Chapter 1
Haar Integration

In this chapter, topological groups and invariant integration are introduced. The existence of a translation invariant measure on a locally compact group, called Haar measure, is a basic fact that makes it possible to apply methods of analysis to study such groups. The Harmonic Analysis of a group is basically concerned with spaces of measurable functions on the group, in particular the spaces $L^1(G)$ and $L^2(G)$, both taken with respect to Haar measure. The invariance of this measure allows to analyze these function spaces by some generalized Fourier Analysis, and we shall see in further chapters of this book how powerful these techniques are.

In this book, we will freely use concepts of set-theoretic topology. For the convenience of the reader we have collected some of these in Appendix A.

1.1 Topological Groups

A *topological group* is a group G, together with a topology on the set G such that the group multiplication and inversion,

$$\begin{aligned} G \times G &\to G \qquad & G &\to G \\ (x, y) &\mapsto xy, \qquad & x &\mapsto x^{-1}, \end{aligned}$$

are both continuous maps.

Remark 1.1.1 It suffices to insist that the map $\alpha : (x, y) \to x^{-1}y$ is continuous. To see this, assume that α is continuous and recall that the map $G \to G \times G$, that maps x to (x, e) is continuous (Example A.5.3), where e is the unit element of the group G. We can thus write the inversion as a composition of continuous maps as follows $x \mapsto (x, e) \mapsto x^{-1}e = x^{-1}$. The multiplication can be written as the map $(x, y) \mapsto \left(x^{-1}, y\right)$ followed by the map α, so is continuous as well, if α is.

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Examples 1.1.2

- Any given group becomes a topological group when equipped with the *discrete topology*, i.e., the topology, in which every subset is open. In this case we speak of a *discrete group*
- The additive and multiplicative groups $(\mathbb{R}, +)$ and $(\mathbb{R}^\times, \times)$ of the field of real numbers are topological groups with their usual topologies. So is the group $\mathrm{GL}_n(\mathbb{R})$ of all real invertible $n \times n$ matrices, which inherits the $\mathbb{R}^{n^2}$-topology from the inclusion $\mathrm{GL}_n(\mathbb{R}) \subset \mathrm{M}_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$, where $\mathrm{M}_n(\mathbb{R})$ denotes the space of all $n \times n$ matrices over the reals. As for the proofs of these statements, recall that in analysis one proves that if the sequences a_i and b_i converge to a and b, respectively, then their difference $a_i - b_i$ converges to $a - b$, and this implies that $(\mathbb{R}, +)$ is a topological group. The proof for the multiplicative group is similar. For the matrix groups recall that matrix multiplication is a polynomial map in the entries of the matrices, and hence continuous. The determinant map also is a polynomial and so the inversion of matrices is given by rational maps, as for an invertible matrix A one has $A^{-1} = \det(A)^{-1} A^{\#}$, where $A^{\#}$ is the adjugate matrix of A; entries of the latter are determinants of sub-matrices of A, therefore the map $A \mapsto A^{-1}$ is indeed continuous.

Let $A, B \subset G$ be subsets of the group G. We write

$$AB = \{ab : a \in A, b \in B\} \qquad \text{and} \qquad A^{-1} = \{a^{-1} : a \in A\},$$

as well as $A^2 = AA$, $A^3 = AAA$ and so on.

Lemma 1.1.3 *Let G be a topological group.*

(a) *For $a \in G$ the translation maps $x \mapsto ax$ and $x \mapsto xa$, as well as the inversion $x \mapsto x^{-1}$ are homeomorphisms of G. A set $U \subset G$ is a neighborhood of $a \in G$ if and only if $a^{-1}U$ is a neighborhood of the unit element $e \in G$. The same holds with Ua^{-1}.*

(b) *If U is a neighborhood of the unit, then $U^{-1} = \{u^{-1} : u \in U\}$ also is a neighborhood of the unit. We call U a symmetric unit-neighborhood if $U = U^{-1}$. Every unit-neighborhood U contains a symmetric one, namely $U \cap U^{-1}$.*

(c) *For a given unit-neighborhood U there exists a unit-neighborhood V with $V^2 \subset U$.*

(d) *If $A, B \subset G$ are compact subsets, then AB is compact.*

(e) *If A,B are subsets of G and A or B is open, then so is AB.*

(f) *For $A \subset G$ the topological closure $\overline{A}$ equals $\overline{A} = \bigcap_V AV$, where the intersection runs over all unit-neighborhoods V in G.*

Proof (a) follows from the continuity of the multiplication and the inversion. (b) follows from the continuity of the inversion. For (c), let U be an open unit-neighborhood, and let $A \subset G \times G$ be the inverse image of U under the continuous map $m : G \times G \to G$ given by the group multiplication. Then A is open in the product topology of $G \times G$. Any set, which is open in the product topology, is a union of sets of the form $W \times X$, where W, X are open in G. Therefore there are unit-neighborhoods W, X with $(e, e) \in W \times X \subset A$. Let $V = W \cap X$. Then V is a unit-neighborhood as well and $V \times V \subset A$, i.e., $V^2 \subset U$. For (d) recall that the set AB is the image of the compact set $A \times B$ under the multiplication map; therefore it is compact. (e) Assume A is open, then $AB = \bigcup_{b \in B} Ab$ is open since every set Ab is open. For (f) let $x \in \overline{A}$, and let V be a unit-neighborhood. Then xV^{-1} is a neighborhood of x, and so $xV^{-1} \cap A \neq \emptyset$. Let $a \in xV^{-1} \cap A$. Then $a = xv^{-1}$ for some $v \in V$, so $x = av \in AV$, which proves the first inclusion. For the other way round let x be in the intersection of all AV as above. Let W be a neighborhood of x. Then $V = x^{-1}W$ is a unit-neighborhood and so is V^{-1}. Hence $x \in AV^{-1}$, so there is $a \in A$, $v \in V$ with $x = av^{-1}$. It follows $a = xv \in xV = W$. This means $W \cap A \neq \emptyset$. As W was arbitrary, this implies $x \in \overline{A}$. □

Lemma 1.1.4 *Let H be a subgroup of the topological group G. Then its closure $\overline{H}$ is also a subgroup of G. If H is normal, then so is $\overline{H}$.*

Proof Let $H \subset G$ be a subgroup. To show that the closure $\overline{H}$ is a subgroup, it suffices to show that $x, y \in \overline{H}$ implies $xy^{-1} \in \overline{H}$. Let m denote the continuous map $\overline{H} \times \overline{H} \to G$ given by $m(x, y) = xy^{-1}$. The pre-image $m^{-1}(\overline{H})$ must be closed and contains the dense set $H \times H$; therefore it contains the whole of $\overline{H} \times \overline{H}$, which proves the first claim. Next assume that H is normal, then for every $g \in G$ the set $g\overline{H}g^{-1}$ is closed and contains $gHg^{-1} = H$; therefore $\overline{H} \subset g\overline{H}g^{-1}$. Conjugating by g one gets $g^{-1}\overline{H}g \subset \overline{H}$. As g varies, the second claim of the lemma follows. □

In functional analysis, people like to use *nets* in topological arguments. These have the advantage of providing very intuitive proofs. We refer the reader to Sect. A.6 for further details on nets and convergence in general. The next lemma is an example, how nets provide intuitive proofs.

Lemma 1.1.5 *Let G be a topological group. Let $A \subset G$ be closed and $K \subset G$ be compact. Then AK is closed.*

Proof Let $\left(x_j = a_j k_j\right)_{j \in J}$ be a net in AK, convergent in G. As K is compact, one can replace it with a subnet so that (k_j) converges in K. Since the composition in G and the inversion are continuous, the net $a_j = x_j k_j^{-1}$ converges too, with limit in $\overline{A} = A$. Therefore the limit of $x_j = a_j k_j$ lies in AK, which therefore is closed. □

Lemma 1.1.6 *Let G be a topological group and $K \subset G$ a compact subset. Let U be an open set containing K. Then there exists a neighborhood V of the unit in G*

such that $KV \cup VK \subset U$. In particular, if U is open and compact, then there exist a neighborhood V of e such that $UV = VU = U$.

Proof For each $x \in K$ choose a unit-neighborhood V_x such that $xV_x^2 \subset U$. By compactness of K we may find $x_1, \dots, x_l \in K$ such that $K \subset \bigcup_{i=1}^l x_i V_{x_i}$ and $K \subset \bigcup_{i=1}^l V_{x_i} x_i$. Set $V = \bigcap_{i=1}^l V_{x_i}$. Then $KV \subset \bigcup_{i=1}^l x_i V_{x_i} V \subset \bigcup_{i=1}^l x_i V_{x_i}^2 \subset U$ and similarly $VK \subset U$. □

Recall (Appendix A) that a topological space X is a T_1-space if for $x \neq y$ in X there are neighborhoods U_x, U_y of x and y, respectively, such that y is not contained in U_x and x is not contained in U_y. So X is T_1 if and only if all singletons $\{x\}$ are closed. The space is called a T_2-space or Hausdorff space if the neighborhoods U_x and U_y can always be chosen disjoint.

Lemma 1.1.7 *Let G be a locally compact group.*

(a) *Let $H \subset G$ be a subgroup. Equip the left coset space $G/H = \{xH : x \in G\}$ with the quotient topology. Then the canonical projection $\pi : G \to G/H$, which sends $x \in G$ to the coset xH, is an open mapping. The space G/H* is a *T_1-space if and only if the group H is closed in G. If H is normal in G, then the quotient group G/H is a topological group.*

(b) *For any open symmetric unit-neighborhood V the set $H = \bigcup_{n=1}^\infty V^n$ is an open subgroup.*

(c) *Every open subgroup of G is closed as well.*

Proof (a) Let $U \subset G$ be open, then $\pi^{-1}(\pi(U)) = UH$ is open by Lemma 1.1.3 (e). As a subset of G/H is open in the quotient topology if and only if its inverse image under π is open in G, the map π is indeed open. So, for every $x \in G$ the set $G \setminus xH$ is mapped to an open set if and only if H is closed. This proves that singletons are closed in *G/H*, if and only if H is closed.

Now suppose that H is normal in G. One has a canonical group isomorphism $(G \times G)/(H \times H) \to G/H \times G/H$ and one realizes that this map also is a homeomorphism, where the latter space is equipped with the product topology. Consider the map $\alpha : G \times G \to G$ and likewise for *G/H*. One gets a commutative diagram

$$\begin{array}{ccc} G \times G & \xrightarrow{\ \alpha\ } & G \\ \downarrow & & \downarrow \\ G/H \times G/H & \xrightarrow{\ \overline{\alpha}\ } & G/H. \end{array}$$

As $G/H \times G/H \cong (G \times G)/(H \times H)$, the map $\overline{\alpha}$ is continuous if and only if the map $G \times G \to G/H$ is continuous, which it is, as α and the projection are continuous.

(b) Let V be a symmetric unit-neighborhood. For $x \in V^n$ and $y \in V^m$ one has $xy \in V^{n+m}$ and as V is symmetric, one also has $x^{-1} \in V^n$, so H is an open subgroup.

(c) Let H be an open subgroup. Writing G as union of left cosets we get $G \setminus H = \bigcup_{g \in G \setminus H} gH$. As H is open, so is gH for every $g \in G$. Hence the complement $G \setminus H$, being the union of open sets, is open, so H is closed. □

Proposition 1.1.8 *Let G be a topological group. Let H be the closure of the set* $\{1\}$.

(a) *The set H is the smallest closed subgroup of G. The group H is a normal subgroup and the quotient G/H with the quotient topology is a* T_1*- space.*

(b) *Every continuous map of G to a* T_1*-space factors over the quotient G/H.*

(c) *Every topological group, which is* T_1*, is already* T_2*, i.e., a Hausdorff space.*

Proof We prove part (a). The set H is a normal subgroup by Lemma 1.1.4. The last assertion follows from Lemma 1.1.7 (a).

