

Chapter 6

Annexes

6.1 Introduction

In this chapter, we collect several results which are used in the book, but whose presentation we have preferred to postpone until now. A first section presents notations and elementary results on matrices. The second section presents some elements of nonlinear and convex analysis. It is mainly used in Chap. 4. The third section presents Gronwall's inequality, both in the forward and in the backward time direction, together with various original extensions of this inequality to stochastic processes. The most important stochastic inequalities are given in Propositions 6.71, 6.74, 6.80. Section four presents the notion of viscosity solutions of nonlinear PDEs, and establishes three different uniqueness results for viscosity solutions of PDEs which appear in previous chapters of this book. These are variants of more or less known results scattered in the literature. We could not possibly cover all types of elliptic and parabolic equations (and systems of equations) with various types of boundary conditions. But we believe that the reader can adapt our arguments to all situations considered in Chaps. 3–5 of the book.

Finally a last section is devoted to giving hints for the solutions to some of the exercises from the book.

6.2 Annex A: Vectors and Matrices

Denote by $\mathbb{R}^{d \times k}$ the linear space of matrices $A = (a_{i,j})_{d \times k}$, where $a_{i,j} \in \mathbb{R}$. If $k = 1$ then $\mathbb{R}^{d \times 1}$ is the Euclidean space \mathbb{R}^d . Denote by $A^* = (a_{j,i})_{k \times d}$ the transposed matrix of A .

Let $x = (x_i)_{i=\overline{1,d}} \in \mathbb{R}^d$ and $y = (y_i)_{i=\overline{1,d}} \in \mathbb{R}^d$. The usual inner product on \mathbb{R}^d is given by

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 + \cdots + x_d y_d = x^* y$$

and the norm

$$|x| = \sqrt{\langle x, x \rangle} = (x_1^2 + x_2^2 + \cdots + x_d^2)^{1/2} = \sqrt{x^* x}.$$

We also introduce the notation $x^+ := (x_i^+)_d \times 1$.

The tensor product of the two vectors x and y is the linear operator $x \otimes y : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ defined by

$$(x \otimes y)(u, v) = \langle x, u \rangle \langle y, v \rangle = u^* (xy^*) v.$$

Hence one can identify

$$x \otimes y = (x_i y_j)_{d \times d} = xy^*.$$

If $A = (a_{i,j})_{d \times d}$ and $\{\mathbf{u}_1, \dots, \mathbf{u}_d\}$ is an orthonormal basis of \mathbb{R}^d , that is

$$\langle \mathbf{u}_i, \mathbf{u}_j \rangle = \delta_{i,j} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

we define

$$\mathbf{Tr} A = \text{Trace}(A) = \sum_{i=1}^d \langle A \mathbf{u}_i, \mathbf{u}_i \rangle.$$

The “Trace” is independent of the basis $\{\mathbf{u}_1, \dots, \mathbf{u}_d\}$ and

$$\mathbf{Tr} A = \sum_{i=1}^d a_{ii} = \mathbf{Tr} A^*.$$

Moreover if $A, B \in \mathbb{R}^{d \times d}$ then one verifies that

$$\mathbf{Tr}(AB) = \mathbf{Tr}(BA) = \mathbf{Tr}(A^* B^*) = \mathbf{Tr}(B^* A^*).$$

Let $A = (a_{i,j})_{d \times k} \in \mathbb{R}^{d \times k}$, $B = (b_{i,j})_{d \times k} \in \mathbb{R}^{d \times k}$. We define the inner product on $\mathbb{R}^{d \times k}$ by

$$\begin{aligned} \langle A, B \rangle &= \mathbf{Tr}(A^* B) = \mathbf{Tr}(AB^*) \\ &= \sum_{i=1}^d \sum_{j=1}^k a_{ij} b_{ij} \end{aligned}$$

and the norm

$$|A| = \sqrt{\mathbf{Tr}(A^*A)} = \sqrt{\mathbf{Tr}(AA^*)} = \left(\sum_{i=1}^d \sum_{j=1}^k a_{ij}^2 \right)^{1/2}.$$

We have

- a) $|AB| \leq |A| |B|,$
- b) $|Ax| \leq |A| |x|,$
- c) $|A| = |A^*|,$
- d) $\mathbf{Tr}(x \otimes y) = \langle x, y \rangle,$
- e) $\mathbf{Tr}[(x \otimes y)AB^*] = \langle x, BA^*y \rangle,$
- f) $\mathbf{Tr}[(x \otimes x)AA^*] = |A^*x|^2,$
- g) $|x \otimes y| = |x| |y|.$

We note that the above matrix norm is not the operator norm

$$\|A\| = \sup \{|Ax| : |x| \leq 1\} \leq |A|,$$

since $\|I_d\| = 1 \neq \sqrt{d} = |I_d|$. Note that

$$\|A\| \leq |A|.$$

We denote by $\mathbb{S}^{d \times d} \subset \mathbb{R}^{d \times d}$ the set of symmetric matrices. If $Q, P \in \mathbb{S}^{d \times d}$, we say that $Q \leq P$ if $\langle Qx, x \rangle \leq \langle Px, x \rangle$, for all $x \in \mathbb{R}^d$; Q is semipositive definite if $Q \geq 0$.

$Q \in \mathbb{S}^{d \times d}$ is semipositive definite if and only if there exists an orthonormal basis $\{\mathbf{v}_1, \dots, \mathbf{v}_d\}$ of \mathbb{R}^d and $\{\lambda_1, \dots, \lambda_d\} \subset [0, \infty[$, such that

$$Q\mathbf{v}_i = \lambda_i \mathbf{v}_i, \quad \forall i \in \overline{1, n}.$$

Then $\mathbf{Tr} Q = \sum_{i=1}^d \lambda_i$ and for all $A \in \mathbb{R}^{d \times d}$ we have

$$\mathbf{Tr}(AQ) = \sum_{i=1}^d \lambda_i \langle A\mathbf{v}_i, \mathbf{v}_i \rangle \leq \|A\| \mathbf{Tr} Q \leq |A| \mathbf{Tr} Q. \quad (6.1)$$

6.3 Annex B: Elements of Nonlinear Analysis

6.3.1 Notations

As references for this Annex, see e.g. [2] or [12]. Throughout in this Annex \mathbb{H} is a real separable Hilbert space with norm $|\cdot|$ and scalar product $\langle \cdot, \cdot \rangle$.

Let $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ be a real Banach space with dual $(\mathbb{X}^*, \|\cdot\|_{\mathbb{X}^*})$. The duality pairing $(\mathbb{X}^*, \mathbb{X})$ is also denoted $\langle \cdot, \cdot \rangle$; hence if $x \in \mathbb{X}$ and $\hat{x} \in \mathbb{X}^*$, then by $\langle \hat{x}, x \rangle$ and $\langle x, \hat{x} \rangle$ we understand the value, $\hat{x}(x)$, of \hat{x} in x .

Given $x \in \mathbb{X}$, $\hat{x} \in \mathbb{X}^*$ and the sequences $x_n \in \mathbb{X}$, $\hat{x}_n \in \mathbb{X}^*$ we say that as $n \rightarrow \infty$

$$x_n \rightarrow x \text{ (strongly) in } \mathbb{X} \text{ if } \|x_n - x\|_{\mathbb{X}} \rightarrow 0,$$

$$x_n \xrightarrow{w} x \text{ (weakly) in } \mathbb{X} \text{ if } \langle \hat{y}, x_n \rangle \rightarrow \langle \hat{y}, x \rangle, \text{ for all } \hat{y} \in \mathbb{X}^*,$$

$$\hat{x}_n \xrightarrow{w^*} \hat{x} \text{ (weak star) in } \mathbb{X}^* \text{ if } \langle \hat{x}_n, y \rangle \rightarrow \langle \hat{x}, y \rangle, \text{ for all } y \in \mathbb{X}.$$

6.3.2 Maximal Monotone Operators

Let \mathbb{X} and \mathbb{Y} be Banach spaces. A multivalued operator $A : \mathbb{X} \rightrightarrows \mathbb{Y}$ (a point-to-set operator $A : \mathbb{X} \rightarrow 2^{\mathbb{Y}}$) will also be regarded as a subset of $\mathbb{X} \times \mathbb{Y}$ setting for $A \subset \mathbb{X} \times \mathbb{Y}$,

$$Ax = \{y \in \mathbb{Y} : (x, y) \in A\}.$$

Define

$$D(A) = \text{Dom}(A) = \{x \in \mathbb{X} : Ax \neq \emptyset\} \text{ – the domain of } A,$$

$$R(A) = \{y \in \mathbb{Y} : \exists x \in D(A), \text{ s.t. } y \in Ax\} \text{ – the range of } A,$$

and define $A^{-1} : \mathbb{Y} \rightrightarrows \mathbb{X}$ to be the point-to-set operator defined by $x \in A^{-1}(y)$ if $y \in A(x)$.

We give some definitions:

- $A : \mathbb{X} \rightrightarrows \mathbb{X}^*$ is *monotone* if

$$\langle y_1 - y_2, x_1 - x_2 \rangle \geq 0, \text{ for all } (x_1, y_1) \in A, (x_2, y_2) \in A.$$

- $A : \mathbb{X} \rightrightarrows \mathbb{X}^*$ is a *maximal monotone* operator if A is a monotone operator and it is maximal in the set of monotone operators: that is,

$$\langle v - y, u - x \rangle \geq 0 \quad \forall (x, y) \in A, \implies (u, v) \in A.$$

- $\mathbf{J}_X : \mathbb{X} \rightrightarrows \mathbb{X}^*$ defined by

$$\begin{aligned} \mathbf{J}_X(x) &= \left\{ \hat{x} : \|\hat{x}\|_{\mathbb{X}^*}^2 = \|x\|^2 = \langle \hat{x}, x \rangle \right\} \\ &= \left\{ \hat{x} : \langle \hat{x}, y - x \rangle + \frac{1}{2} \|x\|^2 \leq \frac{1}{2} \|y\|^2, \quad \forall y \in \mathbb{X} \right\} \end{aligned}$$

is called the *duality mapping*; if $\mathbb{X} = \mathbb{H}$ is a Hilbert space then $\mathbf{J}_X(x) = \mathbf{I}_{\mathbb{H}}(x) = x$ for all $x \in \mathbb{H}$.

- $A : \mathbb{X} \rightrightarrows \mathbb{Y}$ is *locally bounded* at $x_0 \in D(A)$ if there exists a neighborhood V of x_0 such that $A(V) = \{y \in \mathbb{Y} : \exists x \in D(A) \cap V, \text{ s.t. } y \in Ax\}$ is bounded in \mathbb{Y} .

We have:

Proposition 6.1 (Rockafellar). *Let \mathbb{X} be a reflexive Banach space. Then $A : \mathbb{X} \rightrightarrows \mathbb{X}^*$ is maximal monotone operator if and only if A is a monotone operator and*

$$R(\mathbf{J}_X + \varepsilon A) = \mathbb{X}^*, \text{ for all } \varepsilon > 0.$$

Proposition 6.2. *Let $A : \mathbb{H} \rightrightarrows \mathbb{H}$ be a maximal monotone operator. Then:*

- (a) A is a closed subset of $\mathbb{H} \times \mathbb{H}$; moreover if $(x_n, y_n) \in A$ and

$$\begin{aligned} &x_n \rightarrow x \text{ (strongly) in } \mathbb{H} \text{ and } y_n \xrightarrow{w} y \text{ (weakly) in } \mathbb{H}, \text{ or} \\ &x_n \xrightarrow{w} x, \text{ and } y_n \rightarrow y, \text{ or} \\ &x_n \xrightarrow{w} x, \quad y_n \xrightarrow{w} y, \text{ and } \overline{\lim}_n \langle x_n, y_n \rangle \leq \langle x, y \rangle, \end{aligned}$$

then $(x, y) \in A$;

- (b) $D(A)$ and $R(A)$ are convex subsets of \mathbb{H} ;
 (c) Ax is a convex closed subset of \mathbb{H} for all $x \in D(A)$;
 (d) A is locally bounded on $\text{int}(D(A))$ that is: for every $u_0 \in \text{int}(D(A))$ there exists an $r_0 > 0$ such that

$$\bar{B}(u_0, r_0) \stackrel{\text{def}}{=} \{u_0 + r_0 v : |v| \leq 1\} \subset \text{Dom}(A)$$

and

$$A_{u_0, r_0}^\# \stackrel{\text{def}}{=} \sup \{|\hat{u}| : \hat{u} \in A(u_0 + r_0 v), |v| \leq 1\} < \infty.$$

Proposition 6.3. 1. *If $A : \mathbb{H} \rightarrow \mathbb{H}$ is a single-valued monotone hemicontinuous operator then A is maximal monotone ($A : \mathbb{H} \rightarrow \mathbb{H}$ is hemicontinuous if the function $t \rightarrow \langle A(x + tz), y \rangle : \mathbb{R} \rightarrow \mathbb{R}$ is continuous for all $x, y, z \in \mathbb{H}$).*

2. *If $A, B \subset \mathbb{H} \times \mathbb{H}$ are maximal monotone sets and $\text{int}(D(A)) \cap D(B) \neq \emptyset$, then $A + B \stackrel{\text{def}}{=} \{(x, y + z) : (x, y) \in A, (x, z) \in B\}$ is maximal monotone in $\mathbb{H} \times \mathbb{H}$.*

Let $A \subset \mathbb{H} \times \mathbb{H}$ be a maximal monotone operator. Then for each $\varepsilon > 0$ the operators

$$J_\varepsilon x = (I + \varepsilon A)^{-1}(x) \text{ and } A_\varepsilon(x) = \frac{1}{\varepsilon}(x - J_\varepsilon x)$$

from \mathbb{H} to \mathbb{H} are single-valued. The operator A_ε is called Yosida's approximation of the operator A . In [2, 12] we can find the proof of the following properties:

Proposition 6.4. *Let $A : \mathbb{H} \rightrightarrows \mathbb{H}$ be a maximal monotone operator. Then:*

(j) *For all $\varepsilon, \delta > 0$ and for all $x, y \in \mathbb{H}$*

- i) $(J_\varepsilon x, A_\varepsilon x) \in A$,
- ii) $|J_\varepsilon x - J_\varepsilon y| \leq |x - y|$,
- iii) $|A_\varepsilon x - A_\varepsilon y| \leq \frac{1}{\varepsilon} |x - y|$,
- iv) $|J_\varepsilon x - J_\delta x| \leq |\varepsilon - \delta| |A_\delta x|$,
- v) $|J_\varepsilon x| \leq |x| + (1 + |\varepsilon - 1|) |J_1 0|$,
- vi) $A_\varepsilon : \mathbb{H} \rightarrow \mathbb{H}$ is a maximal monotone operator.

(jj) *If $\varepsilon_n \rightarrow 0$, $x_n \xrightarrow{w} x$, $A_{\varepsilon_n} x_n \xrightarrow{w} y$ and*

$$\limsup_{n,m \rightarrow \infty} \langle x_n - x_m, A_{\varepsilon_n} x_n - A_{\varepsilon_m} x_m \rangle \leq 0,$$

then $(x, y) \in A$ and $\lim_{n,m \rightarrow \infty} \langle x_n - x_m, A_{\varepsilon_n} x_n - A_{\varepsilon_m} x_m \rangle = 0$.

(jjj) $\lim_{\varepsilon \searrow 0} J_\varepsilon x = \Pr_{\overline{D(A)}} x$, $\forall x \in \mathbb{H}$ and

$$\lim_{\varepsilon \searrow 0} x_\varepsilon = x \in D(A) \quad \Rightarrow \quad \lim_{\varepsilon \searrow 0} J_\varepsilon x_\varepsilon = x.$$

($\Pr_{\overline{D(A)}} x$ is the orthogonal projection of x on $\overline{D(A)}$.)

(jv) $\lim_{\varepsilon \searrow 0} A_\varepsilon x = \Pr_{A_x} \{0\} \stackrel{\text{def}}{=} A^0 x \in Ax$, for all $x \in D(A)$.

(v) $|A_\varepsilon x|$ is monotone decreasing in $\varepsilon > 0$, and when $\varepsilon \searrow 0$

$$|A_\varepsilon(x)| \nearrow \begin{cases} |A^0(x)|, & \text{if } x \in D(A), \\ +\infty, & \text{if } x \notin D(A). \end{cases}$$

(vj) $|J_\varepsilon x - x| = \varepsilon |A_\varepsilon x| \leq \varepsilon |A^0 x| \leq \varepsilon |z|$, for all $(x, z) \in A$.

(vjj) For all $x \in \mathbb{H}$,

$$\begin{aligned} |J_\varepsilon x - x| &\leq |J_\varepsilon x - J_\varepsilon(J_1 x)| + |J_\varepsilon(J_1 x) - J_1 x| + |J_1 x - x| \\ &\leq 2|J_1 x - x| + \varepsilon |A^0(J_1 x)|. \end{aligned}$$

(vjij) For all $x \in \mathbb{H}$ and $y \in \text{Dom}(A)$

$$\begin{aligned} |J_\varepsilon x - J_\delta y| &\leq |x - y| + |\varepsilon - \delta| |A_\delta y| \\ &\leq |x - y| + |\varepsilon - \delta| |A^0 y|. \end{aligned}$$

The operator A is uniquely defined by its principal section $A^0x \stackrel{\text{def}}{=} \text{Pr}_{Ax} \{0\}$ in the following sense: if $(x, y) \in \overline{D(A)} \times \mathbb{H}$ such that

$$\langle y - A^0u, x - u \rangle \geq 0, \text{ for all } u \in D(A)$$

then $(x, y) \in A$.

Proposition 6.5. *Let $A : \mathbb{H} \rightrightarrows \mathbb{H}$ be a maximal monotone operator.*

I. *If $\bar{B}(x_0, r_0) \subset \text{Dom}(A)$ and*

$$A_{x_0, r_0}^\# \stackrel{\text{def}}{=} \sup \{ |\hat{u}| : \hat{u} \in A(x_0 + r_0v), |v| \leq 1 \},$$

then

$$r_0 |\hat{x}| \leq \langle \hat{x}, x - x_0 \rangle + A_{x_0, r_0}^\# |x - x_0| + r_0 A_{x_0, r_0}^\#, \quad \forall (x, \hat{x}) \in A. \quad (6.2)$$

II. *If there exist $x_0 \in \mathbb{H}$ and $a_0, \hat{a}_0 \geq 0$ such that*

$$r_0 |\hat{x}| \leq \langle \hat{x}, x - x_0 \rangle + a_0 |x - x_0| + \hat{a}_0, \quad \forall (x, \hat{x}) \in A,$$

then there exists a $b_0 \geq 0$ such that for all $x \in \mathbb{H}$, for all $\varepsilon \in]0, 1]$:

$$r_0 |A_\varepsilon x| \leq \langle A_\varepsilon x, x - x_0 \rangle + a_0 |x - x_0| + b_0. \quad (6.3)$$

If $x_0 \in \text{Dom}(A)$ and $0 \in Ax_0$, then $b_0 = \hat{a}_0$.

Proof. I. By monotonicity of A we have $\forall (x, \hat{x}) \in A, \forall |v| \leq 1$:

$$\begin{aligned} r_0 \langle \hat{x}, v \rangle &\leq r_0 \langle \hat{x}, v \rangle + \langle \hat{x} - \hat{y}, x - (x_0 + r_0v) \rangle \\ &= \langle \hat{x}, x - x_0 \rangle - \langle \hat{y}, x - x_0 \rangle + r_0 \langle \hat{y}, v \rangle \\ &\leq \langle \hat{x}, x - x_0 \rangle + A_{x_0, r_0}^\# |x - x_0| + r_0 A_{x_0, r_0}^\#, \end{aligned}$$

which yields (6.2).

II. Since $A_\varepsilon(x) \in A(J_\varepsilon(x))$, it follows that

$$\begin{aligned} r_0 |A_\varepsilon x| &\leq \langle A_\varepsilon x, J_\varepsilon(x) - x_0 \rangle + a_0 |J_\varepsilon(x) - x_0| + \hat{a}_0 \\ &\leq \langle A_\varepsilon x, x - x_0 \rangle + a_0 [|J_\varepsilon(x) - J_\varepsilon(x_0)| + |J_\varepsilon(x_0) - x_0|] + \hat{a}_0 \\ &\leq \langle A_\varepsilon x, x - x_0 \rangle + a_0 |x - x_0| + a_0 |J_\varepsilon(x_0) - x_0| + \hat{a}_0. \end{aligned}$$

Hence the inequality (6.3) holds for $b_0 = a_0 [2 |J_1 x_0 - x_0| + |A^0(J_1 x_0)|] + \hat{a}_0$.
If $0 \in Ax_0$ then $J_\varepsilon(x_0) = x_0$ and $b_0 = \hat{a}_0$. ■

Proposition 6.6. *If A is a maximal monotone set in $\mathbb{H} \times \mathbb{H}$ and $\mathcal{A} \subset L^2(0, T; \mathbb{H}) \times L^2(0, T; \mathbb{H})$ is defined by*

$$\mathcal{A} = \{(x, \hat{x}) \in L^2(0, T; H) \times L^2(0, T; H) : (x(t), \hat{x}(t)) \in A, \text{ a.e. } t \in]0, T[\},$$

then \mathcal{A} is a maximal monotone set in $L^2(0, T; \mathbb{H}) \times L^2(0, T; \mathbb{H})$.

6.3.3 Stochastic Monotone Functions

Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0})$ be a complete stochastic basis and

$$F : \Omega \times [0, +\infty[\times \mathbb{R}^d \times \mathbb{R}^{d \times k} \rightarrow \mathbb{R}^d$$

such that

- ◇ $F(\cdot, \cdot, y, z)$ is \mathcal{P} -m.s.p. for every $(y, z) \in \mathbb{R}^d \times \mathbb{R}^{d \times k}$;
- ◇ for all $y, y' \in \mathbb{R}^d, z, z' \in \mathbb{R}^{d \times k}, t \geq 0$:

$$\langle y - y', F(t, y, z) - F(t, y', z) \rangle \leq 0, \quad \mathbb{P}\text{-a.s.};$$

- ◇ for all $z, z' \in \mathbb{R}^{d \times k}, t \geq 0$:

$$y \mapsto F(t, y, z) : \mathbb{R}^d \rightarrow \mathbb{R}^d \text{ is continuous, } \quad \mathbb{P}\text{-a.s.};$$

- ◇ there exists a \mathcal{P} -m.s.p. $\ell : \Omega \times [0, +\infty[\rightarrow \mathbb{R}_+$ such that for all $y \in \mathbb{R}^d, z, z' \in \mathbb{R}^{d \times k}, t \geq 0$:

$$|F(t, y, z) - F(t, y, z')| \leq \ell_t |z - z'|, \quad \mathbb{P}\text{-a.s.}$$

Since $y \mapsto -F(t, y, z) : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a monotone continuous operator (hence also maximal monotone operator), it follows that for every $\varepsilon > 0$ and $(\omega, t, y, z) \in \Omega \times [0, T] \times \mathbb{R}^d \times \mathbb{R}^{d \times k}$ there exists a unique $J_\varepsilon = J_\varepsilon(\omega, t, y, z) \in \mathbb{R}^d$ such that

$$J_\varepsilon - \varepsilon F(\omega, t, J_\varepsilon, z) = y.$$

The Yosida approximation of F is defined by

$$F_\varepsilon(t, y, z) \stackrel{\text{def}}{=} \frac{1}{\varepsilon} (J_\varepsilon(t, y, z) - y) = F(t, J_\varepsilon(t, y, z), z).$$

Note that $F_\varepsilon = F_\varepsilon(t, y, z)$ is the unique solution of

$$F(\omega, t, y + \varepsilon F_\varepsilon, z) = F_\varepsilon. \tag{6.4}$$

The functions $J_\varepsilon(\cdot, \cdot, y, z), F_\varepsilon(\cdot, \cdot, y, z) : \Omega \times [0, T] \rightarrow \mathbb{R}^d$ are \mathcal{P} -m.s.p. for every $(y, z) \in \mathbb{R}^d \times \mathbb{R}^{d \times k}$ and we have:

Proposition 6.7. For all $\varepsilon, \delta > 0, \forall t \in [0, T], \forall y, y' \in \mathbb{R}^d, \forall z, z' \in \mathbb{R}^{d \times k}$:

$$\begin{aligned}
 (a) \quad & |J_\varepsilon(t, y, z) - J_\varepsilon(t, y', z')| \leq |y - y'| + \varepsilon \ell_t |z - z'|, \\
 (b) \quad & |J_\varepsilon(t, 0, 0)| \leq \varepsilon |F(t, 0, 0)|, \\
 (c) \quad & \langle F_\varepsilon(t, y, z) - F_\varepsilon(t, y', z'), y - y' \rangle \leq \ell_t |z - z'| |y - y'|, \\
 (d) \quad & |F_\varepsilon(t, y, z) - F_\varepsilon(t, y', z')| \leq \frac{2}{\varepsilon} |y - y'| + \ell_t |z - z'|, \\
 (e) \quad & |J_\varepsilon(t, y, z) - y| \leq \varepsilon |F_\varepsilon(t, y, z)| \leq \varepsilon |F(t, y, z)|, \\
 (f) \quad & \lim_{\varepsilon \rightarrow 0} F_\varepsilon(t, y, z) = F(t, y, z),
 \end{aligned} \tag{6.5}$$

$$|J_\varepsilon(t, y, z) - J_\delta(t, y', z')| \leq |y - y'| + \delta \ell_t |z - z'| + |\varepsilon - \delta| |F(t, y, z)| \tag{6.6}$$

and

$$\begin{aligned}
 & \langle y - y', F_\varepsilon(t, y, z) - F_\delta(t, y', z') \rangle + \varepsilon |F_\varepsilon(t, y, z)|^2 + \delta |F_\delta(t, y', z')|^2 \\
 & \leq (\varepsilon + \delta) \langle F_\varepsilon(t, y, z), F_\delta(t, y', z') \rangle \\
 & \quad + \ell_t [|y - y'| + \varepsilon |F(t, y, z)| + \delta |F(t, y', z')|] |z - z'|.
 \end{aligned} \tag{6.7}$$

Proof. (a): If $J = J_\varepsilon(t, y, z), J' = J_\varepsilon(t, y', z')$, then

$$\begin{aligned}
 & |J - J'|^2 \\
 & = \varepsilon \langle F(t, J, z) - F(t, J', z'), J - J' \rangle + \langle y - y', J - J' \rangle \\
 & = \varepsilon \langle F(t, J, z) - F(t, J', z), J - J' \rangle \\
 & \quad + \varepsilon \langle F(t, J', z) - F(t, J', z'), J - J' \rangle + \langle y - y', J - J' \rangle \\
 & \leq \varepsilon [\ell_t |z - z'| |J - J'|] + |y - y'| |J - J'|
 \end{aligned}$$

and (6.5-a) follows.

(b): With the notation $J^0 = J(t, 0, 0)$,

$$|J^0|^2 = \varepsilon \langle F(t, J^0, 0), J^0 \rangle \leq \varepsilon \langle F(t, 0, 0), J^0 \rangle \leq \varepsilon |F(t, 0, 0)| |J^0|$$

which gives (6.5-b).

(c): We have

$$\begin{aligned}
 & \langle F_\varepsilon(t, y, z) - F_\varepsilon(t, y', z'), y - y' \rangle \\
 & = \frac{1}{\varepsilon} \langle J_\varepsilon(t, y, z) - J_\varepsilon(t, y', z'), y - y' \rangle - \frac{1}{\varepsilon} |y - y'|^2
 \end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{\varepsilon} [|y - y'| + \varepsilon \ell_t |z - z'|] |y - y'| - \frac{1}{\varepsilon} |y - y'|^2 \\ &\leq \ell_t |z - z'| |y - y'| \end{aligned}$$

that is (6.5-b).

(d): From (a) and the definition of F_ε the inequality (d) clearly follows.

(e): The properties follow from those of the Yosida approximation, A_ε , of a maximal operator A (here $A_\varepsilon(y) = -F_\varepsilon(t, y, z)$ for (ω, t, z) fixed).

(6.6): Let $J_\varepsilon = J_\varepsilon(t, y, z)$ and $J'_\delta = J'_\delta(t, y', z')$. Then

$$\begin{aligned} |J_\varepsilon - J'_\delta|^2 &= (\varepsilon - \delta) \langle F(t, J_\varepsilon, z), J_\varepsilon - J'_\delta \rangle \\ &\quad + \delta \langle F(t, J_\varepsilon, z) - F(t, J'_\delta, z'), J_\varepsilon - J'_\delta \rangle + \langle y - y', J_\varepsilon - J'_\delta \rangle \\ &\leq |\varepsilon - \delta| |F(t, J_\varepsilon, z)| |J_\varepsilon - J'_\delta| + \delta \ell_t |z - z'| |J_\varepsilon - J'_\delta| + |y - y'| |J_\varepsilon - J'_\delta| \end{aligned}$$

and (6.6) follows.

(6.7): Now, we have

$$\begin{aligned} &\langle J_\varepsilon - J'_\delta, F_\varepsilon(t, y, z) - F_\delta(t, y', z') \rangle \\ &= \langle J_\varepsilon - J'_\delta, F(t, J_\varepsilon, z) - F(t, J'_\delta, z') \rangle \\ &\leq 0 + \langle J_\varepsilon - J'_\delta, F(t, J'_\delta, z) - F(t, J'_\delta, z') \rangle \\ &\leq |J_\varepsilon - J'_\delta| \ell_t |z - z'| \\ &\leq \ell_t [\varepsilon |F(t, y, z)| + \delta |F(t, y', z')| + |y - y'|] |z - z'| \end{aligned}$$

and then

$$\begin{aligned} &\langle y - y', F_\varepsilon(t, y, z) - F_\delta(t, y', z') \rangle \\ &= \langle J_\varepsilon - \varepsilon F_\varepsilon(t, y, z) - J'_\delta + \delta F_\delta(t, y', z'), F_\varepsilon(t, y, z) - F_\delta(t, y', z') \rangle \\ &\leq -\varepsilon |F_\varepsilon(t, y, z)|^2 - \delta |F_\delta(t, y', z')|^2 + (\varepsilon + \delta) \langle F_\varepsilon(t, y, z), F_\delta(t, y', z') \rangle \\ &\quad + \ell_t [\varepsilon |F(t, y, z)| + \delta |F(t, y', z')| + |y - y'|] |z - z'| \end{aligned}$$

that is (6.7). ■

If we define

$$F_R^\#(t) \stackrel{\text{def}}{=} \sup_{|y| \leq R} |F(t, y, 0)|,$$

then we have the following:

Proposition 6.8. For all $\varepsilon > 0$, $p, a > 1$, $r_0 \geq 0$, $y \in \mathbb{R}^d$, $z \in \mathbb{R}^{d \times k}$, $t \in [0, T]$:

$$\begin{aligned} r_0 |F_\varepsilon(t, y, z)| + \langle F_\varepsilon(t, y, z), y \rangle &\leq r_0 \left(F_{r_0}^\#(t) + r_0 \frac{a}{2n_p} (\ell_t)^2 \right) \\ &+ \left(F_{r_0}^\#(t) + r_0 \frac{a}{n_p} (\ell_t)^2 \right) |y| + \frac{a}{2n_p} (\ell_t)^2 |y|^2 + \frac{n_p}{2a} |z|^2, \text{ a.s.}, \end{aligned} \quad (6.8)$$

where

$$n_p \stackrel{\text{def}}{=} 1 \wedge (p - 1).$$

Proof. Let $0 \leq r_0 \leq 1$. The monotonicity property of F_ε implies that for all $|u| \leq 1$:

$$\langle F_\varepsilon(t, r_0 u, z) - F_\varepsilon(t, y, z), r_0 u - y \rangle \leq 0,$$

and, consequently, $\forall |u| \leq 1$:

$$\begin{aligned} &r_0 \langle F_\varepsilon(t, y, z), -u \rangle + \langle F_\varepsilon(t, y, z), y \rangle \\ &\leq |F_\varepsilon(t, r_0 u, z)| |y - r_0 u| \\ &\leq |F_\varepsilon(t, r_0 u, 0)| (|y| + r_0) + \ell_t |z| (|y| + r_0) \\ &\leq |F(t, r_0 u, 0)| (|y| + r_0) + \frac{a}{2n_p} (\ell_t)^2 (|y| + r_0)^2 + \frac{n_p}{2a} |z|^2. \end{aligned}$$

The inequality (6.8) follows by taking the sup of the left-hand side over all vectors u such that $|u| \leq 1$. \blacksquare

Finally we give some convergence results.

Let $F : \Omega \times [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^{d'}$ be a function satisfying

- i) $F(\cdot, \cdot, x)$ is $\mathcal{F} \otimes \mathcal{B}_{[0, T]}$ -measurable, $\forall x \in \mathbb{R}^d$,
 - ii) $F(\omega, t, \cdot)$ is continuous $d\mathbb{P} \otimes dt$ -a.e. $(\omega, t) \in \Omega \times [0, T]$,
 - iii) $\exists \alpha > 0$ such that $\int_0^T (F_R^\#(t))^\alpha dt < +\infty$, \mathbb{P} -a.s., $\forall R > 0$.
- (6.9)

Proposition 6.9. Assume that F satisfies (6.9). Let

$$X^\varepsilon, X \in L^0(\Omega; C([0, T]; \mathbb{R}^d))$$

be such that

$$\sup_{t \in [0, T]} |X_t^\varepsilon - X_t| \xrightarrow[\varepsilon \rightarrow 0]{\text{prob.}} 0.$$

Then

$$\begin{aligned} & \sup_{t \in [0, T]} \left| \int_0^t F(s, X_s^\varepsilon) ds - \int_0^t F(s, X_s) ds \right| \\ & \leq \int_0^T |F(s, X_s^\varepsilon) - F(s, X_s)| ds \xrightarrow[\varepsilon \rightarrow 0]{\text{prob.}} 0. \end{aligned}$$

Moreover if for some $p, \alpha > 0$:

$$C_{p, \alpha} \stackrel{\text{def}}{=} \sup_{0 < \varepsilon \leq \varepsilon_0} \mathbb{E} \left(\int_0^T |F(t, X_t^\varepsilon)|^\alpha dt \right)^p < +\infty, \quad (6.10)$$

then

$$\begin{aligned} c_1) \quad & \mathbb{E} \left(\int_0^T |F(t, X_t)|^\alpha dt \right)^p \leq C_p, \\ c_2) \quad & \mathbb{E} \left(\int_0^T |F(t, X_t^\varepsilon) - F(t, X_t)|^\alpha dt \right)^q \xrightarrow[\varepsilon \rightarrow 0]{} 0, \quad \forall q \in]0, p[. \end{aligned} \quad (6.11)$$

If, in addition, $x \mapsto -F(t, x)$ is a monotone operator and $F_\varepsilon = F_\varepsilon(t, x)$, $\varepsilon > 0$, is the Yosida approximation of F (F_ε is the unique solution of $F(\omega, t, x + \varepsilon F_\varepsilon) = F_\varepsilon$) then $\forall q \in]0, p[$:

$$\mathbb{E} \left(\int_0^T |F_\varepsilon(t, X_t^\varepsilon) - F(t, X_t)|^\alpha dt \right)^q \xrightarrow[\varepsilon \rightarrow 0]{} 0. \quad (6.12)$$

Proof. Let $\varepsilon_n \rightarrow 0$ such that

$$\lim_{\varepsilon_n \rightarrow 0} \sup_{t \in [0, T]} |X_t^{\varepsilon_n} - X_t| = 0, \quad \mathbb{P}\text{-a.s.}$$

Then by the Lebesgue dominated convergence theorem

$$\lim_{\varepsilon_n \rightarrow 0} \int_0^T |F(s, X_s^{\varepsilon_n}) - F(s, X_s)|^\alpha ds = 0, \quad \mathbb{P}\text{-a.s.}$$

Since the convergence in probability is given by a metric, by *reductio ad absurdum* we infer that

$$\int_0^T |F(s, X_s^\varepsilon) - F(s, X_s)|^\alpha ds \xrightarrow[\varepsilon \rightarrow 0]{\text{prob.}} 0.$$

Also, if $C_p < \infty$, then Fatou's lemma clearly yields (6.11-c₁).

I. Denote by C positive constants independent of ε_n . Let

$$(\Delta_n) \Delta_{\varepsilon_n} \stackrel{\text{def}}{=} \int_0^T |F(s, X_s^{\varepsilon_n}) - F(s, X_s)|^\alpha ds.$$

Then by the Lebesgue dominated convergence theorem $\Delta_n \rightarrow 0$, \mathbb{P} -a.s., and

$$\mathbb{E}\Delta_n^p \leq C.$$

Since

$$\begin{aligned} \mathbb{E}\Delta_n^q &\leq \mathbb{E}(\Delta_n^q \mathbf{1}_{\Delta_n \leq R}) + \mathbb{E}\left(\Delta_n^q \frac{\Delta_n^{p-q}}{R^{p-q}} \mathbf{1}_{\Delta_n > R}\right) \\ &\leq \mathbb{E}(\Delta_n^q \mathbf{1}_{\Delta_n \leq R}) + \frac{C}{R^{p-q}}, \end{aligned}$$

it follows that

$$0 \leq \limsup_{\varepsilon_n \rightarrow 0} \mathbb{E}\Delta_n^q \leq \frac{C}{R^{p-q}} \quad \forall R > 0,$$

that is $\lim_{\varepsilon_n \rightarrow 0} \mathbb{E}\Delta_n^q = 0$ and by *reductio ad absurdum* the full sequence Δ_ε has the property (6.11-c₂).

II. Since $|F_\varepsilon(t, X_t^\varepsilon)| \leq |F(t, X_t^\varepsilon)|$, on a subsequence

$$\begin{aligned} \lim_{\varepsilon_n \rightarrow 0} F_{\varepsilon_n}(t, X_t^{\varepsilon_n}) &= \lim_{\varepsilon_n \rightarrow 0} F(t, X_t^\varepsilon + \varepsilon_n F_{\varepsilon_n}(t, X_t^{\varepsilon_n})) \\ &= F(t, X_t), \quad \mathbb{P}\text{-a.s.} \end{aligned}$$

and then the convergence result, (6.12), follows in exactly the same manner with $\Delta_{\varepsilon_n} := \int_0^T |F_{\varepsilon_n}(s, X_s^{\varepsilon_n}) - F(s, X_s)|^\alpha ds$. ■

6.3.4 Compactness Results

Let $I \subset \mathbb{R}$ be an interval. Denote by $C(I; \mathbb{R}^d)$ the space of continuous functions $g : I \rightarrow \mathbb{R}^d$. If $I = [a, b]$ then $C([a, b]; \mathbb{R}^d)$ is a separable Banach space with respect to the norm $\|\cdot\|_{[a,b]}$, where if $g : [a, b] \rightarrow \mathbb{R}^d$ we define

$$\|g\|_{[a,t]} = \sup \{|g(s)| : a \leq s \leq t\}.$$

If $[a, b] = [0, t]$ then

$$\|g\|_t \stackrel{\text{def}}{=} \|g\|_{[0,t]} = \sup \{|g(s)| : 0 \leq s \leq t\}.$$

For $g \in C([0, T]; \mathbb{R}^d)$ we define for $t \in [0, T]$ and $\varepsilon \geq 0$:

$$\mathbf{m}_g(t, \varepsilon) = \sup \{\|g(t) - g(s)\|_{\mathbb{R}^d} : |t - s| \leq \varepsilon, s \in [0, T]\}$$

the modulus of continuity at t , and

$$\mathbf{m}_g(\varepsilon) = \mathbf{m}(\varepsilon; g) = \sup \{|g(t) - g(s)| : |t - s| \leq \varepsilon, t, s \in [0, T]\}$$

the modulus of uniformly continuity.

We also introduce the notation

$$\boldsymbol{\mu}_g(\varepsilon) = \boldsymbol{\mu}(\varepsilon; g) = \varepsilon + \mathbf{m}_g(\varepsilon).$$

Note that

$$\begin{aligned} m_1) \quad & 0 = \mathbf{m}_g(0) \leq \mathbf{m}_g(\varepsilon) \leq \mathbf{m}_g(\delta) \leq 2 \|g\|_T, \quad \forall 0 < \varepsilon < \delta, \\ m_2) \quad & 0 = \boldsymbol{\mu}_g(0) < \boldsymbol{\mu}_g(\varepsilon) < \boldsymbol{\mu}_g(\delta), \quad \forall 0 < \varepsilon < \delta, \\ m_3) \quad & \mathbf{m}_g(\varepsilon + \delta) \leq \mathbf{m}_g(\varepsilon) + \mathbf{m}_g(\delta), \quad \forall \varepsilon, \delta \geq 0, \\ m_4) \quad & \lim_{\varepsilon \searrow 0} \mathbf{m}_g(\varepsilon) = \lim_{\varepsilon \searrow 0} \boldsymbol{\mu}_g(\varepsilon) = 0 \end{aligned} \tag{6.13}$$

and

$$|\mathbf{m}_g(t, \varepsilon) - \mathbf{m}_h(t, \varepsilon)| \leq |\mathbf{m}_g(\varepsilon) - \mathbf{m}_h(\varepsilon)| \leq 2 \|g - h\|_T + m_g(|\varepsilon - \delta|).$$

If $\mathcal{M} \subset C([0, T]; \mathbb{R}^d)$ and $\varepsilon > 0$, then

$$\begin{aligned} \mathbf{m}_{\mathcal{M}}(t, \varepsilon) & \stackrel{\text{def}}{=} \sup \{\mathbf{m}_g(t, \varepsilon) : g \in \mathcal{M}\}, \\ \|\mathcal{M}\|_T & \stackrel{\text{def}}{=} \sup \{\|g\|_T : g \in \mathcal{M}\}, \\ \mathcal{M}(t) & \stackrel{\text{def}}{=} \{g(t) : g \in \mathcal{M}\}, \end{aligned}$$

and

$$\begin{aligned} \mathbf{m}_{\mathcal{M}}(\varepsilon) & \stackrel{\text{def}}{=} \sup \{\mathbf{m}_g(\varepsilon) : g \in \mathcal{M}\}, \\ \boldsymbol{\mu}_{\mathcal{M}}(\varepsilon) & \stackrel{\text{def}}{=} \varepsilon + \mathbf{m}_{\mathcal{M}}(\varepsilon). \end{aligned}$$

Theorem 6.10 (Arzelà–Ascoli). *Let $\mathcal{M} \subset C([0, T]; \mathbb{R}^d)$. Then the following three conditions are equivalent:*

- (A) \mathcal{M} is relatively compact in $C([0, T]; \mathbb{R}^d)$;
- (B) (B_1) (equicontinuity): $\lim_{\varepsilon \rightarrow 0} \mathbf{m}_{\mathcal{M}}(t, \varepsilon) = 0, \forall t \in [0, T]$;
- (B_2) (bounded images) for each $t \in [0, T]$ the set $\mathcal{M}(t) = \{g(t) : g \in \mathcal{M}\}$ is bounded in \mathbb{R}^d ;
- (C) (C_1) (uniform equicontinuity): $\lim_{\varepsilon \rightarrow 0} \mathbf{m}_{\mathcal{M}}(\varepsilon) = 0$;
- (C_2) the set $\{g(t) : t \in [0, T], g \in \mathcal{M}\}$ is bounded in \mathbb{R}^d .

Theorem 6.11 (Kolmogorov–Riesz–Weil). Let $p \in [1, \infty[$. A set $\mathcal{S} \subset L^p(0, T; \mathbb{R}^d)$ is relatively compact in $L^p(0, T; \mathbb{R}^d)$ if and only if:

(j) (p -equi-integrability)

$$\lim_{\varepsilon \searrow 0} \left[\sup_{g \in \mathcal{S}} \int_{\varepsilon}^{T-\varepsilon} \|g(t + \varepsilon) - g(t)\|_{\mathbb{R}^d}^p dt \right] = 0,$$

(jj) (boundedness):

$$\sup_{g \in \mathcal{S}} \int_0^T |g(t)| dt < \infty.$$

(For the proofs of these two last theorems see as example the book of Vrabie [70].)

Clearly we have:

Corollary 6.12. Let $M > 0$ and $\gamma_n \searrow 0, \varepsilon_n \searrow 0$ be two sequences.

a) Then the set

$$\mathcal{K}_1 = \left\{ z \in L^2(0, T; \mathbb{R}^d) : \int_0^T |z(t)|^2 dt \leq M, \right. \\ \left. \sup_{0 \leq \theta \leq \varepsilon_n} \int_0^{T-\varepsilon_n} |z(t + \theta) - z(t)|^2 dt \leq \gamma_n, \forall n \in \mathbb{N}^* \right\}$$

is a compact subset of $L^2(0, T; \mathbb{R}^d)$.

b) If $N_n = \left\lceil \frac{T}{\varepsilon_n} \right\rceil$ and $t_i = \frac{(i-1)T}{N_n}$, for $1 \leq i \leq N_n, n \geq 1$, then the set

$$\mathcal{K}_2 = \left\{ z \in C([0, T]; \mathbb{R}^d) : |z(0)| \leq M, \right. \\ \left. \sup_{1 \leq i \leq N_n} \sup_{0 < \theta \leq \varepsilon_n} |z(t_i + \theta) - z(t_i)| \leq \gamma_n, \forall n \in \mathbb{N}^* \right\}$$

is a compact subset of $C([0, T]; \mathbb{R}^d)$ (here z_t is extended outside of $[0, T]$ by continuity $z_s = z_T$, for $s \geq T$ and $z_s = z_0$, for $s \leq 0$).

6.3.5 Bounded Variation Functions

Let $[a, b]$ be a closed interval from \mathbb{R} and $\mathcal{D}_{[a,b]}$ be the set of all partitions

$$\Delta : a = t_0 < t_1 < \cdots < t_n = b, \quad n = n_\Delta \in \mathbb{N}^*.$$

Define $\|\Delta\| = \sup \{t_{i+1} - t_i : 0 \leq i \leq n - 1\}$.

Let

$$V_\Delta(k) \stackrel{\text{def.}}{=} \sum_{i=0}^{n-1} |k(t_{i+1}) - k(t_i)|$$

be the variation of k corresponding to the partition $\Delta \in \mathcal{D}_{[a,b]}$. We define the total variation of k on $[a, b]$ by

$$\begin{aligned} \Downarrow k \Downarrow_{[a,b]} &= \sup_{\Delta \in \mathcal{D}_{[a,b]}} V_\Delta(k) \\ &= \sup \left\{ \sum_{i=0}^{n_\Delta-1} |k(t_{i+1}) - k(t_i)| : \Delta \in \mathcal{D}_{[a,b]} \right\} \end{aligned}$$

and if $[a, b] = [0, T]$ then

$$\Downarrow k \Downarrow_T := \Downarrow k \Downarrow_{[0,T]}.$$

Proposition 6.13. *If $k \in C([0, T]; \mathbb{R}^d)$ and $\overline{\Delta}_N \in \mathcal{D}_{[0,T]}$*

$$\overline{\Delta}_N : \left\{ 0 = \frac{0}{2^N} T < \frac{1}{2^N} T < \cdots < \frac{2^N - 1}{2^N} T < \frac{2^N}{2^N} T = T \right\},$$

then

$$V_{\overline{\Delta}_N}(k) \nearrow \Downarrow k \Downarrow_T \quad \text{as } N \nearrow \infty.$$

Proof. Clearly $V_{\overline{\Delta}_N}(k)$ is increasing with respect to N and $V_{\overline{\Delta}_N}(k) \leq \Downarrow k \Downarrow_T$.

Let $\Delta \in \mathcal{D}_{[a,b]}$ be arbitrary

$$\Delta : 0 = t_0 < t_1 < \cdots < t_{n_\Delta} = T,$$

and $j_i = \left[\frac{t_i}{T} 2^N \right]$ be the integer part of $\frac{t_i}{T} 2^N$. Then

$$V_\Delta(k) = \sum_{i=1}^{n_\Delta-1} |k(t_{i+1}) - k(t_i)|$$

$$\begin{aligned} &\leq \sum_{i=1}^{n_{\Delta}-1} \left[\left| k(t_{i+1}) - k\left(\frac{j_{i+1}T}{2^N}\right) \right| + \left| k\left(\frac{j_{i+1}T}{2^N}\right) - k\left(\frac{j_i T}{2^N}\right) \right| \right. \\ &\quad \left. + \left| k\left(\frac{j_i T}{2^N}\right) - k(t_i) \right| \right] \\ &\leq 2n_{\Delta} \mathbf{m}_k \left(\frac{T}{2^N}\right) + V_{\overline{\Delta}_N}(k) \end{aligned}$$

and passing to the limit for $N \nearrow \infty$ we obtain

$$V_{\Delta}(k) \leq \lim_{N \nearrow \infty} V_{\overline{\Delta}_N}(k) \leq \Downarrow k \Downarrow_T, \quad \forall \Delta \in \mathcal{D}_{[a,b]}.$$

Hence $\lim_{N \nearrow \infty} V_{\overline{\Delta}_N}(k) = \Downarrow k \Downarrow_T$. ■

Definition 6.14. A function $k : [a, b] \rightarrow \mathbb{R}^d$ has bounded variation on $[a, b]$ if $\Downarrow k \Downarrow_{[a,b]} < \infty$. The space of bounded variation functions on $[a, b]$ will be denoted by $BV([a, b]; \mathbb{R}^d)$.

If $x \in C([a, b]; \mathbb{R}^d)$ and $k \in BV([a, b]; \mathbb{R}^d)$ then the Riemann–Stieltjes integral is defined by

$$\int_a^b \langle x(t), dk(t) \rangle = \lim_{\|\Delta\| \rightarrow 0} \sum_{i=0}^{n_{\Delta}-1} \langle x(\tau_i), k(t_{i+1}) - k(t_i) \rangle,$$

where the integral is independent of the arbitrary choice of $\tau_i \in [t_i, t_{i+1}]$.

The Riemann–Stieltjes integral satisfies

$$\left| \int_a^b \langle x(t), dk(t) \rangle \right| \leq \|x\|_{[a,b]} \Downarrow k \Downarrow_{[a,b]}.$$

Proposition 6.15. *Equipped with the norm*

$$\|k\|_{BV([a,b]; \mathbb{R}^d)} := |k(a)| + \Downarrow k \Downarrow_{[a,b]},$$

the space $BV([a, b]; \mathbb{R}^d)$ is a Banach space. An element k of $BV([a, b]; \mathbb{R}^d)$ can be identified with the following linear continuous mapping on $C([a, b]; \mathbb{R}^d)$:

$$x \mapsto \langle x(a), k(a) \rangle + \int_a^b \langle x(t), dk(t) \rangle.$$

With this identification, $BV([a, b]; \mathbb{R}^d)$ is the dual of the space $C([a, b]; \mathbb{R}^d)$.

Proposition 6.16 (Helly–Bray). *Let $n \in \mathbb{N}^*$, $x_n, x \in C([0, T]; \mathbb{R}^d)$, $k_n \in BV([0, T]; \mathbb{R}^d)$, $k : [0, T] \rightarrow \mathbb{R}^d$, such that*

- (i) $x_n \rightarrow x$ in $C([0, T]; \mathbb{R}^d)$,
- (ii) $k_n(t) \rightarrow k(t)$, $\forall t \in [0, T]$ and
- (iii) $\sup_{n \in \mathbb{N}^*} \downarrow k_n \downarrow_T = M < +\infty$.

Then $k \in BV([0, T]; \mathbb{R}^d)$, $\downarrow k \downarrow_T \leq M$, and $\forall 0 \leq s \leq t \leq T$:

- (j) $\int_s^t \langle x_n(r), dk_n(r) \rangle \rightarrow \int_s^t \langle x(r), dk(r) \rangle$, as $n \rightarrow \infty$,
- (jj) $\int_s^t |x(r)| d \downarrow k \downarrow_r \leq \liminf_{n \rightarrow +\infty} \int_s^t |x_n(r)| d \downarrow k_n \downarrow_r$.

