Chapter 3
Duality for Abelian Groups

In this chapter we are mainly interested in the study of *abelian* locally compact groups A, their dual groups $\widehat{A}$ together with various associated group algebras. Using the Gelfand-Naimark Theorem as a tool, we shall then give a proof of the Plancherel Theorem, which asserts that the Fourier transform extends to a unitary equivalence of the Hilbert spaces $L^2(A)$ and $L^2(\widehat{A})$. We also prove the Pontryagin Duality Theorem that gives a canonical isomorphism between A and its bidual $\widehat{\widehat{A}}$.

3.1 The Dual Group

A locally compact abelian group will be called an *LCA-group* for short. A *character* of an LCA-group A is a continuous group homomorphism

$$\chi : A \to \mathbb{T},$$

where $\mathbb{T}$ is the *circle group*, i.e., the multiplicative group of all complex numbers of absolute value one. The set $\widehat{A}$ of all characters on A forms a group under point-wise multiplication

$$(\chi \cdot \mu)(x) = \chi(x) \cdot \mu(x), \qquad x \in A.$$

The inverse element to a given $\chi \in \widehat{A}$ is given by $\chi^{-1}(x) = \frac{1}{\chi(x)} = \overline{\chi(x)}$. The group $\widehat{A}$ is called the *dual group* of A.

Examples 3.1.1

- As explained in Example 1.7.1, the dual group of $\mathbb{Z}$ is $\mathbb{R}/\mathbb{Z}$ and vice versa.
- The characters of the additive group $\mathbb{R}$ are the maps $\chi_t : x \mapsto e^{2\pi i x t}$, where t varies in $\mathbb{R}$. We then get an isomorphism of groups $\mathbb{R} \cong \widehat{\mathbb{R}}$ mapping t to χ_t.

Definition In what follows next we want to show that $\widehat{A}$ carries a natural topology that makes it a topological group. For a given topological space X let $C(X)$ be the

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complex vector space of all continuous maps from X to $\mathbb{C}$. For a compact set $K \subset X$ and an open set $U \subset \mathbb{C}$ define the set

$$L(K,U) \stackrel{\text{def}}{=} \{f \in C(X) : f(K) \subset U\}.$$

This is the set of all f that map a given compact set into a given open set. The topology generated by the sets $L(K,U)$ as K and U vary, is called the *compact-open topology*.

Lemma 3.1.2 (a) *Let X be a topological space. With the compact-open topology, $C(X)$ is a Hausdorff space.*

(b) *A net (f_i) in $C(X)$ converges in the compact-open topology if and only if it converges uniformly on every compact subset of X.*

(c) *If X is locally compact, then a net (f_i) converges in the compact-open topology if and only if it converges locally uniformly.*

(d) *If X is compact, the compact-open topology on $C(X)$ coincides with the topology given by the sup-norm.*

(e) *If $C(X)$ is endowed with the compact-open topology, then each point evaluation map $\delta_x : C(X) \to \mathbb{C}$; $f \mapsto f(x)$ is continuous.*

Proof (a) Let $f \neq g$ in $C(X)$. Choose $x \in X$ such that $f(x) \neq g(x)$, and choose disjoint open neighborhoods S, T in $\mathbb{C}$ of $f(x)$ and $g(x)$. Then the sets $L(\{x\}, S)$ and $L(\{x\}, T)$ are disjoint open neighborhoods of f and g, so $C(X)$ is a Hausdorff space.

(b) Fix $\varepsilon > 0$, let $f_i \to f$ be a net converging in the compact-open topology and let $K \subset X$ be a compact subset. For $z \in \mathbb{C}$ and $r > 0$ let $B_r(z)$ be the open ball of radius r around z and let $\bar{B}_r(z)$ be its closure. For $x \in X$ let U_x be the inverse image under f of the open ball $B_{\epsilon/3}(f(x))$. Then U_x is an open neighborhood of x and f maps its closure $\bar{U}_x$ into the closed ball $\overline{B_{\epsilon/3}}(f(x))$. As K is compact, there are $x_1, \dots x_n \in K$ such that K is a subset of the union $U_{x_1} \cup \dots \cup U_{x_n}$. Since closed subsets of compact sets are compact, the set $\bar{U}_{x_i} \cap K$ is compact. Let L be the intersection of the sets $L(\bar{U}_{x_i} \cap K, B_{2\varepsilon/3}(f(x_i)))$. Then L is an open neighborhood of f in the compact-open topology. Therefore, there exists an index j_0 such that for $j \geq j_0$ each f_j lies in L. Let $j \geq j_0$ and $x \in K$. Then there exists i such that $x \in U_{x_i}$. Therefore,

$$\begin{aligned} |f_j(x) - f(x)| &\leq |f_j(x) - f(x_i)| + |f(x_i) - f(x)| \\ &< \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

It follows that the net converges uniformly on K. The converse direction is trivial.

(c) Let X be a locally compact space and let (f_j) be a net in $C(X)$ which converges to $f \in C(X)$ in the compact-open topology, i.e., it converges uniformly on compact sets. As every $x \in X$ has a compact neighborhood, (f_j) converges uniformly on a neighborhood of a given x, hence it converges locally uniformly. Conversely, assume that (f_j) converges locally uniformly and let $K \subset X$ be compact. For each $x \in K$ there exists an open neighborhood U_x on which the net (f_j) converges uniformly. These U_x form an open covering of K, hence finitely many suffice, i.e., $K \subset U_{x_1} \cup \cdots \cup U_{x_n}$ for some $x_1, \ldots, x_n \in K$. As (f_j) converges uniformly on each U_{x_i}, in converges uniformly on K.

(d) If X is compact, the compact-open topology and the sup-norm topology generate the same set of convergent nets. Therefore they have the same closed sets, so they are equal. For the last point, (e), let (f_i) be a net in $C(X)$ convergent to f. Then $\delta_x(f_i) = f_i(x)$ converges to $f(x) = \delta_x(f)$, so the evaluation map is continuous. $\square$

By definition, the dual group $\widehat{A}$ is a subset of the set $C(A)$ of all continuous maps from A to $\mathbb{C}$. It is a consequence of Lemma 3.1.2 (e) that $\widehat{A}$ is closed in the compact-open topology of $C(A)$.

Examples 3.1.3.

- The compact-open topology on the dual $\widehat{\mathbb{Z}} \cong \mathbb{T}$ of $\mathbb{Z}$ is the natural topology of the circle group $\mathbb{T}$.
- The compact-open topology on the dual $\widehat{\mathbb{T}} \cong \mathbb{Z}$ of $\mathbb{T}$ is the discrete topology.
- The compact-open topology on the dual $\widehat{\mathbb{R}} \cong \mathbb{R}$ of $\mathbb{R}$ is the usual topology of $\mathbb{R}$.

Proposition 3.1.4 *With the compact-open topology, $\widehat{A}$ is a topological group that is Hausdorff.*

Later we will see that $\widehat{A}$ is also locally compact, i.e., an LCA-group.

Proof We have to show that the map $\alpha : \widehat{A} \times \widehat{A} \to \widehat{A}$, that sends a pair (χ, η) to $\chi\eta^{-1}$, is continuous. For two pairs $(\chi, \eta), (\chi'\eta')$ and $x \in A$ we have

$$\begin{aligned} |\chi(x)\eta^{-1}(x) - \chi'(x)\eta'^{-1}(x)| &\le |\chi(x)\eta^{-1}(x) - \chi(x)\eta'^{-1}(x)| \\ &\quad + |\chi(x)\eta'^{-1}(x) - \chi'(x)\eta'^{-1}(x)| \\ &= |\eta^{-1}(x) - \eta'^{-1}(x)| + |\chi(x) - \chi'(x)|, \end{aligned}$$

Let $K \subset A$ be compact and let $\varepsilon > 0$. Then

$$B_{K,\varepsilon}(\chi\eta^{-1}) = \left\{\gamma \in \widehat{A} : \|\gamma - \chi\eta^{-1}\|_K < \varepsilon\right\}$$

is an open neighborhood of $\chi\eta^{-1}$ and sets of this form are a neighborhood base. The estimate above shows that the open neighborhood $B_{K,\varepsilon/2}(\chi) \times B_{K,\varepsilon/2}(\eta)$ of (χ, η) is mapped to $B_{K,\varepsilon}(\chi\eta^{-1})$, so α is continuous. $\square$

The observation, that the dual group of the compact group $\mathbb{T}$ is the discrete group $\mathbb{Z}$ and vice versa, is an example of the following general principle:

Proposition 3.1.5

(a) *If A is compact, then $\widehat{A}$ is discrete.*

(b) *If A is discrete, then $\widehat{A}$ is compact.*

Proof Let A be compact, and let L be the set of all $\eta \in \widehat{A}$ such that $\eta(A)$ lies in the open set $\{\mathrm{Re}(\cdot) > 0\}$. As A is compact, L is an open unit-neighborhood in $\widehat{A}$. For every $\eta \in \widehat{A}$, the image $\eta(A)$ is a subgroup of $\mathbb{T}$. The only subgroup of $\mathbb{T}$, however, that is contained in $\{\mathrm{Re}(\cdot) > 0\}$, is the trivial group. Therefore $L = \{1\}$, and so $\widehat{A}$ is discrete.

