Chapter 5
Backward Stochastic Differential Equations

5.1 Introduction

In this chapter we discuss so-called "backward stochastic differential equations", BSDEs for short. Linear BSDEs first appeared a long time ago, both as the equations for the adjoint process in stochastic control, as well as the model behind the Black and Scholes formula for the pricing and hedging of options in mathematical finance. These linear BSDEs can be solved more or less explicitly (see Proposition 5.31 below). However, the first published paper on nonlinear BSDEs, appeared only in 1990, see Pardoux and Peng [51]. Since then, the interest in BSDEs has increased regularly, due to the connections of this subject with mathematical finance, stochastic control, and partial differential equations. We refer the interested reader to El Karoui et al. [29] and [30], Pham [60] and the references therein for developments on the use of BSDEs as models in mathematical finance, as well as the connection of BSDEs with stochastic control (see also [28] and [37]). BSDEs are also an efficient tool for constructing Γ-martingales on manifolds with prescribed limit, see Darling [19]. The connection of BSDEs with semi linear PDEs was initiated in Pardoux, Peng [54], see also among the now vast literature on the subject [6, 48, 52] and [53].

We shall present both the abstract theory of BSDEs, and the connection of BSDEs with semilinear PDEs (both parabolic and elliptic). Let us motivate the notion of a BSDE via an associated semilinear parabolic PDE.

To each $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, we associate the Markov diffusion process $\{X_s^{t,x}, s \geq t\}$ which is a solution of the SDE

$$X_s^{t,x} = x + \int_t^s f(r, X_r^{t,x})dr + \int_t^s g(r, X_r^{t,x})dB_r,$$

where the Brownian motion B has dimension k. The associated infinitesimal generator reads

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$$\mathcal{A}_t\varphi(x) = \frac{1}{2}\mathbf{Tr}\left[gg^*(t,x)D^2\varphi(x)\right] + \langle f(t,x), \nabla\varphi(x)\rangle.$$

Let $T > 0$ be an arbitrary final time, $\kappa \in C(\mathbb{R}^d)$ and $F \in C([0,T]\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^k)$. We consider the following backward semilinear second order PDE

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) + \mathcal{A}_t u(t,x) + F(t,x,u(t,x),(\nabla u g)(t,x)) = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad (t,x) \in [0,T]\times\mathbb{R}^d, \\ u(T,x) = \kappa(x), \quad x \in \mathbb{R}^d. \end{cases}$$

Suppose that this equation has a classical solution $u \in C^{1,2}([0,T]\times\mathbb{R}^d)$. It then follows from Itô's formula that for any $0 \le t < s \le T$,

$$u(s,X_s^{t,x}) = \kappa(X_T^{t,x}) + \int_s^T F(r,X_r^{t,x},u(r,X_r^{t,x}),(\nabla u g)(r,X_r^{t,x})) \\ -\int_s^T (\nabla u g)(r,X_r^{t,x})dB_r.$$

Considering the pair of adapted processes

$$\left(Y_r^{t,x}, Z_r^{t,x}\right) = \left(u(r,X_r^{t,x}), (\nabla u g)(r,X_r^{t,x})\right),$$

we have that for each $(t,x) \in [0,T]\times\mathbb{R}^d$,

$$Y_s^{t,x} = \kappa(X_T^{t,x}) + \int_s^T F(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})dr - \int_s^T Z_r^{t,x}dB_r, \ s \le r \le T,$$

and $Y_t^{t,x}$ is a deterministic quantity which equals $u(t,x)$. The solution u of the above semilinear parabolic PDE is expressed in terms of the solution of this last backward stochastic differential equation (BSDE). We will see below that this is indeed an extension of the Feynman–Kac formula (in the sense that if F is affine, then the Feynman–Kac formula is a consequence of the above representation). Note that the above computation can be applied to a system of PDEs, rather than a single PDE. We shall consider only the case where the same second order PDE operator $\mathcal{A}$ is applied to each coordinate u_i of u. A probabilistic representation for more general systems of semilinear PDEs, with a different $\mathcal{A}$ for each coordinate of u, can be found in [55], see also [52] and [58].

Let us now write an abstract version of the above BSDE. Let $t = 0$, and forget about the superscript x. Suppose now that we are given a probability space with filtration $(\Omega,\mathcal{F},\mathcal{F}_t,\mathbb{P})$ and for each $(y,z) \in \mathbb{R}\times\mathbb{R}^k$, a measurable process $\{F(t,y,z),\ 0 \le t \le T\}$ (F being jointly measurable), together with an $\mathcal{F}_T$ random variable η.

We formulate the problem of solving a BSDE as follows: find a pair of adapted processes $\{(Y_t,Z_t),\ 0 \le t \le T\}$ such that

$$Y_t = \eta + \int_t^T F(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s,\ 0 \le t \le T,\ \text{a.s.}$$

Note that, since the boundary condition for $\{Y_t : t \in [0, T]\}$ is given at the terminal time T, it is not really natural for the solution $\{Y_t\}$ to be adapted at each time t to the past of the Brownian motion $\{B_s\}$ before time t. The price we have to pay for such a severe constraint to be satisfied is to have the coefficient of the Brownian motion – the process $\{Z_t\}$ – to be chosen independently of $\{Y_t\}$, hence the solution of the BSDE is a *pair* of processes. Note that in the case $F \equiv 0$, $Y_t = \mathbb{E}(\eta|\mathcal{F}_t)$ and Z is given by the martingale representation theorem from Sect. 2.4.

One may also think of a "backward SDE" as an inverse problem for an SDE, namely we are looking for a point $y \in \mathbb{R}$, and an adapted process $\{Z_t\}$, such that the solution $\{Y_t\}$ of

$$Y_t = y - \int_0^t F(s, Y_s, Z_s)ds + \int_0^t Z_s dB_s$$

satisfies $Y_T = \eta$.

Finally, note that while the above presentation treats T as a deterministic quantity, an important alternative is to replace it by a stopping time (or else by $+\infty$). This is essential when giving probabilistic representations of semilinear elliptic PDEs.

In this chapter, we suppose given a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\ge 0})$ with $\{B_t ; t \ge 0\}$ a k-dimensional Brownian motion and the filtration $\{\mathcal{F}_t\}_{t\ge 0}$ being the natural filtration of $\{B_t : t \ge 0\}$, i.e. for all $t \ge 0$:

$$\mathcal{F}_t = \mathcal{F}_t^B \overset{def}{=} \sigma\left(\{B_s : 0 \le s \le t\}\right) \vee \mathcal{N}.$$

5.2 Basic Inequalities

For convenience we rewrite in this context the Itô formula (2.14) and we give a basic inequality. First we introduce a notation used in this chapter.

Notation 5.1. *For $p \ge 1$ we define*

$$n_p \overset{def}{=} 1 \wedge (p-1)\,.$$

Let $(Y, Z) \in S_m^0\,[0, T] \times \Lambda_{m\times k}^0\,(0, T)$ satisfy for all $t \in [0, T]$, $\mathbb{P}$-*a.s.*:

$$Y_t = Y_T + \int_t^T dK_s - \int_t^T Z_s dB_s,$$

where

◊ $K \in S_m^0$,
◊ $K_\cdot(\omega) \in BV_{loc}([0,\infty[\,;\mathbb{R}^m)$, $\mathbb{P}$-a.s. $\omega \in \Omega$.

5.2.1 Backward Itô's Formula

If $\varphi \in C^{1,2}([0,T] \times \mathbb{R}^m)$, *then* $\mathbb{P}$-*a.s., for all* $t \in [0,T]$:

$$\begin{aligned}\varphi(t,Y_t) + \int_t^T \left\{ \frac{\partial\varphi}{\partial t}(s,Y_s) + \frac{1}{2}\mathbf{Tr}\left[Z_s Z_s^* \varphi''_{xx}(s,Y_s)\right]\right\} ds \\ = \varphi(T,Y_T) + \int_t^T \langle \varphi'_x(s,Y_s), dK_s\rangle - \int_t^T \langle \varphi'_x(s,Y_s), Z_s dB_s\rangle .\end{aligned} \tag{5.1}$$

From Corollary 2.29 we get for all $p \in \mathbb{R}$,

$$\begin{aligned}\left(|Y_t|^2+\varepsilon\right)^{p/2} + \frac{p}{2}\int_t^T R_s^{(p,\varepsilon)} ds + \frac{p}{2}\left(L_T^{(p,\varepsilon)} - L_t^{(p,\varepsilon)}\right) = \left(|Y_T|^2+\varepsilon\right)^{p/2} \\ + p\int_t^T \left\langle U_s^{(p,\varepsilon)}, dK_s\right\rangle - p\int_t^T \left\langle U_s^{(p,\varepsilon)}, Z_s dB_s\right\rangle,\end{aligned} \tag{5.2}$$

where

$$\begin{aligned}(j)\quad & U_s^{(p,\varepsilon)} = \left(|Y_s|^2+\varepsilon\right)^{(p-2)/2} Y_s, \\ (jj)\quad & R_s^{(p,\varepsilon)} = \left[|Z_s|^2|Y_s|^2 + (p-2)|Z_s^* Y_s|^2\right]\left(|Y_s|^2+\varepsilon\right)^{(p-4)/2}, \\ (jjj)\quad & L_t^{(p,\varepsilon)} = \varepsilon\int_0^t |Z_s|^2\left(|Y_s|^2+\varepsilon\right)^{(p-4)/2} ds.\end{aligned}$$

We have $\left|U_s^{(p,\varepsilon)}\right| \le \left(|Y_s|^2+\varepsilon\right)^{(p-1)/2}$ and

$$n_p |Z_s|^2 |Y_s|^2 \left(|Y_s|^2+\varepsilon\right)^{(p-4)/2} \le R_s^{(p,\varepsilon)} \le m_p |Z_s|^2 |Y_s|^2 \left(|Y_s|^2+\varepsilon\right)^{(p-4)/2},$$

where $n_p \overset{def}{=} 1 \wedge (p-1)$ and $m_p \overset{def}{=} 1 \vee (p-1)$.