In particular $k_n \xrightarrow{w^*} k$ in $BV([0, T]; \mathbb{R}^d)$, that is for all $y \in C([0, T]; \mathbb{R}^d)$:

$$\int_0^T \langle y(t), dk_n(t) \rangle \rightarrow \int_0^T \langle y(t), dk(t) \rangle.$$

Proof. First let $\Delta_N \in \mathcal{D}_{[0, T]}$ be a sequence such that

$$V_{\Delta_N}(k) \nearrow \downarrow k \downarrow_T \quad \text{as } N \nearrow \infty.$$

From the definition of $\downarrow \cdot \downarrow_T$ we have

$$V_{\Delta_N}(k_n) \leq \downarrow k_n \downarrow_T \leq M.$$

Since $k_n(t) \rightarrow k(t)$ for all $t \in [0, T]$, it follows that $V_{\Delta_N}(k_n) \rightarrow V_{\Delta_N}(k)$. Hence

$$V_{\Delta_N}(k) \leq M \quad \text{for all } N \in \mathbb{N}^*$$

and passing to the limit as $N \nearrow \infty$ we obtain

$$\downarrow k \downarrow_T \leq M.$$

Let $\varepsilon > 0$

$$\Delta : s = t_0 < t_1 < \cdots < t_N = t, \quad N = N_\Delta \in \mathbb{N}^*,$$

with $t_i \in [0, T]$, $\|\Delta\| = \sup\{t_{i+1} - t_i : 0 \leq i \leq N-1\} \leq \varepsilon$. For $x_i = x(t_i)$, $k_i = k(t_i)$, define

$$S_\Delta(x, k) = \sum_{i=0}^{N-1} \langle x_i, k_{i+1} - k_i \rangle$$

and $\mathbf{m}_x : [0, \infty[\rightarrow [0, \infty[$

$$\mathbf{m}_x(\varepsilon) = \sup \{|x(r) - x(s)| : |r - s| \leq \varepsilon, r, s \in [0, T]\}$$

the modulus of continuity of x on $[0, T]$.

We have

$$\left| \int_s^t \langle x(r), dk(r) \rangle - S_\Delta(x, k) \right| \leq \mathbf{m}_x(\varepsilon) \Downarrow k \Downarrow_T. \quad (6.14)$$

Indeed

$$\begin{aligned} \left| \int_s^t \langle x(r), dk(r) \rangle - S_\Delta(x, k) \right| &= \left| \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \langle x(r), dk(r) \rangle - \int_{t_i}^{t_{i+1}} \langle x_i, dk(r) \rangle \right| \\ &\leq \sum_{i=0}^{N-1} \left| \int_{t_i}^{t_{i+1}} \langle x(r) - x_i, dk(r) \rangle \right| \\ &\leq \mathbf{m}_x(\|\Delta\|) \sum_{i=0}^{N-1} \Downarrow k \Downarrow_{[t_i, t_{i+1}]} \\ &\leq \mathbf{m}_x(\varepsilon) \Downarrow k \Downarrow_T. \end{aligned}$$

Then

$$\begin{aligned} \left| \int_s^t \langle x(r), d(k_n(r) - k(r)) \rangle - S_\Delta(x, k_n - k) \right| &\leq \mathbf{m}_x(\|\Delta\|) \Downarrow k_n - k \Downarrow_T \\ &\leq \mathbf{m}_x(\varepsilon) [\Downarrow k_n \Downarrow_T + \Downarrow k \Downarrow_T]. \end{aligned}$$

Now we obtain the estimate

$$\begin{aligned} &\left| \int_s^t \langle x_n(r), dk_n(r) \rangle - \int_s^t \langle x(r), dk(r) \rangle \right| \\ &= \left| \int_s^t \langle x_n(r) - x(r), dk_n(r) \rangle + \int_s^t \langle x(r), dk_n(r) - dk(r) \rangle \right| \\ &\leq \|x_n - x\|_T \Downarrow k_n \Downarrow_T + \left| \int_s^t \langle x(r), dk_n(r) - dk(r) \rangle \right| \\ &\leq \|x_n - x\|_T \Downarrow k_n \Downarrow_T + \mathbf{m}_x(\varepsilon) [\Downarrow k_n \Downarrow_T + \Downarrow k \Downarrow_T] + |S_\Delta(x, k_n - k)|. \end{aligned}$$

Since $k_n(t) \rightarrow k(t)$ for all $t \in [0, T]$, it follows that $\lim_{n \rightarrow \infty} |S_\Delta(x, k_n - k)| = 0$ and

$$\limsup_{n \rightarrow \infty} \left| \int_s^t \langle x_n(r), dk_n(r) \rangle - \int_s^t \langle x(r), dk(r) \rangle \right| \leq 2M \mathbf{m}_x(\varepsilon), \quad \forall \varepsilon > 0.$$

Hence the limit $\lim_{n \rightarrow \infty} \int_s^t \langle x_n(r), dk_n(r) \rangle$ exists, as does

$$\lim_{n \rightarrow \infty} \int_s^t \langle x_n(r), dk_n(r) \rangle = \int_s^t \langle x(r), dk(r) \rangle.$$

Now, let $\alpha \in C([0, T]; \mathbb{R}^d)$, $\|\alpha\|_T \leq 1$. Then

$$\begin{aligned} \int_s^t |x(r)| \langle \alpha(r), dk(r) \rangle &= \lim_{n \rightarrow \infty} \int_s^t |x_n(r)| \langle \alpha(r), dk_n(r) \rangle \\ &\leq \liminf_{n \rightarrow +\infty} \int_s^t |x_n(r)| d \downarrow k_n \uparrow_r \end{aligned}$$

and passing to $\sup_{\|\alpha\|_T \leq 1}$ we obtain

$$\int_s^t |x(r)| d \downarrow k \uparrow_r \leq \liminf_{n \rightarrow +\infty} \int_s^t |x_n(r)| d \downarrow k_n \uparrow_r.$$

■

We now give some other auxiliary results used in the book:

Proposition 6.17. *Let $A : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ be a maximal monotone operator and $\mathcal{A} : C(\mathbb{R}_+; \mathbb{R}^d) \rightrightarrows BV_{loc}(\mathbb{R}_+; \mathbb{R}^d)$ be defined by:*

$$(x, k) \in \mathcal{A} \quad \text{if} \quad x \in C(\mathbb{R}_+; \overline{D(A)}), \quad k \in BV_{loc}(\mathbb{R}_+; \mathbb{R}^d) \quad \text{and}$$

$$\int_s^t \langle x(r) - z, dk(r) - \hat{z}dr \rangle \geq 0, \quad \forall (z, \hat{z}) \in A, \quad \forall 0 \leq s \leq t. \quad (6.15)$$

Then the relation (6.15) is equivalent to: for all $u, \hat{u} \in C(\mathbb{R}_+; \mathbb{R}^d)$ such that $(u(r), \hat{u}(r)) \in A, \forall r \geq 0$

$$\int_s^t \langle x(r) - u(r), dk(r) - \hat{u}(r)dr \rangle \geq 0, \quad \forall 0 \leq s \leq t, \quad (6.16)$$

and \mathcal{A} is a monotone operator, that is:

for all $(x, k), (y, \ell) \in \mathcal{A}$

$$\int_s^t \langle x(r) - y(r), dk(r) - d\ell(r) \rangle \geq 0, \quad \forall 0 \leq s \leq t.$$

Moreover \mathcal{A} is a maximal monotone operator.

Proof. (6.15) \implies (6.16):

Let $\forall u, \hat{u} \in C(\mathbb{R}_+; \mathbb{R}^d)$ be such that $(u(r), \hat{u}(r)) \in A, \forall r \geq 0$. Then

$$\begin{aligned} & \int_s^t \langle x(r) - u(r), dk(r) - \hat{u}(r) dr \rangle \\ &= \lim_{n \rightarrow \infty} \int_s^t \left\langle x(r) - u\left(\frac{\lfloor nr \rfloor}{n}\right), dk(r) - \hat{u}\left(\frac{\lfloor nr \rfloor}{n}\right) dr \right\rangle \geq 0. \end{aligned}$$

(6.16) \implies (6.15):

The implication is obtained for $u(r) = z$ and $\hat{u}(r) = \hat{z}$.

Let $(x, k), (y, \ell) \in \mathcal{A}$ be arbitrary. Then for all $u, \hat{u} \in C(\mathbb{R}_+; \mathbb{R}^d)$ such that $(u(r), \hat{u}(r)) \in A, \forall r \geq 0$ we have for all $0 \leq s \leq t$,

$$\begin{aligned} & \int_s^t \langle y(r) - u(r), d\ell(r) - \hat{u}(r) dr \rangle \geq 0, \\ & \int_s^t \langle x(r) - u(r), dk(r) - \hat{u}(r) dr \rangle \geq 0. \end{aligned}$$

We put here

$$u(r) = J_\varepsilon \left(\frac{x(r) + y(r)}{2} \right) = \frac{x(r) + y(r)}{2} - \varepsilon A_\varepsilon \left(\frac{x(r) + y(r)}{2} \right)$$

and

$$\hat{u}(r) = A_\varepsilon \left(\frac{x(r) + y(r)}{2} \right),$$

where $J_\varepsilon(z) = (I + \varepsilon A)^{-1}(z)$, $A_\varepsilon(z) = \frac{1}{\varepsilon}(z - J_\varepsilon(z))$. Since A is a maximal operator on \mathbb{R}^d it follows that $\overline{D(A)}$ is convex and $\lim_{\varepsilon \rightarrow 0} \varepsilon A_\varepsilon(u) \rightarrow 0, \forall u \in \overline{D(A)}$. Also for all $a \in D(A)$

$$\varepsilon |A_\varepsilon(u)| \leq \varepsilon |A_\varepsilon(u) - A_\varepsilon(a)| + \varepsilon |A_\varepsilon(a)| \leq |u - a| + \varepsilon |A^0(a)|.$$

Adding the inequalities term by term we obtain:

$$0 \leq \frac{1}{2} \int_s^t \langle y(r) - x(r), d\ell(r) - dk(r) \rangle + \varepsilon \int_s^t \left\langle A_\varepsilon \left(\frac{x(r) + y(r)}{2} \right), d\ell(r) + dk(r) \right\rangle.$$

Passing to $\lim_{\varepsilon \searrow 0}$ we obtain $\int_s^t \langle y(r) - x(r), d\ell(r) - dk(r) \rangle \geq 0$. \mathcal{A} is a maximal monotone operator since if $(y, \ell) \in C(\mathbb{R}_+; \overline{D(A)}) \times BV_{loc}(\mathbb{R}_+; \mathbb{R}^d)$ satisfies

$$\int_s^t \langle y(r) - x(r), d\ell(r) - dk(r) \rangle \geq 0, \quad \forall (x, k) \in \mathcal{A},$$

then this last inequality is satisfied for all (x, k) of the form $(x(t), k(t)) = (z, \hat{z}t)$, where $(z, \hat{z}) \in A$, and consequently (from the definition of \mathcal{A}) $(y, \ell) \in \mathcal{A}$. The proof is complete. ■

Remark 6.18. Often we restrict the realization to

$$C(\mathbb{R}_+; \mathbb{R}^d) \times \left[C(\mathbb{R}_+; \mathbb{R}^d) \cap BV_{0,loc}(\mathbb{R}_+; \mathbb{R}^d) \right]$$

and we write (for this case) $dk(t) \in A(x(t))(dt)$ if

- (a₁) $x \in C(\mathbb{R}_+; \overline{\text{Dom}(A)})$,
- (a₂) $k \in C(\mathbb{R}_+; \mathbb{R}^d) \cap BV_{loc}(\mathbb{R}_+; \mathbb{R}^d)$, $k(0) = 0$,
- (a₃) $\langle x(t) - u, dk(t) - \hat{u}dt \rangle \geq 0$, on \mathbb{R}_+ , $\forall (u, \hat{u}) \in A$.

Proposition 6.19. *Let $A \subset \mathbb{R}^d \times \mathbb{R}^d$ be a maximal subset and \mathcal{A} be the realization of A on $C(\mathbb{R}_+; \mathbb{R}^d) \times BV_{loc}(\mathbb{R}_+; \mathbb{R}^d)$ defined by (6.15). Assume that $\text{int}(\text{Dom}(A)) \neq \emptyset$. Let $u_0 \in \text{int}(\text{Dom}(A))$ and $r_0 > 0$ be such that $\bar{B}(u_0, r_0) \subset \text{Dom}(A)$. Then*

$$A_{u_0, r_0}^\# \stackrel{\text{def}}{=} \sup \{ |\hat{u}| : \hat{u} \in Au, u \in \bar{B}(u_0, r_0) \} < \infty,$$

and for all $(x, k) \in \mathcal{A}$:

$$r_0 d \downarrow k \downarrow_t \leq \langle x(t) - u_0, dk(t) \rangle + (A_{u_0, r_0}^\# |x(t) - u_0| + r_0 A_{u_0, r_0}^\#) dt \quad (6.17)$$

as signed measures on \mathbb{R}_+ . Moreover there exists a constant $b_0 > 0$ such that

$$\begin{aligned} r_0 \int_s^t |A_\varepsilon y(r)| dr &\leq \int_s^t \langle y(r) - u_0, A_\varepsilon y(r) \rangle dr \\ &\quad + A_{u_0, r_0}^\# \int_s^t |y(r) - u_0| dr + b_0(t - s), \end{aligned} \quad (6.18)$$

for all $0 \leq s \leq t \leq T$, $y \in C(\mathbb{R}_+; \mathbb{R}^d)$ and $0 < \varepsilon \leq 1$.

Proof. Since A is locally bounded on $\text{int}(\text{Dom}(A))$, it follows that for $u_0 \in \text{int}(\text{Dom}(A))$, there exists an $r_0 > 0$ such that $u_0 + r_0 v \in \text{int}(\text{Dom}(A))$ for all $|v| \leq 1$ and

$$A_{u_0, r_0}^\# \stackrel{\text{def}}{=} \sup \{ |\hat{z}| : \hat{z} \in Az, z \in \bar{B}(u_0, r_0) \} < \infty.$$

Let $0 \leq s = t_0 < t_1 < \dots < t_n = t \leq T$, $\max_i (t_{i+1} - t_i) = \delta_n \rightarrow 0$.

We put in (6.15) $z = u_0 + r_0 v$. Then

$$\int_{t_i}^{t_{i+1}} \langle x(r) - (u_0 + r_0 v), dk(r) - \hat{z} dr \rangle \geq 0, \quad \forall |v| \leq 1, \quad \forall 0 \leq s \leq t \leq T,$$

and we obtain

$$\begin{aligned} & r_0 \langle k(t_{i+1}) - k(t_i), v \rangle \\ & \leq \int_{t_i}^{t_{i+1}} \langle x(r) - u_0, dk(r) \rangle + A_{u_0, r_0}^\# \int_{t_i}^{t_{i+1}} |x(r) - u_0| dr + r_0 A_{u_0, r_0}^\# (t_{i+1} - t_i), \end{aligned}$$

for all $|v| \leq 1$. Hence

$$\begin{aligned} & r_0 |k(t_{i+1}) - k(t_i)| \\ & \leq \int_{t_i}^{t_{i+1}} \langle x(r) - u_0, dk(r) \rangle + A_{u_0, r_0}^\# \int_{t_i}^{t_{i+1}} |x(r) - u_0| dr + r_0 A_{u_0, r_0}^\# (t_{i+1} - t_i) \end{aligned}$$

and adding term by term for $i = 0$ to $i = n - 1$ the inequality

$$\begin{aligned} r_0 \sum_{i=0}^{n-1} |k(t_{i+1}) - k(t_i)| & \leq \int_s^t \langle x(t) - u_0, dk(t) \rangle \\ & \quad + A_{u_0, r_0}^\# \int_s^t |x(r) - u_0| dr + (t - s) r_0 A_{u_0, r_0}^\#, \end{aligned}$$

holds and clearly (6.17) follows.

Setting in (6.3) $x = y(r)$, $x_0 = u_0$ and integrating from s to t the inequality (6.18) follows. ■

Often in the book we use some energy type equalities that we describe in the next lemma.

Lemma 6.20. *Let $x, k, m \in C([0, \infty[; \mathbb{R}^d)$, $k \in BV_{loc}([0, \infty[; \mathbb{R}^d)$, $k(0) = m(0) = 0$ such that*

$$x(t) + k(t) = x_0 + m(t), \quad \forall t \geq 0.$$

Then

(I): For all $t \geq 0$ and for all $u \in \mathbb{R}^d$:

$$\begin{aligned} & |x(t) - m(t) - u|^2 + 2 \int_0^t \langle x(r) - u, dk(r) \rangle \\ & = |x_0 - u|^2 + 2 \int_0^t \langle m(r), dk(r) \rangle. \end{aligned} \tag{6.19}$$

(II): For all $0 \leq s \leq t$:

$$\begin{aligned} |x(t) - x(s) - m(t) + m(s)|^2 + 2 \int_s^t \langle x(r) - x(s), dk(r) \rangle \\ = 2 \int_s^t \langle m(r) - m(s), dk(r) \rangle. \end{aligned} \quad (6.20)$$

Proof. (I): We have

$$\begin{aligned} |x(t) - m(t) - u|^2 \\ = |x_0 - k(t) - u|^2 \\ = |x_0 - u|^2 + 2 \int_0^t \langle x_0 - k(r) - u, d(x_0 - k - u)(r) \rangle \\ = |x_0 - u|^2 + 2 \int_0^t \langle x(r) - m(r) - u, -dk(r) \rangle \\ = |x_0 - u|^2 + 2 \int_0^t \langle m(r), dk(r) \rangle - 2 \int_0^t \langle x(r) - u, dk(r) \rangle, \end{aligned}$$

that is (6.19).

(II): From (6.19) we have for $u = 0$

$$\begin{aligned} |x(t) - m(t)|^2 - |x(s) - m(s)|^2 + 2 \int_s^t \langle x(r), dk(r) \rangle \\ = 2 \int_s^t \langle m(r), dk(r) \rangle. \end{aligned}$$

But $k(t) - k(s) = m(t) - x(t) - m(s) + x(s)$,

$$\begin{aligned} |x(t) - m(t)|^2 = |x(t) - x(s) - m(t) + m(s)|^2 + |x(s) - m(s)|^2 \\ - 2 \langle x(s) - m(s), k(t) - k(s) \rangle \end{aligned}$$

and

$$\begin{aligned} 2 \int_s^t \langle m(r), dk(r) \rangle &= 2 \int_s^t \langle m(r) - m(s), dk(r) \rangle + 2 \langle m(s), k(t) - k(s) \rangle, \\ 2 \int_s^t \langle x(r), dk(r) \rangle &= 2 \int_s^t \langle x(r) - x(s), dk(r) \rangle + 2 \langle x(s), k(t) - k(s) \rangle. \end{aligned}$$

Hence, the equality (6.20) holds. ■

Finally we give an approximation result via Stieltjes integrals.

Lemma 6.21. *Let*

- $Q : [0, T] \rightarrow \mathbb{R}$ *be a strictly increasing continuous function such that* $Q(0) = 0$,
- $f, \gamma : [0, T] \rightarrow \mathbb{R}^d$ *be bounded measurable functions,*
- $\varphi : \mathbb{R}^d \rightarrow]-\infty, +\infty]$ *be a proper convex lower semicontinuous function.*

If

$$f_\varepsilon(t) = f(0) e^{\frac{-Q(t)}{Q(\varepsilon)}} + \frac{1}{Q(\varepsilon)} \int_0^t e^{\frac{Q(r)-Q(t)}{Q(\varepsilon)}} f(r) dQ(r), \quad t \in [0, T], \varepsilon > 0$$

then as $\varepsilon \rightarrow 0_+$

- (j) $f_\varepsilon(r) \rightarrow f(r)$, *a.e.* $r \in [0, T]$,
- (jj) $\int_t^s \varphi(f_\varepsilon(r)) \gamma(r) dQ(r) \rightarrow \int_t^s \varphi(f(r)) \gamma(r) dQ(r)$, $\forall [t, s] \subset [0, T]$.

If $f : [0, T] \rightarrow \mathbb{R}^d$ *is a continuous function it moreover follows that*

$$\sup_{t \in [0, T]} |f_\varepsilon(t) - f(t)| \rightarrow 0.$$

Remark 6.22. The same conclusions are true if we replace $f_\varepsilon(t)$ by

$$g_\varepsilon(t) = f(T) e^{\frac{Q(t)-Q(T)}{Q(\varepsilon)}} + \frac{1}{Q(\varepsilon)} \int_t^T e^{\frac{Q(t)-Q(r)}{Q(\varepsilon)}} f(r) dQ(r), \quad t \in [0, T].$$

Proof of Lemma 6.21 (j). Obviously we have

$$\begin{aligned} \int_0^t \frac{1}{Q(\varepsilon)} e^{\frac{Q(r)-Q(t)}{Q(\varepsilon)}} f(r) dQ(r) &= \int_{\frac{-Q(t)}{Q(\varepsilon)}}^0 e^u f((Q^{-1}(uQ(\varepsilon) + Q(t)))) du \\ &= \int_{\frac{-Q(t)}{Q(\varepsilon)}}^0 e^u [f(Q^{-1}(uQ(\varepsilon) + Q(t))) - f(Q^{-1}(Q(t)))] du + f(t) \int_{\frac{-Q(t)}{Q(\varepsilon)}}^0 e^u du. \end{aligned} \tag{6.21}$$

But

$$\begin{aligned} &\limsup_{\varepsilon \rightarrow 0} \left| \int_{\frac{-Q(t)}{Q(\varepsilon)}}^0 e^u [f(Q^{-1}(uQ(\varepsilon) + Q(t))) - f(Q^{-1}(Q(t)))] du \right| \\ &\leq \limsup_{\varepsilon \rightarrow 0} \int_{-\infty}^0 e^u |f(Q^{-1}((uQ(\varepsilon) + Q(t)) \vee 0)) - f(Q^{-1}(Q(t)))| du \\ &\leq 2C \int_{-\infty}^{-n} e^u du + \int_{-n}^0 e^u |f(Q^{-1}((uQ(\varepsilon) + Q(t)) \vee 0)) - f(Q^{-1}(Q(t)))| du \\ &\leq 2C e^{-n} + \limsup_{\varepsilon \rightarrow 0} \int_{-n}^0 |f(Q^{-1}((uQ(\varepsilon) + Q(t)) \vee 0)) - f(Q^{-1}(Q(t)))| du \\ &\leq 2C e^{-n}, \text{ for all } n, \end{aligned}$$

since

$$\lim_{\delta \rightarrow 0} \int_{\alpha}^{\beta} |f(Q^{-1}(s + \delta u)) - f(Q^{-1}(s))| du = 0, \text{ a.e.}$$

Therefore the following limit exists

$$\lim_{\varepsilon \rightarrow 0} \left| \int_{\frac{-Q(t)}{Q(\varepsilon)}}^0 e^u [f(Q^{-1}(uQ(\varepsilon) + Q(t))) - f(t)] du \right| = 0,$$

and (j) follows. In the case where f is continuous, it is sufficient to write

$$\begin{aligned} f_{\varepsilon}(t) &= f(0) e^{-\frac{Q(t)}{Q(\varepsilon)}} + \frac{1}{Q(\varepsilon)} \int_0^t e^{\frac{Q(r)-Q(t)}{Q(\varepsilon)}} f(r) dQ(r) \\ &= f(0) e^{-\frac{Q(t)}{Q(\varepsilon)}} + \frac{1}{Q(\varepsilon)} \int_0^{t_{\varepsilon}} e^{\frac{Q(r)-Q(t)}{Q(\varepsilon)}} f(r) dQ(r) + \frac{1}{Q(\varepsilon)} \int_{t_{\varepsilon}}^t e^{\frac{Q(r)-Q(t)}{Q(\varepsilon)}} f(r) dQ(r), \end{aligned}$$

where $t_{\varepsilon} := Q^{-1}(Q(t) - \sqrt{Q(\varepsilon)}) \rightarrow t$, as $\varepsilon \rightarrow 0$, and $t_{\varepsilon} < t$.

(jj) We have

$$\begin{aligned} & \int_t^s \varphi(f_{\varepsilon}(r)) \gamma(r) dQ(r) \\ & \leq \int_t^s e^{-\frac{Q(r)}{Q(\varepsilon)}} \varphi(f(0)) \gamma(r) dQ(r) \\ & \quad + \int_t^s \left(\int_0^r \frac{1}{Q(\varepsilon)} e^{\frac{Q(u)-Q(r)}{Q(\varepsilon)}} \varphi(f(u)) dQ(u) \right) \gamma(r) dQ(r) \\ & = \varphi(f(0)) \int_t^s e^{-\frac{Q(r)}{Q(\varepsilon)}} \gamma(r) dQ(r) \\ & \quad + \int_0^s \left(\int_0^s \frac{1}{Q(\varepsilon)} e^{\frac{Q(u)-Q(r)}{Q(\varepsilon)}} \varphi(f(u)) \mathbf{1}_{[0,r]}(u) dQ(u) \right) \gamma(r) dQ(r) \\ & \quad - \int_0^t \left(\int_0^t \frac{1}{Q(\varepsilon)} e^{\frac{Q(u)-Q(r)}{Q(\varepsilon)}} \varphi(f(u)) \mathbf{1}_{[0,r]}(u) dQ(u) \right) \gamma(r) dQ(r) \\ & = \varphi(f(0)) \int_t^s e^{-\frac{Q(r)}{Q(\varepsilon)}} \gamma(r) dQ(r) \\ & \quad + \int_0^s \left(\varphi(f(u)) \int_0^s \frac{1}{Q(\varepsilon)} e^{\frac{Q(u)-Q(r)}{Q(\varepsilon)}} \mathbf{1}_{[u,s]}(r) \gamma(r) dQ(r) \right) dQ(u) \\ & \quad - \int_0^t \left(\varphi(f(u)) \int_0^t \frac{1}{Q(\varepsilon)} e^{\frac{Q(u)-Q(r)}{Q(\varepsilon)}} \mathbf{1}_{[u,t]}(r) \gamma(r) dQ(r) \right) dQ(u). \end{aligned}$$

Using Remark 6.22 we have

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_u^s \frac{1}{Q(\varepsilon)} e^{\frac{Q(u)-Q(r)}{Q(\varepsilon)}} \gamma(r) dQ(r) &= \lim_{\varepsilon \rightarrow 0} \int_u^t \frac{1}{Q(\varepsilon)} e^{\frac{Q(u)-Q(r)}{Q(\varepsilon)}} \gamma(r) dQ(r). \\ &= \gamma(u), \text{ a.e.} \end{aligned}$$

By Lebesgue’s dominated convergence theorem and the lower semicontinuity of φ we conclude that

$$\begin{aligned} \int_t^s \varphi(f(r)) \gamma(r) dQ(r) &\leq \liminf_{\varepsilon \rightarrow 0} \int_t^s \varphi(f_\varepsilon(r)) \gamma(r) dQ(r) \\ &\leq \limsup_{\varepsilon \rightarrow 0} \int_t^s \varphi(f_\varepsilon(r)) \gamma(r) dQ(r) \\ &\leq \int_t^s \varphi(f(r)) \gamma(r) dQ(r). \end{aligned}$$

■

6.3.6 Semicontinuity

Let (\mathbb{X}, ρ) be a metric space.

Definition 6.23. A function $f : \mathbb{X} \rightarrow \overline{\mathbb{R}}$ is lower semicontinuous (l.s.c.) at $x \in \mathbb{X}$ if

$$f(x) \leq \liminf_{y \rightarrow x} f(y),$$

i.e. for all $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon, x) > 0$ such that $\rho(x, y) < \delta$ implies $f(y) \geq f(x) - \varepsilon$. The function f is l.s.c. if it is l.s.c. at all $x \in \mathbb{X}$.

A function $g : \mathbb{X} \rightarrow \overline{\mathbb{R}}$ is upper semicontinuous (u.s.c.) if $-g$ is l.s.c.

Proposition 6.24. *The following assertions are equivalent:*

- (i) $f : \mathbb{X} \rightarrow \overline{\mathbb{R}}$ is lower semicontinuous;
- (ii) the set $\{x \in \mathbb{X} : f(x) \leq a\}$ is closed in \mathbb{X} , for all $a \in \mathbb{R}$.

It is easy to prove that:

- ▲ If $g_n : \mathbb{X} \rightarrow \overline{\mathbb{R}}, n \in \mathbb{N}$, are l.s.c. functions and

$$g(x) = \sup\{g_n(x) : n \in \mathbb{N}\},$$

then $g : \mathbb{X} \rightarrow \overline{\mathbb{R}}$ is a l.s.c. function.

- ▲ If $f : \mathbb{X} \rightarrow]-\infty, +\infty]$ is a l.s.c. function, then f is bounded from below on compact subsets of \mathbb{X} .

Lemma 6.25. Let (\mathbb{X}, ρ) be a metric space. If $f : \mathbb{X} \rightarrow \overline{\mathbb{R}}$ is bounded below on bounded subsets of \mathbb{X} , then there exists a continuous function $\mu : \mathbb{X} \rightarrow \mathbb{R}$ such that

$$\mu(x) \leq f(x), \quad \text{for all } x \in \mathbb{X}.$$

Proof. Let $n \in \mathbb{N}^*$ and $a \in \mathbb{X}$. Define

$$\mu_n = n \wedge \inf\{f(x) : \rho(x, a) \leq n\}.$$

Then $\mu_n \in \mathbb{R}$. Define $\mu : \mathbb{X} \rightarrow \mathbb{R}$ such that, if $n - 1 \leq \rho(x, a) < n$

$$\mu(x) = \mu_n - 2 \left[\rho(x, a) - \left(n - \frac{1}{2} \right) \right]^+ (\mu_n - \mu_{n+1}).$$

The function μ is continuous on \mathbb{X} and

$$\mu(x) \leq f(x) \quad \text{for all } x \in \mathbb{X}.$$

■

Proposition 6.26. Let (\mathbb{X}, ρ) be a separable metric space. If $f : \mathbb{X} \rightarrow]-\infty, +\infty]$ is a l.s.c. function and $\mu : \mathbb{X} \rightarrow \mathbb{R}$ is a continuous function such that

$$\mu(x) \leq f(x), \quad \text{for all } x \in \mathbb{X}, \quad (6.22)$$

then there exists a sequence of continuous functions $f_n : \mathbb{X} \rightarrow \mathbb{R}$, $n \in \mathbb{N}^*$, such that for all $x \in \mathbb{X}$

$$\mu(x) \leq f_1(x) \leq \dots \leq f_n(x) \leq \dots \leq f(x) \quad \text{and} \quad \lim_{n \rightarrow \infty} f_n(x) = f(x).$$

Proof. Using only the boundedness from below (6.22) we shall show that there exists a sequence of continuous functions $f_n : \mathbb{X} \rightarrow \mathbb{R}$, $n \in \mathbb{N}^*$, such that

$$\mu(y) \leq f_1(y) \leq \dots \leq f_n(y) \leq \dots \leq f(y), \quad \text{for all } y \in \mathbb{X}, \quad (6.23)$$

and such that for all $x \in \mathbb{X}$ there exists a sequence $y_n \rightarrow x$ satisfying

$$n \wedge \left(f(y_n) - \frac{1}{n} \right) \leq \sup_{j \in \mathbb{N}^*} f_j(x) \leq f(x), \quad \text{for all } n \in \mathbb{N}^*. \quad (6.24)$$

Then the result follows using the lower semicontinuity of f :

$$f(x) \leq \liminf_{n \rightarrow +\infty} \left[n \wedge \left(f(y_n) - \frac{1}{n} \right) \right] \leq \sup_{j \in \mathbb{N}^*} f_j(x) \leq f(x).$$

Let us prove (6.23) and (6.24). Let $n, i \in \mathbb{N}^*$ and $a \in \mathbb{X}$. Define

$$\mu_{i,n} = \mu_{i,n}(a) = n \wedge \inf \left\{ f(y) : \rho(y, a) < \frac{i}{n} \right\}.$$

Then $-\infty \leq \mu_{i,n} < +\infty$. Define $\psi_n(\cdot, a) : \mathbb{X} \rightarrow \mathbb{R}$ such that, if $\frac{i-1}{n} \leq \rho(y, a) < \frac{i}{n}$, $i \in \mathbb{N}^*$, then

$$\begin{aligned} \psi_n(y, a) &= \mu(y) \vee \mu_{i,n} - 2n \left[\rho(y, a) - \left(i - \frac{1}{2} \right) \frac{1}{n} \right]^+ \\ &\quad \times [(\mu(y) \vee \mu_{i,n}) - (\mu(y) \vee \mu_{i+1,n})]. \end{aligned}$$

For each $a \in \mathbb{X}$ the function $\psi_n(\cdot, a)$ is continuous on \mathbb{X} and

$$\psi_n(y, a) \leq f(y) \quad \text{for all } y \in \mathbb{X}.$$

Let $A_1 \subset A_2 \subset \dots \subset A_n \subset \dots$ be finite sets such that $A = \bigcup_{n \in \mathbb{N}^*} A_n$ is a dense subset of \mathbb{X} . Define $f_n : \mathbb{X} \rightarrow \mathbb{R}$

$$f_n(y) = \max_{k \in \mathbb{1}, n} \left[\max_{a \in A_n} \psi_k(y, a) \right], \quad y \in \mathbb{X}.$$

Clearly f_n , $n \in \mathbb{N}^*$, are continuous functions and

$$f_1(y) \leq \dots \leq f_n(y) \leq \dots \leq f(y), \quad \forall y \in \mathbb{X}.$$

Let $x \in \mathbb{X}$ be arbitrary. Then there exist $a_n \in A$ and $k_n \geq n$ such that

$$\rho(x, a_n) < \frac{1}{2n} \quad \text{and} \quad a_n \in A_{k_n}.$$

If $\mu_{1,n}(a_n) \in \mathbb{R}$, then from the definition of $\mu_{1,n}(a_n)$, there exists $y_n \in B(a_n, \frac{1}{n})$ such that

$$n \wedge \left(f(y_n) - \frac{1}{n} \right) \leq \mu_{1,n}(a_n) \leq \psi_n(x, a_n) \leq f_{k_n}(x) \leq \sup_{j \in \mathbb{N}^*} f_j(x) \leq f(x).$$

If $\mu_{1,n}(a_n) = -\infty$ then, once again from the definition of $\mu_{1,n}(a_n)$, there exists $y_n \in B(a_n, \frac{1}{n})$ such that

$$n \wedge \left(f(y_n) - \frac{1}{n} \right) \leq f_1(x) \leq \sup_{j \in \mathbb{N}^*} f_j(x) \leq f(x).$$

We remark that

$$\rho(y_n, x) \leq \rho(y_n, a_n) + \rho(a_n, x) \leq \frac{3}{2n}$$

and consequently $y_n \rightarrow x$. The proof is complete. \blacksquare

We also have the following:

Proposition 6.27. *If $f : \mathbb{X} \rightarrow \mathbb{R}$ is a continuous function and $f_n : \mathbb{X} \rightarrow \mathbb{R}$, $n \in \mathbb{N}^*$, are lower semicontinuous functions such that for all $x \in \mathbb{X}$:*

$$f_1(x) \leq f_2(x) \leq \dots \leq f_n(x) \leq \dots \leq f(x) \quad \text{and} \quad \lim_{n \rightarrow \infty} f_n(x) = f(x),$$

then for every compact set $K \subset \mathbb{X}$

$$\lim_{n \rightarrow \infty} \left[\sup_{x \in K} |f_n(x) - f(x)| \right] = 0. \quad (6.25)$$

Proof. For each $\varepsilon > 0$, $G_n = \{x \in \mathbb{X} : f(x) - f_n(x) < \varepsilon\}$ is an open subset of \mathbb{X} and

$$K \subset \mathbb{X} = \bigcup_{n \in \mathbb{N}^*} \uparrow G_n.$$

Hence, by the compactness of K , there exists an $n \in \mathbb{N}^*$ such that $K \subset G_n$ and the uniform convergence (6.25) follows. \blacksquare

We now give some examples (as exercises for the reader) of lower semicontinuous functions that are used in the book.

Example 6.28. Let (\mathbb{X}, ρ) be a separable metric space and $E \subset \mathbb{X}$. Then E is a closed subset of \mathbb{X} if and only if the function

$$I_E(x) = \begin{cases} 0, & \text{if } x \in E, \\ +\infty, & \text{otherwise,} \end{cases}$$

is a l.s.c. function on \mathbb{X} .

Example 6.29. Let (\mathbb{X}, ρ) be a separable metric space. Let $0 \leq s < t \leq T$. If $f : \mathbb{X} \rightarrow]-\infty, +\infty]$ is a l.s.c. function bounded below on bounded subsets of \mathbb{X} and $\Phi : C([0, T]; \mathbb{X}) \rightarrow]-\infty, +\infty]$ is defined by

$$\Phi(x) = \begin{cases} \int_s^t f(x(r)) dr, & \text{if } f(x) \in L^1(0, T) \\ +\infty, & \text{otherwise} \end{cases}$$

then Φ is a l.s.c. function.

Let $0 \leq s < t \leq T$. Let $\mathcal{D}_{[s,t]}$ be the set of partitions $\Delta : s = r_0 < r_1 < \dots < r_n = t$, and

$$V_{\Delta}(k) \stackrel{\text{def.}}{=} \sum_{i=0}^{n-1} \rho(k(r_i), k(r_{i+1})).$$

Define the total variation of k on $[s, t]$ by

$$\uparrow k \downarrow_{[s,t]} = \sup_{\Delta \in \mathcal{D}_{[a,b]}} V_{\Delta}(k).$$

Then as a sup of continuous functions:

Example 6.30. The mapping $k \mapsto \uparrow k \downarrow_{[s,t]} : C([0, T]; \mathbb{X}) \rightarrow [0, \infty]$ is a l.s.c. function.

Finally we present Ekeland’s principle (see [26], or [4], p. 29, Th. 3.2):

Lemma 6.31 (Ekeland). *Let (\mathbb{X}, ρ) be a complete metric space and $J : \mathbb{X} \rightarrow]-\infty, +\infty]$ be a proper lower-semicontinuous function bounded from below. Then for any $\varepsilon > 0$ there exists an $x_{\varepsilon} \in \mathbb{X}$ such that:*

$$\begin{cases} J(x_{\varepsilon}) \leq \inf_{x \in \mathbb{X}} J(x) + \varepsilon & \text{and} \\ J(x_{\varepsilon}) < J(x) + \sqrt{\varepsilon} \rho(x_{\varepsilon}, x), & \forall x \in \mathbb{X} \setminus \{x_{\varepsilon}\}. \end{cases}$$

6.3.7 Convex Functions

6.3.7.1 Definitions: Properties

Let $(\mathbb{X}, \|\cdot\|)$ be a real Banach space and $(\mathbb{X}^*, \|\cdot\|_*)$ its dual. A function $\varphi : \mathbb{X} \rightarrow]-\infty, +\infty]$ is convex if

$$\varphi((1 - \lambda)x + \lambda y) \leq (1 - \lambda)\varphi(x) + \lambda\varphi(y), \text{ for all } x, y \in \mathbb{X} \text{ and } \lambda \in]0, 1[.$$

Denote by

$$\text{Dom}(\varphi) = \{x \in \mathbb{X} : \varphi(x) < +\infty\}$$

the domain of φ and

$$\partial\varphi(x) = \{\hat{x} \in \mathbb{X}^* : \langle \hat{x}, z - x \rangle + \varphi(x) \leq \varphi(z), \forall z \in \mathbb{X}\}$$

the subdifferential of the function φ at x . We say that φ is proper if $\text{Dom}(\varphi) \neq \emptyset$. Clearly if φ is a convex function then $\text{Dom}(\varphi)$ is a convex subset of \mathbb{X} .

Theorem 6.32. *If \mathbb{X} is a Banach space and $\varphi : \mathbb{X} \rightarrow]-\infty, +\infty]$ is a proper convex l.s.c. function then*

$$\text{Dom}(\partial\varphi) \stackrel{\text{def}}{=} \{x \in \mathbb{X} : \partial\varphi(x) \neq \emptyset\}$$

is non-empty and $\partial\varphi : \mathbb{X} \rightrightarrows \mathbb{X}^$ is a maximal monotone operator.*

If K is a convex subset of \mathbb{X} then the function $I_K : \mathbb{X} \rightarrow]-\infty, +\infty]$ defined by

$$I_K(x) = \begin{cases} 0, & \text{if } x \in K, \\ +\infty, & \text{if } x \in \mathbb{X} \setminus K, \end{cases}$$

is a convex function called the *convex indicator function of K* .

Recall, from [71] Chapter 2, the following:

Proposition 6.33. *Let $g : \mathbb{R} \rightarrow]-\infty, +\infty]$ be a convex function. Then:*

- (a) *Dom(g) is an interval in \mathbb{R} ;*
 (b) *the left derivative $g'_- : \text{Dom}(g) \rightarrow]-\infty, +\infty]$ and the right derivative $g'_+ : \text{Dom}(g) \rightarrow]-\infty, +\infty]$ are well defined increasing functions and they satisfy:*

- (j) $g'_+(r) \leq \frac{g(s) - g(r)}{s - r} \leq g'_-(s), \quad \forall r, s \in \text{Dom}(g), r < s;$
 (jj) $g'_-(r) \leq g'_+(r), \quad \forall r \in \text{Dom}(g);$
 (jjj) g'_- *is left continuous and g'_+ is right continuous on $\text{int}(\text{Dom}(g))$;*
 (jv) $u \in [g'_-(r), g'_+(r)] \cap \mathbb{R} \iff u(s - r) \leq g(s) - g(r), \quad \forall s \in \mathbb{R};$
 (v) $\{r \in \text{Dom}(g) : g'_-(r) \neq g'_+(r)\}$ *is at most countable;*

- (c) *g is locally Lipschitz continuous on $\text{int}(\text{Dom}(g))$;*
 (d) *$A \subset \mathbb{R} \times \mathbb{R}$ is a maximal monotone operator if and only if there exists a convex function $j : \mathbb{R} \rightarrow]-\infty, +\infty]$ such that $\beta = \partial j$.*

Note that if φ is a proper convex lower semicontinuous (l.s.c.) function then:

- φ is bounded from below by an affine function, that is $\exists v \in \mathbb{X}^*$ and $a \in \mathbb{R}$ such that

$$\varphi(x) \geq \langle v, x \rangle + a, \quad \text{for all } x \in \mathbb{X},$$

and, moreover, if \mathbb{X} is reflexive and $\lim_{\|x\| \rightarrow \infty} \varphi(x) = +\infty$ then there exists an $x_0 \in \text{Dom}(\varphi)$ such that

$$\varphi(x) \geq \varphi(x_0), \quad \text{for all } x \in \mathbb{X};$$

- (Fenchel–Moreau theorem on biconjugate functions)

$$\varphi(x) = \varphi^{**}(x) = \sup \{ \langle x, x^* \rangle - \varphi^*(x^*) : x^* \in \mathbb{X}^* \},$$

where $\varphi^* : \mathbb{X}^* \rightarrow \bar{\mathbb{R}}$ is the conjugate of the function φ , i.e.

$$\varphi^*(x^*) = \sup \{ \langle u, x^* \rangle - \varphi(u) : u \in \text{Dom}(\varphi) \};$$

- φ is continuous on $\text{int}(\text{Dom}(\varphi))$;
- $\partial\varphi : \mathbb{X} \rightrightarrows \mathbb{X}^*$ is maximal monotone; $\overline{\text{int}(\text{Dom}(\varphi))} = \overline{\text{Dom}(\partial\varphi)}$;
- $\text{int}(\text{Dom}(\varphi)) = \text{int}(\text{Dom}(\partial\varphi))$ and $\text{Dom}(\partial\varphi) = \overline{\text{Dom}(\varphi)}$.

We have the following instance of Jensen’s inequality.

Lemma 6.34. *Let $\varphi : \mathbb{R}^d \rightarrow]-\infty, +\infty]$ be a proper convex lower semicontinuous function. If $a, b \in \mathbb{R}$, $a < b$, $y \in L^\infty(a, b; \mathbb{R}^d)$ and $\rho \in L^1(a, b; \mathbb{R}_+)$ such that $\int_a^b \rho(r) dr = 1$, then*

$$\varphi \left(\int_a^b \rho(r) y(r) dr \right) \leq \int_a^b \rho(r) \varphi(y(r)) dr.$$

Proof. Since there exists a set $\Gamma \subset \mathbb{R}^d \times \mathbb{R}$ such that

$$\varphi(x) = \sup \{ \langle v, x \rangle + \gamma : (v, \gamma) \in \Gamma \},$$

we have

$$\langle v, \int_a^b \rho(r) y(r) dr \rangle + \gamma = \int_a^b \rho(r) [\langle v, y(r) \rangle + \gamma] dr \leq \int_a^b \rho(r) \varphi(y(r)) dr$$

and the result follows passing to $\sup_{(v,\gamma) \in \Gamma}$. ■

6.3.7.2 Regularization of Convex Functions

Let $(\mathbb{H}, |\cdot|)$ be a real separable Hilbert space and $\varphi : \mathbb{H} \rightarrow]-\infty, +\infty]$ be a proper convex l.s.c. function. The Moreau regularization φ_ε of the convex l.s.c. function φ is defined by

$$\varphi_\varepsilon(x) = \inf \left\{ \frac{1}{2\varepsilon} |z - x|^2 + \varphi(z); z \in \mathbb{H} \right\}, \quad \varepsilon > 0.$$

The function φ_ε is a convex function of class C^1 on \mathbb{H} ; the gradient $\nabla\varphi_\varepsilon$ is a Lipschitz function on \mathbb{H} with the Lipschitz constant equal to ε^{-1} . If we define:

$$J_\varepsilon x = x - \varepsilon \nabla \varphi_\varepsilon(x),$$

then one can easily prove (see e.g. Brezis [12], Barbu [2], Rockafellar [65] or Zălinescu [71]) that for all $x \in \mathbb{H}$ and $\varepsilon > 0$:

1. $\varphi_\varepsilon(x) = \frac{\varepsilon}{2} |\nabla \varphi_\varepsilon(x)|^2 + \varphi(J_\varepsilon x)$,
2. $\varphi(J_\varepsilon x) \leq \varphi_\varepsilon(x) \leq \varphi(x)$,
3. $\nabla \varphi_\varepsilon(x) = \partial \varphi_\varepsilon(x)$ and

$$\begin{aligned} \varphi(J_\varepsilon x) &\leq \varphi_\varepsilon(x) \\ &\leq \varphi_\varepsilon(z) + \langle x - z, \nabla \varphi_\varepsilon(x) \rangle \\ &\leq \varphi(z) + \langle x - z, \nabla \varphi_\varepsilon(x) \rangle, \quad \forall z \in \mathbb{H}, \end{aligned}$$

4. $\nabla \varphi_\varepsilon(x) \in \partial \varphi(J_\varepsilon x)$ i.e.

$$\langle \nabla \varphi_\varepsilon(x), z - J_\varepsilon x \rangle + \varphi(J_\varepsilon x) \leq \varphi(z), \quad \forall z \in \mathbb{H}.$$

Hence $J_\varepsilon x = (I + \varepsilon \partial \varphi)^{-1}(x)$ and $\nabla \varphi_\varepsilon(x) = A_\varepsilon(x)$, where A is the maximal monotone operator $\partial \varphi$; $\nabla \varphi_\varepsilon$ is called the Moreau–Yosida approximation of $\partial \varphi$.

5. If $(u_0, \hat{u}_0) \in \partial \varphi$, then for all $y \in \mathbb{H}$

$$\left\{ \begin{array}{l} (a) \quad |\nabla \varphi_\varepsilon(u_0)| \leq |\hat{u}_0|, \\ (b) \quad 0 \leq \varphi(u_0) - \varphi_\varepsilon(u_0) \leq \varphi(u_0) - \varphi(J_\varepsilon u_0) \leq \varepsilon |\hat{u}_0|^2, \\ (c) \quad |J_\varepsilon(y)| \leq |y - u_0| + \varepsilon |\hat{u}_0| + |u_0|, \\ (d) \quad \varphi(J_\varepsilon y) \geq \varphi(u_0) - |\hat{u}_0| |y - u_0| - \varepsilon |\hat{u}_0|^2, \\ (e) \quad \frac{\varepsilon}{2} |\nabla \varphi_\varepsilon(y)|^2 \leq \varphi_\varepsilon(y) - \varphi(u_0) + |\hat{u}_0| |y - u_0| + \varepsilon |\hat{u}_0|^2. \end{array} \right. \quad (6.26)$$

Indeed $|\nabla \varphi_\varepsilon(u_0)| = |A_\varepsilon(u_0)| \leq |A^0(x)|$ and

$$\begin{aligned} -\varepsilon |\hat{u}_0|^2 &\leq -\varepsilon \langle \hat{u}_0, \nabla \varphi_\varepsilon(u_0) \rangle \\ &= \langle \hat{u}_0, J_\varepsilon u_0 - u_0 \rangle \\ &\leq \varphi(J_\varepsilon u_0) - \varphi(u_0) \\ &\leq \varphi_\varepsilon(u_0) - \varphi(u_0) \\ &\leq 0. \end{aligned}$$

For the inequality (c) we have

$$|J_\varepsilon(y)| \leq |J_\varepsilon(y) - J_\varepsilon(u_0)| + |J_\varepsilon(u_0) - u_0| + |u_0|,$$

and therefore

$$\begin{aligned} \varphi(J_\varepsilon y) &\geq \varphi(u_0) + \langle \hat{u}_0, J_\varepsilon(y) - u_0 \rangle \\ &\geq \varphi(u_0) - |\hat{u}_0| |J_\varepsilon(y) - J_\varepsilon(u_0)| - |\hat{u}_0| |J_\varepsilon(u_0) - u_0| \end{aligned}$$

which yields (d).

For the last inequality, (d), we have

$$\begin{aligned} \frac{\varepsilon}{2} |\nabla \varphi_\varepsilon(y)|^2 &= \varphi_\varepsilon(y) - \varphi(J_\varepsilon y) \\ &\leq \varphi_\varepsilon(y) - \varphi(u_0) + |\hat{u}_0| |y - u_0| + \varepsilon |\hat{u}_0|^2. \end{aligned}$$

6. If $0 = \varphi(0) \leq \varphi(x), \forall x \in \mathbb{H}$, it is easy to verify that, moreover

$$\begin{aligned} j) \quad &0 \in \partial\varphi(0), \quad 0 = \varphi_\varepsilon(0) \leq \varphi_\varepsilon(x), \quad J_\varepsilon(0) = \nabla\varphi_\varepsilon(0) = 0, \\ jj) \quad &\frac{\varepsilon}{2} |\nabla\varphi_\varepsilon(x)|^2 \leq \varphi_\varepsilon(x) \leq \langle \nabla\varphi_\varepsilon(x), x \rangle, \quad \forall x \in \mathbb{H}, \\ jjj) \quad &|\nabla\varphi_\varepsilon(x)| \leq \frac{1}{\varepsilon} |x|, \quad \text{and } 0 \leq \varphi_\varepsilon(x) \leq \frac{1}{2\varepsilon} |x|^2, \quad \forall x \in \mathbb{H}, \\ jv) \quad &\langle \nabla\varphi_\varepsilon(x), x - y \rangle \geq -\varphi(J_\varepsilon x) - \varepsilon \langle \nabla\varphi_\varepsilon(x), \nabla\varphi_\varepsilon(y) \rangle, \quad \forall x, y \in \mathbb{H}. \end{aligned} \tag{6.27}$$

If for a fixed $a \geq 0$

$$\langle \hat{x} - \hat{y}, x - y \rangle \geq a|x - y|^2, \quad \forall (x, \hat{x}), (y, \hat{y}) \in \partial\varphi,$$

or equivalently the function

$$\psi(x) = \varphi(x) - \frac{a}{2}|x|^2$$

is convex, too, then by the definition of J_ε and the monotonicity of the operator $\partial\varphi$ we have $\forall r \in]0, 1[$:

$$\begin{aligned} &a \left[(1-r)|x-y|^2 - \frac{1-r}{r} |\varepsilon \nabla\varphi_\varepsilon(x) - \delta \nabla\varphi_\delta(y)|^2 \right] \\ &\leq a |J_\varepsilon x - J_\delta y|^2 \\ &\leq \langle \nabla\varphi_\varepsilon(x) - \nabla\varphi_\delta(y), J_\varepsilon x - J_\delta y \rangle \\ &= \langle \nabla\varphi_\varepsilon(x) - \nabla\varphi_\delta(y), x - y \rangle - \varepsilon |\nabla\varphi_\varepsilon(x)|^2 - \delta |\nabla\varphi_\delta(y)|^2 \\ &\quad + (\varepsilon + \delta) \langle \nabla\varphi_\varepsilon(x), \nabla\varphi_\delta(y) \rangle \end{aligned}$$

and then

$$\begin{aligned} a) \quad & \langle \nabla \varphi_\varepsilon(x) - \nabla \varphi_\varepsilon(y), x - y \rangle \geq a(1-r)|x-y|^2 \\ b) \quad & \langle \nabla \varphi_\varepsilon(x) - \nabla \varphi_\delta(y), x - y \rangle \geq a(1-r)|x-y|^2 \\ & - (\varepsilon + \delta) |\nabla \varphi_\varepsilon(x)| |\nabla \varphi_\delta(y)| \end{aligned} \quad (6.28)$$

for all $x, y \in \mathbb{H}$, $r \in (0, 1)$, $\varepsilon, \delta > 0$ such that $0 \leq a(1-r)\varepsilon \leq r$, $0 \leq a(1-r)\delta \leq r$.

