Chapter 5
Operators on Hilbert Spaces

In this chapter, we will apply the results of Chap. 2 on C^*-algebras to operators on Hilbert spaces. In particular, we will discuss the continuous functional calculus for normal bounded operators on Hilbert space, which turns out to be a powerful tool.

The space $\mathcal{B}(H)$ of all bounded linear operators on a Hilbert space H is a Banach algebra with the operator norm (Example 2.1.1), and, as we have seen in Example 2.6.1, even a C^*-algebra. We will write

$$\sigma(T) = \sigma_{\mathcal{B}(H)}(T)$$

for the spectrum of T with respect to the C^*-algebra $\mathcal{B}(H)$ and call it simply the *spectrum of the operator* T.

5.1 Functional Calculus

Let H be a Hilbert space, and let T be a bounded *normal operator* on H, this means that T commutes with its adjoint T^*, i.e., T is normal as an element of the C^*-algebra $\mathcal{B}(H)$. We then can apply the results of Sect. 2.7, which for any continuous function f on the spectrum $\sigma(T)$ give a unique element $f(T)$ of $\mathcal{B}(H)$ that commutes with T and satisfies

$$\widehat{f(T)} = f \circ \widehat{T},$$

where the hat means the Gelfand transform with respect to the unital C^*-algebra generated by T. Recall that by Lemma 2.7.2 the spectrum of a normal operator T does not depend on the C^*-algebra. The map from $C(\sigma(T))$ to $\mathcal{B}(H)$ mapping f to $f(T)$ is the *continuous functional calculus*. In the next proposition, we summarize some important properties.

Proposition 5.1.1. *Let T be a normal bounded operator on the Hilbert space H and let $\mathcal{A} = C^*(T, 1)$ be the unital C^*-algebra generated by T.*

(a) *The map $f \mapsto f(T)$ is a unital isometric C^*-isomorphism from* $C(\sigma(T))$ to $\mathcal{A}$, *which sends the identity map* $\mathrm{Id}_{\sigma(T)}$ *to T.*

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(b) *Let $V \subset H$ be a closed subspace stable under T and T^*. Then V is stable under $\mathcal{A}$ and $f(T)|_V = f(T|_V)$.*

(c) *Let V be the kernel of $f(T)$. Then V is stable under T and T^*, and the spectrum of $f(T|_V)$ is contained in the zero-set of f.*

(d) *If $f(z) = \sum_{n=0}^\infty a_n z^n$ is a power series that converges for $z = \|T\|$, then $f(T) = \sum_{n=0}^\infty a_n T^n$.*

Proof The first assertion is a direct consequence of Theorem 2.7.3.

To show (b), note first that if V is stable under T and T^*, then V is $\mathcal{A}$-stable, since the linear combinations of operators of the form $T^k(T^*)^l$ are dense in $\mathcal{A}$. We therefore get a well defined $*$-homomorphism $\Psi : \mathcal{A} \to \mathcal{B}(V)$ mapping S to $S|_V$. The assertion in (b) is then a consequence of Corollary 2.7.5.

In (c), the space V is stable under T and T^* as these operators commute with $f(T)$. Further, using Corollary 2.7.5, one has

$$f(\sigma(T|_V)) = \sigma(f(T|_V)) = \sigma(f(T)|_V) = \{0\}.$$

Finally, part (d) is contained in Theorem 2.7.3, since convergence of the power series at $\|T\|$ implies uniform convergence on $\sigma(T) \subseteq B_{\|T\|}(0)$. □

An important class of normal operators is formed by the self-adjoint operators, i.e., operators T with $T = T^*$. It is shown in Corollary 2.7.5 that $\sigma(T) \subseteq \mathbb{R}$ for every self-adjoint T. Another class of interesting normal operators consists of the *unitary operators*. These are operators $U \in \mathcal{B}(H)$ satisfying $UU^* = U^*U = 1$. Note that a normal operator $U \in \mathcal{B}(H)$ is unitary if and only if $\sigma(U) \subseteq \mathbb{T}$. This follows from functional calculus, because if U is normal, then $U^*U = 1$ if and only if $\overline{\mathrm{Id}} \cdot \mathrm{Id} = 1$ for $\mathrm{Id} = \mathrm{Id}_{\sigma(U)}$, which is equivalent to $\sigma(U) \subseteq \mathbb{T}$.

Recall the Schwartz space $\mathcal{S}(\mathbb{R})$ consisting of all functions $f : \mathbb{R} \to \mathbb{C}$ such that for any two integers $m, n \geq 0$ the function $x^n f^{(m)}(x)$ is bounded. So a *Schwartz function on* $\mathbb{R}$ is a smooth function on $\mathbb{R}$, which, together with all its derivatives, is rapidly decreasing.

For $f \in \mathcal{S}(\mathbb{R})$ the *Fourier inversion formula* says that

$$f(x) = \int_{\mathbb{R}} \hat{f}(y) e^{2\pi i xy} \, dy,$$

where $\hat{f}(y) = \int_{\mathbb{R}} f(x) e^{-2\pi i xy} \, dx$ is the *Fourier transform* (See Exercise 3.14 or [Dei05] Sect. 3.4).

Proposition 5.1.2. *Let T be a self-adjoint bounded operator on the Hilbert space H. Then for every $f \in \mathcal{S}(\mathbb{R})$,*

$$f(T) = \int_{\mathbb{R}} \hat{f}(y) e^{2\pi i y T} \, dy,$$

where the unitary operator $e^{2\pi i y T}$ *is defined by the continuous functional calculus and the integral is a vector-valued integral in the Banach space* $\mathcal{B}(H)$ *as in* Sect. B.6.