For part (b) let $x \in G$. As the translation by x is a homeomorphism, the closure of the set $\{x\}$ is the set $xH = Hx$. So, if $A \subset G$ is a closed set, then $A = AH = HA$. Let $f : G \to Y$ be a continuous map into a T_1-space Y. For $y \in Y$ the singleton $\{y\}$ is closed, so $f^{-1}(\{y\})$ is closed, hence of the form AH for some set $A \subset G$. This implies that $f(gh) = f(g)$ for every $g \in G$ and every $h \in H$.

To show part (c), let G be a topological group that is T_1. Let $x \neq y$ in G and set $U = G \setminus \{xy^{-1}\}$. Then U is an open neighborhood of the unit. Let V be a symmetric unit-neighborhood with $V^2 \subset U$. Then $V \cap Vxy^{-1} = \emptyset$, for otherwise there would be $a, b \in V$ with $a = b^{-1}xy^{-1}$, so $xy^{-1} = ab \in V^2$, a contradiction. So it follows that $Vx \cap Vy = \emptyset$, i.e., Vx and Vy are disjoint neighborhoods of x and y, which means that G is a Hausdorff space. □

The following observation is often useful.

Lemma 1.1.9 *Suppose that* $\phi : G \to H$ *is a homomorphism between topological groups G and H. Then* ϕ *is continuous if and only if it is continuous at the unit* 1_G.

Proof Assume that ϕ is continuous at 1_G. Let $x \in G$ be arbitrary and let (x_j) be a net with $x_j \to x$ in G. Then $x^{-1}x_j \to x^{-1}x = 1_G$ and we have $\phi(x)^{-1}\phi(x_j) = \phi(x^{-1}x_j) \to \phi(1_G) = 1_H$, which then implies $\phi(x_j) \to \phi(x)$. Thus ϕ is continuous. □

Notation In the preceding proof we have used the notation $x_j \to x$ indicating that the net (x_j) converges to the point x.

1.2 Locally Compact Groups

A topological space is called *locally compact* if every point possesses a compact neighborhood. A topological group is called a *locally compact group* if it is Hausdorff and locally compact.

Note that by Proposition 1.1.8, every topological group has a biggest Hausdorff quotient group, and every continuous function to the complex numbers factors through that quotient. So, as far as continuous functions are concerned, a topological group is indistinguishable from its Hausdorff quotient. Thus it makes sense to restrict the attention to Hausdorff groups.

A subset $A \subset X$ of a topological space X is called *relatively compact* if its closure $\overline{A}$ is compact in X. Note that in a locally compact Hausdorff space X, every point has a neighborhood base consisting of compact sets. A subset S of G is called *σ-compact* if it can be written as a countable union of compact sets.

Proposition 1.2.1 *Let G be a locally compact group.*

(a) *For a closed subgroup H the quotient space G/H is a locally compact Hausdorff space.*

(b) *The group G possesses an open subgroup, which is σ-compact.*

(c) *The union of countably many open σ-compact subgroups generates an open σ-compact subgroup.*

Proof For (a) let $xH \neq yH$ in G/H. Choose an open, relatively compact neighborhood $U \subset G$ of x with $\overline{U} \cap yH = \emptyset$. The set $\overline{U}H$ is closed by Lemma 1.1.3, so there is an open, relatively compact neighborhood V of y such that $V \cap UH = \emptyset$. This implies $VH \cap UH = \emptyset$, and we have found disjoint open neighborhoods of xH and yH, which means that G/H is a Hausdorff space. It is locally compact, as for given $x \in H$, and a compact neighborhood U of x the set $UH \subset G/H$ is the image of the continuous map $G \to G/H$ of the compact set U; therefore it is a compact neighborhood of xH in G/H.

To show (b), let V be a symmetric, relatively compact open unit-neighborhood. For every $n \in \mathbb{N}$ one has $\overline{V}^n = \overline{V}^n \subset V \cdot V^n = V^{n+1}$. Therefore $H = \bigcup_n \overline{V}^n = \bigcup_n V^n$. An iterated application of Lemma 1.1.3 (d) shows that $\overline{V}^n$ is compact, so H is σ-compact. By Lemma 1.1.7 (b), H is an open subgroup.

Finally, for (c) let L_n be a sequence of σ-compact open subgroups. Then each L_n is the union of a sequence $\left(K_{n,j}\right)_j$ of compact sets. The group L generated by all L_n is also generated by the family $\left(K_{n,j}\right)_{n,j\in\mathbb{N}}$ and is therefore σ-compact. It is also open since it contains the open subgroup L_n for any n. □

1.3 Haar Measure

For a topological space X, we naturally have a σ-algebra $\mathcal{B}$ on X, the smallest σ-algebra containing all open sets. This σ-algebra also contains all closed sets and is generated by either class. It is called the *Borel σ-algebra*. Any element of this σ-algebra is called a *Borel set*.

Fix a measure space $(X, \mathcal{A}, \mu)$, so $\mathcal{A} \subset \mathcal{P}(X)$ is a σ-algebra and $\mu : \mathcal{A} \to [0, \infty]$ is a measure. One calls μ a *complete measure* if every subset of a μ-null-set is an element of $\mathcal{A}$. If μ is not complete, one can extend μ in a unique way to the σ-algebra $\overline{\mathcal{A}}$ generated by $\mathcal{A}$ and all subsets of μ-null-sets; this is called the *completion* of $\mathcal{A}$ with respect to μ. A function $f : X \to \mathbb{C}$ will be called μ*-measurable* if $f^{-1}(S)$ lies in $\overline{\mathcal{A}}$ for every Borel set $S \subset \mathbb{C}$.

Any measure $\mu : \mathcal{A} \to [0, \infty]$ defined on a σ-algebra $\mathcal{A} \supset \mathcal{B}$ is called a *Borel measure*. Unless specified otherwise, we will always assume $\mathcal{A}$ to be the completion of $\mathcal{B}$ with respect to μ. A Borel measure μ is called *locally finite* if every point $x \in X$ possesses a neighborhood U with $\mu(U) < \infty$.

Example 1.3.1 The Lebesgue measure on $\mathbb{R}$ is a Borel measure. So is the counting measure #, which for any set A is defined by

$$\#(A) \stackrel{\text{def}}{=} \begin{cases} \text{cardinality of } A & \text{if } A \text{ is finite} \\ \infty & \text{otherwise.} \end{cases}$$

The Lebesgue measure is locally finite; the counting measure is not.

Definition A locally finite Borel measure μ on $\mathcal{B}$ is called an *outer Radon measure* if

- $\mu(A) = \inf_{U \supset A} \mu(U)$ holds for every $A \in \mathcal{B}$, where the infimum is taken over all open sets U containing A, and
- $\mu(U) = \sup_{K \subset U} \mu(K)$ holds for every open set U, where the supremum is extended over all compact sets K contained in U.

For the first property one says that an outer Radon measure is *outer regular*. The second says that an outer Radon measure is *weakly inner regular*. For simplicity, we will use the term *Radon measure* for an outer Radon measure. In the literature, one will sometimes find the notion of Radon measure used for what we call an inner Radon measure; see Appendix B.2 for a discussion.

Note that for an outer Radon measure μ one has $\mu(A) = \sup_{K \subset A} \mu(K)$ for every measurable A with $\mu(A) < \infty$, where the supremum is taken over all subsets of A which are compact in X. This is proved in Lemma B.2.1.

Example 1.3.2

- The Lebesgue measure on the Borel sets of $\mathbb{R}$ is a Radon measure.
- A locally finite measure, which is not outer regular, is given by the following example. Let X be an uncountable set equipped with the *cocountable topology*, i.e., a non-empty set A is open if and only if its complement $X \setminus A$ is countable. The Borel σ-algebra consists of all sets that are either countable or have a countable

complement. On this σ-algebra define a measure μ by $\mu(A) = 0$ if A is countable and $\mu(A) = 1$ otherwise. Then μ is finite, but not outer regular, since every open subset U of X is either empty or satisfies $\mu(U) = 1$.

The following assertion is often used in the sequel.

Proposition 1.3.3 *Let μ be an outer Radon measure on a locally compact Hausdorff space X. Then the space $C_c(X)$ is dense in $L^p(\mu)$ for every $1 \le p < \infty$.*

Proof Fix p as in the lemma and let $V \subset L^p(\mu)$ be the closure of $C_c(X)$ inside $L^p = L^p(\mu)$. We have to show $V = L^p$. By integration theory, the space of Lebesgue step functions is dense in L^p and any such is a linear combination of functions of the form $\mathbf{1}_A$, where $A \subset X$ is of finite measure. So we have to show $\mathbf{1}_A \in V$. By outer regularity, there exists a sequence $U_n \supset A$ of open sets such that $\mathbf{1}_{U_n}$ converges to $\mathbf{1}_A$ in L^p. So it suffices to assume that A is open. By weak inner regularity we similarly reduce to the case when A is compact. For given $\varepsilon > 0$ there exists an open set $U \supset A$ with $\mu(U \smallsetminus A) < \varepsilon$. By Urysohn's Lemma (A.8.1) there is $g \in C_c(X)$ with $0 \le g \le 1$, the function vanishes outside U and is constantly equal to 1 on A. Then the estimate

$$\|1_A - g\|_p^p = \int_{U \smallsetminus A} |g(x)|^p \, dx \le \mu(U \smallsetminus A) < \varepsilon$$

shows the claim. □

Let G be a locally compact group. A measure μ on the Borel σ-algebra of G is called a *left-invariant measure*, or simply *invariant* if $\mu(xA) = \mu(A)$ holds for every measurable set $A \subset G$ and every $x \in G$. Here xA stands for the set of all xa, where a ranges over A.

Examples 1.3.4

- The counting measure is invariant on any group.
- For the group $(\mathbb{R}, +)$ the Lebesgue measure dx is invariant under translations, so it is invariant in the sense above.
- For the multiplicative group $(\mathbb{R}^\times, \cdot)$ the measure $\frac{dx}{|x|}$ is invariant as follows from the change of variables rule.

Theorem 1.3.5 *Let G be a locally compact group. There exists a non-zero left-invariant outer Radon measure on G. It is uniquely determined up to positive multiples. Every such measure is called a* Haar measure. *The corresponding integral is called* Haar-integral.

The existence of an invariant measure can be made plausible as follows. Given an open set U in a topological group G one can measure the *relative size* of a set $A \subset G$

by the minimal number $(A : U)$ of translates xU needed to cover A. This relative measure is clearly invariant under left translation, and is finite, if A is compact. One can compare the sizes of sets and the quotient $\frac{(A:U)}{(K:U)}$, where K is a given fixed compact, should converge as U shrinks to a point. The limit is the measure in question. It is, however, hard to verify that the limit exists and defines a measure. We circumvent this problem by considering functionals on continuous functions of compact support instead of measures. Before giving the proof of the theorem, we will draw a few immediate conclusions.

For a function f on a topological space X the *support* is the closure of the set $\{x \in X : f(x) \neq 0\}$.

Corollary 1.3.6 *Let μ be a Haar measure on the locally compact group G.*

(a) *Every non-empty open set has strictly positive* (> 0) *measure.*

(b) *Every compact set has finite measure.*

(c) *Every continuous positive function $f \geq 0$ with $\int_G f(x)\,d\mu(x) = 0$ vanishes identically.*

(d) *Let f be a measurable function on G, which is integrable with respect to a Haar measure. Then the support of f is contained in a σ-*compact *open subgroup of G.*

Proof For (a) assume there is a non-empty open set U of measure zero. Then every translate xU of U has measure zero by invariance. As every compact set can be covered by finitely many translates of U, every compact set has measure zero. Being a Radon measure, μ is zero, a contradiction.

For (b) recall that the local-finiteness implies the existence of an open set U of finite measure. Then every translate of U has finite measure. A given compact set can be covered by finitely many translates, hence has finite measure.

For (c) let f be as above, then the measure of the open set $f^{-1}(0, \infty)$ must be zero, so it is empty by part (a).