For the second part, assume that A is discrete. Then $\widehat{A} = \mathrm{Hom}(A, \mathbb{T})$ is a subset of the set $\mathrm{Map}(A, \mathbb{T})$ of all maps from A to $\mathbb{T}$. The set $\mathrm{Map}(A, \mathbb{T})$ can be identified with the product $\prod_{a \in A} \mathbb{T}$. By Tychonov's Theorem, the latter is a compact Hausdorff space in the product topology and $\widehat{A}$ forms a closed subspace. As A is discrete, the inclusion $\widehat{A} \hookrightarrow \prod_{a \in A} \mathbb{T}$ induces a homeomorphism of $\widehat{A}$ onto its image in the product space. Hence $\widehat{A}$ is compact. □

3.2 The Fourier Transform

Let A be an LCA-group and consider its convolution algebra $L^1(A)$. In this section we want to show that the topological space $\widehat{A}$ is canonically homeomorphic to the structure space $\Delta_{L^1(A)}$ of the commutative Banach algebra $L^1(A)$. Since this structure space is locally compact, this will show that the dual group $\widehat{A}$ is an LCA-group. Recall that the Fourier transform $\hat{f} : \widehat{A} \to \mathbb{C}$ of a function $f \in L^1(A)$ is defined as

$$\hat{f}(\chi) = \int_A f(x)\overline{\chi(x)}\, dx.$$

Theorem 3.2.1 *The map $\chi \mapsto d_\chi$ from the dual group $\widehat{A}$ to the structure space $\Delta_{L^1(A)}$ defined by*

$$d_\chi(f) = \hat{f}(\chi)$$

is a homeomorphism. In particular, $\widehat{A}$ is a locally compact Hausdorff space, so $\widehat{A}$ is an LCA-group.

It follows that for every $f \in L^1(A)$ the Fourier transform $\hat{f}$ is a continuous function on the dual group $\widehat{A}$, which vanishes at infinity.

Proof By Lemma 1.7.2 it follows that d_χ indeed lies in the structure space of the Banach algebra $L^1(A)$.

Injectivity Assume $d_\chi = d_{\chi'}$, then $\int_A f(x)\overline{(\chi(x) - \chi'(x))}\,dx = 0$ for every $f \in C_c(A)$. This implies that the continuous functions χ and χ' coincide.

Surjectivity Let $m \in \Delta_{L^1(A)}$. As $C_c(A)$ is dense in $L^1(A)$, there exists an element $g \in C_c(A)$ with $m(g) \neq 0$. For $x \in A$ define $\chi(x) = \overline{m(L_x g)}/\overline{m(g)}$. The continuity of m and Lemma 1.4.2 implies that χ is a continuous function on A. One computes,

$$m(L_x g)m(L_y g) = m(L_x g * L_y g) = m(L_{xy} g * g) = m(L_{xy} g)m(g).$$

Dividing by $m(g)^2$ and taking complex conjugates, one gets the identity $\chi(x)\chi(y) = \chi(xy)$, so χ is a multiplicative map from A to $\mathbb{C}^\times$. Let $f \in C_c(A)$. Then one can write the convolution $f * g$ as $\int_A f(x) L_x g\, dx$, and this integral may be viewed as a vector-valued integral with values in the Banach space $L^1(A)$ as in Sect. B.6. One uses the continuity of the linear functional m and Lemma B.6.5 to get

$$\begin{aligned}\int_A f(x)\overline{\chi(x)}\,dx &= \frac{1}{m(g)}\int_A f(x)m(L_x g)\,dx = \frac{1}{m(g)} m\left(\int_A f(x)L_x g\,dx\right)\\ &= \frac{1}{m(g)} m(f * g) = \frac{m(f)m(g)}{m(g)} = m(f).\end{aligned}$$

Let (ϕ_U) be a Dirac net in $C_c(A)$. Then $\phi_U * \overline{\chi}$ converges point-wise to $\overline{\chi}$ and so for $x \in A$ and $\varepsilon > 0$ there exists a unit-neighborhood U, such that

$$\begin{aligned}|\chi(x)| &\le \left|\phi_U * \overline{\chi}(x)\right| + \varepsilon = \left|\int_A L_x\phi_U(y)\overline{\chi(y)}\,dy\right| + \varepsilon\\ &= \left|m(L_x\phi_U)\right| \ \le\ \lim_U \|L_x\phi_U\|_1 + \varepsilon = 1 + \varepsilon.\end{aligned}$$

As ε is arbitrary, we get $|\chi(x)| \le 1$ for every $x \in A$. By $\chi(x^{-1}) = \chi(x)^{-1}$ we infer $|\chi(x)| = 1$ for every $x \in A$. So the map χ lies in $\widehat{A}$, and the map d is surjective.

Continuity Let $\chi_j \to \chi$ be a net in $\widehat{A}$ which converges locally uniformly on A. Let $f \in L^1(A)$ and choose $\varepsilon > 0$. We have to show that there exists j_0 such that for $j \ge j_0$ one has $|\hat{f}(\chi_j) - \hat{f}(\chi)| < \varepsilon$. Let $g \in C_c(A)$ with $\|f - g\|_1 < \varepsilon/3$. Since $\chi_j \to \chi$ uniformly on $\operatorname{supp}(g)$, there exists j_0 such that for $j \ge j_0$ it holds $|\hat{g}(\chi_j) - \hat{g}(\chi)| < \varepsilon/3$. For $j \ge j_0$ one has

$$\begin{aligned}|\hat{f}(\chi_j) - \hat{f}(\chi)| &\le |\hat{f}(\chi_j - \hat{g}(\chi_j)| + |\hat{g}(\chi_j) - \hat{g}(\chi)| + |\hat{g}(\chi) - \hat{f}(\chi)|\\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.\end{aligned}$$

The *continuity of the inverse map* d^{-1} is a direct consequence of the following lemma.

Lemma 3.2.2 *Let $\chi_0 \in \widehat{A}$. Let K be a compact subset of A, and let $\varepsilon > 0$. Then there exist $l \in \mathbb{N}$, functions $f_0, f_1, \dots, f_l \in L^1(A)$, and $\delta > 0$ such that for $\chi \in \widehat{A}$ the condition $|\hat{f}_j(\chi) - \hat{f}_j(\chi_0)| < \delta$ for every $j = 0, \dots, l$ implies $|\chi(x) - \chi_0(x)| < \varepsilon$ for every $x \in K$.*

Proof For $f \in L^1(A)$ we have

$$\begin{aligned}\hat{f}(\chi) - \hat{f}(\chi_0) &= \int_A f(x)\overline{(\chi(x) - \chi_0(x))}\, dx \\ &= \int_A f(x)\overline{\chi_0(x)}\,(\overline{\chi(x)}\chi_0(x) - 1)\, dx \\ &= \widehat{f\bar{\chi}_0}(\chi\bar{\chi}_0) - \widehat{f\bar{\chi}_0}(1).\end{aligned}$$

So without loss of generality we can assume $\chi_0 = 1$.

Let $f \in L^1(A)$ with $\hat{f}(1) = \int_A f(x)\, dx = 1$. Then there is a unit-neighborhood U in A with $\|L_u f - f\|_1 < \varepsilon/3$ for every $u \in U$. As K is compact, there are finitely many $x_1, \dots, x_l \in A$ such that K is a subset of $x_1U \cup \dots \cup x_lU$. Set $f_j = L_{x_j} f$ as well as $f_0 = f$ and let $\delta = \varepsilon/3$. Let $\chi \in \widehat{A}$ with $|\hat{f}_j(\chi) - 1| < \varepsilon/3$ for every $j = 0, \dots, l$. Now let $x \in K$. Then there exists $1 \le j \le l$ and $u \in U$ such that $x = x_j u \in x_j U$. One gets

$$\begin{aligned}|\chi(x) - 1| &= |\overline{\chi(x)} - 1| \\ &\le |\overline{\chi(x)} - \overline{\chi(x)}\hat{f}(\chi)| + |\hat{f}(\chi)\overline{\chi(x)} - \hat{f}_j(\chi)| + |\hat{f}_j(\chi) - 1| \\ &= |1 - \hat{f}(\chi)| + |\widehat{L_x f}(\chi) - \widehat{L_{x_j} f}(\chi)| + |\hat{f}_j(\chi) - 1| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,\end{aligned}$$

where the last inequality uses

$$\begin{aligned}|\widehat{L_x f}(\chi) - \widehat{L_{x_j} f}(\chi)| &\le \|L_x f - L_{x_j} f\|_1 = \|L_{x_j}(L_u f - f)\|_1 \\ &= \|L_u f - f\|_1 < \varepsilon/3.\end{aligned}$$

The lemma and the theorem are proven. □

3.3 The C^*-Algebra of an LCA-Group

In this section we introduce the C^*-algebra $C^*(A)$ of the LCA-group A as a certain completion of the convolution algebra $L^1(A)$. We show that restriction of multiplicative functionals from $C^*(A)$ to the dense subalgebra $L^1(A)$ defines a homeomorphism between $\Delta_{C^*(A)}$ and $\Delta_{L^1(A)}$. Hence by the results of the previous section, $\Delta_{C^*(A)}$ is canonically homeomorphic to the dual group $\widehat{A}$. The Gelfand-Naimark Theorem then implies that the Fourier transform on $L^1(A)$ extends to an isometric $*$-isomorphism between $C^*(A)$ and $C_0(\widehat{A})$. These results will play an important role in the proof of the Plancherel Theorem in the following section.