Proof To see that the operator $e^{2\pi i y T}$ is unitary, we compute

$$\left(e^{2\pi i y T}\right)^* = e^{-2\pi i y T^*} = e^{-2\pi i y T} = \left(e^{2\pi i y T}\right)^{-1}.$$

The Bochner integral exists by Lemma B.6.2 and Proposition B.6.3. Next, let $\Phi : C(\sigma(T)) \to C^*(T, 1), g \mapsto g(T)$ denote the isometric $*$-homomorphism underlying the Functional Calculus for T. The Fourier inversion formula implies that

$$f|_{\sigma(T)} = \int_{\mathbb{R}} \hat{f}(y) e^{2\pi i y \mathrm{Id}_{\sigma(T)}}\, dy.$$

By continuity of Φ we therefore get

$$\begin{aligned} f(T) = \Phi(f|_{\sigma(T)}) &= \Phi\left(\int_{\mathbb{R}} \hat{f}(y) e^{2\pi i y \mathrm{Id}_{\sigma(T)}}\, dy\right) \\ &= \int_{\mathbb{R}} \hat{f}(y) \Phi\left(e^{2\pi i y \mathrm{Id}_{\sigma(T)}}\right) dy = \int_{\mathbb{R}} \hat{f}(y) e^{2\pi i y T}\, dy, \end{aligned}$$

where the last equation follows from Corollary 2.7.5, which implies $\Phi\left(e^{2\pi i y \mathrm{Id}_{\sigma(T)}}\right) = e^{2\pi i y \Phi\left(\mathrm{Id}_{\sigma(T)}\right)} = e^{2\pi i T}$. □

Definition A self-adjoint operator $T \in \mathcal{B}(H)$ is called *positive* if

$$\langle T v, v\rangle \geq 0 \quad \forall v \in H.$$

In what follows, we want to use the spectral theorem to compute a positive square root for any positive operator T. For this we need to know that positive operators have positive spectrum.

Theorem 5.1.3 *Let T be a self-adjoint bounded operator on the Hilbert space H. Then the following are equivalent*:

(a) T is positive.

(b) *The spectrum $\sigma(T)$ is contained in the interval* $[0, \infty)$.

(c) *There exists an operator $R \in \mathcal{B}(H)$ with $T = R^* R$.*

(d) *There exists a unique positive operator S with $T = S^2$. In this case we write $S = \sqrt{T}$.*

Proof The implications (d) ⇒ (c) and (c) ⇒ (a) are trivial. So it is enough to show that (a) ⇒ (b) and (b) ⇒ (d) hold.

For (a) $\Rightarrow$ (b) assume without loss of generality that $\|T\| = 1$. Then $\sigma(T) \subseteq [-1, 1]$ since T is self-adjoint. We show that $T_\mu \stackrel{\text{def}}{=} T + \mu 1$ is invertible for every $\mu > 0$, which will imply that there are no negative spectral values for T. By assumption we have

$$\|T_\mu v\| \|v\| \geq \langle T_\mu v, v \rangle = \langle T v, v \rangle + \mu \langle v, v \rangle \geq \mu \|v\|^2,$$

which implies that $\|T_\mu v\| \geq \mu \|v\|$ for every $v \in H$. It follows that T_μ is injective. Since T_μ is self-adjoint we also get $(T_\mu(H))^\perp = \ker T_\mu = \{0\}$, since if $w \in (T_\mu(H))^\perp$, then $0 = \langle T_\mu v, w \rangle = \langle v, T_\mu w \rangle$ for every $v \in H$, which implies that $T_\mu w = 0$. Thus we get $\overline{T_\mu(H)} = H$ and for each $w \in H$ we find a sequence v_n in H with $T v_n \to w$. Since $\|v_n - v_m\| \leq \frac{1}{\mu} \|T_\mu v_n - T_\mu v_m\|$ for all $n, m \in \mathbb{N}$, it follows that (v_n) is a Cauchy-sequence and hence converges to some $v \in H$. Then $T_\mu v = w$, which shows that T_μ is also surjective. The Open Mapping Theorem C.1.5 implies that T_μ^{-1} is continuous, so T_μ is invertible in $\mathcal{B}(H)$.

Assume finally that (b) holds. Then $t \to \sqrt{t}$ is a continuous function on $\sigma(T)$, and by functional calculus we can build the operator $S = \sqrt{T}$. Since $\sqrt{\cdot}$ is real and positive, it follows from Corollary 2.7.5 that S is self-adjoint, $\sigma(S) \subset [0, \infty)$, and $S^2 = T$. For uniqueness assume that $\tilde{S}$ is another such operator. Then T lies in the commutative C^*-algebra $C^*(\tilde{S}, 1)$. But then $S \in C^*(T, 1) \subseteq C^*(\tilde{S}, 1) \cong C(\sigma(\tilde{S}))$, and the result follows from the fact that a positive real function has a unique positive square root. □

Definition Let T be a bounded operator on a Hilbert space H. Define the operator $|T|$ by $|T| \stackrel{\text{def}}{=} \sqrt{T^* T}$, which exists and is well defined by the above theorem.

Proposition 5.1.4 *Let T be a bounded operator on H. Then the norm of* $|T|v$ *coincides with* $\|Tv\|$. *There is an isometric operator U from the closure of* $\operatorname{Im}(|T|)$ *to the closure of* $\operatorname{Im}(T)$ *such that* $T = U|T|$. *This decomposition of T is called* polar decomposition. *It is unique in the following sense. If* $T = U'P$, *where P is self-adjoint and positive, and* $U' : \overline{\operatorname{Im}(P)} \to H$ *is isometric, then* $U' = U$ *and* $P = |T|$.

