(x)v - v)\, dx\|$ and is therefore less than or equal to $\int_G \phi_U(x)\|\pi(x)v - v\|\, dx$. For given $\varepsilon > 0$ there exists a unit-neighborhood U_0 in G such that for $x \in U_0$ one has $\|\pi(x)v - v\| < \varepsilon$. For $U \subset U_0$ it follows $\|\pi(\phi_U)v - v\| < \varepsilon$. □

Definition We say that a $*$-representation $\pi : L^1(G) \to \mathcal{B}(V)$ of $L^1(G)$ on a Hilbert space V is *non-degenerate* if the vector space

$$\pi(L^1(G))V \stackrel{\text{def}}{=} \text{span}\{\pi(f)v : f \in L^1(G), v \in V\}$$

is dense in V. It follows from the above lemma that every representation of $L^1(G)$ that comes from a representation (π, V_π) as in Proposition 6.2.1, is non-degenerate. The next proposition gives a converse to this.

Proposition 6.2.3 *Let $\pi : L^1(G) \to \mathcal{B}(V)$ be a non-degenerate $*$- representation on a Hilbert space V. Then there exists a unique unitary representation $(\tilde{\pi}, V)$ of G such that $\langle \pi(f)v, w\rangle = \int_G f(x)\langle \tilde{\pi}(x)v, w\rangle\, dx$ holds for all $f \in L^1(G)$ and all $v, w \in V$.*

Proof Note first that π is continuous by Lemma 2.7.1. We want to define an operator $\tilde{\pi}(x)$ on the dense subspace $\pi(L^1(G))V$ of V. This space is made up of sums of the form $\sum_{i=1}^n \pi(f_i)v_i$ for $f_i \in L^1(G)$ and $v_i \in V$. We propose to define $\tilde{\pi}(x)\sum_{i=1}^n \pi(f_i)v_i \stackrel{\text{def}}{=} \sum_{i=1}^n \pi(L_x f_i)v_i$. We have to show well-definedness, which amounts to show that if $\sum_{i=1}^n \pi(f_i)v_i = 0$, then $\sum_{i=1}^n \pi(L_x f_i)v_i = 0$ for every $x \in G$. For $x \in G$ and $f, g \in L^1(G)$ a short computation shows that $g^* * (L_x f) = (L_{x^{-1}} g)^* * f$. Based on this, we compute for $v, w \in V$ and $f_1, \dots, f_n \in L^1(G)$,

$$\left\langle \sum_{i=1}^n \pi(L_x f_i)v, \pi(g)w \right\rangle = \sum_{i=1}^n \left\langle \pi(g^* * (L_x f_i))v, w \right\rangle$$
$$= \sum_{i=1}^n \left\langle \pi((L_{x^{-1}} g)^* * f_i)v, w \right\rangle = \left\langle \sum_{i=1}^n \pi(f_i)v, \pi(L_{x^{-1}} g)w \right\rangle.$$

Now for the well-definedness of $\tilde{\pi}(x)$ assume $\sum_{i=1}^n \pi(f_i)v_i = 0$, then the above computation shows that the vector $\sum_{i=1}^n \pi(L_x f_i)v_i$ is orthogonal to all vectors of the form $\pi(g)w$, which span the dense subspace $\pi(L^1(G))V$, hence $\sum_{i=1}^n \pi(L_x f_i)v_i = 0$ follows. The computation also shows that this, now well-defined operator $\tilde{\pi}(x)$ is unitary on the space $\pi(L^1(G))V$ and since the latter is dense in V, the operator $\tilde{\pi}(x)$ extends to a unique unitary operator on V with inverse $\tilde{\pi}(x^{-1})$, and we clearly have $\tilde{\pi}(xy) = \tilde{\pi}(x)\tilde{\pi}(y)$ for all $x, y \in G$. Since for each $f \in L^1(G)$ the map $G \to L^1(G)$ sending x to $L_x f$ is continuous by Lemma 1.4.2, it follows that $x \mapsto \tilde{\pi}(x)v$ is continuous for every $v \in V$. Thus $(\tilde{\pi}, V)$ is a unitary representation of G.

It remains to show that $\pi(f)$ equals $\tilde{\pi}(f)$ for every $f \in L^1(G)$. By continuity it is enough to show that $\langle \tilde{\pi}(f)\pi(g)v, w\rangle = \langle \pi(f)\pi(g)v, w\rangle$ for all $f, g \in C_c(G)$ and $v, w \in V$. Since $g \mapsto \langle \pi(g)v, w\rangle$ is a continuous linear functional on $L^1(G)$ we can use Lemma B.6.5 to get

$$\begin{aligned}
\langle \tilde{\pi}(f)\pi(g)v, w\rangle &= \int_G f(x)\langle \tilde{\pi}(x)(\pi(g)v), w\rangle \, dx \\
&= \int_G \langle \pi(f(x)L_x g)v, w\rangle \, dx \\
&= \left\langle \pi\left(\int_G f(x)L_x g \, dx\right) v, w \right\rangle \\
&= \langle \pi(f * g)v, w\rangle = \langle \pi(f)\pi(g)v, w\rangle,
\end{aligned}$$

which completes the proof. □

Remark 6.2.4 If we define unitary equivalence and irreducibility for representations of $L^1(G)$ in the same way as we did for unitary representations of G, then it is easy to see that the one-to-one correspondence between unitary representations of G and non-degenerate $*$-representations of $L^1(G)$ preserves unitary equivalence and irreducibility in both directions. Note that an irreducible representation π of $L^1(G)$ is automatically non-degenerate, since the closure of $\pi(L^1(G))V_\pi$ is an invariant subspace of V_π. Thus, we obtain a bijection between the space $\widehat{G}$ of equivalence classes of irreducible representations of G and the set $L^1(G)\widehat{}$ of irreducible $*$-representations of $L^1(G)$.