To show (d), let f be an integrable function. It suffices to show that the set $A = \{x \in X : f(x) \neq 0\}$ is contained in an open σ-compact subgroup L, as the closure will then also be in L, which is closed by Lemma 1.1.7 (c). The set A is the union of the sets $A_n = \{x \in X : |f(x)| > 1/n\}$ for $n \in \mathbb{N}$, each of which is of finite measure. By Proposition 1.2.1 (c), it suffices to show that a set A of finite measure is contained in an open σ-compact subgroup L. By the outer regularity there exists an open set $U \supset A$ with $\mu(U) < \infty$. It suffices to show that U lies in a σ-compact open subgroup. Let $H \subset G$ be any open σ-compact subgroup of G, which exists by Proposition 1.2.1 (b). Then G is the disjoint union of the open cosets xH, where $x \in G$ ranges over a set of representatives of G/H. The set U can only meet countably many cosets xH, since for every coset one has either $xH \cap U = \emptyset$ or $\mu(xH \cap U) > 0$ by part (a) of this corollary. Let L be the group generated by H and the countably many cosets xH with $xH \cap U \neq \emptyset$. Then $L \supset U \supset A$ and L is σ-compact and open by Proposition 12.1 (c). □

Proof of the theorem Let $C_c(G)$ denote the space of all continuous functions from G to $\mathbb{C}$ of compact support.

Definition We say that a map $f : G \to X$ to a metric space (X, d) is *uniformly continuous*, if for every $\varepsilon > 0$ there exists a unit-neighborhood U such that for $x^{-1}y \in U$ or $yx^{-1} \in U$ one has d $(f(x), f(y)) < \varepsilon$.

Lemma 1.3.7 *Any function $f \in C_c(G)$ is uniformly continuous.*

Proof We only show the part with $x^{-1}y \in U$ because the other part is proved similarly and to obtain both conditions, one simply intersects the two unit-neighborhoods. Let K be the support of f. Fix $\varepsilon > 0$ and a compact unit-neighborhood V. As f is continuous, for every $x \in G$ there exists an open unit-neighborhood $V_x \subset V$ such that $y \in xV_x \Rightarrow |f(x) - f(y)| < \varepsilon/2$. Let U_x be a symmetric open unit-neighborhood with $U_x^2 \subset V_x$. Then the sets xU_x, for $x \in KV$, form an open covering of the compact set KV, so there are $x_1, \dots x_n \in KV$ such that $KV \subset x_1U_1 \cap \dots \cap x_nU_n$, where we have written U_j for U_{x_j}. Let $U = U_1 \cap \dots \cap U_n$. Then U is a symmetric open unit-neighborhood. Let now $x, y \in G$ with $x^{-1}y \in U$. If $x \notin KV$, then $y \notin K$ as $x \in yU^{-1} = yU \subset yV$. So in this case we conclude $f(x) = f(y) = 0$. It remains to consider the case when $x \in KV$. Then there exists j with $x \in x_jU_j$, and so $y \in x_jU_jU \subset x_jV_j$. It follows that

$$|f(x) - f(y)| \le |f(x) - f(x_j)| + |f(x_j) - f(y)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

as claimed. □

In order to prove Theorem 1.3.5, we use Riesz's Representation Theorem B.2.2. It suffices to show that up to positive multiples there is exactly one positive linear map $I : C_c(G) \to \mathbb{C}$, $I \neq 0$, which is invariant in the sense that $I(L_x f) = I(f)$ holds for every $x \in G$ and every $f \in C_c(G)$, where the *left translation* is defined by $L_x f(y) \stackrel{\text{def}}{=} f\left(x^{-1}y\right)$. Likewise, the *right translation* is defined by $R_x f(y) \stackrel{\text{def}}{=} f(yx)$. Note that $L_{xy} f = L_x L_y f$ and likewise for R.

Definition We say that a function f on G is a *positive function* if $f(x) \ge 0$ for every $x \in G$. We then write $f \ge 0$. Write $C_c^+(G)$ for the set of all positive functions $f \in C_c(G)$. For any two functions $f, g \in C_c^+(G)$ with $g \neq 0$ there are finitely many elements $s_j \in G$ and positive numbers c_j such that for every $x \in G$ one has $f(x) \le \sum_{j=1}^n c_j g(s_j^{-1}x)$. We can also write this inequality without arguments as $f \le \sum_{j=1}^n c_j L_{s_j} g$. Put

$$(f:g) \overset{\text{def}}{=} \inf \left\{ \sum_{j=1}^{n} c_j : \begin{array}{c} \text{there are } s_j \in G \\ \text{such that } f \le \sum_{j=1}^{n} c_j L_{s_j} g \end{array} \right\}.$$

Lemma 1.3.8 *For $f, f_1, f_2, g, h \in C_c^+(G)$ with $g, h \neq 0$, $c > 0$ and $y \in G$ one has*

(a) $(L_y f : g) = (f : g)$, *so the index is translation-invariant,*

(b) $(f_1 + f_2 : g) \le (f_1 : g) + (f_2 : g)$, *sub-additive,*

(c) $(cf : g) = c(f, g)$, *homogeneous,*

(d) $f_1 \le f_2 \Rightarrow (f_1 : g) \le (f_2 : g)$, *monotonic,*

(e) $(f : h) \le (f : g)(g : h)$,

(f) $(f : g) \ge \frac{\max f}{\max g}$, *where* $\max f \overset{\text{def}}{=} \max\{f(x) : x \in G\}$.

Proof We only prove (e) and (f), as the other assertions are easy exercises. For (e) assume $f \le \sum_j c_j L_{s_j} g$ and $g \le \sum_l d_l L_{t_l} h$. Then $f \le \sum_j \sum_l c_j d_l L_{s_j t_l} h$, which implies the claim. For (f) choose $x \in G$ with $\max f = f(x)$. Then $\max f = f(x)$ is less than or equal to $\sum_j c_j g\left(s_j^{-1} x\right) \le \sum_j c_j \max g$. □

Fix a non-zero $f_0 \in C_c^+(G)$. For $f, \phi \in C_c^+(G)$ with $\phi \neq 0$ let

$$J(f, \phi) = J_{f_0}(f, \phi) = \frac{(f : \phi)}{(f_0 : \phi)}.$$

Lemma 1.3.9 *For $f, g, \phi \in C_c^+(G)$ with $f, \phi \neq 0$, $c > 0$ and $s \in G$ one has*

(a) $\frac{1}{(f_0 : f)} \le J(f, \phi) \le (f : f_0)$,

(b) $J(L_s f, \phi) = J(f, \phi)$,

(c) $J(f + g, \phi) \le J(f, \phi) + J(g, \phi)$,

(d) $J(cf, \phi) = cJ(f, \phi)$.

Proof This follows from Lemma 1.3.8. □

The map $J(\cdot, \phi)$ will approximate the Haar-integral as the support of ϕ shrinks to $\{e\}$. Directly from Lemma 1.3.9 we only get sub-additivity, but in the limit this function will become additive as the following lemma shows. This is the central point of the proof of the existence of the Haar integral.

Lemma 1.3.10 *Let $f_1, f_2 \in C_c^+(G)$ and $\varepsilon > 0$. Then there is a unit-neighborhood V in G such that*

$$J(f_1, \phi) + J(f_2, \phi) \le J(f_1 + f_2, \phi)(1 + \varepsilon)$$

holds for every $\phi \in C_c^+(G) \setminus \{0\}$ *with support in* V.

Proof Choose $f' \in C_c^+(G)$ such that $f' \equiv 1$ on the support of $f_1 + f_2$. Let $\varepsilon, \delta > 0$ be arbitrary. Set

$$f = f_1 + f_2 + \delta f', \quad h_1 = \frac{f_1}{f}, \quad h_2 = \frac{f_2}{f},$$

where we set $h_j(x) = 0$ if $f(x) = 0$. Then $h_j \in C_c^+(G)$ for $j = 1, 2$.

According to Lemma 1.3.7, every function in $C_c(G)$ is left uniformly continuous, so in particular, for h_j this means that there is a unit-neighborhood V such that for $x, y \in G$ with $x^{-1}y \in V$ and $j = 1, 2$ one has $|h_j(x) - h_j(y)| < \frac{\varepsilon}{2}$. Let $\phi \in C_c^+(G) \setminus \{0\}$ with support in V. Choose finitely many $s_k \in G, c_k > 0$ with $f \leq \sum_k c_k L_{s_k}\phi$. Then $\phi(s_k^{-1}x) \neq 0$ implies $|h_j(x) - h_j(s_k)| < \frac{\varepsilon}{2}$, and for all x one has

$$\begin{aligned} f_j(x) = f(x)h_j(x) &\leq \sum_k c_k\phi\left(s_k^{-1}x\right)h_j(x) \\ &\leq \sum_k c_k\phi\left(s_k^{-1}x\right)\left(h_j(s_k) + \frac{\varepsilon}{2}\right), \end{aligned}$$

so that $(f_j : \phi) \leq \sum_k c_k(h_j(s_k) + \frac{\varepsilon}{2})$, implying that $(f_1 : \phi) + (f_2 : \phi)$ is less than or equal to $\sum_k c_k(1 + \varepsilon)$, which yields

$$\begin{aligned} J(f_1, \phi) + J(f_2, \phi) &\leq J(f, \phi)(1 + \varepsilon) \\ &\leq (J(f_1 + f_2, \phi) + \delta J(f', \phi))(1 + \varepsilon). \end{aligned}$$

For $\delta \to 0$ we get the claim. □

Lemma 1.3.8(e) together with $(f : f) = 1$ implies $\frac{1}{(f_0:f)} \leq (f : f_0)$. For $f \in C_c^+(G) \setminus \{0\}$ let S_f be the compact interval $\left[\frac{1}{(f_0:f)}, (f : f_0)\right]$. The space $S \stackrel{\text{def}}{=} \prod_{f \neq 0} S_f$, where the product runs over all non-zero $f \in C_c^+(G)$, is compact by Tychonov's Theorem A.7.1. Recall from Lemma 1.3.9 (a) that for every $\phi \in C_c^+(G) \setminus \{0\}$ we get an element $J(f, \phi) \in S_f$ and hence an element $(J(f, \phi))_f$ of the product space S. For a unit-neighborhood V let L_V be the closure in S of the set of all $(J(f, \phi))_f$ where ϕ ranges over all ϕ with support in V. As S is compact, the intersection $\bigcap_V L_V$ over all unit-neighborhoods V is non-empty. Choose an element $(I_{f_0}(f))_f$ in this intersection. By Lemma 1.3.9 and 1.3.10, it follows that $I = I_{f_0}$ is a positive invariant homogeneous and additive map on $C_c^+(G)$. Any real valued function $f \in C_c(G)$ can be written as the difference $f_+ - f_-$ of two positive functions. Setting $I(f) = I(f_+) - I(f_-)$, and for complex-valued functions $I(f) = I(\text{Re}(f)) + iI(\text{Im}(f))$, one gets a well-defined positive linear map that is invariant. This proves the existence of the Haar integral. For the proof of the uniqueness we need the following lemma.

Lemma 1.3.11 *Let ν be a Haar measure on G. Then for every $f \in C_c(G)$ the function $s \mapsto \int_G f(xs)\,d\nu(x)$ is continuous on G.*

Proof We have to show that for a given $s_0 \in G$ and given $\varepsilon > 0$ there exists a neighborhood U of s_0 such that for every $s \in U$ one has $\left|\int_G f(xs) - f(xs_0)\,d\nu(x)\right| < \varepsilon$. Replacing f by $R_{s_0} f(x) = f(xs_0)$, we are reduced to the case $s_0 = e$. Let K be the support of f, and let V be a compact symmetric unit-neighborhood. For $s \in V$ one has $\operatorname{supp}(R_s f) \subset KV$. Let $\varepsilon > 0$. As f is uniformly continuous, there is a symmetric unit-neighborhood W such that for $s \in W$ one has $|f(xs) - f(x)| < \frac{\varepsilon}{\nu(KV)}$. For $s \in U = W \cap V$ one therefore gets

$$\begin{aligned}\left|\int_G f(xs) - f(x)\,d\nu(x)\right| &\le \int_{KV} |f(xs) - f(x)|\,d\nu(x)\\ &< \frac{\varepsilon}{\nu(KV)}\nu(KV) = \varepsilon.\end{aligned}$$

The lemma is proven. □

Suppose now that μ, ν are two non-zero invariant Radon measures. We have to show that there is $c > 0$ with $\nu = c\mu$. For $f \in C_c(G)$ with $\int_G f(t)\,d\mu(t) = I_\mu(f) \neq 0$ set $D_f(s) \stackrel{\text{def}}{=} \int_G f(ts)\,d\nu(t)\frac{1}{I_\mu(f)}$. Then the function D_f is continuous by the lemma.