Proof For $v \in H$ the square of the norm $\|Tv\|^2$ equals

$$\langle Tv, Tv \rangle = \langle T^* T v, v \rangle = \langle |T|^2 v, v \rangle = \langle |T|v, |T|v \rangle,$$

and the latter is $\||T|v\|^2$. For $v \in H$ we define $U(|T|v) = Tv$, then U is a well-defined isometry from $\operatorname{Im}(|T|)$ to $\operatorname{Im}(T)$, which extends to the closure, and satisfies the claim. For the uniqueness let $T = U|T| = U'P$. Extend U to a bounded operator on H by setting $U \equiv 0$ on $\operatorname{Im}(|T|)^\perp$ and do likewise for U'. Then U^*U is the orthogonal projection to $\overline{\operatorname{Im}(|T|)}$ and $(U')^*U'$ is the orthogonal projection to $\operatorname{Im}(P)$, so that $(U')^*U'P = P$. Note $|T| = \sqrt{T^*T} = \sqrt{(U'P)^*U'P} = \sqrt{P^*(U')^*U'P} = \sqrt{P^*P} = \sqrt{P^2} = P$. This also implies $U = U'$. □

An important application of the functional calculus for operators on Hilbert space is Schur's Lemma, which we shall use quite frequently in the remaining part of this book. We first state

Lemma 5.1.5 *Let H be a Hilbert space, and let T be a bounded normal operator, the spectrum of which consists of a single point $\{\lambda\} \subset \mathbb{C}$. Then $T = \lambda \mathrm{Id}$.*

Proof If $\sigma(T) = \{\lambda\}$, then $\mathrm{Id}_{\sigma(T)} = \lambda 1_{\sigma(T)}$ and therefore $T = \mathrm{Id}_{\sigma(T)}(T) = \lambda \cdot \mathrm{Id}_H$. □

Theorem 5.1.6 (Schur's Lemma) *Suppose that $A \subseteq \mathcal{B}(H)$ is a self-adjoint set of bounded operators on the Hilbert space H* (i.e., *$S \in A$ implies $S^* \in A$*). *Then the following are equivalent:*

(a) A is *topologically irreducible, i.e., if $\{0\} \neq L \subseteq H$ is any A-invariant closed subspace of H then $L = H$.*

(b) *If $T \in \mathcal{B}(H)$ commutes with every $S \in A$, then $T = \mu \mathrm{Id}$ for some $\mu \in \mathbb{C}$.*

Proof Assume first that the second assertion holds. Then, if $\{0\} \neq L \subseteq H$ is any A-invariant closed subspace of H, the orthogonal complement $L^\perp$ is A-invariant as well, for with $v \in L$, $u \in L^\perp$, and $S \in A$ we have

$$\langle v, Su\rangle = \langle \underbrace{S^* v}_{\in L}, u\rangle = 0.$$

So the orthogonal projection $P_L : H \to L$ commutes with A, so P_L must be a multiple of the identity. But this implies that $P_L = \mathrm{Id}$ and $L = H$.

For the converse, assume that (a) holds, and let $T \in \mathcal{B}(H)$ commute with A. Then also T^* commutes with A since A is self-adjoint. Thus, writing $T = \frac{1}{2}(T + T^*) - i\frac{1}{2}(iT - iT^*)$ we may assume without loss of generality that T is self-adjoint and $T \neq 0$. We want to show that the spectrum of T consists of a single point. Note that an operator S, which commutes with T, also commutes with $f(T)$ for every $f \in C(\sigma(T))$. Assume that there are $x, y \in \sigma(T)$ with $x \neq y$. Then there are two functions $f, g \in C(\sigma(T))$ with $f(x) \neq 0 \neq g(y)$ and $f \cdot g = 0$. Then $f(T) \neq 0 \neq g(T)$ and $f(T)g(T) = f \cdot g(T) = 0$. Since $g(T)$ commutes with A, the space $L = \overline{g(T)H}$ is a non-zero A-invariant subspace of H. By (a) we get $L = H$. But then $\{0\} \neq f(T)H = f(T)\overline{g(T)H} \subset \overline{f(T)g(T)H} = \{0\}$, a contradiction. □

5.2 Compact Operators

An operator T on a Hilbert space H is called a *compact operator* if T maps bounded sets to relatively compact ones. It is clear from the definition that if T is compact and S a bounded operator on H, then ST and TS are compact. The definition can be rephrased as follows. An operator T is compact if and only if for a given bounded sequence $v_j \in H$ the sequence Tv_j has a convergent subsequence. If the v_j lie in a finite dimensional space, then this is true for every bounded operator. So one may restrict to sequences v_j that are linearly independent.

Definition A bounded linear map $F : H \to H$ on a Hilbert space H is said to be a *finite rank operator* if the image $F(H)$ is finite-dimensional.