Moreover

$$\frac{1}{\sqrt{\varepsilon}}\int_0^t |Z_s|^2 \mathbf{1}_{|Y_s|\le\sqrt{\varepsilon}} ds \le 2\sqrt{2} L_t^{(1,\varepsilon)}.$$

In particular for $p \geq 1$ and $\varepsilon \searrow 0$ we obtain

$$\begin{aligned} &|Y_t|^p + \frac{1}{2}\int_t^T R_s^{(p)} ds + \frac{1}{2}(L_T - L_t)\,\mathbf{1}_{p\neq 1} = |Y_T|^p \\ &+\int_t^T |Y_s|^{p-1}\langle \operatorname{sgn}(Y_s), dK_s\rangle - \int_t^T |Y_s|^{p-1}\langle \operatorname{sgn}(Y_s), Z_s dB_s\rangle, \end{aligned} \tag{5.3}$$

where

$$\operatorname{sgn} : \mathbb{R}^m \to \mathbb{R}^m, \quad \operatorname{sgn}(x) = \begin{cases} 0, & \text{if } x = 0, \\ \dfrac{x}{|x|}, & \text{if } x \neq 0, \end{cases}$$

$$R_s^{(p)} = \begin{cases} 0, & \text{if } Y_s = 0, \\ \left(|Z_s|^2 + (p-2)\,|Z_s^* \operatorname{sgn}(Y_s)|^2\right)|Y_s|^{p-2}, & \text{if } Y_s \neq 0, \end{cases}$$

and $\{L_t : t \geq 0\}$ is an increasing continuous progressively measurable stochastic process such that for all $t \geq 0$ (in the sense of convergence in probability)

$$L_t = \frac{1}{2}\lim_{\varepsilon\to 0_+}\int_0^t \frac{\varepsilon\,|Z_s|^2}{\left(|Y_s|^2+\varepsilon\right)^{3/2}} ds.$$

The stochastic process $\{L_t : t \geq 0\}$ has the following property:

$$L_t(\omega) = L_s(\omega), \quad \mathbb{P}\text{-a.s.},$$

for every interval $[s,t] \subset \{r \geq 0 : Y_r(\omega) = 0\}$, or $[s,t] \subset \operatorname{int}\{r \geq 0 : Y_r(\omega) \neq 0\}$.

Moreover, we have

$$\limsup_{\delta\to 0_+}\frac{1}{\delta}\int_0^T |Z_s|^2\,\mathbf{1}_{|Y_s|\leq\delta} ds \leq 2\sqrt{2}L_T \quad \text{and} \quad \int_0^T |Z_s|^2\,\mathbf{1}_{Y_s=0} ds = 0.$$

Since

$$0 \leq \frac{p}{2}n_p\int_s^t |Y_r|^{p-2}\,\mathbf{1}_{Y_r\neq 0}\,|Z_r|^2\,dr \leq \int_s^t R_r^{(p)} dr < \infty, \text{ for all } 0 \leq s < t \leq T,\ a.s.,$$

it follows that for every $p \geq 1$ and $0 \leq t \leq T$:

$$\begin{aligned} &|Y_t|^p + \frac{p}{2}n_p\int_t^T |Y_s|^{p-2}\,\mathbf{1}_{Y_s\neq 0}\,|Z_s|^2\,ds \leq |Y_T|^p \\ &+p\int_t^T |Y_s|^{p-2}\,\mathbf{1}_{Y_s\neq 0}\langle Y_s, dK_s\rangle - p\int_t^T |Y_s|^{p-2}\,\mathbf{1}_{Y_s\neq 0}\langle Y_s, Z_s dB_s\rangle,\ a.s. \end{aligned} \tag{5.4}$$

In fact we deduce from Lemma 2.37 a more general inequality:
◊ *if* $\psi : [0,T]\times\mathbb{R}^m \to \mathbb{R}$ *is a function of class* C^1, *convex in its second argument, then a.s.,* $\forall\, t \in [0,T]$:

$$\psi(t,Y_t) + \int_t^T \frac{\partial\psi}{\partial t}(s,Y_s)\,ds \le \psi(T,Y_T) + \int_t^T \langle\nabla\psi(s,Y_s),\, dK_s\rangle \\ - \int_t^T \langle\nabla\psi(s,Y_s),\, Z_s dB_s\rangle\,. \tag{5.5}$$

5.2.2 A Fundamental Inequality

Let $(Y,Z) \in S_m^0[0,T] \times \Lambda_{m\times k}^0(0,T)$ satisfy an identity of the form

$$Y_t = Y_T + \int_t^T dK_s - \int_t^T Z_s dB_s, \quad t \in [0,T], \quad \mathbb{P}\text{-a.s.}, \tag{5.6}$$

where

◊ $K \in S_m^0([0,T])$ and $K_\cdot(\omega) \in BV([0,T];\mathbb{R}^m)$, $\mathbb{P}$-a.s. $\omega \in \Omega$.
◊ *Assume that there exist*

(*a*) D, R, N $\mathcal{P}$*-m.i.c.s.p.,* $D_0 = R_0 = N_0 = 0$;
(*b*) V $\mathcal{P}$*–m.b-v.c.s.p.* $V_0 = 0$;
(*c*) $\lambda < 1 \le p$,

such that as measures on $[0,T]$, *a.s.*

$$dD_t + \langle Y_t, dK_t\rangle \le \left[\mathbf{1}_{p\ge 2}dR_t + |Y_t|dN_t + |Y_t|^2 dV_t\right] + \frac{n_p}{2}\lambda\,|Z_t|^2\,dt, \tag{5.7}$$

where

$$n_p \overset{def}{=} 1 \wedge (p-1)\,.$$

By Proposition 6.80, Corollary 6.81 and Corollary 6.82 from Annex C we have:

Proposition 5.2. *Let (5.6) and (5.7) be satisfied and moreover*

$$\mathbb{E}\,\left\|Ye^V\right\|_T^p < \infty.$$

(A) If $p > 1$, *then there exists a positive constant* $C_{p,\lambda}$, *depending only upon* (p,λ), *such that,* $\mathbb{P}$*-a.s., for all* $t \in [0,T]$:

$$\mathbb{E}^{\mathcal{F}_t} \sup_{r\in[t,T]} \left|e^{V_r}Y_r\right|^p + \mathbb{E}^{\mathcal{F}_t}\left(\int_t^T e^{2V_r}dD_r\right)^{p/2} + \mathbb{E}^{\mathcal{F}_t}\left(\int_t^T e^{2V_r}|Z_r|^2\,dr\right)^{p/2} \tag{5.8}$$

$$\leq C_{p,\lambda}\,\mathbb{E}^{\mathcal{F}_t}\left[\left|e^{V_T}Y_T\right|^p+\left(\int_t^T e^{2V_r}\mathbf{1}_{p\geq 2}dR_r\right)^{p/2}+\left(\int_t^T e^{V_r}dN_r\right)^p\right].$$

(B) *If* $p=1$ *(and* $n_p=0$*), then* $\mathbb{P}$*-a.s., for all* $0\leq t\leq T$

$$e^{V_t}\,|Y_t|\leq \mathbb{E}^{\mathcal{F}_t}e^{V_T}\,|Y_T|+\mathbb{E}^{\mathcal{F}_t}\int_t^T e^{V_r}dN_r$$

and for all $0<\alpha<1$ *there exists a positive constant* C_α*, depending only upon* α *such that*

$$\sup_{r\in[t,T]}\left[\mathbb{E}\left(e^{V_r}\,|Y_r|\right)\right]^\alpha+\mathbb{E}\left(\sup_{r\in[t,T]}\left|e^{V_r}Y_r\right|^\alpha\right)+\mathbb{E}\left(\int_t^T e^{2V_r}\,|Z_r|^2\,dr\right)^{\alpha/2}$$
$$+\mathbb{E}\left(\int_t^T e^{2V_r}\,|D_r|^2\,dr\right)^{\alpha/2}$$
$$\leq C_\alpha\left[\left(\mathbb{E}\left(e^{V_T}\,|Y_T|\right)\right)^\alpha+\left(\mathbb{E}\int_t^T e^{V_r}dN_r\right)^\alpha\right].$$

(C) *If* $p\geq 1$ *and* $R=N=0$*, then* $\mathbb{P}$*-a.s., for all* $t\in[0,T]$*:*

$$e^{pV_t}\,|Y_t|^p\leq \mathbb{E}^{\mathcal{F}_t}e^{pV_T}\,|Y_T|^p\,. \tag{5.9}$$

Corollary 5.3. *Under the assumptions of Proposition 5.2, if there exists a* $c\geq 0$ *such that* $\sup_{s\in[0,T]}|V_s|\leq c$*, then* $\mathbb{P}$*-a.s., for all* $t\in[0,T]$*:*

$$\mathbb{E}^{\mathcal{F}_t}\sup_{s\in[t,T]}|Y_s|^p+\mathbb{E}^{\mathcal{F}_t}\left(\int_t^T|Z_s|^2\,ds\right)^{p/2}$$
$$\leq C_{p,\lambda}\,e^{2c}\,\mathbb{E}^{\mathcal{F}_t}\left[|Y_T|^p+\left(\int_t^T\mathbf{1}_{p\geq 2}dR_s\right)^{p/2}+\left(\int_t^T dN_s\right)^p\right].$$

5.3 BSDEs with Deterministic Final Time

Our main goal in this section is to study backward stochastic differential equations (abbreviated BSDEs) of the form

$$\begin{cases}-dY_t=F\left(t,Y_t,Z_t\right)dt+G\left(t,Y_t\right)dA_t-Z_t dB_t, & 0\leq t<T,\\ Y_T=\eta,\end{cases} \tag{5.10}$$

or equivalently, a.s. for all $t\in[0,T]$:

$$Y_t = \eta + \int_t^T F(s, Y_s, Z_s)\, ds + \int_t^T G(s, Y_s)\, dA_s - \int_t^T Z_s dB_s,$$

whose solution $(Y_t, Z_t)_{t\in[0,T]}$ takes values in $\mathbb{R}^m \times \mathbb{R}^{m\times k}$, and where we assume in this section that:

- $T > 0$ is a fixed final deterministic time;
- $\eta : \Omega \to \mathbb{R}^m$, the final condition, is an $\mathcal{F}_T$-measurable random vector;
- $F : \Omega\times[0,T]\times\mathbb{R}^m\times\mathbb{R}^{m\times k} \to \mathbb{R}^m$ is a $\left(\mathcal{P}, \mathbb{R}^m \times \mathbb{R}^{m\times k}\right)$-Carathéodory function, that is

$$\begin{aligned}&F(\cdot,\cdot,y,z) \text{ is } \mathcal{P}\text{-m.s.p., } \forall\ (y,z) \in \mathbb{R}^m\times\mathbb{R}^{m\times k};\\ &F(\omega,t,\cdot,\cdot) \text{ is a continuous function, } d\mathbb{P}\otimes dt\text{-}a.e.;\end{aligned}$$

- $G : \Omega \times [0,T] \times \mathbb{R}^m \to \mathbb{R}^m$ is a $(\mathcal{P}, \mathbb{R}^m)$-Carathéodory function, i.e.

$$\begin{aligned}&G(\cdot,\cdot,y) \text{ is } \mathcal{P}\text{-m.s.p., } \forall\ y \in \mathbb{R}^m;\\ &G(\omega,t,\cdot,\cdot) \text{ is a continuous function, } d\mathbb{P}\otimes dt\text{-}a.e.;\end{aligned}$$

- A is a $\mathcal{P}$-m.i.c.s.p., $A_0 = 0$.