Let $g \in C_c(G)$. Using Fubini's Theorem (B.3.3) and the invariance of the measures μ, ν we get

$$\begin{aligned}I_\mu(f)I_\nu(g) &= \int_G\int_G f(s)g(t)\,d\nu(t)\,d\mu(s)\\ &= \int_G\int_G f(s)g(s^{-1}t)\,d\nu(t)\,d\mu(s) = \int_G\int_G f(ts)g(s^{-1})\,d\mu(s)\,d\nu(t)\\ &= \int_G\int_G f(ts)g(s^{-1})\,d\nu(t)\,d\mu(s) = \int_G\int_G f(ts)\,d\nu(t)\,g(s^{-1})\,d\mu(s)\\ &= I_\mu(f)\int_G D_f(s)g(s^{-1})\,d\mu(s).\end{aligned}$$

Since $I_\mu(f) \neq 0$ one concludes $I_\nu(g) = \int_G D_f(s)g(s^{-1})\,d\mu(s)$. Let f' be another function in $C_c(G)$ with $I_\mu(f') \neq 0$, so it follows $\int_G (D_f(s) - D_{f'}(s))g(s^{-1})\,d\mu(s) = 0$ for every $g \in C_c(G)$. Replacing g with the function $\tilde{g}$ given by $\tilde{g}(s) = |g(s)|^2(D_f(s^{-1}) - D_{f'}(s^{-1}))$ one gets $\int_G |(D_f(s) - D_{f'}(s))g(s^{-1})|^2\,d\mu(s) = 0$. Corollary 1.3.6 (c) implies that $(D_f(s) - D_{f'}(s))g(s^{-1}) = 0$ holds for every $s \in G$. As g is arbitrary, one gets $D_f = D_{f'}$. Call this function D. For every f with $I_\mu(f) \neq 0$ one has $\int_G f(t)\,d\mu(t)D(e) = \int_G f(t)\,d\nu(t)$. By linearity, it follows that this equality holds everywhere. This finishes the proof of the theorem. □

Example 1.3.12 Let B be the subgroup of $\mathrm{GL}_2(\mathbb{R})$ defined by

$$B = \left\{ \begin{pmatrix} 1 & x \\ & y \end{pmatrix} : x, y \in \mathbb{R},\ y \neq 0 \right\}.$$

Then $I(f) = \int_{\mathbb{R}^\times} \int_{\mathbb{R}} f\left(\begin{pmatrix} 1 & x \\ & y \end{pmatrix} \right) dx \frac{dy}{y}$ is a Haar-integral on B. (See Exercise 1.8.)

1.4 The Modular Function

From now on, if not mentioned otherwise, for a given locally compact group G, we will always assume a fixed choice of Haar measure. For the integral we will then write $\int_G f(x)\,dx$, and for the measure of a set $A \subset G$ we write $\mathrm{vol}(A)$. If the group G is compact, any Haar measure is finite, so, if not mentioned otherwise, we will then assume the measure to be the *normalized Haar measure*, i.e., we assume $\mathrm{vol}(G) = 1$ in that case. Also, for $p \geq 1$ we write $L^p(G)$ for the L^p-space of G with respect to a Haar measure, see Appendix B.4. Note that this space does not depend on the choice of a Haar measure.

Definition Let G be a locally-compact group, and let μ be a Haar measure on G. For $x \in G$ the measure μ_x, defined by $\mu_x(A) = \mu(Ax)$, is a Haar measure again, as for $y \in G$ one has $\mu_x(yA) = \mu(yAx) = \mu(Ax) = \mu_x(A)$. Therefore, by the uniqueness of the Haar measure, there exists a number $\Delta(x) > 0$ with $\mu_x = \Delta(x)\mu$. In this way one gets a map $\Delta : G \to \mathbb{R}_{>0}$, which is called the *modular function* of the group G. If $\Delta \equiv 1$, then G is called a *unimodular group*. In this case every left Haar measure is right invariant as well.

Let X be any set, and let $f : X \to \mathbb{C}$ be a function. The *sup-norm* or *supremum-norm* of f is defined by

$$\|f\|_X \stackrel{\text{def}}{=} \sup_{x \in X} |f(x)|.$$

Note that some authors use $\|\cdot\|_\infty$ to denote the sup-norm. This, however, is in conflict with the equally usual and better justified notation for the norm on the space L^∞ (See Appendix B.4).

Theorem 1.4.1

(a) *The modular function $\Delta : G \to \mathbb{R}_{>0}^\times$ is a continuous group homomorphism.*

(b) *One has $\Delta \equiv 1$ if G is abelian or compact.*

(c) *For $y \in G$ and $f \in L^1(G)$ one has $R_y f \in L^1(G)$ and*

$$\int_G R_y f(x)\,dx = \int_G f(xy)\,dx = \Delta(y^{-1}) \int_G f(x)\,dx.$$

(d) *The equality* $\int_G f(x^{-1})\,\Delta(x^{-1})\,dx = \int_G f(x)\,dx$ *holds for every integrable function f.*

Proof Part (c) is clear if f is the characteristic function $\mathbf{1}_A$ of a measurable set A. It follows generally by the usual approximation argument.

We now prove part (a) of the theorem. For $x, y \in G$ and a measurable set $A \subset G$, one computes

$$\begin{aligned}\Delta(xy)\mu(A) = \mu_{xy}(A) = \mu(Axy) = \mu_y(Ax)\\ = \Delta(y)\mu(Ax) = \Delta(y)\Delta(x)\mu(A).\end{aligned}$$

Choose A with $0 < \mu(A) < \infty$ to get $\Delta(xy) = \Delta(x)\Delta(y)$. Hence Δ is a group homomorphism.

Continuity: Let $f \in C_c(G)$ with $c = \int_G f(x)\,dx \neq 0$. By part (c) we have

$$\Delta(y) = \frac{1}{c}\int_G f\left(xy^{-1}\right)\,dx = \frac{1}{c}\int_G R_{y^{-1}} f(x)\,dx.$$

So the function Δ is continuous in y by Lemma 1.3.11.

For part (b), if G is abelian, then every right translation is a left translation, and so every left Haar measure is right-invariant.

If G is compact, then so is the image of the continuous map Δ. As Δ is a group homomorphism, the image is also a subgroup of $\mathbb{R}_{>0}$. But the only compact subgroup of the latter is the trivial group $\{1\}$, which means that $\Delta \equiv 1$.

Finally, for part (d) of the theorem let $f \in C_c(G)$ and set $I(f) = \int_G f\left(x^{-1}\right)\Delta\left(x^{-1}\right)\,dx$. Then, by part (c),

$$\begin{aligned}I(L_z f) &= \int_G f(z^{-1}x^{-1})\,\Delta(x^{-1})\,dx = \int_G f((xz)^{-1})\,\Delta(x^{-1})\,dx\\ &= \Delta(z^{-1})\int_G f(x^{-1})\,\Delta((xz^{-1})^{-1})\,dx = \int_G f(x^{-1})\,\Delta(x^{-1})\,dx = I(f).\end{aligned}$$

It follows that I is an invariant integral; hence there is $c > 0$ with $I(f) = c\int_G f(x)\,dx$. To show that $c = 1$ let $\varepsilon > 0$ and choose a symmetric unit-neighborhood V with $|1 - \Delta(s)| < \varepsilon$ for every $s \in V$. Choose a nonzero symmetric function $f \in C_c^+(V)$. Then

$$|1-c|\int_G f(x)\,dx = \left|\int_G f(x)\,dx - I(f)\right| \le \int_G |f(x) - f(x^{-1})\Delta(x^{-1})|\,dx$$
$$= \int_V f(x)\,|1-\Delta(x^{-1})|\,dx < \varepsilon \int_G f(x)\,dx.$$

So one gets $|1-c| < \varepsilon$, and as ε was arbitrary, it follows $c = 1$ as claimed. The proof of the theorem is finished. □

Lemma 1.4.2 *For given* $1 \le p < \infty$, *and* $g \in L^p(G)$ *the maps* $y \mapsto L_y g$ *and* $y \mapsto R_y g$ *are continuous maps from G to* $L^p(G)$. *In particular, for every* $\varepsilon > 0$ *there exists a neighborhood U of the unit such that*

$$y \in U \quad \Rightarrow \quad \begin{array}{l} \|L_y g - g\|_p \;<\; \varepsilon, \\ \|R_y g - g\|_p \;<\; \varepsilon. \end{array}$$

The proof will show that L is even uniformly continuous and in case that G is unimodular, so is R.

Proof Note that by invariance of the Haar integral we have

$$\|L_y g - L_x g\|_p = \|L_{x^{-1}y} g - g\|_p,$$

so uniform continuity as claimed follows from continuity at 1, which is the displayed formula in the lemma. Likewise, for the right translation we have $\|R_y g - R_x g\|_p = \Delta(x^{-1})^{1/p}\|R_{x^{-1}y} g - g\|_p$ as follows from part (c) of the theorem. It remains to show continuity at the unit element.We first assume that $g \in C_c(G)$. Choose $\varepsilon > 0$. Let K be the support of g. Then the support of $L_y g$ is yK. Let U_0 be a compact symmetric unit-neighborhood. Then for $y \in U_0$ one has $\mathrm{supp} L_y g \subset U_0 K$.

Let $\delta > 0$. By Lemma 1.3.7 there exists a unit-neighborhood $U \subset U_0$ such that for $y \in U$, the sup-norm $\|L_y g - g\|_G$ is less than δ.

In particular, for every $y \in U$ one has

$$\|L_y g - g\|_p = \left(\int_G |g(y^{-1}x) - g(x)|^p\,dx\right)^{\frac{1}{p}} < \delta\,\mathrm{vol}(U_0 K)^{\frac{1}{p}}.$$

By setting δ equal to $\varepsilon/\mathrm{vol}(U_0 K)^{1/p}$, one gets the claim for $g \in C_c(G)$.

For general g, choose $f \in C_c(G)$ such that $\|f - g\|_p < \varepsilon/3$. Choose a unit-neighborhood U with $\|f - L_y f\|_p < \varepsilon/3$ for every $y \in U$. Then for $y \in U$ one has

$$\|g - L_y g\|_p \le \|g - f\|_p + \|f - L_y f\|_p + \|L_y f - L_y g\|_p < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

In the last step we have used $\|L_y f - L_y g\|_p = \|f - g\|_p$, i.e., the left- invariance of the p-norm. This implies the claim for the left translation. The proof for the right-translation R_y is similar, except for the very last step, where instead of the invariance we use the continuity of the modular function and the equality $\|R_y f - R_y g\|_p = \Delta\left(y^{-1}\right)^{1/p}\|f - g\|_p$, which follows from part (c) of the theorem. □

Example 1.4.3 Let B be the group of real matrices of the form $\left(\begin{smallmatrix} 1 & x \\ 0 & y \end{smallmatrix}\right)$ with $y \neq 0$. Then the modular function Δ is given by $\Delta \left(\begin{smallmatrix} 1 & x \\ 0 & y \end{smallmatrix}\right) = |y|$ (See Exercise 1.8).

Proposition 1.4.4 *Let G be a locally compact group. The following assertions are equivalent.*

(a) *There exists $x \in G$ such that the singleton $\{x\}$ has non-zero measure.*

(b) *The set $\{1\}$ has non-zero measure.*

(c) *The Haar measure is a multiple of the counting-measure.*

(d) *G is a discrete group.*

Proof The equivalence of (a) and (b) is clear by the invariance of the measure. Assume (b) holds. Let $c > 0$ be the measure of $\{1\}$. Then for every finite set $E \subset G$ one has $\mathrm{vol}(E) = \sum_{e \in E} \mathrm{vol}(\{e\}) = c\#E$. Since the measure is monotonic, every infinite set gets measure ∞, and so the Haar measure equals c times the counting measure.