Proposition 5.2.1 *For a bounded operator T on a Hilbert space H the following are equivalent.*

(a) *T is compact.*

(b) *For every orthonormal sequence e_j the sequence Te_j has a convergent subsequence.*

(c) *There exists a sequence F_n of finite rank operators such that $\|T - F_n\|_{\text{op}}$ tends to zero, as $n \to \infty$.*

Proof The implication (a)⇒(b) is trivial. For (b)⇒(c) let $T : H \to H$ be compact and let $B \subset H$ denote the closed unit ball. Then $\overline{T(B)}$ is compact, hence has a vector v_1 of maximal norm. Next suppose the vectors $v_1, \dots v_n$ are already constructed and let V_n be their span. Choose a vector v_{n+1} of maximal norm in $\overline{T(B)} \cap V_n^\perp$. The vectors $v_1, v_2, \dots$ are pairwise orthogonal and for their norms we have $\|v_1\| \geq \|v_2\| \geq \dots$. We claim that the sequence v_n tends to zero. Assume not, then there exists $\delta > 0$ such that $\|v_n\| \geq \delta$ for all n. For $i \neq j$ if follows $\|v_i - v_j\|^2 = \|v_i\|^2 + \|v_j\|^2 \geq 2\delta^2$, hence the sequence has no convergent subsequence, in contradiction to the compactness of $\overline{T(B)}$. So the sequence does tend to zero. Let P_n be the orthogonal projection onto V_n. Then

$$\|T - P_n T\| = \sup_{v \in \overline{T(B)}} \|v - P_n v\| \leq \|v_n\| \to 0.$$

So with $F_n = P_n T$ the claim follows.

For (c)⇒(a) let v_j be a bounded sequence, and let T be the norm-limit of a sequence of finite rank operators F_n. We can assume $\|v_j\|, \|T\| \leq 1$. Then v_j has a subsequence v_j^1 such that $F_1(v_j^1)$ converges. Next, v_j^1 has a subsequence v_j^2 such that $F_2(v_j^2)$ converges, and so on. Let $w_j = v_j^j$. Then for every $n \in \mathbb{N}$, the sequence $(F_n(w_j))_{j \in \mathbb{N}}$ converges. As T is the uniform limit of the F_n, the sequence Tw_j converges as well. □

Theorem 5.2.2 (Spectral Theorem). *Let T be a compact normal operator on the Hilbert space H. Then there exists a sequence λ_n of non-zero complex numbers, which is either finite or tends to zero, such that one has an orthogonal decomposition*

$$H = \ker(T) \oplus \overline{\bigoplus_n \text{Eig}(T, \lambda_n)}.$$

Each eigenspace $\text{Eig}(T, \lambda_n) = \{v \in H : Tv = \lambda_n v\}$ *is finite dimensional, and the eigenspaces are pairwise orthogonal.*

Proof We first show that a given compact normal operator $T \neq 0$ has an eigenvalue $\lambda \neq 0$. We show that it suffices to assume that T is self-adjoint. Note that $T =$

$\frac{1}{2}(T+T^*) - \frac{i}{2}(iT+(iT)^*) = T_1 + iT_2$ is a linear combination of two commuting compact self-adjoint operators. If $T_2 = 0$, then T is self-adjoint and we are done. Otherwise, T_2 has a non-zero eigenvalue $\nu \in \mathbb{R} \smallsetminus \{0\}$. The corresponding eigenspace is left stable by T_1, which therefore induces a self-adjoint compact operator on that space, hence has an eigenvalue $\mu \in \mathbb{R}$. Then $\lambda = \mu + i\nu$ is a non-zero eigenvalue of T.

We have to show that a compact self-adjoint operator $T \neq 0$ has an eigenvalue $\lambda \neq 0$.

Lemma 5.2.3 *For a bounded self-adjoint operator T on a Hilbert space H we have* $\|T\| = \sup\{|\langle Tv, v\rangle| : \|v\| = 1\}$.

Proof Let C be the right hand side. By the Cauchy-Schwarz inequality we have $C \leq \|T\|$. On the other hand, for $v, w \in H$ with $\|v\|, \|w\| \leq 1$ one has

$$\begin{aligned} C &\geq \frac{1}{2}\, C\left(\|v\|^2 + \|w\|^2\right) = \frac{1}{4}\, C\left(\|v+w\|^2 + \|v-w\|^2\right) \\ &\geq \frac{1}{4}|\langle T(v+w), v+w\rangle - \langle T(v-w), v-w\rangle| \\ &= \frac{1}{2}|\langle Tv, w\rangle + \langle Tw, v\rangle| = \frac{1}{2}|\langle Tv, w\rangle + \langle w, Tv\rangle| \\ &= |\mathrm{Re}\langle Tv, w\rangle|. \end{aligned}$$

Replacing v with θv for some $\theta \in \mathbb{C}$ with $|\theta| = 1$ we get $C \geq |\langle Tv, w\rangle|$ for all $\|v\|, \|w\| \leq 1$ and so $\|T\| \leq C$. □

We continue the proof that a compact self-adjoint operator $T \neq 0$ has an eigenvalue $\lambda \neq 0$. Indeed, we prove that either $\|T\|$ or $-\|T\|$ is an eigenvalue for T. By the lemma there is a sequence $v_j \in H$ with $\|v_j\| = 1$ and $\langle Tv_j, v_j\rangle \to \pm\|T\|$. Replacing T with $-T$ if necessary, we assume $\langle Tv_j, v_j\rangle \to \|T\|$. Since T is compact, there exists a norm-convergent subsequence, i.e., we can assume that $Tv_j \to u$ in norm. Then $\|u\| \leq \|T\|$ and we get

$$\begin{aligned} 0 \leq \|Tv_j - \|T\|v_j\|^2 &= \|Tv_j\|^2 - 2\|T\|\langle Tv_j, v_j\rangle + \|T\|^2\|v_j\|^2 \\ &\to \|u\|^2 - \|T\|^2 \leq 0, \end{aligned}$$

which implies that $\|Tv_j - \|T\|v_j\| \to 0$. Thus $v := \lim_j v_j = \frac{1}{\|T\|}u$ exists and $Tv = \lim_j Tv_j = u = \|T\|v$.