Let $f \in L^1(A)$ and $\phi, \psi \in L^2(A)$. For every $y \in A$ one has

$$|\langle L_y\phi, \psi\rangle| \leq \|L_y\phi\|_2\|\psi\|_2 = \|\phi\|_2\|\psi\|_2.$$

This implies that the integral $\int_A f(y)\langle L_y\phi, \psi\rangle\, dy$ exists, and one has the estimate

$$\left|\int_A f(y)\langle L_y\phi, \psi\rangle\, dy\right| \leq \|f\|_1\|\phi\|_2\|\psi\|_2.$$

In other words, the anti-linear map that sends ψ to the integral $\int_A f(y)\langle L_y\phi, \psi\rangle\, dy$ is bounded, hence continuous by Lemma C.1.2. As every continuous anti-linear map on a Hilbert space is represented by a unique vector, there exists a unique element $L(f)\phi$ in $L^2(A)$ such that $\langle L(f)\phi, \psi\rangle = \int_A f(y)\langle L_y\phi, \psi\rangle\, dy$ for every $\psi \in L^2(A)$. The above estimate gives $|\langle L(f)\phi, \psi\rangle| \leq \|f\|_1\|\phi\|_2\|\psi\|_2$. In particular, for $\psi = L(f)\phi$ one concludes $\|L(f)\phi\|_2^2 \leq \|f\|_1\|\phi\|_2\|L(f)\phi\|_2$, hence $\|L(f)\phi\|_2 \leq \|f\|_1\|\phi\|_2$, which implies that the linear map $\phi \mapsto L(f)\phi$ is bounded, hence continuous. Note that for $\phi \in C_c(G)$ one has $L(f)\phi = f * \phi$ by Lemma 3.3.1 below.

Lemma 3.3.1 *If $f \in L^1(A)$ and $\phi \in L^1(A) \cap L^2(A)$, then $L(f)\phi = f * \phi = \phi * f$.*

Proof Let $\psi \in C_c(A)$. Then the inner product $\langle L(f)\phi, \psi\rangle$ equals $\int_A f(y) \int_A \phi(y^{-1}x)\overline{\psi(x)}\, dx\, dy$. This integral exists if f, ϕ, ψ are replaced with their absolute values. Therefore we can apply Fubini's Theorem to get $\langle L(f)\phi, \psi\rangle = \langle f * \phi, \psi\rangle$, whence the claim.

Lemma 3.3.2 *The map L from $L^1(A)$ to the space $\mathcal{B}(L^2(A))$ is an injective, continuous homomorphism of Banach-$*$-algebras.*

Proof The map is linear and satisfies $\|L(f)\|_{\text{op}} \leq \|f\|_1$, therefore is continuous. For $f, g \in L^1(A)$ and ϕ in the dense subspace $C_c(A)$ of $L^2(A)$ the above lemma and the associativity of convolution implies

$$L(f * g)\phi = (f * g) * \phi = f * (g * \phi) = L(f)L(g)\phi,$$

so L is multiplicative. For $\phi, \psi \in C_c(A)$ we get

$$\begin{aligned}
\langle f * \phi, \psi\rangle &= \int_A \int_A f(y)\phi(y^{-1}x)\overline{\psi(x)}\, dy\, dx \\
&= \int_A \int_A f(y)\phi(x)\overline{\psi(yx)}\, dx\, dy \\
&= \int_A \int_A \phi(x)\overline{\Delta(y^{-1})\overline{f(y^{-1})}\psi(y^{-1}x)}\, dy\, dx \\
&= \langle \phi, f^* * \psi\rangle,
\end{aligned}$$

where we used the transformation $x \mapsto yx$ followed by the transformation $y \mapsto y^{-1}$. This shows $L(f^*) = L(f)^*$.

For the injectivity, let $f \in L^1(G)$ with $L(f) = 0$. Then in particular $f * \phi = 0$ for every $\phi \in C_c(A)$. Using Lemma 1.6.6 this implies $f = 0$.

Definition We define the *group C^*-algebra* $C^*(A)$ of the LCA-group A to be the norm-closure of $(L^1(A))$ in the C^*-algebra $\mathcal{B}(L^2(A))$. As $L^1(A)$ is a commutative Banach algebra, $C^*(A)$ is a commutative C^*-algebra.

Theorem 3.3.3 *The map $L^* : \Delta_{C^*(A)} \to \Delta_{L^1(A)}$ given by $m \mapsto m \circ L$ is a homeomorphism. It follows $\Delta_{C^*(A)} \cong \widehat{A}$ and $C^*(A) \cong C_0(\widehat{A})$.*

Proof As the image of L is dense in $C^*(A)$, it follows that $m \circ L \neq 0$ for every $m \in \Delta_{C^*(A)}$ and that L^* is injective. Therefore by Lemma 2.4.7 it suffices to show that L^* is surjective.

To prove this, let $m \in \Delta_{L^1(A)}$ and $\chi \in \widehat{A}$ such that $m(f) = \hat{f}(\chi)$ for every $f \in L^1(A)$. We have to show that m is continuous in the C^*-norm, because then it has a unique extension to $C^*(A)$. For this let $\mu_0 \in \Delta_{C^*(A)}$ be fixed. Then there is $\chi_0 \in \widehat{A}$ such that for $f \in L^1(A)$ the identity $\hat{f}(\chi_0) = \mu_0(f)$ holds, where we have written $\mu_0(L(f)) = \mu_0(f)$. For $f \in L^1(A)$, one has

$$m(f) = \int_A f(x)\overline{\chi(x)}\,dx = \int_A f(x)\overline{\chi(x)}\chi_0(x)\overline{\chi_0(x)}\,dx = \mu_0(f\bar{\chi}\chi_0).$$

It follows that $|m(f)| = |\mu_0(f\bar{\chi}\chi_0)| \le \|f\bar{\chi}\chi_0\|_{C^*(A)}$. So we have to show that for $f \in L^1(A)$ the C^*-norm of f equals the C^*-norm of $f\eta$ for any $\eta \in \widehat{A}$. As the C^*-norm is the operator norm in $\mathcal{B}(L^2(A))$, we consider $\phi, \psi \in L^2(A)$, and we compute

$$\begin{aligned}
\langle L(\eta f)\phi, \psi\rangle &= \int_A \eta(x) f(x) \langle L_x\phi, \psi\rangle\,dx \\
&= \int_A \eta(x) f(x) \int_A \phi(x^{-1}y)\overline{\psi(y)}\,dy\,dx \\
&= \int_A f(x) \int_A (\bar{\eta}\phi)(x^{-1}y)\overline{(\bar{\eta}\psi)(y)}\,dy\,dx \\
&= \langle L(f)(\bar{\eta}\phi), \bar{\eta}\psi\rangle.
\end{aligned}$$

Putting $\psi = L(\eta f)\phi$, we get

$$\|L(\eta f)\phi\|_2^2 = \langle L(f)(\bar{\eta}\phi), \bar{\eta}L(\eta f)\phi\rangle \le \|L(f)(\bar{\eta}\phi)\|_2 \|\bar{\eta}L(\eta f)\phi\|_2.$$

Since $\|\bar{\eta}L(\eta f)\phi\|_2 = \|L(\eta f)\phi\|_2$ it follows $\|L(\eta f)\phi\|_2 \le \|L(f)(\bar{\eta}\phi)\|_2$ and so the operator norm of $L(\eta f)$ is less than or equal to the operator norm of $L(f)$. By symmetry we get equality and the theorem follows. $\square$

Corollary 3.3.4 *Let A be an LCA-group. Then the Fourier transform $L^1(A) \to C_0(\widehat{A})$, mapping f to $\hat{f}$, is injective.*

Proof Let $\mathcal{A} = L^1(A)$. As $\widehat{A} \cong \Delta_{\mathcal{A}} \cong \Delta_{C^*(A)}$, the Fourier transform is the composition of the injective maps $\mathcal{A} \to C^*(A) \to C_0(\widehat{A})$. □

3.4 The Plancherel Theorem

In this section we will construct the Plancherel measure on the dual group $\widehat{A}$ relative to a given Haar measure on the LCA group A and we will state the Plancherel Theorem, which says that the Fourier transform extends to a unitary equivalence

$$L^2(A) \cong L^2(\widehat{A}).$$

The proof of the Plancherel theorem will be postponed to the following section, where it will be shown as a consequence of Pontryagin duality.

Lemma 3.4.1 *Let $\phi, \psi \in L^2(A)$. Then the convolution integral $\phi * \psi(x) = \int_A \phi(y)\psi(y^{-1}x)\,dy$ exists for every $x \in A$ and defines a continuous function in x. The convolution product $\phi * \psi$* lies in *$C_0(A)$ and its sup-norm satisfies $\|\phi * \psi\|_A \le \|\phi\|_2\|\psi\|_2$. Finally one has $\phi * \phi^*(1) = \|\phi\|_2^2$.*

Proof With ψ, also the function $L_x\psi^*$ lies in $L^2(A)$, as A is abelian, hence unimodular. The convolution integral is the same as the inner product $\langle \phi, L_x\psi^* \rangle$, hence the integral exists for every $x \in A$. The continuity follows from Lemma 1.4.2 and the fact that the map $L^2(A) \to \mathbb{C}$, given by $\psi \mapsto \langle \phi, \psi \rangle$ is continuous. Next use the Cauchy-Schwarz inequality to get

$$\|\phi * \psi\|_A = \sup_{x \in A} \left|\left\langle \phi, L_x\psi^* \right\rangle\right| \le \|\phi\|_2\|\psi\|_2.$$

Choose sequences (ϕ_n) and (ψ_n) in $C_c(A)$ with $\|\phi_n - \phi\|_2, \|\psi_n - \psi\|_2 \to 0$. Then it follows from the above inequality that $\phi_n * \psi_n \in C_c(A)$ converges uniformly to $\phi * \psi$. It follows that $\phi * \psi \in C_0(A)$ since $C_0(A)$ is complete. The final assertion $\phi * \phi^*(1) = \|\phi\|_2^2$ is clear by definition. □

The space $\mathcal{C} = C_0(A) \times C_0(\widehat{A})$ is a Banach space with the norm

$$\|(f, \eta)\|_0^* = \max\left(\|f\|_A, \|\eta\|_{\widehat{A}}\right).$$

We embed $C_0(A) \cap L^1(A)$ into this product space by mapping f to $(f, \hat{f})$ and we denote the closure of $C_0(A) \cap L^1(A)$ inside $\mathcal{C}$ by

$$C_0^*(A).$$

This is a Banach space the norm of which we write as $\|f\|_0^*$.