To see that (c) implies (d) recall that every compact set has finite measure, and by locally compactness, there exists an open set of finite measure, i.e., a finite set U that is open. By the Hausdorff axiom one can separate the elements of U by open sets, so the singletons in U are open; hence every singleton, and so every set, is open, i.e., G is discrete. Finally, if G is discrete, then each singleton is open, hence has strictly positive measure by Corollary 1.3.6. □

Proposition 1.4.5 *Let G be a locally compact group. Then G has finite volume under a Haar measure if and only if G is compact.*

Proof If G is compact, it has finite volume by Corollary 1.3.2. For the other direction suppose G has finite Haar measure. Let U be a compact unit-neighborhood. As the Haar measure of G is finite, there exists a maximal number $n \in \mathbb{N}$ of pairwise disjoint translates xU of U. Let $z_1 U, \dots, z_n U$ be such pairwise disjoint translates, and set K equal to the union of these finitely many translates. Then K is compact, and for every $x \in G$ one has $K \cap xK \neq \emptyset$. This means that $G = KK^{-1}$, which is a compact set. □

1.5 The Quotient Integral Formula

Let G be a locally compact group and let H be a closed subgroup. Then G/H is a locally compact Hausdorff space by Proposition 1.2.1. For $f \in C_c(G)$ let $f^H(x) \stackrel{\text{def}}{=} \int_H f(xh)\,dh$. For any x the function mapping h to $f(xh)$ is continuous of compact support, so the integral exists.

Lemma 1.5.1 *The function f^H lies in $C_c(G/H)$, and the support of f^H is contained in* $(\mathrm{supp}(f)H)/H$. *The map $f \mapsto f^H$ from $C_c(G)$ to $C_c(G/H)$ is surjective.*

Proof Let K be the support of f. Then KH/H is compact in G/H and contains the support of f^H, which therefore is compact. To prove continuity, let $x_0 \in G$ and U a compact neighborhood of x_0. For every $x \in U$, the function $h \mapsto f(xh)$ is supported in the compact set $U^{-1}K \cap H$. Put $d = \mu_H(U^{-1}K \cap H)$, where μ_H denotes the given Haar measure on H. Given $\varepsilon > 0$ it follows from uniform continuity of f (Lemma 1.3.7) that there exists a neighborhood $V \subseteq U$ of x_0 such that $|f(xh) - f(x_0h)| < \frac{\varepsilon}{d}$ for every $x \in V$, from which it follows that

$$|f^H(x) - f^H(x_0)| \;\le\; \int_{U^{-1}K\cap H} |f(xh) - f(x_0h)|\,dh \;<\; \varepsilon$$

for every $x \in V$, which proves continuity of f^H.

Write π for the natural projection $G \to G/H$. To show surjectivity of the map $f \mapsto f^H$, we first show that for a given compact subset C of the quotient G/H there exists a compact subset K of G such that $C \subset \pi(K)$. To this end choose a pre-image $y_c \in G$ to every $c \in C$ and an open, relatively compact neighborhood $U_c \subset G$ of y_c. As π is open, the images $\pi(U_c)$ form an open covering of C, so there are $c_1, \dots c_n \in C$ such that $C \subset \pi(K)$ with K being the compact set $\overline{U}_{c_1} \cup \dots \cup \overline{U}_{c_n}$.

Apply this construction to the set C being the support of a given $g \in C_c(G/H)$. Let $\phi \in C_c(G)$ be such that $\phi \ge 0$ and $\phi \equiv 1$ on K, which exists by Urysohn's Lemma (Lemma A.8.1). Then set $f = g\phi/\phi^H$ where g is non-zero and $f = 0$ otherwise. This definition makes sense as $\phi^H > 0$ on the support of g. One gets $f \in C_c(G)$ and $f^H = g\phi^H/\phi^H = g$. □

Remark 1.5.2 For later use we note that in the proof of the above lemma we also showed that for any compact set $C \subset G/H$ there exists a compact set $K \subset G$ such that $C \subseteq \pi(K)$. By passing to $\pi^{-1}(C) \cap K$ if necessary, we can even choose K such that $\pi(K) = C$.

A measure ν on the Borel σ-algebra of G/H is called an *invariant measure* if $\nu(xA) = \nu(A)$ holds for every $x \in G$ and every measurable $A \subset G/H$. Let Δ_G be the modular function of G and Δ_H the modular function of H.

Theorem 1.5.3 (Quotient Integral Formula) *Let G be a locally compact group, and let H be a closed subgroup. There exists an invariant Radon measure $\nu \ne 0$ on the quotient G/H if and only if the modular functions* Δ_G and Δ_H *agree on H. In this case, the measure ν is unique up to a positive scalar factor. Given Haar measures on G and H, there is a unique choice for ν, such that for every $f \in C_c(G)$ one has the* quotient integral formula

$$\int_G f(x)\,dx = \int_{G/H} \int_H f(xh)\,dh\,dx.$$

We will always assume this normalization and call the ensuing measure on G/H the quotient measure.

The quotient integral formula is valid for every $f \in L^1(G)$.

The last assertion says that if f is an integrable function on G, then the integral $f^H(x) = \int_H f(xh)\,dx$ exists almost everywhere in x and defines a ν-measurable, indeed integrable function on G/H, such that the integral formula holds.

See Exercise 1.10 for a generalization of the quotient integral formula.

Proof Assume first that there exists an invariant Radon measure $\nu \neq 0$ on the quotient space G/H. Define a linear functional I on $C_c(G)$ by $I(f) = \int_{G/H} f^H(x)\,d\nu(x)$. Then $I(f)$ is a non-zero invariant integral on G, so it is given by a Haar measure. We write $I(f) = \int_G f(x)\,dx$. For $h_0 \in H$ one gets

$$\begin{aligned}\Delta_G(h_0)\int_G f(x)\,dx &= \int_G f\left(xh_0^{-1}\right)\,dx = \int_G R_{h_0^{-1}} f(x)\,dx\\ &= \int_{G/H}\int_H f\left(xhh_0^{-1}\right)\,dh\,d\nu(x)\\ &= \Delta_H(h_0)\int_{G/H}\int_H f(xh)\,dh\,d\nu(x)\\ &= \Delta_H(h_0)\int_G f(x)\,dx.\end{aligned}$$

As f can be chosen with $\int_G f(x)\,dx \neq 0$, it follows that $\Delta_G|_H = \Delta_H$.

For the converse direction assume $\Delta_G|_H = \Delta_H$, and let $f \in C_c(G)$ with $f^H = 0$. We want to show that $\int_G f(x)\,dx = 0$. For let ϕ be another function in $C_c(G)$. We use Fubini's Theorem to get

$$\begin{aligned}0 &= \int_G\int_H f(xh)\phi(x)\,dh\,dx = \int_H\int_G \phi(x)f(xh)\,dx\,dh\\ &= \int_H \Delta_G(h^{-1})\int_G \phi\left(xh^{-1}\right) f(x)\,dx\,dh\\ &= \int_G\int_H \Delta_H\left(h^{-1}\right)\phi\left(xh^{-1}\right)\,dh\,f(x)\,dx\\ &= \int_G\int_H \phi(xh)\,dh\,f(x)\,dx = \int_G \phi^H(x)f(x)\,dx.\end{aligned}$$

As we can find ϕ with $\phi^H \equiv 1$ on the support of f, it follows that $\int_G f(x)\,dx = 0$. This means that we can unambiguously define a non-zero invariant integral on G/H by $I(g) = \int_G f(x)\,dx$, whenever $g \in C_c(G/H)$ and $f \in C_c(G)$ with $f^H = g$. By Riesz's Theorem, this integral comes from an invariant Radon measure. In particular, it follows that the quotient integral formula is valid for every $f \in C_c(G)$. All but the last assertion of the theorem is proven.

We want to prove the quotient integral formula for an integrable function f on G. It suffices to assume $f \geq 0$. Then f is a monotone limit of step-functions, so by monotone convergence one may assume f is a step-function itself and by linearity one reduces to the case of f being the characteristic function of a measurable set A with finite Haar measure. We have to show that $\mathbf{1}_A^H$ is measurable on G/H and that its integral equals $\int_G \mathbf{1}_A(x)\,dx$. We start with the case of $A = U$ being open. Note that by Lemma B.3.2 the function $g(xH) = \sup_{\substack{\phi\in C_c(G)\\ 0\leq\phi\leq\mathbf{1}_U}} \int_H \phi(xh)\,dh$ is measurable on G/H and coincides with $\mathbf{1}_U^H$. A repeated use of the Lemma of Urysohn and Lemma B.3.2 shows the claim for $A = U$,

$$\begin{aligned}\int_{G/H}\int_H \mathbf{1}_U(xh)\,dh\,dx &= \int_{G/H}\int_H \sup_{0\leq\phi\leq\mathbf{1}_U} \phi(xh)\,dh\,dx\\ &= \sup_{0\leq\phi\leq\mathbf{1}_U}\int_{G/H}\int_H \phi(xh)\,dh\,dx\\ &= \sup_{0\leq\phi\leq\mathbf{1}_U}\int_G \phi(x)\,dx = \int_G \sup_{0\leq\phi\leq\mathbf{1}_U}\phi(x)\,dx\\ &= \int_G \mathbf{1}_U(x)\,dx.\end{aligned}$$

If $A = K$ is a compact set, then let V be a relatively compact open neighborhood of K. Then $\mathbf{1}_K = \mathbf{1}_V - \mathbf{1}_{V\setminus K}$. The claim follows for $A = K$. For general A of finite measure and given $n \in \mathbb{N}$, by regularity and Lemma B.2.1, there are a compact set K_n and an open set U_n such that $K_n \subset A \subset U_n$ and $\mu(U_n \setminus K_n) < 1/n$. We can further assume that the sequence U_n is decreasing and K_n is increasing. Let g be the pointwise limit of the increasing sequence $\mathbf{1}_{K_n}^H$ and let h be the limit of $\mathbf{1}_{U_n}$. Then g and h are integrable on G/H, one has $0 \leq g \leq \mathbf{1}_A^H \leq h$ and $h - g$ is a positive function of integral zero, hence a nullfunction. This means that $\mathbf{1}_A^H$ coincides with g up to a nullfunction and thus is measurable. One has

$$\begin{aligned}\int_{G/H}\mathbf{1}_A^H(x)\,dx &= \int_{G/H} g(x)\,dx = \lim_n \int_{G/H}\mathbf{1}_{K_n}^H(x)\,dx\\ &= \lim_n \int_G \mathbf{1}_{K_n}(x)\,dx = \int_G \mathbf{1}_A(x)\,dx.\end{aligned}$$ □

The quotient integral formula should be understood as a one-sided version of Fubini's Theorem for product spaces. As for Fubini, it has a partial converse, which we give now. Let μ be a measure on a set X. Recall that a measurable subset $A \subset X$ is called *σ-finite* if A can be written as a countable union $A = \bigcup_{j=1}^{\infty} A_j$ of sets with $\mu(A_j) < \infty$ for every j. If X itself is σ-finite, one also says that the measure μ is σ-finite.