We have proven that every compact normal operator T has an eigenvalue $\lambda \neq 0$. Let $U \subset V$ be the closure of the sum of all eigenspaces of T corresponding to non-zero eigenvalues. By Lemma C.3.3 every eigenvector for T is also an eigenvector for T^*, so U is stable under T and T^* and hence the orthogonal complement $U^\perp$ is stable under T and T^* as well. The operator T induces a compact normal operator on $U^\perp$; as this operator cannot have a non-zero eigenvalue, it is zero and $U^\perp$ is the kernel of T. We have shown that H is a direct sum of eigenspaces of T.

It remains to show that every eigenspace for a non-zero eigenvalue is finite dimensional and that the eigenvalues do not accumulate away from zero. For this let f be a continuous function on $\mathbb{C}$ whose zero set is the closed ε-neighborhood $\bar{B}_\varepsilon(\lambda)$ of a given $\lambda \in \mathbb{C}$, where $0 < \varepsilon < |\lambda|$. Let V be the kernel of $f(T)$. By Proposition 5.1.1, the space V is stable under T and T^*, and $\sigma(T|_V) \subset \bar{B}_\varepsilon(\lambda)$. It follows from Functional Calculus that $\|T - \lambda\|_V = \|\mathrm{Id}_{\sigma(T|_V)} - \lambda 1_{\sigma(T|_V)}\|_{\sigma(T|_V)} \le \varepsilon$, which implies that for $v \in V$ one has $\|Tv\| \ge (|\lambda| - \varepsilon)\|v\|$. We want to deduce that V is finite dimensional. Assume the contrary, so there exists an orthonormal sequence $(f_j)_{j\in\mathbb{N}}$ in V. Then $\|f_i - f_j\| = \sqrt{2}$ for $i \neq j$ and so $\|Tf_j - Tf_i\| \ge (|\lambda| - \varepsilon)\sqrt{2}$, which means that no subsequence of (Tf_j) is a Cauchy sequence, hence (Tf_j) does not contain a convergent subsequence, a contradiction to the compactness of T. So V is finite dimensional, hence it is a direct orthogonal sum of T-eigenspaces. It now follows that no spectral values of T can accumulate away from zero, and all spectral values apart from zero are eigenvalues of finite multiplicity. Finally, the fact that the eigenspaces are pairwise orthogonal is in Lemma C.3.3. The theorem is proven. □

Definition Let T be a compact operator on a Hilbert space H. Then T^*T is a self-adjoint compact operator with positive eigenvalues. The operator $|T| = \sqrt{T^*T}$ also is a compact operator. Let $s_j(T)$ be the family of non-zero eigenvalues of $|T|$ repeated with multiplicities and such that $s_{j+1}(T) \le s_j(T)$ for all j. These $s_j(T)$ are called the *singular values* of T.

Proposition 5.2.4 *Let T be a compact operator.*

(a) *We have* $s_1(T) = \|T\|$ *and*

$$s_{j+1}(T) = \inf_{v_1,\dots,v_j \in H} \sup\{\|Tw\| : w \perp v_1, \dots, v_j,\ \|w\| = 1\},$$

where the vectors $v_1, \dots, v_j$ *are unit eigenvectors for the eigenvalues* $s_1(T), \dots, s_j(T)$, *respectively.*

(b) *For any bounded operator S on H one has* $s_j(ST) \le \|S\| s_j(T)$.

Proof The formulas in (a) follow from the fact that the s_j are the eigenvalues of the self-adjoint operator $|T|$ and $\|T\| = \||T|\|$. Part (b) is a consequence of (a). We leave the details as an exercise (See Exercise 5.4). □

5.3 Hilbert-Schmidt and Trace Class

Let $T \in \mathcal{B}(H)$, and let (e_j) be an orthonormal basis of H. The *Hilbert-Schmidt norm* $\|T\|_{\mathrm{HS}}$ of T is defined by

$$\|T\|_{\mathrm{HS}}^2 \stackrel{\text{def}}{=} \sum_j \langle Te_j, Te_j \rangle.$$

This number is ≥ 0 but can be $+\infty$. It does not depend on the choice of the orthonormal basis, as we will prove now. Along the way we also show that $\|T\|_{\mathrm{HS}} = \|T^*\|_{\mathrm{HS}}$ holds for every bounded operator T. First recall that for any two vectors $v, w \in H$ and any orthonormal basis (e_j) one has

$$\langle v, w \rangle = \sum_j \langle v, e_j \rangle \langle e_j, w \rangle.$$

Let now (ϕ_α) be another orthonormal basis; then, not knowing the independence yet, we write $\|T\|^2_{\mathrm{HS}}(e_i)$ and $\|T\|^2_{\mathrm{HS}}(\phi_\alpha)$, respectively. We compute

$$\begin{aligned}\|T\|^2_{\mathrm{HS}}(e_j) &= \sum_j \sum_\alpha \langle Te_j, \phi_\alpha \rangle \langle \phi_\alpha, Te_j \rangle = \sum_j \sum_\alpha \langle e_j, T^*\phi_\alpha \rangle \langle T^*\phi_\alpha, e_j \rangle \\ &= \sum_\alpha \sum_j \langle e_j, T^*\phi_\alpha \rangle \langle T^*\phi_\alpha, e_j \rangle = \|T^*\|^2_{\mathrm{HS}}(\phi_\alpha).\end{aligned}$$

The interchange of summation order is justified by the fact that all summands are positive. Applying this to $(e_j) = (\phi_\alpha)$ first and then to T^* instead of T we get $\|T\|^2_{\mathrm{HS}}(e_j) = \|T^*\|^2_{\mathrm{HS}}(e_j) = \|T\|^2_{\mathrm{HS}}(\phi_\alpha)$, as claimed.