Lemma 3.4.2 *Let p_0 and p_* be the projections from $\mathcal{C}$ to $C_0(A)$ and $C_0(\widehat{A})$, respectively. Then the restrictions of p_0 and p_* to $C_0^*(A)$ are both injective. Hence we can consider $C_0^*(A)$ as a subspace of $C_0(A)$ as well as of $C_0(\widehat{A})$.*

Proof Let $f \in C_0^*(A)$ and write f_* for $p_*(f)$ and f_0 for $p_0(f)$. We have to show that if one of these two is zero, then so is the other. Let (f_n) be a sequence in $C_0(A) \cap L^1(A)$ converging to f in $C_0^*(A)$. Then f_n converges to f_* in $C^*(A)$ and to f_0 uniformly on A. So for $\psi \in L^2(A)$ the sequence $f_n * \psi$ converges to $f_*(\psi)$ in $L^2(A)$. If ψ is in $C_c(A)$, then $f_n * \psi$ also converges uniformly to $f_0 * \psi$. So for every $\phi \in C_c(A)$, the sequence $\langle f_n * \psi, \phi\rangle$ converges to $\langle f^*(\psi), \phi\rangle$ and by uniform convergence also to $\langle f_0 * \psi, \phi\rangle$, i.e., we have $\langle f_*(\psi), \phi\rangle = \langle f_0 * \psi, \phi\rangle$. As this holds for all $\psi, \phi \in C_c(G)$, we conclude $f_* = 0 \Leftrightarrow f_0 = 0$ as claimed. □

A given element f of $C_0^*(A)$ can be viewed as an element of $C_0(A)$, or of $C^*(A) \cong C_0(\widehat{A})$. We will freely switch between these two viewpoints in the sequel. If we want to emphasize the distinction, we write f for the function on A and $\hat{f}$ for its Fourier transform, the function on $\widehat{A}$.

For $g \in C^*(A)$ and $\phi \in L^2(A)$ we from now on write $L(g)\phi$ for the element $g(\phi)$ of $L^2(A)$.

Lemma 3.4.3 *Let $f \in C_0^*(A)$. If the Fourier transform $\hat{f}$ is real-valued, then $f(1)$ is real. If $\hat{f} \geq 0$, then $f(1) \geq 0$. Here 1 denotes the unit element of A.*

Proof Suppose that $\hat{f}$ is real-valued. Then $\hat{f} = \overline{\hat{f}} = \widehat{f^*}$, so we get $f = f^*$, and therefore $f(1) = f^*(1) = \overline{f(1)}$. Now suppose $\hat{f} \geq 0$. Then there exists $g \in C_0(\widehat{A}) \cong C^*(A)$ with $g \geq 0$ and $\hat{f} = g^2$. Let $\phi = \phi^* \in C_c(A)$. Then $L(g)\phi \in L^2(A)$, so $(L(g)\phi) * (L(g)\phi)^*(1) = \|L(g)\phi\|_2^2 \geq 0$. Now g is a limit in $C^*(A)$ of a sequence (g_n) in $L^1(A)$. We can assume $g_n = g_n^*$ for every $n \in \mathbb{N}$. Using Lemma 3.3.1 we have

$$\begin{aligned}
(L(g)\phi) * (L(g)\phi)^* &= \lim_n (L(g_n)\phi) * (L(g_n)\phi)^* = \lim_n (g_n * \phi) * (g_n * \phi)^* \\
&= \lim_n g_n * \phi * \phi * g_n = \lim_n g_n * g_n * \phi * \phi \\
&= \lim_n L(g_n * g_n)(\phi * \phi) = L(f)(\phi * \phi) = f * \phi * \phi.
\end{aligned}$$

We get $f * \phi * \phi(1) \geq 0$, and therefore $f(1) \geq 0$ by Lemma 1.6.6. since we can let $\phi * \phi$ run through a Dirac net. □

Lemma 3.4.4 (a) *The space $L^1(A) * C_c(A)$ is a subspace of $C_0(A)$.*

(b) *Let $f \in C^*(A)$, and let $\phi, \psi \in C_c(A)$. Then $L(f)(\phi * \psi)$ lies in $C_0^*(A) \cap L^2(A)$, viewed as a subspace of $C_0(A)$. One has $L(f)(\phi * \psi)\widehat{\ } = \hat{f}\hat{\phi}\hat{\psi}$.*

Proof (a) Let $f \in L^1(A)$ and $\phi \in C_c(A)$. Choose a sequence $f_n \in C_c(A)$ such that $\|f_n - f\|_1 \to 0$. Then $f_n * \phi \in C_c(A)$, and for every $x \in A$ we have $|f * \phi(x) - f_n * \phi(x)| \le \|f - f_n\|_1 \|\phi\|_\infty$. This shows that $f * \phi$ is a uniform limit of functions in $C_0(A)$. Since $C_0(A)$ is complete with respect to $\|\cdot\|_A$, the result follows.

For (b) let now $f \in C^*(A)$. There is a sequence $f_n \in L^1(A)$ converging to f in $C^*(A)$. Then $L(f_n)(\phi * \psi) = f_n * \phi * \psi$ lies in $C_0(A) \cap L^1(A)$. We have to show that the ensuing sequence $f_n * \phi * \psi$ is a Cauchy sequence in $C_0^*(A)$. This means that the sequence, as well as its Fourier transform, are both Cauchy sequences in $C_0(A)$ and $C_0(\widehat{A})$, respectively. Observe first that $(f_n * \phi * \psi)\widehat{} = \hat{f}_n \hat{\phi} \hat{\psi}$. Now $\hat{f}_n$ converges uniformly on $\widehat{A}$, so $(f_n * \phi * \psi)\widehat{}$ converges uniformly to $\hat{f}\hat{\phi}\hat{\psi}$, hence is Cauchy in $C_0(\widehat{A})$. By Lemma 3.4.1 we conclude that for $m, n \in \mathbb{N}$ one has $\|(f_m - f_n) * \phi * \psi\|_A \le \|(f_m - f_n) * \phi\|_2 \|\psi\|_2$. The right hand side tends to zero as m, n grow large, so $f_n * \phi * \psi$ is a Cauchy sequence in $C_0(A)$. Since $L(f)(\phi * \psi) \in L^2(A)$, the result follows. □

Lemma 3.4.5 *Let (ϕ_U) be a Dirac net in $C_c(A)$. Then*

(a) *$(f * \phi_U)$ converges to f in $C^*(A)$ for every $f \in C^*(A)$,*

(b) *$(f * \phi_U)$ converges uniformly to f for every $f \in C_0(A)$,*

(c) *$(f * \phi_U)$ converges to f in $C_0^*(A)$ for every $f \in C_0^*(A)$,*

(d) *$(\widehat{\phi_U})$ converges locally uniformly to 1 on $\widehat{A}$.*

Proof For (a) observe that the result holds for the dense subspace $L^1(A)$ by Lemma 1.6.6. Then a standard $\varepsilon/3$-argument extends it to all of $C^*(A)$. For (b) we can use the same argument with $L^1(A)$ replaced by the dense subspace $C_c(A)$ of $C_0(A)$. Then (c) is a consequence of (a) and (b). For the proof of (d) let $C \subseteq \widehat{A}$ be a compact set. Choose a positive $\psi \in C_c(\widehat{A})$ with $\psi \equiv 1$ on C and let $f \in C^*(A)$ with $\hat{f} = \psi$. Then $\|\hat{\phi}_U \psi - \psi\|_{\widehat{A}} = \|\phi_U * f - f\|_{op} \to 0$ by (a) and the result follows. □

Lemma 3.4.6 *Let $\eta \in C_c(\widehat{A})$ be real-valued, and let $\varepsilon > 0$. Then there are $f_1, f_2 \in C_0^*(A) \cap L^2(A)$, considered as subspace of $C_0(A)$, such that*

- *the Fourier transforms $\hat{f}_1, \hat{f}_2$ lie in $C_c(\widehat{A})$,*
- *they satisfy $\hat{f}_1 \le \eta \le \hat{f}_2$, further $\|\hat{f}_1 - \hat{f}_2\|_{\widehat{A}} < \varepsilon$, and $\operatorname{supp}(\hat{f}_i) \subset \operatorname{supp}(\eta)$ for $i = 1, 2$,*
- *as well as $0 \le f_2(1) - f_1(1) < \varepsilon$.*

In particular, every $\eta \in C_c(\widehat{A})$ is the uniform limit of functions of the form $\hat{f}$ with $f \in C_0^*(A)$ of support contained in $\operatorname{supp}(\eta)$.