Corollary 1.5.4 *Suppose that H is a closed subgroup of G such that there exists an invariant Radon measure $\neq 0$ on G/H. Let $f : G \to \mathbb{C}$ be a measurable function such that the set $A = \{x \in G : f(x) \neq 0\}$ is σ-finite. If the iterated integral $\int_{G/H}\int_H |f(xh)|\,dh\,dx$ exists, then f is integrable.*

Proof It suffices to show that $|f|$ is integrable. So choose a sequence $(A_n)_{n\in\mathbb{N}}$ of measurable sets in G with finite Haar measure such that $A = \bigcup_{n=1}^{\infty} A_n$, and define $f_n : G \to \mathbb{C}$ by $f_n = \min(|f| \cdot \mathbf{1}_{A_n}, n)$. Then $(f_n)_{n\in\mathbb{N}}$ is an increasing sequence of integrable functions that converges point-wise to $|f|$. It follows from Theorem 1.5.3 that $\int_G f_n(x)\,dx = \int_{G/H}\int_H f_n(xh)\,dh\,dx \le \int_{G/H}\int_H |f(xh)|\,dh\,dx$ for every $n \in \mathbb{N}$. The result follows then from the Monotone Convergence Theorem. □

Corollary 1.5.5

(a) *If H is a normal closed subgroup of G, then the modular functions Δ_G and Δ_H agree on H.*

(b) *Let H be the kernel of Δ_G. Then H is unimodular.*

Proof (a) The Haar measure of the group G/H is an invariant Radon measure, so (a) follows from the theorem. Part (b) follows from part (a). □

Proposition 1.5.6 *Let G be a locally compact group, $K \subset G$ a compact subgroup and $H \subset G$ a closed subgroup such that $G = HK$. Then one can arrange the Haar measures on G, H, K in a way that for every $f \in L^1(G)$ one has*

$$\int_G f(x)\,dx = \int_H \int_K f(hk)\,dk\,dh.$$

Proof The group $H \times K$ acts on G by $(h,k).g = hgk^{-1}$. As this operation is transitive, G can be identified with $H \times K/H \cap K$, where we embed $H \cap K$ diagonally into $H \times K$. The group $H \cap K$ is compact; therefore it has trivial modular function and the modular function of $H \times K$ is trivial on this subgroup. By Theorem 1.5.3 there is a unique $H \times K$-invariant Radon measure on G up to scaling. We show that the Haar measure on G also is $H \times K$-invariant, so the uniqueness implies our claim. Obviously, the Haar measure is invariant under the action of H as the latter is the left multiplication. As K is compact, we have $\Delta_G(k) = 1$ for every $k \in K$ and so $\int_G f(xk)\,dx = \int_G f(x)\,dx$ for every $f \in C_c(G)$ by Theorem 1.4.1 (c). □

Lemma 1.5.7 *Let H be a closed subgroup of the locally compact group G such that there exists a G-invariant Radon measure on G/H. Fix such a measure. For given $1 \le p < \infty$, and $g \in L^p(G/H)$ the map $y \mapsto L_y g$ is a uniformly continuous map from G* to $L^p(G/H)$. *In particular, for every $\varepsilon > 0$ there exists a neighborhood U of the unit such that*

$$y \in U \quad \Rightarrow \quad \|L_y g - g\|_p < \varepsilon.$$

Proof The lemma is a generalization of Lemma 1.4.2 and the proof of the latter extends to give a proof of the current lemma. □

1.6 Convolution

An *algebra* over $\mathbb{C}$ is a complex vector space $\mathcal{A}$ together with a map $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$, called product or multiplication and written $(a, b) \mapsto ab$, which is bilinear, i.e., it satisfies

$$a(b + c) = ab + ac, \quad (a + b)c = ab + ac, \quad \lambda(ab) = (\lambda a)b = a(\lambda b)$$

for $a, b, c \in \mathcal{A}$ and $\lambda \in \mathbb{C}$, and it is *associative*, i.e., one has

$$a(bc) = (ab)c$$

for all $a, b, c \in \mathcal{A}$. The algebra $\mathcal{A}$ is called a *commutative algebra* if in addition for all $a, b \in \mathcal{A}$ one has $ab = ba$.

Example 1.6.1

- The vector space $\mathcal{A} = \mathrm{M}_n(\mathbb{C})$ of complex $n \times n$ matrices forms an algebra with matrix multiplication as product. This algebra is not commutative unless $n = 1$.
- For a set S the vector space $\mathrm{Map}(S, \mathbb{C})$ of all maps from S to $\mathbb{C}$ forms a commutative algebra with the point-wise product, i.e., for $f, g \in \mathrm{Map}(S, \mathbb{C})$ the product fg is the function given by $(fg)(s) = f(s)g(s)$ for $s \in S$.

Definition Let G be a locally-compact group. For two measurable functions $f, g : G \to \mathbb{C}$ define the *convolution product* as

$$f * g(x) = \int_G f(y)g(y^{-1}x)\, dy,$$

whenever the integral exists.

Theorem 1.6.2 *If $f, g \in L^1(G)$, then the integral $f * g$ exists almost everywhere in x and defines a function in $L^1(G)$. The L^1-norm satisfies $\|f * g\|_1 \le \|f\|_1 \|g\|_1$. The convolution product endows $L^1(G)$ with the structure of an algebra.*

Proof Let f, g be integrable functions on G. Then f and g are measurable in the sense that pre-images of Borel-sets are in the completed Borel-σ-algebra. Let the function ψ be defined by $\psi(y, x) = f(y)g(y^{-1}x)$. We write ψ as a composition of the map $\alpha : G \times G \to G \times G$; $(y, x) \mapsto \left(y, y^{-1}x\right)$ followed by $f \times g$ and multiplication, which are measurable. We show that α is measurable. Recall that we need measurability here with respect to the completion of the Borel σ-algebra. Since α is continuous, it is measurable with respect to the Borel σ-algebra, so we need to know that the pre-image of a null-set is a null-set. This however is clear, as α preserves the Haar measure on $G \times G$, as follows from the formula

$$\int_G \int_G \phi(y,x)\,dx\,dy = \int_G \int_G \phi\left(y, y^{-1}x\right)\,dx\,dy, \quad \phi \in C_c(G \times G),$$

and Fubini's Theorem. Being a composition of measurable maps, ψ is measurable. Let $S(f)$ and $S(g)$ be the supports of f and g, respectively. Then the sets $S(f)$ and $S(g)$ are σ-compact by Corollary 1.3.6 (d). The support of ψ is contained in the σ-compact set $S(f) \times S(f)S(g)$, and therefore is σ-compact itself. We can apply the Theorem of Fubini to calculate

$$\begin{aligned} \|f * g\|_1 &\le \int_G \int_G |f(y)g\left(y^{-1}x\right)|\,dy\,dx = \int_G \int_G |f(y)g\left(y^{-1}x\right)|\,dx\,dy \\ &= \int_G \int_G |f(y)g(x)|\,dx\,dy = \|f\|_1\,\|g\|_1 \,<\, \infty. \end{aligned}$$

The function $\psi(x,\cdot)$ is therefore integrable almost everywhere in x, and the function $f * g$ exists and is measurable. Further, the norm $\|f * g\|_1$ is less than or equal to $\int_{G\times G} |\psi(x,y)|\,dx\,dy = \|f\|_1\|g\|_1$. Associativity and distributivity are proven by straightforward calculations. □

Recall that for a function $f : G \to \mathbb{C}$ and $y \in G$ we have defined

$$R_y(f)(x) = f(xy) \quad \text{and} \quad L_y(f)(x) = f(y^{-1}x).$$

Lemma 1.6.3 *For $f, g \in L^1(G)$ and $y \in G$ one has $R_y(f * g) = f * (R_y g)$ and $L_y(f * g) = (L_y f) * g$.*

Proof We compute

$$R_y(f * g)(x) = \int_G f(z)g\left(z^{-1}xy\right)\,dz = \int_G f(z)R_y g\left(z^{-1}x\right)\,dz = f * (R_y g)(x),$$

and likewise for L. □

Theorem 1.6.4 *The algebra $L^1(G)$ is commutative if and only if G is abelian.*

Proof Assume $L^1(G)$ is commutative. Let $f, g \in L^1(G)$. For $x \in G$ we have

$$\begin{aligned} 0 = f * g(x) - g * f(x) &= \int_G f(xy)g\left(y^{-1}\right) - g(y)f\left(y^{-1}x\right)\,dy \\ &= \int_G g(y)\left(\Delta(y^{-1})f(xy^{-1}) - f(y^{-1}x)\right)\,dy. \end{aligned}$$

Since this is valid for every g, one concludes that $\Delta\left(y^{-1}\right) f\left(xy^{-1}\right) = f\left(y^{-1}x\right)$ holds for every $f \in C_c(G)$. For $x = 1$ one gets $\Delta \equiv 1$, so G is unimodular and $f\left(xy^{-1}\right) = f\left(y^{-1}x\right)$ for every $f \in C_c(G)$ and all $x, y \in g$. This implies that G is abelian. The converse direction is trivial. □

Definition By a *Dirac function* we mean a function $\phi \in C_c(G)$, which

- is positive, i.e., $\phi \geq 0$,
- has integral equal to one, $\int_G \phi(x)\,dx = 1$, and
- is symmetric, $\phi(x^{-1}) = \phi(x)$.

A *Dirac family* is a family $(\phi_U)_U$ of Dirac functions indexed by the set $\mathcal{U}$ of all unit-neighborhoods U such that ϕ_U has support inside U. Note that the set $\mathcal{U}$ can be partially ordered by reversed inclusion which makes it a directed set. So a Dirac-family is a net, which we also refer to as a *Dirac net*

Lemma 1.6.5 *If ϕ and ψ are Dirac functions, then so is their convolution product $\phi * \psi$. To every unit neighborhood U their exists a Dirac function ϕ_U such that ϕ_U as well $\phi_U * \phi_U$ have support inside U.*

Proof If ϕ and ψ are positive, then so is their convolution product. For the integral we have $\int_G \phi * \psi(x)\,dx = \int_G \phi(x)\,dx \int_G \psi(x)\,dx = 1$ and symmetry is preserved by convolution. For the second assertion, let U be a given unit neighborhood. Then their exists a symmetric unit neighborhood $W \subset U$ such that $W^2 \subset U$ as well. The Lemma of Urysohn (A.8.1) yields a function $h \in C_c(G)$ with $0 \neq h \geq 0$ and $\operatorname{supp}(h) \subset W$. Set $\phi_U(x) = h(x) + h\left(x^{-1}\right)$ and scale this function so it has integral one. Then $\operatorname{supp}(\phi_U * \phi_U) \subset \operatorname{supp}(\phi_U)\operatorname{supp}(\phi_U) \subset W^2 \subset U$, so ϕ_U satisfies the claim. □

Lemma 1.6.6 *Let $\varepsilon > 0$. For every $f \in L^1(G)$ there exists a unit-neighborhood U such that for every Dirac function ϕ_U with support in U one has*

$$\|f * \phi_U - f\|_1 < \varepsilon, \quad \|\phi_U * f - f\|_1 < \varepsilon.$$

For every continuous function f on G and every compact set $K \subset G$ there exists a unit-neighborhood U such that for every Dirac function ϕ_U with support in U one has

$$\|f * \phi_U - f\|_K < \varepsilon, \quad \|\phi_U * f - f\|_K < \varepsilon,$$

where $\|g\|_K = \sup_{x \in K} |g(x)|$.

In other words this means that the net $(\phi_U * f)_U$ indexed by the set of all unit-neighborhoods, converges to f in the L^1 norm if $f \in L^1(G)$ and compactly uniformly, if f is continuous.

Proof It suffices to consider $\phi_U * f$, as the other side is treated similarly. We compute

$$\begin{aligned}\|\phi_U * f - f\|_1 &= \int_G \left| \int_G \phi_U(y)(f\left(y^{-1}x\right) - f(x))\,dy \right| dx \\ &\le \int_G \int_G \phi_U(y)|f\left(y^{-1}x\right) - f(x)|\,dy\,dx \\ &= \int_G \phi_U(y)\|L_y f - f\|_1\,dy.\end{aligned}$$

The claim nor follows from Lemma 1.4.2.

For the last statement let f be continuous and let $K \subset G$ be compact. Since a continuous function on a compact set is uniformly continuous, for every $\varepsilon > 0$ there exists a unit-neighborhood U, such that for $x \in K$, $y^{-1}x \in U$ one has $|f(y) - f(x)| < \varepsilon$. Let now ϕ_U be a Dirac function with support in U. Then $|\phi_U * f(x) - f(x)| \le \int_G \phi_U(y^{-1}x)|f(y) - f(x)|\,dy < \varepsilon$. □

1.7 The Fourier Transform

A locally compact abelian group will be called an *LCA-group* for short. A *character* of an LCA-group A is a continuous group homomorphism

$$\chi : A \to \mathbb{T},$$

where $\mathbb{T}$ is the *circle group*, i.e., the multiplicative group of all complex numbers of absolute value one.