Let $u_0 \in \mathbb{H}$ and $r_0 \geq 0$ be such that

$$\{u_0 + r_0 v : |v| \leq 1\} \subset \text{Dom} \varphi.$$

Note that if

$$\varphi_{u_0, r_0}^\# \left(\stackrel{\text{def}}{=} \sup \{ \varphi(u_0 + r_0 v) : |v| \leq 1 \} \right) < \infty,$$

then we have for all $(x, \hat{x}) \in \partial \varphi$

$$\begin{aligned} a) \quad & r_0 |\hat{x}| + \varphi(x) \leq \langle \hat{x}, x - u_0 \rangle + \varphi_{u_0, r_0}^\#, \quad \forall (x, \hat{x}) \in \partial \varphi, \\ b) \quad & r_0 |\hat{x}| + |\varphi(x) - \varphi(u_0)| \leq \langle \hat{x}, x - u_0 \rangle \\ & + 2 \left| (\partial \varphi)^0(u_0) \right| |x - u_0| + \varphi_{u_0, r_0}^\# - \varphi(u_0) \end{aligned} \quad (6.29)$$

and in particular for $r_0 = 0$

$$|\varphi(x) - \varphi(u_0)| \leq \langle \hat{x}, x - u_0 \rangle + 2 \left| (\partial \varphi)^0(u_0) \right| |x - u_0|. \quad (6.30)$$

Let us prove (6.29). For $(x, \hat{x}) \in \partial \varphi$ and $|v| \leq 1$ we have

$$\langle \hat{x}, u_0 + r_0 v - x \rangle + \varphi(x) \leq \varphi(u_0 + r_0 v) \leq \varphi_{u_0, r_0}^\#$$

and consequently

$$r_0 \langle \hat{x}, v \rangle + \varphi(x) \leq \langle \hat{x}, x - u_0 \rangle + \varphi_{u_0, r_0}^\#$$

which yields (6.29-a) taking the $\sup_{|v| \leq 1}$.

On the other hand for all arbitrary $\hat{u}_0 \in \partial \varphi(u_0)$,

$$\langle \hat{u}_0, x - u_0 \rangle + \varphi(u_0) \leq \varphi(x),$$

which yields

$$|\varphi(x) - \varphi(u_0)| \leq \varphi(x) - \varphi(u_0) + 2 |\hat{u}_0| |x - u_0|.$$

Hence for all $|v| \leq 1$:

$$\begin{aligned} r_0 \langle \hat{x}, v \rangle + |\varphi(x) - \varphi(u_0)| &\leq \langle \hat{x}, x - u_0 \rangle + 2|\hat{u}_0| |x - u_0| \\ &\quad + \varphi_{u_0, r_0}^\# - \varphi(u_0) \end{aligned}$$

which yields (6.29-b).

Observing that $\nabla\varphi_\varepsilon(x) \in \partial\varphi(J_\varepsilon x)$, we have

$$\begin{aligned} r_0 |\nabla\varphi_\varepsilon(x)| + |\varphi(J_\varepsilon x) - \varphi(u_0)| &\leq \langle \nabla\varphi_\varepsilon(x), J_\varepsilon x - u_0 \rangle + 2|\hat{u}_0| |J_\varepsilon x - u_0| \\ &\quad + \varphi_{u_0, r_0}^\# - \varphi(u_0). \end{aligned}$$

But

$$\langle \nabla\varphi_\varepsilon(x), J_\varepsilon x - u_0 \rangle = \langle \nabla\varphi_\varepsilon(x), x - u_0 - \varepsilon|\nabla\varphi_\varepsilon(x)|^2 \rangle$$

and

$$\begin{aligned} |J_\varepsilon x - u_0| &\leq |J_\varepsilon x - J_\varepsilon u_0| + |J_\varepsilon u_0 - u_0| \\ &\leq |x - u_0| + \varepsilon|\hat{u}_0|. \end{aligned}$$

Hence for all $\varepsilon \in]0, 1]$, $x \in \mathbb{H}$ and $\hat{u}_0 \in \partial\varphi(u_0)$:

$$\begin{aligned} r_0 |\nabla\varphi_\varepsilon(x)| + |\varphi(J_\varepsilon x) - \varphi(u_0)| + \varepsilon |\nabla\varphi_\varepsilon(x)|^2 \\ \leq \langle \nabla\varphi_\varepsilon(x), x - u_0 \rangle + 2|\hat{u}_0| |x - u_0| + \left[2|\hat{u}_0|^2 + \varphi_{u_0, r_0}^\# - \varphi(u_0) \right]. \end{aligned} \quad (6.31)$$

In particular for $u_0 = 0$ and $\hat{u}_0 = 0$ we obtain

◆ If $\varphi(x) \geq \varphi(0) = 0$, for all $x \in \mathbb{H}$ and

$$\varphi_{r_0}^\# = \sup \{ \varphi(r_0 v) : |v| \leq 1 \} < \infty,$$

then:

$$\begin{aligned} a) \quad r_0 |\hat{x}| + \varphi(x) &\leq \langle \hat{x}, x \rangle + \varphi_{r_0}^\#, \quad \forall (x, \hat{x}) \in \partial\varphi, \\ b) \quad r_0 |\nabla\varphi_\varepsilon(x)| + \varphi(J_\varepsilon x) + \varepsilon |\nabla\varphi_\varepsilon(x)|^2 &\leq \langle \nabla\varphi_\varepsilon(x), x \rangle + \varphi_{r_0}^\#, \\ &\quad \forall \varepsilon > 0, \forall x \in \mathbb{H}. \end{aligned} \quad (6.32)$$

6.3.7.3 Convex Functions on $C([0, T]; \mathbb{R}^d)$

Proposition 6.35. *If $\varphi : \mathbb{R}^d \rightarrow]-\infty, +\infty]$ is a proper convex l.s.c. function and $\Phi : C([0, T]; \mathbb{R}^d) \rightarrow]-\infty, +\infty]$,*

$$\Phi(x) = \begin{cases} \int_0^T \varphi(x(t))dt, & \text{if } \varphi(x) \in L^1(0, T), \\ +\infty, & \text{otherwise,} \end{cases} \tag{6.33}$$

then

c₁) Φ is a proper convex l.s.c. function,

c₂) $\partial\Phi(x) \stackrel{\text{def}}{=} \left\{ k \in BV([0, T]; \mathbb{R}^d) : \right.$

$$\left. \int_0^T \langle y(r) - x(r), dk(r) \rangle + \Phi(x) \leq \Phi(y), \quad \forall y \in C([0, T]; \mathbb{R}^d) \right\}$$

is a maximal monotone operator:

Proof. We shall prove only the maximal property of the operator $\partial\Phi$, since the other properties are immediate. Let $\mathbb{X} = C([0, T]; \mathbb{R}^d)$. Then the dual space is $\mathbb{X}^* = BV([0, T]; \mathbb{R}^d)$. Let

$$\langle k - \zeta, x - z \rangle \geq 0, \text{ for all } (z, \zeta) \in \partial\Phi. \tag{6.34}$$

The function $\psi(z) = \Phi(z) + \frac{1}{2} \|z - x\|_{\mathbb{X}}^2 - \langle k, z \rangle$ defined on \mathbb{X} is a proper convex l.s.c. function. Furthermore, there exists a $c \in \mathbb{R}$ such that $\Phi(z) \geq c, \forall z \in \mathbb{X}$. By Ekeland's principle there exists a $z_\varepsilon \in \mathbb{X}$ such that

$$\begin{aligned} \psi(z_\varepsilon) &\rightarrow \inf \{ \psi(z) : z \in \mathbb{X} \}, \\ \psi(z_\varepsilon) &\leq \psi(z) + \sqrt{\varepsilon} \|z - z_\varepsilon\|_{\mathbb{X}} = \tilde{\psi}(z), \quad \forall z \in \mathbb{X}. \end{aligned}$$

Then $0 \in \partial\tilde{\psi}(z_\varepsilon)$, which means

$$\partial\Phi(z_\varepsilon) + F(z_\varepsilon - x) - k + \sqrt{\varepsilon}\theta_\varepsilon \ni 0, \tag{6.35}$$

where $F : \mathbb{X} \rightrightarrows \mathbb{X}^*$ is the duality mapping and $\|\theta_\varepsilon\|_{\mathbb{X}^*} \leq 1$. Multiplying by $z_\varepsilon - x$ we have $\langle \zeta_\varepsilon - k, z_\varepsilon - x \rangle + \|z_\varepsilon - x\|_{\mathbb{X}}^2 + \sqrt{\varepsilon} \langle \theta_\varepsilon, z_\varepsilon - x \rangle = 0$, for some $\zeta_\varepsilon \in \partial\Phi(z_\varepsilon)$, which implies by (6.34) $\|z_\varepsilon - x\|_{\mathbb{X}} \leq \sqrt{\varepsilon}$. Hence $z_\varepsilon \xrightarrow{\mathbb{X}} x$, and by (6.35) $\zeta_\varepsilon \xrightarrow{\mathbb{X}^*} k$, as $\varepsilon \rightarrow 0$. From the definition of the subdifferential operator: $\langle \zeta_\varepsilon, y - z_\varepsilon \rangle + \Phi(z_\varepsilon) \leq \Phi(y), \forall y \in \mathbb{X}$ and passing to the limit as $\varepsilon \rightarrow 0$ we obtain $(x, k) \in \partial\Phi$. ■

Proposition 6.36. *If $\varphi : \mathbb{R}^d \rightarrow]-\infty, +\infty]$ is a proper convex l.s.c. function, Φ is defined by (6.33), $x \in C([0, T]; \mathbb{R}^d)$ and $k \in C([0, T]; \mathbb{R}^d) \cap BV([0, T]; \mathbb{R}^d)$, then the following assertions are equivalent:*

$$\begin{aligned}
 a_1) \quad & \int_s^t \langle z - x(r), dk(r) \rangle + \int_s^t \varphi(x(r))dr \leq (t - s)\varphi(z), \\
 & \qquad \qquad \qquad \forall z \in \mathbb{R}^d, \forall 0 \leq s \leq t \leq T, \\
 a_2) \quad & \int_s^t \langle y(r) - x(r), dk(r) \rangle + \int_s^t \varphi(x(r))dr \leq \int_s^t \varphi(y(r))dr, \\
 & \qquad \qquad \qquad \forall y \in C([0, T]; \mathbb{R}^d), \forall 0 \leq s \leq t \leq T, \\
 a_3) \quad & \int_s^t \langle x(r) - z, dk(r) - \hat{z}dr \rangle \geq 0, \quad \forall (z, \hat{z}) \in \partial\varphi, \\
 & \qquad \qquad \qquad \forall 0 \leq s \leq t \leq T, \\
 a_4) \quad & \int_s^t \langle x(r) - y(r), dk(r) - \hat{y}(r)dr \rangle \geq 0, \quad \forall y, \hat{y} \in C([0, T]; \mathbb{R}^d), \\
 & \qquad \qquad \qquad (y(r), \hat{y}(r)) \in \partial\varphi, \quad \forall r \in [0, T], \quad \forall 0 \leq s \leq t \leq T, \\
 a_5) \quad & (x, k) \in \partial\Phi, \text{ that is, } \forall y \in C([0, T], \mathbb{R}^d) : \\
 & \qquad \int_0^T \langle y(r) - x(r), dk(r) \rangle + \int_0^T \varphi(x(r))dr \leq \int_0^T \varphi(y(r))dr.
 \end{aligned} \tag{6.36}$$

Proof. We shall show that $a_1 \Leftrightarrow a_2 \Rightarrow a_3 \Rightarrow a_4 \Rightarrow a_5 \Rightarrow a_2$.

$a_2 \Rightarrow a_1$: is evident.

$a_1 \Rightarrow a_2$: Let $y \in C([0, T]; \mathbb{R}^d)$. We extend $y(t) = y(0)$ for $t \leq 0$ and $y(t) = y(T)$ for $t \geq T$. The same extension will be considered for the functions x and k .

To prove a_2) it is sufficient to consider the case $0 < s < t < T$.

Since φ is bounded from below by an affine function, from a_1) we deduce that $\varphi(x) \in L^1(0, T)$.

Let $n_0 \in \mathbb{N}^*$ be such that $0 < \frac{1}{n_0} < s < t < t + \frac{1}{n_0} < T$ and $n \geq n_0$. Let $u \in [s, t]$. From (a_1) we have for $z = y(u)$

$$\int_{u-1/n}^u \langle y(u) - x(r), dk(r) \rangle + \int_{u-1/n}^u \varphi(x(r)) dr \leq \frac{1}{n} \varphi(y(u)).$$

Integrating on $[s, t]$ with respect to u we deduce that

$$\begin{aligned}
 \int_s^t \left(n \int_{u-1/n}^u \langle y(u) - x(r), dk(r) \rangle \right) du + \int_s^t \left(n \int_{u-1/n}^u \varphi(x(r)) dr \right) du \\
 \leq \int_s^t \varphi(y(u)) du.
 \end{aligned} \tag{6.37}$$

By Fatou's Lemma we have

$$\int_s^t \varphi(x(u)) du \leq \liminf_{n \rightarrow +\infty} \int_s^t \left(n \int_{u-1/n}^u \varphi(x(r)) dr \right) du.$$

On the other hand by the Lebesgue dominated convergence theorem

$$\begin{aligned}
 & \int_s^t \left(n \int_{u-1/n}^u \langle y(u) - x(r), dk(r) \rangle \right) du \\
 &= \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} n \mathbf{1}_{[s,t]}(u) \mathbf{1}_{[u-1/n,u]}(r) \langle y(u) - x(r), dk(r) \rangle \right) du \\
 &= \int_{-\infty}^{+\infty} \left\langle \int_{-\infty}^{+\infty} n \mathbf{1}_{[s,t]}(u) \mathbf{1}_{[r,r+1/n]}(u) [y(u) - x(r)] du, dk(r) \right\rangle \\
 &= \int_{-\infty}^{+\infty} \left\langle n \int_r^{r+1/n} \mathbf{1}_{[s,t]}(u) [y(u) - x(r)] du, dk(r) \right\rangle \\
 &\rightarrow \int_{-\infty}^{+\infty} \mathbf{1}_{[s,t]}(r) \langle y(r) - x(r), dk(r) \rangle, \quad \text{as } n \rightarrow \infty.
 \end{aligned}$$

Passing to $\liminf_{n \rightarrow +\infty}$ in (6.37) (a_2) follows.

$a_2 \Rightarrow a_3$: is obtained by adding the following inequalities term by term:

$$\begin{aligned}
 \int_s^t \langle z - x(r), dk(r) \rangle + \int_s^t \varphi(x(r)) dr &\leq \int_s^t \varphi(z) dr \\
 \int_s^t \langle x(r) - z, \hat{z} \rangle dr + \int_s^t \varphi(z) dr &\leq \int_s^t \varphi(x(r)) dr.
 \end{aligned}$$

$a_3 \Rightarrow a_4$: is proved in Proposition 6.17 since $A = \partial\varphi$ is maximal monotone.

$a_4 \Rightarrow a_5$: Let $(\tilde{x}, \tilde{k}) \in \partial\Phi$ be arbitrary. Hence for all $y, \hat{y} \in C([0, T]; \mathbb{R}^d)$, $(y(r), \hat{y}(r)) \in \partial\varphi$ we have: $(y, \int_0^T \hat{y} dt) \in \partial\Phi$ and

$$\begin{aligned}
 \int_0^T \langle \tilde{x}(r) - y(r), d\tilde{k}(r) - \hat{y}(r) dr \rangle &\geq 0 \quad (\partial\Phi \text{ is monotone}), \\
 \int_0^T \langle x(r) - y(r), dk(r) - \hat{y}(r) dr \rangle &\geq 0 \quad (\text{by } a_4).
 \end{aligned}$$

Since $A = \partial\varphi$ is maximal monotone, by Proposition 6.17 we have

$$\int_0^T \langle \tilde{x}(t) - x(t), d\tilde{k}(t) - dk(t) \rangle \geq 0,$$

where $(\tilde{x}, \tilde{k}) \in \partial\Phi$ is arbitrary. But by Proposition 6.35, $\partial\Phi$ is a maximal monotone operator. Hence $(x, k) \in \partial\Phi$.

$a_5 \Rightarrow a_2$: Let $a, b \geq 0$ such that $\varphi(y) + a|y| + b \geq 0$. From a_5) it follows that $\varphi(x) \in L^1(0, T)$. Let $\alpha_n \in C([0, T]; \mathbb{R})$, $0 \leq \alpha_n \leq 1$, and $\alpha_n \nearrow \mathbf{1}_{[s,t]}$. In a_5) we put $y(r) := (1 - \alpha_n(r))x(r) + \alpha_n(r)y$. So we have

$$\int_0^T \langle (y-x)\alpha_n, dk(r) \rangle + \int_0^T \varphi(x) dr \leq \int_0^T ((1-\alpha_n)\varphi(x) + \alpha_n\varphi(y)) dr$$

and furthermore

$$\begin{aligned} & \int_0^T \langle (y-x)\alpha_n, dk(r) \rangle + \int_0^T \alpha_n\varphi(x) dr \\ & \leq \int_0^T \alpha_n\varphi(y) dr \\ & \leq \int_0^T \alpha_n(\varphi(y) + a|y| + b) dr - \int_0^T \alpha_n(a|y| + b) dr \\ & \leq \int_0^T 1_{[s,t]}(r)(\varphi(y) + a|y| + b) dr - \int_0^T \alpha_n(a|y| + b) dr \\ & \leq \int_s^t \varphi(y) dr + \int_0^T (1_{[s,t]} - \alpha_n)(a|y| + b) dr. \end{aligned}$$

Passing to the limit as $n \rightarrow \infty$, $a_2)$ follows. ■

Proposition 6.37. *If $\varphi : \mathbb{R}^d \rightarrow]-\infty, +\infty]$ is a proper convex l.s.c. function and $\tilde{\Phi} : L^2(\Omega; C([0, T]; \mathbb{R}^d)) \rightarrow]-\infty, +\infty]$,*

$$\tilde{\Phi}(x) = \begin{cases} \mathbb{E} \int_0^T \varphi(x(t)) dt, & \text{if } \varphi(x) \in L^1(\Omega \times]0, T]), \\ +\infty, & \text{otherwise} \end{cases} \quad (6.38)$$

then

a) $\tilde{\Phi}$ is a proper convex l.s.c. function,

b) $\partial\tilde{\Phi}(x) \stackrel{\text{def}}{=} \left\{ K \in L^2(\Omega; BV([0, T]; \mathbb{R}^d)) : \mathbb{E} \int_0^T \langle Y_t - X_t, dK_t \rangle \right.$

$$\left. + \mathbb{E} \int_0^T \varphi(X_t) dt \leq \mathbb{E} \int_0^T \varphi(Y_t) dt, \forall Y \in L^2(\Omega; C([0, T]; \mathbb{R}^d)) \right\}$$

is a maximal monotone operator,

c) $K \in \partial\tilde{\Phi}(x)$ iff $K(\omega) \in \partial\Phi(X(\omega))$, \mathbb{P} -a.s. $\omega \in \Omega$, with $\partial\Phi$ characterized in Proposition 6.36.

Proof. The assertions a) and b) are obtained in the same manner as $c_1)$ and $c_2)$ from Proposition 6.35. The point c) follows from b) putting $Y := X1_{A^c} + Y1_A$, where $A \in \mathcal{F}$ is arbitrary. ■

Proposition 6.38. *Let $\varphi : \mathbb{R}^d \rightarrow]-\infty, +\infty]$ be a proper convex l.s.c. function such that $\text{int}(\text{Dom}(\varphi)) \neq \emptyset$. Let*

$\Phi: C([0, T]; \mathbb{R}^d) \rightarrow]-\infty, +\infty]$ be defined by (6.33). Let $(u_0, \hat{u}_0) \in \partial\varphi$, $r_0 \geq 0$ and

$$\varphi_{u_0, r_0}^\# \stackrel{\text{def}}{=} \sup \{ \varphi(u_0 + r_0 v) : |v| \leq 1 \}.$$

Then for all $0 \leq s \leq t \leq T$ and $(x, k) \in \partial\Phi$:

$$\begin{aligned} r_0 (\Downarrow k \Downarrow_t - \Downarrow k \Downarrow_s) + \int_s^t \varphi(x(r)) dr \\ \leq \int_s^t \langle x(r) - u_0, dk(r) \rangle + (t-s) \varphi_{u_0, r_0}^\#. \end{aligned} \quad (6.39)$$

Moreover for all $0 \leq s \leq t \leq T$ and for all $(x, k) \in \partial\Phi$:

$$\begin{aligned} r_0 (\Downarrow k \Downarrow_t - \Downarrow k \Downarrow_s) + \int_s^t |\varphi(x(r)) - \varphi(u_0)| dr \leq \int_s^t \langle x(r) - u_0, dk(r) \rangle \\ + \int_s^t (2|\hat{u}_0| |x(r) - u_0| + \varphi_{u_0, r_0}^\# - \varphi(u_0)) dr. \end{aligned} \quad (6.40)$$

Proof. Let $0 \leq s = t_0 < t_1 < \dots < t_n = t \leq T$, $\max_i (t_{i+1} - t_i) = \delta_n \rightarrow 0$. By (6.36-a₁) for $z = u_0 + r_0 v$. We obtain

$$\begin{aligned} r_0 \langle k(t_{i+1}) - k(t_i), v \rangle + \int_{t_i}^{t_{i+1}} \varphi(x(r)) dr \leq \int_{t_i}^{t_{i+1}} \langle x(r) - u_0, dk(r) \rangle \\ + (t_{i+1} - t_i) \varphi_{u_0, r_0}^\#, \end{aligned}$$

for all $|v| \leq 1$. Hence

$$\begin{aligned} r_0 |k(t_{i+1}) - k(t_i)| + \int_{t_i}^{t_{i+1}} \varphi(x(r)) dr \leq \int_{t_i}^{t_{i+1}} \langle x(r) - u_0, dk(r) \rangle \\ + (t_{i+1} - t_i) \varphi_{u_0, r_0}^\# \end{aligned}$$

and adding term by term for $i = 0$ to $i = n - 1$ we have

$$r_0 \sum_{i=0}^{n-1} |k(t_{i+1}) - k(t_i)| + \int_s^t \varphi(x(t)) dt \leq \int_s^t \langle x(t) - u_0, dk(t) \rangle + (t-s) \varphi_{u_0, r_0}^\#,$$

which clearly yields (6.39). The second inequality (6.40) now follows, using the fact that

$$|\varphi(x) - \varphi(u_0)| \leq \varphi(x) - \varphi(u_0) + 2|\hat{u}_0| |x - u_0|,$$

for all $x \in \mathbb{R}^d$ and $(u_0, \hat{u}_0) \in \partial\varphi$. ■

Remark 6.39. Since φ is locally bounded on $\text{int}(\text{Dom}\varphi)$, it follows that for

$$u_0 \in \text{int}(\text{Dom}\varphi) [= \text{int}(\text{Dom}(\partial\varphi))],$$

there exists $r_0 > 0$ and $M_0 \geq 0$, such that

$$\sup \{ |\varphi(u_0 + r_0 v)| : |v| \leq 1 \} \leq M_0.$$

6.3.8 Semiconvex Functions

Let $\varphi : \mathbb{R}^d \rightarrow]-\infty, +\infty]$.

Define

$$\text{Dom}(\varphi) = \{v \in \mathbb{R}^d : \varphi(v) < +\infty\}.$$

We say that φ is a proper function if $\text{Dom}(\varphi) \neq \emptyset$ and $\text{Dom}(\varphi)$ has no isolated points.

Definition 6.40. The (Fréchet) subdifferential of φ at $x \in \mathbb{R}^d$ is defined by

$$\partial^- \varphi(x) = \left\{ \hat{x} \in \mathbb{R}^d : \liminf_{y \rightarrow x} \frac{\varphi(y) - \varphi(x) - \langle \hat{x}, y - x \rangle}{|y - x|} \geq 0 \right\},$$

if $x \in \text{Dom}(\varphi)$, and $\partial^- \varphi(x) = \emptyset$, if $x \notin \text{Dom}(\varphi)$.

Example 6.41. If E is a non-empty closed subset of \mathbb{R}^d and

$$\varphi(x) = I_E(x) = \begin{cases} 0, & \text{if } x \in E, \\ +\infty, & \text{if } x \notin E, \end{cases}$$

then φ is l.s.c. and (by a result of Colombo and Goncharov [17] we have for any closed subset E of a Hilbert space)

$$\begin{aligned} \partial^- I_E(x) &= \{ \hat{x} \in \mathbb{R}^d : \limsup_{y \rightarrow x, y \in E} \frac{\langle \hat{x}, y - x \rangle}{|y - x|} \leq 0 \} \\ &= \begin{cases} 0, & \text{if } x \in \text{int}(E), \\ N_E(x), & \text{if } x \in \text{Bd}(E), \\ \emptyset, & \text{if } x \notin E, \end{cases} \end{aligned}$$

where $N_E(x)$ is the closed normal cone at E in $x \in \text{Bd}(E)$

$$N_E(x) \stackrel{\text{def}}{=} \left\{ u \in \mathbb{R}^d : \lim_{\varepsilon \searrow 0} \frac{d_E(x + \varepsilon u)}{\varepsilon} = |u| \right\}$$

and

$$d_E(z) \stackrel{\text{def}}{=} \inf \{|z - x| : x \in E\}$$

is the distance of a point $z \in \mathbb{R}^d$ to E .

Denote

- a) $\text{Dom}(\partial^- \varphi) = \{x \in \mathbb{R}^d : \partial^- \varphi(x) \neq \emptyset\}$,
- b) $\partial^- \varphi = \{(x, \hat{x}) : x \in \text{Dom}(\partial^- \varphi), \hat{x} \in \partial^- \varphi(x)\}$.

Definition 6.42. A closed set $E \subset \mathbb{R}^d$ is γ -semiconvex, $\gamma \geq 0$, if for all $x \in \text{Bd}(E)$ there exists an $\hat{x} \neq 0$ such that

$$\langle \hat{x}, y - x \rangle \leq \gamma |\hat{x}| |y - x|^2, \quad \text{for all } y \in E.$$

Note that if E is a semiconvex set, then

$$\partial^- I_E(x) = \{\hat{x} \in \mathbb{R}^d : \langle \hat{x}, y - x \rangle \leq \gamma |\hat{x}| |y - x|^2, \quad \text{for all } y \in E\}.$$

Definition 6.43. $\varphi : \mathbb{R}^d \rightarrow]-\infty, +\infty]$ is a semiconvex function if there exist $\rho, \gamma \geq 0$ such that

- (a) $\overline{\text{Dom}(\varphi)}$ is γ -semiconvex;
- (b) $\text{Dom}(\partial^- \varphi) \neq \emptyset$;
- (c) for all $y \in \mathbb{R}^d$ and for all $(x, \hat{x}) \in \partial^- \varphi$

$$\langle \hat{x}, y - x \rangle + \varphi(x) \leq \varphi(y) + (\rho + \gamma |\hat{x}|) |y - x|^2.$$

A function φ satisfying the properties of this definition will sometimes be called a (ρ, γ) -semiconvex function, or a γ -semiconvex function (since the second parameter is the most important one).

Note that a convex function is a (ρ, γ) -semiconvex function for all $\rho, \gamma \geq 0$.

A set E is γ -semiconvex iff I_E is a $(0, \gamma)$ -semiconvex function.

If we write the definition of semiconvexity for a fixed $(x_0, \hat{x}_0) \in \partial^- \varphi$, then it is clear that we have:

Proposition 6.44. *If $\varphi : \mathbb{R}^d \rightarrow]-\infty, +\infty]$ is a semiconvex function, then there exists an $a \geq 0$ such that*

$$\varphi(y) + a |y|^2 + a \geq 0, \quad \forall y \in \mathbb{R}^d.$$

In particular φ is bounded below on bounded subsets of \mathbb{R}^d .

The following properties also hold:

Proposition 6.45. *Let $\varphi : \mathbb{R}^d \rightarrow]-\infty, +\infty]$ be a semiconvex function. If there exist $u_0 \in \text{Dom}(\varphi)$, $r_0, M_0 > 0$ such that*

$$\varphi(u_0 + r_0 v) \leq M_0, \quad \forall |v| \leq 1,$$

then there exist $\rho_0 > 0$ and $b \geq 0$ such that

$$\rho_0 |\hat{x}| \leq \langle \hat{x}, x - u_0 \rangle + b + b(1 + |\hat{x}|) |x - u_0|^2, \quad \forall (x, \hat{x}) \in \partial^- \varphi \quad (6.41)$$

and moreover there exist $M \geq 0$ and $\delta_0 \in]0, r_0]$ such that

$$|\hat{x}| \leq M, \quad \forall x \in \bar{B}(u_0, \delta_0) \subset \text{Dom}(\varphi) \text{ and } \hat{x} \in \partial^- \varphi(x). \quad (6.42)$$

Proof. Let $(x, \hat{x}) \in \partial^- \varphi$. Then for all $|v| \leq 1$ and $\lambda \in [0, 1]$:

$$\langle \hat{x}, (u_0 + r_0 \lambda v) - x \rangle + \varphi(x) \leq \varphi(u_0 + r_0 \lambda v) + (\rho + \gamma |\hat{x}|) |(u_0 + r_0 \lambda v) - x|^2,$$

which yields

$$r_0 \lambda \langle \hat{x}, v \rangle \leq \langle \hat{x}, x - u_0 \rangle + (a |x|^2 + a) + M_0 + 2(\rho + \gamma |\hat{x}|) [|x - u_0|^2 + r_0^2 \lambda^2].$$

Taking the $\sup_{|v| \leq 1}$, we deduce for $\lambda = 1/(1 + 2\gamma r_0)$:

$$\frac{r_0}{(1 + 2\gamma r_0)^2} |\hat{x}| \leq \langle \hat{x}, x - u_0 \rangle + C + C(1 + |\hat{x}|) |x - u_0|^2$$

that is (6.41).

Moreover if $|x - u_0| \leq \delta_0 = 1 \wedge \frac{\rho_0}{2(1+b)} \wedge r_0$, then

$$\begin{aligned} \rho_0 |\hat{x}| &\leq \langle \hat{x}, x - u_0 \rangle + b + b(1 + |\hat{x}|) |x - u_0|^2 \\ &\leq (\delta_0 + b\delta_0^2) |\hat{x}| + b + b\delta_0^2 \\ &\leq \delta_0(1 + b) |\hat{x}| + 2b \\ &\leq \frac{\rho_0}{2} |\hat{x}| + 2b \end{aligned}$$

and (6.42) follows. ■

Let E be a non-empty closed subset of \mathbb{R}^d and $\varepsilon > 0$. We denote by

$$U_\varepsilon(E) \stackrel{\text{def}}{=} \{y \in \mathbb{R}^d : d_E(y) < \varepsilon\}$$

the open ε -neighbourhood of E and

$$\overline{U}_\varepsilon(E) \stackrel{\text{def}}{=} \{z \in \mathbb{R}^d : d_E(z) \leq \varepsilon\}$$

the closed ε -neighbourhood of E .

Given $z \in \mathbb{R}^d$, we denote by $\Pi_E(z)$ the set of elements $x \in E$ with $|z - x| = d_E(z)$. We remark that $\Pi_E(z)$ is always non-empty since E is non-empty and closed. We also note that if $z \in \mathbb{R}^d$ and $\hat{z} \in \Pi_E(z)$, then $z - \hat{z} \in N_E(\hat{z})$. This follows from the fact that for $0 < \varepsilon < 1$ we have

$$\begin{aligned} d_E(\hat{z} + \varepsilon(z - \hat{z})) &= d_E(z) + d_E(\hat{z} + \varepsilon(z - \hat{z})) - d_E(z) \\ &\geq |z - \hat{z}| - |\hat{z} + \varepsilon(z - \hat{z}) - z| \\ &= \varepsilon |z - \hat{z}| \end{aligned}$$

and

$$\begin{aligned} d_E(\hat{z} + \varepsilon(z - \hat{z})) &= d_E(\hat{z} + \varepsilon(z - \hat{z})) - d_E(\hat{z}) \\ &\leq \varepsilon |z - \hat{z}|. \end{aligned}$$

We recall the notations

$$\begin{aligned} B(y, r) &= \{u \in \mathbb{R}^d : |u - y| < r\}, \quad \text{and} \\ \overline{B}(y, r) &= \{u \in \mathbb{R}^d : |u - y| \leq r\}. \end{aligned}$$

Definition 6.46. We say that E satisfies the “uniform exterior ball condition” (abbreviated UEBC) if

- $N_E(x) \neq \{0\}$ for all $x \in \text{Bd}(E)$,
- $\exists r_0 > 0$ such that, $\forall x \in \text{Bd}(E)$ and $\forall u \in N_E(x)$, $|u| = r_0$:

$$d_E(x + u) = r_0 \quad \text{or equivalently} \quad B(x + u, r_0) \cap E = \emptyset,$$

(in this case we say that E satisfies r_0 -UEBC).

Note that for all $v \in N_E(x)$, $|v| \leq r_0$, we also have

$$d_E(x + v) = |v|. \tag{6.43}$$

Indeed since

$$0 \leq d_E(x + v) = d_E(x + v) - d_E(x) \leq |v|$$

and

$$\begin{aligned}
 |v| &= r_0 + (|v| - r_0) \\
 &= d_E \left(x + \frac{r_0}{|v|} v \right) + (|v| - r_0) \\
 &\leq \left| d_E \left(x + \frac{r_0}{|v|} v \right) - d_E(x + v) \right| + d_E(x + v) + (|v| - r_0) \\
 &\leq \left(\frac{r_0}{|v|} - 1 \right) |v| + d_E(x + v) + (|v| - r_0) \\
 &= d_E(x + v),
 \end{aligned}$$

then (6.43) follows.

It is clear that, under the uniform exterior ball condition with ball radius r_0 , for all $z \in \mathbb{R}^d$ with $d_E(z) < r_0$, the set $\Pi_E(z)$ is a singleton. The unique element of $\Pi_E(z)$ is called the projection of z on E , and it is denoted by $\pi_E(z)$.

We have the following characterization of the notion of the uniform exterior ball condition:

Lemma 6.47. *Let E be a non-empty closed subset of \mathbb{R}^d . The following assertions are equivalent:*

- (i) E satisfies the uniform exterior ball condition;
- (ii) E is a semiconvex subset of \mathbb{R}^d , that is $\exists \gamma \geq 0$ and for all $x \in \text{Bd}(E)$ there exists an $\hat{x} \neq 0$ such that

$$\langle \hat{x}, y - x \rangle \leq \gamma |\hat{x}| |y - x|^2, \quad \text{for all } y \in E,$$

(in this case $\hat{x} \in N_E(x)$ follows);

- (iii) $\exists \gamma \geq 0, \forall x, y \in \text{Bd}(E), \forall \lambda \in]0, 1[$:

$$d_E((1 - \lambda)x + \lambda y) \leq 4\lambda(1 - \lambda)\gamma |x - y|^2;$$

- (iii') $\exists \gamma \geq 0, \forall x, y \in E, \forall \lambda \in]0, 1[$:

$$d_E((1 - \lambda)x + \lambda y) \leq 4\lambda(1 - \lambda)\gamma |x - y|^2;$$

- (iv) $\exists \gamma \geq 0, \forall x, y \in \text{Bd}(E)$:

$$d_E\left(\frac{x + y}{2}\right) \leq \gamma |x - y|^2,$$

- (iv') $\exists \gamma \geq 0, \forall x, y \in E$:

$$d_E\left(\frac{x + y}{2}\right) \leq \gamma |x - y|^2,$$

(v) $\exists \delta > 0$ and $\mu > 0$ such that the function

$$x \longrightarrow \psi_E^\mu(x) \stackrel{\text{def}}{=} d_E(x) + \mu |x|^2 : U_\delta(E) \rightarrow \mathbb{R}$$

is convex on each convex subset of $\overline{U_\delta(E)}$.

Proof. We first remark that the conditions (ii), (iii), (iii'), (iv), (iv') are satisfied for $\gamma = 0$ if and only if E is convex; the convex sets satisfy the r -UEBC for all $r > 0$.

Step I. (i) \Leftrightarrow (ii)

(i) \Rightarrow (ii): Let $x \in \text{Bd}(E)$ and $\hat{x} \in N_E(x)$, $\hat{x} \neq 0$. Then there exists an $r_0 > 0$ such that

$$d_E\left(x + \frac{r_0}{|\hat{x}|}\hat{x}\right) = r_0.$$

We have for all $y \in E$ and $\gamma = \frac{1}{2r_0}$

$$\begin{aligned} \gamma |\hat{x}| |y - x|^2 - \langle \hat{x}, y - x \rangle &= \frac{1}{2r_0} |\hat{x}| \left[\left| y - \left(x + \frac{r_0}{|\hat{x}|}\hat{x} \right) \right|^2 - r_0^2 \right] \\ &\geq \frac{1}{2r_0} |\hat{x}| \left[d_E^2\left(x + \frac{r_0}{|\hat{x}|}\hat{x}\right) - r_0^2 \right] \\ &= 0. \end{aligned}$$

(ii) \Rightarrow (i): Let $r_0 > 0$ be such that $2\gamma r_0 \leq 1$. Let $x \in \text{Bd}(E)$ be arbitrary and $u = r_0 \frac{\hat{x}}{|\hat{x}|}$. Then

$$\begin{aligned} |u|^2 &= r_0^2 \\ &\leq r_0^2 + \frac{2r_0}{|\hat{x}|} \left[\gamma |\hat{x}| |y - x|^2 - \langle \hat{x}, y - x \rangle \right] \\ &\leq |y - (x + u)|^2, \quad \forall y \in E. \end{aligned}$$

Hence

$$|u| = r_0 \leq d_E(x + u) \leq |u|,$$

that is E satisfies the r_0 -uniform exterior ball condition.

From this equivalence we have that

$$E \text{ is } r_0\text{-UEBC} \Leftrightarrow E \text{ is } \frac{1}{2r_0}\text{-semiconvex.} \quad (6.44)$$

Step II. (iii) \Leftrightarrow (iii').

We have to prove only (iii) \Rightarrow (iii'). Let $x, y \in E$ and $0 < \lambda < 1$. Let $u_\lambda = (1 - \lambda)x + \lambda y = x + \lambda(y - x)$. If $u_\lambda \in E$, then

$$d_E((1 - \lambda)x + \lambda y) = 0 \leq 4\lambda(1 - \lambda)\gamma|x - y|^2.$$

If $u_\lambda \notin E$, then there exist $0 < \alpha < \lambda < \beta < 1$ such that

$$u_\rho = x + \rho(y - x) \notin E, \quad \text{for all } \alpha < \rho < \beta$$

and

$$u_\alpha = x + \alpha(y - x) \in E, \quad u_\beta = x + \beta(y - x) \in E.$$

We have

$$u_\lambda = u_\alpha + \frac{\lambda - \alpha}{\beta - \alpha}(u_\beta - u_\alpha)$$

and consequently

$$\begin{aligned} d_E((1 - \lambda)x + \lambda y) &= d_E(u_\lambda) \\ &\leq 4\frac{\lambda - \alpha}{\beta - \alpha}\left(1 - \frac{\lambda - \alpha}{\beta - \alpha}\right)\gamma|u_\beta - u_\alpha|^2 \\ &\leq 4(\lambda - \alpha)(\beta - \lambda)\gamma|y - x|^2 \\ &\leq 4\lambda(1 - \lambda)\gamma|x - y|^2. \end{aligned}$$

Step II. (iii') \Rightarrow (iv') \Rightarrow (iv) \Rightarrow (i) \Rightarrow (iii).

(iii') \Rightarrow (iv') \Rightarrow (iv) as particular cases: (iv') for (iii') and (iv) for (iv').

(iv) \Rightarrow (i): We prove by contradiction. We can assume $\gamma > 0$. We suppose that there is some $z \in \mathbb{R}^d$ in the r_0 -neighbourhood of E such that, for two different $x, y \in \text{Bd}(E)$,

$$|z - x| = |z - y| = d_E(z) < r_0 = \frac{1}{2\gamma}.$$

Under this hypothesis the vectors $z - \frac{1}{2}(x + y) = \frac{1}{2}[(z - y) + (z - x)]$ and $2(x - y) = 2[(z - y) - (z - x)]$ are orthogonal and, consequently,

$$d_E^2(z) = |z - x|^2 = \left|z - \frac{1}{2}(x + y)\right|^2 + 4|y - x|^2.$$

Let $u \in \Pi_E\left(\frac{1}{2}(x+y)\right)$. Then, from condition (iv) we obtain

$$\begin{aligned} \gamma |x-y|^2 &\geq d_E\left(\frac{1}{2}(x+y)\right) = \left|\left(\frac{1}{2}(x+y)\right) - u\right| \\ &\geq |z-u| - \left|z - \left(\frac{1}{2}(x+y)\right)\right| \\ &\geq d_E(z) - \left|z - \left(\frac{1}{2}(x+y)\right)\right| \\ &= \sqrt{\left|z - \frac{1}{2}(x+y)\right|^2 + 4|y-x|^2} - \left|z - \left(\frac{1}{2}(x+y)\right)\right|. \end{aligned}$$

Hence, we have

$$\left|z - \frac{1}{2}(x+y)\right|^2 + 4|y-x|^2 \leq \left[\gamma |x-y|^2 + \left|z - \left(\frac{1}{2}(x+y)\right)\right|\right]^2,$$

from which we easily deduce that

$$\begin{aligned} 4 &\leq \gamma^2 |y-x|^2 + 2\gamma \left|z - \frac{1}{2}(x+y)\right| \\ &\leq \gamma^2 [|z-x| + |z-y|]^2 + \gamma [|z-x| + |z-y|] \\ &< 2, \end{aligned}$$

which is a contradiction. Consequently, condition (iv) implies the $\frac{1}{2\gamma}$ -uniform exterior ball condition.

(i) \implies (iii): Let us now suppose that E satisfies the uniform exterior ball condition with an r_0 -ball. Let $x, y \in \text{Bd}(E)$. In a first step we assume that x, y are two different elements such that $0 < |x-y| \leq r_0$. Let $\lambda \in]0, 1[$ be such that $x_\lambda = x + \lambda(y-x) \notin E$ (if there is not such a λ , we are done), and let $\bar{x}_\lambda \in \Pi_E(x_\lambda)$. We fix any $u_\lambda \in N_E(\bar{x}_\lambda)$, $|u_\lambda| = r_0$ and put $z_\lambda = \bar{x}_\lambda + u_\lambda$. Then, due to condition (i), $|v - z_\lambda| \geq r_0$, for all $v \in E$. In particular, we have

$$|x - z_\lambda| \geq r_0, \quad \text{and} \quad |y - z_\lambda| \geq r_0.$$

We also observe that

$$|x_\lambda - \bar{x}_\lambda| = d_E(x_\lambda) \leq |x_\lambda - x| = \lambda |y-x| \leq r_0 = |z_\lambda - \bar{x}_\lambda|,$$

and

$$\alpha_\lambda = \frac{\langle x - z_\lambda, y - z_\lambda \rangle}{|x - z_\lambda| |y - z_\lambda|} \in [0, 1].$$

Hence,

$$\begin{aligned}
& |x_\lambda - \bar{x}_\lambda| \\
&= r_0 - |z_\lambda - x_\lambda| \\
&= r_0 - \sqrt{(1-\lambda)^2 |x - z_\lambda|^2 + \lambda^2 |y - z_\lambda|^2 + 2\lambda(1-\lambda) |x - z_\lambda| |y - z_\lambda| \alpha} \\
&\leq r_0 \left(1 - \sqrt{(1-\lambda)^2 + \lambda^2 + 2\lambda(1-\lambda)\alpha} \right) \\
&= r_0 \left(1 - \sqrt{1 - 2\lambda(1-\lambda)(1-\alpha)} \right) \\
&\leq r_0 [1 - (1 - 2\lambda(1-\lambda)(1-\alpha))] = 2r_0\lambda(1-\lambda)(1-\alpha).
\end{aligned}$$

On the other hand, for $\gamma \geq 1/(2r_0)$,

$$\begin{aligned}
& 4\lambda(1-\lambda)\gamma |x - y|^2 \\
&= 4\lambda(1-\lambda)\gamma \left(|x - z_\lambda|^2 + |y - z_\lambda|^2 - 2|x - z_\lambda| |y - z_\lambda| \alpha \right) \\
&\geq 8\lambda(1-\lambda)\gamma |x - z_\lambda| |y - z_\lambda| (1-\alpha) \\
&\geq 8\lambda(1-\lambda)\gamma r_0^2 (1-\alpha) \geq 2r_0\lambda(1-\lambda)(1-\alpha).
\end{aligned}$$

Consequently, $d_E(x_\lambda) = |x_\lambda - \bar{x}_\lambda| \leq 4\lambda(1-\lambda)\gamma |x - y|^2$, if $\gamma \geq 1/(4r_0)$.

In order to complete the proof, we still have to consider the case of $x, y \in \text{Bd}(E)$ with $|x - y| > r_0$. In this case, for $\gamma \geq 1/(2r_0)$, we have

$$\begin{aligned}
& d_E(x + \lambda(y - x)) (= d_E(y - (1-\lambda)(y - x))) \\
&\leq [\lambda \wedge (1-\lambda)] |x - y| \leq 2\lambda(1-\lambda) |x - y| \\
&\leq 4\lambda(1-\lambda) \frac{1}{2r_0} |x - y|^2.
\end{aligned}$$

This proves that under the r_0 -uniform exterior ball condition the statement (iii) holds with $\gamma \geq 1/(2r_0)$.

Step III. (v) \Rightarrow (iii) \Rightarrow (v).

(v) \Rightarrow (iii): Let $\lambda \in (0, 1)$ and $x, y \in \text{Bd}(E)$ with $|x - y| < \delta$. Then $x, y \in B(x; \delta) = \{z \in \mathbb{R}^d : |z - x| < \delta\} \subset U_\delta(E)$, and, consequently,

$$\begin{aligned}
& d_E(\lambda x + (1-\lambda)y) + \mu |\lambda x + (1-\lambda)y|^2 \\
&= \psi_E^\mu(\lambda x + (1-\lambda)y) \leq \lambda \psi_E^\mu(x) + (1-\lambda) \psi_E^\mu(y) \\
&= \lambda \mu |x|^2 + (1-\lambda) \mu |y|^2.
\end{aligned}$$

By subtracting $\mu |\lambda x + (1 - \lambda) y|^2$ on the left-hand and the right-hand sides of this inequality we obtain

$$d_E(\lambda x + (1 - \lambda) y) \leq \lambda(1 - \lambda)\mu |x - y|^2.$$

On the other hand, if $x, y \in \text{Bd}(E)$ are such that $|x - y| \geq \delta$, then

$$d_E(\lambda x + (1 - \lambda) y) \leq [\lambda \wedge (1 - \lambda)] |x - y| \leq \frac{2}{\delta} \lambda(1 - \lambda) |x - y|^2.$$

This shows that (iii) is fulfilled for $\gamma \geq \frac{1}{2\delta} \vee \frac{\mu}{4}$.

(iii) \implies (v): We fix any $\delta \in (0, r_0)$, and we recall that $\pi_E : \overline{U}_\delta(E) \rightarrow E$ is Lipschitz continuous with Lipschitz constant $L_\delta = r_0 / (r_0 - \delta)$. Let $\lambda \in (0, 1)$ and $u, v \in \overline{U}_\delta(E)$ be such that $(1 - \lambda)u + \lambda v \in \overline{U}_\delta(E)$. For simplicity of notation we put $x = \pi_E(u)$, $y = \pi_E(v)$, $z_\lambda = (1 - \lambda)u + \lambda v$, and $\bar{z}_\lambda = (1 - \lambda)x + \lambda y$. Then,

$$\begin{aligned} d_E((1 - \lambda)u + \lambda v) &= d_E(z_\lambda) \\ &\leq |z_\lambda - \pi_E(\bar{z}_\lambda)| \\ &\leq |z_\lambda - \bar{z}_\lambda| + |\bar{z}_\lambda - \pi_E(\bar{z}_\lambda)| \\ &\leq (1 - \lambda)d_E(u) + \lambda d_E(v) + d_E(\bar{z}_\lambda) \\ &\leq (1 - \lambda)d_E(u) + \lambda d_E(v) + 4\lambda(1 - \lambda)\gamma |x - y|^2 \\ &\leq (1 - \lambda)d_E(u) + \lambda d_E(v) + 4\lambda(1 - \lambda)\gamma L_\delta^2 |u - v|^2. \end{aligned}$$

Hence, for $\mu \geq 4\gamma L_\delta^2$,

$$\begin{aligned} \psi_E^\mu((1 - \lambda)u + \lambda v) &= d_E((1 - \lambda)u + \lambda v) + \mu |(1 - \lambda)u + \lambda v|^2 \\ &\leq (1 - \lambda) \left[d_E(u) + \mu |u|^2 \right] + \lambda \left[d_E(v) + \mu |v|^2 \right] \\ &= (1 - \lambda) \psi_E^\mu(u) + \lambda \psi_E^\mu(v). \end{aligned}$$

This proves that ψ_E^μ is convex on each convex subset of $\overline{U}_\delta(E)$. ■

Corollary 6.48. *If E is a closed subset of \mathbb{R}^d and satisfies the r_0 -uniform exterior ball condition, then for all $x \in E$*

$$N_E(x) = \left\{ \hat{x} \in \mathbb{R}^d : \langle \hat{x}, y - x \rangle \leq \frac{1}{2r_0} |\hat{x}| |y - x|^2; \quad \forall y \in E \right\}$$

and $\varphi = I_E$ is a $(0, \frac{1}{2r_0})$ -semiconvex l.s.c. function. Moreover $N_E(x) = \partial^- I_E(x)$.

Let $r_0 > 0$. The set E satisfies the r_0 -uniform exterior ball condition if and only if E is $\frac{1}{2r_0}$ -semiconvex.

We recall the following well-known property of the projection.

Lemma 6.49. *Suppose that E satisfies the uniform exterior ball condition with ball radius r_0 and $\varepsilon \in]0, r_0[$. Then the projection π_E restricted to $\overline{U}_\varepsilon(E)$ (the closed ε -neighbourhood of E) is Lipschitz with Lipschitz constant $L_\varepsilon = r_0 / (r_0 - \varepsilon)$, and the function d_E^2 is of class C^1 on $\overline{U}_\varepsilon(E)$ with*

$$\frac{1}{2} \nabla d_E^2(z) = z - \pi_E(z), \quad \text{and} \quad z - \pi_E(z) \in N_E(\pi_E(z)),$$

for all $z \in \overline{U}_\varepsilon(E)$.