Proof For any Dirac function ϕ in $C_c(A)$ one has $\hat{\phi} \in C_0(\widehat{A})$ by Theorem 3.2.1 and by Lemma 3.4.5 the ensuing function $\hat{\phi}$ can be chosen to approximate the constant 1 arbitrarily close on any compact set. Note that the Fourier transform of a function of the form $h * h^*$ is ≥ 0. Let $K \subset \widehat{A}$ be the support of η. As $C_c(A)$ contains Dirac functions of arbitrary small support, we conclude that for every $\delta > 0$ there exists a function $\phi_\delta \in C_c^+(A)$ such that the function $\psi_\delta = \phi_\delta * \phi_\delta^*$ satisfies

$$1 - \delta \;\leq\; \hat{\psi}_\delta(\chi) \;\leq\; 1 + \delta \qquad \text{for every } \chi \in K.$$

Fix $\phi \in C_c^+(A)$ such that $\psi = \phi * \phi^*$ satisfies $\hat{\psi}(\chi) \geq 1$ for every $\chi \in K$. Let $f \in C^*(A)$ with $\hat{f} = \eta$ and set

$$f_1 = f * (\psi_\delta - \delta\psi), \qquad f_2 = f * (\psi_\delta + \delta\psi).$$

According to Lemma 3.4.4, the functions f_1 and f_2 lie in the space $C_0(A) \cap L^2(A)$. For every $\chi \in \widehat{A}$ it holds,

$$\hat{f}_1(\chi) = \hat{f}(\chi)\big(\hat{\psi}_\delta(\chi) - \delta\hat{\psi}(\chi)\big) \;\leq\; \eta(\chi) \;\leq\; \hat{f}_2(\chi).$$

Further, as $\hat{f}(\chi) = \eta(\chi)$, one has $\operatorname{supp}(\hat{f}_i) \subset \operatorname{supp}(\eta)$. The other properties follow by choosing δ small enough. □

Proposition 3.4.7 *Let $\psi \in C_c(\widehat{A})$ be real-valued. Then the supremum of the set*

$$\{f(1) : f \in C_0^*(A),\ \hat{f} \leq \psi\}$$

equals the infimum of the set

$$\{f(1) : f \in C_0^*(A),\ \hat{f} \geq \psi\}.$$

We denote this common value by $I(\psi)$. We extend I to all of $C_c(\widehat{A})$ by setting $I(u + iv) = I(u) + iI(v)$, where u and v are real-valued. Then I is a Haar integral on $C_c(\widehat{A})$.

We write this integral as

$$I(\psi) = \int_{\widehat{A}} \psi(\chi)\, d\chi.$$

Proof It follows from Lemma 3.4.3 that the supremum is less or equal to the infimum and Lemma 3.4.6 implies that they coincide. Thus I exists. It is clearly linear and it is positive by Lemma 3.4.3. For the invariance let $\psi \in C_c(\widehat{A})$ be real-valued, and let $f \in C_0^*(A)$ with $\hat{f} \leq \psi$. For $\chi \in \widehat{A}$ we then have $L_\chi \hat{f} \leq L_\chi \psi$. Further, $L_\chi \hat{f} = \widehat{\chi f}$ as well as $\chi f(1) = f(1)$. This implies the invariance of I. The proof of the proposition is finished. □

We close this section with formulating the Plancherel theorem for LCA groups. The proof will be given as a consequence of the Pontryagin Duality Theorem in the following section.

Theorem 3.4.8 (Plancherel Theorem). *For a given Haar measure on A there exists a uniquely determined Haar measure on $\widehat{A}$, called the* Plancherel measure, *such that for $f \in L^1(A) \cap L^2(A)$ one has*

$$\|f\|_2 = \|\hat{f}\|_2.$$

This implies that the Fourier transform extends to an isometry from $L^2(A)$ to $L^2(\widehat{A})$. Indeed, it is also surjective, so the Fourier transform extends to a canonical unitary equivalence $L^2(A) \cong L^2(\widehat{A})$.

In the special case of a compact group we derive from this, that the characters form an orthonormal basis of $L^2(A)$.

Corollary 3.4.9 *Let A be a compact LCA-group. Then the elements of the dual group $\widehat{A}$ form an orthonormal basis of $L^2(A)$.*

Proof According to our conventions, we assume the Haar measure of A to be normalized in a way that the total volume is one. As A is compact, any continuous function on A, in particular every character, lies in $L^2(A)$. We show that the characters of A form an orthonormal system, i.e., that for $\chi, \eta \in \widehat{A}$ we have

$$\langle \chi, \eta \rangle = \delta_{\chi,\eta} = \begin{cases} 1 & \chi = \eta, \\ 0 & \chi \neq \eta. \end{cases}$$

If $\chi = \eta$, then

$$\langle \chi, \eta \rangle = \int_A \underbrace{\chi(x)\overline{\chi(x)}}_{=1}\, dx = \int_A dx = 1.$$

If $\chi \neq \eta$, then pick $x_0 \in A$ with $\chi(x_0) \neq \eta(x_0)$. We obtain

$$\chi(x_0)\langle \chi, \eta \rangle = \int_A \chi(x_0 x)\overline{\eta(x)}\, dx = \int_A \chi(x)\overline{\eta(x_0^{-1}x)}\, dx = \eta(x_0)\langle \chi, \eta \rangle,$$

which implies $\langle \chi, \eta \rangle = 0$ as claimed. It follows that the Fourier transform of a character χ is the map δ_χ with $\delta_\chi(\eta) = \delta_{\chi,\eta}$. These maps form an orthonormal basis of the Hilbert space $L^2(\widehat{A})$ for the discrete group $\widehat{A}$. Since the Fourier transform is a unitary equivalence, the characters form an orthonormal basis of $L^2(A)$. $\square$

3.5 Pontryagin Duality

In the previous sections we saw that the dual group $\widehat{A}$ of an LCA group A, which consists of all continuous homomorphisms of A into the circle group $\mathbb{T}$, is again an LCA group. So we can also consider the dual group $\widehat{\widehat{A}}$ of $\widehat{A}$. There is a canonical homomorphism $\delta : A \to \widehat{\widehat{A}}$, which we write as $x \mapsto \delta_x$, and which is given by

$$\delta_x(\chi) = \chi(x).$$

We call δ the *Pontryagin map*. To see that for each $x \in A$ the map $\delta_x : \widehat{A} \to \mathbb{T}$ is indeed a continuous group homomorphism, and hence an element of $\widehat{\widehat{A}}$, we first observe that

$$\delta_x(\chi\mu) = \chi\mu(x) = \chi(x)\mu(x) = \delta_x(\chi)\delta_x(\mu)$$

for all $\chi, \mu \in \widehat{A}$, which implies that δ_x is a homomorphism. Since convergence in $\widehat{A}$ with respect to the compact open topology implies point-wise convergence we see that if a net $\chi_j \to \chi$ converges in $\widehat{A}$, then the net $\delta_x(\chi_j) = \chi_j(x)$ converges to $\chi(x) = \delta_x(\chi)$, which proves continuity of δ_x for each $x \in A$.

Examples 3.5.1.

- If $A = \mathbb{R}$ we know that $\mathbb{R} \cong \widehat{\mathbb{R}}$ via $t \mapsto \chi_t$ with $\chi_t(s) = e^{2\pi i st}$. Thus we can also identify $\mathbb{R}$ with its bidual by mapping $s \in \mathbb{R}$ to a character $\mu_s : \widehat{\mathbb{R}} \to \mathbb{T}$, $\mu_s(\chi_t) = e^{2\pi i ts}$. It is easy to check that the map $\mu_s = \delta_s$ with $\delta : \mathbb{R} \to \widehat{\widehat{\mathbb{R}}}$ coincides with the above defined Pontryagin map. So we see in particular that the Pontryagin map is an isomorphism of groups in the case $A = \mathbb{R}$.
- Very similarly, we see that the Pontryagin maps $\delta : \mathbb{T} \to \widehat{\widehat{\mathbb{T}}}$ and $\delta : \mathbb{Z} \to \widehat{\widehat{\mathbb{Z}}}$ transform to the identity maps under the identifications $\mathbb{Z} \cong \widehat{\mathbb{T}}$ and $\mathbb{T} \cong \widehat{\mathbb{Z}}$ as explained in Example 1.7.1.

Proposition 3.5.2 *Let A be an LCA-group. The Pontryagin map is an injective continuous group homomorphism from A to $\widehat{\widehat{A}}$. In particular, if $1 \neq x \in A$ there exists some $\chi \in \widehat{A}$ such that $\chi(x) \neq 1$.*

Proof Note first that the Pontryagin map δ is a group homomorphism, since $\delta_{xy}(\chi) = \chi(xy) = \chi(x)\chi(y) = \delta_x(\chi)\delta_y(\chi)$. It suffices to show continuity at the unit element 1. So let V be an open unit-neighborhood in $\widehat{\widehat{A}}$. Then there exists a compact set $K^* \subset \widehat{A}$ and an $\varepsilon > 0$, such that V contains the open unit-neighborhood

$$B_{K^*,\varepsilon} = \left\{\alpha \in \widehat{\widehat{A}} : |\alpha(\chi) - 1| < \varepsilon \ \forall_{\chi \in K^*}\right\}.$$

Let $L \subset A$ be a compact unit-neighborhood. As K^* is compact, there are $\chi_1 \dots, \chi_n \in K^*$ such that $K^* \subset B_{L,\varepsilon/2}(\chi_1) \cup \dots \cup B_{L,\varepsilon/2}(\chi_n)$, where

$$B_{L,\varepsilon}(\chi) = \left\{\chi' \in \widehat{A} : \|\chi' - \chi\|_L < \varepsilon\right\}.$$

For $j = 1, \dots, n$ let $U_j = \{x \in A : |\chi_j(x) - 1| < \varepsilon/2\}$. Let $U = \mathring{L} \cap U_1 \cap \dots \cap U_n$. Then U is a unit-neighborhood for which we have $x \in U \Rightarrow |\chi(x) - 1| < \varepsilon \ \forall_{\chi \in K^*}$. So $\delta(U) \subset V$ and δ is continuous.