Example 1.7.1

- The characters of the group $\mathbb{Z}$ are the maps $k \mapsto e^{2\pi i k x}$, where x varies in $\mathbb{R}/\mathbb{Z}$ (See [Dei05] Sect. 7.1).
- The characters of $\mathbb{R}/\mathbb{Z}$ are the maps $x \mapsto e^{2\pi i k x}$, where k varies in $\mathbb{Z}$ (See [Dei05] Sect. 7.1).

Definition The set of characters forms a group under point-wise multiplication, called the *dual group* and denoted $\widehat{A}$. Later, we will equip the group $\widehat{A}$ with a topology that makes it an LCA-group.

Let $f \in L^1(A)$ and define its *Fourier transform* to be the map $\hat{f} : \widehat{A} \to \mathbb{C}$ given by

$$\hat{f}(\chi) \stackrel{\text{def}}{=} \int_A f(x)\overline{\chi(x)}\,dx.$$

This integral exists as χ is bounded.

Lemma 1.7.2 *For* $f, g \in L^1(A)$ *and* $\chi \in \widehat{A}$ *one has* $|\hat{f}(\chi)| \le \|f\|_1$ *and* $\widehat{f * g} = \hat{f}\hat{g}$.

Proof The first assertion is clear. Using Fubini's Theorem, one computes,

$$\begin{aligned}\widehat{f * g}(\chi) &= \int_A f * g(x)\overline{\chi(x)}\,dx = \int_A \int_A f(y) g\left(y^{-1}x\right)\,dy\,\overline{\chi(x)}\,dx \\ &= \int_A f(y) \int_A g\left(y^{-1}x\right)\,\overline{\chi(x)}\,dx\,dy \\ &= \int_A f(y)\overline{\chi(y)}\,dy \int_A g(x)\,\overline{\chi(x)}\,dx = \hat{f}(\chi)\hat{g}(\chi).\end{aligned}$$

□

1.8 Limits

In this section we shall give a construction principle for locally compact groups as *limits*, i.e., direct and projective limits. The reader mostly interested in developing the theory may proceed to the next chapter.

We first recall the notion of a *partial order* on a set I. This is a relation $\le$ such that for all $a, b, c \in I$ one has $a \le a$, (reflexivity); $a \le b$ and $b \le a$ implies $a = b$ (antisymmetry); $a \le b$ and $b \le c$ implies $a \le c$ (transitivity).

Definition A *directed set* is a tuple $(I, \le)$ consisting of a non-empty set I and a partial order $\le$ on I, such that any two elements of I have a common upper bound, which means that for any two $a, b \in I$ there exists an element $c \in I$ with $a \le c$ and $b \le c$.

Examples 1.8.1

- The set $\mathbb{N}$ of natural numbers is an example with the natural order $\le$. In this case the order is even *linear*, which means that any two elements on $\mathbb{N}$ can be compared. Every linear order is directed.
- Let Ω be an infinite set and let I be the set of all finite subsets of Ω, ordered by inclusion, so $A \le B \Leftrightarrow A \subset B$. Then I is directed, as for $A, B \in I$ the union $C = A \cup B$ is an upper bound.
- Directed sets are precisely the index sets of nets $(x_i)_{i \in I}$.

Direct Limits

A *direct system* of groups consists of the following data

- a directed set $(I, \leq)$,
- a family $(G_i)_{i \in I}$ of groups and
- a family of group homomorphisms $\phi_i^j : G_i \to G_j$, if $i \leq j$,

such that the following axioms are satisfied:

$$\phi_i^i = \mathrm{Id}_{G_i} \quad \text{and} \quad \phi_j^k \circ \phi_i^j = \phi_i^k, \text{ if } i \leq j \leq k.$$

Examples 1.8.2

- Let G be a group and let $(G_i)_{i \in I}$ be a family of subgroups, such that for any two indices $i, j \in I$ there exists an index $k \in I$, such that $G_i, G_j \subset G_k$. Then the G_i form a direct system, if on I one installs the partial order

$$i \leq j \quad \Leftrightarrow \quad G_i \subset G_j,$$

  and if for group homomorphisms ϕ_i^j one takes the inclusions.
- Let X be a topological space, fix $x_0 \in X$ and let I be the set of all neighborhoods of x_0 in X. For $U \in I$ let G_U be the group $C(U)$ of all continuous functions from U to $\mathbb{C}$. We order I by the *inverse inclusion*, i.e., $U \leq V \;\Leftrightarrow\; U \supset V$. The *restriction homomorphisms*

$$\phi_U^V : C(U) \to C(V), \quad \phi_U^V(f) = f|_V$$

  form a direct system.

Definition Let $\big((G_i)_{i \in I}, (\phi_i^j)_{i \leq j}\big)$ be a direct system of groups. The *direct limit* of the system is the set

$$\varinjlim_{i \in I} G_i \stackrel{\text{def}}{=} \coprod_{i \in I} G_i \,\big/ \sim,$$

where $\coprod$ denotes the disjoint union and $\sim$ the following equivalence relation: For $a \in G_i$ and $b \in G_j$ we say $a \sim b$, if there is $k \in I$ with $k \geq i, j$ and $\phi_i^k(a) = \phi_j^k(b)$.

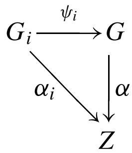
On the set $G = \varinjlim G_i$ we define a group multiplication as follows. Let $a \in G_i$ and $b \in G_j$ and let $[a]$ and $[b]$ denote their equivalence classes in G. Then there is $k \in I$ with $k \geq i$ and $k \geq j$. We define $[a][b]$ to be the equivalence class of the element $\phi_i^k(a)\phi_j^k(b)$ in G_k, so $[a][b] = [\phi_i^k(a)\phi_j^k(b)]$. Some authors also use the notion *inductive limit* instead of direct limit.

Proposition 1.8.3 *The multiplication is well-defined and defines a group structure on the set G. This group is called the* direct limit *of the system* $\big(G_i, \phi_j^i\big)$. *For every* $i \in I$ *the map*

$$\psi_i : G_i \hookrightarrow \coprod_{j \in I} G_j \to G$$

is a group homomorphism.

The direct limit has the following universal property: Let Z be a group and for every $i \in I$ *let a group homomorphism* $\alpha_i : G_i \to Z$ *be given, such that* $\alpha_i = \alpha_j \circ \phi_i^j$ *holds if* $i \leq j$*. Then there exists exactly one group homomorphism* $\alpha : G \to Z$ *making all diagrams*



commutative.

Note that in this construction the word "group" can be replaced with other algebraic structures, like rings. Then one assumes that the structure homomorphisms ϕ_i^j are ring homomorphisms and gets a ring as direct limit.

Proof To show well-definedness, we need to show that the product is independent of the choice of k. If k' is another element of I with $k' \geq i, j$, there exists a common upper bound l for k and k', so $l \geq k, k'$. We show that the construction gives the same element with l as with k. Then we apply the same argument to k' and l. Note that by definition for every $c \in G_k$ one has $[c] = \left[\phi_k^l(c)\right]$. Gs ϕ_k^l is a group homomorphism, it follows that

$$\left[\phi_i^k(a)\phi_j^k(b)\right] = \left[\phi_k^l\left(\phi_i^k(a)\phi_j^k(b)\right)\right] = \left[\phi_k^l(\phi_i^k(a))\phi_k^l\left(\phi_j^k(b)\right)\right] = \left[\phi_i^l(a)\phi_j^l(b)\right].$$

This proves well-definedness. The rest is left as an exercise to the reader. □

Examples 1.8.4

- In the case of the direct system $(C(U))_U$, where U runs through all neighborhoods of a point in a topological space, one calls the elements of $\lim\limits_{\rightarrow} C(U)$ *germs of continuous functions*.
- A special example of a direct limit is the *direct sum* of groups. So let $S \neq \emptyset$ be an index set and for each $s \in S$ let G_s be a group. Let I be the directed set of all finite subsets of S. For each $E \in I$ we let G_E be the finite product of groups,

  $$G_E = \prod_{s \in E} G_s.$$

  For $E \subset F$ in I we have the natural group homomorphism $\phi_E^F : G_E \to G_F$ sending x to $(x, 1, \ldots, 1)$. The direct limit constructed in this way is called the direct sum of the groups G_s and is denoted as

  $$\bigoplus_{s \in S} G_s.$$

Since all groups G_E can also be embedded into the product $\prod_{s\in S} G_s$ we find that the direct sum is isomorphic to the subgroup of $\prod_{s\in S} G_s$ consisting of those elements x with $x_s = 1$ for almost all $s \in S$.

Definition We say that a direct system $\big((G_i)_{i\in I}, \big(\phi_i^j\big)_{i\le j}\big)$ is a *Mittag-Leffler direct system*, if the kernel of the homomorphism ϕ_i^k stabilizes as k grows. More precisely, if for every $i \in I$ there is a $k_0 \ge i$ such that for every $k \ge k_0$ one has

$$\ker(\phi_i^k) = \ker\big(\phi_i^{k_0}\big).$$

In particular, it follows that $\ker(\psi_i) = \ker\big(\phi_i^{k_0}\big)$.

Examples 1.8.5

- The case of a family of subgroups provides an example of a Mittag-Leffler system, as here the structure homomorphisms are indeed injective.
- The system of germs of continuous functions at a point is in general not a Mittag-Leffler system.

Definition Suppose that $\big(G_i, \phi_i^j\big)$ is a direct system of topological groups, i.e., each G_i is a topological group and each ϕ_i^j is continuous. Then one defines the *direct product topology* on the limit $G = \varinjlim G_i$ to be the topology generated by the maps $\psi_i : G_i \to G$, i.e., it is the finest topology that makes all maps $G_i \to G$ continuous. Recall that a map $f : G \to X$ into some topological space is continuous if and only if all compositions $f \circ \psi_i$ are continuous (see Appendix A.5).

Proposition 1.8.6 *Let $\big(G_i, \phi_i^j\big)$ be a direct system of topological groups with limit G. Assume that all structure homomorphisms ϕ_i^j are open maps.*

(a) *The limit G is a topological group, when equipped with the inductive limit topology. The natural homomorphisms $\psi_i : G_i \to G$ are open maps.*

(b) *Suppose that all the groups G_i are Hausdorff, then the limit G is Hausdorff if and only if each of the kernels of the maps $\psi_i : G_i \to G$ is closed.*

(c) *If the system is Mittag-Leffler and all G_i are Hausdorff, then G is Hausdorff.*

(d) *If all G_i are locally compact groups and* $\ker(\psi_i)$ *is closed for each $i \in I$, then G is a locally compact group.*

Proof (a) A subset U of G is open if and only if the pre-image $\psi_i^{-1}(U) \subset G_i$ is open in G_i for every $i \in I$. Since the structure homomorphisms are open, the maps $\psi_i : G_i \to G$ are open as well and a set $U \subset G$ is open if and only if it can be written as $U = \bigcup_{i\in I} \psi_i(U_i)$ for some open sets $U_i \subset G_i$. We use this to show that the natural continuous bijection

$$\lim_{\substack{\rightarrow \\ i,j}} G_i \times G_j \to G \times G$$

is also open, hence a homeomorphism. As any open subset of the left hand side is a union of images of open subsets of $G_i \times G_j$, it suffices to show that the image of an open subset of $G_i \times G_j$ in $G \times G$ is open. For this it suffices to assume that the open set be a rectangle, i.e., of the form $U_i \times U_j$ for open sets $U_i \subset G_i$ and $U_j \subset G_j$. But then the images of U_i and U_j in G are open, hence the image of $U_i \times U_j$ in $G \times G$ is open.

We need to show continuity of the multiplication map $G \times G \to G$. As $G \times G$ is homeomorphic with the direct limit of the $G_i \times G_j$, it suffices to show that the composite map $\alpha : G_i \times G_j \to G \times G \to G$ is continuous, where the second map is multiplication. Choose some $k \in I$ with $k \geq i, j$. Then α also equals the map $G_i \times G_j \to G_k \times G_k \to G_k \to G$. In the second description the continuity follows from the continuity of the multiplication map of G_k. The inversion is dealt with in a similar way. This shows that G is a topological group if all G_i are.