Note that, by Exercise 1.1, F is $(\mathcal{P}\otimes\mathcal{B}_m\otimes\mathcal{B}_{m\times k}, \mathcal{B}_m)$-measurable and G is $(\mathcal{P}\otimes\mathcal{B}_m, \mathcal{B}_m)$-measurable.

We state the following definition:

Definition 5.4. A pair $(Y,Z) \in S_m^0[0,T] \times \Lambda^0_{m\times k}(0,T)$ is a solution of (5.10) if

$$\int_0^T |F(t,Y_t,Z_t)|\, dt + \sum_{i=1}^N \int_0^T |G(t,Y_t)|\, dA_t < \infty, \quad \mathbb{P}\text{-a.s.}$$

and, a.s. for all $t \in [0,T]$:

$$Y_t = \eta + \int_t^T F(s, Y_s, Z_s)\, ds + \int_t^T G(s, Y_s)\, dA_s - \int_t^T Z_s dB_s. \tag{5.11}$$

5.3.1 A Priori Estimates and Uniqueness

We now consider the BSDE

$$Y_t = \eta + \int_t^T \Phi(s, Y_s, Z_s)\, dQ_s - \int_t^T Z_s dB_s,,\ t \in [0,T],\ a.s., \tag{5.12}$$

where

- $\eta : \Omega \to \mathbb{R}^m$, the final condition, is an $\mathcal{F}_T$-measurable random vector;
- $(\omega, t, y, z) \longmapsto \Phi(\omega, t, y, z) : \Omega \times [0, T] \times \mathbb{R}^m \times \mathbb{R}^{m\times k} \to \mathbb{R}^m$;
- $(\omega, t) \longmapsto Q_t(\omega) : \Omega \times [0, T] \to \mathbb{R}$ is a $\mathcal{P}$-m.i.c.s.p. such that $Q_0 = 0$.

Note that the BSDE (5.10) can be written in this form with

$$\begin{aligned}\Phi(\omega, t, y, z) &= \alpha_t(\omega) F(\omega, t, y, z) + \beta_t(\omega) G(\omega, y), \quad \text{and}\\ Q_t(\omega) &= t + A_t(\omega)\end{aligned}$$

where $\{\alpha_t : t \geq 0\}$ and $\{\beta_t : t \geq 0\}$ are two real positive $\mathcal{P}$-m.s.p. (given by the Radon–Nikodym representation theorem), $\alpha_t + \beta_t = 1$, such that

$$dt = \alpha_t dQ_t \quad \text{and} \quad dA_t = \beta_t dQ_t.$$

We define for any $\rho \geq 0$

$$\Phi^\#_\rho(t) \stackrel{def}{=} \sup_{|y|\leq\rho} |\Phi(t, y, 0)|; \quad \text{in particular } \Phi^\#_0(t) = |\Phi(t, 0, 0)|.$$

The basic assumptions on Φ are the following

$$(\mathbf{BSDE\text{-}H_\Phi}): \tag{5.13}$$

♦ *$\forall\ y \in \mathbb{R}^m$, $z \in \mathbb{R}^{m\times k}$, the function $\Phi(\cdot, \cdot, y, z) : \Omega \times [0, T] \to \mathbb{R}^m$ is $\mathcal{P}$-measurable;*

♦ *there exist three $\mathcal{P}$-m.s.p. $\mu : \Omega \times [0, T] \to \mathbb{R}$ and $\ell, \alpha : \Omega \times [0, T] \to \mathbb{R}_+$ such that, $\mathbb{P}$-a.s.*

$$\begin{aligned} &(i) \quad \alpha_t dQ_t = dt,\\ &(ii) \quad \int_0^T \left[|\mu_t| dQ_t + (\ell_t)^2 dt\right] < \infty, \end{aligned} \tag{5.14}$$

and for all $y, y' \in \mathbb{R}^m$ and *$z, z' \in \mathbb{R}^{m\times k}$, $d\mathbb{P} \otimes dQ_t$-a.e.:*

$$\begin{aligned} & && \textit{Continuity:}\\ &(C_y) && y \longrightarrow \Phi(t, y, z) : \mathbb{R}^m \to \mathbb{R}^m \ \textit{is continuous};\\ & && \textit{Monotonicity condition:}\\ &(M_y) && \langle y' - y, \Phi(t, y', z) - \Phi(t, y, z)\rangle \leq \mu_t |y' - y|^2;\\ & && \textit{Lipschitz condition:}\\ &(L_z) && |\Phi(t, y, z') - \Phi(t, y, z)| \leq \alpha_t\ \ell_t |z' - z|;\\ & && \textit{Boundedness condition:}\\ &(B_y) && \int_0^T \Phi^\#_\rho(s)\, dQ_s < \infty, \quad \forall\ \rho \geq 0. \end{aligned} \tag{5.15}$$

The assumptions on Φ yield a continuity behaviour result which we leave as an exercise for the reader.

Lemma 5.5. *Under the assumption (5.15)*

$$\int_0^T |\Phi(t,Y_t,Z_t)|\,dQ_t < \infty, \quad \mathbb{P}\text{-a.s.}, \quad \forall\ (Y,Z) \in S_m^0[0,T] \times \Lambda_{m\times k}^0(0,T),$$

and the mapping

$$(Y,Z) \longrightarrow \int_0^\cdot \Phi(s,Y_s,Z_s)dQ_s$$

is continuous from $S_m^0[0,T] \times \Lambda_{m\times k}^0(0,T)$ into $S_m^0[0,T]$.

We shall show that the monotonicity of Φ yields an inequality of the form (5.7). Let (with $a > 1$ arbitrary)

$$n_p \overset{def}{=} 1 \wedge (p-1) \quad \text{and} \quad \gamma_s \overset{def}{=} \mu_s + \frac{a}{2n_p}(\ell_s)^2 \alpha_s.$$

We have:

Lemma 5.6. *Let $a,p > 1$, $r_0 \geq 0$ and the assumptions (5.13-**BSDE-H**$_\Phi$) be satisfied. Let (Y,Z), $(\tilde{Y},\tilde{Z}) \in S_m^0[0,T] \times \Lambda_{m\times k}^0(0,T)$. Then, in the sense of signed measures on $[0,T]$:*

$$dD_t^{(r_0)} + \langle Y_t, \Phi(t,Y_t,Z_t)\,dQ_t\rangle \leq \left[dR_t^{(r_0)} + |Y_t|dN_t^{(r_0)} + |Y_t|^2 dV_t\right] + \frac{n_p}{2a}|Z_t|^2\,dt, \tag{5.16}$$

and

$$\left\langle Y_t - \tilde{Y}_t, \Phi(t,Y_t,Z_t) - \Phi(t,\tilde{Y}_t,\tilde{Z}_t)\right\rangle dQ_t \leq |Y_t - \tilde{Y}_t|^2 dV_t + \frac{n_p}{2a}\left|Z_t - \tilde{Z}_t\right|^2 dt \tag{5.17}$$

where

$$D_t^{(r_0)} = r_0\int_0^t |\Phi(s,Y_s,Z_s)|\,dQ_s, \quad R_t^{(r_0)} = r_0\int_0^t \Phi_{r_0}^{\#}(s)\,dQ_s + r_0^2\int_0^t \gamma_s^+ dQ_s,$$
$$V_t = \int_0^t \gamma_s dQ_s, \quad N_t^{(r_0)} = \int_0^t \Phi_{r_0}^{\#}(s)\,dQ_s + 2r_0\int_0^t |\gamma_s|\,dQ_s. \tag{5.18}$$

Proof. The monotonicity property of Φ implies that for any $\mathbb{R}^m$-valued stochastic process $\{U_s : s \geq 0\}$, $|U_s| \leq 1$:

$$\langle r_0U_s - Y_s, \Phi(s,r_0U_s,Z_s) - \Phi(s,Y_s,Z_s)\rangle\,dQ_s \leq \mu_s\,|r_0U_s - Y_s|^2\,dQ_s.$$

Since

$$\left|\Phi\left(s, r_0 U_s, Z_s\right)\right| dQ_s \leq \left[\Phi_{r_0}^{\#}(s) + \alpha_s \ell_s \left|Z_s\right|\right] dQ_s = \Phi_{r_0}^{\#}(s)\, dQ_s + \ell_s \left|Z_s\right| ds$$

it follows that

$$\begin{aligned}
& r_0 \left\langle U_s, -\Phi\left(s, Y_s, Z_s\right)\right\rangle dQ_s + \left\langle Y_s, \Phi\left(s, Y_s, Z_s\right)\right\rangle dQ_s \\
& \quad \leq \left|r_0 U_s - Y_s\right|^2 \mu_s dQ_s + \left|r_0 U_s - Y_s\right| \left[\Phi_{r_0}^{\#}(s)\, dQ_s + \ell_s \left|Z_s\right| ds\right] \\
& \quad \leq \left|r_0 U_s - Y_s\right|^2 \mu_s dQ_s + \left(r_0 + \left|Y_s\right|\right) \Phi_{r_0}^{\#}(s)\, dQ_s \\
& \qquad + \frac{a}{2n_p} \left|r_0 U_s - Y_s\right|^2 \left(\ell_s\right)^2 ds + \frac{n_p}{2a} \left|Z_s\right|^2 ds.
\end{aligned}$$

Hence

$$\begin{aligned}
& r_0 \left\langle U_s, -\Phi\left(s, Y_s, Z_s\right)\right\rangle dQ_s + \left\langle Y_s, \Phi\left(s, Y_s, Z_s\right)\right\rangle dQ_s \\
& \quad \leq \left(r_0 + \left|Y_s\right|\right) \Phi_{r_0}^{\#}(s)\, dQ_s \\
& \qquad + \left(r_0^2 \left|U_s\right|^2 - 2 r_0 \left\langle U_s, Y_s\right\rangle + \left|Y_s\right|^2\right) \gamma_s dQ_s + \frac{n_p}{2a} \left|Z_s\right|^2 ds \\
& \quad \leq \left[r_0 \Phi_{r_0}^{\#}(s) + r_0^2 \gamma_s^{+}\right] dQ_s + \left|Y_s\right| \left(\Phi_{r_0}^{\#}(s) + 2 r_0 \left|\gamma_s\right|\right) dQ_s + \left|Y_s\right|^2 \gamma_s dQ_s \\
& \qquad + \frac{n_p}{2a} \left|Z_s\right|^2 ds.
\end{aligned}$$

(5.16) follows if we choose

$$U_s = \begin{cases} \frac{-\Phi(s, Y_s, Z_s)}{\left|\Phi(s, Y_s, Z_s)\right|}, & \text{if } \Phi\left(s, Y_s, Z_s\right) \neq 0, \\ 0, & \text{if } \Phi\left(s, Y_s, Z_s\right) = 0. \end{cases}$$