We still have to show that $\delta : A \to \widehat{\widehat{A}}$ is injective. So assume that $1 \neq x \in A$ with $\delta_x = 1_{\widehat{A}}$. Then $\chi(x) = 1$ for every $\chi \in \widehat{A}$. Choose $g \in C_c(A)$ with $g(1) = 1$ and $g(x^{-1}) = 0$. Then $L_x(g) \neq g$, but by Lemma 1.7.2 we have $\widehat{L_x(g)}(\chi) = \bar{\chi}(x)\hat{g}(\chi) = \hat{g}(\chi)$ for every $\chi \in \widehat{A}$. This contradicts the fact that the Fourier transform is injective by Corollary 3.3.4. □

Lemma 3.5.3 *Let $f \in C_0^*(A)$ be such that its Fourier transform lies in $C_c(\widehat{A})$. Then for every $x \in A$ one has $f(x) = \hat{\hat{f}}(\delta_{x^{-1}})$.*

Proof One has for $x \in A$,

$$f(x) = L_{x^{-1}} f(1) = \int_{\widehat{A}} \widehat{L_{x^{-1}} f}(\chi)\, d\chi = \int_{\widehat{A}} \hat{f}(\chi)\delta_x(\chi)\, d\chi = \hat{\hat{f}}(\delta_{x^{-1}}). \qquad □$$

Lemma 3.5.4 *For an LCA-group A the following hold.*

(a) *$C_c(A)$ is dense in $C_0^*(A)$.*

(b) *$C_c(\widehat{A}) \cap \{\hat{f} : f \in C_0^*(A) \cap L^2(A)\}$ is dense in $C_0^*(\widehat{A})$.*

(c) *$C_c(\widehat{A}) \cap \{\hat{f} : f \in C_0^*(A) \cap L^2(A)\}$ is dense in $L^2(\widehat{A})$.*

Proof (a) As $C_0(A) \cap L^1(A)$ is dense in $C_0^*(A)$ by definition, it suffices to show that for a given f in this space there exists a sequence in $C_c(A)$ converging to f in the norms $\|\cdot\|_A$ and $\|\cdot\|_1$ simultaneously. Let $n \in \mathbb{N}$, and let $K_n \subset A$ be a compact set with $|f| < 1/n$ outside K_n. Choose a function χ_n in $C_c(A)$ with $0 \leq \chi_n \leq 1$, which is constantly equal to 1 on K_n. Set $f_n = \chi_n f$. Then the sequence f_n converges to f in both norms. Parts (b) and (c) follow from part (a) and Lemma 3.4.6. □

Theorem 3.5.5 (Pontryagin Duality). *The Pontryagin map $\delta : A \to \widehat{\widehat{A}}$ is an isomorphism of LCA groups.*

Proof We already know that δ is an injective continuous group homomorphism. We will demonstrate that it has a dense image. Assume this is not the case. Then there is an open subset U of $\widehat{\widehat{A}}$, which is disjoint from $\delta(A)$. By Lemma 3.4.6 applied to $\widehat{A}$, there exists $\psi \in C_0^*(\widehat{A})$, which is non-zero such that $\hat{\psi}$ is supported in U, i.e., it satisfies $\hat{\psi}(\delta(A)) = 0$. By Lemma 3.5.4, there exists a sequence (f_n) in $C_0^*(A)$ such that $\psi_n \stackrel{\text{def}}{=} \hat{f}_n$ lies in $C_c(\widehat{A})$ and converges to ψ in $C_0^*(\widehat{A})$. The inversion formula of Lemma 3.5.3 shows that $f_n(x) = \hat{\psi}_n(\delta_{x^{-1}})$ for every $x \in A$. This implies that the sequence f_n tends to zero uniformly on A. On the other hand $\hat{f}_n$ converges to ψ uniformly on $\widehat{A}$. This implies that (f_n) is a Cauchy sequence in $C_0^*(A)$ so it converges in this space. As the limit is unique, it follows from Lemma 3.4.2 that $\psi = 0$ in contradiction to our assumption. So the image of δ is indeed dense in $\widehat{\widehat{A}}$.

We next show that δ is a proper map, i.e., that the inverse image of a compact set is compact. For this let $K \subset \widehat{\widehat{A}}$ be compact. It suffices to show that the function $\breve{\delta}(x) = \delta(x^{-1})$ is proper. By Lemma 3.4.6, there exists $\psi \in C_0^*(\widehat{A})$ such that $\hat{\psi}$ has compact support, is ≥ 0 on $\widehat{\widehat{A}}$ and ≥ 1 on K. As above, there is a sequence (f_n) in $C_0^*(A)$ such that $\psi_n \stackrel{\text{def}}{=} \hat{f}_n \geq 0$ lies in $C_c(\widehat{A})$ and converges to ψ in $C_0^*(\widehat{A})$. Fix n with $\|\hat{\psi}_n - \hat{\psi}\|_{\widehat{\widehat{A}}} < 1/2$. We also have $f_n(x) = \hat{\psi}_n(\delta_{x^{-1}})$ for every $x \in A$ again and, as f_n is in $C_0(A)$, there exists a compact set $C \subset A$ such that $|f_n| < 1/2$ outside C. As $\hat{\psi}_n$ is $\geq 1/2$ on K, it follows that the pre-image of K under $\breve{\delta}$ is contained in C. As δ is continuous, this pre-image is closed, hence compact, so δ is proper.

It remains to show that δ is a *closed map*, i.e., that it maps closed sets to closed sets. Then δ is a homeomorphism, i.e., the theorem follows. So we finish our proof with the following lemma.

Lemma 3.5.6 *Let $\phi : X \to Y$ be a continuous map between locally compact Hausdorff spaces. If ϕ is proper, then it is closed.*

Proof Let T be a closed subset of X. We show first that

$(*)$ For every compact set $L \subset Y$ the intersection $\phi(T) \cap L$ is closed.

For this recall that $\phi^{-1}(L)$ is compact and therefore $T \cap \phi^{-1}(L)$ is compact and so $\phi(T) \cap L = \phi(T \cap \phi^{-1}(L))$ is compact and therefore closed.

Now we use $(*)$ to deduce that $\phi(T)$ is closed. Let y be in the closure of $\phi(T)$. Let L be a compact neighborhood of y. For every neighborhood U of y one has $U \cap (L \cap \phi(T)) \neq \emptyset$, so y is in $\overline{L \cap \phi(T)} = L \cap \phi(T) \subset \phi(T)$. This means that $\phi(T)$ is closed. □

Proposition 3.5.7 *The Fourier transform induces an isometric isomorphism of Banach spaces $\mathcal{F} : C_0^*(A) \to C_0^*(\widehat{A})$ with inverse map given by the* dual Fourier transform $\widehat{\mathcal{F}} : C_0^*(\widehat{A}) \to C_0^*(A)$; $\widehat{\mathcal{F}}(\psi)(x) \stackrel{\text{def}}{=} \hat{\psi}(\delta_{x^{-1}})$.

Proof Let B be the space of all $f \in C_0^*(A)$ such that $\hat{f}$ lies in $C_c(\widehat{A})$. For $f \in B$ we have $\widehat{\mathcal{F}} \circ \mathcal{F}(f) = f$ by Lemma 3.5.3. Further, one has

$$\begin{aligned} \|f\|_0^* &= \max(\|\hat{f}\|_{\widehat{A}}, \|f\|_A) = \max(\|\hat{f}\|_{\widehat{A}}, \|\widehat{\mathcal{F}} \circ \mathcal{F}(f)\|_A) \\ &= \max(\|\hat{f}\|_{\widehat{A}}, \|\hat{\hat{f}}\|_{\widehat{\widehat{A}}}) = \|\mathcal{F}(f)\|_0^*. \end{aligned}$$

As the set $\mathcal{F}(B)$ is dense in $C_0^*(\widehat{A})$ by Lemma 3.5.4, the Fourier transform defines a surjective isometry from the closure of B to $C_0^*(\widehat{A})$. Conversely, this means that $\widehat{\mathcal{F}}$ is an isometry from $C_0^*(\widehat{A})$ to $C_0^*(A)$. Since $\widehat{\mathcal{F}} = \mathcal{F}_{\widehat{A}} \circ \delta^{-1}$, where $\mathcal{F}_{\widehat{A}}$ denotes the Fourier transform on $\widehat{A}$ and since $\mathcal{F}_{\widehat{A}}\left(C_0^*(\widehat{A})\right)$ contains a subset of $C_c(\widehat{\widehat{A}})$ that is dense in $C_0^*(\widehat{\widehat{A}})$ by Lemma 3.5.4, it follows from Pontryagin duality that $\widehat{\mathcal{F}}(C_0^*(\widehat{A}))$ is dense in $C_0^*(A)$. Since it is isometric it must be an isomorphism of Banach spaces as claimed. □

Theorem 3.5.8 (Inversion Formula). *Let $f \in L^1(A)$ be such that its Fourier transform $\hat{f}$ lies in $L^1(\widehat{A})$. Then f is a continuous function, and for every $x \in A$ one has*

$$f(x) = \hat{\hat{f}}(\delta_{x^{-1}}).$$

Proof Let $f \in L^1(A)$ with $\hat{f} \in L^1(\widehat{A})$. Then $\hat{f}$ lies in $C_0(\widehat{A}) \cap L^1(\widehat{A})$, which is a subspace of $C_0^*(\widehat{A})$. By Proposition 3.5.7, there exists $g \in C_0^*(A)$ with $\hat{g} = \hat{f}$ and $g(x) = \hat{f}(\delta_{x^{-1}})$ for every $x \in A$. Since the Fourier transform is injective on $C^*(A)$, we have $f = g$. □

We are now ready for the proof of the Plancherel Theorem.