Proof. To simplify we denote $\pi = \pi_E$ and $d = d_E$. Let $x, y \in \overline{U}_\varepsilon(E)$. Then we have $x - \pi(x) \in N_E(\pi(x))$, $y - \pi(y) \in N_E(\pi(y))$ and

$$\begin{aligned} |\pi(x) - \pi(y)|^2 &= \langle y - \pi(y), \pi(x) - \pi(y) \rangle + \langle x - \pi(x), \pi(y) - \pi(x) \rangle \\ &\quad + \langle x - y, \pi(x) - \pi(y) \rangle \\ &\leq \frac{\varepsilon}{r_0} |\pi(x) - \pi(y)|^2 + |x - y| |\pi(x) - \pi(y)|. \end{aligned}$$

Hence

$$|\pi(x) - \pi(y)| \leq \frac{r_0}{r_0 - \varepsilon} |x - y|. \quad (6.45)$$

To obtain the second part of lemma it is sufficient to show that there exist a positive constant $C = C_{\varepsilon, r_0}$ such that

$$-C |y - x|^2 \leq d^2(y) - d^2(x) - 2 \langle x - \pi(x), y - x \rangle \leq C |y - x|^2. \quad (6.46)$$

We have

$$\begin{aligned} &d^2(y) - d^2(x) - 2 \langle x - \pi(x), y - x \rangle \\ &= |(y - x) + (x - \pi(x)) + \pi(x) - \pi(y)|^2 - |x - \pi(x)|^2 \\ &\quad - 2 \langle x - \pi(x), y - x \rangle \\ &= |y - x|^2 + |\pi(x) - \pi(y)|^2 + 2 \langle y - x, \pi(x) - \pi(y) \rangle \\ &\quad + 2 \langle x - \pi(x), \pi(x) - \pi(y) \rangle. \end{aligned}$$

Since

$$\langle x - \pi(x), \pi(x) - \pi(y) \rangle \geq -\frac{\varepsilon}{2r_0} |\pi(y) - \pi(x)|^2$$

and

$$\begin{aligned}
 & \langle x - \pi(x), \pi(x) - \pi(y) \rangle \\
 & \leq \langle y - \pi(y), \pi(x) - \pi(y) \rangle + \langle x - y, \pi(x) - \pi(y) \rangle \\
 & \leq \frac{\varepsilon}{2r_0} |\pi(x) - \pi(y)|^2 + |x - y| |\pi(x) - \pi(y)|
 \end{aligned}$$

the inequality (6.46) follows from this and (6.45). ■

6.3.9 Differential Equations

Let \mathbb{H} be a separable real Hilbert space. If $A : \mathbb{H} \rightrightarrows \mathbb{H}$ is a maximal monotone operator, $u_0 \in \overline{D(A)}$, $f \in L^1(0, T; \mathbb{H})$, then the strong solution of the Cauchy problem

$$\begin{cases} \frac{du(t)}{dt} + Au(t) \ni f(t), & a.e. t \in]0, T[, \\ u(0) = u_0, \end{cases} \quad (6.47)$$

is defined as a function $u \in C([0, T]; \mathbb{H})$ satisfying:

- i) $u(t) \in D(A)$ a.e. $t \in]0, T[$,
- ii) $\exists h = h^{(u)} \in L^1(0, T; \mathbb{H})$ such that $h(t) \in Au(t)$, a.e. $t \in]0, T[$, and

$$u(t) + \int_0^t h(s) ds = u_0 + \int_0^t f(s) ds, \quad \forall t \in [0, T],$$

and we shall write $u = \mathcal{S}(A; u_0, f)$. Note that the strong solution is unique when it exists. Indeed if u, v are two solutions corresponding to $(u_0, f), (v_0, g)$, respectively, then

$$\begin{aligned}
 & |u(t) - v(t)|^2 + 2 \int_0^t \langle h^{(u)}(s) - h^{(v)}(s), u(s) - v(s) \rangle ds \\
 & = |u_0 - v_0|^2 + 2 \int_0^t \langle f(s) - g(s), u(s) - v(s) \rangle ds
 \end{aligned}$$

and by the monotonicity of A it follows that

$$|u(t) - v(t)|^2 \leq |u_0 - v_0|^2 + 2 \int_0^t |f(s) - g(s)| |u(s) - v(s)| ds.$$

Using Gronwall’s inequality (Lemma 6.63, Annex C) we obtain

$$|u(t) - v(t)| \leq |u_0 - v_0| + \int_0^t |f(s) - g(s)| ds. \tag{6.48}$$

We recall from Barbu [3], p. 31, that the following proposition holds:

Proposition 6.50. *If A is maximal monotone operator on \mathbb{H} , $u_0 \in D(A)$ and $f \in W^{1,1}([0, T]; \mathbb{H})$, then the Cauchy problem (6.47) has a unique strong solution $u \in W^{1,\infty}([0, T]; \mathbb{H})$. Moreover if A_ε is the Yosida approximation of the operator A and u_ε is the solution of the approximate equation*

$$\frac{du_\varepsilon}{dt} + A_\varepsilon u_\varepsilon = f, \quad u_\varepsilon(0) = u_0,$$

then for all $(x_0, y_0) \in A$ there exists a constant $C = C(\alpha, T, x_0, y_0) > 0$ such that

- c1) $\|u_\varepsilon\|_{C([0, T]; \mathbb{H})} \leq C(1 + |u_0| + \|f\|_{L^1(0, T; \mathbb{H})})$, and
- c2) $\lim_{\varepsilon \searrow 0} u_\varepsilon = u$ in $C([0, T]; \mathbb{H})$.

We introduce the notation

$$W^{1,p}([0, T]; \mathbb{H}) = \left\{ f : \exists a \in \mathbb{H}, g \in L^p(0, T; \mathbb{H}) \text{ such that } f(t) = a + \int_0^t g(s) ds, \forall t \in [0, T] \right\}.$$

From Barbu [2] (Chap. IV, p. 197, Theorem 2.5) we recall:

Proposition 6.51. *Let A be a maximal monotone operator on \mathbb{H} such that*

$$\text{int}(D(A)) \neq \emptyset.$$

If $u_0 \in \overline{D(A)}$ and $f \in W^{1,1}([0, T]; \mathbb{H})$, then the Cauchy problem (6.47) has a unique strong solution $u \in W^{1,1}([0, T]; \mathbb{H})$.

By the continuity property (6.48) one can generalize the notion of the solution of Eq. (6.47) as follows:

◆ u is a generalized solution of the Cauchy problem (6.47) with

$$u_0 \in \overline{D(A)}, \quad f \in L^1(0, T; \mathbb{H}),$$

(and we shall write $u = \mathcal{GS}(A; u_0, f)$) if

- ◇ $u \in C([0, T]; \mathbb{H})$ and
- ◇ there exist $u_{0n} \in D(A)$, $f_n \in W^{1,1}([0, T]; \mathbb{H})$ such that

- a) $u_{0n} \rightarrow u_0$ in \mathbb{H} ,
- b) $f_n \rightarrow f$ in $L^1(0, T; \mathbb{H})$,
- c) $u_n = \mathcal{S}(A; u_{0n}, f_n) \rightarrow u$ in $C([0, T]; \mathbb{H})$.

Clearly we have:

Proposition 6.52. *If A is a maximal monotone operator on \mathbb{H} , $u_0 \in \overline{D(A)}$ and $f \in L^1(0, T; \mathbb{H})$, then the Cauchy problem (6.47) has a unique generalized solution $u \in C([0, T]; \mathbb{H})$. Moreover if $u = \mathcal{GS}(A; u_0, f)$ and $v = \mathcal{GS}(A; v_0, g)$ then*

$$|u(t) - v(t)| \leq |u_0 - v_0| + \int_0^t |f(s) - g(s)| ds \tag{6.49}$$

and for all $(x_0, \hat{x}_0) \in A$ there exists a constant $C = C(T, x_0, \hat{x}_0) > 0$ such that

$$\|u\|_{C([0, T]; \mathbb{H})} \leq C(1 + |u_0| + \|f\|_{L^1(0, T; \mathbb{H})}). \tag{6.50}$$

In the case when $\text{int}(D(A)) \neq \emptyset$ one can give supplementary properties of generalized solutions.

Proposition 6.53. *Let $A \subset \mathbb{H} \times \mathbb{H}$ be a maximal monotone operator such that*

$$\text{int}(D(A)) \neq \emptyset.$$

Let $u_0 \in \overline{D(A)}$ and $f \in L^1(0, T; \mathbb{H})$. Then:

I. *there exists a unique pair (u, k) such that*

$$(P_A) : \begin{cases} a) & u \in C([0, T]; \mathbb{H}), \quad u(t) \in \overline{D(A)} \quad \forall t \in [0, T], \quad u(0) = u_0, \\ b) & k \in C([0, T]; \mathbb{H}) \cap BV([0, T]; \mathbb{H}), \quad k(0) = 0, \\ c) & u(t) + k(t) = u_0 + \int_0^t f(s) ds, \quad \forall t \in [0, T], \\ d) & \int_s^t \langle u(r) - x, dk(r) - \hat{x} dr \rangle \geq 0, \\ & \forall 0 \leq s \leq t \leq T, \quad \forall (x, \hat{x}) \in A; \end{cases}$$

II. $u = \mathcal{GS}(A; u_0, f)$ if and only if u is solution of the problem (P_A) ;

III. *the following estimate holds:*

$$\|u\|_{C([0, T]; \mathbb{H})}^2 + \|k\|_{BV([0, T]; \mathbb{H})} \leq C \left(1 + |u_0|^2 + \|f\|_{L^1(0, T; \mathbb{H})}^2 \right),$$

where C is a positive constant independent of u_0 and f .

Proof. Uniqueness. If (u, k) and (v, ℓ) are two solutions of the problem (P_A) corresponding to (u_0, f) , (v_0, g) respectively, then

$$\begin{aligned}
 &|u(t) - v(t)|^2 + 2 \int_0^t \langle dk(s) - d\ell(s), u(s) - v(s) \rangle ds \\
 &= |u_0 - v_0|^2 + 2 \int_0^t \langle f(s) - g(s), u(s) - v(s) \rangle ds.
 \end{aligned}$$

But by Proposition 6.17, the monotonicity of A and $(P_A - d)$ we have

$$\int_0^t \langle dk(s) - d\ell(s), u(s) - v(s) \rangle ds \geq 0.$$

Hence

$$|u(t) - v(t)|^2 \leq |u_0 - v_0|^2 + 2 \int_0^t |f(s) - g(s)| |u(s) - v(s)| ds,$$

which yields (6.49) and, in particular, the uniqueness follows.

Existence. Let $u_{0n} \in D(A)$, $f_n \in W^{1,1}([0, T]; \mathbb{H})$ such that

$$u_{0n} \rightarrow u_0 \quad \text{in } \mathbb{H} \quad \text{and} \quad f_n \rightarrow f \quad \text{in } L^1(0, T; \mathbb{H}).$$

Let $u_n = \mathcal{S}(A; u_{0n}, f_n)$ be the strong solution corresponding to $(A; u_{0n}, f_n)$. Hence there exists an $h_n \in L^1(0, T; \mathbb{H})$ such that $h_n(t) \in Au_n(t)$, a.e. $t \in]0, T[$ and denoting $k_n(t) = \int_0^t h_n(s) ds$ we have

$$\begin{aligned}
 a) \quad &u_n(t) + k_n(t) = u_{0n} + \int_0^t f_n(s) ds, \quad \forall t \in [0, T], \\
 b) \quad &\int_s^t \langle u_n(r) - x, dk_n(r) - \hat{x}dr \rangle \geq 0, \\
 &\forall 0 \leq s \leq t \leq T, \quad \forall (x, \hat{x}) \in A.
 \end{aligned} \tag{6.51}$$

Let $x_0 \in \text{int}(D(A))$ and $\hat{x}_0 \in A(x_0)$. Then

$$\begin{aligned}
 &|u_n(t) - x_0|^2 + 2 \int_0^t \langle h_n(s), u_n(s) - x_0 \rangle ds \\
 &= |u_{0n} - x_0|^2 + 2 \int_0^t \langle f_n(s), u_n(s) - x_0 \rangle ds.
 \end{aligned}$$

Since

$$\langle h_n(s), u_n(s) - x_0 \rangle \geq \langle \hat{x}_0, u_n(s) - x_0 \rangle,$$

we infer

$$|u_n(t) - x_0|^2 \leq |u_{0n} - x_0|^2 + 2 \int_0^t (|f_n(s)| + |\hat{x}_0|) |u_n(s) - x_0| ds.$$

By the Gronwall type inequality from Lemma 6.63, Annex C, we obtain

$$\begin{aligned} |u_n(t) - x_0| &\leq |u_{0n} - x_0| + \int_0^T |f_n(s)| ds + T |\hat{x}_0| \\ &\leq C \left[1 + |u_{0n}| + \int_0^T |f_n(s)| ds \right], \end{aligned}$$

where $C = C(x_0, \hat{x}_0, T) > 0$.

By Proposition 6.5 we have *a.e.* $t \in]0, T[$:

$$r_0 |h_n(t)| \leq \langle h_n(t), u_n(t) - x_0 \rangle + M_0 |u_n(t) - x_0| + r_0 M_0,$$

and then

$$\begin{aligned} 2r_0 \int_0^t |h_n(s)| ds &\leq |u_{0n} - x_0|^2 + 2 \int_0^t (|f_n(s)| + M_0) |u_n(s) - x_0| ds + 2r_0 M_0 T \\ &\leq C \left[1 + |u_{0n}|^2 + \left(\int_0^T |f_n(s)| ds \right)^2 \right] \end{aligned}$$

with C a constant depending on $x_0, \hat{x}_0, T, M_0, r_0$.

Hence $k_n(t) = \int_0^t h_n(s) ds$ is bounded in $BV([0, T]; \mathbb{H})$. Then there exists a $k \in BV([0, T]; \mathbb{H})$ such that on a subsequence also denoted by k_n we have

$$k_n \xrightarrow{w^*} k \quad \text{in } BV([0, T]; \mathbb{H}).$$

The sequence $(u_n)_{n \in \mathbb{N}^*}$ is a Cauchy sequence in $C([0, T]; \mathbb{H})$ since if $u_m = S(A; u_{0m}, f_m)$ then

$$\sup_{t \in [0, T]} |u_n(t) - u_m(t)| \leq |u_{0n} - u_{0m}| + \int_0^T |f_n(s) - f_m(s)| ds.$$

Then there exists a $u \in C([0, T]; \mathbb{H})$ such that

$$u_n \rightarrow u \quad \text{in } C([0, T]; \mathbb{H}).$$

Passing to the limit in (6.51), we obtain that (u, k) satisfies (P_A) . The proof is complete. ■

If the assumption $\text{int}(D(A)) \neq \emptyset$ has a smoothing effect as we saw in Proposition 6.51, the maximal monotone $A = \partial\varphi$ also has a smoothing effect.

Consider the differential equation

$$\begin{cases} \frac{du(t)}{dt} + \partial\varphi u(t) \ni f(t), & a.e. t \in]0, T[, \\ u(0) = u_0, \end{cases} \tag{6.52}$$

where

$$\varphi : \mathbb{H} \rightarrow]-\infty, +\infty] \text{ is a proper convex l.s.c. function.}$$

Proposition 6.54. *If $u_0 \in \overline{D(\partial\varphi)} (= \overline{\text{Dom}(\varphi)})$ and $f \in L^2(0, T; \mathbb{H})$, then the Cauchy problem (6.52) has a unique strong solution. Moreover $u \in W^{1,2}(\delta, T; \mathbb{H})$, $\forall \delta > 0$, $\sqrt{t} \frac{du}{dt} \in L^2(0, T; \mathbb{H})$, $\varphi(u) \in L^1(0, T)$ and if $u_0 \in \text{Dom}(\varphi)$, then $\frac{du}{dt} \in L^2(0, T; \mathbb{H})$ and $\varphi(u) \in L^\infty(0, T)$.*

Consider now the Cauchy problem

$$\begin{cases} \frac{dy(t)}{dt} + \partial^-\varphi(x(t)) \ni g(t), & a.e. t \in [0, T] \\ x(0) = x_0, \end{cases} \tag{6.53}$$

where

- (i) $\varphi : \mathbb{R}^d \rightarrow]-\infty, +\infty]$ is a proper l.s.c. (ρ, γ) -semiconvex function,
 - (ii) $\text{Dom}(\varphi)$ is a locally closed subset of \mathbb{R}^d ,
- (6.54)

and

- (i) $x_0 \in \text{Dom}(\varphi)$,
 - (ii) $g \in L^2(0, T; \mathbb{R}^d)$.
- (6.55)

Hence for all $(x, \hat{x}) \in \partial^-\varphi$

$$\langle \hat{x}, z - x \rangle + \varphi(x) \leq \varphi(z) + (\rho + \gamma |\hat{x}|) |z - x|^2, \quad \forall z \in \mathbb{R}^d.$$

We denote here by $\partial^-\varphi(x)$ the Fréchet subdifferential given in Definition 6.40. Recall that $E \subset \mathbb{R}^d$ is locally closed if for all $x \in E$, there exists a $\delta > 0$ such that $E \cap \overline{B}(x, \delta)$ is closed.

From Degiovanni–Marino–Tosques [21] and Rossi–Savaré [66] we have:

Proposition 6.55. *Let the assumptions (6.54) and (6.55) be satisfied. Then there exist $h \in L^2(0, T; \mathbb{R}^d)$ and a unique absolutely continuous function $x : [0, T] \rightarrow \text{Dom}(\varphi)$ such that:*

- (a) $\int_0^T [|x'(t)|^2 + |\varphi(x(t))|] dt < \infty,$
- (b) $x(t) \in \text{Dom}(\partial^-\varphi), \quad \text{a.e. } t \in]0, T[,$
- (c) $h(t) \in \partial^-\varphi(x(t)), \quad \text{a.e. } t \in]0, T[,$

and

$$(P_g) : \begin{cases} x'(t) + h(t) = g(t), & \text{a.e. } t \in]0, T[\\ x(0) = x_0. \end{cases}$$

Moreover a.e. $t, s \in]0, T[, s < t$:

$$\int_s^t |x'(r)|^2 dr = \varphi(x(s)) - \varphi(x(t)) + \int_s^t (g(r), x'(r)) dr$$

and there exists a positive constant C_T (independent of x_0 and g) such that

$$\|x\|_T + \|\varphi(x)\|_T + \int_0^T |x'(r)|^2 dr \leq C_T \left(|x_0|^2 + \varphi^+(x_0) + \int_0^T |g(r)|^2 dr \right).$$

Remark 6.56. If we put

$$k(t) = \int_0^t h(s) ds$$

then

$$(GSP) : \begin{cases} j) & k \in BV([0, T]; \mathbb{R}^d), \quad k(0) = 0, \\ jj) & x(t) + k(t) = x_0 + \int_0^t g(s) ds, \quad \forall t \in [0, T], \\ jv) & \forall 0 \leq s \leq t, \forall y : [0, \infty[\rightarrow \mathbb{R}^d \text{ continuous:} \\ & \int_s^t \langle y(r) - x(r), dk(r) \rangle + \int_s^t \varphi(x(r)) dr \\ & \leq \int_s^t \varphi(y(r)) dr + \int_s^t |y(r) - x(r)|^2 (\rho dr + \gamma d \downarrow k \uparrow_r), \end{cases}$$

that is (x, k) is the solution of the generalized Skorohod problem $(x_0, m, \partial^-\varphi)$ with $m(t) = \int_0^t g(s) ds$ (see Definition 4.29).

6.3.10 Auxiliary Results

Proposition 6.57. *If $g \in L^1(0, T)$ and*

$$\rho_\lambda(t) = e^{-\lambda t} \int_0^t |g(s)| e^{\lambda s} ds, \quad t \in [0, T], \lambda > 0,$$

then

$$\lim_{\lambda \rightarrow \infty} \left[\sup_{t \in [0, T]} \rho_\lambda(t) \right] = 0.$$

Proof. Let the continuous function $t \mapsto G(t) = \int_0^t |g(s)| ds$ and $\mathbf{m}_G(\varepsilon)$ be the modulus of continuity of G on $[0, T]$. We have for all $t \in [0, T]$ and $\lambda > 0$:

$$\begin{aligned} 0 \leq \rho_\lambda(t) &= e^{-\lambda t} \left[\int_0^{(t-\sqrt{1/\lambda})^+} |g(s)| e^{\lambda s} ds + \int_{(t-\sqrt{1/\lambda})^+}^t |g(s)| e^{\lambda s} ds \right] \\ &\leq e^{-\lambda t} e^{\lambda(t-\sqrt{1/\lambda})^+} \int_0^{(t-\sqrt{1/\lambda})^+} |g(s)| ds + e^{-\lambda t} e^{\lambda t} \mathbf{m}_G(\sqrt{1/\lambda}) \\ &\leq e^{-\sqrt{\lambda}} G(T) + \mathbf{m}_G(\sqrt{1/\lambda}), \end{aligned}$$

which yields the result. ■

We now give a variant of the Banach fixed point theorem.

Let $\{(\mathbb{V}_a, d_a) : a \geq 0\}$ be a family of complete metric spaces such that for all $0 \leq a \leq b$:

$$\mathbb{V}_b \subset \mathbb{V}_a$$

with a continuous embedding. Let

$$\mathbb{V} = \bigcap_{a \geq 0} \mathbb{V}_a = \bigcap_{a \in \mathbb{N}^*} \mathbb{V}_a,$$

and assume $\mathbb{V} \neq \emptyset$. Then \mathbb{V} is a complete metric space with respect to the metric

$$\rho(x, y) = \sum_{a \in \mathbb{N}} \frac{1}{2^a} \frac{d_a(x, y)}{1 + d_a(x, y)},$$

and if $x_n, x \in \mathbb{V}$, $n \in \mathbb{N}^*$, then as $n \rightarrow \infty$,

$$x_n \rightarrow x \text{ in } \mathbb{V} \iff x_n \rightarrow x \text{ in } \mathbb{V}_a, \forall a \geq 0.$$

Lemma 6.58. *Let $\Gamma : \mathbb{V} \rightarrow \mathbb{V}$ be a mapping satisfying: there exists an $a_0 \geq 0$ and for all $a \geq a_0$ there exists a $\delta_a \in]0, 1[$ such that*

$$d_a(\Gamma(x), \Gamma(y)) \leq \delta_a d_a(x, y), \text{ for all } x, y \in \mathbb{V}.$$

Then Γ has a unique fixed point, i.e. there exists a unique $x \in \mathbb{V}$ such that

$$x = \Gamma(x).$$

(Banach's fixed point theorem corresponds to the case $(\mathbb{V}_a, d_a) \equiv (\mathbb{V}_0, d_0)$ for all $a \geq 0$.)

Proof. We define

$$x_0 \in \mathbb{V}, \quad x_{n+1} = \Gamma(x_n).$$

Then by recurrence we deduce that

$$x_n \in \mathbb{V}, \quad \text{for all } n \in \mathbb{N},$$

and

$$d_a(x_{n+p}, x_n) \leq \frac{\delta_a^n}{1-\delta_a} d_a(x_1, x_0),$$

for all $a \geq a_0, n, p \in \mathbb{N}^*$. Hence there exists a unique $x^{(a)} \in \mathbb{V}_a$ such that as $n \rightarrow \infty$

$$x_n \rightarrow x^{(a)} \text{ in } \mathbb{V}_a.$$

Moreover by the continuity of the embedding $\mathbb{V}_a \subset \mathbb{V}_b$ for $0 \leq b \leq a$, we infer

$$x_n \rightarrow x^{(a)} \text{ in } \mathbb{V}_b.$$

Consequently $x^{(a)} = x^{(a_0)}$ for all $a \geq a_0, x \stackrel{\text{def}}{=} x^{(a_0)} \in \mathbb{V}$ and for $a \geq a_0$

$$\begin{aligned} d_a(x, \Gamma(x)) &\leq d_a(x, x_{n+1}) + d_a(\Gamma(x_n), \Gamma(x)) \\ &\leq d_a(x, x_{n+1}) + \delta_a d_a(x_n, x) \\ &\rightarrow 0, \text{ as } n \rightarrow \infty, \end{aligned}$$

which yields

$$x = \Gamma(x).$$

The fixed point x is unique, since if $x, y \in \mathbb{V}$ are two fixed points, then for $a \geq a_0$

$$d_a(x, y) = d_a(\Gamma(x), \Gamma(y)) \leq \delta_a d_a(x, y)$$

and $x = y$ follows. ■

6.4 Annex C: Deterministic and Stochastic Inequalities

6.4.1 Deterministic Inequalities

Proposition 6.59 (Stieltjes–Gronwall Inequality). *Let $K : [0, T] \rightarrow \mathbb{R}$ be a continuous increasing function, $a : [0, T] \rightarrow [0, \infty[$ be an increasing function and $x : [0, T] \rightarrow \mathbb{R}$ be a measurable function such that*

$$\int_0^T |x(r)| dK(r) < \infty.$$

If

$$x(t) \leq a(t) + \int_0^t x(r) dK(r), \quad \forall t \in [0, T],$$

then

$$x(t) \leq a(t) e^{K(t)-K(0)}, \quad \forall t \in [0, T]. \quad (6.56)$$

Proof. I. Note that if $\alpha, \beta_0, \beta_1, \dots, \beta_n$ and $z_0, z_1, \dots, z_n \in \mathbb{R}$ satisfy

$$\begin{aligned} z_0 &\leq \alpha, \\ z_i &\leq \alpha + \beta_0 z_0 + \beta_1 z_1 + \dots + \beta_{i-1} z_{i-1}, \quad 1 \leq i \leq n, \end{aligned}$$

then

$$z_i \leq \alpha e^{\beta_0 + \beta_1 + \dots + \beta_{i-1} + \beta_i}.$$

Indeed, associating the sequence

$$x_0 = \alpha, \quad x_i = \alpha + \beta_0 x_0 + \beta_1 x_1 + \dots + \beta_{i-1} x_{i-1}, \quad 1 \leq i \leq n,$$

by recurrence

$$z_i \leq x_i = \alpha (1 + \beta_0) (1 + \beta_1) \dots (1 + \beta_{i-1}) \leq \alpha e^{\beta_0 + \beta_1 + \dots + \beta_{i-1}}$$

follows.

Let

$$g(t) = a(t) + \int_0^t x^+(r) dK(r).$$

Clearly g is an increasing function and

$$x(t) \leq x^+(t) \leq g(t) \leq a(t) + \int_0^t g(r) dK(r).$$

Let $0 < t_1 < \dots < t_n = t$ be such that

$$1 > \max \{K(t_i) - K(t_{i-1}) : i \in \overline{1, n}\} \stackrel{\text{def}}{=} \gamma_n \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Let $g_i = g(t_i)$, $c_0 = 0$, $c_i = \int_{t_{i-1}}^{t_i} dK(r) = K(t_i) - K(t_{i-1}) \leq \gamma_n$. We have

$$\begin{aligned} g_i &\leq a(t_i) + \sum_{j=1}^i \int_{t_{j-1}}^{t_j} g(r) dK(r) \\ &\leq a(t) + \sum_{j=1}^i g_j \int_{t_{j-1}}^{t_j} dK(r) \\ &\leq a(t) + (c_0 g_0 + c_1 g_1 + \dots + c_{i-1} g_{i-1}) + \gamma_n g_i, \end{aligned}$$

which yields

$$g_i \leq \frac{a(t)}{1 - \gamma_n} + \frac{c_0}{1 - \gamma_n} g_0 + \frac{c_1}{1 - \gamma_n} g_1 + \dots + \frac{c_{i-1}}{1 - \gamma_n} g_{i-1}$$

for all $i \in \{1, 2, \dots, n\}$. Hence

$$\begin{aligned} x(t) \leq g(t) = g_n &\leq \frac{a(t)}{1 - \gamma_n} \exp \left[\frac{1}{1 - \gamma_n} \sum_{j=0}^n c_j \right] \\ &= \frac{a(t)}{1 - \gamma_n} \exp \left[\frac{1}{1 - \gamma_n} [K(t) - K(0)] \right]. \end{aligned}$$

The inequality (6.56) follows by letting $n \rightarrow \infty$. ■

For $K(t) = \int_0^t b(r) dr$, where $b : [0, \infty[\rightarrow [0, \infty[$ is a locally integrable function, the following lemma holds.

Corollary 6.60 (Gronwall Inequality). *Let $a : [0, T] \rightarrow [0, \infty[$ be an increasing function and $x, b : [0, T] \rightarrow \mathbb{R}$, $b \geq 0$, be integrable functions such that*

$$\int_0^T b(t) |x(t)| dt < \infty.$$

If

$$x(t) \leq a(t) + \int_0^t b(s) x(s) ds, \quad \forall t \in [0, T],$$

then

$$x(t) \leq a(t) \exp\left(\int_0^t b(s) ds\right), \quad \forall t \in [0, T]. \quad (6.57)$$

Corollary 6.61 (Backward Stieltjes–Gronwall Inequality). Let $\tilde{K} : [0, T] \rightarrow \mathbb{R}$ be a continuous increasing function, $\tilde{a} : [0, T] \rightarrow [0, \infty[$ be a decreasing function and $y : [0, T] \rightarrow \mathbb{R}$ be a measurable function such that

$$\int_0^T |y(r)| d\tilde{K}(r) < \infty.$$

If

$$y(t) \leq \tilde{a}(t) + \int_t^T y(r) d\tilde{K}(r), \quad \forall t \in [0, T],$$

then

$$y(t) \leq \tilde{a}(t) e^{\tilde{K}(T) - \tilde{K}(t)}, \quad \forall t \in [0, T]. \quad (6.58)$$

Proof. Let $x(t) = y(T - t)$, $a(t) = \tilde{a}(T - t)$ and $K(t) = \tilde{K}(T) - \tilde{K}(T - t)$. Then

$$x(t) \leq a(t) + \int_0^t x(r) dK(r), \quad \forall t \in [0, T],$$

and by Proposition 6.59

$$x(t) \leq a(t) e^{K(t) - K(0)}, \quad \forall t \in [0, T],$$

that is (6.58) replacing t by $T - t$. ■

In particular for $K(t) = \int_0^t b(r) dr$, we have:

Corollary 6.62 (Backward Gronwall Inequality). Let $\tilde{a} : [0, T] \rightarrow [0, \infty[$ be a decreasing function and $y, b : [0, T] \rightarrow \mathbb{R}$, $b \geq 0$, be integrable functions such that

$$\int_0^T b(t) |y(t)| dt < \infty.$$

If

$$y(t) \leq \tilde{a}(t) + \int_t^T b(s) y(s) ds, \quad \forall t \in [0, T],$$

then

$$y(t) \leq \tilde{a}(t) \exp\left(\int_t^T b(s) ds\right), \quad \forall t \in [0, T]. \quad (6.59)$$

We now give some other deterministic inequalities used in the book.

Lemma 6.63. Let $\alpha, \beta \in L^1_{loc}([0, \infty[)$.

I. If $\alpha \geq 0$ a.e. and $x : [0, \infty[\rightarrow \mathbb{R}^d$ is an absolutely continuous function such that

$$\langle x'(t), x(t) \rangle \leq \alpha(t) |x(t)| + \beta(t) |x(t)|^2, \quad \text{a.e. } t \geq 0,$$

then

$$|x(t)| \leq |x(\tau)| e^{\int_\tau^t \beta(s) ds} + \int_\tau^t \alpha(s) e^{\int_s^t \beta(r) dr} ds \quad (6.60)$$

for all $0 \leq \tau \leq t$.

II. If $\alpha, \beta \geq 0$ a.e., $a : [0, \infty[\rightarrow [0, \infty[$ is an increasing function and $\varphi : [0, \infty[\rightarrow [0, \infty[$ is a continuous function such that $\forall t \geq 0$

$$\varphi^2(t) \leq a(t) + 2 \int_0^t \alpha(s) \varphi(s) ds + 2 \int_0^t \beta(s) \varphi^2(s) ds,$$

then

$$\varphi(t) \leq \sqrt{a(t)} e^{\int_0^t \beta(s) ds} + \int_0^t \alpha(s) e^{\int_s^t \beta(r) dr} ds, \quad \forall t \geq 0. \quad (6.61)$$

Proof. I. Let $u_\varepsilon(t) = |x(t)|^2 e^{-2 \int_0^t \beta(s) ds} + \varepsilon$, $\varepsilon > 0$. Then

$$\begin{aligned} u'_\varepsilon(t) &= 2 \langle x'(t), x(t) \rangle e^{-2 \int_0^t \beta(s) ds} - 2\beta(t) |x(t)|^2 e^{-2 \int_0^t \beta(s) ds} \\ &\leq 2\alpha(t) |x(t)| e^{-2 \int_0^t \beta(s) ds} \\ &\leq 2\alpha(t) \sqrt{u_\varepsilon(t)} e^{-\int_0^t \beta(s) ds}, \end{aligned}$$

which yields

$$\begin{aligned} \frac{d}{dt} \left(\sqrt{u_\varepsilon(t)} \right) &= \frac{u'_\varepsilon(t)}{2\sqrt{u_\varepsilon(t)}} \\ &\leq \alpha(t) e^{-\int_0^t \beta(s) ds}. \end{aligned}$$

Hence

$$\sqrt{u_\varepsilon(t)} \leq \sqrt{u_\varepsilon(\tau)} + \int_\tau^t \alpha(s) e^{-\int_0^s \beta(r) dr} ds.$$

Passing to the limit as $\varepsilon \searrow 0$ the inequality (6.60) follows.

II. Let $\theta \in [0, T]$ be fixed and

$$x(t) = \left(a(\theta) + 2 \int_0^t \alpha(s) \varphi(s) ds + 2 \int_0^t \beta(s) \varphi^2(s) ds \right)^{1/2}.$$

Then for all $t \in [0, \theta]$:

$$\varphi^2(t) \leq a(\theta) + 2 \int_0^t \alpha(s) \varphi(s) ds + 2 \int_0^t \beta(s) \varphi^2(s) ds = x^2(t),$$

and

$$\begin{aligned} x'(t) x(t) &= \alpha(t) \varphi(t) + \beta(t) \varphi^2(t) \\ &\leq \alpha(t) x(t) + \beta(t) x^2(t), \end{aligned}$$

which implies, by the first part, that for $t \in [0, \theta]$:

$$\varphi(t) \leq x(t) \leq x(0) e^{\int_0^t \beta(s) ds} + \int_0^t \alpha(s) e^{\int_s^t \beta(r) dr} ds,$$

which is (6.61) if we choose $t = \theta$. ■

Corollary 6.64. *If $\alpha, \beta \geq 0$ a.e., $\tilde{a} : [0, T] \rightarrow [0, \infty[$ is a decreasing function and $\psi : [0, T] \rightarrow [0, \infty[$ is a continuous function such that $\forall t \in [0, T]$:*

$$\psi^2(t) \leq \tilde{a}(t) + 2 \int_t^T \alpha(s) \psi(s) ds + 2 \int_t^T \beta(s) \psi^2(s) ds,$$

then

$$\psi(t) \leq \sqrt{\tilde{a}(t)} e^{\int_t^T \beta(s) ds} + \int_t^T \alpha(s) e^{\int_t^s \beta(r) dr} ds, \quad \forall t \in [0, T]. \quad (6.62)$$

Proof. Note that $\forall t \in [0, T]$:

$$\begin{aligned} & \psi^2(T-t) \\ & \leq \tilde{a}(T-t) + 2 \int_{T-t}^T \alpha(s) \psi(s) ds + 2 \int_{T-t}^T \beta(s) \psi^2(s) ds \\ & = \tilde{a}(T-t) + 2 \int_0^t \alpha(T-s) \psi(T-s) ds + 2 \int_0^t \beta(T-s) \psi^2(T-s) ds. \end{aligned}$$

Hence by (6.61)

$$\psi(T-t) \leq \tilde{a}(T-t) e^{\int_0^t \beta(T-s) ds} + \int_0^t \alpha(T-s) e^{\int_s^t \beta(T-r) dr} ds,$$

which clearly yields (6.62) replacing $T-t$ by t . ■

If $f, g \in BV_{loc}([0, \infty[) (= BV_{loc}([0, \infty[; \mathbb{R}))$, we say that $df(s) \leq dg(s)$ as signed measures on $[0, \infty[$ if

d1.

$$\int_t^s \varphi(r) df(r) \leq \int_t^s \varphi(r) dg(r),$$

for all $0 \leq t \leq s$ and for all continuous function $\varphi : [0, \infty[\rightarrow [0, \infty[$, or equivalently

d2. $f(s) - f(t) = \int_t^s df(r) \leq \int_t^s dg(r) = g(s) - g(t)$, $\forall 0 \leq t \leq s$, or equivalently

d3. $h(s) = f(s) - g(s)$ is a decreasing function on $[0, \infty[$.

Lemma 6.65. Let $x, N, V \in BV_{loc}([0, \infty[)$. If

$$x(s) \leq x(t) + \int_t^s [dN(r) + x(r) dV(r)], \quad \forall 0 \leq t \leq s,$$

or equivalently

$$dx(r) \leq dN(r) + x(r) dV(r)$$

as signed measures on $[0, \infty[$, then for all $0 \leq t \leq s$:

$$e^{-V_s} x(s) \leq x(t) e^{-V_t} + \int_t^s e^{-V_r} dN(r). \quad (6.63)$$

Proof. We have

$$\begin{aligned} d(x(r) e^{-V_r}) &= e^{-V_r} dx(r) - e^{-V_r} x(r) dV(r) \\ &\leq e^{-V_r} dN(r) \end{aligned}$$

and the result follows. ■

Corollary 6.66. *Let $\alpha, \beta \in L^1_{loc}([0, \infty[)$ and $y : [0, \infty[\rightarrow \mathbb{R}$ be a continuous function such that*

$$y(t) \leq y(s) + \int_t^s [\alpha(r) + \beta(r) y(r)] dr, \quad \forall 0 \leq t \leq s,$$

then

$$e^{\int_0^t \beta(u) du} y(t) \leq y(s) e^{\int_0^s \beta(u) du} + \int_t^s \alpha(r) e^{\int_0^r \beta(u) du} dr. \tag{6.64}$$

Proof. By Lemma 6.65 and

$$(-y(s)) \leq (-y(t)) + \int_t^s [\alpha(r) - \beta(r) (-y(r))] dr,$$

the result follows. ■

Finally we have:

Proposition 6.67. *Let $x \in BV_{loc}([0, \infty[; \mathbb{R}^d)$ and $V \in BV_{loc}([0, \infty[; \mathbb{R})$ be continuous functions. Let $R, N : [0, \infty[\rightarrow [0, \infty[$ be continuous increasing functions. If*

$$\langle x(t), dx(t) \rangle \leq dR(t) + |x(t)| dN(t) + |x(t)|^2 dV(t)$$

as signed measures on $[0, \infty[$, then for all $0 \leq t \leq T$:

$$\|e^{-V} x\|_{[t, T]} \leq 2 \left[\left| e^{-V(t)} x(t) \right| + \left(\int_t^T e^{-2V(s)} dR(s) \right)^{1/2} + \int_t^T e^{-V(s)} dN(s) \right]$$

and

$$\|x\|_{[t, T]} \leq 2e^{\uparrow V \downarrow [t, T]} \left[|x(t)| + \sqrt{R(T) - R(t)} + (N(T) - N(t)) \right].$$

If $R = 0$ then for all $0 \leq t \leq s$:

$$|x(s)| \leq e^{V(s)-V(t)} |x(t)| + \int_t^s e^{V(s)-V(r)} dN(r). \tag{6.65}$$

Proof. Let $u_\varepsilon(r) = |x(r)|^2 e^{-2V_r} + \varepsilon$, $\varepsilon > 0$. We have as signed measures on $[0, \infty[$

$$\begin{aligned} du_\varepsilon(r) &= -2e^{-2V(r)} |x(r)|^2 dV(r) + 2e^{-2V(r)} \langle x(r), dx(r) \rangle \\ &\leq 2e^{-2V(r)} dR(r) + 2e^{-2V(r)} |x(r)| dN(r) \\ &\leq 2e^{-2V(r)} dR(r) + 2e^{-V(r)} \sqrt{u_\varepsilon(r)} dN(r). \end{aligned}$$

If $R = 0$ then

$$d\left(\sqrt{u_\varepsilon(r)}\right) = \frac{du_\varepsilon(r)}{2\sqrt{u_\varepsilon(r)}} \leq e^{-V(r)} dN(r),$$

and consequently

$$\sqrt{u_\varepsilon(s)} \leq \sqrt{u_\varepsilon(t)} + \int_t^s e^{-V(r)} dN(r),$$

which yields (6.65) passing to the limit as $\varepsilon \rightarrow 0$.

If $R \neq 0$ then

$$\begin{aligned} &e^{-2V(s)} |x(s)|^2 \\ &\leq e^{-2V(t)} |x(t)|^2 + 2 \int_t^s e^{-2V(r)} dR(r) + 2 \int_t^s e^{-2V(r)} |x(r)| dN(r) \\ &\leq e^{-2V(t)} |x(t)|^2 + 2 \int_t^s e^{-2V(r)} dR(r) + 2 \|e^{-V} x\|_{[t,T]}^2 \int_t^s e^{-V(r)} dN(r) \\ &\leq \left|e^{-V(t)} x(t)\right|^2 + 2 \int_t^T e^{-2V(r)} dR(r) + \frac{1}{2} \|e^{-V} x\|_{[t,T]}^2 + 2 \left(\int_t^T e^{-V(r)} dN(r)\right)^2. \end{aligned}$$

Hence for all $t \leq \tau \leq T$

$$\begin{aligned} e^{-V(\tau)} |x(\tau)| &\leq \|e^{-V} x\|_{[t,T]}^2 \\ &\leq 2e^{-2V(t)} |x(t)|^2 + 4 \int_t^T e^{-2V(s)} dR(s) + 4 \left(\int_t^T e^{-V(s)} dN(s)\right)^2 \end{aligned}$$

and the results follow. ■

6.4.2 Stochastic Inequalities

In this subsection $\{B_t : t \geq 0\}$ is a k -dimensional Brownian motion with respect to a given stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0})$.

Proposition 6.68 (Stochastic Gronwall Inequality). *Let*

- ◇ $a, b : [0, \infty[\rightarrow [0, \infty[$ be measurable deterministic functions and
- ◇ $H, \alpha, \beta, \gamma, \delta : \Omega \times [0, \infty[\rightarrow [0, \infty[$ be stochastic processes, where H is a continuous stochastic processes. If for all $t \geq 0$

$$|X_t| + |U_t| \leq |H_t| + \int_0^t (\alpha_s + a(s) |X_s|) ds + \left| \int_0^t G_s dB_s \right|, \quad \mathbb{P}\text{-a.s.}, \quad (6.66)$$

where

- i) $X, U \in S_d^0, \quad G \in \Lambda_{d \times k}^0,$
- ii) $|G_t| \leq \beta_t + b(t) |X_t|, \quad d\mathbb{P} \otimes dt\text{-a.e.},$

then for all $q \geq 1$ there exists a positive constant C_q such that for all $T \geq 0$:

$$\begin{aligned} \mathbb{E} \sup_{t \in [0, T]} |X_t|^q + \mathbb{E} \sup_{t \in [0, T]} |U_t|^q &\leq \left[\mathbb{E} \|H\|_T^q + \mathbb{E} \left(\int_0^T \alpha_s ds \right)^q \right. \\ &\quad \left. + \mathbb{E} \left(\int_0^T \beta_s^2 ds \right)^{q/2} \right] \times \exp \left\{ C_q \left[1 + T^{q-1} \int_0^T (a^q(s) + b^{2q}(s)) ds \right] \right\}. \end{aligned} \quad (6.67)$$

In particular if the right-hand side of the inequality (6.67) is finite then

$$X, U \in S_d^q, \quad G \in \Lambda_{d \times k}^q.$$

Proof. Clearly we can assume that the right-hand side of the inequality (6.67) is finite. Denote by C_q different constants depending only q and which can be changed from one line to another. For each $n \geq 1$, we define the stopping time

$$\tau_n(\omega) = \inf \{t \geq 0 : |X_t(\omega)| \geq n\} \wedge n.$$

Note that for all positive stochastic processes Z ,

$$\int_0^{t \wedge \tau_n} |X_s|^p Z_s ds = \int_0^{t \wedge \tau_n} |X_{s \wedge \tau_n}|^p Z_s ds \leq \int_0^t \sup_{r \in [0, s]} |X_{r \wedge \tau_n}|^p Z_s ds.$$

By the convexity of the function $\varphi(r) = |r|^q$ we have

$$\begin{aligned} |X_{t \wedge \tau_n}|^q + |U_{t \wedge \tau_n}|^q &\leq 2^q \|H\|_{t \wedge \tau_n}^q + 4^q \left| \int_0^{t \wedge \tau_n} (\alpha_s + a_s |X_s|) ds \right|^q \\ &\quad + 4^q \left| \int_0^{t \wedge \tau_n} G_s dB_s \right|^q. \end{aligned}$$

By the Burkholder–Davis–Gundy and Hölder inequalities:

$$\begin{aligned}
 4^q \mathbb{E} \sup_{s \in [0, t]} \left| \int_0^{s \wedge \tau_n} G_r dB_r \right|^q &\leq C_q \mathbb{E} \left[\left(\int_0^{t \wedge \tau_n} |G_s|^2 ds \right)^{q/2} \right] \\
 &\leq C_q \mathbb{E} \left(2 \int_0^t \beta_s^2 ds + 2 \int_0^t b^2(s) |X_{s \wedge \tau_n}|^2 ds \right)^{q/2} \\
 &\leq C_q \mathbb{E} \left(\int_0^t \beta_s^2 ds \right)^{q/2} + C_q \mathbb{E} \left[\sup_{s \in [0, t]} |X_{s \wedge \tau_n}|^{q/2} \left(\int_0^t b^2(s) \sup_{r \in [0, s]} |X_{r \wedge \tau_n}| ds \right)^{q/2} \right] \\
 &\leq \frac{1}{4} \mathbb{E} \sup_{s \in [0, t]} |X_{s \wedge \tau_n}|^q + C_q \mathbb{E} \left(\int_0^t \beta_s^2 ds \right)^{q/2} + C_q \mathbb{E} \left(\int_0^t b^2(s) \sup_{r \in [0, s]} |X_{r \wedge \tau_n}| ds \right)^q \\
 &\leq \frac{1}{4} \mathbb{E} \sup_{s \in [0, t]} |X_{s \wedge \tau_n}|^q + C_q \mathbb{E} \left(\int_0^t \beta_s^2 ds \right)^{q/2} \\
 &\quad + C_q t^{q-1} \int_0^t b^{2q}(s) \mathbb{E} \sup_{r \in [0, s]} |X_{r \wedge \tau_n}|^q ds.
 \end{aligned}$$

Also

$$\begin{aligned}
 4^q \left| \int_0^{t \wedge \tau_n} (\alpha_s + a_s |X_s|) ds \right|^q &\leq C_q \left(\int_0^t \alpha_s ds \right)^q + C_q \left(\int_0^t a(s) |X_{s \wedge \tau_n}| ds \right)^q \\
 &\leq C_q \left(\int_0^t \alpha_s ds \right)^q + C_q t^{q-1} \int_0^t a^q(s) \sup_{r \in [0, s]} |X_{r \wedge \tau_n}|^q ds.
 \end{aligned}$$

Hence, defining

$$K_{q,t} = \mathbb{E} \|H\|_t^q + \mathbb{E} \left[\left(\int_0^t \alpha_s ds \right)^q + \left(\int_0^t \beta_s^2 ds \right)^{q/2} \right],$$

we have

$$\begin{aligned}
 \mathbb{E} \sup_{s \in [0, t]} |X_{s \wedge \tau_n}|^q + \mathbb{E} \sup_{s \in [0, t]} |U_{s \wedge \tau_n}|^q &\leq 2 \mathbb{E} \sup_{s \in [0, t]} [|X_{s \wedge \tau_n}|^q + |U_{s \wedge \tau_n}|^q] \\
 &\leq C_q K_{q,t} + C_q t^{q-1} \int_0^t (a^q(s) + b^{2q}(s)) \mathbb{E} \sup_{r \in [0, s]} |X_{r \wedge \tau_n}|^q ds.
 \end{aligned}$$

Using Gronwall's inequality (6.57) we obtain

$$\mathbb{E} \sup_{s \in [0, t]} |X_{s \wedge \tau_n}|^q \leq C_q K_{q,t} e^{C_q A_q(t)} < \infty, \quad (6.68)$$

where

$$A_q(t) = t^{q-1} \int_0^t (a^q(s) + b^{2q}(s)) ds.$$

Since $1 + xe^{ax} \leq e^{(a+1)x}$, for all $x \geq 0$, it follows that

$$\begin{aligned} \mathbb{E} \sup_{s \in [0,t]} |U_{s \wedge \tau_n}|^q &\leq C_q [K_{q,t} + A_q(t) C_q K_{q,t} e^{C_q A_q(t)}] \\ &\leq C_q K_{q,t} e^{2C_q A_q(t)} < \infty. \end{aligned} \tag{6.69}$$

We also have

$$\mathbb{E} \left[\left(\int_0^{t \wedge \tau_n} |G_s|^2 ds \right)^{q/2} \right] \leq \hat{C}_{q,t} < \infty \tag{6.70}$$

for some $\hat{C}_{q,t}$ independent of n . Passing to the limit in (6.68)–(6.70) as $n \rightarrow \infty$, we obtain $X, U \in S_d^q[0, T]$, $G \in \Lambda_{d \times k}^q(0, T)$ and (6.67) follows. \blacksquare

Proposition 6.69. *Let $\delta \in \{-1, 1\}$. Let $\{B_t : t \geq 0\}$ be a k -dimensional Brownian motion. Let $Y, K, V : \Omega \times \mathbb{R}_+ \rightarrow \mathbb{R}$ and $G : \Omega \times \mathbb{R}_+ \rightarrow \mathbb{R}^k$ be progressively measurable stochastic processes such that*

- i) Y, K, V are continuous stochastic processes,
- ii) $V, K \in BV_{loc}([0, \infty[; \mathbb{R})$, $V_0 = K_0 = 0$, \mathbb{P} -a.s.,
- iii) $\int_t^s |G_r|^2 dr < \infty$, \mathbb{P} -a.s., $\forall 0 \leq t \leq s$.

If for all $0 \leq t \leq s$,

$$\delta(Y_t - Y_s) \leq \int_t^s (dK_r + Y_r dV_r) + \int_t^s \langle G_r, dB_r \rangle, \quad \mathbb{P}\text{-a.s.},$$

then

$$\delta(Y_t e^{\delta V_t} - Y_s e^{\delta V_s}) \leq \int_t^s e^{\delta V_r} dK_r + \int_t^s e^{\delta V_r} \langle G_r, dB_r \rangle, \quad \mathbb{P}\text{-a.s.}$$

Proof. Denoting

$$M_s = \int_0^s \langle G_r, dB_r \rangle, \quad \tilde{Y}_s = -M_s - \delta Y_s, \tag{6.71}$$

we obtain

$$\tilde{Y}_s \leq \tilde{Y}_t + \int_t^s [dK_r + (-\delta \tilde{Y}_r - \delta M_r) dV_r].$$

Hence

$$s \mapsto L_s \stackrel{\text{def}}{=} \tilde{Y}_s - \int_0^s [dK_r + (-\delta\tilde{Y}_r - \delta M_r) dV_r]$$

is a decreasing function and then

$$\begin{aligned} d(\tilde{Y}_s e^{\delta V_s}) &= \{dL_s + [dK_s + (-\delta\tilde{Y}_s - \delta M_s) dV_s]\} e^{\delta V_s} + \delta \tilde{Y}_s e^{\delta V_s} dV_s \\ &\leq -\delta M_s e^{\delta V_s} dV_s + e^{\delta V_s} dK_s \end{aligned}$$

and integrating from t to s

$$\begin{aligned} \tilde{Y}_s e^{\delta V_s} &\leq \tilde{Y}_t e^{\delta V_t} - \int_t^s \delta M_r e^{\delta V_r} dV_r + \int_t^s e^{\delta V_r} dK_r \\ &= \tilde{Y}_t e^{\delta V_t} - M_s e^{\delta V_s} + M_t e^{\delta V_t} + \int_t^s e^{\delta V_r} \langle G_r, dB_r \rangle + \int_t^s e^{\delta V_r} dK_r. \end{aligned}$$

Now by (6.71) we obtain the conclusions. ■

6.4.3 Forward Stochastic Inequalities

In this subsection $\{B_t : t \geq 0\}$ is a k -dimensional Brownian motion with respect to a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0})$.

We shall derive some estimates on the local semimartingale $X \in S_d^0$ of the form

$$X_t = X_0 + K_t + \int_0^t G_s dB_s, \quad t \geq 0, \quad \mathbb{P}\text{-a.s.}, \quad (6.72)$$

where

- ◇ $K \in S_d^0$; $K \in BV_{loc}([0, \infty[; \mathbb{R}^d)$, $K_0 = 0$, \mathbb{P} -a.s.;
- ◇ $G \in \Lambda_{d \times k}^0$.

Notation 6.70. Let $p \geq 1$ and $m_p \stackrel{\text{def}}{=} 1 \vee (p - 1)$.