The inequality (5.17) is obtained as follows:

$$\begin{aligned}
& \left\langle Y_s - \tilde{Y}_s, \Phi\left(s, Y_s, Z_s\right) - \Phi(s, \tilde{Y}_s, \tilde{Z}_s)\right\rangle dQ_s \\
& \leq \left[\mu_s \left|Y_s - \tilde{Y}_s\right|^2 + \ell_s \alpha_s \left|Y_s - \tilde{Y}_s\right| \left|Z_s - \tilde{Z}_s\right|\right] dQ_s \\
& \leq \mu_s \left|Y_s - \tilde{Y}_s\right|^2 dQ_s + \left|Y_s - \tilde{Y}_s\right| \left|Z_s - \tilde{Z}_s\right| \ell_s ds \\
& \leq \left(\mu_s dQ_s + \frac{a}{2n_p} \left(\ell_s\right)^2 ds\right) \left|Y_s - \tilde{Y}_s\right|^2 + \frac{n_p}{2a} \left|Z_s - \tilde{Z}_s\right|^2 ds.
\end{aligned}$$

■

Taking into account Proposition 5.2 with $dK_s = \Phi\left(s, Y_s, Z_s\right) dQ_s$, we deduce from (5.16), first with $r_0 = 0$ and then with $r_0 > 0$:

Proposition 5.7. *Let the assumptions (5.13-**BSDE-H**$_\Phi$) be satisfied. Then for every $a,p>1$ there exists a constant $C_{a,p}$ such that for all solutions $(Y,Z)\in S_m^0[0,T]\times \Lambda^0_{m\times k}(0,T)$ of the BSDE (5.12) satisfying*

$$\mathbb{E}\left\|Ye^V\right\|_T^p<\infty,$$

where again

$$V_t\overset{def}{=}V_t^{a,p}=\int_0^t\mu_s dQ_s+\frac{a}{2n_p}\int_0^t(\ell_s)^2\,ds,$$

the following inequality holds, $\mathbb{P}$-a.s., for all $t\in[0,T]$:

$$\begin{aligned}&\mathbb{E}^{\mathcal{F}_t}\left(\sup_{s\in[t,T]}\left|e^{V_s}Y_s\right|^p\right)+\mathbb{E}^{\mathcal{F}_t}\left(\int_t^T e^{2V_s}|Z_s|^2\,ds\right)^{p/2}\\&\le C_{a,p}\left[\mathbb{E}^{\mathcal{F}_t}\left|e^{V_T}\eta\right|^p+\mathbb{E}^{\mathcal{F}_t}\left(\int_t^T e^{V_s}|\Phi(s,0,0)|\,dQ_s\right)^p\right].\end{aligned}\tag{5.19}$$

Moreover, if $p\ge 2$, then for all $r_0>0$:

$$\begin{aligned}&\mathbb{E}\left(r_0\int_0^T e^{2V_s}|\Phi(s,Y_s,Z_s)|\,dQ_s\right)^{p/2}\le C_{a,p}\left[\mathbb{E}\left|e^{V_T}\eta\right|^p\right.\\&\left.+\mathbb{E}\left(\int_0^T e^{2V_s}dR_s^{(r_0)}\right)^{p/2}+\mathbb{E}\left(\int_0^T e^{V_s}dN_s^{(r_0)}\right)^p\right].\end{aligned}\tag{5.20}$$

Corollary 5.8. *Let $p=1$. Let the assumptions (5.13-**BSDE-H**$_\Phi$) be satisfied and Φ be independent of $z\in\mathbb{R}^{m\times k}$ ($\ell_t\equiv 0$ and $V_t=\bar{\mu}_t=\int_0^t\mu_s dQ_s$). If $(Y,Z)\in S_m^0[0,T]\times\Lambda^0_{m\times k}(0,T)$ is a solution of the BSDE (5.12) satisfying*

$$\mathbb{E}\sup_{s\in[0,T]}e^{\bar{\mu}_s}|Y_s|<\infty,$$

then the following inequality holds $\mathbb{P}$-a.s., for all $t\in[0,T]$:

$$e^{\bar{\mu}_t}|Y_t|\le\mathbb{E}^{\mathcal{F}_t}e^{\bar{\mu}_T}|\eta|+\mathbb{E}^{\mathcal{F}_t}\int_t^T e^{\bar{\mu}_s}|\Phi(s,0)|\,dQ_s.$$

Moreover for all $0<q<1$

$$\begin{aligned}&\sup_{s\in[0,T]}\left(\mathbb{E}\left(e^{\bar{\mu}_s}|Y_s|\right)\right)^q+\mathbb{E}\sup_{s\in[0,T]}\left|e^{\bar{\mu}_s}Y_s\right|^q+\mathbb{E}\left(\int_0^T e^{2\bar{\mu}_s}|Z_s|^2\,ds\right)^{q/2}\\&\le C_q\left[\left(\mathbb{E}\left(e^{\bar{\mu}_T}|\eta|\right)\right)^{q/2}+\left(\mathbb{E}\int_0^T e^{\bar{\mu}_s}|\Phi(s,0)|\,dQ_s\right)^{q/2}\right].\end{aligned}$$

Proof. Since

$$\langle Y_t, \Phi(t, Y_t, Z_t)\, dQ_t\rangle \leq |Y_t|\,|\Phi(t,0)|\, dQ_t + |Y_t|^2\, d\bar{\mu}_t$$

the conclusions follow by Corollary 6.81. ■

From (5.19) we immediately have:

Corollary 5.9. *Let $a, p > 1$. If*

$$\mathbb{E} \sup_{t\in[0,T]} \left|Y_t e^{V_t}\right|^p < \infty$$

and there exists a constant $A \geq 0$ such that for all $t \in [0,T]$:

$$\mathbb{E}^{\mathcal{F}_t}\left[\left|e^{V_T - V_t}\eta\right|^p + \left(\int_t^T e^{V_s - V_t}\,|\Phi(s,0,0)|\, dQ_s\right)^p\right] \leq A,\ \ a.s.,$$

then for all $t \in [0,T]$:

$$|Y_t|^p + \mathbb{E}^{\mathcal{F}_t}\left(\int_t^T e^{2(V_s - V_t)}\,|Z_s|^2\, ds\right)^{p/2} \leq A\, C_{a,p},\ a.s.$$

Let $(Y,Z) \in S_m^0[0,T] \times \Lambda^0_{m\times k}(0,T)$ be a solution of the BSDE

$$Y_t = \eta + \int_t^T \Phi(s, Y_s, Z_s)\, dQ_s - \int_t^T Z_s dB_s, \tag{5.21}$$

where Φ satisfies (5.13–**BSDE-H**$_\Phi$) and $(\hat{Y}, \hat{Z}) \in S_m^0[0,T] \times \Lambda^0_{m\times k}(0,T)$ is a solution of the BSDE

$$\hat{Y}_t = \hat{\eta} + \int_t^T \hat{\Phi}(s, \hat{Y}_s, \hat{Z}_s) dQ_s - \int_t^T \hat{Z}_s dB_s, \tag{5.22}$$

where $\hat{\Phi}(\cdot,\cdot,\cdot,\cdot) : \Omega \times [0,T] \times \mathbb{R}^m \times \mathbb{R}^{m\times k} \to \mathbb{R}^m$ is $\mathcal{P}$-measurable with respect to $(\omega, t) \in \Omega \times [0,T]$ and continuous with respect to $(y,z) \in \mathbb{R}^m \times \mathbb{R}^{m\times k}$. We clearly need to assume that

$$\int_0^T \left|\hat{\Phi}(s, \hat{Y}_s, \hat{Z}_s)\right| dQ_s < \infty,\ \mathbb{P}\text{-a.s.}$$

Note that

$$Y_t - \hat{Y}_t = (\eta - \hat{\eta}) + \int_t^T dK_s - \int_t^T (Z_s - \hat{Z}_s) dB_s,$$

where

$$K_t = \int_0^t \left[\Phi\left(s, Y_s, Z_s\right) - \hat{\Phi}(s, \hat{Y}_s, \hat{Z}_s) \right] dQ_s,$$

and by the assumptions (5.13−**BSDE-H**$_\Phi$)

$$\left\langle Y_t - \hat{Y}_t, dK_t \right\rangle \le \left| Y_t - \hat{Y}_t \right| \left| \Phi(t, \hat{Y}_t, \hat{Z}_t) - \hat{\Phi}(t, \hat{Y}_t, \hat{Z}_t) \right| dQ_t + \left| Y_t - \hat{Y}_t \right|^2 dV_t + \frac{n_p}{2a} \left| Z_t - \tilde{Z}_t \right|^2 dt$$

with, as above,

$$dV_t = \mu_t dQ_t + \frac{a}{2n_p} \left(\ell_t\right)^2 dt.$$

Hence by Proposition 5.2 we have:

Theorem 5.10 (Continuity and Uniqueness). *Let* $a, p > 1$ *and the assumptions* (*5.13*−**BSDE-H**$_\Phi$) *be satisfied. Let*

$$\left(Y, Z\right), \ (\hat{Y}, \hat{Z}) \in S_m^0 \left[0, T\right] \times \Lambda^0_{m \times k} \left(0, T\right)$$

be solutions of the BSDEs (5.21) and (5.22) respectively. If

$$\mathbb{E} \sup_{t \in [0,T]} \left(e^{pV_t} \left| Y_t - \hat{Y}_t \right|^p \right) < \infty, \tag{5.23}$$

then there exists a positive $C_{a,p}$ *such that:*

$$\begin{aligned} &\mathbb{E} \left(\sup_{s \in [0,T]} e^{pV_s} \left| Y_s - \hat{Y}_s \right|^p \right) + \mathbb{E} \left[\left(\int_0^T e^{2V_s} \left| Z_s - \hat{Z}_s \right|^2 ds \right)^{p/2} \right] \\ &\le C_{a,p}\, \mathbb{E} \left[e^{pV_T} \left| \eta - \hat{\eta} \right|^p + \left(\int_0^T e^{V_s} \left| \Phi(s, \hat{Y}_s, \hat{Z}_s) - \hat{\Phi}(s, \hat{Y}_s, \hat{Z}_s) \right| dQ_s \right)^p \right]. \end{aligned} \tag{5.24}$$

If $\Phi = \hat{\Phi}$, *then for all* $0 \le t \le s \le T$,

$$e^{pV_t} \left| Y_t - \hat{Y}_t \right|^p \le \mathbb{E}^{\mathcal{F}_t} \left(e^{pV_s} \left| Y_s - \hat{Y}_s \right|^p \right), \quad \mathbb{P}\text{-}a.s. \tag{5.25}$$