(b) Suppose that G is Hausdorff. Then for given $i \in I$ the map $\psi_i : G_i \to G$ is continuous, hence its kernel is closed, as it is the pre-image of the closed set $\{1\}$. Conversely, assume all kernels $\ker(\psi_i)$ are closed and let $y \neq 1$ in G. Then there exists $i \in I$ and $y_i \in G_i$ such that $y = \psi_i(y_i)$. Now ψ_i is open, and so $U = \psi_i(G_i \smallsetminus \ker(\psi_i))$ is an open neighborhood of y which does not contain 1. Therefore, G is Hausdorff.

(c) Now suppose the system is a Mittag-Leffler direct system and that all G_i are Hausdorff. Let $i \in I$ and fix $k_0 \in I$ such that $H_i = \ker\left(\phi_i^k\right) = \ker\left(\phi_i^{k_0}\right)$ holds for every $k \geq k_0$. Then the closed subgroup H_i is also the kernel of ψ_i, so G is Hausdorff by part (b).

(d) Finally, suppose that all G_i are locally compact groups and the kernels $\ker(\psi_i)$ are closed. Then G is Hausdorff by (b), further, as each $\psi_i : G_i \to G$ is open as well, a compact unit neighborhood U inside G_i maps to a compact unit neighborhood in G, which therefore is locally compact. □

Examples 1.8.7

- If all G_i are open subgroups of a given topological group H with their subspace topology, then the limit is their union and the limit topology is the subspace topology as well.
- If $\left(G_i, \phi_i^j\right)$ is a direct system of discrete groups, then the limit is a discrete, hence a locally compact group.
- This example shows that the Hausdorff property in the direct limit can fail if the system does not satisfy the Mittag-Leffler condition. Let V be an infinite-dimensional Hilbert space and let $D \neq V$ be a dense subspace of V. Let I be the

set of all finite subsets of D, for each $\alpha \in I$ let V_α denote the (finite-dimensional) linear span of α and set

$$G_\alpha = V/V_\alpha.$$

We order I by set inclusion, then if $\alpha \leq \beta$ there is a natural projection $\phi_\alpha^\beta : G_\alpha \to G_\beta$. This family of maps forms a direct system. Each V_α is a Hilbert space, hence a Hausdorff topological group and the structure maps are open, but the direct limit, which can be identified with V/D is no longer Hausdorff, indeed, it carries the trivial topology.

Projective Limits

There is a dual construction to the direct limit, called the projective limit.

Definition A *projective system* of groups consists of the following data

- a directed set $(I, \leq)$,
- a family $(G_i)_{i\in I}$ of groups and
- a family of group homomorphisms

$$\pi_i^j : G_j \to G_i, \qquad \text{if } i \leq j,$$

such that the following axioms are met:

$$\pi_i^i = \mathrm{Id}_{G_i} \quad \text{and} \quad \pi_i^j \circ \pi_j^k = \pi_i^k, \text{ if } i \leq j \leq k.$$

Note that, in comparison to a direct system, the homomorphisms now run in the opposite direction.

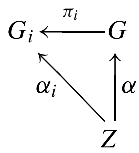
Example 1.8.8 Let p be a prime number. Let $I = \mathbb{N}$ with the usual order. For $n \in \mathbb{N}$ let $G_n = \mathbb{Z}/p^n\mathbb{Z}$ and for $m \geq n$ let $\pi_n^m : \mathbb{Z}/p^m\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}$ be the canonical projection. Then $\left(G_n, \pi_n^m\right)$ form a projective system of groups.

Definition Let $\left(G_i, \pi_i^j\right)$ be a projective system of groups. The *projective limit* of the system is the set

$$G = \varprojlim G_i$$

of all $a \in \prod_{i\in I} G_i$ such that $a_i = \pi_i^j(a_j)$ holds for every pair $i \leq j$ in I.

Proposition 1.8.9 *The projective limit G of the system (G_i) is a subgroup of the product $\prod_{i\in I} G_i$. Let $\pi_i : G \to G_i$ be the map given by the projection to the i-th coordinate. Then π_i is a group homomorphism. The projective limit has the following* *universal property: If Z is a group with group homomorphisms $\alpha_i : Z \to G_i$, such that $\alpha_i = \pi_i^j \circ \alpha_j$ holds for all $i \leq j$ in I, then there exists exactly one group homomorphism $\alpha : Z \to G$, such that all diagrams*



commute.

As in the case of direct limits, one can replace the word "group" with, say, the word "ring". Then one assumes that the structure homomorphisms π_i^j are ring homomorphisms and gets a ring as projective limit.

Proof The proof is left to the reader. □

Definition Again assume that the groups G_i in a given projective system are topological groups and that all structure homomorphisms π_i^j are continuous. Then one equips $G = \varprojlim G_i$ with the topology induced by the projections $p_i : G \to G_i$ and calls this the *projective limit topology*.

Since the topology of the product $\prod_i G_i$ is induced by the projections as well, the projective limit G carries the subspace topology of the product. Hence the question of locally compactness is connected to the same question for products.

Lemma 1.8.10 *Let I be an index set and for every $i \in I$ let there be given a non-empty locally compact space X_i. Then the product space $X = \prod_{i \in I} X_i$ is locally compact if and only if almost all the spaces X_i are compact.*

Proof Let $E \subset I$ be a finite subset and for each $i \in E$ let $U_i \subset X_i$ be a subset. These data define a *rectangle*

$$R = R((U_i)_{i \in E}) = \prod_{i \in E} U_i \times \prod_{i \in I \setminus E} X_i.$$

A rectangle is open if and only if every U_i is open.

By definition of the product topology, every open set is a union of open rectangles. The intersection of two open rectangles is again an open rectangle. If $X \neq \emptyset$ is locally compact, there therefore exists a non-empty open rectangle with compact closure. The closure of a rectangle $R((U_i)_i)$ is the rectangle $R((\overline{U_i})_i)$, and for this to be compact, almost all X_i must be compact.

The converse direction follows from Tychonov's Theorem and the simple observation that a finite product of topological spaces is locally compact if and only if all factors are locally compact. □

Proposition 1.8.11 *Let $\left(G_i, \pi_i^j\right)$ be a projective system of topological groups with limit G. Then G is a closed subgroup of the product $\prod_i G_i$ and carries the subspace topology, hence it is a topological group. If all G_i are Hausdorff, then G is Hausdorff. If all G_i are locally compact and all but finitely many are compact, then G is locally compact.*

Proof The assertions of this propositions are clear by what has been said above. □

Definition A *profinite group* is a locally compact group isomorphic to a projective limit of finite groups.

Example 1.8.12 Let p be a prime number. The profinite group

$$\mathbb{Z}_p = \lim_{\substack{\leftarrow \\ n}} \mathbb{Z}/p^n\mathbb{Z}$$

is called the group of *p-adic integers*, see Sect. 14.1.

1.9 Exercises

Exercise 1.1 Determine the Haar measures of the groups $\mathbb{Z}, \mathbb{R}, (\mathbb{R}^\times, \cdot), \mathbb{T}$.

Exercise 1.2 Give an example of a locally compact group G and two closed subsets A, B of G such that AB is not closed.

(Hint: There is an example with $G = \mathbb{R}$.)

Exercise 1.3 Let G be a topological group and suppose there exists a compact subset K of G such that $xK \cap K \neq \emptyset$ for every $x \in G$. Show that G is compact.

Exercise 1.4 Let G be a locally compact group with Haar measure μ, and let $S \subset G$ be a measurable subset with $0 < \mu(S) < \infty$. Show that the map $x \mapsto \mu(S \cap xS)$ from G to $\mathbb{R}$ is continuous.

(Hint: Note that $\mathbf{1}_S \in L^2(G)$. Write the map as $\langle \mathbf{1}_S, L_{x^{-1}} \mathbf{1}_S \rangle$ and use the Cauchy-Schwarz inequality.)

Exercise 1.5 Let G be a locally compact group with Haar measure μ, and let S be a measurable subset with $0 < \mu(S) < \infty$. Show that the set K of all $k \in G$ with $\mu(S \cap kS) = \mu(S)$ is a closed subgroup of G.

Exercise 1.6 Let G be a locally compact group, H a dense subgroup, and μ a Radon measure on G such that $\mu(hA) = \mu(A)$ holds for every measurable set $A \subset G$ and every $h \in H$. Show that μ is a Haar measure.

Exercise 1.7 Let G be a locally compact group, H a dense subgroup, and μ a Haar measure. Let $S \subset G$ be a measurable subset such that for each $h \in H$ the sets

$$hS \cap (G \smallsetminus S) \quad \text{and} \quad S \cap (G \smallsetminus S)$$

are both null-sets. Show that either S or its complement $G \smallsetminus S$ is a null-set.

(Hint: Show that the measure $\nu(A) = \mu(A \cap S)$ is invariant.)

Exercise 1.8 Let B be the subgroup of $\mathrm{GL}_2(\mathbb{R})$ defined by

$$B = \left\{ \begin{pmatrix} 1 & x \\ & y \end{pmatrix} : x, y \in \mathbb{R},\ y \neq 0 \right\}.$$

Show that $I(f) = \int_{\mathbb{R}^\times} \int_{\mathbb{R}} f\left(\begin{smallmatrix} 1 & x \\ & y \end{smallmatrix}\right) dx \frac{dy}{y}$ is a Haar-integral on B. Show that the modular function Δ of B satisfies: $\Delta\left(\begin{smallmatrix} 1 & x \\ & y \end{smallmatrix}\right) = |y|$.

Exercise 1.9 Let G be a locally-compact group. Show that the convolution satisfies $f * (g * h) = (f * g) * h$, $f * (g + h) = f * g + f * h$.

Exercise 1.10 Let G be a locally compact group, and let $\chi : G \to \mathbb{R}^\times_{>0}$ be a continuous group homomorphism.

(a) Show that there exists a unique Radon measure μ on G, which is χ-quasi-invariant in the sense that $\mu(xA) = \chi(x)\mu(A)$ holds for every $x \in G$ and every measurable subset $A \subset G$.

(b) Let $H \subset G$ be a closed subgroup. Show that there exists a Radon measure ν on G/H with $\nu(xA) = \chi(x)\nu(A)$ for every $x \in G$ and every measurable $A \subset G/H$, if and only if for every $h \in H$ one has $\chi(h)\Delta_G(h) = \Delta_H(h)$.

(Hint: Verify that the measure μ is χ-quasi-invariant if and only if the corresponding integral J satisfies $J(L_x f) = \chi(x)J(f)$. If I is a Haar-integral, consider $J(f) = I(\chi f)$.)

Exercise 1.11 Let G, H be locally compact groups and assume that G acts on H by group homomorphisms $h \mapsto {}^g h$, such that the ensuing map $G \times H \to H$ is continuous.

(a) Show that the product $(h, g)(h', g') = (h\, {}^g h', gg')$ gives $H \times G$ (with the product topology) the structure of a locally compact group, called the *semi-direct product* $H \rtimes G$.

(b) Show that there is a unique group homomorphism $\delta : G \to (0, \infty)$ such that $\mu_H({}^g A) = \delta(g)\mu_H(A)$, where μ_H is a Haar measure on H and A is a measurable subset of H.

(c) Show that $\int_H f({}^g x)\,d\mu_H(x) = \delta(g)\int_H f(x)\,d\mu_H(x)$ for $f \in C_c(H)$ and deduce that δ is continuous.

(d) Show that a Haar integral on $H \rtimes G$ is given by

$$\int_H \int_G f(h,g)\delta(g)\,d\mu_H(h)\,d\mu_G(g).$$

Exercise 1.12 For a finite group G define the *group algebra* $\mathbb{C}[G]$ to be a vector space of dimension equal to the group order $|G|$, with a special basis $(v_g)_{g\in G}$, and equipped with a multiplication $v_g v_{g'} \stackrel{\text{def}}{=} v_{gg'}$. Show that $\mathbb{C}[G]$ indeed is an algebra over $\mathbb{C}$. Show that the linear map $v_g \mapsto \mathbf{1}_{\{g\}}$ is an isomorphism of $\mathbb{C}[G]$ to the convolution algebra $L^1(G)$.