Proposition 6.71. Let $X \in S_d^0$ be a local semimartingale of the form (6.72). Assume there exist $p \geq 1$, a \mathcal{P} -m.i.c.s.p. D and a \mathcal{P} -m.b.v.c.s.p. V , $D_0 = V_0 = 0$, such that as signed measures on $[0, \infty[$

$$dD_t + \langle X_t, dK_t \rangle + \frac{1}{2} m_p |G_t|^2 dt \leq |X_t|^2 dV_t, \quad \mathbb{P}\text{-a.s.}, \quad (6.73)$$

then for all $0 \leq t \leq s$:

$$\mathbb{E}^{\mathcal{F}_t} |e^{-V_s} X_s|^p + p \mathbb{E}^{\mathcal{F}_t} \int_t^s e^{-pV_r} |X_r|^{p-2} dD_r \leq |e^{-V_t} X_t|^p, \quad \mathbb{P}\text{-a.s.} \quad (6.74)$$

Moreover for all $\delta \geq 0, 0 \leq t \leq s$:

$$\begin{aligned} \mathbb{E}^{\mathcal{F}_t} \frac{|e^{-V_s} X_s|^p}{(1+\delta|e^{-V_s} X_s|^2)^{p/2}} + p \mathbb{E}^{\mathcal{F}_t} \int_t^s \frac{e^{-pV_r} |X_r|^{p-2}}{(1+\delta|e^{-V_r} X_r|^2)^{(p+2)/2}} dD_r \\ \leq \frac{|e^{-V_t} X_t|^p}{(1+\delta|e^{-V_t} X_t|^2)^{p/2}}, \quad \mathbb{P}\text{-a.s.} \end{aligned} \quad (6.75)$$

The proof of this Proposition is contained in the proof of the next Proposition.

Remark 6.72. Since by (2.27)

$$\mathbf{1}_{X_t=0} |G_t|^2 dt = 0,$$

we see that the condition (6.73) yields

$$\mathbf{1}_{X_t=0} dD_t = 0.$$

We now formulate a more general assumption.

(FB) *There exist*

- $p \geq 1, \lambda \geq 0,$
- *three \mathcal{P} -m.i.c.s.p. $D, R, N, D_0 = R_0 = N_0 = 0,$ and*
- *a \mathcal{P} -m.b-v.c.s.p. $V, V_0 = 0,$*

such that, as signed measures on $[0, \infty[$:

$$\begin{aligned} dD_t + \langle X_t, dK_t \rangle + \left(\frac{1}{2} m_p + 9p\lambda \right) |G_t|^2 dt \\ \leq \mathbf{1}_{p \geq 2} dR_t + |X_t| dN_t + |X_t|^2 dV_t. \end{aligned} \quad (6.76)$$

Remark 6.73. From the condition (6.76), we deduce that

$$\begin{aligned} \mathbf{1}_{X_t=0} dD_t = 0, \quad \text{if } 1 \leq p < 2, \text{ and} \\ \mathbf{1}_{X_t=0} dD_t \leq \mathbf{1}_{X_t=0} dR_t \leq dR_t, \quad \text{if } p \geq 2. \end{aligned}$$

Proposition 6.74. *Let $X \in S_d^0$ be a local semimartingale of the form (6.72). Assume that there exist $p \geq 1$ and $\lambda > 1$ such that **(FB)** is satisfied. Then there exists a positive constant $C_{p,\lambda}$ depending only on (p, λ) such that for all $\delta \geq 0,$ and $0 \leq t \leq s$:*

$$\begin{aligned}
& \mathbb{E}^{\mathcal{F}_t} \frac{\|e^{-V} X\|_{[t,s]}^p}{(1+\delta\|e^{-V} X\|_{[t,s]}^2)^{p/2}} + \mathbb{E}^{\mathcal{F}_t} \int_t^s \frac{e^{-pV_r} |X_r|^{p-2}}{(1+\delta|e^{-V_r} X_r|^2)^{(p+2)/2}} dD_r \\
& \quad + \mathbb{E}^{\mathcal{F}_t} \left(\int_t^s \frac{e^{-2V_r}}{(1+\delta|e^{-V_r} X_r|^2)^2} dD_r \right)^{p/2} \\
& \quad + \mathbb{E}^{\mathcal{F}_t} \left(\int_t^s \frac{e^{-2V_r}}{(1+\delta|e^{-V_r} X_r|^2)^2} |G_r|^2 dr \right)^{p/2} \tag{6.77} \\
& \leq C_{p,\lambda} \left[\frac{|e^{-V_t} X_t|^p}{(1+\delta|e^{-V_t} X_t|^2)^{p/2}} + \mathbb{E}^{\mathcal{F}_t} \left(\int_t^s e^{-2V_r} \mathbf{1}_{p \geq 2} dR_r \right)^{p/2} \right. \\
& \quad \left. + \mathbb{E}^{\mathcal{F}_t} \left(\int_t^s e^{-V_r} dN_r \right)^p \right], \quad \mathbb{P}\text{-a.s.}
\end{aligned}$$

If we set $\delta = 0$ in (6.77), we obtain the following:

Corollary 6.75. *Under the assumption (FB), for all $0 \leq t \leq s$:*

$$\begin{aligned}
& \mathbb{E}^{\mathcal{F}_t} \|e^{-V} X\|_{[t,s]}^p + \mathbb{E}^{\mathcal{F}_t} \int_t^s e^{-pV_r} |X_r|^{p-2} dD_r \\
& \quad + \mathbb{E}^{\mathcal{F}_t} \left(\int_t^s e^{-2V_r} dD_r \right)^{p/2} \\
& \quad + \mathbb{E}^{\mathcal{F}_t} \left(\int_t^s e^{-2V_r} |G_r|^2 dr \right)^{p/2} \tag{6.78} \\
& \leq C_{p,\lambda} \left[|e^{-V_t} X_t|^p + \mathbb{E}^{\mathcal{F}_t} \left(\int_t^s e^{-2V_r} \mathbf{1}_{p \geq 2} dR_r \right)^{p/2} \right. \\
& \quad \left. + \mathbb{E}^{\mathcal{F}_t} \left(\int_t^s e^{-V_r} dN_r \right)^p \right], \quad \mathbb{P}\text{-a.s.}
\end{aligned}$$

Proof (of Proposition 6.74). In view of the monotone convergence theorem it suffices to treat the case $\delta > 0$, which we assume from now on.

To simplify, we define

$$\begin{aligned}
J_r & \stackrel{\text{def}}{=} \frac{|e^{-V_r} X_r|}{(1 + \delta |e^{-V_r} X_r|^2)^{1/2}} \\
& \leq \frac{1}{\sqrt{\delta}},
\end{aligned}$$

and

$$\begin{aligned}
\hat{j}_r^{(p)} & \stackrel{\text{def}}{=} \frac{|e^{-V_r} X_r|^{p-2} \mathbf{1}_{X_r \neq 0}}{(1 + \delta |e^{-V_r} X_r|^2)^{(p+2)/2}} \\
& = J_r^{p-2} \frac{\mathbf{1}_{X_r \neq 0}}{(1 + \delta |e^{-V_r} X_r|^2)^2}.
\end{aligned}$$

We remark that

$$\hat{J}_r^{(p)} e^{-V_r} |X_r| \leq J_r^{p-1} \mathbf{1}_{X_r \neq 0} \quad \text{and} \quad \hat{J}_r^{(p)} e^{-2V_r} |X_r|^2 \leq J_r^p.$$

Step 1. General calculation.

We begin by assuming a condition which is more general than the assumptions (6.73) and (6.76), namely that there exists a $\gamma \geq 0$ such that

$$\begin{aligned} dD_r + \langle X_r, dK_r \rangle + \left(\frac{m_p}{2} + \gamma \right) |G_r|^2 dr \\ \leq \mathbf{1}_{p \geq 2} dR_r + |X_r| dN_r + |X_r|^2 dV_r. \end{aligned} \quad (6.79)$$

Since by Itô's formula

$$e^{-V_t} X_t = X_0 + \int_0^t (e^{-V_r} dK_r - e^{-V_r} X_r dV_r) + \int_0^t e^{-V_r} G_r dB_r,$$

it follows from the inequality (2.28) in Corollary 2.28 that for all $0 \leq t \leq s$ and any stopping time θ

$$\begin{aligned} J_{s \wedge \theta}^p &\leq J_{t \wedge \theta}^p + p \int_t^s \mathbf{1}_{r < \theta} \hat{J}_r^{(p)} e^{-2V_r} \langle X_r, G_r dB_r \rangle \\ &\quad + p \int_t^s \mathbf{1}_{r < \theta} \hat{J}_r^{(p)} e^{-2V_r} \left[\langle X_r, dK_r - X_r dV_r \rangle + \frac{1}{2} m_p |G_r|^2 \right] dr, \quad a.s. \end{aligned}$$

But

$$J_s^p \mathbf{1}_{s < \theta} \leq J_{s \wedge \theta}^p;$$

hence we deduce that

$$\begin{aligned} J_s^p \mathbf{1}_{s < \theta} + p \int_t^s \mathbf{1}_{r < \theta} \hat{J}_r^{(p)} e^{-2V_r} dD_r + p\gamma \int_t^s \mathbf{1}_{r < \theta} \hat{J}_r^{(p)} e^{-2V_r} |G_r|^2 dr \\ \leq J_{t \wedge \theta}^p + p \int_t^s \mathbf{1}_{r < \theta} \hat{J}_r^{(p)} e^{-2V_r} \langle X_r, G_r dB_r \rangle \\ + p \int_t^s \mathbf{1}_{r < \theta} \hat{J}_r^{(p)} e^{-2V_r} \left[dD_r + \langle X_r, dK_r - X_r dV_r \rangle + \left(\frac{1}{2} m_p + \gamma \right) |G_r|^2 \right] dr, \end{aligned}$$

and using the assumption (6.79) it follows that for any stopping time θ and for all $0 \leq t \leq s$, \mathbb{P} -a.s.:

$$\begin{aligned}
& J_s^p \mathbf{1}_{s < \theta} + p \int_t^s \mathbf{1}_{r < \theta} \hat{J}_r^{(p)} e^{-2V_r} dD_r + p\gamma \int_t^s \mathbf{1}_{r < \theta} \hat{J}_r^{(p)} e^{-2V_r} |G_r|^2 dr \\
& \leq J_{t \wedge \theta}^p + p \int_t^s \mathbf{1}_{r < \theta} \hat{J}_r^{(p)} e^{-2V_r} \langle X_r, G_r dB_r \rangle \\
& \quad + p \int_t^s \mathbf{1}_{r < \theta} J_r^{p-2} \mathbf{1}_{X_r \neq 0} e^{-2V_r} \mathbf{1}_{p \geq 2} dR_r + p \int_t^s \mathbf{1}_{r < \theta} J_r^{p-1} \mathbf{1}_{X_r \neq 0} e^{-V_r} dN_r.
\end{aligned} \tag{6.80}$$

Since for all $T > 0$:

$$\begin{aligned}
\int_0^T \left| \hat{J}_r^{(p)} e^{-2V_r} X_r^* G_r \right|^2 dr & \leq \sup_{r \in [0, T]} \left[e^{-\rho V_r} |X_r|^{p-1} \right] \int_0^T |G_r|^2 dr \\
& < \infty, \quad \mathbb{P}\text{-a.s.},
\end{aligned}$$

it follows that for all $0 \leq t \leq s$:

$$\int_t^s \hat{J}_r^{(p)} e^{-2V_r} dD_r + \gamma \int_t^s \hat{J}_r^{(p)} e^{-2V_r} |G_r|^2 dr < \infty, \quad a.s.$$

For each $n \in \mathbb{N}^*$ we define the stopping time

$$\begin{aligned}
\theta_n = \inf \left\{ t \geq 0 : \int_0^t J_r^{p-2} \mathbf{1}_{X_r \neq 0} e^{-2V_r} \mathbf{1}_{p \geq 2} dR_r + \int_0^t J_r^{p-1} \mathbf{1}_{X_r \neq 0} e^{-V_r} dN_r \right. \\
\left. + \int_0^t \left| \hat{J}_r^{(p)} e^{-2V_r} X_r^* G_r \right|^2 dr \geq n \right\}.
\end{aligned} \tag{6.81}$$

Note that for $\theta = \theta_n$

$$M_t^n = p \int_0^t \mathbf{1}_{r < \theta_n} \hat{J}_r^{(p)} \langle e^{-V_r} X_r, e^{-V_r} G_r dB_r \rangle$$

is a martingale and consequently, for all $0 \leq t \leq s$:

$$\mathbb{E}^{\mathcal{F}_t} \int_t^s \mathbf{1}_{r < \theta_n} \hat{J}_r^{(p)} e^{-2V_r} dD_r + \gamma \mathbb{E}^{\mathcal{F}_t} \int_t^s \mathbf{1}_{r < \theta_n} \hat{J}_r^{(p)} e^{-2V_r} |G_r|^2 dr < \infty, \quad a.s.$$

Step 2. Proof of the inequality (6.75).

In view of the first step, the assumption (6.73) yields (6.80) with $\gamma = 0$ and $R = N = 0$, from which we deduce

$$\mathbb{E}^{\mathcal{F}_t} J_s^p \mathbf{1}_{s < \theta_n} + p \mathbb{E}^{\mathcal{F}_t} \int_t^s \mathbf{1}_{r < \theta_n} \hat{J}_r^{(p)} e^{-2V_r} dD_r \leq J_{t \wedge \theta_n}^p, \quad a.s., \tag{6.82}$$

and passing to the limit as $n \rightarrow \infty$ (the first two terms converge monotonically and the third one converges *a.s.*) the estimate (6.75) follows in view of Remark 6.73, since $R = 0$.

Step 3. Proof of the inequality (6.77).

(A) Let $\gamma > 0$. From (6.80) we have

$$\begin{aligned}
& \mathbb{E}^{\mathcal{F}_t} \sup_{r \in [t, s]} (J_r^p \mathbf{1}_{r < \theta_n}) + p \mathbb{E}^{\mathcal{F}_t} \int_t^s \mathbf{1}_{r < \theta_n} \hat{J}_r^{(p)} e^{-2V_r} dD_r \\
& \quad + p\gamma \mathbb{E}^{\mathcal{F}_t} \int_t^s \mathbf{1}_{r < \theta_n} \hat{J}_r^{(p)} |e^{-V_r} G_r|^2 dr \\
& \leq 2J_{t \wedge \theta_n}^p + 2p \mathbb{E}^{\mathcal{F}_t} \int_t^s \mathbf{1}_{r < \theta_n} J_r^{p-2} \mathbf{1}_{X_r \neq 0} e^{-2V_r} \mathbf{1}_{p \geq 2} dR_r \\
& \quad + 2p \mathbb{E}^{\mathcal{F}_t} \int_t^s \mathbf{1}_{r < \theta_n} J_r^{p-1} \mathbf{1}_{X_r \neq 0} e^{-V_r} dN_r \\
& \quad + 2p \mathbb{E}^{\mathcal{F}_t} \sup_{u \in [t, s]} \left| \int_t^u \mathbf{1}_{r < \theta_n} \hat{J}_r^{(p)} e^{-2V_r} X_r^* G_r dB_r \right|.
\end{aligned}$$

By the Burkholder–Davis–Gundy inequality

$$\begin{aligned}
& 2p \mathbb{E}^{\mathcal{F}_t} \sup_{u \in [t, s]} \left| \int_t^u \mathbf{1}_{r < \theta_n} \hat{J}_r^{(p)} \langle e^{-V_r} X_r, e^{-V_r} G_r dB_r \rangle \right| \\
& \leq 6p \mathbb{E}^{\mathcal{F}_t} \sqrt{\int_t^s \mathbf{1}_{r < \theta_n} \hat{J}_r^{(p)} e^{-2V_r} X_r^* G_r|^2 dr} \\
& \leq 6p \mathbb{E}^{\mathcal{F}_t} \left[\sqrt{\sup_{r \in [t, s]} \hat{J}_r^{(p)} e^{-2V_r} |X_r|^2 \mathbf{1}_{r < \theta_n}} \sqrt{\int_t^s \mathbf{1}_{r < \theta_n} \hat{J}_r^{(p)} |e^{-V_r} G_r|^2 dr} \right] \\
& \leq \frac{1}{\lambda} \mathbb{E}^{\mathcal{F}_t} \sup_{r \in [t, s]} (J_r^p \mathbf{1}_{r < \theta_n}) + 9p^2 \lambda \mathbb{E}^{\mathcal{F}_t} \int_t^s \mathbf{1}_{r < \theta_n} \hat{J}_r^{(p)} |e^{-V_r} G_r|^2 dr,
\end{aligned}$$

for all $\lambda > 0$. Hence

$$\begin{aligned}
& \left(1 - \frac{1}{\lambda}\right) \mathbb{E}^{\mathcal{F}_t} \sup_{r \in [t, s]} (J_r^p \mathbf{1}_{r < \theta_n}) + p \mathbb{E}^{\mathcal{F}_t} \int_t^s \mathbf{1}_{r < \theta_n} \hat{J}_r^{(p)} e^{-2V_r} dD_r \\
& \quad + p(\gamma - 9p\lambda) \mathbb{E}^{\mathcal{F}_t} \int_t^s \mathbf{1}_{r < \theta_n} \hat{J}_r^{(p)} |e^{-V_r} G_r|^2 dr \\
& \leq 2J_{t \wedge \theta_n}^p + 2p \mathbb{E}^{\mathcal{F}_t} \int_t^s \mathbf{1}_{r < \theta_n} J_r^{p-2} \mathbf{1}_{X_r \neq 0} e^{-2V_r} \mathbf{1}_{p \geq 2} dR_r \\
& \quad + 2p \mathbb{E}^{\mathcal{F}_t} \int_t^s \mathbf{1}_{r < \theta_n} J_r^{p-1} \mathbf{1}_{X_r \neq 0} e^{-V_r} dN_r.
\end{aligned}$$

Let $\gamma = 9p\lambda$, $\lambda > 1$. By Hölder's inequality

$$\begin{aligned}
& 2p \mathbb{E}^{\mathcal{F}_t} \int_t^s \mathbf{1}_{r < \theta_n} J_r^{p-2} \mathbf{1}_{X_r \neq 0} e^{-2V_r} \mathbf{1}_{p \geq 2} dR_r + 2p \mathbb{E}^{\mathcal{F}_t} \int_t^s \mathbf{1}_{r < \theta_n} J_r^{p-1} \mathbf{1}_{X_r \neq 0} e^{-V_r} dN_r \\
& \leq 2p \mathbb{E}^{\mathcal{F}_t} \left[\sup_{r \in [t, s]} (J_r^{p-2} \mathbf{1}_{X_r \neq 0} \mathbf{1}_{p \geq 2} \mathbf{1}_{r < \theta_n}) \int_t^s e^{-2V_r} \mathbf{1}_{p \geq 2} dR_r \right] \\
& \quad + 2p \mathbb{E}^{\mathcal{F}_t} \left[\sup_{r \in [t, s]} (J_r^{p-1} \mathbf{1}_{X_r \neq 0} \mathbf{1}_{r < \theta_n}) \int_t^s e^{-V_r} dN_r \right] \\
& \leq \frac{1}{2} \left(1 - \frac{1}{\lambda} \right) \mathbb{E}^{\mathcal{F}_t} \sup_{r \in [t, s]} (J_r^p \mathbf{1}_{r < \theta_n}) + C_{p, \lambda} \mathbb{E}^{\mathcal{F}_t} \left(\int_t^s e^{-2V_r} \mathbf{1}_{p \geq 2} dR_r \right)^{p/2} \\
& \quad + C_{p, \lambda} \mathbb{E}^{\mathcal{F}_t} \left(\int_t^s e^{-V_r} dN_r \right)^p.
\end{aligned}$$

We deduce from the above that

$$\begin{aligned}
& \mathbb{E}^{\mathcal{F}_t} \sup_{r \in [t, s]} (J_r^p \mathbf{1}_{r < \theta_n}) + \mathbb{E}^{\mathcal{F}_t} \int_t^s \mathbf{1}_{r < \theta_n} \hat{J}_r^{(p)} e^{-2V_r} dD_r \\
& \leq C_{p, \lambda} \left[J_{t \wedge \theta_n}^p + \mathbb{E}^{\mathcal{F}_t} \left(\int_t^s e^{-2V_r} \mathbf{1}_{p \geq 2} dR_r \right)^{p/2} + \mathbb{E}^{\mathcal{F}_t} \left(\int_t^s e^{-V_r} dN_r \right)^p \right].
\end{aligned} \tag{6.83}$$

The argument used in order to take the limit in (6.82) yields as $n \rightarrow \infty$:

$$\begin{aligned}
& \mathbb{E}^{\mathcal{F}_t} \sup_{r \in [t, s]} J_r^p + \mathbb{E}^{\mathcal{F}_t} \int_t^s \hat{J}_r^{(p)} e^{-2V_r} dD_r \leq C_{p, \lambda} \left[J_t^p \right. \\
& \quad \left. + \mathbb{E}^{\mathcal{F}_t} \left(\int_t^s e^{-2V_r} \mathbf{1}_{p \geq 2} dR_r \right)^{p/2} + \mathbb{E}^{\mathcal{F}_t} \left(\int_t^s e^{-V_r} dN_r \right)^p \right].
\end{aligned} \tag{6.84}$$

(B) From (6.80) for $p = 2$, $\gamma = 1$ and $\theta = \theta_n$ we have

$$\begin{aligned}
& J_{s \wedge \theta_n}^2 + 2 \int_t^s \mathbf{1}_{r < \theta_n} \hat{J}_r^{(2)} e^{-2V_r} dD_r + 2 \int_t^s \mathbf{1}_{r < \theta_n} \hat{J}_r^{(2)} |e^{-V_r} G_r|^2 dr \\
& \leq J_{t \wedge \theta_n}^2 + 2 \int_t^s \mathbf{1}_{r < \theta_n} \mathbf{1}_{X_r \neq 0} e^{-2V_r} dR_r + 2 \int_t^s \mathbf{1}_{r < \theta_n} J_r \mathbf{1}_{X_r \neq 0} e^{-V_r} dN_r \\
& \quad + 2 \int_t^s \mathbf{1}_{r < \theta_n} \hat{J}_r^{(2)} \langle e^{-V_r} X_r, e^{-V_r} G_r dB_r \rangle,
\end{aligned}$$

which yields

$$\begin{aligned}
& \mathbb{E}^{\mathcal{F}_t} \left(\int_t^s \mathbf{1}_{r < \theta_n} \hat{J}_r^{(2)} e^{-2V_r} dD_r \right)^{p/2} + \mathbb{E}^{\mathcal{F}_t} \left(\int_t^s \mathbf{1}_{r < \theta_n} \hat{J}_r^{(2)} |e^{-V_r} G_r|^2 dr \right)^{p/2} \\
& \leq C_p J_t^p + C_p \mathbb{E}^{\mathcal{F}_t} \left(\int_t^s e^{-2V_r} \mathbf{1}_{p \geq 2} dR_r \right)^{p/2}
\end{aligned}$$

$$\begin{aligned}
& + C_p \mathbb{E}^{\mathcal{F}_t} \sup_{r \in [t, s]} (J_r^{p/2} \mathbf{1}_{r < \theta_n}) \left(\int_t^s e^{-V_r} dN_r \right)^{p/2} \\
& + C_p \mathbb{E}^{\mathcal{F}_t} \sup_{u \in [t, s]} \left| \int_t^u \mathbf{1}_{r < \theta_n} \hat{J}_r^{(2)} \left(e^{-V_r} X_r, e^{-V_r} G_r dB_r \right) \right|^{p/2}.
\end{aligned}$$

By the Burkholder–Davis–Gundy inequality (2.8)

$$\begin{aligned}
& C_p \mathbb{E}^{\mathcal{F}_t} \sup_{u \in [t, s]} \left| \int_t^u \mathbf{1}_{r < \theta_n} \hat{J}_r^{(2)} \left(e^{-V_r} X_r, e^{-V_r} G_r dB_r \right) \right|^{p/2} \\
& \leq C'_p \mathbb{E}^{\mathcal{F}_t} \left(\int_t^s \mathbf{1}_{r < \theta_n} \left| \hat{J}_r^{(2)} e^{-2V_r} |X_r^* G_r|^2 \right| dr \right)^{p/4} \\
& \leq C'_p \mathbb{E}^{\mathcal{F}_t} \sup_{r \in [t, s]} (J_r^{p/2} \mathbf{1}_{r < \theta_n}) \left(\int_t^s \mathbf{1}_{r < \theta_n} \hat{J}_r^{(2)} |e^{-V_r} G_r|^2 dr \right)^{p/4} \\
& \leq C''_p \mathbb{E}^{\mathcal{F}_t} \sup_{r \in [t, s]} (J_r^p \mathbf{1}_{r < \theta_n}) + \frac{1}{2} \mathbb{E}^{\mathcal{F}_t} \left(\int_t^s \mathbf{1}_{r < \theta_n} \hat{J}_r^{(2)} |e^{-V_r} G_r|^2 dr \right)^{p/2}.
\end{aligned}$$

Hence

$$\begin{aligned}
& \mathbb{E}^{\mathcal{F}_t} \left(\int_t^s \mathbf{1}_{r < \theta_n} \hat{J}_r^{(2)} e^{-2V_r} dD_r \right)^{p/2} + \frac{1}{2} \mathbb{E}^{\mathcal{F}_t} \left(\int_t^s \mathbf{1}_{r < \theta_n} \hat{J}_r^{(2)} |e^{-V_r} G_r|^2 dr \right)^{p/2} \\
& \leq C_p \left[\mathbb{E}^{\mathcal{F}_t} \sup_{r \in [t, s]} J_r^p + \mathbb{E}^{\mathcal{F}_t} \left(\int_t^s e^{-2V_r} \mathbf{1}_{p \geq 2} dR_r \right)^{p/2} + \mathbb{E}^{\mathcal{F}_t} \left(\int_t^s e^{-V_r} dN_r \right)^p \right].
\end{aligned} \tag{6.85}$$

We take the limit as $n \rightarrow \infty$ in the last inequality and the estimate (6.77) follows from (6.84), (6.85), Remark 6.73 and the identity

$$\sup_{r \in [t, s]} J_r^p = \frac{\|e^{-V} X\|_{[t, s]}^p}{(1 + \delta \|e^{-V} X\|_{[t, s]}^2)^{p/2}}.$$

This last fact follows from the increasing monotonicity of the function

$$r \mapsto \frac{r^p}{(1 + \delta r^2)^{p/2}} : [0, \infty[\rightarrow [0, \infty[.$$

The proof is complete. ■

We shall give a supplementary result in the case when R, N, V are deterministic functions.

Corollary 6.76. *Let $X \in S_d^0$ be a local semimartingale of the form*

$$X_t = X_0 + K_t + \int_0^t G_s dB_s, \quad t \geq 0, \quad \mathbb{P}\text{-a.s.},$$

where

- ◇ $K \in S_d^0$; $K \in BV_{loc}([0, \infty[; \mathbb{R}^d)$, $K_0 = 0$, \mathbb{P} -a.s.;
- ◇ $G \in \Lambda_{d \times k}^0$.

Assume that there exist

- $p \geq 1, m_p \stackrel{\text{def}}{=} 1 \vee (p - 1)$;
- two continuous increasing deterministic functions $R, N : [0, \infty[\rightarrow [0, \infty[$, $R(0) = N(0) = 0$, and
- a continuous deterministic function with bounded variation $V : [0, \infty[\rightarrow \mathbb{R}$, $V(0) = 0$,

such that as signed measures on $[0, \infty[$:

$$(X_t, dK_t) + \frac{1}{2} m_p |G_t|^2 dt \leq \mathbf{1}_{p \geq 2} dR(t) + |X_t| dN(t) + |X_t|^2 dV(t). \quad (6.86)$$

Define

$$\begin{aligned} Q(t) &= 2R(t) \mathbf{1}_{p \geq 2} + N(t), \\ P(t) &= (p - 2) R(t) \mathbf{1}_{p \geq 2} + (p - 1) N(t) + pV(t) \quad \text{and} \\ M(t) &= \int_0^t e^{-P(r)} dQ(r). \end{aligned}$$

Then for all $\delta \geq 0$ and $0 \leq t \leq s$:

$$\mathbb{E} \frac{|X_s|^p e^{-P(s)}}{\left(1 + \delta |X_s|^2\right)^{p/2}} \leq \mathbb{E} \frac{|X_t|^p e^{-P(t)}}{\left(1 + \delta |X_t|^2\right)^{p/2}} + M(s) - M(t). \quad (6.87)$$

In particular for $\delta \searrow 0$ and $0 = t \leq s$:

$$\begin{aligned} (a) \quad & e^{-P(s)} \mathbb{E} |X_s|^p \leq \mathbb{E} |X_0|^p + M(s), \\ (b) \quad & \int_0^\infty e^{-P(s) - \alpha M(s) - \lambda s} (\mathbb{E} |X_s|^p) ds \leq \frac{1}{\lambda} \left(\mathbb{E} |X_0|^p + \frac{1}{\alpha} \right) \end{aligned} \quad (6.88)$$

for all $\alpha, \lambda > 0$.

Proof. We follow from (6.80) the first steps from the proof of Proposition 6.74 but now

$$J_r = \frac{|X_r|}{(1 + \delta |X_r|^2)^{1/2}} \quad \text{and} \quad \hat{J}_r^{(p)} = \frac{|X_r|^{p-2} \mathbf{1}_{X_r \neq 0}}{(1 + \delta |X_r|^2)^{(p+2)/2}}$$

and $\theta = \theta_n$ is defined similarly.

From the inequality (2.28) in Corollary 2.28, we have for all $0 \leq t \leq s$

$$\begin{aligned} & \mathbb{E} (J_s^p \mathbf{1}_{s < \theta_n}) \\ & \leq \mathbb{E} J_{s \wedge \theta_n}^p \\ & \leq \mathbb{E} J_{t \wedge \theta_n}^p + p \mathbb{E} \int_t^s \mathbf{1}_{r < \theta_n} \hat{J}_r^{(p)} \left[\langle X_r, dK_r \rangle + \frac{1}{2} m_p |G_r|^2 \right] dr \\ & \leq \mathbb{E} J_{t \wedge \theta_n}^p \\ & \quad + p \mathbb{E} \int_t^s \mathbf{1}_{r < \theta_n} [J_r^{p-2} \mathbf{1}_{X_r \neq 0}, \mathbf{1}_{p \geq 2} dR(r) + J_r^{p-1} \mathbf{1}_{X_r \neq 0} dN(r) + J_r^p dV(r)]. \end{aligned}$$

Taking into account that

$$J_r^{p-2} \mathbf{1}_{X_r \neq 0} \leq \frac{2}{p} + \frac{p-2}{p} J_r^p \quad \text{and} \quad J_r^{p-1} \mathbf{1}_{X_r \neq 0} \leq \frac{1}{p} + \frac{p-1}{p} J_r^p,$$

and passing to the limit as $n \rightarrow \infty$ we have for all $0 \leq t \leq s$:

$$\begin{aligned} \mathbb{E} J_s^p & \leq \mathbb{E} J_t^p + 2 \int_t^s \mathbf{1}_{p \geq 2} dR(r) + \int_t^s dN(r) \\ & \quad + \int_t^s [(p-2) \mathbf{1}_{p \geq 2} dR(r) + (p-1) dN(r) + p dV(r)] \mathbb{E} (J_r^p), \quad \text{a.s.,} \end{aligned}$$

that is

$$\mathbb{E} J_s^p \leq \mathbb{E} J_t^p + \int_t^s dQ(r) + \int_t^s \mathbb{E} (J_r^p) dP(r).$$

By Gronwall's inequality (Proposition 6.69), we have for all $0 \leq t \leq s$:

$$e^{-P(s)} \mathbb{E} J_s^p \leq e^{-P(t)} \mathbb{E} J_t^p + \int_t^s e^{-P(r)} dQ(r),$$

and the inequality (6.87) follows. The inequality (6.88-b) clearly follows from (6.88-a) using the elementary inequality

$$y e^{-x - \alpha y - \lambda s} \leq \frac{1}{\alpha} e^{-\lambda s}, \quad \text{for all } x, y, s, \lambda \geq 0 \text{ and } \alpha > 0.$$

■

Let $X, \hat{X} \in S_d^0$ be two semimartingales given by

$$\begin{aligned} X_t &= X_0 + K_t + \int_0^t G_s dB_s, \quad t \geq 0, \\ \hat{X}_t &= \hat{X}_0 + \hat{K}_t + \int_0^t \hat{G}_s dB_s, \quad t \geq 0, \end{aligned} \quad (6.89)$$

where

- ◇ $K, \hat{K} \in S_d^0$;
 - ◇ $K, (\omega), \hat{K}, (\omega) \in BV_{loc}([0, \infty[; \mathbb{R}^d)$, $K_0(\omega) = \hat{K}_0(\omega) = 0$, \mathbb{P} -a.s. $\omega \in \Omega$;
 - ◇ $G, \hat{G} \in \Lambda_{d \times k}^0$.
- (FB'): Assume there exist $p \geq 1$ and $\lambda \geq 0$ and a \mathcal{P} -m.b.v.c.s.p. $V, V_0 = 0$, such that as measures on $[0, \infty[$:

$$\left\langle X_t - \hat{X}_t, dK_t - d\hat{K}_t \right\rangle + \left(\frac{1}{2} m_p + 9p\lambda \right) \left| G_t - \hat{G}_t \right|^2 dt \leq |X_t - \hat{X}_t|^2 dV_t. \quad (6.90)$$

Corollary 6.77. Let $p \geq 1$ and A be a \mathcal{P} -m.i.c.s.p., $A_0 = 0$.

(I) If the assumption (6.90) is satisfied with $\lambda = 0$, then for all $\delta \geq 0, 0 \leq t \leq s$:

$$\begin{aligned} \mathbb{E}^{\mathcal{F}_t} \frac{e^{-p(V_s + A_s)} |X_s - \hat{X}_s|^p}{(1 + \delta e^{-2(V_r + A_r)} |X_r - \hat{X}_r|^2)^{p/2}} + \mathbb{E}^{\mathcal{F}_t} \int_t^s \frac{e^{-p(V_r + A_r)} |X_r - \hat{X}_r|^p}{(1 + \delta e^{-2(V_r + A_r)} |X_r - \hat{X}_r|^2)^{(p+2)/2}} dA_r \\ \leq \frac{e^{-p(V_t + A_t)} |X_t - \hat{X}_t|^p}{(1 + \delta e^{-2(V_t + A_t)} |X_t - \hat{X}_t|^2)^{p/2}}, \quad \mathbb{P} - a.s. \end{aligned}$$

In particular for $\delta = 0$

$$\begin{aligned} \mathbb{E}^{\mathcal{F}_t} e^{-p(V_s + A_s)} |X_s - \hat{X}_s|^p + \mathbb{E}^{\mathcal{F}_t} \int_t^s e^{-p(V_r + A_r)} |X_r - \hat{X}_r|^p dA_r \\ \leq e^{-p(V_t + A_t)} |X_t - \hat{X}_t|^p, \quad \mathbb{P} - a.s., \end{aligned}$$

for all $0 \leq t \leq s$.

(II) If the assumption (6.90) is satisfied with $\lambda > 1$, then there exists a positive constant $C_{p,\lambda}$ depending only on (p, λ) such that for all $\delta \geq 0, 0 \leq t \leq s$:

$$\begin{aligned} \mathbb{E}^{\mathcal{F}_t} \frac{\|e^{-V-A}(X-\hat{X})\|_{[t,s]}^p}{(1+\delta\|e^{-V-A}(X-\hat{X})\|_{[t,s]}^2)^{p/2}} + \mathbb{E}^{\mathcal{F}_t} \left(\int_t^s \frac{e^{-2(V_r+A_r)} |X_r - \hat{X}_r|^2}{(1 + \delta e^{-2(V_r + A_r)} |X_r - \hat{X}_r|^2)^2} dA_r \right)^{p/2} \\ \leq C_{p,\lambda} \frac{e^{-p(V_t + A_t)} |X_t - \hat{X}_t|^p}{(1 + \delta e^{-2(V_t + A_t)} |X_t - \hat{X}_t|^2)^{p/2}}, \quad \mathbb{P} - a.s. \end{aligned}$$

In particular for $\delta = 0$

$$\begin{aligned} \mathbb{E}^{\mathcal{F}_t} \left\| e^{-V-A} \left(X - \hat{X} \right) \right\|_{[t,s]}^p + \mathbb{E}^{\mathcal{F}_t} \left(\int_t^s e^{-2(V_r+A_r)} \left| X_r - \hat{X}_r \right|^2 dA_r \right)^{p/2} \\ \leq C_{p,\lambda} e^{-p(V_t+A_t)} \left| X_t - \hat{X}_t \right|^p, \quad \mathbb{P}\text{-a.s.}, \end{aligned}$$

for all $0 \leq t \leq s$.

Proof. Since the assumption (6.90) is equivalent to

$$\begin{aligned} dD_t + \left\langle X_t - \hat{X}_t, dK_t - d\hat{K}_t \right\rangle + \left(\frac{1}{2}m_p + 9p\lambda \right) \left| G_t - \hat{G}_t \right|^2 dt \\ \leq \left| X_t - \hat{X}_t \right|^2 d(V_t + A_t), \end{aligned}$$

with

$$D_t = \int_0^t \left| X_r - \hat{X}_r \right|^2 dA_r,$$

the results clearly follow from Propositions 6.71 and 6.74 applied to the identity

$$X_t - \hat{X}_t = X_0 - \hat{X}_0 + \left(K_t - \hat{K}_t \right) + \int_0^t \left(G_s - \hat{G}_s \right) dB_s.$$

■

Since

$$\frac{1}{2}(r \wedge 1) \leq \frac{r}{(1+r^2)^{1/2}} \leq r \wedge 1, \quad \forall r \geq 0,$$

we have:

Corollary 6.78. *If the assumption (6.90) is satisfied with $\lambda > 1$ and $p \geq 1$, then there exists a positive constant $C_{p,\lambda}$ depending only on (p, λ) such that \mathbb{P} -a.s.*

$$\mathbb{E}^{\mathcal{F}_t} \left[1 \wedge \left\| e^{-V} \left(X - \hat{X} \right) \right\|_{[t,s]}^p \right] \leq C_{p,\lambda} \left[1 \wedge \left| e^{-V_t} \left(X_t - \hat{X}_t \right) \right|^p \right],$$

for all $0 \leq t \leq s$.

6.4.4 Backward Stochastic Inequalities

Let $\{B_t : t \geq 0\}$ be a k -dimensional Brownian motion with respect to a given stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t^B\}_{t \geq 0})$, where \mathcal{F}_t^B is the natural filtration associated to $\{B_t : t \geq 0\}$.

Notation 6.79. For $p > 1$ define

$$n_p \stackrel{\text{def}}{=} 1 \wedge (p - 1).$$

In this subsection we shall derive some estimates on $(Y, Z) \in S_m^0 \times \Lambda_{m \times k}^0$ satisfying for all $T \geq 0$ and $t \in [0, T]$:

$$Y_t = Y_T + (K_T - K_t) - \int_t^T Z_s dB_s, \quad \mathbb{P}\text{-a.s.}, \tag{6.91}$$

where $K \in S_m^0$ and $K.(\omega) \in BV_{loc}(\mathbb{R}_+; \mathbb{R}^m)$, $\mathbb{P}\text{-a.s. } \omega \in \Omega$.

We note that if the interval $[0, T]$ is fixed then the equality (6.91) will be extended to \mathbb{R}_+ by $Y_s = Y_T$, $K_s = K_T$ and $Z_s = 0$ for all $s > T$.

Proposition 6.80. Let $(Y, Z) \in S_m^0 \times \Lambda_{m \times k}^0$ satisfy

$$Y_t = Y_T + \int_t^T dK_s - \int_t^T Z_s dB_s, \quad 0 \leq t \leq T, \quad \mathbb{P}\text{-a.s.},$$

where $K \in S_m^0$ and $K.(\omega) \in BV_{loc}(\mathbb{R}_+; \mathbb{R}^m)$, $\mathbb{P}\text{-a.s. } \omega \in \Omega$.

Assume given

- ▲ three \mathcal{P} -m.i.c.s.p. $D, R, N, D_0 = R_0 = N_0 = 0$,
- ▲ a \mathcal{P} -m.b.v.c.s.p. $V, V_0 = 0$,
- ▲ two stopping times τ and σ such that $0 \leq \tau \leq \sigma < \infty$.

(A) If $\lambda < 1, q > 0$ and

$$dD_t + \langle Y_t, dK_t \rangle \leq dR_t + |Y_t| dN_t + |Y_t|^2 dV_t + \frac{\lambda}{2} |Z_t|^2 dt,$$

then there exists a positive constant $C_{q,\lambda}$, depending only on (q, λ) , such that

$$\begin{aligned} & \mathbb{E}^{\mathcal{F}_\tau} \left(\int_\tau^\sigma e^{2V_r} dD_r \right)^{q/2} + \mathbb{E}^{\mathcal{F}_\tau} \left(\int_\tau^\sigma e^{2V_r} |Z_r|^2 dr \right)^{q/2} \\ & \leq C_{q,\lambda} \mathbb{E}^{\mathcal{F}_\tau} \left[\sup_{s \in [\tau, \sigma]} |e^{V_s} Y_s|^q + \left(\int_\tau^\sigma e^{2V_s} dR_s \right)^{q/2} + \left(\int_\tau^\sigma e^{V_s} dN_s \right)^q \right], \end{aligned} \tag{6.92}$$

$\mathbb{P}\text{-a.s.}$

(B) If $\lambda < 1 < p$,

- (i) $dD_t + \langle Y_t, dK_t \rangle \leq (\mathbf{1}_{p \geq 2} dR_t + |Y_t| dN_t + |Y_t|^2 dV_t) + \frac{n_p}{2} \lambda |Z_t|^2 dt$,
- (ii) $\mathbb{E} \sup_{s \in [\tau, \sigma]} e^{pV_s} |Y_s|^p < \infty$,

$$\tag{6.93}$$

then there exists a positive constant $C_{p,\lambda}$, depending only on (p, λ) , such that \mathbb{P} -a.s.,

$$\begin{aligned} & \mathbb{E}^{\mathcal{F}_\tau} \left(\sup_{s \in [\tau, \sigma]} |e^{V_s} Y_s|^p \right) + \mathbb{E}^{\mathcal{F}_\tau} \left(\int_\tau^\sigma e^{2V_s} dD_s \right)^{p/2} \\ & \quad + \mathbb{E}^{\mathcal{F}_\tau} \left(\int_\tau^\sigma e^{2V_s} |Z_s|^2 ds \right)^{p/2} \\ & \quad + \mathbb{E}^{\mathcal{F}_\tau} \int_\tau^\sigma e^{pV_s} |Y_s|^{p-2} \mathbf{1}_{Y_s \neq 0} dD_s + \mathbb{E}^{\mathcal{F}_\tau} \int_\tau^\sigma e^{pV_s} |Y_s|^{p-2} \mathbf{1}_{Y_s \neq 0} |Z_s|^2 ds \\ & \leq C_{p,\lambda} \mathbb{E}^{\mathcal{F}_\tau} \left[|e^{V_\sigma} Y_\sigma|^p + \left(\int_\tau^\sigma e^{2V_s} \mathbf{1}_{p \geq 2} dR_s \right)^{p/2} + \left(\int_\tau^\sigma e^{V_s} dN_s \right)^p \right]. \end{aligned} \quad (6.94)$$

Proof. Step I.

By the Itô formula, we have for all $0 \leq t \leq s$:

$$\begin{aligned} |Y_t|^2 e^{2V_t} + \int_t^s e^{2V_r} |Z_r|^2 dr &= |Y_s|^2 e^{2V_s} + 2 \int_t^s e^{2V_r} (\langle Y_r, dK_r \rangle - |Y_r|^2 dV_r) \\ &\quad - 2 \int_t^s e^{2V_r} \langle Y_r, Z_r dB_r \rangle, \quad a.s. \end{aligned}$$

Since

$$\langle Y_r, dK_r \rangle - |Y_r|^2 dV_r \leq -dD_r + dR_r + |Y_r| dN_r + \frac{\lambda}{2} |Z_r|^2 dr,$$

we get

$$\begin{aligned} & |Y_t|^2 e^{2V_t} + 2 \int_t^s e^{2V_r} dD_r + (1 - \lambda) \int_t^s e^{2V_r} |Z_r|^2 dr \\ & \leq |Y_s|^2 e^{2V_s} + 2 \int_t^s e^{2V_r} dR_r + 2 \int_t^s e^{V_r} |Y_r| dN_r - 2 \int_t^s e^{2V_r} \langle Y_r, Z_r dB_r \rangle. \end{aligned} \quad (6.95)$$

Let the stopping times $0 \leq \tau \leq \sigma < \infty$ and

$$\begin{aligned} \theta_n = \sigma \wedge \inf \left\{ s \geq \tau : \|e^V Y - e^{V_\tau} Y_\tau\|_s + \int_\tau^{s \vee \tau} e^{2V_r} dD_r + \int_\tau^s e^{2V_r} |Z_r|^2 dr \right. \\ \left. + \int_\tau^{s \vee \tau} e^{2V_r} dR_r + \int_\tau^{s \vee \tau} e^{V_r} dN_r \geq n \right\}. \end{aligned}$$

We have $\tau \leq \theta_n \leq \sigma$ and $\theta_n \nearrow \sigma$ \mathbb{P} -a.s. Replacing in (6.95) t by τ and s by θ_n we obtain

$$\begin{aligned}
& 2 \int_{\tau}^{\theta_n} e^{2V_r} dD_r + (1 - \lambda) \int_{\tau}^{\theta_n} e^{2V_r} |Z_r|^2 dr \\
& \leq |Y_{\theta_n}|^2 e^{2V_{\theta_n}} + 2 \int_{\tau}^{\theta_n} e^{2V_r} (dR_r + |Y_r| dN_r) \\
& \quad - 2 \int_{\tau}^{\theta_n} e^{2V_r} \langle Y_r, Z_r dB_r \rangle \\
& \leq |Y_{\theta_n}|^2 e^{2V_{\theta_n}} + \sup_{r \in [\tau, \sigma]} \mathbf{1}_{[\tau, \theta_n]}(r) |e^{V_r} Y_r|^2 + 2 \int_{\tau}^{\theta_n} e^{2V_r} dR_r \\
& \quad + \left(\int_{\tau}^{\theta_n} e^{V_r} dN_r \right)^2 - 2 \int_{\tau}^{\sigma} \mathbf{1}_{[\tau, \theta_n]}(r) e^{2V_r} \langle Y_r, Z_r dB_r \rangle.
\end{aligned}$$

Moreover, by Minkowski's inequality we infer for all $q > 0$

$$\begin{aligned}
& \mathbb{E}^{\mathcal{F}_{\tau}} \left(\int_{\tau}^{\theta_n} e^{2V_r} dD_r \right)^{q/2} + \mathbb{E}^{\mathcal{F}_{\tau}} \left(\int_{\tau}^{\theta_n} e^{2V_r} |Z_r|^2 dr \right)^{q/2} \\
& \leq C_{q,\lambda} \mathbb{E}^{\mathcal{F}_{\tau}} \sup_{r \in [\tau, \sigma]} |e^{V_r} Y_r|^q + C_{q,\lambda} \mathbb{E}^{\mathcal{F}_{\tau}} \left(\int_{\tau}^{\sigma} e^{2V_r} dR_r \right)^{q/2} \\
& + C_{q,\lambda} \mathbb{E}^{\mathcal{F}_{\tau}} \left(\int_{\tau}^{\sigma} e^{V_r} dN_r \right)^q + C_{q,\lambda} \mathbb{E}^{\mathcal{F}_{\tau}} \left| \int_{\tau}^{\sigma} \mathbf{1}_{[\tau, \theta_n]}(r) e^{2V_r} \langle Y_r, Z_r dB_r \rangle \right|^{q/2}.
\end{aligned} \tag{6.96}$$

But by the Burkholder–Davis–Gundy and Cauchy–Schwarz inequalities, we get

$$\begin{aligned}
& C_{q,\lambda} \mathbb{E}^{\mathcal{F}_{\tau}} \left| \int_{\tau}^{\sigma} \mathbf{1}_{[\tau, \theta_n]}(r) e^{2V_r} \langle Y_r, Z_r dB_r \rangle \right|^{q/2} \\
& \leq C_{q,\lambda} \mathbb{E}^{\mathcal{F}_{\tau}} \left(\int_{\tau}^{\sigma} \mathbf{1}_{[\tau, \theta_n]}(r) e^{4V_r} |Y_r|^2 |Z_r|^2 dr \right)^{q/4} \\
& \leq C_{q,\lambda} \mathbb{E}^{\mathcal{F}_{\tau}} \left[\sup_{r \in [\tau, \sigma]} \left(\mathbf{1}_{[\tau, \theta_n]}(r) |e^{V_r} Y_r|^{q/2} \right) \left(\int_{\tau}^{\sigma} \mathbf{1}_{[\tau, \theta_n]}(r) e^{2V_r} |Z_r|^2 dr \right)^{q/4} \right] \\
& \leq C'_{q,\lambda} \mathbb{E}^{\mathcal{F}_{\tau}} \sup_{r \in [\tau, \theta_n]} |e^{V_r} Y_r|^q + \frac{1}{2} \mathbb{E}^{\mathcal{F}_{\tau}} \left(\int_{\tau}^{\theta_n} e^{2V_r} |Z_r|^2 dr \right)^{q/2}.
\end{aligned}$$

Since $\int_{\tau}^{\theta_n} e^{2V_r} |Z_r|^2 dr$ is finite, from (6.96) we infer

$$\begin{aligned} & \mathbb{E}^{\mathcal{F}_\tau} \left(\int_\tau^{\theta_n} e^{2V_r} dD_r \right)^{q/2} + \frac{1}{2} \mathbb{E}^{\mathcal{F}_\tau} \left(\int_\tau^{\theta_n} e^{2V_r} |Z_r|^2 dr \right)^{q/2} \\ & \leq C_{q,\lambda} \left[\mathbb{E}^{\mathcal{F}_\tau} \sup_{r \in [\tau, \sigma]} |e^{V_r} Y_r|^q + \mathbb{E}^{\mathcal{F}_\tau} \left(\int_\tau^\sigma e^{2V_r} dR_r \right)^{q/2} + \mathbb{E}^{\mathcal{F}_\tau} \left(\int_\tau^\sigma e^{V_r} dN_r \right)^q \right]. \end{aligned} \quad (6.97)$$

By the monotone convergence theorem as $n \rightarrow \infty$ the inequality (6.92) follows.

Step II. Let us first assume that $p \geq 1$.