Proof of Theorem 3.4.8 Let $f \in L^1(A) \cap L^2(A)$. By Lemma 3.4.1 one has $f * f^* \in L^1(A) \cap C_0(A)$. The continuous function $h = \widehat{f * f^*} = |\hat{f}|^2$ is positive. Let $\phi \in C_c(\widehat{A})$ satisfy $0 \le \phi \le h$. Then

$$\int_{\widehat{A}} \phi(\chi)\, d\chi \le f * f^*(1) = \|f\|_2^2 < \infty.$$

Therefore h is integrable, so $\widehat{f * f^*} \in L^1(A)$. By Theorem 3.5.8 it follows that $\|f\|_2^2 = f * f^*(1) = \widehat{\widehat{f * f^*}}(1) = \widehat{|\hat{f}|^2}(1) = \|\hat{f}\|_2^2$. As $L^1(A) \cap L^2(A)$ is dense in $L^2(A)$, the Fourier-transform $f \mapsto \hat{f}$ extends uniquely to an isometric linear map $L^2(A) \to L^2(\widehat{A})$. By Lemma 3.4.6 the image in $L^2(\widehat{A})$ is dense, so the map is surjective. □

With the help of the Plancherel theorem, we can see that there are indeed many functions *f*, to which the inversion formula applies.

Proposition 3.5.9 *Let $\phi, \psi \in L^1(A) \cap L^2(A)$, and let $f = \phi * \psi$. Then $f \in L^1(A)$ and $\hat{f} \in L^1(\widehat{A})$, so the inversion formula applies to f.*

Proof We have $\hat{f} = \widehat{\phi * \psi} = \hat{\phi}\hat{\psi}$ is the point-wise product of L^2-functions on $\widehat{A}$, hence $\hat{f} \in L^1(\widehat{A})$. □

3.6 The Poisson Summation Formula

Let A be an LCA group, and let B be a closed subgroup of A. We want to study the relations between the dual group $\widehat{A}$ of A and the dual groups $\widehat{B}$ and $\widehat{A/B}$ of the subgroup B and the quotient group A/B. The Poisson Summation Formula relates the Fourier transform of A to the transforms on B and A/B.

We introduce some further notation: If E is a subset of A we denote by $E^\perp$ the *annihilator of E in $\widehat{A}$*, i.e., the set of all characters $\chi \in \widehat{A}$ with $\chi(E) = 1$. Similarly, if $L \subset \widehat{A}$, we denote by $L^\perp$ the *annihilator of* L *in* A, i.e., the set of all $x \in A$ such that $\chi(x) = 1$ for every $\chi \in L$. In short, we have

$$E^\perp = \{\chi \in \widehat{A} : \chi(x) = 1 \ \forall x \in E\}$$
$$L^\perp = \{x \in A : \chi(x) = 1 \ \forall \chi \in L\}.$$

It is easy to see that $E^\perp$ is a closed subgroup of $\widehat{A}$, and $L^\perp$ is a closed subgroup of A. Recall that the Pontryagin isomorphism $\delta : A \to \widehat{\widehat{A}}$ is defined by putting $\delta_x(\chi) = \chi(x)$ for every $x \in A$.

Proposition 3.6.1 *Let A be an LCA group, and let B be a closed subgroup of A. Then the following are true*:

(a) $B^\perp$ *is isomorphic to* $\widehat{A/B}$ *via* $\chi \mapsto \widetilde{\chi}$ *with* $\widetilde{\chi} \in \widehat{A/B}$ *defined by* $\widetilde{\chi}(xB) \stackrel{\text{def}}{=} \chi(x)$.

(b) $(B^\perp)^\perp = B$.

(c) $\widehat{A}/B^\perp$ *is isomorphic to* $\widehat{B}$ via $\chi \cdot B^\perp \mapsto \chi|_B$.

Proof This is a straightforward verification (See Exercise 3.10). □

As a direct corollary we get

Corollary 3.6.2 *Let B be a closed subgroup of the LCA-group A. Then the restriction map* $\mathrm{res}_B^A : \widehat{A} \to \widehat{B}$ *defined by* $\chi \mapsto \chi|_B$ *is surjective with kernel* $\widehat{A/B}$.

Note that one could formulate the above result in more fancy language as follows: If B is a closed subgroup of A, then we get the short exact sequence

$$1 \longrightarrow B \stackrel{\iota}{\longrightarrow} A \stackrel{q}{\longrightarrow} A/B \longrightarrow 1$$

of LCA groups. The above result then says that the dual sequence

$$1 \longrightarrow \widehat{A/B} \stackrel{\hat{q}}{\longrightarrow} \widehat{A} \stackrel{\hat{\iota}}{\longrightarrow} \widehat{B} \longrightarrow 1$$

is also an exact sequence of LCA groups, where for any continuous homomorphism $\psi : A_1 \to A_2$ between two LCA groups A_1, A_2, we denote by $\hat{\psi} : \widehat{A_2} \to \widehat{A_1}$ the homomorphism defined by $\hat{\psi}(\chi) = \chi \circ \psi$ for $\chi \in \widehat{A_2}$. One should not mistake this notation with the notion of the Fourier transform of a function. Note that if $\iota : B \to A$ is the inclusion map, then $\hat{\iota}(\chi) = \chi \circ \iota = \chi|_B$, so $\hat{\iota} = \mathrm{res}_B^A$.

We now come to Poisson's summation formula. Recall from Theorem 1.5.3 together with Corollary 1.5.5 that for any closed subgroup B of the LCA group A we can choose Haar measures on A, B and A/B in such a way that for every $f \in C_c(A)$ we get the quotient integral formula

$$\int_{A/B} \int_B f(xb)\,db\,dxB = \int_A f(x)\,dx.$$

In what follows we shall always assume that the Haar measures are chosen this way.

Theorem 3.6.3 (Poisson's Summation Formula). *Let B be a closed subgroup of the LCA group A. For $f \in L^1(A)$ define $f^B \in L^1(A/B)$ as $f^B(xB) = \int_B f(xb)\,db$. Then, if we identify $\widehat{A/B}$ with $B^\perp$ as in Proposition* 3.6.1, *we get $\widehat{f^B} = \hat{f}\,|_{B^\perp}$. If, in addition, $\hat{f}\,|_{B^\perp} \in L^1(B^\perp)$, then we get*

$$\int_B f(xb)\,db = \int_{B^\perp} \hat{f}(\chi)\chi(x)\,d\chi,$$

for almost all $x \in A$, where Haar measure on $\widehat{B}^\perp \cong \widehat{A/B}$ is the Plancherel measure with respect to the chosen Haar measure on A/B. If f^B is everywhere defined and continuous, the above equation holds for all $x \in A$.

Proof For $\chi \in B^\perp$ we have $\chi(xb) = \chi(x)$ for every $x \in A$ and $b \in B$. We therefore get from Theorem 1.5.3,

$$\begin{aligned}\widehat{f^B}(\chi) &= \int_{A/B} f^B(xB)\bar{\chi}(x)\,dxB = \int_{A/B}\int_B f(xb)\bar{\chi}(xb)\,db\,dxB\\ &= \int_A f(x)\bar{\chi}(x)\,dx = \hat{f}(\chi)\end{aligned}$$

for every $\chi \in B^\perp$. Moreover, if $\hat{f}\,|_{B^\perp} \in L^1(B^\perp) = L^1(\widehat{A/B})$, then the inversion formula of Theorem 1.5.3 implies that

$$\begin{aligned}\int_B f(xb)\,db = f^B(xB) &= \widehat{\widehat{f^B}}\,(\delta_{x^{-1}B})\\ &= \widehat{\hat{f}\,|_{B^\perp}}(\delta_{x^{-1}B}) = \int_{B^\perp} \hat{f}(\chi)\overline{\chi(x)}\,d\chi.\end{aligned}$$

almost everywhere. It holds everywhere if, in addition, the defining integral for f^B exists everywhere and f^B is continuous. □

Example 3.6.4. (The Poisson Summation formula for $\mathbb{R}$) Let A be the group $(\mathbb{R}, +)$ with the usual topology. Then $A \cong \widehat{A}$ via the map $y \mapsto \chi_y$ where $\chi_y(x) = e^{2\pi i xy}$. Let B be the closed subgroup $\mathbb{Z}$. Then the above identification maps B bijectively to $B^\perp$. For $f \in L^1(\mathbb{R})$ such that $\hat{f}|_{\mathbb{Z}} \in L^1(\mathbb{Z})$, the equality

$$\sum_{k\in\mathbb{Z}} f(x+k) = \sum_{k\in\mathbb{Z}} \hat{f}(k)e^{2\pi i kx}$$

holds almost everywhere in x, where $\hat{f}(x) = \int_{\mathbb{R}} f(y)e^{-2\pi i xy}\,dy$. In particular, define the *Schwartz space* $\mathcal{S}(\mathbb{R})$ as the space of all C^∞- functions $f : \mathbb{R} \to \mathbb{C}$ such that for any two integers $m, n \geq 0$ the function $x^n f^{(m)}(x)$ is bounded. Then the Fourier transform maps $\mathcal{S}(\mathbb{R})$ bijectively to itself (Exercise 3.14). For $f \in \mathcal{S}(\mathbb{R})$, both sums in the Poisson summation formula converge uniformly and define continuous functions, which then must be equal in every point. For $x = 0$ we get the elegant formula

$$\sum_{k\in\mathbb{Z}} f(k) = \sum_{k\in\mathbb{Z}} \hat{f}(k).$$

For applications of this formula to theta series and the Riemann zeta function, see [Dei05].