Example 6.2.5 Consider the left regular representation on G. Then the corresponding representation $L : L^1(G) \to \mathcal{B}(L^2(G))$ is given by the convolution operators $L(f)\phi = f * \phi$ whenever the convolution $f * \phi$ makes sense.

6.3 Exercises

Exercise 6.1 Let G be a topological group and let V be a Banach space. We equip the group $\mathrm{GL}_{\mathrm{cont}}(V)$ with the topology induced by the operator norm. Show that any continuous group homomorphism $G \to \mathrm{GL}_{\mathrm{cont}}(V)$ is a representation but that not every representation is of this form.

Exercise 6.2 If π is a unitary representation of the locally compact group G, then $\|\pi(g)\| = 1$. Give an example of a representation π, for which the map $g \mapsto \|\pi(g)\|$ is not bounded on G.

Exercise 6.3 Let I be an index set, and for $i \in I$ let (π_i, V_i) be a unitary representation of the locally compact group G. Let $V = \bigoplus_{i \in I} V_i$ be the Hilbert direct sum (See Appendix C.3). Define the map $\pi : G \to \mathcal{B}(V)$ by

$$\pi(g) \sum_i v_i = \sum_i \pi_i(g) v_i.$$

Show that this is a unitary representation of the group G. It is called the *direct sum representation*.

Exercise 6.4 Show that the Fourier transform on $\mathbb{R}$ induces a unitary equivalence between the the unitary representations π and η of $\mathbb{R}$ on $L^2(\mathbb{R})$ given by $\pi(x)\phi(y) = \phi(x + y)$ and $\eta(x)\phi(y) = e^{2\pi i x y}\phi(y)$.

Exercise 6.5 Show that the natural representation of $\mathrm{U}(n)$ on $\mathbb{C}^n$ is irreducible.

Exercise 6.6 (a) For $t \in \mathbb{R}$ let $A(t) = \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$. Show that $A(t)$ is not conjugate to a unitary matrix for $t \neq 0$.

(b) Let P be the group of upper triangular matrices in $\mathrm{SL}_2(\mathbb{R})$. The injection $\eta : P \hookrightarrow \mathrm{GL}_2(\mathbb{C})$ can be viewed as a representation on $V = \mathbb{C}^2$. Show that η is not the sum of irreducible representations. Determine all irreducible subrepresentations.

Exercise 6.7 Let G be a locally compact group and H a closed subgroup. Let (π, V_π) be an irreducible unitary representation of G, and let

$$V_\pi^H = \{v \in V_\pi : \pi(h)v = v \; \forall h \in H\}$$

be the space of H-fixed vectors. Show: If H is normal in G, then V_π^H is either zero or the whole space V_π.

Exercise 6.8 Show that $G = \mathrm{SL}_2(\mathbb{R})$ has no finite dimensional unitary representations except the trivial one.

Instructions:

- For $m \in \mathbb{N}$ show
$$\begin{pmatrix} m & \\ & m^{-1} \end{pmatrix} A(t) \begin{pmatrix} m & \\ & m^{-1} \end{pmatrix}^{-1} = A(m^2 t) = A(t)^{m^2}.$$
Let $\phi : G \to \mathrm{U}(n)$ be a representation. Show that the eigenvalues of $\phi(A(t))$ are a permutation of their m-th powers for every $m \in \mathbb{N}$. Conclude that they all must be equal to 1.
- Show that the normal subgroup of G generated by $\{A(t) : t \in \mathbb{R}\}$ is the whole group.

Exercise 6.9 Let (π, V_π) be a unitary representation of the locally compact group G. Let $f \in L^1(G)$. Show that the Bochner integral
$$\int_G f(x)\pi(x)\,dx \ \in \ \mathcal{B}(V_\pi)$$
exists and that the so defined operator coincides with $\pi(f)$ as defined in Proposition 6.2.1.

(Hint: Use Corollary 1.3.6 (d) and Lemma B.6.2 as well as Proposition B.6.3.)

Exercise 6.10 In Lemma 6.2.2 we have shown that for a representation π and Dirac functions ϕ_U the numbers $\|\pi(\phi_U)v - v\|$ become arbitrarily small for fixed $v \in V_\pi$. Give an example, in which $\|\pi(\phi_U) - \mathrm{Id}\|_{\mathrm{op}}$ does not become small as the support of the Dirac function ϕ_U shrinks.

Exercise 6.11 Give an example of a representation that possesses cyclic vectors without being irreducible.

Notes

As for abelian groups, one can associate to each locally compact group G the group C^*-algebra $C^*(G)$. It is defined as the completion of $L^1(G)$ with respect to the norm
$$\|f\|_{C^*} \stackrel{\mathrm{def}}{=} \sup\{\|\pi(f)\| : \pi \text{ a unitary representation of } G\},$$
which is finite since $\|\pi(f)\| \le \|f\|_1$ for every unitary representation π of G. By definition of the norm, every unitary representation π of G extends to a $*$-representation of $C^*(G)$, and, as for $L^1(G)$, this extension provides a one-to-one correspondence between the unitary representation of G to the non-degenerate $*$-representations of $C^*(G)$. Therefore, the rich representation theory of general C^*-algebras, as explained beautifully in Dixmier's classic book [Dix96] can be used for the study of unitary representations of G. For a more recent treatment of C^*-algebras related to locally compact groups we also refer to Dana William's book [Wil07].