In particular uniqueness follows in the space $S_m^p\left([0,T]\,; e^V\right) \times \Lambda^0_{m \times k}\left(0, T\right)$, *where*

$$S_m^p\left([0,T]\,; e^V\right) \overset{def}{=} \left\{ Y \in S_m^0\left[0,T\right] : \mathbb{E} \sup_{s \in [0,T]} \left| e^{V_s} Y_s \right|^p < \infty \right\}.$$

Recall the notation

$$\bar{\mu}_t = \int_0^t \mu_s dQ_s.$$

Theorem 5.11 (Continuity and Uniqueness). *Let $p = 1$. Assume that $\Phi, \hat{\Phi} : \Omega \times [0,T] \times \mathbb{R}^m \times \mathbb{R}^{m\times k} \to \mathbb{R}^m$ satisfy assumptions (5.13) and both are independent of $z \in \mathbb{R}^{m\times k}$ ($\ell_t = \hat{\ell}_t \equiv 0$). If $(Y,Z), (\hat{Y},\hat{Z}) \in S_m^0[0,T] \times \Lambda_{m\times k}^0(0,T)$ are two solutions of the BSDE (5.107) corresponding respectively to (η, Φ) and $(\hat{\eta}, \hat{\Phi})$ such that*

$$\mathbb{E} \sup_{s\in[0,T]} e^{\bar{\mu}_s} \left|Y_s - \hat{Y}_s\right| < \infty,$$

and $\Delta_s \overset{def}{=} \Phi(s, \hat{Y}_s, \hat{Z}_s) - \hat{\Phi}(s, \hat{Y}_s, \hat{Z}_s)$, then $\mathbb{P}$-a.s., for all $t \in [0,T]$:

$$e^{\bar{\mu}_t} \left|Y_t - \hat{Y}_t\right| \le \mathbb{E}^{\mathcal{F}_t}\left(e^{\bar{\mu}_T} |\eta - \hat{\eta}|\right) + \mathbb{E}^{\mathcal{F}_t} \int_t^T e^{\bar{\mu}_s} |\Delta_s| \, dQ_s$$

and for every $q \in (0,1)$ there exists a constant C_q such that

$$\begin{aligned} \sup_{s\in[0,T]} \left(\mathbb{E}\left(e^{\bar{\mu}_s}\left|Y_s - \hat{Y}_s\right|\right)\right)^q + \mathbb{E} \sup_{s\in[0,T]} e^{q\bar{\mu}_s}\left|Y_s - \hat{Y}_s\right|^q \\ + \mathbb{E}\left(\int_0^T e^{2\bar{\mu}_s}\left|Z_s - \hat{Z}_s\right|^2 ds\right)^{q/2} \\ \le C_q \left[\left(\mathbb{E}\left(e^{\bar{\mu}_T}|\eta - \hat{\eta}|\right)\right)^q + \left(\mathbb{E}\int_0^T e^{\bar{\mu}_s}|\Delta_s| \, dQ_s\right)^q\right]. \end{aligned}$$

Proof. Since

$$\begin{aligned} &\left\langle Y_t - \hat{Y}_t, \left[\Phi(t, Y_t, Z_t) - \hat{\Phi}(t, \hat{Y}_t, \hat{Z}_t)\right] dQ_t \right\rangle \\ &\le \left|Y_t - \hat{Y}_t\right| \left|\Phi(t, \hat{Y}_t, \hat{Z}_t) - \hat{\Phi}(t, \hat{Y}_t, \hat{Z}_t)\right| dQ_t + \left|Y_t - \hat{Y}_t\right|^2 d\bar{\mu}_t, \end{aligned}$$

the conclusions follow by Corollary 6.81. ■

5.3.2 Complementary Results

In this subsection we generalize the uniqueness result and we shall give a scheme to obtain the solution as a limit of uniformly bounded solutions of approximate BSDEs.

Let $a, p > 1$ and

$$V_t^{a,p} = \int_0^t \mu_s dQ_s + \frac{a}{2n_p}\int_0^t (\ell_s)^2\, ds.$$

Define

$$S_m^p\left([0,T]\,; e^{V^{a,p}}\right) \overset{def}{=} \left\{ Y \in S_m^0\,[0,T] : \mathbb{E} \sup_{s\in[0,T]} \left| e^{V_s^{a,p}} Y_s \right|^p < \infty \right\}.$$

Note that if $1 < a_1 < a_2$ then $V_t^{a_1,p} \le V_t^{a_2,p}$ and consequently

$$S_m^p\left([0,T]\,; e^{V^{a_2,p}}\right) \subset S_m^p\left([0,T]\,; e^{V^{a_1,p}}\right). \tag{5.26}$$

Let

$$S_m^{1+,p}\left([0,T]\,; e^{V}\right) \overset{def}{=} \bigcup_{a>1} S_m^p\left([0,T]\,; e^{V^{a,p}}\right) \quad \text{and}$$
$$S_m^{1+,1+}\left([0,T]\,; e^{V}\right) \overset{def}{=} \bigcup_{a,\ p>1} S_m^p\left([0,T]\,; e^{V^{a,p}}\right).$$

Remark 5.12. If Q, μ and ℓ are deterministic functions, then for all $a,\ p > 1$:

$$S_m^{1+,p}\left([0,T]\,; e^{V}\right) = S_m^p\left([0,T]\,; e^{V^{a,p}}\right) = S_m^p\,[0,T]$$

and

$$S_m^{1+,1+}\left([0,T]\,; e^{V}\right) = S_m^{1+}\,[0,T] \overset{def}{=} \bigcup_{p>1} S_m^p\,[0,T].$$

Corollary 5.13. *Let the assumptions* (**BSDE-H**$_\Phi$) *be satisfied. Then for each* $p > 1$, *the BSDE (5.12) has at most one solution*

$$(Y,Z) \in S_m^{1+,p}\left([0,T]\,; e^{V}\right) \times \Lambda_{m\times k}^0\,(0,T).$$

If, moreover,

$$\mathbb{E}\exp\left(\lambda\int_0^T (\ell_s)^2\, ds\right) < \infty, \quad \textit{for all } \lambda > 0,$$

then the BSDE (5.12) has at most one solution

$$(Y,Z) \in S_m^{1+,1+}\left([0,T]\,; e^{V}\right) \times \Lambda_{m\times k}^0\,(0,T).$$

Proof. Let $(Y,Z),(\hat{Y},\hat{Z}) \in S_m^0([0,T]) \times \Lambda^0_{m\times k}(0,T)$ be two solutions of the BSDE (5.12) corresponding to η.

(A) Let $p>1$ be such that $(Y,Z),(\hat{Y},\hat{Z}) \in S_m^{1+,p}\left([0,T];e^V\right) \times \Lambda^0_{m\times k}(0,T)$. Then from (5.26) and the definition of $S_m^{1+,p}\left([0,T];e^V\right)$ there exists an $a>1$ such that

$$\mathbb{E}\sup_{t\in[0,T]}\left|e^{V_t^{a,p}}Y_t\right|^p<\infty \quad\text{and}\quad \mathbb{E}\sup_{t\in[0,T]}\left|e^{V_t^{a,p}}\hat{Y}_t\right|^p<\infty,$$

i.e. the condition (5.23) is satisfied; consequently the estimate (5.24) follows and uniqueness too.

(B) If $(Y,Z),(\hat{Y},\hat{Z}) \in S_m^{1+,1+}\left([0,T];e^V\right) \times \Lambda^0_{m\times k}(0,T)$ then there exist a_1,a_2, $p_1,p_2>1$ such that

$$\mathbb{E}\sup_{t\in[0,T]}\left|e^{V_t^{a_1,p_1}}Y_t\right|^{p_1}<\infty \quad\text{and}\quad \mathbb{E}\sup_{t\in[0,T]}\left|e^{V_t^{a_2,p_2}}\hat{Y}_t\right|^{p_2}<\infty.$$

Let $a>1$ and $1<p<p_1\wedge p_2$. Put

$$b_i=\frac{a}{2n_p}-\frac{a_i}{2n_{p_i}}.$$

Since

$$\begin{aligned}V_t^{a,p}&=\int_0^t\mu_s dQ_s+\frac{a}{2n_p}\int_0^t(\ell_s)^2\,ds\\&=V_t^{a_i,p_i}+b_i\int_0^t(\ell_s)^2\,ds,\end{aligned}$$

we get

$$\begin{aligned}\mathbb{E}\sup_{t\in[0,T]}\left|e^{V_t^{a,p}}Y_t\right|^p&=\mathbb{E}\left\{\sup_{t\in[0,T]}\left|e^{V_t^{a_i,p_i}}Y_t\right|^p\exp\left[pb_i\int_0^T(\ell_s)^2\,ds\right]\right\}\\&\le\left(\mathbb{E}\sup_{t\in[0,T]}\left|e^{V_t^{a_i,p_i}}Y_t\right|^{p_i}\right)^{\frac{p}{p_i}}\left\{\mathbb{E}\exp\left[\frac{p_i\,pb_i}{p_i-p}\int_0^T(\ell_s)^2\,ds\right]\right\}^{\frac{p_i-p}{p_i}}\\&<\infty.\end{aligned}$$

Similar we have

$$\mathbb{E}\sup_{t\in[0,T]}\left|e^{V_t^{a,p}}\hat{Y}_t\right|^p<\infty.$$

Hence the estimate (5.24) holds and the uniqueness follows. ■

The next Proposition will allow us to extend existence results from situations where the data satisfy the following strong boundedness condition: there exists a positive constant C such that for all $t \in [0,T]$:

$$|\eta| + |\Phi(t,0,0)| + \left|e^{\hat{V}_T}\eta\right| + \int_0^T e^{\hat{V}_s} |\Phi(s,0,0)|\, dQ_s \le C < \infty, \quad \mathbb{P}\text{-a.s.}$$

where

$$\hat{V}_t = \int_0^t \mu_s^+ dQ_s + \frac{a}{2n_p}\int_0^t (\ell_s)^2\, ds.$$

Let

$$V_t \overset{def}{=} V_t^{a,p} = \int_0^t \mu_s dQ_s + \frac{a}{2n_p}\int_0^t (\ell_s)^2\, ds \quad \text{and}$$
$$\beta_t \overset{def}{=} Q_t + \int_0^t |\mu_s|\, dQ_s + \int_0^t (\ell_s)^2\, ds + \int_0^t |\Phi(s,0,0)|\, dQ_s.$$

We have $V_s - V_t \le \hat{V}_s - \hat{V}_t$ for all $0 \le t \le s \le T$.