Noting that

$$e^{V_t} Y_t = Y_0 - \int_0^t e^{V_r} (dK_r - Y_r dV_r) + \int_0^t e^{V_r} Z_r dB_r,$$

then by the inequality (2.30) from Corollary (2.30) we get, for $p \geq 1$ and for all stopping times $t \in [\tau, \sigma]$

$$\begin{aligned} & e^{pV_t} |Y_t|^p + \frac{p}{2} n_p \int_t^{\theta_n} e^{pV_r} |Y_r|^{p-2} \mathbf{1}_{Y_r \neq 0} |Z_r|^2 dr \leq e^{pV_{\theta_n}} |Y_{\theta_n}|^p \\ & + p \int_t^{\theta_n} e^{pV_r} |Y_r|^{p-2} \mathbf{1}_{Y_r \neq 0} \left(\langle Y_r, dK_r \rangle - |Y_r|^2 dV_r \right) \\ & - p \int_t^{t \vee \theta_n} e^{pV_r} |Y_r|^{p-2} \mathbf{1}_{Y_r \neq 0} \langle Y_r, Z_r dB_r \rangle. \end{aligned} \quad (6.98)$$

We note that the right-hand side of (6.98) is finite \mathbb{P} -a.s. and consequently

$$0 \leq n_p \int_\tau^{\theta_n} e^{pV_r} |Y_r|^{p-2} \mathbf{1}_{Y_r \neq 0} |Z_r|^2 dr < \infty, \quad \mathbb{P}\text{-a.s.}$$

By the assumption (6.93)

$$\langle Y_r, dK_r \rangle - |Y_r|^2 dV_r \leq -dD_r + (\mathbf{1}_{p \geq 2} dR_r + |Y_r| dN_r) + \frac{np}{2} \lambda |Z_r|^2 dr.$$

It follows that

$$\begin{aligned} & e^{pV_t} |Y_t|^p + p \int_t^\sigma \mathbf{1}_{[\tau, \theta_n]}(r) e^{pV_r} |Y_r|^{p-2} \mathbf{1}_{Y_r \neq 0} dD_r \\ & + \frac{p}{2} n_p (1 - \lambda) \int_t^\sigma \mathbf{1}_{[\tau, \theta_n]}(r) e^{pV_r} |Y_r|^{p-2} \mathbf{1}_{Y_r \neq 0} |Z_r|^2 dr \\ & \leq e^{pV_{\theta_n}} |Y_{\theta_n}|^p + (U_{\theta_n} - U_t) - (M_{\theta_n} - M_t), \end{aligned} \quad (6.99)$$

where

$$U_s = p \int_0^s \mathbf{1}_{[\tau, \theta_n]}(r) e^{pV_r} |Y_r|^{p-2} \mathbf{1}_{Y_r \neq 0} (\mathbf{1}_{p \geq 2} dR_r + |Y_r| dN_r) \quad (6.100)$$

and

$$M_s = p \int_0^s \mathbf{1}_{[\tau, \theta_n]}(r) e^{pV_r} |Y_r|^{p-2} \mathbf{1}_{Y_r \neq 0} \langle Y_r, Z_r dB_r \rangle.$$

Note that $\{M_s : s \in [0, T]\}$ is a martingale since

$$\begin{aligned} \mathbb{E} \sqrt{\langle M \rangle_T} &\leq p \mathbb{E} \left(\int_{\tau}^{\theta_n} e^{2pV_r} |Y_r|^{2p-4} \mathbf{1}_{Y_r \neq 0} |Y_r|^2 |Z_r|^2 dr \right)^{1/2} \\ &\leq p \mathbb{E} \left[(|e^{V_{\tau}} Y_{\tau}| + n)^{p-1} \left(\int_{\tau}^{\theta_n} e^{2V_r} |Z_r|^2 dr \right)^{1/2} \right] \\ &\leq C_p \left(\mathbb{E} |e^{V_{\tau}} Y_{\tau}|^{p-1} + n^{p-1} \right) \sqrt{n}. \end{aligned}$$

Therefore from (6.99),

$$e^{pV_{\tau}} |Y_{\tau}|^p \leq \mathbb{E}^{\mathcal{F}_{\tau}} e^{pV_{\theta_n}} |Y_{\theta_n}|^p + \mathbb{E}^{\mathcal{F}_{\tau}} (U_{\theta_n} - U_{\tau}). \quad (6.101)$$

From here we assume that $p > 1$. From (6.99) we also get

$$\begin{aligned} p \mathbb{E}^{\mathcal{F}_{\tau}} \int_{\tau}^{\theta_n} e^{pV_r} |Y_r|^{p-2} \mathbf{1}_{Y_r \neq 0} dD_r \\ + \frac{p}{2} n_p (1 - \lambda) \mathbb{E}^{\mathcal{F}_{\tau}} \int_{\tau}^{\theta_n} e^{pV_r} |Y_r|^{p-2} \mathbf{1}_{Y_r \neq 0} |Z_r|^2 dr \\ \leq \mathbb{E}^{\mathcal{F}_{\tau}} e^{pV_{\theta_n}} |Y_{\theta_n}|^p + \mathbb{E}^{\mathcal{F}_{\tau}} U_{\theta_n}. \end{aligned} \quad (6.102)$$

Since

$$\sup_{t \in [\tau, \sigma]} |M_{\sigma} - M_t| \leq 2 \sup_{t \in [\tau, \sigma]} |M_t - M_{\tau}| = 2 \sup_{t \in [\tau, \sigma]} |M_t|,$$

we obtain from (6.99) that

$$\begin{aligned} \mathbb{E}^{\mathcal{F}_{\tau}} \sup_{t \in [\tau, \theta_n]} (e^{pV_t} |Y_t|^p) \\ \leq \mathbb{E}^{\mathcal{F}_{\tau}} e^{pV_{\theta_n}} |Y_{\theta_n}|^p + \mathbb{E}^{\mathcal{F}_{\tau}} (U_{\theta_n} - U_{\tau}) + 2 \mathbb{E}^{\mathcal{F}_{\tau}} \sup_{t \in [\tau, \theta_n]} |M_t|. \end{aligned} \quad (6.103)$$

By the Burkholder–Davis–Gundy inequality (2.8) and (6.102):

$$\begin{aligned} \mathbb{E}^{\mathcal{F}_{\tau}} \sup_{t \in [\tau, \sigma]} |M_t| \\ \leq 3p \mathbb{E}^{\mathcal{F}_{\tau}} \left(\int_{\tau}^{\sigma} \mathbf{1}_{[\tau, \theta_n]}(r) e^{2pV_r} |Y_r|^{2p-4} \mathbf{1}_{Y_r \neq 0} |Y_r|^2 |Z_r|^2 dr \right)^{1/2} \end{aligned}$$

$$\begin{aligned}
&\leq 3p \mathbb{E}^{\mathcal{F}_\tau} \left[\sup_{r \in [\tau, \theta_n]} e^{(p/2)V_r} |Y_r|^{p/2} \left(\int_\tau^{\theta_n} e^{pV_r} |Y_r|^{p-2} \mathbf{1}_{Y_r \neq 0} |Z_r|^2 dr \right)^{1/2} \right] \\
&\leq \frac{1}{4} \mathbb{E}^{\mathcal{F}_\tau} \sup_{r \in [\tau, \theta_n]} (e^{pV_r} |Y_r|^p) + C_p \mathbb{E}^{\mathcal{F}_\tau} \int_\tau^{\theta_n} e^{pV_r} |Y_r|^{p-2} \mathbf{1}_{Y_r \neq 0} |Z_r|^2 dr \\
&\leq \frac{1}{4} \mathbb{E}^{\mathcal{F}_\tau} \sup_{r \in [\tau, \theta_n]} e^{pV_r} |Y_r|^p + C_{p,\lambda} \mathbb{E}^{\mathcal{F}_\tau} e^{pV_{\theta_n}} |Y_{\theta_n}|^p + C_{p,\lambda} \mathbb{E}^{\mathcal{F}_\tau} (U_{\theta_n} - U_\tau).
\end{aligned}$$

Plugging this last estimate into (6.103) we obtain with another constant $C_{p,\lambda}$

$$\mathbb{E}^{\mathcal{F}_\tau} \sup_{r \in [\tau, \theta_n]} e^{pV_r} |Y_r|^p \leq C_{p,\lambda} \mathbb{E}^{\mathcal{F}_\tau} e^{pV_{\theta_n}} |Y_{\theta_n}|^p + C_{p,\lambda} \mathbb{E}^{\mathcal{F}_\tau} (U_{\theta_n} - U_\tau). \quad (6.104)$$

We deduce from (6.102) and (6.104)

$$\begin{aligned}
\mathbb{E}^{\mathcal{F}_\tau} \sup_{r \in [\tau, \theta_n]} e^{pV_r} |Y_r|^p &+ \mathbb{E}^{\mathcal{F}_\tau} \int_\tau^{\theta_n} e^{pV_r} |Y_r|^{p-2} \mathbf{1}_{Y_r \neq 0} dD_r \\
&+ \mathbb{E}^{\mathcal{F}_\tau} \int_\tau^{\theta_n} e^{pV_r} |Y_r|^{p-2} \mathbf{1}_{Y_r \neq 0} |Z_r|^2 dr \\
&\leq C_{p,\lambda} \mathbb{E}^{\mathcal{F}_\tau} e^{pV_{\theta_n}} |Y_{\theta_n}|^p + C_{p,\lambda} \mathbb{E}^{\mathcal{F}_\tau} U_{\theta_n}.
\end{aligned}$$

But

$$\begin{aligned}
&C_{p,\lambda} \mathbb{E}^{\mathcal{F}_\tau} (U_{\theta_n} - U_\tau) \\
&\leq C_{p,\lambda} \mathbb{E}^{\mathcal{F}_\tau} \left[\sup_{r \in [\tau, \theta_n]} \left[e^{(p-2)V_r} |Y_r|^{p-2} \mathbf{1}_{Y_r \neq 0} \mathbf{1}_{p \geq 2} \right] \int_\tau^{\theta_n} e^{2V_r} \mathbf{1}_{p \geq 2} dR_r \right] \\
&\quad + \mathbb{E}^{\mathcal{F}_\tau} \left[\sup_{r \in [\tau, \theta_n]} \left[e^{(p-1)V_r} |Y_r|^{p-1} \mathbf{1}_{Y_r \neq 0} \right] \int_\tau^{\theta_n} e^{V_r} dN_r \right] \\
&\leq \frac{1}{2} \mathbb{E}^{\mathcal{F}_\tau} \sup_{r \in [\tau, \theta_n]} e^{pV_r} |Y_r|^p + C'_{p,\lambda} \mathbb{E}^{\mathcal{F}_\tau} \left(\int_\tau^\sigma e^{2V_r} \mathbf{1}_{p \geq 2} dR_r \right)^{p/2} \\
&\quad + C'_{p,\lambda} \mathbb{E}^{\mathcal{F}_\tau} \left(\int_\tau^\sigma e^{V_r} dN_r \right)^p.
\end{aligned}$$

Hence

$$\begin{aligned}
\mathbb{E}^{\mathcal{F}_\tau} \sup_{r \in [\tau, \theta_n]} e^{pV_r} |Y_r|^p &+ \mathbb{E}^{\mathcal{F}_\tau} \int_\tau^{\theta_n} e^{pV_r} |Y_r|^{p-2} \mathbf{1}_{Y_r \neq 0} dD_r \\
&+ \mathbb{E}^{\mathcal{F}_\tau} \int_\tau^{\theta_n} e^{pV_r} |Y_r|^{p-2} \mathbf{1}_{Y_r \neq 0} |Z_r|^2 dr
\end{aligned}$$

$$\begin{aligned} &\leq C_{p,\lambda} \mathbb{E}^{\mathcal{F}_\tau} e^{pV_{\theta_n}} |Y_{\theta_n}|^p + C_{p,\lambda} \mathbb{E}^{\mathcal{F}_\tau} \left(\int_\tau^\sigma e^{2V_r} \mathbf{1}_{p \geq 2} dR_r \right)^{p/2} \\ &\quad + C_{p,\lambda} \mathbb{E}^{\mathcal{F}_\tau} \left(\int_\tau^\sigma e^{V_r} dN_r \right)^p. \end{aligned}$$

Now letting $n \rightarrow \infty$, by the Beppo Levi monotone convergence theorem for the first member and by the Lebesgue dominated convergence theorem for the right-hand side of the inequality, we conclude (6.94) (using of course the first step: inequality (6.92)).

The proof is complete. ■

Corollary 6.81. *Let $(Y, Z) \in S_m^0 \times \Lambda_{m \times k}^0$ satisfy*

$$Y_t = Y_T + \int_t^T dK_s - \int_t^T Z_s dB_s, \quad 0 \leq t \leq T, \quad \mathbb{P}\text{-a.s.},$$

where $K \in S_m^0$ and $K.(\omega) \in BV_{loc}(\mathbb{R}_+; \mathbb{R}^m)$, $\mathbb{P}\text{-a.s. } \omega \in \Omega$.

Assume given

- ▲ D and N are \mathcal{P} -m.i.c.s.p., $N_0 = 0$,
- ▲ V a \mathcal{P} -m.b-v.c.s.p., $V_0 = 0$,
- ▲ τ, θ and σ are three stopping times such that $0 \leq \tau \leq \theta \leq \sigma < \infty$.

If

- (a) $dD_t + \langle Y_t, dK_t \rangle \leq |Y_t| dN_t + |Y_t|^2 dV_t$,
- (b) $\mathbb{E} \sup_{s \in [\tau, \sigma]} |e^{V_s} Y_s| < \infty$,

then

$$e^{V_\tau} |Y_\tau| \leq \mathbb{E}^{\mathcal{F}_\tau} e^{V_\sigma} |Y_\sigma| + \mathbb{E}^{\mathcal{F}_\tau} \int_\tau^\sigma e^{V_r} dN_r \tag{6.105}$$

and for all $0 < \alpha < 1$

$$\begin{aligned} &\sup_{\theta \in [\tau, \sigma]} \left[\mathbb{E} (e^{V_\theta} |Y_\theta|)^\alpha + \mathbb{E} \left(\sup_{s \in [\tau, \sigma]} |e^{V_s} Y_s|^\alpha \right) + \mathbb{E} \left(\int_\tau^\sigma e^{2V_r} |Z_r|^2 dr \right)^{\alpha/2} \right. \\ &\quad \left. + \mathbb{E} \left(\int_\tau^\sigma e^{2V_r} |D_r|^2 dr \right)^{\alpha/2} \right] \\ &\leq C_\alpha \left[\left(\mathbb{E} (e^{V_\sigma} |Y_\sigma|) \right)^\alpha + \left(\mathbb{E} \int_\tau^\sigma e^{V_r} dN_r \right)^\alpha \right]. \end{aligned} \tag{6.106}$$

Proof. From (6.101) for $p = 1$ we deduce, using the definition (6.100) of U_s , that

$$\begin{aligned} e^{V_\tau} |Y_\tau| &\leq \mathbb{E}^{\mathcal{F}_\tau} e^{V_{\theta_n}} |Y_{\theta_n}| + \mathbb{E}^{\mathcal{F}_\tau} U_{\theta_n} \\ &\leq \mathbb{E}^{\mathcal{F}_\tau} e^{V_{\theta_n}} |Y_{\theta_n}| + \mathbb{E}^{\mathcal{F}_\tau} \int_\tau^\sigma e^{V_r} dN_r \end{aligned}$$

and the inequality (6.105) follows as $n \rightarrow \infty$. Moreover

$$\sup_{\theta \in [\tau, \sigma]} \mathbb{E} (e^{V_\theta} |Y_\theta|) \leq \mathbb{E} (e^{V_\sigma} |Y_\sigma|) + \mathbb{E} \int_\tau^\sigma e^{V_r} dN_r$$

and by the martingale inequality (1.11- A_3) from Theorem 1.60 we infer

$$\mathbb{E} \left(\sup_{s \in [\tau, \sigma]} |e^{V_s} Y_s|^\alpha \right) \leq \frac{1}{1 - \alpha} \left[\mathbb{E} \left(e^{V_\sigma} |Y_\sigma| + \int_\tau^\sigma e^{V_r} dN_r \right)^\alpha \right].$$

The inequality (6.106) is now a consequence of (6.92). ■

Corollary 6.82. *Let $(Y, Z) \in S_m^0 \times \Lambda_{m \times k}^0$ satisfy*

$$Y_t = Y_T + \int_t^T dK_s - \int_t^T Z_s dB_s, \quad 0 \leq t \leq T, \quad \mathbb{P}\text{-a.s.},$$

where $K \in S_m^0$ and $K.(\omega) \in BV_{loc}(\mathbb{R}_+; \mathbb{R}^m)$, $\mathbb{P}\text{-a.s.}$ $\omega \in \Omega$.

Assume given

- ▲ a \mathcal{P} -m.b.v.c.s.p. $V, V_0 = 0$,
- ▲ τ, θ and σ are three stopping times such that $0 \leq \tau \leq \theta \leq \sigma < \infty$.

If $\lambda < 1 \leq p, n_p = 1 \wedge (p - 1)$ and

- (a) $\langle Y_t, dK_t \rangle \leq |Y_t|^2 dV_t + \frac{n_p \lambda}{2} |Z_t|^2 dt$,
- (b) $\mathbb{E} \sup_{s \in [\tau, \sigma]} e^{pV_s} |Y_s|^p < \infty$,

then for all $1 \leq q \leq p$,

$$e^{qV_\tau} |Y_\tau|^q \leq \mathbb{E}^{\mathcal{F}_\tau} e^{qV_\sigma} |Y_\sigma|^q, \quad \mathbb{P}\text{-a.s.} \tag{6.107}$$

If $p > 1$ then

$$\mathbb{E} \sup_{s \in [\tau, \sigma]} e^{pV_s} |Y_s|^p + \mathbb{E} \left(\int_\tau^\sigma e^{2V_r} |Z_r|^2 dr \right)^{p/2} \leq C_{p, \lambda} \mathbb{E} (e^{pV_\sigma} |Y_\sigma|^p),$$

and if $p = 1$ (and $n_p = 0$) then for all $0 < \alpha < 1$,

$$\sup_{\theta \in [\tau, \sigma]} \left(\mathbb{E} e^{V_\theta} |Y_\theta| \right)^\alpha + \mathbb{E} \sup_{s \in [\tau, \sigma]} e^{\alpha V_s} |Y_s|^\alpha + \mathbb{E} \left(\int_\tau^\sigma e^{2V_r} |Z_r|^2 dr \right)^{\alpha/2} \leq C_\alpha \left(\mathbb{E} e^{V_\sigma} |Y_\sigma| \right)^\alpha. \quad (6.108)$$

Proof. Since

$$\mathbb{E} \sup_{s \in [\tau, \sigma]} e^{qV_s} |Y_s|^q \leq \left(\mathbb{E} \sup_{s \in [\tau, \sigma]} e^{pV_s} |Y_s|^p \right)^{q/p} < \infty,$$

the inequality (6.101) with p replaced by q yields (6.107). The next two inequalities follow from Proposition 6.80 and Corollary 6.81, respectively. ■

Corollary 6.83. *Let $p \geq 1$ and $(V_t)_{t \geq 0}$ be a bounded variation continuous progressively measurable stochastic process with $V_0 = 0$. Let $T > 0$ and $\eta : \Omega \rightarrow \mathbb{R}^m$ be a random variable such that $\mathbb{E} (\sup_{r \in [0, T]} e^{pV_r} |\eta|^p) < \infty$. If $(\xi, \zeta) \in S_m^p[0, T] \times \Lambda_{m \times k}^p[0, T]$ satisfies*

$$\xi_s = \mathbb{E}^{\mathcal{F}_T} \eta - \int_s^T \zeta_r dB_r, \quad s \in [0, T], \quad a.s.$$

(the pair (ξ, ζ) exists and it is unique by the martingale representation: Corollary 2.44), then there exists a $C = C(p) > 0$ such that for all $t \in [0, T]$, for $p > 1$

$$\mathbb{E}^{\mathcal{F}_t} \sup_{s \in [t, T]} e^{pV_s} |\xi_s|^p + \mathbb{E}^{\mathcal{F}_t} \left(\int_t^T e^{2V_s} |\zeta_s|^2 ds \right)^{p/2} \leq C_p \mathbb{E}^{\mathcal{F}_t} \left(\sup_{r \in [0, T]} e^{pV_r} |\eta|^p \right) \quad (6.109)$$

and for $p = 1$

$$\sup_{s \in [0, T]} \left(\mathbb{E} e^{V_s} |\xi_s| \right)^\alpha + \mathbb{E} \sup_{t \in [0, T]} |e^{V_t} \xi_t|^\alpha + \mathbb{E} \left(\int_0^T e^{2V_s} |\zeta_s|^2 ds \right)^{\alpha/2} \leq C_\alpha \left(\mathbb{E} (\sup_{t \in [0, T]} e^{V_t} |\eta|) \right)^\alpha, \quad \text{for all } 0 < \alpha < 1. \quad (6.110)$$

Proof. We see at once that the stochastic pair (ξ, ζ) satisfy the equation

$$\xi_t = \xi_T - \int_t^T \zeta_s dB_s, \quad t \in [0, T], \quad a.s.$$

The stochastic process $\tilde{V}_t = \sup_{s \in [0, t]} V_s$; \tilde{V} is increasing continuous progressively measurable and $\tilde{V}_0 = 0$. Since for all $t \in [0, T]$

$$\mathbb{E}^{\mathcal{F}_t} \left| e^{\tilde{V}_t} \xi_t \right| = \left| e^{\tilde{V}_t} \xi_t \right| = e^{\tilde{V}_t} \left| \mathbb{E}^{\mathcal{F}_t} \eta \right| \leq \mathbb{E}^{\mathcal{F}_t} e^{\tilde{V}_t} |\eta| \leq \mathbb{E}^{\mathcal{F}_t} e^{\tilde{V}_T} |\eta| \quad (6.111)$$

by Proposition 1.56 we infer for all $p > 1$

$$\mathbb{E} \left\| \xi e^{\tilde{V}} \right\|_{[0, T]}^p \leq \left(\frac{p}{p-1} \right)^p \mathbb{E} \left(e^{p \tilde{V}_T} |\eta|^p \right) < \infty$$

and consequently by Proposition 6.80-B (for $(Y, Z) = (\xi, \zeta)$ with $\lambda = 0$, $K = R = N = 0$, $dD_t = |\xi_t|^2 d\tilde{V}$) the inequality (6.109) follows; we also use that $V \leq \tilde{V}$ and

$$\mathbb{E}^{\mathcal{F}_t} \left| e^{\tilde{V}_T} \xi_T \right|^p = \mathbb{E}^{\mathcal{F}_t} \left[\left| e^{\tilde{V}_T} \mathbb{E}^{\mathcal{F}_T} \eta \right|^p \right] \leq \mathbb{E}^{\mathcal{F}_t} \left| e^{\tilde{V}_T} \eta \right|^p.$$

In the case $p = 1$ we have for all $0 < \alpha < 1$, by Proposition 1.56

$$\mathbb{E} \sup_{t \in [0, T]} e^{\alpha V_t} |\xi_t|^\alpha \leq \mathbb{E} \sup_{t \in [0, T]} \left| e^{\tilde{V}_t} \xi_t \right|^\alpha \leq \frac{1}{1-\alpha} \left(\mathbb{E} e^{\tilde{V}_T} |\eta| \right)^\alpha \quad (6.112)$$

and by Proposition 6.80-A

$$\mathbb{E} \left(\int_0^T e^{2\tilde{V}_s} |\zeta_s|^2 ds \right)^{\alpha/2} \leq C_1 \mathbb{E} \sup_{t \in [0, T]} \left| e^{\tilde{V}_t} \xi_t \right|^\alpha. \quad (6.113)$$

Also we can see that from (6.111) $\mathbb{E} \left| e^{\tilde{V}_t} \xi_t \right| \leq \mathbb{E} \left(e^{\tilde{V}_T} |\eta| \right)$ and therefore

$$\sup_{t \in [0, T]} \mathbb{E} \left| e^{\tilde{V}_t} \xi_t \right| \leq \mathbb{E} \left(e^{\tilde{V}_T} |\eta| \right). \quad (6.114)$$

From (6.112)–(6.114) the inequality (6.110) follows. ■

Let $(Y, Z), (\hat{Y}, \hat{Z}) \in S_m^0[0, T] \times \Lambda_{m \times k}^0(0, T)$ satisfying for all $t \in [0, T]$:

$$Y_t = Y_T + \int_t^T dK_s - \int_t^T Z_s dB_s, \quad \mathbb{P}\text{-a.s.},$$

and respectively

$$\hat{Y}_t = \hat{Y}_T + \int_t^T d\hat{K}_s - \int_t^T \hat{Z}_s dB_s, \quad \mathbb{P}\text{-a.s.},$$

where

- ◇ $K, \hat{K} \in S_m^0$,
- ◇ $K.(\omega), \hat{K}.(\omega) \in BV_{loc}([0, \infty[; \mathbb{R}^m)$, \mathbb{P} -a.s. $\omega \in \Omega$.

Assume there exist $\lambda < 1 \leq p$ and V a \mathcal{P} -m.b.v.c.s.p., $V_0 = 0$, such that as signed measures on $[0, T]$:

$$\langle Y_t - \hat{Y}_t, dK_t - d\hat{K}_t \rangle \leq |Y_t - \hat{Y}_t|^2 dV_t + \frac{n_p \lambda}{2} \left| Z_t - \hat{Z}_t \right|^2 dt, \quad (6.115)$$

where $n_p = 1 \wedge (p - 1)$.

Corollary 6.84. Let $\lambda < 1 \leq p$ be given. Let the assumption (6.115) be satisfied and $\{A_t : t \geq 0\}$ be a \mathcal{P} -m.i.c.s.p., $A_0 = 0$, such that

$$\mathbb{E} \sup_{t \in [0, T]} \left(e^{p(A_t + V_t)} \left| Y_t - \hat{Y}_t \right|^p \right) < \infty.$$

Then for all $0 \leq t \leq T$,

$$e^{pV_t} \left| Y_t - \hat{Y}_t \right|^p \leq \mathbb{E}^{\mathcal{F}_t} \left(e^{pV_T} \left| Y_T - \hat{Y}_T \right|^p \right), \quad \mathbb{P}\text{-a.s.}$$

Moreover if $p > 1$, then

$$\begin{aligned} & \mathbb{E}^{\mathcal{F}_t} \left(\sup_{s \in [t, T]} e^{p(A_s + V_s)} \left| Y_s - \hat{Y}_s \right|^p \right) + \mathbb{E}^{\mathcal{F}_t} \int_t^T e^{p(A_s + V_s)} \left| Y_s - \hat{Y}_s \right|^p dA_s \\ & + \mathbb{E}^{\mathcal{F}_t} \left(\int_t^T e^{2(A_s + V_s)} \left| Y_s - \hat{Y}_s \right|^2 dA_s \right)^{p/2} + \mathbb{E}^{\mathcal{F}_t} \left(\int_t^T e^{2(A_s + V_s)} \left| Z_s - \hat{Z}_s \right|^2 ds \right)^{p/2} \\ & \leq C_{p, \lambda} \mathbb{E}^{\mathcal{F}_t} e^{p(A_T + V_T)} \left| Y_T - \hat{Y}_T \right|^p, \quad \mathbb{P}\text{-a.s.}, \end{aligned}$$

where $C_{p, \lambda}$ is a positive constant depending only on (p, λ) .

Proof. The results clearly follow from Corollary 6.82 and the inequality (6.94) from Proposition 6.80, applied to

$$Y_t - \hat{Y}_t = Y_T - \hat{Y}_T + \int_t^T d(K_s - \hat{K}_s) - \int_t^T (Z_s - \hat{Z}_s) dB_s,$$

satisfying

$$dD_t + \langle Y_t - \hat{Y}_t, dK_t - d\hat{K}_t \rangle \leq |Y_t - \hat{Y}_t|^2 d(A_t + V_t) + \frac{n_p \lambda}{2} \left| Z_t - \hat{Z}_t \right|^2 dt$$

with

$$dD_t = |Y_t - \hat{Y}_t|^2 dA_t.$$

■

6.5 Annex D: Viscosity Solutions

The aim of this section is to introduce the notion of viscosity solutions to second order elliptic and parabolic PDEs and give uniqueness results for such solutions. This notion, which was invented by Crandall and Lions, allows us to state that a continuous function satisfies a PDE, without any differentiability requirement on that function. This notion has been invented specifically for nonlinear equations, for which the notion of weak solutions in the sense of distributions is not convenient. We use this notion here for linear and semilinear equations.

This section is divided into four parts. In the first part, we state the main definitions of viscosity solutions to elliptic and parabolic PDEs (or systems of PDEs). We prove three uniqueness results in the next three parts. We do not prove any existence results, since such results for the equations considered in this book are provided by our probabilistic formulas. Concerning uniqueness, it would be too long and repetitive to give a uniqueness result for each PDE considered in this book. The last three parts of this section give uniqueness results, corresponding to three large classes of semilinear PDEs or systems of PDEs. All other relevant results can be proved by combining the arguments given in those three proofs.

The first uniqueness result concerns an elliptic PDE with Dirichlet boundary condition at the boundary of a bounded set. We shall also explain how the proof can be adapted to the parabolic case. The second result treats the case of a system of parabolic PDEs in the whole space. Finally the third result concerns a parabolic PDE with subdifferential operators and nonlinear Neumann boundary condition.

We refer to the well-known “user’s guide” of Crandall et al. [18] for more details, which complements the material presented here.

6.5.1 Definitions

Let \mathcal{O} be a locally closed subset of \mathbb{R}^d , that is for all $x \in \mathcal{O}$ there exists a $\delta > 0$ such that $\mathcal{O} \cap \overline{B}(x, \delta)$ is closed.

A function $h : \mathcal{O} \subset \mathbb{R}^d \rightarrow \mathbb{R}$ is lower semicontinuous and we write $h \in LSC(\mathcal{O})$ if there exist $\{h_n, n \geq 1\} \subset C(\mathcal{O})$ such that

$$h_1(x) \leq \dots \leq h_n(x) \leq \dots \leq h(x) \quad \text{and} \quad \lim_{n \rightarrow \infty} h_n(x) = h(x), \quad \forall x \in \mathcal{O}.$$

The function $h : \mathcal{O} \subset \mathbb{R}^d \rightarrow \mathbb{R}$ is upper semicontinuous and we write $h \in USC(\mathcal{O})$ if $-h$ is lower semicontinuous.

In particular for all $R > 0$ we have

$$\begin{aligned} (i) \quad & \inf_{x \in \mathcal{O}, |x| \leq R} h(x) > -\infty, \text{ if } h \in LSC(\mathcal{O}), \\ (ii) \quad & \sup_{x \in \mathcal{O}, |x| \leq R} h(x) < \infty, \text{ if } h \in USC(\mathcal{O}). \end{aligned}$$

6.5.1.1 Elliptic PDE

Consider the PDE

$$\Phi(x, u(x), Du(x), D^2u(x)) = 0, \quad x \in \mathcal{O}, \quad (6.116)$$

where

$$\Phi : \mathcal{O} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d \rightarrow \mathbb{R},$$

and \mathbb{S}^d denotes the set of symmetric $d \times d$ matrices.

Definition 6.85. (i) $u \in USC(\mathcal{O})$ is a *viscosity sub-solution* of (6.116) if for any $\varphi \in C^2(\mathcal{O})$ and $\hat{x} \in \mathcal{O}$ a local maximum of $u - \varphi$:

$$\Phi(\hat{x}, u(\hat{x}), D\varphi(\hat{x}), D^2\varphi(\hat{x})) \leq 0.$$

(ii) $u \in LSC(\mathcal{O})$ is a *viscosity super-solution* of (6.116) if for any $\varphi \in C^2(\mathcal{O})$ and $\hat{x} \in \mathcal{O}$ a local minimum of $u - \varphi$:

$$\Phi(\hat{x}, u(\hat{x}), D\varphi(\hat{x}), D^2\varphi(\hat{x})) \geq 0.$$

(iii) $u \in C(\mathcal{O})$ is a *viscosity solution* if it is both a viscosity sub- and super-solution.

In these definitions we can also assume that $u(\hat{x}) = \varphi(\hat{x})$ since we can translate φ .

Note that the class of PDEs for which probabilistic formulas are given in this book is the class of semilinear equations, where the function Φ has the following particular form

$$\Phi(x, r, p, X) = -\frac{1}{2} \text{Tr} [g(x)g^*(x)X] - \langle f(x), p \rangle - F(x, r, p). \quad (6.117)$$

In the Definition 6.85 we can replace local maximum (minimum) by strict global maximum (minimum).

Remark 6.86. Let \mathcal{O} be an open subset of \mathbb{R}^d and $u \in C^2(\mathcal{O})$.

- (i) If u is a *viscosity solution* of (6.116), then u is a classical solution.
- (ii) If u is a classical solution of (6.116) and Φ satisfies the degenerate ellipticity condition

$$X \leq Y \Rightarrow \Phi(x, r, p, X) \geq \Phi(x, r, p, Y), \quad \forall x, r, p,$$

then u is a *viscosity solution*.

Definition 6.87. A function $u \in USC(\mathcal{O})$ satisfies the maximum principle if for all $\varphi \in C^2(\mathcal{O})$ and all open subsets $D \subseteq \mathcal{O}$ the inequality

$$\Phi(x, \varphi(x), D\varphi(x), D^2\varphi(x)) > 0, \quad \forall x \in D$$

implies that at every $\hat{x} \in D$ which is a local maximum of $u - \varphi$:

$$u(\hat{x}) < \varphi(\hat{x}).$$

Proposition 6.88. Let \mathcal{O} be an open subset of \mathbb{R}^d and

$$r \leq s \Rightarrow \Phi(x, r, p, X) \leq \Phi(x, s, p, X), \quad \forall x, p, X.$$

Then each viscosity sub-solution u satisfies the maximum principle.

Proof. If we assume that there exist $\varphi \in C^2(\mathcal{O})$, an open subset $D \subseteq \mathcal{O}$ such that

$$\Phi(x, u(x), D\varphi(x), D^2\varphi(x)) > 0, \quad \forall x \in D,$$

and $\hat{x} \in D$ a local maximum of $u - \varphi$ such that $u(\hat{x}) \geq \varphi(\hat{x})$ then

$$\Phi(\hat{x}, \varphi(\hat{x}), D\varphi(\hat{x}), D^2\varphi(\hat{x})) \leq \Phi(\hat{x}, u(\hat{x}), D\varphi(\hat{x}), D^2\varphi(\hat{x})) \leq 0,$$

since u is a sub-solution. Hence necessarily $u(\hat{x}) < \varphi(\hat{x})$. ■

We next introduce the notion of a proper function (in the sense of the theory of viscosity solutions, which should not be confused with the notion of proper convex function), for which the notion of a viscosity solution makes sense.

Definition 6.89. A continuous function

$$\Phi : \mathcal{O} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d \rightarrow \mathbb{R}$$

is said to be proper, if Φ satisfies:

(1) Monotonicity condition

$$r \leq s \Rightarrow \Phi(x, r, p, X) \leq \Phi(x, s, p, X), \forall x, p, X,$$

and

(2) Degenerate ellipticity condition

$$X \leq Y \Rightarrow \Phi(x, r, p, X) \geq \Phi(x, r, p, Y), \forall x, r, p. \quad (6.118)$$

Definition 6.85 of a viscosity solution can be reformulated in terms of subjets and superjets of u .

Definition 6.90. Let \mathcal{O} be a locally closed subset of \mathbb{R}^d , $u : \mathcal{O} \rightarrow \mathbb{R}$ and $x \in \mathcal{O}$.

(i) $(p, X) \in \mathbb{R}^d \times \mathbb{S}^d$ is a superjet to u at x if

$$\limsup_{\mathcal{O} \ni y \rightarrow x} \frac{u(y) - u(x) - \langle p, y - x \rangle - \frac{1}{2} \langle X(y - x), y - x \rangle}{|y - x|^2} \leq 0.$$

The set of superjets to u at x will be denoted $J_{\mathcal{O}}^{2,+} u(x)$.

(ii) $(p, X) \in \mathbb{R}^d \times \mathbb{S}^d$ is a subjet to u at x if

$$\liminf_{\mathcal{O} \ni y \rightarrow x} \frac{u(y) - u(x) - \langle p, y - x \rangle - \frac{1}{2} \langle X(y - x), y - x \rangle}{|y - x|^2} \geq 0.$$

The set of subjets to u at x will be denoted $J_{\mathcal{O}}^{2,-} u(x)$.

If $\mathcal{O} = \mathbb{R}^d$, then the index \mathcal{O} will be omitted.

Proposition 6.91. Let \mathcal{O} be a locally closed subset of \mathbb{R}^d .

(i) Let $u \in USC(\mathcal{O})$ and $\tilde{x} \in \mathcal{O}$.

(a) If $(p, X) \in J_{\mathcal{O}}^{2,+} u(\tilde{x})$, then there exists a $\varphi \in C^2(\mathcal{O})$ such that $u(\tilde{x}) = \varphi(\tilde{x})$,

$$(p, X) = (\varphi'_x(\tilde{x}), \varphi''_{xx}(\tilde{x}))$$

and \tilde{x} is a strict global maximum of $u - \varphi$ in \mathcal{O} .

(b) If $\varphi \in C^2(\mathcal{O})$ and \tilde{x} is a local maximum of $u - \varphi$ in \mathcal{O} , then

$$(\varphi'_x(\tilde{x}), \varphi''_{xx}(\tilde{x})) \in J_{\mathcal{O}}^{2,+} u(\tilde{x}).$$

(ii) Let $u \in LSC(\mathcal{O})$ and $\tilde{x} \in \mathcal{O}$.

(a) If $(p, X) \in J_{\mathcal{O}}^{2,-} u(\tilde{x})$, then there exists a $\varphi \in C^2(\mathcal{O})$ such that $u(\tilde{x}) = \varphi(\tilde{x})$,

$$(p, X) = (\varphi'_x(\tilde{x}), \varphi''_{xx}(\tilde{x}))$$

and \tilde{x} is a strict global minimum of $u - \varphi$ in \mathcal{O} .

(b) If $\varphi \in C^2(\mathcal{O})$ and \tilde{x} is a local minimum of $u - \varphi$ in \mathcal{O} , then

$$(\varphi'_x(\tilde{x}), \varphi''_{xx}(\tilde{x})) \in J_{\mathcal{O}}^{2,-}u(\tilde{x}).$$

Proof. It is sufficient to prove (i) since $J_{\mathcal{O}}^{2,-}u(\tilde{x}) = -J_{\mathcal{O}}^{2,+}(-u)(\tilde{x})$. Also the equivalence is clear if \tilde{x} is an isolated point of \mathcal{O} .

Let \tilde{x} be a non-isolated point of \mathcal{O} .

(\Rightarrow): Let $(p, X) \in J_{\mathcal{O}}^{2,+}u(\tilde{x})$. Then there exists a strictly increasing function $\rho = \rho(\tilde{x}) : [0, +\infty[\rightarrow [0, +\infty[$, $\rho(0+) = 0$ such that $\forall y \in \mathcal{O}$

$$u(y) \leq u(\tilde{x}) + \langle p, y - \tilde{x} \rangle + \frac{1}{2} \langle X(y - \tilde{x}), y - \tilde{x} \rangle + \rho(|y - \tilde{x}|)|y - \tilde{x}|^2. \quad (6.119)$$

One can define ρ by

$$\rho(r) = r + \sup_{y \in \mathcal{O}, |y - \tilde{x}| \leq r} \frac{(u(y) - u(\tilde{x}) - \langle p, y - \tilde{x} \rangle - \frac{1}{2} \langle X(y - \tilde{x}), y - \tilde{x} \rangle)^+}{|y - \tilde{x}|^2}.$$

Let

$$\beta(r) = \frac{1}{r^2} \int_r^{2r} \int_{r_2}^{2r_2} \int_{r_1}^{2r_1} \rho(\sqrt{\tau}) d\tau dr_1 dr_2, \quad \text{for } r > 0,$$

and $\beta(r) = 0$ if $r \leq 0$. Then $r\rho(\sqrt{r}) < \beta(r) < 8r\rho(8\sqrt{r})$ for all $r > 0$, $\beta \in C^2(]0, \infty[)$, $\beta(0+) = \beta'(0+) = 0$ and

$$\lim_{r \searrow 0} r\beta''(r) = 0.$$

Define $\varphi \in C^2(\mathbb{R}^d)$ by

$$\varphi(y) \stackrel{\text{def}}{=} u(\tilde{x}) + \langle p, y - \tilde{x} \rangle + \frac{1}{2} \langle X(y - \tilde{x}), y - \tilde{x} \rangle + \beta(|y - \tilde{x}|^2).$$

Then

$$\varphi'_x(\tilde{x}) = p \quad \text{and} \quad \varphi''_{xx}(\tilde{x}) = X$$

and \tilde{x} is a strict global maximum of $u - \varphi$ since for $y \in \mathcal{O} \setminus \{\tilde{x}\}$:

$$\begin{aligned} u(y) - \varphi(y) &\leq \rho(|y - \tilde{x}|)|y - \tilde{x}|^2 - \beta(|y - \tilde{x}|^2) \\ &< 0 = \varphi(\tilde{x}) - u(\tilde{x}). \end{aligned}$$

(\Leftarrow): Let $\varphi \in C^2(\mathcal{O})$ and \tilde{x} be a local maximum of $u - \varphi$. Let

$$\psi(y) = \varphi(y) - \varphi(\tilde{x}) + u(\tilde{x}).$$

By Taylor's formula

$$\begin{aligned} 0 &= \lim_{y \rightarrow \tilde{x}} \frac{\psi(y) - \psi(\tilde{x}) - \langle \psi'_x(\tilde{x}), y - \tilde{x} \rangle - \frac{1}{2} \langle \psi''_{xx}(\tilde{x})(y - \tilde{x}), y - \tilde{x} \rangle}{|y - \tilde{x}|^2} \\ &\geq \limsup_{y \rightarrow \tilde{x}, y \in \mathcal{O}} \frac{u(y) - u(\tilde{x}) - \langle \varphi'_x(\tilde{x}), y - \tilde{x} \rangle - \frac{1}{2} \langle \varphi''_{xx}(\tilde{x})(y - \tilde{x}), y - \tilde{x} \rangle}{|y - \tilde{x}|^2}. \end{aligned}$$

■

Corollary 6.92. *Let \mathcal{O} be a locally closed subset of \mathbb{R}^d .*

(i) *$u \in USC(\mathcal{O})$ is a viscosity sub-solution of (6.116) iff for any $x \in \mathcal{O}$ and $(p, X) \in J_{\mathcal{O}}^{2,+}u(x)$*

$$\Phi(x, u(x), p, X) \leq 0.$$

(ii) *$u \in LSC(\mathcal{O})$ is a viscosity super-solution of (6.116) iff for any $x \in \mathcal{O}$ and $(p, X) \in J_{\mathcal{O}}^{2,-}u(x)$*

$$\Phi(x, u(x), p, X) \geq 0.$$

Definition 6.93. *Let $u : \mathcal{O} \rightarrow \mathbb{R}$ and $x \in \mathcal{O}$.*

$\overline{J_{\mathcal{O}}^{2,+}u(x)}$ (respect. $\overline{J_{\mathcal{O}}^{2,-}u(x)}$) is the set of $(p, X) \in \mathbb{R}^d \times \mathbb{S}^d$ such that there exists a sequence $(x_n, p_n, X_n) \in \mathcal{O} \times \mathbb{R}^d \times \mathbb{S}^d$, $n \in \mathbb{N}^*$, with the properties

$$(p_n, X_n) \in J_{\mathcal{O}}^{2,+}u(x_n), \text{ (respect. } (p_n, X_n) \in J_{\mathcal{O}}^{2,-}u(x_n)), \forall n \in \mathbb{N}^*,$$

and

$$(x_n, u(x_n), p_n, X_n) \rightarrow (x, u(x), p, X), \text{ as } n \rightarrow \infty.$$

6.5.1.2 Systems of PDEs

Backward stochastic differential equations naturally give probabilistic formulas for systems of PDEs, not just for single PDEs.

Let \mathcal{O} be an open subset of \mathbb{R}^d , $\Phi \in C(\overline{\mathcal{O}} \times \mathbb{R}^m \times \mathbb{R}^d \times \mathbb{S}^d; \mathbb{R}^m)$. We want to explain what we mean by the fact that $u \in C(\mathcal{O}, \mathbb{R}^m)$ solves in the viscosity sense the following systems of PDEs

$$\Phi_i(x, u(x), Du_i(x), D^2u_i(x)) = 0, \quad 1 \leq i \leq m, \quad x \in \mathcal{O}. \quad (6.120)$$

Note that the various equations are coupled only through the vector $u(x)$. The i -th equation depends upon all coordinates of $u(x)$, but only on the i -th coordinate of $Du(x)$ and $D^2u(x)$. This is essential for the following definition to make sense.

Definition 6.94. Let \mathcal{O} be a locally closed subset of \mathbb{R}^d .

- (i) $u \in USC(\mathcal{O})$ is a viscosity sub-solution of (6.120) if

$$\Phi_i(x, u(x), p, X) \leq 0 \text{ for } x \in \mathcal{O}, 1 \leq i \leq m, (p, X) \in \overline{J}_{\mathcal{O}}^{2,+} u_i(x).$$

- (ii) $u \in LSC(\mathcal{O})$ is a viscosity super-solution of (6.120) if

$$\Phi_i(x, u(x), p, X) \geq 0 \text{ for } x \in \mathcal{O}, 1 \leq i \leq m, (p, X) \in \overline{J}_{\mathcal{O}}^{2,-} u_i(x).$$

- (iii) $u \in C(\mathcal{O})$ is a viscosity solution of (6.120) if it is both a viscosity sub- and super-solution.

6.5.1.3 Boundary Conditions

We now discuss the formulation of the boundary condition in the framework of viscosity solutions. Suppose for simplicity that the boundary $\partial\mathcal{O}$ of the open set \mathcal{O} is of class C^1 and that \mathcal{O} satisfies the uniform exterior ball condition. We shall consider two types of boundary conditions, namely:

- Dirichlet boundary conditions, of the form

$$u(x) - \kappa(x) = 0, \quad x \in \partial\mathcal{O};$$

- Nonlinear Neumann boundary conditions, of the form

$$\langle n(x), Du(x) \rangle + G(x, u(x)) = 0, \quad x \in \partial\mathcal{O},$$

where $n(x)$ denotes the outward normal vector to the boundary $\partial\mathcal{O}$ at x .

Consider the function

$$\Gamma : \partial\mathcal{O} \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$$

defined in the case of the Dirichlet boundary condition by

$$\Gamma(x, r, p) = r - \kappa(x),$$

and in the case of the Neumann boundary condition by

$$\Gamma(x, r, p) = \langle n(x), p \rangle - G(x, r),$$

where $G \in C(\partial\mathcal{O} \times \mathbb{R})$ and $r \rightarrow G(x, r)$ is assumed to be nonincreasing for all $x \in \partial\mathcal{O}$. The correct formulation of the boundary value problem

$$\begin{cases} \Phi(x, u(x), Du(x), D^2u(x)) = 0, & x \in \mathcal{O}, \\ \Gamma(x, u(x), Du(x)) = 0, & x \in \partial\mathcal{O}, \end{cases} \quad (6.121)$$

is as follows.

Definition 6.95. Let \mathcal{O} be an open subset of \mathbb{R}^d , $\Phi \in C(\overline{\mathcal{O}} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d)$ be proper and $\Gamma \in C(\mathcal{O} \times \mathbb{R} \times \mathbb{R}^d)$ be as defined above.

- (i) $u \in USC(\overline{\mathcal{O}})$ is a viscosity sub-solution of (6.121) if

$$\begin{cases} \Phi(x, u(x), p, X) \leq 0 \text{ for } x \in \mathcal{O}, (p, X) \in \overline{J}_{\mathcal{O}}^{2,+} u(x), \\ \Phi(x, u(x), p, X) \wedge \Gamma(x, u(x), p) \leq 0 \text{ for } x \in \partial\mathcal{O}, (p, X) \in \overline{J}_{\mathcal{O}}^{2,+} u(x). \end{cases}$$

- (ii) $u \in LSC(\overline{\mathcal{O}})$ is a viscosity super-solution of (6.121) if

$$\begin{cases} \Phi(x, u(x), p, X) \geq 0 \text{ for } x \in \mathcal{O}, (p, X) \in \overline{J}_{\mathcal{O}}^{2,-} u(x), \\ \Phi(x, u(x), p, X) \vee \Gamma(x, u(x), p) \geq 0 \text{ for } x \in \partial\mathcal{O}, (p, X) \in \overline{J}_{\mathcal{O}}^{2,-} u(x). \end{cases}$$

- (iii) $u \in C(\overline{\mathcal{O}})$ is a viscosity solution of (6.121) if it is both a viscosity sub- and super-solution.

6.5.1.4 Parabolic PDEs

One might think that a parabolic PDE is an elliptic PDE with one more variable, namely time t . However, because we are considering equations with first derivatives in t only, the variable t plays a specific role. In particular, there will be a boundary condition either at the initial point or at the final point of the time interval, not at both.

Given $\mathcal{O} \subset \mathbb{R}^d$ and $\Phi \in C([0, T] \times \mathcal{O} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d)$, we consider the parabolic equation

$$\begin{cases} \frac{\partial u}{\partial t}(t, x) + \Phi(t, x, u(t, x), Du(t, x), D^2u(t, x)), & 0 < t < T, x \in \mathcal{O}, \\ u(0, x) = \kappa(x), & x \in \mathcal{O}, \end{cases} \quad (6.122)$$

where as previously, Du stands for the vector of first order partial derivatives with respect to the x_i 's, and D^2u for the matrix of second order derivatives with respect to x_i and x_j , $1 \leq i, j \leq d$. Only in the case $\mathcal{O} = \mathbb{R}^d$ can we hope that the above parabolic PDE is well posed. If $\mathcal{O} \neq \mathbb{R}^d$, some boundary condition is needed. This will be discussed later.

We denote by $\mathcal{P}_{\mathcal{O}}^{2,+}$ and $\mathcal{P}_{\mathcal{O}}^{2,-}$ the parabolic analogs of $J_{\mathcal{O}}^{2,+}$ and $J_{\mathcal{O}}^{2,-}$. More specifically, for \mathcal{O} a locally compact subset of \mathbb{R}^d , $T > 0$, denoting $\mathcal{O}_T = (0, T) \times \mathcal{O}$, if $u : \mathcal{O}_T \rightarrow \mathcal{R}$, $0 < s, t < T$, $x, y \in \mathcal{O}$, $(p, q, X) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d$, we say that $(p, q, X) \in \mathcal{P}_{\mathcal{O}}^{2,+}u(t, x)$, whenever

$$u(s, y) \leq u(t, x) + p(s - t) + \langle q, y - x \rangle + \frac{1}{2} \langle X(y - x), y - x \rangle + o(|s - t| + |y - x|^2).$$

Moreover $\mathcal{P}_{\mathcal{O}}^{2,-}u = -\mathcal{P}_{\mathcal{O}}^{2,+}(-u)$. The corresponding definitions of $\overline{\mathcal{P}}_{\mathcal{O}}^{2,+}u(t, x)$ and $\overline{\mathcal{P}}_{\mathcal{O}}^{2,-}u(t, x)$ are now clear.

We now give a definition of the notion of a viscosity solution of equation (6.122).

Definition 6.96. With the above notation:

- (i) $u \in USC([0, T) \times \mathcal{O})$ is a viscosity sub-solution of Eq. (6.122) if $u(0, x) \leq \kappa(x)$, $x \in \mathcal{O}$ and

$$p + \Phi(t, x, u(t, x), q, X) \leq 0, \text{ for } (t, x) \in \mathcal{O}_T, (p, q, X) \in \mathcal{P}_{\mathcal{O}}^{2,+}u(t, x).$$

- (ii) $u \in LSC([0, T) \times \mathcal{O})$ is a viscosity super-solution of Eq. (6.122) if $u(0, x) \geq \kappa(x)$, $x \in \mathcal{O}$ and

$$p + \Phi(t, x, u(t, x), q, X) \geq 0, \text{ for } (t, x) \in \mathcal{O}_T, (p, q, X) \in \mathcal{P}_{\mathcal{O}}^{2,-}u(t, x).$$

- (iii) $u \in C([0, T) \times \mathcal{O})$ is a viscosity solution of (6.122) if it is both a sub- and a super-solution.

We remark that $u(t, x)$ solves the parabolic PDE (6.122) if and only if $v(t, x) = e^{\lambda t}u(t, x)$ solves the same equation with Φ replaced by $\Phi + \lambda r$, which in the case where Φ has the form (6.117) is proper iff $r \rightarrow \lambda r - F(t, x, r, q)$ is increasing for any (t, x, q) . The fact that this is true for some λ is one of our standing assumptions on F for existence and uniqueness of the solution to the associated BSDE.

Note that we also consider parabolic PDEs with a final condition (at time $t = T$) rather than an initial condition (at time $t = 0$). In that case, the equation becomes

$$-\frac{\partial u}{\partial t}(t, x) + \Phi(t, x, u(t, x), Du(t, x), D^2u(t, x)) = 0,$$

and the condition $u(0, x) \leq \kappa(x)$ (resp. $u(0, x) \geq \kappa(x)$) becomes $u(T, x) \leq \kappa(x)$ (resp. $u(T, x) \geq \kappa(x)$).

Finally we explain what we mean by a viscosity solution of the parabolic PDE

$$\frac{\partial u}{\partial t}(t, x) + \Phi(t, x, u(t, x), Du(t, x), D^2u(t, x)) + \partial\varphi(u(t, x)) \ni 0,$$

where $\partial\varphi$ is the subdifferential of the convex lower semicontinuous function $\varphi : \mathbb{R} \rightarrow (-\infty, +\infty]$.

A sub-solution is a function $u \in USC(\mathcal{O}_T)$ which is such that for any $(t, x) \in \mathcal{O}_T, u(t, x) \in \text{Dom}(\varphi)$ and whenever $(p, q, X) \in \mathcal{P}_{\mathcal{O}}^{2,+}u(t, x)$,

$$p + \Phi(t, x, u(t, x), q, X) + \varphi'_-(u(t, x)) \leq 0,$$

where $\varphi'_-(r)$ is the left derivative of φ at the point r . A super-solution is defined similarly with the usual changes, the left derivative of φ being replaced by its right derivative.