We say that the operator T is a *Hilbert-Schmidt operator* if the Hilbert-Schmidt norm $\|T\|_{HS}$ is finite.

Lemma 5.3.1 *For any two bounded operators S,T on H one has* $\|ST\|_{HS} \leq \|S\|_{\mathrm{op}} \|T\|_{HS}$, *and* $\|ST\|_{HS} \leq \|S\|_{HS} \|T\|_{\mathrm{op}}$, *as well as* $\|T\|_{\mathrm{op}} \leq \|T\|_{HS}$. *For every unitary operator U we have* $\|UT\|_{HS} = \|TU\|_{HS} = \|T\|_{HS}$.

Proof Let (e_j) be an orthonormal basis. We have $\|ST\|^2_{HS} = \sum_j \|STe_j\|^2 \leq \|S\|^2_{\mathrm{op}} \sum_j \|Te_j\|^2$, which implies the first estimate. The second follows by using $\|T\|_{HS} = \|T^*\|_{HS}$ and the same assertion for the operator norm.

Let $v \in H$ with $\|v\| = 1$. Then there is an orthonormal basis (e_j) with $e_1 = v$. We get $\|Tv\|^2 = \|Te_1\|^2 \leq \sum_j \|Te_j\|^2 = \|T\|^2_{HS}$. The invariance under multiplication by unitary operators is clear, since (Ue_j) is an orthonormal basis when (e_j) is. □

Example 5.3.2 The main example we are interested in is the following. For a measure space $(X, \mathcal{A}, \mu)$ consider the Hilbert space $L^2(X)$. Assume that μ is either σ-finite or that X is locally compact and μ is an outer Radon measure, so that Fubini's Theorem holds with respect to the product measure $\mu \otimes \mu$ on $L^2(X \times X)$. Let k be a function in $L^2(X \times X)$. Then we call k an L^2-*kernel*.

Proposition 5.3.3 *Suppose* $k(x, y)$ *is an* L^2*-kernel on X. For* $\phi \in L^2(X)$ *define*

$$K\phi(x) \stackrel{\mathrm{def}}{=} \int_X k(x, y)\phi(y)\, d\mu(y).$$

Then this integral exists almost everywhere in x. The function $K\phi$ lies in $L^2(X)$, and K extends to a Hilbert-Schmidt operator $K : L^2(X) \to L^2(X)$ with

$$\|K\|_{HS}^2 = \int_X \int_X |k(x,y)|^2 \, d\mu(x) \, d\mu(y).$$

Proof To see that the integral exists for almost all $x \in X$ let ψ be any element in $L^2(X)$. Then $(x, y) \mapsto \psi(x)\phi(y)$ lies in $L^2(X \times X)$, and therefore the function $(x, y) \to k(x, y)\phi(y)\psi(x)$ is integrable over $X \times X$. By Fubini, it follows that

$$\int_X \psi(x)k(x,y)\phi(y)\,dy = \psi(x)\int_X k(x,y)\phi(y)\,dy$$

exists for almost all $x \in X$. Since $k(x, y)$ vanishes for every x outside some σ-finite subset A of X, we may let ψ run through the characteristic functions of an increasing sequence of finite measurable sets that exhaust A, to conclude that the integral $\int_X k(x,y)\phi(y)\,dy$ exists for almost all $x \in X$.

We use the Cauchy-Schwarz inequality to estimate

$$\begin{aligned} \|K\phi\|^2 &= \int_X |K\phi(x)|^2\,dx \\ &= \int_X \left|\int_X k(x,y)\phi(y)\,dy\right|^2 dx \\ &\le \int_X \int_X |k(x,y)|^2\,dx\,dy \ \int_X |\phi(y)|^2\,dy \\ &= \int_X \int_X |k(x,y)|^2\,dx\,dy \ \|\phi\|^2. \end{aligned}$$

So K extends to a bounded operator on $L^2(X)$. Let (e_j) be an orthonormal basis of $L^2(X)$. Then

$$\begin{aligned} \|K\|_{HS}^2 &= \sum_j \left\langle Ke_j, Ke_j \right\rangle = \sum_j \int_X Ke_j(x)\overline{Ke_j(x)}\,dx \\ &= \sum_j \int_X \int_X k(x,y)e_j(y)\,dy \overline{\int_X k(x,y)e_j(y)\,dy}\,dx \\ &= \sum_j \int_X \left\langle k(x,.), e_j \right\rangle \left\langle e_j, k(x,.) \right\rangle dx \\ &= \int_X \sum_j \left\langle k(x,.), e_j \right\rangle \left\langle e_j, k(x,.) \right\rangle dx \\ &= \int_X \langle k(x,.), k(x,.) \rangle\,dx = \int_X \int_X |k(x,y)|^2\,dx\,dy. \end{aligned}$$

□

Proposition 5.3.4 *The operator T is Hilbert-Schmidt if and only if it is compact and its singular values satisfy $\sum_j s_j(T)^2 < \infty$. Indeed, then one has $\sum_j s_j(T)^2 = \|T\|_{\mathrm{HS}}^2$.*