Define, for $n \in \mathbb{N}^*$,

$$\eta^n = \eta\, \mathbf{1}_{[0,n]}(\beta_T + |\eta|),$$
$$\Phi^n(t,y,z) = \Phi(t,y,z) - \Phi(t,0,0)\ \mathbf{1}_{[n,\infty[}(\beta_t + |\Phi(t,0,0)|),$$

and the stochastic processes

$$H_t^n = \left|e^{V_T - V_t}\eta^n\right| + \int_t^T e^{V_s - V_t} |\Phi^n(s,0,0)|\, dQ_s,$$
$$\hat{H}_t^n = \left|e^{\hat{V}_T - \hat{V}_t}\eta^n\right| + \int_t^T e^{\hat{V}_s - \hat{V}_t} |\Phi^n(s,0,0)|\, dQ_s.$$

It is easy to verify that there exists a positive constant $M_{n,p,a}$ such that

$$0 \le H_0^n \le \|H^n\|_T \le \left\|\hat{H}^n\right\|_T \le M_{n,T}, \ \ \mathbb{P}\text{-a.s.}$$

Proposition 5.14. *Let $a, p > 1$ and the assumptions (5.13-**BSDE-H**$_\Phi$) be satisfied. Also assume that*

$$\mathbb{E}e^{pV_T}|\eta|^p + \mathbb{E}\Big(\int_0^T e^{V_s}|\Phi(s,0,0)|\, dQ_s\Big)^p < \infty. \tag{5.27}$$

If for each $n \in \mathbb{N}^*$, $(Y^n, Z^n) \in S_m^0[0,T] \times \Lambda_{m\times k}^0(0,T)$ *is a solution of the BSDE*

$$Y_t^n = \eta^n + \int_t^T \Phi^n\left(s, Y_s^n, Z_s^n\right) dQ_s - \int_t^T Z_s^n dB_s$$

such that $e^V Y^n \in S_m^p[0,T]$, *then*

$$\|Y^n\|_T + \left\|e^V Y^n\right\|_T \le M'_{n,p,a}, \quad a.s., \tag{5.28}$$

and there exists (a unique!) $(Y,Z) \in S_m^p\left([0,T]; e^V\right) \times \Lambda_{m\times k}^p\left(0,T; e^V\right)$ *such that*

$$\lim_{n\to\infty}\left[\mathbb{E}\ \left\|e^V(Y^n - Y)\right\|_T^p + \mathbb{E}\ \left(\int_0^T e^{2V_s}\left|Z_s^n - Z_s\right|^2 ds\right)^{p/2}\right] = 0 \tag{5.29}$$

and, $\mathbb{P}$*-a.s. for all* $t \in [0,T]$*:*

$$Y_t = \eta + \int_t^T \Phi\left(s, Y_s, Z_s\right) dQ_s - \int_t^T Z_s dB_s. \tag{5.30}$$

Proof. In view of (5.19) we have for all $t \in [0,T]$:

$$e^{pV_t}\left|Y_t^n\right|^p + \left|Y_t^n\right|^p \le M'_{n,p,a}, \quad \mathbb{P}\text{-a.s.}$$

and (5.28) follows.

For all $n, i \in \mathbb{N}^*$:

$$Y_t^n - Y_t^{n+i} = \eta^n - \eta^{n+i} + \int_t^T d\left(K_s^n - K_s^{n+i}\right) - \int_t^T \left(Z_s^n - Z_s^{n+i}\right) dB_s,$$

where

$$K_t^n = \int_0^t \Phi^n\left(s, Y_s^n, Z_s^n\right) dQ_s$$

and similarly for K_t^{n+i}.

Since

$$\begin{aligned}\left\langle Y_s^n - Y_s^{n+i}, d\left(K_s^n - K_s^{n+i}\right)\right\rangle \le \left|Y_s^n - Y_s^{n+i}\right| \left|\Phi(s,0,0)\right| \mathbf{1}_{\beta_s + |\Phi(s,0,0)| \ge n} dQ_s \\ + \left|Y_s^n - Y_s^{n+i}\right|^2 dV_s + \frac{n_p}{2a}\left|Z_s^n - Z_s^{n+i}\right|^2 ds,\end{aligned}$$

we deduce from Proposition 5.2 that

$$\mathbb{E} \sup_{s\in[0,T]} e^{pV_s} \left|Y_s^n - Y_s^{n+i}\right|^p + \mathbb{E} \left(\int_0^T e^{2V_s} \left|Z_s^n - Z_s^{n+i}\right|^2 ds\right)^{p/2}$$

$$\leq C_{a,p}\mathbb{E}\left(e^{pV_T} |\eta|^p \mathbf{1}_{\beta_T+|\eta|\geq n}\right) + C_{a,p}\mathbb{E}\left(\int_0^T e^{V_s}\mathbf{1}_{\beta_s+|\Phi(s,0,0)|\geq n} |\Phi(s,0,0)|\, dQ_s\right)^p.$$

Hence there exists $(Y,Z) \in S_m^p\left([0,T];e^V\right) \times \Lambda_{m\times k}^p\left(0,T;e^V\right)$ such that (5.29) holds. The last assertion follows from Lemma 5.16 below, whose proof is left as an exercise for the reader. ■

Clearly from the construction in Proposition 5.14 we have:

Corollary 5.15. *Suppose that the assumptions from (5.13-**BSDE-H**$_\Phi$) are satisfied. Then the existence of a solution under the conditions (5.27) with $p = 2$ and some $a > 1$ implies existence under the same conditions for any $p > 1$.*

We end this subsection with a continuity result, the easy proof of which is left as an exercise for the reader.

Lemma 5.16. *Let the assumptions (5.13-**BSDE-H**$_\Phi$) be satisfied. If $(Y,Z) \in S_m^0[0,T] \times \Lambda_{m\times k}^0(0,T)$, then*

$$\int_0^T |\Phi(t,Y_t,Z_t)|\, dQ_t < \infty, \quad \mathbb{P}\text{-a.s.}$$

and the mapping

$$(U,V) \longrightarrow \int_0^\cdot \Phi(s,U_s,V_s)dQ_s : S_m^0[0,T] \times \Lambda_{m\times k}^0(0,T) \to S_m^0[0,T]$$

is continuous.

5.3.3 BSDEs with Lipschitz Coefficients

5.3.3.1 BSDEs with Deterministic Lipschitz Conditions

Consider the backward stochastic differential equation: $\mathbb{P}$-$a.s.$, for all $t \in [0,T]$

$$Y_t = \eta + \int_t^T F(s,Y_s,Z_s)\, ds - \int_t^T Z_s dB_s, \tag{5.31}$$

under the assumptions

◊ $p > 1$,

$$\eta \in L^p\left(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^m\right), \tag{5.32}$$

◊ *the function* $F(\cdot, \cdot, y, z) : \Omega \times [0, T] \to \mathbb{R}^m$ *is* $\mathcal{P}$*-measurable for every* $(y, z) \in \mathbb{R}^m \times \mathbb{R}^{m\times k}$,

◊ *there exist* $L \in L^1(0, T)$, $\ell \in L^2(0, T)$ *such that*

$$\begin{cases} (I) \quad \textit{Lipschitz conditions:} \\ \textit{for all } y, y' \in \mathbb{R}^m,\ z, z' \in \mathbb{R}^{m\times k},\ d\mathbb{P} \otimes dt\text{-a.e.:} \\ \qquad (L_y) \quad |F(t, y', z) - F(t, y, z)| \le L(t)\,|y' - y|\,, \\ \qquad (L_z) \quad |F(t, y, z') - F(t, y, z)| \le \ell(t)\,|z' - z|; \\ (II) \quad \textit{Boundedness condition:} \\ \qquad (B_F) \quad \mathbb{E}\left(\int_0^T |F(t, 0, 0)|\, dt\right)^p < \infty. \end{cases} \tag{5.33}$$

We recall the notation

$$S_m^{1+}[0, T] \overset{def}{=} \bigcup_{p>1} S_m^p[0, T]\,.$$

Theorem 5.17. *Let* $p > 1$ *and the assumptions (5.32) and (5.33) be satisfied. Then the BSDE (5.31) has a unique solution* $(Y, Z) \in S_m^p[0, T] \times \Lambda^p_{m\times k}(0, T)$. *Moreover uniqueness holds in* $S_m^{1+}[0, T] \times \Lambda^0_{m\times k}(0, T)$.

Proof. We first remark that if $(Y, Z) \in S_m^p[0, T] \times \Lambda^p_{m\times k}(0, T)$ then

$$K_\cdot \overset{def}{=} \int_0^\cdot F(r, Y_r, Z_r)\, dr \ \in\ S_m^p[0, T] \text{ and } \mathbb{E}\ \updownarrow K \updownarrow_T^p < \infty.$$

Indeed, since

$$|F(r, Y_r, Z_r)| \le |F(r, 0, 0)| + L(r)\,|Y_r| + \ell(r)\,|Z_r|\,,$$

then

$$\begin{aligned} \mathbb{E} \sup_{t\in[0,T]} |K_t|^p &\le \mathbb{E} \updownarrow K \updownarrow_T^p \\ &= \mathbb{E}\left(\int_0^T |F(r, Y_r, Z_r)|\, dr\right)^p \\ &\le C_p \mathbb{E}\left(\int_0^T |F(r, 0, 0)|\, dr\right)^p + C_p \left(\int_0^T L(r)\, dr\right)^p \mathbb{E}\ \|Y\|_T^p \end{aligned}$$

$$+ C_p \left(\int_0^T \ell^2(r)\, dr \right)^{p/2} \mathbb{E} \left(\int_0^T |Z_r|^2\, dr \right)^{p/2}$$
$$< \infty.$$

Uniqueness follows from Corollary 5.13.

We prove existence.

Note that a solution of the Eq. (5.31) is a fixed point of the mapping $\Gamma : S_m^p[0,T] \times \Lambda_{m\times k}^p(0,T) \to S_m^p[0,T] \times \Lambda_{m\times k}^p(0,T)$ defined by

$$(Y, Z) = \Gamma(X, U),$$

where

$$Y_t = \eta + \int_t^T F(r, X_r, U_r)\, dr - \int_t^T Z_r dB_r,\ a.s.\ \ t \in [0,T].$$

By Corollary 2.45 the mapping Γ is well defined.

Let $M \in \mathbb{N}^*$ and $0 = T_0 < T_1 < \cdots < T_M = T$, with $T_i = \dfrac{iT}{M}$. Then

$$\boldsymbol{\alpha}\left(\tfrac{T}{M}\right) \overset{def}{=} \sup_{0<s-t<\frac{T}{M}} \int_t^s \left[L(r) + \ell^2(r)\right] dr \to 0, \quad \text{as } M \to \infty.$$

We show that Γ is a strict contraction on the Banach space $S_m^p[T_{M-1}, T] \times \Lambda_{m\times k}^p(T_{M-1}, T)$ with the norm

$$|||(X,U)|||_M = \left[\mathbb{E} \sup_{r\in[T_{M-1},T]} |X_r|^p + \mathbb{E}\left(\int_{T_{M-1}}^T |U_r|^2\, dr \right)^{p/2} \right]^{1/p}$$

for M large enough.