6.5.2 A First Uniqueness Result

Let \mathcal{O} be an open subset of \mathbb{R}^d and $\Phi \in C(\mathcal{O} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d)$.

The basic assumptions of this subsection are:

(A₁) *Super-monotonicity*: there exists a $\delta > 0$ such that for all $x \in \mathcal{O}, p \in \mathbb{R}^d, X \in \mathbb{S}^d, r, s \in \mathbb{R}$:

$$r_1 \leq r_2 \Rightarrow \Phi(x, r_2, p, X) - \Phi(x, r_1, p, X) \geq (r_2 - r_1) \delta,$$

and

(A₂) *Super-degenerate-ellipticity*: for all $R > 0$ there exists an increasing function $\mathbf{m}_R : \mathbb{R}_+ \rightarrow \mathbb{R}_+, \mathbf{m}_R(0+) = 0$ such that if $\alpha > 0, X, Y \in \mathbb{S}^d$ and

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}, \tag{6.123}$$

or equivalently

$$\langle Xz, z \rangle - \langle Yw, w \rangle \leq 3\alpha |z - w|^2, \quad \forall z, w \in \mathbb{R}^d,$$

then for all $x, y \in \mathcal{O} \cap \overline{B(0, R)}, r \in \mathbb{R}$:

$$\Phi(y, r, \alpha(x - y), Y) - \Phi(x, r, \alpha(x - y), X) \leq \mathbf{m}_R \left(|x - y| + \alpha |x - y|^2 \right). \tag{6.124}$$

Note that if X and Y satisfy (6.123) then $Y \leq X$ (setting $z = w$).

In the particular case of the function Φ given by (6.117), the super-monotonicity of Φ is a consequence of the same property for $-F$. As for the super degenerate ellipticity, we have the following:

Lemma 6.97. *If g is globally Lipschitz, f is globally monotone, and $-F$ satisfies (6.124), then Φ is super-degenerate-elliptic.*

Proof. The global monotonicity of f implies that

$$-(f(y) - f(x), \alpha(x - y)) \leq \mu\alpha|x - y|^2.$$

Now consider the term involving g . We take advantage of (6.123) and the Lipschitz continuity of g :

$$\begin{aligned} \mathbf{Tr} [gg^*(x)X] - \mathbf{Tr} [gg^*(y)Y] &= \mathbf{Tr} [g^*(x)Xg(x) - g^*(y)Yg(y)] \\ &= \sum_{i=1}^d [\langle Xg(x)e_i, g(x)e_i \rangle - \langle Yg(y)e_i, g(y)e_i \rangle] \\ &\leq 3\alpha \sum_{i=1}^d |g(x)e_i - g(y)e_i|^2 \\ &\leq C|x - y|^2. \end{aligned}$$

■

Theorem 6.98 (Comparison Principle). *Let \mathcal{O} be a bounded open subset of \mathbb{R}^d and assume that $\Phi : \mathcal{O} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d \rightarrow \mathbb{R}$ satisfies (A_1) and (A_2) . If*

- (j) $u \in USC(\overline{\mathcal{O}})$ is a sub-solution of $\Phi = 0$ in \mathcal{O} ,
- (jj) $v \in LSC(\overline{\mathcal{O}})$ is a super-solution of $\Phi = 0$ in \mathcal{O} ,
- (jjj) $u(x) \leq v(x), \forall x \in \partial\mathcal{O}$,

then

$$u(x) \leq v(x) \quad \forall x \in \overline{\mathcal{O}}.$$

We first prove auxiliary results.

Lemma 6.99. *Given $u, v \in C(\overline{\mathcal{O}})$, $\alpha > 0$, we define*

$$\psi_\alpha(x, y) = u(x) - v(y) - \frac{\alpha}{2}|x - y|^2.$$

Let (\hat{x}, \hat{y}) be a local maximum in $\mathcal{O} \times \mathcal{O}$ of ψ_α . Then there exist $X, Y \in \mathbb{S}^d$ such that

- (j) $(\alpha(\hat{x} - \hat{y}), X) \in \bar{J}_\mathcal{O}^{2,+}u(\hat{x})$,
- (jj) $(\alpha(\hat{x} - \hat{y}), Y) \in \bar{J}_\mathcal{O}^{2,-}v(\hat{y})$,
- (jjj) $\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$.

Proof. We shall use the notation

$$A = \alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

It is sufficient to prove the proposition in case $\mathcal{O} = \mathbb{R}^d$, $\hat{x} = \hat{y} = 0$, $u(0) = v(0) = 0$, $(0, 0)$ is a global maximum of ψ_α , u and $-v$ are bounded from above. Hence we may assume that for all $x, y \in \mathbb{R}^d$,

$$u(x) - v(y) \leq \frac{1}{2} \left\langle A \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle, \quad (6.125)$$

and we need to show that there exist $X, Y \in S_d$ such that

- (j') $(0, X) \in \bar{J}^{2,+}u(0)$,
- (jj') $(0, Y) \in \bar{J}^{2,+}v(0)$,
- (jjj') $\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq 3A$.

With the notations $\bar{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, $\bar{\xi} = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$, we deduce from Schwarz's inequality that (with the notation $\|A\| \stackrel{\text{def}}{=} \sup\{|\langle A\bar{\xi}, \bar{\xi} \rangle|; |\bar{\xi}| \leq 1\}$):

$$\begin{aligned} \langle A\bar{x}, \bar{x} \rangle &= \langle A\bar{\xi}, \bar{\xi} \rangle + \langle A(\bar{x} - \bar{\xi}), \bar{x} - \bar{\xi} \rangle + 2\langle \bar{x} - \bar{\xi}, A\bar{\xi} \rangle \\ &\leq \langle A\bar{\xi}, \bar{\xi} \rangle + \frac{1}{\alpha} |A\bar{\xi}|^2 + (\alpha + \|A\|) |\bar{x} - \bar{\xi}|^2 \\ &\leq \left\langle \left(A + \frac{1}{\alpha} A^2 \right) \bar{\xi}, \bar{\xi} \right\rangle + (\alpha + \|A\|) |\bar{x} - \bar{\xi}|^2. \end{aligned}$$

Hence if $B \stackrel{\text{def}}{=}} 3A = A + \frac{1}{\alpha} A^2$, $\lambda \stackrel{\text{def}}{=} \alpha + \|A\|$, and $w(\bar{x}) \stackrel{\text{def}}{=} u(x) - v(y)$, (6.125) implies

$$w(\bar{x}) - \frac{\lambda}{2} |\bar{x} - \bar{\xi}|^2 \leq \frac{1}{2} \langle B\bar{\xi}, \bar{\xi} \rangle. \quad (6.126)$$

We now introduce inf- and sup-convolutions. Let

$$\begin{aligned} \hat{w}(\bar{\xi}) &\stackrel{\text{def}}{=} \sup_{\bar{x}} (w(\bar{x}) - \frac{\lambda}{2} |\bar{x} - \bar{\xi}|^2) \\ &= \hat{u}(\xi) - \hat{v}(\eta), \end{aligned}$$

where

$$\hat{u}(\xi) = \sup_x (u(x) - \frac{\lambda}{2}|x - \xi|^2),$$

$$\hat{v}(\eta) = \inf_y (v(y) + \frac{\lambda}{2}|y - \eta|^2).$$

Since a supremum (resp. an infimum) of convex (resp. concave) functions is convex (resp. concave), the mappings

$$\bar{\xi} \rightarrow \hat{w}(\bar{\xi}) + \frac{\lambda}{2}|\bar{\xi}|^2, \quad \text{and} \quad \xi \rightarrow \hat{u}(\xi) + \frac{\lambda}{2}|\xi|^2$$

are convex, while

$$\eta \rightarrow \hat{v}(\eta) - \frac{\lambda}{2}|\eta|^2$$

is concave. Hence \hat{w} , \hat{u} and $-\hat{v}$ are “semiconvex”, i.e. they are the sum of a convex function and a function of class C^2 . Note that the hyphen is here on purpose, in order to distinguish this notion from the notion of semiconvex functions, as introduced in Chap. 4.3.

Moreover:

$$\hat{w}(0) \leq w(0) = 0,$$

and from (6.126)

$$\hat{w}(\bar{\xi}) \leq \frac{1}{2} \langle B\bar{\xi}, \bar{\xi} \rangle,$$

hence

$$\hat{w}(0) \leq 0,$$

and consequently

$$\hat{w}(0) = \max_{\bar{\xi}} \left(\hat{w}(\bar{\xi}) - \frac{1}{2} \langle B\bar{\xi}, \bar{\xi} \rangle \right).$$

If \hat{w} is smooth, we could deduce that there exists an $\mathcal{X} \in S_{2d}$ such that $(0, \mathcal{X}) \in J^2\hat{w}(0)$, and $\mathcal{X} \leq B$. Since \hat{w} is semiconvex, it is possible to show, using Alexandrov’s theorem (which says that a semiconvex function is a.e. twice differentiable), and a lemma due to R. Jensen, which states that the above is essentially true in the sense that it is true provided the first condition is changed to $(0, \mathcal{X}) \in \bar{J}^2\hat{w}(0)$. We refer to the user’s guide [18] for more details. Now,

since $\hat{w}(\bar{\xi}) = \hat{u}(\xi) - \hat{v}(\eta)$, it is not hard to deduce that $\mathcal{X} = \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix}$, and $(0, X) \in \bar{J}^2 \hat{u}(0)$, $(0, Y) \in \bar{J}^2 \hat{v}(0)$.

The magical property of sup-convolution is that this is enough to conclude that $(0, X) \in \bar{J}^{2,+} u(0)$ and $(0, Y) \in \bar{J}^{2,-} v(0)$, which is a consequence of the next Lemma. \blacksquare

Lemma 6.100. *Let $\lambda > 0$, $u \in C(\mathbb{R}^d)$ be bounded from above, and*

$$\hat{u}(\zeta) = \sup_{x \in \mathbb{R}^d} \left(u(x) - \frac{\lambda}{2} |x - \zeta|^2 \right).$$

If $\eta, q \in \mathbb{R}^d$, $X \in S_d$ and $(\eta, X) \in J^{2,+} \hat{u}(\eta)$, then $(q, X) \in J^{2,+} u(\eta + q/\lambda)$.

Proof. We assume that $(q, X) \in J^{2,+} \hat{u}(\eta)$. Let $y \in \mathbb{R}^d$ be such that

$$\hat{u}(\eta) = u(y) - \frac{\lambda}{2} |y - \eta|^2.$$

Then for any $x, \zeta \in \mathbb{R}^d$,

$$\begin{aligned} u(x) - \frac{\lambda}{2} |x - \zeta|^2 &\leq \hat{u}(\zeta) \\ &\leq \hat{u}(\eta) + \langle q, \zeta - \eta \rangle + \frac{1}{2} \langle X(\zeta - \eta), \zeta - \eta \rangle + o(|\zeta - \eta|^2) \\ &= u(y) - \frac{\lambda}{2} |y - \eta|^2 + \langle q, \zeta - \eta \rangle \\ &\quad + \frac{1}{2} \langle X(\zeta - \eta), \zeta - \eta \rangle + o(|\zeta - \eta|^2) \\ &= u(y) - \frac{\lambda}{2} |y - \eta|^2 + \langle q, \zeta - \eta \rangle + O(|\zeta - \eta|^2). \end{aligned}$$

If we choose $\zeta = x - y + \eta$, then we deduce from the above that

$$u(x) \leq u(y) + \langle q, x - y \rangle + \frac{1}{2} \langle X(x - y), x - y \rangle + o(|x - y|^2).$$

On the other hand, choosing $x = y$ and $\zeta = \eta + \alpha(\lambda(\eta - y) + q)$, we obtain that

$$0 \leq \alpha |\lambda(\eta - y) + q|^2 + O(\alpha^2).$$

The first inequality says that $(q, X) \in J^{2,+} u(y)$, while the second, with $\alpha < 0$ small enough in absolute value, implies that $y = \eta + \frac{q}{\lambda}$. The result is proved. \blacksquare

We shall also need the following:

Lemma 6.101. *Let \mathcal{O} be locally closed subset of \mathbb{R}^d , $\Phi \in USC(\mathcal{O})$, $\Psi \in LSC(\mathcal{O})$, $\Psi \geq 0$, $\varepsilon > 0$ and*

$$M_\varepsilon = \sup_{x \in \mathcal{O}} \left\{ \Phi(x) - \frac{1}{\varepsilon} \Psi(x) \right\}.$$

If $\lim_{\varepsilon \rightarrow 0} M_\varepsilon$ exists in \mathbb{R} and $x_\varepsilon \in \mathcal{O}$ satisfies

$$\lim_{\varepsilon \rightarrow 0} \left[M_\varepsilon - \Phi(x_\varepsilon) + \frac{1}{\varepsilon} \Psi(x_\varepsilon) \right] = 0,$$

then

$$\lim_{\varepsilon \rightarrow 0} \frac{\Psi(x_\varepsilon)}{\varepsilon} = 0. \quad (6.127)$$

Moreover if $\hat{x} \in \mathcal{O}$ and there exists an $\varepsilon_n \rightarrow 0$ such that $x_{\varepsilon_n} \rightarrow \hat{x}$, then

$$\Psi(\hat{x}) = 0, \text{ and } \lim_{\varepsilon \rightarrow 0} M_\varepsilon = \Phi(\hat{x}) = \sup \{ \Phi(x) : x \in \mathcal{O}, \Psi(x) = 0 \}. \quad (6.128)$$

Proof. Let $\alpha_\varepsilon = M_\varepsilon - \Phi(x_\varepsilon) + \frac{1}{\varepsilon} \Psi(x_\varepsilon)$. Note that for $0 < \varepsilon < \delta$ we have $M_\varepsilon \leq M_\delta$ and

$$M_{2\varepsilon} \geq \Phi(x_\varepsilon) - \frac{1}{2\varepsilon} \Psi(x_\varepsilon) = M_\varepsilon - \alpha_\varepsilon + \frac{1}{2\varepsilon} \Psi(x_\varepsilon).$$

Then

$$\frac{\Psi(x_\varepsilon)}{\varepsilon} \leq 2(M_{2\varepsilon} - M_\varepsilon + \alpha_\varepsilon)$$

and (6.127) follows. Moreover by the lower semicontinuity of Ψ

$$0 \leq \Psi(\hat{x}) \leq \liminf_{\varepsilon_n \rightarrow 0} \Psi(x_{\varepsilon_n}) = 0.$$

Using now the upper semicontinuity of Φ we have

$$\begin{aligned} \Phi(\hat{x}) &\geq \limsup_{\varepsilon_n \rightarrow 0} \Phi(x_{\varepsilon_n}) \\ &= \limsup_{\varepsilon_n \rightarrow 0} \left[M_{\varepsilon_n} - \alpha_{\varepsilon_n} + \frac{1}{\varepsilon_n} \Psi(x_{\varepsilon_n}) \right] \\ &= \lim_{\varepsilon \rightarrow 0} M_\varepsilon \end{aligned}$$

$$\begin{aligned} &\geq \sup \{ \Phi(x) : x \in \mathcal{O}, \Psi(x) = 0 \} \\ &\geq \Phi(\hat{x}). \end{aligned}$$

The result follows. ■

Proof of the comparison principle. Assume that

$$M \stackrel{\text{def}}{=} \sup_{x \in \overline{\mathcal{O}}} \{ u(x) - v(x) \} > 0.$$

Let $\varepsilon > 0$ and

$$M_\varepsilon \stackrel{\text{def}}{=} \sup_{(x,y) \in \overline{\mathcal{O}} \times \overline{\mathcal{O}}} \left[u(x) - v(y) - \frac{1}{2\varepsilon} |x - y|^2 \right].$$

Clearly for $\delta > \varepsilon$, $M_\delta \geq M_\varepsilon \geq u(x) - v(x)$, $\forall x \in \overline{\mathcal{O}}$ and consequently M_ε converges in \mathbb{R} as $\varepsilon \rightarrow 0$,

$$M_\varepsilon \geq M > 0 \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} M_\varepsilon \geq M.$$

Since $\overline{\mathcal{O}}$ is compact and $(x, y) \mapsto u(x) - v(y)$ is upper semicontinuous on $\overline{\mathcal{O}} \times \overline{\mathcal{O}}$, there exists $(x_\varepsilon, y_\varepsilon) \in \overline{\mathcal{O}} \times \overline{\mathcal{O}}$ such that

$$u(x_\varepsilon) - v(y_\varepsilon) - \frac{1}{2\varepsilon} |x_\varepsilon - y_\varepsilon|^2 = M_\varepsilon.$$

By Lemma 6.101, with $\Phi(x, y) = u(x) - v(y)$ and $\Psi(x, y) = \frac{1}{2} |x - y|^2$ we obtain

$$\lim_{\varepsilon \rightarrow \infty} \frac{1}{\varepsilon} |x_\varepsilon - y_\varepsilon|^2 = 0.$$

We now conclude that there exists an $\varepsilon_0 > 0$ such that

$$x_\varepsilon, y_\varepsilon \in \mathcal{O}, \quad \text{for all } 0 < \varepsilon \leq \varepsilon_0.$$

Since $u(x) \leq v(x)$, $\forall x \in \partial\mathcal{O}$ and whenever $\varepsilon_n \rightarrow 0$ and $x_{\varepsilon_n}, y_{\varepsilon_n} \rightarrow \hat{x}$, it follows that

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} M_\varepsilon &= u(\hat{x}) - v(\hat{x}) \\ &= \sup \{ \Phi(x, y) : (x, y) \in \overline{\mathcal{O}} \times \overline{\mathcal{O}}, \Psi(x, y) = 0 \} \\ &= \sup_{x \in \overline{\mathcal{O}}} \{ u(x) - v(x) \} \\ &> 0. \end{aligned}$$

By Lemma 6.99, for $0 < \varepsilon \leq \varepsilon_0$ there exist $X_\varepsilon, Y_\varepsilon \in \mathbb{S}^d$ such that

$$\begin{aligned} \left(\frac{1}{\varepsilon} \Psi'_x(x_\varepsilon, y_\varepsilon), X_\varepsilon \right) &\in \bar{\mathcal{J}}_{\mathcal{O}}^{2,+} u(x_\varepsilon), \quad \text{and} \\ \left(-\frac{1}{\varepsilon} \Psi'_y(x_\varepsilon, y_\varepsilon), Y_\varepsilon \right) &\in \bar{\mathcal{J}}_{\mathcal{O}}^{2,-} v(y_\varepsilon) \end{aligned} \quad (6.129)$$

and the inequality (jjj) in Lemma 6.99 reads here

$$\begin{pmatrix} X_\varepsilon & 0 \\ 0 & -Y_\varepsilon \end{pmatrix} \leq \frac{3}{\varepsilon} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

Let $R > 0$ such that $\mathcal{O} \subset B(0, R)$. From (A_2) with $\alpha = \varepsilon^{-1}$, we deduce that

$$\begin{aligned} &\Phi\left(y_\varepsilon, v(y_\varepsilon), \frac{x_\varepsilon - y_\varepsilon}{\varepsilon}, Y_\varepsilon\right) - \Phi\left(x_\varepsilon, v(y_\varepsilon), \frac{x_\varepsilon - y_\varepsilon}{\varepsilon}, X_\varepsilon\right) \\ &\leq \mathbf{m}_R \left(|x_\varepsilon - y_\varepsilon| + \frac{1}{\varepsilon} |x_\varepsilon - y_\varepsilon|^2 \right), \end{aligned}$$

and since $u(x_\varepsilon) > v(y_\varepsilon)$ for ε small enough, we deduce from (A_1) that

$$\begin{aligned} &\Phi\left(x_\varepsilon, v(y_\varepsilon), \frac{x_\varepsilon - y_\varepsilon}{\varepsilon}, X_\varepsilon\right) - \Phi\left(x_\varepsilon, u(x_\varepsilon), \frac{x_\varepsilon - y_\varepsilon}{\varepsilon}, X_\varepsilon\right) \\ &\leq \delta [v(y_\varepsilon) - u(x_\varepsilon)] \\ &= -\delta \left[\frac{1}{2\varepsilon} |x_\varepsilon - y_\varepsilon|^2 + M_\varepsilon \right]. \end{aligned}$$

It follows that

$$\begin{aligned} &\Phi\left(y_\varepsilon, v(y_\varepsilon), \frac{x_\varepsilon - y_\varepsilon}{\varepsilon}, Y_\varepsilon\right) - \Phi\left(x_\varepsilon, u(x_\varepsilon), \frac{x_\varepsilon - y_\varepsilon}{\varepsilon}, X_\varepsilon\right) \\ &\leq \mathbf{m}_R \left(|x_\varepsilon - y_\varepsilon| + \frac{1}{\varepsilon} |x_\varepsilon - y_\varepsilon|^2 \right) - \delta \left[\frac{1}{2\varepsilon} |x_\varepsilon - y_\varepsilon|^2 + M_\varepsilon \right]. \end{aligned}$$

Since u is a viscosity sub-solution and v is a viscosity super-solution of the equation $\Phi = 0$, we deduce from (6.129) that

$$\Phi\left(x_\varepsilon, u(x_\varepsilon), \frac{x_\varepsilon - y_\varepsilon}{\varepsilon}, X_\varepsilon\right) \leq 0 \leq \Phi\left(y_\varepsilon, v(y_\varepsilon), \frac{x_\varepsilon - y_\varepsilon}{\varepsilon}, Y_\varepsilon\right).$$

Hence

$$0 \leq \mathbf{m}_R \left(|x_\varepsilon - y_\varepsilon| + \frac{1}{\varepsilon} |x_\varepsilon - y_\varepsilon|^2 \right) - \delta \left[\frac{1}{2\varepsilon} |x_\varepsilon - y_\varepsilon|^2 + M_\varepsilon \right],$$

then also

$$0 < \delta M \leq \delta M_\varepsilon \leq \mathbf{m}_R \left(|x_\varepsilon - y_\varepsilon| + \frac{1}{\varepsilon} |x_\varepsilon - y_\varepsilon|^2 \right) - \frac{\delta}{2\varepsilon} |x_\varepsilon - y_\varepsilon|^2$$

and letting $\varepsilon \rightarrow 0$, we infer the contradiction

$$0 < \delta M \leq 0.$$

The Theorem is established. ■

We deduce from this theorem the uniqueness of the viscosity solution for the Dirichlet problem.

Corollary 6.102. *Under the assumptions of Theorem 6.98, if $u, v \in C(\overline{\mathcal{O}})$ are two viscosity solutions of $\Phi = 0$ on \mathcal{O} then*

$$u(x) = v(x), \forall x \in \partial\mathcal{O} \implies u(x) = v(x), \forall x \in \overline{\mathcal{O}}.$$

This Corollary proves that our probabilistic formula provides the unique solution of the corresponding elliptic PDE, satisfying the Dirichlet boundary condition in the classical sense. However it follows from Theorem 7.9 in [18] that it is also the unique solution in the larger class of those solutions satisfying the Dirichlet boundary condition in the (relaxed) viscosity sense.

Let us now indicate how the above proof can be modified, in order to treat the case of a parabolic PDE with Dirichlet condition at the boundary of a bounded set.

Let \mathcal{O} be a bounded open subset of \mathbb{R}^d . Consider the Cauchy–Dirichlet problem

$$\begin{cases} \frac{\partial u}{\partial t} + \Phi(t, x, u, u'_x, u''_{xx}) = 0 & \text{in }]0, T[\times \mathcal{O}, \\ u(t, x) = \kappa(t, x), & (t, x) \in]0, T[\times \partial\mathcal{O}, \\ u(0, x) = \kappa(0, x) & x \in \overline{\mathcal{O}}, \end{cases} \tag{6.130}$$

where $\kappa \in C([0, T] \times \overline{\mathcal{O}})$.

The notion of a viscosity solution to (6.130) is expressed as in Definition 6.96, adding the requirement $u(t, x) \leq \kappa(t, x)$ (resp. \geq) for $(t, x) \in (0, T) \times \partial\mathcal{O}$ for u to be a sub-solution (resp. a super-solution).

We have the comparison principle:

Theorem 6.103. *Let $\Phi \in C([0, T] \times \overline{\mathcal{O}} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d)$ be a proper function satisfying (A_1) and (A_2) for each fixed $t \in [0, T[$, with the same δ and \mathbf{m}_R . If $u \in USC([0, T) \times \overline{\mathcal{O}})$ is a viscosity sub-solution of (6.130) and $v \in LSC([0, T) \times \overline{\mathcal{O}})$ is a viscosity super-solution of (6.130) then*

$$u(t, x) \leq v(t, x), \text{ for all } (t, x) \in [0, T) \times \overline{\mathcal{O}}.$$

An essential tool for the proof of this Theorem is the parabolic analog of Lemma 6.99, which is as follows:

Lemma 6.104. *Given $u, v \in C(\mathcal{O}_T)$, $\alpha > 0$, let*

$$\psi_\alpha(t, x, y) = u(t, x) - v(t, y) - \frac{\alpha}{2}|x - y|^2.$$

Let $(\hat{t}, \hat{x}, \hat{y})$ be a local maximum of ψ_α in $(0, T) \times \mathcal{O} \times \mathcal{O}$. Suppose moreover that there is an $r > 0$ such that for every $M > 0$ there is a C with the property that whenever $(p, q, X) \in \mathcal{P}_{\mathcal{O}}^{2,+}u(t, x)$, $|x - \hat{x}| + |t - \hat{t}| \leq r$ and $|u(t, x)| + |q| + |X| \leq M$, then $p \leq C$, and the same is true if we replace $\mathcal{P}_{\mathcal{O}}^{2,+}u(t, x)$ by $-\mathcal{P}_{\mathcal{O}}^{2,-}v(t, x)$. Then there exist $p \in \mathbb{R}$, $X, Y \in \mathbb{S}^d$ such that

- (j) $(p, \alpha(\hat{x} - \hat{y}), X) \in \overline{\mathcal{P}}_{\mathcal{O}}^{2,+}u(\hat{t}, \hat{x})$,
- (jj) $(-p, \alpha(\hat{x} - \hat{y}), Y) \in \overline{\mathcal{P}}_{\mathcal{O}}^{2,-}v(\hat{t}, \hat{x})$,
- (jjj) $\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$.

Proof of the Theorem. We only sketch the proof. We first observe that it suffices to prove that $\tilde{u}(t, x) = u(t, x) - \varepsilon/(T - t) \leq v(t, x)$ for all $(t, x) \in (0, T) \times \mathcal{O}$ and all $\varepsilon > 0$. Now \tilde{u} satisfies

$$\begin{cases} \frac{\partial \tilde{u}}{\partial t}(t, x) + \Phi(t, x, \tilde{u}(t, x), D\tilde{u}(t, x), D^2\tilde{u}(t, x)) \leq -\frac{\varepsilon}{(T - t)^2}, \\ \lim_{t \rightarrow T} \tilde{u}(t, x) = -\infty. \end{cases}$$

From now on we write u instead of \tilde{u} . We want to contradict the assumption that $\max_{(0, T) \times \mathcal{O}}[u - v] = \delta > 0$. Let $(\hat{t}, \hat{x}, \hat{y})$ be a local maximum of $\psi_\alpha(t, x, y)$ from Lemma 6.104, and write

$$M_\alpha = u(\hat{t}, \hat{x}) - v(\hat{t}, \hat{y}) - \frac{\alpha}{2}|\hat{x} - \hat{y}|^2.$$

From our standing assumption, $M_\alpha \geq \delta > 0$. It is not hard to show that for α large enough, $0 < \hat{t} < T$, $\hat{x}, \hat{y} \in \mathcal{O}$. Arguing as in the proof of Theorem 6.98 with the help this time of Lemma 6.104, we conclude that there exist $p \in \mathbb{R}$, $X, Y \in \mathbb{S}^d$ $c > 0$ such that

$$\begin{aligned} p + \Phi(\hat{t}, \hat{x}, u(\hat{t}, \hat{x}), \alpha(\hat{x} - \hat{y}), X) &\leq -c, \\ -p + \Phi(\hat{t}, \hat{y}, v(\hat{t}, \hat{y}), \alpha(\hat{x} - \hat{y}), Y) &\geq 0, \end{aligned}$$

while

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

We deduce that

$$\begin{aligned}
 c &\leq \Phi(\hat{t}, \hat{y}, v(\hat{t}, \hat{y}), \alpha(\hat{x} - \hat{y}), Y) - \Phi(\hat{t}, \hat{x}, u(\hat{t}, \hat{x}), \alpha(\hat{x} - \hat{y}), X) \\
 &\leq m(\alpha|\hat{x} - \hat{y}|^2 + |\hat{x} - \hat{y}|),
 \end{aligned}$$

from which a contradiction follows. ■

6.5.3 A Second Uniqueness Result

We are given a continuous and globally monotone $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ and a globally Lipschitz $g : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ together with

$$\kappa \in C(\mathbb{R}^d; \mathbb{R}^m), \quad \text{and } F \in C([0, T] \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{m \times d}; \mathbb{R}^m)$$

such that, for each $1 \leq i \leq k$, $F_i(t, x, y, z)$ depends on the matrix z only through its i -th column z_i . As already explained, this assumption is essential for the notion of a viscosity solution of the system of partial differential equations to be considered below to make sense. We assume specifically that for some constants $C, p > 0$:

(A.2i) $|F(t, x, 0, 0, 0)| \leq C(1 + |x|^p), \quad |\kappa(x)| \leq C(1 + |x|^p),$

(A.2ii) $F = F(t, x, y, z)$ is globally Lipschitz in (y, z) , uniformly in (t, x) .

Remark 6.105. In the case of systems of equations, it does not seem possible to weaken the Lipschitz continuity of F in y to a monotonicity condition as we do in the case $m = 1$.

Under the assumptions (A.2i) and (A.2ii), for each $t \in [0, T]$ and $x \in \mathbb{R}^d$, we consider the system of PDEs

$$\begin{cases}
 -\frac{\partial u_i}{\partial t}(t, x) + \Phi_i(t, x, u(t, x), Du_i(t, x), D^2u_i(t, x)) = 0, \\
 \hspace{10em} (t, x) \in [0, T] \times \mathbb{R}^d, \quad 1 \leq i \leq k, \\
 u_i(T, x) = \kappa_i(x), \quad x \in \mathbb{R}^d, \quad 1 \leq i \leq m,
 \end{cases} \tag{6.131}$$

where

$$\Phi_i(t, x, r, q, X) = -\frac{1}{2}Tr[gg^*(x)X] - \langle f(x), q \rangle - F_i(t, x, r, q).$$

The notion of a viscosity solution for such a system is easily deduced from a combination of Definitions 6.94 and 6.96.

We can replace “global maximum point” or “global minimum point” by “strict global maximum point” or “strict global minimum point”. The proof of this claim is very simple and we leave it as an exercise for the reader.

Now we give a uniqueness result for (6.131). This result is obtained under the following additional assumption:

$$(A.2 \text{ iii}) \quad |F(t, x, r, p) - F(t, y, r, p)| \leq \mathbf{m}_R(|x - y|(1 + |p|)),$$

for all $x, y \in \mathbb{R}^d$ such that $|x| \leq R, |y| \leq R, r \in \mathbb{R}^m, p \in \mathbb{R}^d$, where for each $R > 0, \mathbf{m}_R \in C(\mathbb{R}_+)$ is increasing and $\mathbf{m}_R(0) = 0$.

Our result is the following:

Theorem 6.106. *Assume that f, g satisfy (A2). Then there exists at most one viscosity solution u of (6.131) such that*

$$\lim_{|x| \rightarrow +\infty} |u(t, x)|e^{-\delta[\log(|x|)]^2} = 0, \tag{6.132}$$

uniformly for $t \in [0, T]$, for some $\delta > 0$.

Remark 6.107. Notice that any function which has at most a polynomial growth at infinity satisfies (6.132).

The growth condition (6.132) is optimal to obtain such a uniqueness result for (6.131). Indeed, consider the equation

$$\frac{\partial u}{\partial t} - \frac{x^2}{2} \frac{\partial^2 u}{\partial x^2} - \frac{x}{2} \frac{\partial u}{\partial x} = 0 \quad \text{in } (0, T) \times (0, +\infty), \tag{6.133}$$

then u is a solution of (6.133) if and only if the function $v(t, y) = u(t, e^y)$ is a solution of the Heat Equation

$$\frac{\partial v}{\partial t} - \frac{1}{2} \frac{\partial^2 v}{\partial x^2} = 0 \quad \text{in } (0, T) \times \mathbb{R}. \tag{6.134}$$

But it is well-known that, for the Heat Equation, the uniqueness holds in the class of solutions v satisfying

$$\lim_{|y| \rightarrow +\infty} |v(t, y)|e^{-\delta|y|^2} = 0, \tag{6.135}$$

uniformly for $t \in [0, T]$, for some $\delta > 0$. And (6.135) gives back (6.132) for (6.133) since $y = \log(x)$.

Let us finally mention that, in our case, the growth condition (6.132) is mainly a consequence of the assumptions on the coefficients of the differential operator and in particular on $a = gg^*$; under the assumptions of Theorem 6.106, the matrix a has, a priori, a quadratic growth at infinity. If a is assumed to have a linear growth at infinity, an easy adaptation of the proof of Theorem 6.106 shows that the uniqueness holds in the class of solutions satisfying

$$\lim_{|x| \rightarrow +\infty} |u(t, x)|e^{-\delta|x|} = 0,$$

uniformly for $t \in [0, T]$, for some $\delta > 0$.

Proof of Theorem 6.106. Let u and v be two viscosity solutions of (6.131). The proof consists of two steps. We first show that $u - v$ and $v - u$ are viscosity sub-solutions of an integral partial differential system; then we build a suitable sequence of smooth super-solutions of this system to show that $|u - v| = 0$ in $[0, T] \times \mathbb{R}^d$. Here and below, we denote by $|\cdot|$ the sup norm in \mathbb{R}^m .

Lemma 6.108. *Let u be a sub-solution and v a super-solution of (6.131). Then the function $\omega := u - v$ is a viscosity sub-solution of the system*

$$-\frac{\partial \omega_i}{\partial t} - \mathcal{A}\omega_i - \tilde{K} [|\omega| + |\nabla \omega_i g|] = 0 \text{ in } [0, T] \times \mathbb{R}^d, \tag{6.136}$$

for $1 \leq i \leq k$, where \tilde{K} is the Lipschitz constant of F in (r, p) .

Proof. Let $\varphi \in C^2([0, T] \times \mathbb{R}^d)$ and let $(t_0, x_0) \in (0, T) \times \mathbb{R}^d$ be a strict global maximum point of $\omega_i - \varphi$ for some $1 \leq i \leq k$.

We introduce the function

$$\psi_n(t, x, y) = u_i(t, x) - v_i(t, y) - n|x - y|^2 - \varphi(t, x),$$

where n is devoted to tend to infinity.

Since (t_0, x_0) is a strict global maximum point of $u_i - v_i - \varphi$, by a classical argument in the theory of viscosity solutions, there exists a sequence (t_n, x_n, y_n) such that:

- (i) (t_n, x_n, y_n) is a global maximum point of ψ_n in $[0, T] \times (\overline{B_R})^2$, where B_R is a ball with a large radius R ;
- (ii) $(t_n, x_n), (t_n, y_n) \rightarrow (t_0, x_0)$ as $n \rightarrow \infty$;
- (iii) $n|x_n - y_n|^2$ is bounded and tends to zero as $n \rightarrow \infty$.

It follows from a variant of Lemma 6.104, see also Theorem 8.3 in the user’s guide [18], that there exist $X, Y \in \mathbb{S}^d$ such that

$$\left(\frac{\partial \varphi}{\partial t}(t_n, x_n), q_n + D\varphi(t_n, x_n), X \right) \in \mathcal{P}^{2,+} u_i(t_n, x_n)$$

$$(0, q_n, Y) \in \mathcal{P}^{2,-} v_i(t_n, y_n)$$

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq 4n \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \begin{pmatrix} D^2\varphi(t_n, x_n) & 0 \\ 0 & 0 \end{pmatrix},$$

where

$$q_n = 2n(x_n - y_n).$$

Modifying if necessary ψ_n by adding terms of the form $\chi(x)$ and $\chi(y)$ with supports in $B_{R/2}^c$, we may assume that (t_n, x_n, y_n) is a global maximum point of ψ_n in $([0, T] \times \mathbb{R}^d)^2$. Since u and v are respectively sub and super-solutions of (6.131), we have

$$\begin{aligned} & -\frac{\partial\varphi}{\partial t}(t_n, x_n) - \frac{1}{2}\text{Tr}(a(x_n)X) - \langle f(x_n), q_n + D\varphi(t_n, x_n) \rangle \\ & - F_i(t_n, x_n, u(t_n, x_n), (q_n + D\varphi(t_n, x_n))g(x_n)) \leq 0 \end{aligned}$$

and

$$-\frac{1}{2}\text{Tr}(a(y_n)Y) - \langle f(y_n), q_n \rangle - F_i(t_n, y_n, v(t_n, y_n), p_n g(y_n)) \geq 0.$$

The computation of Lemma 6.97 yields

$$\begin{aligned} & \frac{1}{2}\text{Tr}[gg^*(x_n)X] - \frac{1}{2}\text{Tr}[gg^*(y_n)Y] + \langle f(x_n) - f(y_n), q_n \rangle \\ & \leq n|x_n - y_n|^2 + \text{Tr}[gg^*(x_n)D^2\varphi(t_n, x_n)]. \end{aligned}$$

Finally, we consider the difference between the nonlinear terms

$$\begin{aligned} & F_i(t_n, x_n, u(t_n, x_n), (q_n + D\varphi(t_n, x_n))g(x_n)) - F_i(t_n, y_n, v(t_n, y_n), q_n g(y_n)) \\ & \leq \mathbf{m}(|x_n - y_n|(1 + |p_n g(y_n)|)) + \tilde{K}|u(t_n, x_n) - v(t_n, y_n)| \\ & \quad + \tilde{K}|q_n(g(x_n) - g(y_n)) + D\varphi(t_n, x_n)g(x_n)|. \end{aligned}$$

The first term on the right-hand side comes from (A.2 iii): we have denoted by \mathbf{m} the modulus \mathbf{m}_R which appears in this assumption for R large enough. The two last terms come from the Lipschitz continuity of F_i with respect to the two last variables.

We notice that

$$|q_n(g(x_n) - g(y_n))| \leq Cn|x_n - y_n|^2,$$

because of the Lipschitz continuity of g and that

$$|x_n - y_n| \cdot |q_n g(y_n)| \leq Cn|x_n - y_n|^2.$$

Now we subtract the viscosity inequalities for u and v : thanks to the above estimates, we can write the obtained inequality in the following way

$$-\frac{\partial\varphi}{\partial t}(t_n, x_n) - \mathcal{A}\varphi(t_n, x_n) - \tilde{K}|u(t_n, x_n) - v(t_n, y_n)| \leq \omega_1(n),$$

where we have gathered in $\omega_1(n)$ all the terms of the form $n|x_n - y_n|^2$ and $|x_n - y_n|$; $\omega_1(n) \rightarrow 0$ when n tends to ∞ . To conclude we let $n \rightarrow \infty$. Since $(t_n, x_n), (t_n, y_n) \rightarrow (t_0, x_0)$, we obtain:

$$-\frac{\partial \varphi}{\partial t}(t_0, x_0) - \mathcal{A}\varphi(t_0, x_0) - \tilde{K}|\omega(t_0, x_0)| - \tilde{K}|D\varphi(t_0, x_0)g(x_0)| \leq 0,$$

and therefore ω is a sub-solution of the desired equation. ■

Now we are going to build suitable smooth super-solutions for the equation (6.136).

Lemma 6.109. *For any $\delta > 0$, there exists a $C_1 > 0$ such that the function*

$$\chi(t, x) = \exp[(C_1(T - t) + \delta)\psi(x)]$$

where

$$\psi(x) = [\log(|x|^2 + 1)^{1/2} + 1]^2,$$

satisfies

$$-\frac{\partial \chi}{\partial t} - \mathcal{A}\chi - \tilde{K}\chi - \tilde{K}|D\chi g| > 0 \text{ in } [t_1, T] \times \mathbb{R}^d$$

for $1 \leq i \leq k$ where $t_1 = T - \delta/C_1$.

Proof. We first estimate the term $K\chi$, the main point being its dependence in x . For the sake of simplicity of notation, we denote below by C all the positive constants which enter into these estimates. These constants depend only on δ and on the bounds on the coefficients of the equations.

We first give estimates on the first and second derivatives of ψ : easy computations yield

$$|D\psi(x)| \leq \frac{2[\psi(x)]^{1/2}}{(|x|^2 + 1)^{1/2}} \leq 4 \text{ in } \mathbb{R}^d,$$

and

$$|D^2\psi(x)| \leq \frac{C(1 + [\psi(x)]^{1/2})}{|x|^2 + 1} \text{ in } \mathbb{R}^d.$$

These estimates imply that, if $t \in [t_1, T]$

$$\begin{aligned} |D\chi(t, x)| &\leq (C_1(T - t) + \delta)\chi(t, x)|D\psi(x)| \\ &\leq C\chi(t, x) \frac{[\psi(x)]^{1/2}}{(|x|^2 + 1)^{1/2}}, \end{aligned}$$

and, in the same way

$$|D^2\chi(t, x)| \leq C\chi(t, x) \frac{\psi(x)}{|x|^2 + 1}.$$

It is worth noticing that, because of our choice of t_1 , the above estimates do not depend on C_1 .

Since gg^* and $\langle f(x), x \rangle$ grow at most quadratically at infinity, we have

$$\begin{aligned} & -\frac{\partial\chi}{\partial t}(t, x) - \mathcal{A}\chi(t, x) - \tilde{K}\chi(t, x) - \tilde{K}|D\chi(t, x)g(x)| \\ & \geq \chi \left[C_1\psi(x) - C\psi(x) - C\frac{\psi(x)}{|x|^2 + 1} - \tilde{K} - C\tilde{K}[\psi(x)]^{1/2} - C\tilde{K}\frac{[\psi(x)]^{1/2}}{(|x|^2 + 1)^{1/2}} \right]. \end{aligned}$$

Since $\psi(x) \geq 1$ in \mathbb{R}^d , by using the Cauchy–Schwartz inequality, it is clear enough that for C_1 large enough the quantity in the brackets is positive and the proof is complete. ■

To conclude the proof, we are going to show that $\omega = u - v$ satisfies

$$|\omega(t, x)| \leq \alpha\chi(t, x) \text{ in } [0, T] \times \mathbb{R}^d$$

for any $\alpha > 0$. Then we will let α tend to zero.

To prove this inequality, we first remark that because of (6.132)

$$\lim_{|x| \rightarrow +\infty} |\omega(t, x)| e^{-\delta[\log((|x|^2+1)^{1/2})]^2} = -\infty$$

uniformly for $t \in [0, T]$, for some $\delta > 0$. From now on we choose δ in the definition of χ such that this holds. Then $|\omega_i| - \alpha\chi$ is bounded from above in $[t_1, T] \times \mathbb{R}^d$ for any $1 \leq i \leq k$ and

$$M = \max_{1 \leq i \leq m} \max_{[t_1, T] \times \mathbb{R}^d} (|\omega_i| - \alpha\chi)(t, x) e^{-\tilde{K}(T-t)}$$

is attained at some point (t_0, x_0) and for some i_0 .

We first remark that, since $|\cdot|$ is the sup norm in \mathbb{R}^m , we have

$$M = \max_{[t_1, T] \times \mathbb{R}^d} (|\omega| - \alpha\chi)(t, x) e^{-\tilde{K}(T-t)}$$

and $|\omega_{i_0}(t_0, x_0)| = |\omega(t_0, x_0)|$. We may assume without loss of generality that $|\omega_{i_0}(t_0, x_0)| > 0$, otherwise we are done.

There are two cases: either $\omega_{i_0}(t_0, x_0) > 0$ or $\omega_{i_0}(t_0, x_0) < 0$. We treat the first case, the second one is treated in a similar way since the roles of u and v are symmetric.

From the maximum point property, we deduce that

$$\omega_{i_0}(t, x) - \alpha\chi(t, x) \leq (\omega_{i_0} - \alpha\chi)(t_0, x_0)e^{-\tilde{K}(t-t_0)}$$

and this inequality can be interpreted as the property for the function $\omega_{i_0} - \phi$ to have a global maximum point at (t_0, x_0) , where

$$\phi(t, x) = \alpha\chi(t, x) + (\omega_{i_0} - \alpha\chi)(t_0, x_0)e^{-\tilde{K}(t-t_0)}.$$

Since ω is a viscosity sub-solution of (6.136), if $t_0 \in [t_1, T[$, we have

$$-\frac{\partial\phi}{\partial t}(t_0, x_0) - \mathcal{A}\phi(t_0, x_0) - \tilde{K}|\omega(t_0, x_0)| - \tilde{K}|D\phi(t_0, x_0)g(x_0)| \leq 0.$$

But the left-hand side of this inequality is nothing but

$$\alpha \left[-\frac{\partial\chi}{\partial t}(t_0, x_0) - \mathcal{A}\chi(t_0, x_0) - \tilde{K}\chi(t_0, x_0) - \tilde{K}|D\chi(t_0, x_0)g(x_0)| \right],$$

since $\omega_{i_0}(t_0, x_0) = |\omega(t_0, x_0)|$; so, by Lemma 6.109, we have a contradiction. Therefore $t_0 = T$ and since $|\omega(T, x)| = 0$, we have

$$|\omega(t, x)| - \alpha\chi(t, x) \leq 0 \text{ in } [t_1, T] \times \mathbb{R}^d.$$

Letting α tend to zero, we obtain

$$|\omega(t, x)| = 0 \text{ in } [t_1, T] \times \mathbb{R}^d.$$

Applying successively the same argument on the intervals $[t_2, t_1]$ where $t_2 = (t_1 - \delta/C_1)^+$ and then, if $t_2 > 0$, on $[t_3, t_2]$ where $t_3 = (t_2 - \delta/C_1)^+ \dots$ etc, we finally obtain that

$$|\omega(t, x)| = 0 \text{ in } [0, T] \times \mathbb{R}^d$$

and the proof is complete.

6.5.4 A Third Uniqueness Result

Let D be an open connected bounded subset of \mathbb{R}^d of the form

$$D = \{x \in \mathbb{R}^d : \phi(x) < 0\}, \quad \text{Bd}(D) = \{x \in \mathbb{R}^d : \phi(x) = 0\},$$

where $\phi \in C_b^3(\mathbb{R}^d)$, $|\nabla\phi(x)| = 1$, for all $x \in \text{Bd}(D)$.

We define the outward normal derivative of v at the point $x \in \text{Bd}(D)$ by

$$\frac{\partial v(x)}{\partial n} = \sum_{j=1}^d \frac{\partial \phi(x)}{\partial x_j} \frac{\partial v(x)}{\partial x_j} = \langle \nabla \phi(x), \nabla v(x) \rangle.$$

The aim of this section is to prove uniqueness of a viscosity solution for the following parabolic variational inequality (PVI) with a mixed nonlinear multivalued Neumann–Dirichlet boundary condition:

$$\begin{cases} \frac{\partial u(t, x)}{\partial t} - \mathcal{A}_t u(t, x) + \partial \varphi(u(t, x)) \ni F(t, x, u(t, x), (\nabla u g)(t, x)), & t > 0, x \in D, \\ \frac{\partial u(t, x)}{\partial n} + \partial \psi(u(t, x)) \ni G(t, x, u(t, x)), & t > 0, x \in \text{Bd}(D), \\ u(0, x) = \kappa(x), & x \in \overline{D}, \end{cases} \quad (6.137)$$

where the operator \mathcal{A}_t is given by

$$\mathcal{A}_t v(x) = \frac{1}{2} \text{Tr}[g(t, x)g^*(t, x)D^2v(x)] + \langle f(t, x), \nabla v(x) \rangle.$$

We will make the following assumptions:

(I) The functions

$$\begin{aligned} f &: [0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R}^d, \\ g &: [0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}, \\ F &: [0, \infty) \times \overline{D} \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}, \\ G &: [0, \infty) \times \text{Bd}(D) \times \mathbb{R} \rightarrow \mathbb{R}, \\ \kappa &: \overline{D} \rightarrow \mathbb{R} \end{aligned} \quad (6.138)$$

are continuous.

We assume that for all $T > 0$, there exist $\alpha \in \mathbb{R}$ and $L, \beta, \gamma \geq 0$ (which can depend on T) such that $\forall t \in [0, T], \forall x, \tilde{x} \in \mathbb{R}^d$:

$$\langle f(t, x) - f(t, \tilde{x}), \frac{x - \tilde{x}}{|x - \tilde{x}|} \rangle^+ + |g(t, x) - g(t, \tilde{x})| \leq L|x - \tilde{x}|, \quad (6.139)$$

and $\forall t \in [0, T], \forall x \in \overline{D}, x' \in \text{Bd}(D), y, \tilde{y} \in \mathbb{R}, z, \tilde{z} \in \mathbb{R}^d$:

$$\begin{aligned} (i) & \quad (y - \tilde{y})(F(t, x, y, z) - F(t, x, \tilde{y}, z)) \leq \alpha|y - \tilde{y}|^2, \\ (ii) & \quad |F(t, x, y, z) - F(t, x, y, \tilde{z})| \leq \beta|z - \tilde{z}|, \\ (iii) & \quad |F(t, x, y, 0)| \leq \gamma(1 + |y|), \\ (iv) & \quad (y - \tilde{y})(G(t, x', y) - G(t, x', \tilde{y})) \leq \alpha|y - \tilde{y}|^2, \\ (v) & \quad |G(t, x', y)| \leq \gamma(1 + |y|). \end{aligned} \quad (6.140)$$

In fact, the conditions (6.140-*i*) and (6.140-*iv*) mean that, for all $t \in [0, T]$, $x \in \overline{D}$, $x' \in \text{Bd}(D)$, $z \in \mathbb{R}^d$,

$$r \mapsto \alpha y - F(t, x, ry, z) \quad \text{and} \quad r \mapsto \alpha r - G(t, x', r)$$

are increasing functions.

(II) We assume that

$$\begin{aligned} (i) \quad & \varphi, \psi : \mathbb{R} \rightarrow (-\infty, +\infty] \text{ are proper convex l.s.c. functions,} \\ (ii) \quad & \varphi(y) \geq \varphi(0) = 0 \text{ and } \psi(y) \geq \psi(0) = 0, \quad \forall y \in \mathbb{R}, \end{aligned} \quad (6.141)$$

and there exists a positive constant M such that

$$\begin{aligned} (i) \quad & \left| \varphi(\kappa(x)) \right| \leq M, \quad \forall x \in \overline{D}, \\ (ii) \quad & \left| \psi(\kappa(x)) \right| \leq M, \quad \forall x \in \text{Bd}(D). \end{aligned} \quad (6.142)$$

Remark 6.110. Condition (6.141-ii) is generally satisfied after a translation of both the functions φ , ψ and their arguments.

We define

$$\begin{aligned} \text{Dom}(\varphi) &= \{u \in \mathbb{R} : \varphi(u) < \infty\}, \\ \partial\varphi(u) &= \{\hat{u} \in \mathbb{R} : \hat{u}(v - u) + \varphi(u) \leq \varphi(v), \forall v \in \mathbb{R}\}, \\ \text{Dom}(\partial\varphi) &= \{u \in \mathbb{R} : \partial\varphi(u) \neq \emptyset\}, \\ (u, \hat{u}) \in \partial\varphi &\Leftrightarrow u \in \text{Dom}\partial\varphi, \quad \hat{u} \in \partial\varphi(u) \end{aligned}$$

and we will use the same notions with φ replaced by ψ .

At every point $y \in \text{Dom}(\varphi)$ we have

$$\partial\varphi(y) = \mathbb{R} \cap [\varphi'_-(y), \varphi'_+(y)],$$

where $\varphi'_-(y)$ and $\varphi'_+(y)$ are resp. the left and right derivatives of φ at y .

For the reader's convenience we recall here from Sect. 5.8 the definition of a viscosity solution of the parabolic variational inequality (6.137). We define

$$\Phi(t, x, r, q, X) := -\frac{1}{2} \text{Tr}((gg^*)(t, x)X) - \langle f(t, x), q \rangle - F(t, x, r, qg(t, x)),$$

$$\Gamma(t, x, r, q) := \langle \nabla\phi(x), q \rangle - G(t, x, r).$$

Definition 6.111. Let $u : [0, \infty) \times \overline{D} \rightarrow \mathbb{R}$ be a continuous function, which satisfies $u(0, x) = \kappa(x)$, $\forall x \in \overline{D}$.