Proof We show that a bounded operator T is Hilbert-Schmidt if and only if $|T| = \sqrt{T^*T}$ is. This follows from

$$\sum_j \langle Te_j, Te_j\rangle = \sum_j \langle T^*Te_j, e_j\rangle = \sum_j \langle |T|^2 e_j, e_j\rangle$$
$$= \sum_j \langle |T|e_j, |T|e_j\rangle.$$

Let T be Hilbert-Schmidt. To see that T is compact, it suffices to show that if e_j is an orthonormal sequence, then Te_j has a convergent subsequence. But indeed, extend e_j to an orthonormal basis, then the Hilbert-Schmidt criterion shows that Te_j tends to zero. So T is compact. The operator $|T|$ is Hilbert-Schmidt if and only if $\sum_j s_j(T)^2$ converges, as one sees by applying the Hilbert-Schmidt criterion to an orthonormal basis consisting of eigenvectors of $|T|$. Finally, it is clear that $\sum_j s_j(T)^2 = \||T|\|_{\mathrm{HS}}^2$, but by the above computation the latter equals $\|T\|_{\mathrm{HS}}^2$. □

A compact operator T is called a *trace class operator* if the *trace norm*,

$$\|T\|_{\mathrm{tr}} \stackrel{\mathrm{def}}{=} \sum_j s_j(T),$$

is finite. It follows that every trace class operator is also Hilbert-Schmidt.

Lemma 5.3.5 *Let T be a trace class operator and S a bounded operator.*

(a) The norms $\|ST\|_{\mathrm{tr}}$, $\|TS\|_{\mathrm{tr}}$ are both $\le \|S\|\|T\|_{\mathrm{tr}}$.

(b) *Let T be a compact operator on H. One has*

$$\|T\|_{\mathrm{tr}} = \sup_{(e_i),(h_i)} \sum_i |\langle Te_i, h_i\rangle|,$$

where the supremum runs over all orthonormal bases (e_i) and (h_i).

Proof The inequality $\|ST\|_{\mathrm{tr}} \le \|S\|\|T\|_{\mathrm{tr}}$ is a consequence of Proposition 5.2.4 (b). The other follows from $\|T\| = \|T^*\|$ and the same for the trace norm.

For the second part we use the Spectral Theorem for compact operators to find an orthonormal sequence (f_j) such that

$$|T|v = \sum_j s_j \langle v, f_j\rangle f_j.$$

We then write $T = U|T|$, where U is an isometric operator on the image of $|T|$ to get

$$Tv = U \sum_j s_j \langle v, f_j \rangle f_j = \sum_j s_j \langle v, f_j \rangle g_j,$$

where (g_j) is the orthonormal sequence $g_j = Uf_j$. Therefore, we can use the Cauchy-Schwarz inequality to get for any two orthonormal bases e, h,

$$\begin{aligned}
\sum_i |\langle Te_i, h_i\rangle| &= \sum_i \left| \sum_j s_j \langle e_i, f_j \rangle \langle g_j, h_i \rangle \right| \\
&\le \sum_j s_j \sum_i |\langle e_i, f_j \rangle \langle g_j, h_i \rangle| \\
&\le \sum_j s_j \left(\sum_i |\langle e_i, f_j \rangle|^2 \right)^{\frac{1}{2}} \left(\sum_i |\langle g_j, h_i \rangle|^2 \right)^{\frac{1}{2}} \\
&= \sum_j s_j \|f_j\| \|g_j\| = \sum_j s_j.
\end{aligned}$$

This implies the $\ge$ part of the claim. The other part is obtained by taking e to be any orthonormal basis that prolongs the orthonormal sequence f and h any orthonormal basis that prolongs the orthonormal sequence g, because then $\sum_i |\langle Te_i, h_i \rangle| = \sum_j s_j$. □

Theorem 5.3.6 *For a trace class operator T the trace*

$$\mathrm{tr}\,(T) \stackrel{\text{def}}{=} \sum_j \langle Te_j, e_j \rangle$$

does not depend on the choice of an orthonormal base (e_j)*. If T is trace class and normal, we have* $\mathrm{tr}\,(T) = \sum_n \lambda_n \dim \mathrm{Eig}(T, \lambda_n)$*, where the sum runs over the sequence of non-zero eigenvalues* (λ_n) *of T. The sum converges absolutely.*

Proof Let $T = U|T|$ be the polar decomposition of T. It follows from the Spectral Theorem that the image of the operator $S_2 = \sqrt{|T|}$ equals the image of $|T|$ and therefore we can define the operator $S_1 = U\sqrt{|T|}$. The operators S_1, S_2 and S_1^* are Hilbert-Schmidt operators, and $T = S_1 S_2$. Therefore $\sum_i \langle Te_i, e_i \rangle = \sum_i \langle S_2 e_i, S_1^* e_i \rangle$, and the latter does not depend on the choice of the orthonormal basis as can be seen in a similar way as in the beginning of this section. Choose a basis of eigenvectors to prove the second statement. □

Theorem 5.3.7 *Let H be a Hilbert space,* $\mathcal{F}$ *the space of bounded operators T of finite rank (i.e., finite dimensional image),* $\mathcal{T}$ *the set of trace class operators,* $\mathcal{HS}$ *the set of Hilbert-Schmidt operators, and* $\mathcal{K}$ *the set of compact operators. Further,* *we write $\mathcal{HS}^2$ for the linear span of all opertors of the form ST, where S and T are both in $\mathcal{HS}$.*

(a) *The spaces $\mathcal{F}, \mathcal{T}, \mathcal{HS}$ and $\mathcal{K}$ are ideals in the algebra $\mathcal{B}(H)$, which are stable under $*$.*

(b) *The space $\mathcal{K}$ is the norm closure of $\mathcal{F}$.*

(c) *One has*

$$\mathcal{F} \subset \mathcal{T} = \mathcal{HS}^2 \subset \mathcal{HS} \subset \mathcal{K},$$

where the inclusions are strict if $\dim(H) = \infty$.