Let $(X,U), (X',U') \in S_m^p[T_{M-1},T] \times \Lambda_{m\times k}^p(T_{M-1},T)$. Then

$$Y_t - Y_t' = \int_t^T dK_r - \int_t^T \left(Z_r - Z_r'\right) dB_r, \quad t \in [0,T],$$

where

$$K_t = \int_0^t \left[F(r, X_r, U_r) - F\left(r, X_r', U_r'\right)\right] dr.$$

Since

$$\begin{aligned}\langle Y_r - Y_r', dK_r\rangle &\le \left|F(r, X_r, U_r) - F\left(r, X_r', U_r'\right)\right| |Y_r - Y_r'|\, dr \\ &\le \left[L(r)\,|X_r - X_r'| + \ell(r)\,|U_r - U_r'|\right] |Y_r - Y_r'|\, dr\end{aligned}$$

and

$$\mathbb{E}\left(\sup_{r\in[T_{M-1},T]}\left|Y_r - Y_r'\right|^p\right) < \infty,$$

we have by (5.8), with $D = 0$, $R = V = 0$, $\lambda = 0$,

$$\begin{aligned}
&\left\|\left|(Y,Z) - \left(Y',Z'\right)\right\|\right|_M^p \\
&\quad = \mathbb{E}\left(\sup_{r\in[T_{M-1},T]}\left|Y_r - Y_r'\right|^p\right) + \mathbb{E}\left(\int_{T_{M-1}}^T \left|Z_r - Z_r'\right|^2 dr\right)^{p/2} \\
&\quad \le C_p \mathbb{E}\left(\int_{T_{M-1}}^T \left[L(r)\left|X_r - X_r'\right| + \ell(r)\left|U_r - U_r'\right|\right] dr\right)^p \\
&\quad \le C_p'\left(\int_{T_{M-1}}^T L(r)\,dr\right)^p \mathbb{E}\sup_{r\in[T_{M-1},T]}\left|X_r - X_r'\right|^p \\
&\qquad + C_p'\left(\int_{T_{M-1}}^T \ell^2(r)\,dr\right)^{p/2} \mathbb{E}\left(\int_{T_{M-1}}^T \left|U_r - U_r'\right|^2 dr\right)^{p/2} \\
&\quad \le C_p'\left[\boldsymbol{\alpha}^p\left(\tfrac{T}{M}\right) + \boldsymbol{\alpha}^{p/2}\left(\tfrac{T}{M}\right)\right]\left\|\left|(X,U) - \left(X',U'\right)\right\|\right|_M^p .
\end{aligned}$$

Let $M_0 \in \mathbb{N}^*$ be such that

$$C_p'\left[\boldsymbol{\alpha}^p\left(\tfrac{T}{M_0}\right) + \boldsymbol{\alpha}^{p/2}\left(\tfrac{T}{M_0}\right)\right] \le \frac{1}{2^p}.$$

Then Γ is a strict contraction on $S_m^p\,[T_{M_0-1},T]\times\Lambda_{m\times k}^p\,(T_{M_0-1},T)$ and consequently the Eq. (5.31) has a unique solution $(Y,Z) \in S_m^p\,[T_{M_0-1},T]\times\Lambda_{m\times k}^p\,(T_{M_0-1},T)$. The next step is to solve the equation on the interval $[T_{M_0-2},T_{M_0-1}]$ with the final value $Y\,(T_{M_0-1})$. Repeating the same arguments, the proof is completed in M_0 steps. ■

Corollary 5.18. *Consider the BSDE:* $\forall\ t \in [0,T]\,,\ \ \mathbb{P}$-*a.s.*

$$Y_t = \eta + S_T - S_t + \int_t^T F\,(s,Y_s,Z_s)\,ds - \int_t^T Z_s dB_s. \tag{5.34}$$

If $p > 1$*,* $S \in S_m^p\,[0,T]$*,* $\eta \in L^p\,(\Omega,\mathcal{F}_T,\mathbb{P};\mathbb{R}^m)$ *and* F *satisfies the assumptions (5.33), then the Eq. (5.34) has a unique solution* $(Y,Z) \in S_m^p\,[0,T] \times \Lambda_{m\times k}^p\,(0,T)$.

Proof. By the substitutions $\hat{Y}_t = Y_t + S_t$, $\hat{\eta} = \eta + S_T$ and $\hat{F}\,(t,y,z) = F\,(t,y-S_t,z)$ the Eq. (5.34) is transformed into

$$\hat{Y}_t = \hat{\eta} + \int_t^T \hat{F}\left(s, \hat{Y}_s, Z_s\right) ds - \int_t^T Z_s dB_s,$$

which satisfies the assumptions of Theorem 5.17. ■

We now study the case $p = 1$, where we restrict ourselves to the case where F does not depend on z.

Corollary 5.19. *If $S \in S_d^1[0,T]$, $\eta \in L^1(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^m)$ and $F(t,y,z) \equiv F(t,y)$ satisfies the assumptions (5.33) with $p = 1$, then the BSDE*

$$Y_t = \eta + S_T - S_t + \int_t^T F(s, Y_s)\, ds - \int_t^T Z_s dB_s \tag{5.35}$$

has a unique solution $(Y,Z) \in S_m^0[0,T] \times \Lambda_{m\times k}^0(0,T)$ such that

$$M_t = \int_0^t Z_s dB_s \quad \text{is a martingale}$$

and

$$\sup_{t\in[0,T]} \mathbb{E}|Y_t| + \mathbb{E} \sup_{t\in[0,T]} |Y_t|^q + \mathbb{E}\left(\int_0^T |Z_t|^2\, dt\right)^{q/2} < \infty, \ \forall\, 0 < q < 1.$$

Proof. As in the proof of Corollary 5.18 we can reduce the problem to the case $S = 0$.

Let $n, i \in \mathbb{N}^*$. By Theorem 5.17 there exists a unique pair (Y^n, Z^n) such that for all $p \geq 1$

$$(Y^n, Z^n) \in S_m^p[0,T] \times \Lambda_{m\times k}^p(0,T) \tag{5.36}$$

and (Y^n, Z^n) is solution of the equation

$$Y_t^n = \eta \mathbf{1}_{|\eta|\leq n} + \int_t^T \left[F\left(r, Y_r^n\right) - \mathbf{1}_{|F(r,0)|\geq n} F(r,0)\right] dr - \int_t^T Z_r^n dB_r. \tag{5.37}$$

Note that

$$\beta_n \overset{def}{=} \mathbb{E}|\eta|\, \mathbf{1}_{|\eta|>n} + \mathbb{E}\int_0^T |F(s,0)|\, \mathbf{1}_{|F(r,0)|\geq n} ds \to 0 \quad \text{as } n \to \infty.$$

By (5.4) for $Y_s^n - Y_s^{n+i}$ and $p = 1$ we infer

$$\begin{aligned} \left|Y_t^n - Y_t^{n+i}\right| &\leq \beta_n + \int_t^T L(s)\left|Y_s^n - Y_s^{n+i}\right| ds \\ &\quad - \int_t^T \left|Y_s^n - Y_s^{n+i}\right|^{-1} \mathbf{1}_{Y_s^n - Y_s^{n+i} \neq 0} \left\langle Y_s^n - Y_s^{n+i}, \left(Z_s^n - Z_s^{n+i}\right) dB_s\right\rangle. \end{aligned} \tag{5.38}$$

Denote by C, C' generic constants independent of n and i.

From (5.36)

$$M_t^{n,n+i} = \int_0^t \left|Y_s^n - Y_s^{n+i}\right|^{-1} \mathbf{1}_{Y_s^n - Y_s^{n+i} \neq 0} \left\langle Y_s^n - Y_s^{n+i}, \left(Z_s^n - Z_s^{n+i}\right) dB_s \right\rangle$$

is a martingale. Then $\mathbb{E}M_t^{n,n+i} = 0$ and, taking the expectation in (5.38) we deduce from the backward Gronwall inequality (Corollary 6.62):

$$\mathbb{E}\left|Y_t^n - Y_t^{n+i}\right| \leq C\ \beta_n. \tag{5.39}$$

Using the Burkholder–Davis–Gundy inequality (1.18) and Doob's inequality (1.11-A_3), we deduce for $0 < q < 1$:

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^T \left|Z_s^n - Z_s^{n+i}\right|^2 ds\right)^{q/2}\right] \\
&\leq \frac{1}{c_q}\mathbb{E}\left(\sup_{0\leq t\leq T}\left|\int_0^t \left(Z_s^n - Z_s^{n+i}\right) dB_s\right|^q\right) \\
&\leq \frac{1}{c_q\,(1-q)}\left[\mathbb{E}\left|\int_0^T \left(Z_s^n - Z_s^{n+i}\right) dB_s\right|\right]^q \\
&\leq \frac{1}{c_q\,(1-q)}\left[\mathbb{E}\left|Y_0^n - Y_0^{n+i}\right| + \mathbb{E}\int_0^T \left|F\left(s, Y_s^n\right) - F\left(s, Y_s^{n+i}\right)\right|\right]^q \\
&\leq \frac{1}{c_q\,(1-q)}\left[\mathbb{E}\left|Y_0^n - Y_0^{n+i}\right| + L\left(s\right)\int_0^T \mathbb{E}\left|Y_s^n - Y_s^{n+i}\right| ds\right]^q \\
&\leq C\ \beta_n^q.
\end{aligned}$$

Recalling that $M_t^{n,n+i}$ is a martingale then, once again by the Burkholder–Davis–Gundy inequality

$$\begin{aligned}
\mathbb{E}\sup_{t\in[0,T]}\left|M_T^{n,n+i} - M_t^{n,n+i}\right|^q &\leq 2^q\mathbb{E}\sup_{t\in[0,T]}\left|M_t^{n,n+i}\right|^q \\
&\leq C_q\mathbb{E}\left[\left(\int_0^T \left|Z_s^n - Z_s^{n+i}\right|^2 ds\right)^{q/2}\right] \\
&\leq C'\ \beta_n^q,
\end{aligned}$$

then from (5.38) and (5.39) we obtain for every $0 < q < 1$

$$\begin{aligned}\mathbb{E}\sup_{t\in[0,T]}\left|Y_t^n - Y_t^{n+i}\right|^q &\leq \beta_n^q + \mathbb{E}\left(\int_0^T L(s)\left|Y_s^n - Y_s^{n+i}\right| ds\right)^q + C'\,\beta_n^q \\ &\leq \beta_n^q + \left(\int_0^T L(s)\,\mathbb{E}\left|Y_s^n - Y_s^{n+i}\right| ds\right)^q + C'\,\beta_n^q \\ &\leq C\,\beta_n^q.\end{aligned}$$

Hence, there exist Y and Z such that

$$\begin{aligned}&Y^n \to Y, \quad \text{in } S_m^q[0,T] \cap C\left([0,T]; L^1(\Omega,\mathcal{F},\mathbb{P})\right), \text{ and} \\ &Z^n \to Z, \quad \text{in } \Lambda_{m\times k}^q(0,T).\end{aligned}$$

Passing to the limit in (5.37) we deduce that the pair (Y,Z) solves the problem.