3.7 Exercises and Notes

Exercise 3.1. Let $\mathcal{U}$ be a basis for the topology on the LCA-group A. Let $\mathcal{U}_c$ denote the set of all $U \in \mathcal{U}$ that are relatively compact. Show that the set $\mathcal{B}$ of all $L(\bar{U}, V)$, where $U \in \mathcal{U}_c$ and V is open in $\mathbb{T}$, generates the topology of $\widehat{A}$.

Exercise 3.2. Show that if an LCA-group A is second countable, then so is its dual $\widehat{A}$.

Exercise 3.3. Let b be the map $b : \mathbb{R} \to \prod_{t\in\mathbb{R}} \mathbb{T}$ sending $x \in \mathbb{R}$ to the element $b(x)$ with coordinates $b(x)_t = e^{2\pi i t x}$. Let B denote the closure of $b(\mathbb{R})$ in the product space. By Tychonov's Theorem the product is compact; therefore B is a compact group called the *Bohr compactification of* $\mathbb{R}$. Show that B is separable but not second countable.

(Hint: Use the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$. Show that B is isomorphic to the dual group of $\mathbb{R}_{\text{disc}}$, which is the group $(\mathbb{R}, +)$ with the discrete topology. Then use Exercise 3.2)

Exercise 3.4. Verify the statements in Example 3.1.3.

Exercise 3.5. Let A and B be two LCA groups. Show that $\widehat{A \times B} = \widehat{A} \times \widehat{B}$.

Exercise 3.6. Show that the multiplicative group $\mathbb{C}^\times$ is locally compact with the topology of $\mathbb{C}$ and that

$$\widehat{\mathbb{C}^\times} \cong \mathbb{Z} \times \mathbb{R}.$$

Exercise 3.7. Let $(A_j)_{j\in J}$ be a family of discrete groups. Show that there is a canonical isomorphism

$$\widehat{\bigoplus_{j\in J} A_j} \cong \prod_{j\in J} \widehat{A_j}.$$

Exercise 3.8.

(a) Let (A_j, p_i^j) be a projective system of compact groups. Show that there is a canonical isomorphism of topological groups,

$$\widehat{\lim_{\leftarrow} A_j} \cong \lim_{\rightarrow} \widehat{A_j}.$$

(b) Let (B_j, d_i^j) be a direct system of discrete groups satisfying the Mittag-Leffler condition. Show that there is a canonical isomorphism

$$\widehat{\lim_{\rightarrow} B_j} \cong \lim_{\leftarrow} \widehat{B_j}.$$

Exercise 3.9. Let A be an LCA group, and let $f \in L^1(A)$ such that $\widehat{f} \in L^1(\widehat{A})$. Show $f \in L^2(A)$.

Exercise 3.10. For a closed subgroup B of the LCA-group A and a closed subgroup L of $\widehat{A}$ let

$$B^\perp = \{\chi \in \widehat{A} : \chi(B) = 1\}$$
$$L^\perp = \{x \in A : \delta_x(L) = 1\}.$$

Show that $B^\perp$ is canonically isomorphic to $\widehat{A/B}$, $(B^\perp)^\perp = B$, and $\widehat{A}/B^\perp$ is canonically isomorphic to $\widehat{B}$.

Exercise 3.11. For a continuous group homomorphism $\phi : A \to B$ between LCA groups, define $\hat{\phi} : \widehat{B} \to \widehat{A}$ by

$$\hat{\phi}(\chi) \stackrel{\text{def}}{=} \chi \circ \phi.$$

Show that for any two composable homomorphisms ϕ and ψ one has $\widehat{\phi \circ \psi} = \hat{\psi} \circ \hat{\phi}$. This means that $A \mapsto \widehat{A}$ defines a contravariant functor on the category of LCA groups and continuous group homomorphisms.

Exercise 3.12. A *short exact sequence* of LCA groups is a sequence of continuous group homomorphisms

$$A \stackrel{\alpha}{\hookrightarrow} B \stackrel{\beta}{\twoheadrightarrow} C$$

such that α is injective, β is surjective, the image of α is the kernel of β, the group A carries the subspace topology and C carries the quotient topology. Show that a short exact sequence like this induces a short exact sequence of the dual groups

$$\widehat{C} \stackrel{\hat{\beta}}{\hookrightarrow} \widehat{B} \stackrel{\hat{\alpha}}{\twoheadrightarrow} \widehat{A}.$$

Exercise 3.13 Let $A = \mathbb{R}$, and choose the Lebesgue measure as Haar measure. Identify $\widehat{A}$ with $\mathbb{R}$ via $x \mapsto \chi_x$ with $\chi_x(y) = e^{2\pi i xy}$. Show that via this identification, the Lebesgue measure is the Plancherel measure on $\widehat{A}$.

(Hint: Use the fact that $\int_{\mathbb{R}} e^{-\pi x^2}\, dx = 1$ and compute the Fourier transform of $f(x) = e^{-\pi x^2}$.)

Exercise 3.14. Show that $\hat{f} \in \mathcal{S}(\mathbb{R})$ for every $f \in \mathcal{S}(\mathbb{R})$ and that the map $\mathcal{F} : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ defined by $\mathcal{F}(f) = \hat{f}$ is a bijective linear map with

$$\mathcal{F}^{-1}(g)(y) = \int_{\mathbb{R}} g(x) e^{2\pi i xy}\, dx.$$

Exercise 3.15. Let $f \in \mathcal{S}(\mathbb{R})$ and set $g(x) = \sum_{k\in\mathbb{Z}} f(x+k)$. Show that g is a smooth function on $\mathbb{R}$.

(Hint: The estimate $|f(x)| \le C/(1+x^2)$ for a constant C shows point-wise convergence. The same holds for the n-th derivative $f^{(n)}$ instead of f. Now integrate n times.)

Exercise 3.16. As an application of Theorem 3.6.3, show that for every Schwartz function $f \in \mathcal{S}(\mathbb{R})$,

$$\sum_{k\in\mathbb{Z}} f(k) = \sum_{k\in\mathbb{Z}} \hat{f}(k)$$

holds, where $\hat{f}(x) = \int_{\mathbb{R}} f(y)\, e^{-2\pi i xy}\, dy$.

Exercise 3.17. Let $A = \mathbb{R}^n$ with Lebesgue measure as Haar measure and identify $\mathbb{R}^n$ with $\widehat{\mathbb{R}^n}$ via $x \mapsto \chi_x$ with $\chi_x(y) = e^{-2\pi i \langle x,y\rangle}$, where $\langle x, y\rangle$ denotes the standard inner product on $\mathbb{R}^n$. Let $\mathcal{S}(\mathbb{R}^n)$ denote the space of all C^∞- functions $f : \mathbb{R}^n \to \mathbb{C}$ such that for any two multi-indices $\alpha, \beta \in \mathbb{N}_0^n$ the function

$$x^\alpha \partial^\beta(f) \stackrel{\text{def}}{=} x_1^{\alpha_1} \dots x_n^{\alpha_n} \frac{\partial^{|\beta|} f}{x_1^{\beta_1} \dots x_n^{\beta_n}}$$

is bounded, where $|\beta| = \beta_1 + \cdots + \beta_n$. Formulate and prove the analogues of Exercise 3.14 and Exercise 3.16 in this setting.

Exercise 3.18. (Parseval's equation) Let A be an LCA group, and let $\widehat{A}$ be equipped with the Plancherel measure with respect to a given Haar measure on A. Show that the equation

$$\langle f, g\rangle = \int_A f(x)\bar{g}(x)\, dx = \int_{\widehat{A}} \hat{f}(\chi)\bar{\hat{g}}(\chi)\, d\chi = \langle \hat{f}, \hat{g}\rangle$$

holds for all $f, g \in L^1(A) \cap L^2(A)$.

Exercise 3.19. A finite abelian group A can be equipped either with the counting measure or with the normalized Haar measure that gives A the volume 1. What is the Plancherel measure in either case?

Exercise 3.20. For a finite abelian group A, let $C(A)$ be the space of all function from A to $\mathbb{C}$. For a group homomorphism $\phi : A \to B$ between finite abelian groups let $\phi^* : C(B) \to C(A)$ be defined by $\phi^* f = f \circ \phi$, and let $\phi_* : C(A) \to C(B)$ be defined by

$$\phi_* g(b) \stackrel{\text{def}}{=} \sum_{a:\phi(a)=b} g(a),$$

where the empty sum is interpreted as zero. Show that for composable homomorphisms one has $(\phi\psi)_* = \phi_*\psi_*$ and $(\phi\psi)^* = \psi^*\phi^*$.

Exercise 3.21. For a finite abelian group A let $\mathbb{F} : C(A) \to C(\widehat{A})$ be the Fourier transform. Show that for every group homomorphism $\phi : A \to B$ between finite abelian groups the diagram

$$\begin{array}{ccc} C(A) & \xrightarrow{\mathbb{F}} & C(\widehat{A}) \\ \uparrow \phi^* & & \uparrow \hat{\phi}_* \\ C(B) & \xrightarrow{\mathbb{F}} & C(\widehat{B}) \end{array}$$

Exercise 3.22. An LCA-group A is called *monothetic*, if it contains a dense cyclic subgroup. Show that a compact LCA-group A is monothetic if and only if its dual $\widehat{A}$ is isomorphic to a subgroup of $\mathbb{T}_d$, where $\mathbb{T}_d$ is the circle group with the discrete topology.

Notes

In principle, the ideas for the proofs of the Plancherel Theorem and the Pontryagin Duality Theorem given in this chapter goes back to the paper [Wil62] of J.H. Williamson. However, to our knowledge, this book is the first that exploits the very natural isomorphism $C_0^*(A) \cong C_0^*(\widehat{A})$.