Proof (a) The *-ideal property is clear for $\mathcal{F}$ and $\mathcal{K}$. The space $\mathcal{T}$ is an ideal by Lemma 5.3.5 and $\mathcal{HS}$ by Lemma 5.3.1. Part (b) of the theorem is contained in Proposition 5.2.1. The first inclusion of (c) is clear. Let T be in $\mathcal{T}$ and write $T = U|T|$ as in Proposition 5.1.4. With $S = \sqrt{|T|}$ one has $T = (US)S$ and by Proposition 5.3.4, the operators S and US are in $\mathcal{HS}$, so $T \in \mathcal{HS}^2$. If $S \in \mathcal{HS}$, then by definition $S^*S \in \mathcal{T}$ and by polarization we find $\mathcal{HS}^2 \subset \mathcal{T}$. The remaining inclusions are clear and we leave the strictness as an exercise. □

5.4 Exercises

Exercise 5.1 Let A and B be bounded operators on a Hilbert space H. Show that $AB - BA \neq \mathrm{Id}$, where Id is the identity operator.

(Hint: Assume the contrary and show that $AB^n - B^nA = nB^{n-1}$ holds for every $n \in \mathbb{N}$. Then take norms.)

Exercise 5.2 Let H be a Hilbert space, and let $T \in \mathcal{B}(H)$ be a normal operator. Show that the map $\psi : t \mapsto \exp(tT)$ satisfies $\psi(t+s) = \psi(t)\psi(s)$, that it is differentiable as a map from $\mathbb{R}$ to the Banach space $\mathcal{B}(H)$, which satisfies $\psi(0) = \mathrm{Id}$ and $\psi'(t) = T\psi(t)$.

Exercise 5.3 Show that for a bounded operator T on a Hilbert space H the following are equivalent:

- T is compact,
- T^*T is compact,
- T^* is compact.

Exercise 5.4 Check the details of the proof of Proposition 5.2.4.

Exercise 5.5 Show that a continuous invertible operator T on a Hilbert space H can only be compact if H is finite dimensional.

Exercise 5.6 Show that $T \in \mathcal{B}(H)$ for a Hilbert space H is compact if and only if the image of the closed unit ball is compact (as opposed to relatively compact).

Exercise 5.7 Let H be a Hilbert space.

(a) Show that for a trace class operator T on H one has $\mathrm{tr}\,(T^*) = \overline{\mathrm{tr}\,(T)}$.

(b) Show that for two Hilbert-Schmidt operators S, T on H one has

$$\mathrm{tr}\,(ST) = \mathrm{tr}\,(TS).$$

Exercise 5.8 Let H be the real Hilbert space $\ell^2(\mathbb{N}, \mathbb{R})$ and let $(e_j)_{j\in \mathrm{N}}$ be the standard orthonormal basis. Define a linear operator T on H by

$$T(e_j) = \frac{(-1)^{j+1}}{j} e_{j+(-1)^{j+1}}.$$

Show that for every orthonormal basis (f_n) of H one has

$$\sum_n |\langle Tf_n, f_n\rangle| \le \sum_{j=1}^{\infty} \frac{1}{2j(2j-1)}.$$

Exercise 5.9 Show that the set $\mathcal{HS}(V)$ of Hilbert-Schmidt operators on a given Hilbert space V becomes a Hilbert space with the inner product $\langle S, T\rangle = \mathrm{tr}\,(ST^*)$. Show that the map $\psi : V\hat{\otimes} V' \to \mathcal{HS}(V)$ given by $\psi(v \otimes \alpha)(w) = \alpha(w)v$ defines a Hilbert space isomorphism (Compare Appendix C.3 for the notation).

Exercise 5.10 Let H be a Hilbert space. For $p > 0$ let $S_p(H)$ be the set of all compact operators T on H such that

$$\|T\|_p \stackrel{\text{def}}{=} \left(\sum_j s_j(T)^p\right)^{\frac{1}{p}} < \infty.$$

Show that $S_p(H)$ is a vector space. It is called the p-th *Schatten class*.

Exercise 5.11 Let H be a Hilbert space. An operator $T \in \mathcal{B}(H)$ is called *nilpotent* if $T^k = 0$ for some $k \in \mathbb{N}$. Show that if T is nilpotent, then $\sigma(T) = \{0\}$. Show also that the converse is not generally true.

Exercise 5.12 Let H be a Hilbert space. Show that an operator T is invertible in $\mathcal{B}(H)$ if and only if $|T|$ is invertible.

Exercise 5.13 Let H be a Hilbert space, $T \in \mathcal{B}(H)$ invertible. Let $T = U|T|$ be the polar decomposition. Show that T is normal if and only if $U|T| = |T|U$.

Exercise 5.14 Let $G = \mathrm{SL}_n(\mathbb{R})$, and let H be the subgroup of upper triangular matrices in G. Let $K = \mathrm{SO}(n)$. Show that $G = HK$.

(Hint: For $g \in G$ apply the spectral theorem to the positive definite matrix $g^t g$.)