(a) u is a viscosity sub-solution of (6.137) if:

$$\begin{cases} u(t, x) \in \text{Dom}(\varphi), & \forall (t, x) \in (0, \infty) \times \overline{D}, \\ u(t, x) \in \text{Dom}(\psi), & \forall (t, x) \in (0, \infty) \times \text{Bd}(D), \end{cases}$$

and for any $(t, x) \in (0, \infty) \times \overline{D}$, any $(p, q, X) \in \mathcal{P}^{2,+}u(t, x)$:

$$\begin{cases} p + \Phi(t, x, u(t, x), q, X) + \varphi'_-(u(t, x)) \leq 0 & \text{if } x \in D, \\ \min \left\{ p + \Phi(t, x, u(t, x), q, X) + \varphi'_-(u(t, x)), \right. \\ \left. \Gamma(t, x, u(t, x), q) + \psi'_-(u(t, x)) \right\} \leq 0 & \text{if } x \in \text{Bd}(D). \end{cases} \quad (6.143)$$

- (b) The viscosity super-solution of (6.137) is defined in a similar manner as above, with $\mathcal{P}^{2,+}$ replaced by $\mathcal{P}^{2,-}$, the left derivative replaced by the right derivative, min by max, and the inequalities \leq by \geq .
- (c) A continuous function $u : [0, \infty) \times \overline{D}$ is a viscosity solution of (6.137) if it is both a viscosity sub- and super-solution.

We now present the main result of this section.

Theorem 6.112. *Let the assumptions (6.138)–(6.142) be satisfied. If moreover the function*

$$r \rightarrow G(t, x, r) \text{ is decreasing for all } t \geq 0, x \in \text{Bd}(D), \quad (6.144)$$

and there exists a continuous function $\mathbf{m} : [0, \infty) \rightarrow [0, \infty)$, $\mathbf{m}(0) = 0$, such that

$$\begin{aligned} |F(t, x, r, q) - F(t, y, r, q)| &\leq \mathbf{m}(|x - y|(1 + |q|)), \\ \forall t \geq 0, x, y \in \overline{D}, q \in \mathbb{R}^d, \end{aligned} \quad (6.145)$$

then the parabolic variational inequality (6.137) has at most one viscosity solution.

Proof. It is sufficient to prove uniqueness on a fixed arbitrary interval $[0, T]$.

Also, it suffices to prove that if u is a sub-solution and v is a super-solution such that $u(0, x) = v(0, x) = \kappa(x)$, $x \in \overline{D}$, then $u \leq v$.

Clearly by adding a constant we may assume that $\phi(x) \geq 0$ on \overline{D} .

For $\lambda = \alpha^+ + 1$ and $\delta, \varepsilon, c > 0$ let

$$\begin{aligned} \bar{u}(t, x) &= e^{-\lambda t} u(t, x) - \delta \phi(x) - c \\ \bar{v}(t, x) &= e^{-\lambda t} v(t, x) + \delta \phi(x) + c + \frac{\varepsilon}{T - t}. \end{aligned}$$

Let

$$\begin{aligned}\tilde{\Phi}(t, x, r, q, X) &= \lambda r - \frac{1}{2} \text{Tr}[(gg^*)(t, x)X] - \langle f(t, x), q \rangle \\ &\quad - e^{-\lambda t} F(t, x, e^{\lambda t} r, e^{\lambda t} qg(t, x)), \\ \tilde{\Gamma}(t, x, r, q) &= \langle \nabla\phi(x), q \rangle - e^{-\lambda t} G(t, x, e^{\lambda t} r).\end{aligned}\tag{6.146}$$

Clearly $r \rightarrow \tilde{\Phi}(t, x, r, q, X)$ is an increasing function for all $(t, x, q, X) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^d$. Moreover, since

$$\sup_{(t,x) \in [0,T] \times \bar{D}} \{|\phi(x)| + |D\phi(x)| + |D^2\phi(x)| + |f(t, x)| + |g(t, x)|\} < \infty,$$

then for any $\delta > 0$, we can choose $c = c(\delta) > 0$ such that $c(\delta) \rightarrow 0$ as $\delta \rightarrow 0$ and for all $\delta, \varepsilon > 0$,

$$\begin{aligned}\tilde{\Phi}(t, x, r, q, X) &\leq \tilde{\Phi}(t, x, r + \delta\phi + c, q + \delta D\phi, X + \delta D^2\phi), \\ \tilde{\Phi}(t, x, r - \delta\phi - c - \frac{\varepsilon}{T-t}, q - \delta D\phi, X - \delta D^2\phi) &\leq \tilde{\Phi}(t, x, r, q, X).\end{aligned}$$

We will prove that $\bar{u} \leq \bar{v}$ for all $\delta > 0, \varepsilon > 0, c = c(\delta)$. This will imply $u \leq v$ on $[0, T) \times \bar{D}$ by letting $\delta, \varepsilon \rightarrow 0$. The result will follow, since T is arbitrary.

Using the two last properties, assumption (6.144) and the fact that the left and right derivative of φ and ψ are increasing we infer that \bar{u} satisfies in the viscosity sense:

$$\left\{ \begin{aligned} \frac{\partial \bar{u}}{\partial t}(t, x) + \tilde{\Phi}(t, x, \bar{u}(t, x), D\bar{u}(t, x), D^2\bar{u}(t, x)) + e^{-\lambda t} \varphi'_-(e^{\lambda t} \bar{u}(t, x)) &\leq 0 \\ &\text{if } x \in D, t > 0 \\ \min \left\{ \frac{\partial \bar{u}}{\partial t}(t, x) + \tilde{\Phi}(t, x, \bar{u}(t, x), D\bar{u}(t, x), D^2\bar{u}(t, x)) + e^{-\lambda t} \varphi'_-(e^{\lambda t} \bar{u}(t, x)), \right. \\ \left. \tilde{\Gamma}(t, x, \bar{u}(t, x), D\bar{u}(t, x)) + \delta + e^{-\lambda t} \psi'_-(e^{\lambda t} \bar{u}(t, x)) \right\} &\leq 0 \\ &\text{if } x \in \text{Bd}(D), t > 0. \end{aligned} \right.\tag{6.147}$$

Analogously we see that \bar{v} satisfies in the viscosity sense:

$$\left\{ \begin{aligned} \frac{\partial \bar{v}}{\partial t}(t, x) + \tilde{\Phi}(t, x, \bar{v}(t, x), D\bar{v}(t, x), D^2\bar{v}(t, x)) \\ + e^{-\lambda t} \varphi'_+(e^{\lambda t} \bar{v}(t, x)) - \frac{\varepsilon}{(T-t)^2} &\geq 0, \text{ if } x \in D, t > 0, \\ \max \left\{ \frac{\partial \bar{v}}{\partial t}(t, x) + \tilde{\Phi}(t, x, \bar{v}(t, x), D\bar{v}(t, x), D^2\bar{v}(t, x)) + e^{-\lambda t} \varphi'_+(e^{\lambda t} \bar{v}(t, x)) \right. \\ \left. - \frac{\varepsilon}{(T-t)^2}, \tilde{\Gamma}(t, x, \bar{v}(t, x), D\bar{v}(t, x)) - \delta + e^{-\lambda t} \psi'_+(e^{\lambda t} \bar{v}(t, x)) \right\} &\geq 0 \\ &\text{if } x \in \text{Bd}(D), t > 0. \end{aligned} \right.\tag{6.148}$$

For simplicity of notation we write from now on u, v instead of \bar{u}, \bar{v} respectively.

We now assume that

$$\max_{[0, T] \times \bar{D}} (u - v)^+ > 0. \tag{6.149}$$

By an argument similar to that of Theorem 6.103, see Theorem 4.2 in [56] for more details, there exists $(\hat{t}, \hat{x}) \in (0, T] \times \text{Bd}(D)$ such that

$$u(\hat{t}, \hat{x}) - v(\hat{t}, \hat{x}) = \max_{[0, T] \times \bar{D}} (u - v)^+ > 0.$$

We now let

$$\psi_n(t, x, y) = u(t, x) - v(t, y) - \rho_n(t, x, y), \text{ with } (t, x, y) \in [0, T] \times \bar{D} \times \bar{D},$$

where

$$\begin{aligned} \rho_n(t, x, y) &= \frac{n}{2} |x - y|^2 + e^{-\lambda \hat{t}} G(\hat{t}, \hat{x}, e^{\lambda \hat{t}} u(\hat{t}, \hat{x})) \langle \nabla \phi(\hat{x}), x - y \rangle + |x - \hat{x}|^4 \\ &+ |t - \hat{t}|^4 - e^{\lambda \hat{t}} \psi'_-(e^{-\lambda \hat{t}} u(\hat{t}, \hat{x})) \langle \nabla \phi(\hat{x}), x - y \rangle. \end{aligned} \tag{6.150}$$

Let (t_n, x_n, y_n) be a maximum point of ψ_n .

We observe that $u(t, x) - v(t, x) - |x - \hat{x}|^4 - |t - \hat{t}|^4$ has (\hat{t}, \hat{x}) as its unique maximum point. Then, by Lemma 6.101, we have that as $n \rightarrow \infty$

$$\begin{aligned} t_n &\rightarrow \hat{t}, \quad x_n \rightarrow \hat{x}, \quad y_n \rightarrow \hat{x}, \quad n |x_n - y_n|^2 \rightarrow 0, \\ u(t_n, x_n) &\rightarrow u(\hat{t}, \hat{x}), \quad v(t_n, x_n) \rightarrow v(\hat{t}, \hat{x}). \end{aligned} \tag{6.151}$$

But the domain D satisfies the uniform exterior sphere condition:

$$\exists r_0 > 0 \text{ such that } S(x + r_0 \nabla \phi(x), r_0) \cap D = \emptyset, \text{ for all } x \in \text{Bd}(D),$$

where $S(x, r_0)$ denotes the closed ball of radius r_0 centered at x .

Then

$$|y - x - r_0 \nabla \phi(x)|^2 > r_0^2, \text{ for } x \in \text{Bd}(D), y \in \bar{D},$$

or equivalently

$$\langle \nabla \phi(x), y - x \rangle < \frac{1}{2r_0} |y - x|^2 \text{ for } x \in \text{Bd}(D), y \in \bar{D}. \tag{6.152}$$

If $x_n \in \text{Bd}(D)$, we have, using the form of ρ_n given by (6.150) and (6.152), that

$$\begin{aligned} \tilde{\Gamma}(t_n, x_n, u(t_n, x_n), D_x \rho_n(t_n, x_n, y_n)) &= \tilde{\Gamma}(t_n, x_n, u(t_n, x_n), n(x_n - y_n)) \\ &\quad + e^{-\lambda \hat{t}} G(\hat{t}, \hat{x}, e^{\lambda \hat{t}} u(\hat{t}, \hat{x})) \nabla \phi(\hat{x}) + 4|x_n - \hat{x}|^2 (x_n - \hat{x}) \\ &\quad - e^{-\lambda \hat{t}} \psi'_-(e^{\lambda \hat{t}} u(\hat{t}, \hat{x})) \nabla \phi(\hat{x}) \\ &\geq -\frac{n}{2r_0} |x_n - y_n|^2 + e^{-\lambda \hat{t}} G(\hat{t}, \hat{x}, e^{\lambda \hat{t}} u(\hat{t}, \hat{x})) \langle \nabla \phi(\hat{x}), \nabla \phi(x_n) \rangle \\ &\quad - e^{-\lambda t_n} G(t_n, x_n, e^{\lambda t_n} u(t_n, x_n)) + 4|x_n - \hat{x}|^2 \langle \nabla \phi(x_n), x_n - \hat{x} \rangle \\ &\quad - e^{-\lambda \hat{t}} \psi'_-(e^{\lambda \hat{t}} u(\hat{t}, \hat{x})) \langle \nabla \phi(\hat{x}), \nabla \phi(x_n) \rangle. \end{aligned}$$

Then (6.151) and the lower semicontinuity property of ψ'_- implies that along a subsequence $\{x_n\}$ which belongs to ∂D :

$$\liminf_{n \rightarrow \infty} \left[\tilde{\Gamma}(t_n, x_n, u(t_n, x_n), D_x \rho_n(t_n, x_n, y_n)) + \delta + e^{-\lambda t_n} \psi'_-(e^{\lambda t_n} u(t_n, x_n)) \right] > 0. \quad (6.153)$$

Analogously if $y_n \in \partial D$ we infer

$$\limsup_{n \rightarrow \infty} \left[\tilde{\Gamma}(t_n, y_n, v(t_n, y_n), -D_y \rho_n(t_n, x_n, y_n)) - \delta + e^{-\lambda t_n} \psi'_+(e^{\lambda t_n} v(t_n, x_n)) \right] < 0. \quad (6.154)$$

From Lemma 6.104 we deduce that there exists

$$(p, X, Y) \in \mathbb{R} \times \mathbb{S}^d \times \mathbb{S}^d,$$

such that

$$\begin{aligned} (p, D_x \rho_n(t_n, x_n, y_n), X) &\in \overline{\mathcal{P}}^{2,+} u(t_n, x_n), \\ (p, -D_y \rho_n(t_n, x_n, y_n), Y) &\in \overline{\mathcal{P}}^{2,-} v(t_n, y_n), \end{aligned}$$

and

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq A + \frac{1}{n} A^2, \quad (6.155)$$

where $A = D_{x,y}^2 \rho_n(t_n, x_n, y_n)$. From (6.150) we have

$$\begin{aligned} A &= n \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + O(|x_n - \hat{x}|^2), \\ A^2 &= 2n^2 \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + O(n|x_n - \hat{x}|^2 + |x_n - \hat{x}|^4). \end{aligned}$$

Then (6.155) becomes

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq 3n \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \delta_n \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \quad (6.156)$$

where $\delta_n \rightarrow 0$.

Then from (6.147), (6.148) together with (6.154) and (6.153), we deduce that for n large enough

$$p + \tilde{\Phi}(t_n, x_n, u(t_n, x_n), D_x \rho_n(t_n, x_n, y_n), X) + e^{-\lambda t_n} \varphi'_-(e^{\lambda t_n} u(t_n, x_n)) \leq 0,$$

and

$$\begin{aligned} p + \tilde{\Phi}(t_n, y_n, v(t_n, y_n), -D_y \rho_n(t_n, x_n, y_n), Y) + e^{-\lambda t_n} \varphi'_+(e^{\lambda t_n} v(t_n, y_n)) \\ \geq \frac{\varepsilon}{(T - t_n)^2}. \end{aligned}$$

Subtracting the last two inequalities, we deduce that

$$\begin{aligned} \frac{\varepsilon}{(T - t_n)^2} \\ \leq \tilde{\Phi}(t_n, y_n, v(t_n, y_n), -D_y \rho_n(t_n, x_n, y_n), Y) + e^{-\lambda t_n} \varphi'_+(e^{\lambda t_n} v(t_n, y_n)) \\ - \tilde{\Phi}(t_n, x_n, u(t_n, x_n), D_x \rho_n(t_n, x_n, y_n), X) - e^{-\lambda t_n} \varphi'_-(e^{\lambda t_n} u(t_n, x_n)). \end{aligned} \quad (6.157)$$

By (6.149) and (6.151) there exists an $N \geq 1$ such that for all $n \geq N$, the above holds together with

$$u(t_n, x_n) > v(t_n, y_n), \quad (6.158)$$

and consequently

$$e^{-\lambda t_n} \varphi'_-(e^{\lambda t_n} u(t_n, x_n)) \geq e^{-\lambda t_n} \varphi'_+(e^{\lambda t_n} v(t_n, y_n)).$$

Combining this with (6.157), we deduce that

$$\begin{aligned} \frac{\varepsilon}{(T - t_n)^2} &\leq \tilde{\Phi}(t_n, y_n, v(t_n, y_n), -D_y \rho_n(t_n, x_n, y_n), Y) \\ &\quad - \tilde{\Phi}(t_n, x_n, u(t_n, x_n), D_x \rho_n(t_n, x_n, y_n), X) \\ &\leq \frac{1}{2} \text{Tr}[(gg^*)(t_n, x_n)X - (gg^*)(t_n, y_n)Y] + Cn|x_n - y_n|^2 + \omega_n, \end{aligned}$$

where $\omega_n \rightarrow 0$ as $n \rightarrow \infty$. Note that we have used the assumption (6.145), (6.151), (6.158), the fact that $r \rightarrow \lambda r - F(t, x, r, z)$ is increasing, and the Lipschitz continuity of F with respect to its last variable.

From (6.156), $\forall q, \tilde{q} \in \mathbb{R}^d$,

$$\langle Xq, q \rangle - \langle Y\tilde{q}, \tilde{q} \rangle \leq 3n|q - \tilde{q}|^2 + (|q|^2 + |\tilde{q}|^2)\delta_n.$$

Hence by the same computation as in Lemma 6.97 we obtain

$$\begin{aligned} &\text{Tr}[(gg^*)(t_n, x_n)X - (gg^*)(t_n, y_n)Y] \\ &\leq 3Cn|x_n - y_n|^2 + (|g(t_n, x_n)|^2 + |g(t_n, y_n)|^2)\delta_n, \end{aligned}$$

and consequently taking the limit in the above set of inequalities yields

$$\frac{\varepsilon}{(T - \hat{t})^2} \leq 0,$$

which is a contradiction.

Then

$$u(t, x) \leq v(t, x), \quad \forall (t, x) \in [0, T] \times \overline{D}.$$

■

6.6 Annex E: Hints for Some Exercises

Chapter 1

Exercise 1.7

By Proposition 1.34 we have

$$\begin{aligned} \mathbb{E}(g(B_T) | \mathcal{F}_t) &= \mathbb{E}\left(g\left(\frac{B_T - B_t}{\sqrt{T-t}}\sqrt{T-t} + B_t\right) | \mathcal{F}_t\right) \\ &= \int_{\mathbb{R}} g(x\sqrt{T-t} + B_t) \rho(x) dx. \end{aligned}$$

Setting here $g(u) = \mathbf{1}_{(-\infty, a]}(u)$, the second assertion follows.

Exercise 1.15: Let $N > 0$ and a sequence $\varepsilon_n \searrow 0$ as $n \rightarrow \infty$. Then

$$\begin{aligned} \mathbb{P} \left(\limsup_{n \rightarrow +\infty} \frac{|B_{t+\varepsilon_n} - B_t|}{\varepsilon_n} > N \right) &= \mathbb{P} \left(\bigcap_{n \geq 1} \downarrow \left(\bigcup_{k \geq n} (|B_{t+\varepsilon_k} - B_t| > N \varepsilon_k) \right) \right) \\ &= \lim_{n \rightarrow \infty} \mathbb{P} \left(\bigcup_{k \geq n} (|B_{t+\varepsilon_k} - B_t| > N \varepsilon_k) \right) \\ &\geq \liminf_{n \rightarrow +\infty} \mathbb{P} (|B_{t+\varepsilon_n} - B_t| > N \varepsilon_n) \\ &= \liminf_{n \rightarrow +\infty} \mathbb{P} (|B_1| > N \sqrt{\varepsilon_n}) \\ &= \mathbb{P} (|B_1| > 0) \\ &= 1. \end{aligned}$$

Exercise 1.16: Let us write $S_n^{(p)} = S_{\Delta_n}^{(p)}(B; [s, t])$. The results are consequences of the following inequalities (see Proposition 1.86 for the first one) combined with Proposition 1.14 and Proposition 1.7:

$$\mathbb{E} \left[S_n^{(2)} - (t - s) \right]^2 = \text{Var} \left(S_n^{(2)} \right) \leq 2 \|\Delta_n\| (t - s)$$

and

$$\begin{aligned} S_{\Delta_n}^{(p)} &\leq S_{\Delta_n}^{(2)} \times (\mathbf{m}_B(\|\Delta_n\|))^{p-2}, \quad \text{for } p > 2, \quad \text{and} \\ S_{\Delta_n}^{(2)} &\leq S_{\Delta_n}^{(p)} \times (\mathbf{m}_B(\|\Delta_n\|))^{2-p}, \quad \text{for } 1 \leq p < 2, \end{aligned}$$

where

$$\mathbf{m}_B(\delta) = \sup \{ |B_u - B_v| : u, v \in [s, t], |u - v| \leq \delta \},$$

is the modulus of continuity of $\{B_u : u \in [s, t]\}$.

Exercise 1.17: Applying the inequality (1.25) with $\alpha = \frac{1}{2} - \frac{\varepsilon}{2}$ and $p = \frac{2}{\varepsilon}$, we deduce that for all $s, t \in [0, T]$

$$|X_t(\omega) - X_s(\omega)| \leq \xi(\omega) T^\varepsilon |t - s|^{\frac{1}{2} - \varepsilon},$$

where

$$\xi(\omega) = \xi_{\varepsilon, T}(\omega) = \begin{cases} 0, & \text{if } T = 0, \\ \frac{C_\varepsilon}{T^\varepsilon} \left(\int_0^T \int_0^T \frac{|B_u(\omega) - B_r(\omega)|^{\frac{2}{\varepsilon}}}{|u - r|^{\frac{1}{\varepsilon}}} dudr \right)^{\frac{\varepsilon}{2}}, & \text{if } T > 0. \end{cases}$$

Let $1 \leq q \leq \frac{2}{\varepsilon} \leq p$. By Lyapunov's inequality and Minkowski's inequality (1.24) from Exercise 1.2 we obtain

$$\begin{aligned} \|\xi\|_{L^q(\Omega, \mathcal{F}, \mathbb{P})} &\leq \|\xi\|_{L^p(\Omega, \mathcal{F}, \mathbb{P})} \\ &= \frac{C_\varepsilon}{T^\varepsilon} \left\| \int_0^T \int_0^T \frac{|B_u - B_r|^{\frac{2}{\varepsilon}}}{|u - r|^{\frac{1}{\varepsilon}}} dudr \right\|_{L^{\varepsilon p/2}(\Omega, \mathcal{F}, \mathbb{P})}^{\frac{\varepsilon}{2}} \\ &\leq \frac{C_\varepsilon}{T^\varepsilon} \left(\int_0^T \int_0^T \frac{\| |B_u - B_r|^{\frac{2}{\varepsilon}} \|_{L^{\varepsilon p/2}(\Omega, \mathcal{F}, \mathbb{P})}}{|u - r|^{\frac{1}{\varepsilon}}} dudr \right)^{\frac{\varepsilon}{2}} \\ &= C_{\varepsilon, p}, \end{aligned}$$

since

$$\begin{aligned} \left\| |B_u - B_r|^{\frac{2}{\varepsilon}} \right\|_{L^{\varepsilon p/2}(\Omega, \mathcal{F}, \mathbb{P})} &= (\mathbb{E} |B_u - B_r|^p)^{\frac{2}{\varepsilon p}} \\ &= (C_p |u - r|^{p/2})^{\frac{2}{\varepsilon p}}. \end{aligned}$$

Exercise 1.19: Deduce from the proof of Theorem 1.40 that for any $0 < \delta < b/a$, there exists a constant $K = K(M, T, a, b, \delta)$ such that for all $\varepsilon, \lambda > 0$,

$$\mathbb{P}(\mathbf{m}_{X^n}(\varepsilon; [0, T]) \geq \lambda) \leq \frac{1}{\lambda^a} \mathbb{E}(\mathbf{m}_{X^n}^a(\varepsilon; [0, T])) \leq \frac{K}{\lambda^a} \varepsilon^{b-a\delta}$$

and conclude that (ii) in Theorem 1.46 is satisfied.

Exercise 1.20 (2) By Lemma 1.73 and Proposition 1.65, we infer that $(U_t^{(\lambda)})_{t \in [0, T]}$ and $(Z_t^{(\lambda)})_{t \in [0, T]}$ are continuous martingales.

(3) Let the stopping time $\tau_n = \inf\{t \geq 0 : |M_t| + \langle M \rangle_t \geq n\}$. Then $\{Z_{t \wedge \tau_n}^{(\lambda)}; t \geq 0\}$ is a martingale and for all $0 \leq s \leq t$

$$\mathbb{E}^{\mathcal{F}_s} Z_t^{(\lambda)} \leq \liminf_{n \rightarrow +\infty} \mathbb{E}^{\mathcal{F}_s} Z_{t \wedge \tau_n}^{(\lambda)} = \liminf_{n \rightarrow +\infty} Z_{s \wedge \tau_n}^{(\lambda)} = Z_s^{(\lambda)}.$$

(5) By Proposition 1.59 with $\varphi(x) = e^{ax}$, $\{e^{\alpha M_{t \wedge \theta_n}}; t \geq 0\}$ is a sub-martingale and the inequality follows by Doob's inequality (Theorem 1.60) and Hölder's inequality.

(6) The inequality yields that $\{Z_{t \wedge \theta_n}^{(\lambda)}; n \in \mathbb{N}^*\}$ is uniformly integrable and consequently $\mathbb{E} Z_t^{(\lambda)} = \lim_{n \rightarrow \infty} \mathbb{E} Z_{t \wedge \theta_n}^{(\lambda)} = 1$.

(7) In the inequality from (6) with $A = \Omega$, one passes to the limit as $n \rightarrow \infty$ and then $\lambda \nearrow 1$.

(8) We have $\mathbb{E}\left(e^{\frac{1}{2}M_T}\right) = \mathbb{E}\left(\sqrt{Z_T}e^{\frac{1}{4}\langle M \rangle_T}\right) \leq (\mathbb{E} Z_T)^2 \mathbb{E}\left(e^{\frac{1}{2}\langle M \rangle_T}\right) \leq \mathbb{E}\left(e^{\frac{1}{2}\langle M \rangle_T}\right) < \infty$.

Chapter 2

Exercise 2.1: (\Leftarrow): From the theory of the Riemann–Stieltjes integral we know that if $g \in BV[0, T]$, then $S_n(f)$ converges (to the Riemann–Stieltjes integral $\int_0^T f(t) dg(t)$).

(\Rightarrow): Let $S_n(f)$ be convergent for all $f \in C[0, T]$. Then $S_n : C[0, T] \rightarrow R$ is a bounded linear operator such that

$$\sup_{n \geq 1} |S_n(f)| < \infty,$$

and by the Banach–Steinhaus Theorem

$$\sup_{n \geq 1} \|S_n\| = M < \infty,$$

where $\|S_n\| = \sup\{|S_n(f)| : \|f\|_T \leq 1\}$. For a fixed n we can construct $h_n \in C[0, T]$ such that $h_n(t_i^n) = \text{sign}\{g(t_{i+1}^n) - g(t_i^n)\}$ and $\|h_n\|_T = 1$. Hence

$$\sum_{i=0}^{n-1} |g(t_{i+1}^n) - g(t_i^n)| = S_n(h_n) \leq \|S_n\| \leq M,$$

and as a consequence g is of finite variation.

Note (Banach–Steinhaus Theorem). Let X be a Banach space and let Y be a normed linear space. Let $S_i : X \rightarrow Y, i \in I$, be a family of bounded linear operators. If for each $x \in X$ the set $\{S_i(x) : i \in I\}$ is bounded then the set $\{\|S_i\| : i \in I\}$ is bounded.

Remark: This is not a contradiction since the subsequence $\{n_k\}$ depends on f .

Exercise 2.3: If \mathcal{E} is the linear subspace of $L^2(\mathbb{R}_+)$ consisting of those functions f of the form:

$$f = \sum_{i=0}^{n-1} a_i \mathbf{1}_{[t_i, t_{i+1}[}, \quad n \in \mathbb{N}^*; 0 = t_0 < t_1 < \dots < t_n; a_i \in \mathbb{R}, i \leq n,$$

then $H[B]$ is the closure of $\{\mathbb{B}(f), f \in \mathcal{E}\}$, which coincides with $\{\mathbb{B}(f), f \in L^2(\mathbb{R}_+)\}$. Moreover the set $\{B_t = \mathbb{B}(\mathbf{1}_{[0, t]}, t > 0\}$ is total in $H[B]$.

Exercise 2.4: Let $s \in [0, T]$. We have

$$\begin{aligned} \mathbb{E}\left[\left(\int_0^T f(t) dB_t + \int_0^T f'(t) B_t dt\right) B_s\right] &= \int_0^s f(t) dt + \int_0^T f'(t)(s \wedge t) dt \\ &= f(T)s. \end{aligned}$$

Since $\mathbb{E}|B_t| = \sqrt{\frac{2t}{\pi}}$, it follows that $\int_0^\infty |f'(t)||B_t| dt < \infty$ a.s., and

$$\begin{aligned} \int_0^\infty f^2(t) dt &= \int_0^\infty \left(\int_0^\infty \mathbf{1}_{[t, \infty[}(u) f'(u) du \int_0^\infty \mathbf{1}_{[t, \infty[}(v) f'(v) dv \right) dt \\ &\leq \int_0^\infty \int_0^\infty (u \wedge v) |f'(u)||f'(v)| du dv \\ &\leq \int_0^\infty \int_0^\infty \sqrt{uv} |f'(u)||f'(v)| du dv \\ &= \left(\int_0^\infty \sqrt{u} |f'(u)| du \right)^2 < \infty. \end{aligned}$$

Exercise 2.5: Note that

$$g'(x) = 30(x-1)^2(2-x)^2 \geq 0 \quad \text{and} \quad g''(x) = 60(x-1)(2-x)(3-2x).$$

and for $x \in [1, 2]$

$$0 \leq (x-1)(2-x) \leq \left(\frac{x-1+2-x}{2} \right)^2 = \frac{1}{4},$$

and therefore for all $x \in [1, 2]$,

$$0 \leq g'(x) \leq 2, \quad |g''(x)| \leq 15.$$

The relation (2.67) follows by taking the limit as $\varepsilon \rightarrow 0$ in Itô's formula for $\varphi_\varepsilon(X_t)$.

Chapter 3

Exercise 3.1: Consider the equation

$$\begin{aligned} X_t &= \xi + \int_0^t F(s, X_s) ds \\ &\quad + \int_0^t \left(-\mu(s) - \frac{m_p}{2} \ell^2(s) - \frac{a}{p} \right) X_s ds + \int_0^t G(s, X_s) dB_s. \end{aligned} \quad (6.159)$$

By Theorem 3.27, it has a unique solution $X \in S_d^0$ and from the inequality (3.18) we clearly have (3.131)

$$U_t = \left(-\mu(t) - \frac{m_p}{2} \ell^2(t) - \frac{a}{p} \right) X_t, \quad (6.160)$$

where $X \in S_d^0$ is the solution of the Eq. (6.159). The inequality (3.132) shows us that $Y_t = e^{at} \frac{|X_t|^p}{(1+\delta|X_t|^2)^{p/2}}$ is a super-martingale and then (3.134).

Exercise 3.3: First deduce the following from the stochastic Gronwall inequalities (Annex C)

$$\mathbb{E} \sup_{t \in [0, T]} |X_t^\varepsilon - X_t|^p \leq C_p \mathbb{E} \left(\int_0^T |F_\varepsilon(r, X_r) - F(r, X_r)| dr \right)^p e^{C_p \int_0^T [\mu^+(r) + \ell^2(r)] dr}.$$

Exercise 3.9: (1i) We clearly have

$$\mathbb{E} \exp \left(C |x + B_t|^b \right) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \exp \left(C |x + \sqrt{t}u|^b - \frac{|u|^2}{2} \right) du < \infty,$$

for all $C, t \geq 0$ if and only if $0 \leq b < 2$.

(1ii) If $0 \leq a < 2$, then by Jensen's inequality

$$\mathbb{E} \exp \left(C \int_0^t |x + B_s|^a ds \right) \leq \frac{1}{t} \int_0^t \mathbb{E} \exp (Ct |x + B_s|^a) ds < \infty.$$

If $-1 < a < 0$, then by Corollary 2.30 we have

$$|x + B_t|^{a+2} = (a + 2) \int_0^t |x + B_s|^a \langle x + B_s, dB_s \rangle + \frac{(a+2)(a+1)}{2} \int_0^t |x + B_s|^a ds.$$

Hence by (2.62-b)

$$\begin{aligned} & \mathbb{E} \exp \left(C \int_0^t |x + B_s|^a ds \right) \\ & \leq \left[\mathbb{E} \exp \left(C_1 |x + B_t|^{a+2} \right) \right]^{1/2} \left[\mathbb{E} \exp \left(C_2 \int_0^t |x + B_s|^a \langle x + B_s, dB_s \rangle \right) \right]^{1/2} \\ & \leq \left[\mathbb{E} \exp \left(C_1 |x + B_t|^{a+2} \right) \right]^{1/2} \left[\mathbb{E} \exp \left(2C_2 \int_0^t |x + B_s|^{2a+2} ds \right) \right]^{1/2} \\ & < \infty. \end{aligned}$$

(1iii)

$$\begin{aligned} & \mathbb{E} \left\{ \exp \left[C \log^2 (|x + B_t|) \right] \right\} \\ & \geq \int_{[0,1]^k} e^{C \log^2 |u|} \frac{1}{(2\pi t)^{k/2}} e^{-|u-x|^2/2t} du \\ & \geq \frac{1}{(2\pi t)^{k/2}} e^{-(k+|x|)^2/2t} \int_{[0,1]^k} e^{\frac{C}{4} \log^2 (u_1^2 + \dots + u_k^2)} du_1 \dots du_k \end{aligned}$$

$$\begin{aligned} &\geq C_{k,t,x} \int_0^1 e^{C \log^2 u_1} du_1 \\ &= C_{k,t,x} \int_0^\infty e^{Cy^2} e^{-y} dy \\ &= \infty. \end{aligned}$$

(1iv) Observe that for every $\alpha \in]0, 1[$, there exists a $C_\alpha > 0$ such that

$$\log^2 |x| \leq C_\alpha + |x| + |x|^{-\alpha}$$

and consequently (iii) follows from (ii).

(2). Existence follows in both cases from Lemma 2.49 and Girsanov’s Theorem 2.51. Uniqueness in law on $(\Omega, \mathcal{F}_{n \wedge \tilde{T}_n})$ (resp. $(\Omega, \mathcal{F}_{n \wedge \hat{T}_n})$) follows again from Girsanov’s Theorem, where

$$\begin{aligned} \tilde{T}_n &= \inf \left\{ t > 0 : \int_0^t |g(X_s)|^2 \log^2(|X_s|) ds > n \right\}, \\ \hat{T}_n &= \inf \left\{ t > 0 : \int_0^t |g(X_s)|^2 |X_s|^a ds > n \right\}. \end{aligned}$$

It remains to note that $\tilde{T}_n \rightarrow \infty, \hat{T}_n \rightarrow \infty$, as $n \rightarrow \infty$.

Exercise 3.10 The function $F : \mathbb{R} \rightarrow \mathbb{R}, F(x) = f(x) \sqrt{|x|} \text{sign}(x)$ is locally monotone and $x F(x) \leq 0$, but it is not locally Lipschitz.

Chapter 4

Exercise 4.1

1. The existence and the uniqueness of the solution $X^n \in S^2[0, T]$ follows from Theorem 3.17; by the comparison result from Proposition 3.12 we have $X_t^{n+1} \geq X_t^n$, for all $t \in [0, T]$, \mathbb{P} -a.s.
2. Let L and ℓ be the Lipschitz constants of f and, respectively, g . We have

$$X_t^n - 1 = (x - 1) + \int_0^t dK_s^n + \int_0^t g(X_s^n) dB_s,$$

with $dK_s^n = [f(X_s^n) + n(X_s^n)^-] ds$ and $G_s^n = g(X_s^n)$. Since

$$dD_t^n + (X_t^n - 1) dK_t^n + |G_t^n|^2 dt \leq dR_t + |X_t^n - 1|^2 dV_t,$$

where $D_t^n = n \int_0^t [(X_s^n)^-]^2 ds + n \int_0^t (X_s^n)^- ds, R_t = \left(\frac{1}{2} |f(1)|^2 + 2 |g(1)|^2 \right) t$

and $V_t = \left(L + \frac{1}{2} + 2\ell^2 \right) t$, it follows by (6.78) (with $p = 2$ and $\lambda = 1/18$) that

$$\mathbb{E} \sup_{t \in [0, T]} |X_t^n - 1|^2 + \mathbb{E} \left(\int_0^T n [(X_t^n)^-]^2 dt \right) + \mathbb{E} \left(\int_0^T n (X_t^n)^- dt \right) \leq C_2.$$

3. Since, moreover, $(X_t^n)^- \geq (X_t^{n+1})^-$ for all $t \in [0, T]$, \mathbb{P} -a.s., it follows that $\lim_{n \rightarrow \infty} (X_t^n)^- = 0$, $d\mathbb{P} \otimes dt$ - a.e. By Itô's formula for $[(X_t^n)^-]^2$ (see Proposition 2.35), we deduce $\mathbb{E} \sup_{0 \leq t \leq T} |(X_t^n)^-|^2 \rightarrow 0$, as $n \rightarrow \infty$.

4. Since

$$\begin{aligned} & (X_t^n - X_t^m) [f(X_t^n) - f(X_t^m) + n(X_t^n)^- - m(X_t^m)^-] dt \\ & \quad + |g(X_t^n) - g(X_t^m)|^2 dt \\ & \leq (n + m) [(X_t^n)^- (X_t^m)^-] dt + (L + \ell^2) |X_t^n - X_t^m| dt, \end{aligned}$$

we see, by (3.138), that

$$\begin{aligned} \mathbb{E} \sup_{t \in [0, T]} |X_t^n - X_t^m|^2 & \leq C \mathbb{E} \int_0^T (n + m) [(X_t^n)^- (X_t^m)^-] dt \\ & \leq C \mathbb{E} \left(\mathbb{E} \sup_{t \in [0, T]} [(X_t^m)^-]^2 \right)^{1/2} \left[\mathbb{E} \left(\int_0^T n (X_t^n)^- dt \right)^2 \right]^{1/2} \\ & \quad + C \mathbb{E} \left(\mathbb{E} \sup_{t \in [0, T]} [(X_t^n)^-]^2 \right)^{1/2} \left[\mathbb{E} \left(\int_0^T m (X_t^m)^- dt \right)^2 \right]^{1/2} \\ & \rightarrow 0, \quad \text{as } n, m \rightarrow \infty. \end{aligned}$$

9. It is sufficient to prove that the SDE

$$X_t = x + \int_0^t (f(X_s) + [f(0)]^- \mathbf{1}_{X_s=0}) ds + \int_0^t g(X_s) dB_s$$

has a unique positive solution $X \in S^2[0, T]$. The uniqueness of positive solutions follows from

$$\begin{aligned} & \left(X_s - \hat{X}_s \right) \left[f(X_s) + [f(0)]^- \mathbf{1}_{X_s=0} - f(\hat{X}_s) - [f(0)]^- \mathbf{1}_{\hat{X}_s=0} \right] \\ & \quad + \left| g(X_s) - g(\hat{X}_s) \right|^2 \\ & \leq (L + \ell^2) \left| X_s - \hat{X}_s \right|^2 \end{aligned}$$

and Corollary 6.77. The existence of a positive solution follows from the approximating equation

$$X_t^\varepsilon = x + \int_0^t \left[f(X_s^\varepsilon) + [f(0)]^- \left(1 - \frac{|X_s^\varepsilon|}{\varepsilon} \right)^+ \right] ds + \int_0^t g(X_s^\varepsilon) dB_s.$$

Note that $\tilde{X}^\varepsilon = 0$ is the unique solution of the SDE

$$\tilde{X}_t^\varepsilon = 0 + \int_0^t \left[f(\tilde{X}_s^\varepsilon) - f(0) \left(1 - \frac{|\tilde{X}_s^\varepsilon|}{\varepsilon} \right)^+ \right] ds + \int_0^t g(\tilde{X}_s^\varepsilon) dB_s,$$

and $f(0) + [f(0)]^- \left(1 - \frac{|0|}{\varepsilon} \right)^+ \geq 0$, which yields (by Proposition 3.12) $X_t^\varepsilon \geq 0$.

10. By Remark 2.27 we have for all $t \geq 0$,

$$0 = \int_0^t \mathbf{1}_{X_s=y} g^2(X_s) ds = g^2(y) \int_0^t \mathbf{1}_{X_s=y} ds.$$

Exercise 4.2: On each interval \mathbf{I}_i^n the equations from the schema (4.149) have unique adapted solutions U^n, V^n and Y^n , respectively; U^n is absolutely continuous; $H_t^n = F_1(\cdot, U_t^n) - \frac{d}{dt} U_t^n \in L^1(\Omega \times]0, T])$. Let $K_t^n = \int_0^t H_s^n ds$. To prove (4.150) the steps are:

1. $\mathbb{E} \left(|U_t^n|^4 + |V_t^n|^4 + |Y_t^n|^4 + |X_t^n|^4 + \diamond K_t^n \diamond^2 \right) \leq C (1 + \mathbb{E}|H_0|^4)$;
2. $\mathbb{E} \sup_{t \in [0, T]} |V_t^n - U_t^n|^4 \leq \frac{C}{n^3} (1 + \mathbb{E}|H_0|^4)$;
3. $\mathbb{E} \sup_{t \in [0, T]} |Y_t^n - U_t^n|^4 + \mathbb{E} \sup_{t \in [0, T]} |X_t^n - U_t^n|^4 \leq \frac{C}{n} (1 + \mathbb{E}|H_0|^4)$;
4. $\mathbb{E} \sup_{t \in [0, T]} |Y_t^n - U_t^n|^2 \leq \frac{C}{\sqrt{n}} (1 + \mathbb{E}|H_0|^4)$;
5. Let $t \in \mathbf{I}_i^n$. By Itô's formula for $|Y_t^n - X_t|^2$ and the above estimates we obtain (4.150).

Exercise 4.3: In the same manner as the estimate from Proposition 4.8 is obtained, we derive using Proposition 6.74 the boundedness of approximating quantities. Then estimating, via the same Proposition 4.8, $X^\varepsilon - X$ and $\tilde{X}^\varepsilon - X$ and using Proposition 6.9 the convergence results follow.

Exercise 4.4: For the first four questions, choose the control in feedback form as follows:

$$U_s = - \left(\mu(s) + \frac{1}{2} m_p \ell^2(s) + \frac{a}{p} \right) (X_s - x_0).$$

For the last question, choose

$$\tilde{U}_t = - \left[\mu(s) + \left(\frac{1}{2} m_p + 9p\lambda \right) \ell^2(s) + \frac{a}{p} \right] (\tilde{X}_s - x_0).$$

Exercise 4.5: The equivalence follows easily from Example 4.79.

Exercise 4.6: 1&2 Let $\hat{x} \in \Pi_E(x)$ and $\hat{y} \in \Pi_E(y)$. Then

$$\begin{aligned} d_K^2(x) - d_K^2(y) &\leq |x - \hat{y}|^2 - |y - \hat{y}|^2 \\ &= |x - y|^2 + 2 \langle x - y, y - \hat{y} \rangle \\ &\leq |x - y| (|x - y| + 2|y - a|). \end{aligned}$$

3. Let $0 < \lambda < 1$ and $x, y \in \mathbb{R}^d$. Put $z = \lambda x + (1 - \lambda)y$. Then there exists a $\hat{z} \in E$ such that $d_K(z) = \|z - \hat{z}\|$. Hence

$$\begin{aligned} |z|^2 - d_K^2(z) &= |z|^2 - |z - \hat{z}|^2 \\ &= 2 \langle z, \hat{z} \rangle - |\hat{z}|^2 \\ &= \lambda \left(2 \langle x, \hat{z} \rangle - |\hat{z}|^2 \right) + (1 - \lambda) \left(2 \langle y, \hat{z} \rangle - |\hat{z}|^2 \right) \\ &= \lambda (|x|^2 - |x - \hat{z}|^2) + (1 - \lambda) (|y|^2 - |y - \hat{z}|^2) \\ &\leq \lambda (|x|^2 - d_E^2(x)) + (1 - \lambda) (|y|^2 - d_E^2(y)). \end{aligned}$$

4. According to Alexandrov's Theorem (1939),¹ the function $x \mapsto |x|^2 - d_K^2(x)$ is almost everywhere twice differentiable, consequently so is $x \mapsto d_K^2(x)$.

Chapter 5

Exercise 5.1

Let $p \geq 2$, $\delta \geq 0$ and the Banach space

$$\mathbb{V}_{m,k}^{\delta,p}(0, T) \stackrel{\text{def}}{=} \{(Y, Z) \in S_m^0[0, T] \times \Lambda_{m \times k}^0(0, T) : \|(Y, Z)\|_{\delta V} < \infty\},$$

where

$$\begin{aligned} \|(Y, Z)\|_{\delta V}^p &\stackrel{\text{def}}{=} \mathbb{E} \sup_{s \in [0, T]} e^{\delta p V_s} |Y_s|^p + \mathbb{E} \left(\int_0^T e^{2\delta V_s} |Y_s|^2 L_s dQ_s \right)^{p/2} \\ &\quad + \mathbb{E} \left(\int_0^T e^{2\delta V_s} |Z_s|^2 ds \right)^{p/2}, \end{aligned}$$

and the complete metric space $\mathbb{V}_{m,k}^p(0, T) = \bigcap_{\delta \geq 0} \mathbb{V}_{m,k}^{\delta,p}(0, T)$.

¹Alexandrov, Alexandr Danilovich (1939) *The existence almost everywhere of the second differential of a convex function and some associated properties of convex surfaces*. (in Russian), Ucenye Zapiski Leningrad. Gos. Univ. Ser. Math. Vol. 37, N. 6, pp. 3–35.

Using Lemma 6.58 we show that the mapping $\Gamma : \mathbb{V}_{m,k}^p(0, T) \rightarrow \mathbb{V}_{m,k}^p(0, T)$ given by

$$\begin{cases} (Y, Z) = \Gamma(X, U) \\ Y_t = \eta + \int_t^T \Phi(s, X_s, U_s) dQ_s - \int_t^T Z_s dB_s \end{cases}$$

has a unique fixed point in $\mathbb{V}_{m,k}^p(0, T)$. First Γ is well defined because by Corollary 2.45 $\mathbb{E} \sup_{t \in [0, T]} e^{p\delta V_t} |Y_t|^p < \infty$ and by the inequality

$$\begin{aligned} & |Y_s|^2 L_s dQ_s + \langle Y_s, \Phi(s, X_s, U_s) \rangle dQ_s \\ & \leq \frac{1}{4(\delta-1)} |X_s|^2 L_s dQ_s + \frac{1}{2\delta} |U_s|^2 ds + |Y_s| |\Phi(s, 0, 0)| dQ_s + |Y_s|^2 \delta dV_s, \quad \forall \delta > 1 \end{aligned}$$

and Proposition 5.2 we get $\|(Y, Z)\|_{\delta V}^p < \infty$ for all $\delta > 1$.

From the inequality

$$\begin{aligned} & |Y_s - Y'_s|^2 L_s dQ_s + \langle Y_s - Y'_s, \Phi(s, X_s, U_s) - \Phi(s, X'_s, U'_s) \rangle dQ_s \\ & \leq \frac{1}{2\delta} |U_s - U'_s|^2 ds + \frac{1}{4(\delta-1)} |X_s - X'_s|^2 L_s dQ_s + |Y_s - Y'_s|^2 \delta dV_s, \quad \forall \delta > 1 \end{aligned}$$

and Proposition 5.2 we obtain

$$\|(Y - Y', Z - Z')\|_{\delta V}^p \leq \frac{C_p}{(\delta - 1)^{p/2}} \|(X, U) - (X', U')\|_{\delta V}^p, \quad \forall \delta > 1$$

which tells us there exists a $\delta_0 > 1$ such that Γ is a *strict contraction* on $(\mathbb{V}_{m,k}^p(0, T), \|\cdot\|_{\delta V})$, for all $\delta \geq \delta_0$, and consequently, by Lemma 6.58, Γ has a unique fixed point in $\mathbb{V}_{m,k}^p(0, T)$.

Exercise 5.3: Since

$$\begin{aligned} & (Y_t^\varepsilon - Y_t^\delta) (G_\varepsilon(t, Y_t^\varepsilon, Z_t^\varepsilon) - G_\delta(t, Y_t^\delta, Z_t^\delta)) \\ & \leq L |Y_t^\varepsilon - Y_t^\delta| (2 + |Y_t^\varepsilon| + |Y_t^\delta| + |Z_t^\varepsilon| + |Z_t^\delta|) \end{aligned}$$

we obtain, by Proposition 5.2, with $N = 0$, $V = 0$, $\lambda = 0$, that

$$\begin{aligned} & \mathbb{E} \left(\sup_{s \in [0, T]} |Y_s^\varepsilon - Y_s^\delta|^p \right) + \mathbb{E} \left(\int_0^T |Z_s^\varepsilon - Z_s^\delta|^2 ds \right)^{p/2} \\ & \leq C_p \mathbb{E} \left(\int_0^T L |Y_s^\varepsilon - Y_s^\delta| (2 + |Y_s^\varepsilon| + |Y_s^\delta| + |Z_s^\varepsilon| + |Z_s^\delta|) ds \right)^{p/2} \end{aligned}$$

$$\begin{aligned} &\leq C_p \left[\mathbb{E} \left(\sup_{s \in [0, T]} |Y_s^\varepsilon - Y_s^\delta|^p \right) \right]^{1/2} \\ &\times \left[\mathbb{E} \left(\int_0^T L (2 + |Y_s^\varepsilon| + |Y_s^\delta| + |Z_s^\varepsilon| + |Z_s^\delta|) ds \right)^p \right]^{1/2}. \end{aligned}$$

Exercise 5.5: We apply the existence and uniqueness result from Theorem 5.27 and the comparison result from Theorem 5.33 for the BSDE

$$Y_t = \eta + \int_t^T Y_s (1 - Y_s^+) ds - \int_t^T \langle Z_s, dB_s \rangle$$

with $0 \leq \eta \leq 1$.

Exercise 5.7: Assume that E is not convex. We shall show there exists a bounded continuous function $g : \mathbb{R}^k \rightarrow E$ such that

$$P(\{Y_t \notin E\}) > 0, \text{ for some } t \in [0, T].$$

If E is not convex, we can find $a, b \in \text{Bd}(E)$ such that $a \neq b$ and $a + \lambda(b - a) \notin E$ for all $\lambda \in]0, 1[$. Let $\delta = \frac{1}{4}d_E(\frac{a+b}{2}) > 0$. Define $g : \mathbb{R}^k \rightarrow E$ by $g(x^{(1)}, x^{(2)}, \dots, x^{(k)}) = a + (b - a) \mathbf{1}_{(-\infty, 1]}(x^{(1)})$. By Exercise 1.7 we have

$$\mathbb{E} \left(g \left(B_T^{(1)} \right) \middle| \mathcal{F}_t \right) = a + (b - a) \Phi \left(\frac{1 - B_t^{(1)}}{\sqrt{T - t}} \right),$$

where

$$\Phi(r) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^r e^{-\frac{x^2}{2}} dx, \quad r \in \mathbb{R}.$$

We also have

$$\left| Y_t - \mathbb{E}[g(B_T^{(1)}) | \mathcal{F}_t] \right| \leq \mathbb{E} \left| \int_t^T F_s ds \middle| \mathcal{F}_t \right| \leq M(T - t) \leq \delta,$$

if $t \in [T - \frac{\delta}{M}, T]$, where $M > 0$ denotes the bound of F .

Then for all $t \in [T - \frac{\delta}{M}, T]$,

$$\begin{aligned} \left| Y_t - \frac{a+b}{2} \right| &\leq \left| Y_t - \mathbb{E}[g(B_T^{(1)}) | \mathcal{F}_t] \right| + \left| \mathbb{E}[g(B_T^{(1)}) | \mathcal{F}_t] - \frac{a+b}{2} \right| \\ &\leq \delta + |b - a| \left| \Phi \left(\frac{1 - B_t^{(1)}}{\sqrt{T - t}} \right) - \frac{1}{2} \right|. \end{aligned}$$

Therefore

$$\begin{aligned} 0 &< \mathbb{P} \left[\left| \Phi \left(\frac{1 - B_t^{(1)}}{\sqrt{T-t}} \right) - \frac{1}{2} \right| \leq \frac{\delta}{|b-a|} \right] \\ &\leq \mathbb{P} \left(\left| Y_t - \frac{a+b}{2} \right| \leq 2\delta \right) \\ &\leq \mathbb{P}(Y_t \notin E). \end{aligned}$$