Now, by Corollary 2.47, $M_t = \int_0^t Z_s dB_s$ is a martingale, because

$$S_{\cdot} + \int_0^{\cdot} F(s,Y_s)\,ds \in S_m^1[0,T].$$

Finally, the uniqueness is obtained in the same manner as the estimates for $Y^n - Y^{n+i}$ and $Z^n - Z^{n+i}$. ■

5.3.3.2 BSDEs with Random Lipschitz Conditions

We now generalize Theorem 5.17 to a class of BSDEs with random Lipschitz constants.

We consider the BSDE (5.31) in a more general form:

$$Y_t = \eta + \int_t^T \Phi(s,Y_s,Z_s)\,dQ_s - \int_t^T Z_s dB_s,\ t\in[0,T],\ a.s. \tag{5.40}$$

We assume that

(BSDE-A0) :

(i) $\eta : \Omega \to \mathbb{R}^m$ *is an* $\mathcal{F}_T$*-measurable random vector,*
(ii) Q is a $\mathcal{P}$*-m.i.c.s.p. such that* $Q_0 = 0$;

and the function $\Phi : \Omega\times[0,T]\times\mathbb{R}^m \to \mathbb{R}^m$ satisfies

(BSDE-LH$_\Phi$) ▲ *for all* $y\in\mathbb{R}^m$, $z\in\mathbb{R}^{m\times k}$, *the function* $\Phi(\cdot,\cdot,y,z) : \Omega\times[0,T]\to\mathbb{R}^m$ *is* $\mathcal{P}$*-measurable*;

▲ *there exist $\mathcal{P}$-m.s.p. $L, \ell, \alpha : \Omega \times [0, T] \to \mathbb{R}_+$, such that*

$$\alpha_t dQ_t = dt \qquad \textit{and} \qquad \int_0^T \left(L_t dQ_t + (\ell_t)^2 \, dt \right) < \infty, \quad \mathbb{P}\text{-a.s.};$$

for all $t \in [0, T]$, $y, y' \in \mathbb{R}^m$ and *$z, z' \in \mathbb{R}^{m \times k}$, $\mathbb{P}$-a.s.:*

$$\begin{array}{ll} & \textit{Lipschitz conditions} \\ (i) & |\Phi(t, y', z) - \Phi(t, y, z)| \le L_t |y' - y|, \\ (ii) & |\Phi(t, y, z') - \Phi(t, y, z)| \le \alpha_t \ell_t |z' - z|, \\ & \textit{Boundedness condition:} \\ (iii) & \displaystyle\int_0^T \Phi_\rho^{\#}(t) \, dQ_t < \infty, \quad \forall \, \rho \ge 0, \end{array} \tag{5.41}$$

where

$$\Phi_\rho^{\#}(t) \overset{def}{=} \sup_{|y| \le \rho} |\Phi(t, y, 0)| \, .$$

Note that the condition $\alpha_t dQ_t = dt$ implies that $\Phi(t, Y_t, Z_t) \, dQ_t = F(t, Y_t, Z_t) \, dt + G(t, Y_t) \, dA_t$, where G does not depend upon z.

We recall the following notations. For each fixed $p > 1$ let $n_p = 1 \wedge (p - 1)$ and

$$V_t = V_t^{(p)} = \int_0^t L_s dQ_s + \frac{1}{n_p} \int_0^t (\ell_s)^2 \, ds. \tag{5.42}$$

The stochastic process V is that from Lemma 5.6 with $a = 2$. Therefore for all (Y, Z), $\left(\tilde{Y}, \tilde{Z}\right) \in S_m^0 [0, T] \times \Lambda_{m \times k}^0 (0, T)$ we have

$$\langle Y_t, \Phi(t, Y_t, Z_t) \, dQ_t \rangle \le |Y_t| \, |\Phi(t, 0, 0)| \, dQ_t + |Y_t|^2 dV_t + \frac{n_p}{4} \, |Z_t|^2 \, dt, \tag{5.43}$$

and

$$\left\langle Y_t - \tilde{Y}_t, \Phi(t, Y_t, Z_t) - \Phi(t, \tilde{Y}_t, \tilde{Z}_t) \right\rangle dQ_t \le |Y_t - \tilde{Y}_t|^2 dV_t + \frac{n_p}{4} \left| Z_t - \tilde{Z}_t \right|^2 dt. \tag{5.44}$$

Lemma 5.20. *Let $p \ge 2$ and the assumptions* (**BSDE-A0**)*,* (**BSDE-LH$_\Phi$**) *be satisfied. If moreover there exists a constant $b > 0$ such that*

$$\begin{array}{ll} (i) & \displaystyle\int_0^T L_s dQ_s \le b \quad \textit{and} \quad \int_0^T (\ell_s)^2 \, ds \le b, \quad \mathbb{P}\text{-a.s.}, \\ (ii) & \mathbb{E} \, |\eta|^p + \mathbb{E} \left(\displaystyle\int_0^T |\Phi(s, 0, 0)| \, dQ_s \right)^p < \infty, \end{array} \tag{5.45}$$

then the BSDE (5.40) has a unique solution $(Y,Z) \in S_m^p[0,T] \times \Lambda_{m\times k}^p(0,T)$.

Proof. We have

$$V_t = \int_0^t \left(L_s dQ_s + \frac{1}{n_p}(\ell_s)^2 ds \right) = \int_0^t \left(L_s dQ_s + (\ell_s)^2 ds \right).$$

Since

$$0 \le V_t \le 2b, \quad \text{for all } t \in [0,T],$$

it follows that for every $\delta > 0$ we can define on $S_m^p[0,T] \times \Lambda_{m\times k}^p(0,T)$ an equivalent norm by

$$\|(Y,Z)\|_{\delta V} \stackrel{def}{=} \Bigg[\mathbb{E} \sup_{s\in[0,T]} e^{\delta p V_s} |Y_s|^p + \mathbb{E}\left(\int_0^T e^{2\delta V_s} |Y_s|^2 L_s dQ_s \right)^{p/2} + \mathbb{E}\left(\int_0^T e^{2\delta V_s} |Z_s|^2 ds \right)^{p/2} \Bigg]^{1/p}.$$

Let $\Gamma : S_m^p[0,T] \times \Lambda_{m\times k}^p(0,T) \to S_m^p[0,T] \times \Lambda_{m\times k}^p(0,T)$ be defined by

$$(Y,Z) = \Gamma(X,U)$$
$$Y_t = \eta + \int_t^T \Phi(s,X_s,U_s)\, dQ_s - \int_t^T Z_s dB_s.$$

We remark that for all $X, U \in S_m^p[0,T] \times \Lambda_{m\times k}^p(0,T)$,

$$\begin{aligned}\int_0^t |\Phi(s,X_s,U_s)|\, dQ_s &\le \int_0^t |\Phi(s,0,0)|\, dQ_s + \int_0^t |X_s|\, L_s dQ_s + \int_0^t |U_s|\, \ell_s ds \\ &\le \int_0^t |\Phi(s,0,0)|\, dQ_s + b \sup_{s\in[0,t]} |X_s| + b\left(\int_0^t |U_s|^2 ds \right)^{1/2}\end{aligned}$$

and consequently $S_\cdot = \int_0^\cdot \Phi(s,X,U)\, dQ_s \in S_m^p[0,T]$. By the martingale representation result from Corollary 2.45 it follows that Γ is well defined.

The fact that BSDE (5.40) has a unique solution $(Y,Z) \in S_m^p[0,T] \times \Lambda_{m\times k}^p(0,T)$ will be a consequence of the fact that Γ is a strict contraction on the Banach space $\left(S_m^p[0,T] \times \Lambda_{m\times k}^p(0,T), \|\cdot\|_\delta\right)$, for some $\delta > 0$.

Let $(Y,Z) = \Gamma(X,U)$ and $(Y',Z') = \Gamma(X',U')$. We have

$$Y_t - Y_t' = \int_t^T dK_s - \int_t^T \left(Z_s - Z_s'\right) dB_s,$$

where $K_t = \int_0^t \left[\Phi\left(s, X_s, U_s\right) - \Phi\left(s, X'_s, U'_s\right)\right] dQ_s$ and for all $\delta > 1$

$$\begin{aligned}
&\left|Y_s - Y'_s\right|^2 L_s dQ_s + \left\langle Y_s - Y'_s, dK_s\right\rangle \\
&\quad\le \left|Y_s - Y'_s\right|^2 L_s dQ_s + \left|Y_s - Y'_s\right|\left[\left|X_s - X'_s\right| L_s dQ_s + \left|U_s - U'_s\right| \ell_s ds\right] \\
&\quad\le \left|Y_s - Y'_s\right|^2 L_s dQ_s + \left[\frac{1}{4\left(\delta-1\right)}\left|X_s - X'_s\right|^2 + \left(\delta - 1\right)\left|Y_s - Y'_s\right|^2\right] L_s dQ_s \\
&\qquad + \left(\frac{1}{4\delta}\left|U_s - U'_s\right|^2 + \delta\left|Y_s - Y'_s\right|^2 \ell_s^2\right) ds \\
&\quad\le \frac{1}{4\delta}\left|U_s - U'_s\right|^2 ds + \frac{1}{4\left(\delta-1\right)}\left|X_s - X'_s\right|^2 L_s dQ_s + \left|Y_s - Y'_s\right|^2 \delta dV_s.
\end{aligned}$$

Then by Proposition 5.2-A,

$$\begin{aligned}
&\mathbb{E}\left(\sup_{s\in[0,T]} e^{p\delta V_s}\left|Y_s - Y'_s\right|^p\right) + \mathbb{E}\left(\int_0^T e^{2\delta V_s}\left|Y_s - Y'_s\right|^2 L_s dQ_s\right)^{p/2} \\
&\qquad\qquad + \mathbb{E}\left(\int_0^T e^{2\delta V_s}\left|Z_s - Z'_s\right|^2 ds\right)^{p/2} \\
&\le \frac{C_p}{\delta^{p/2}}\mathbb{E}\left(\int_0^T e^{2\delta V_s}\left|U_s - U'_s\right|^2 ds\right)^{p/2} \\
&\qquad\qquad + \frac{C_p}{\left(\delta-1\right)^{p/2}}\mathbb{E}\left(\int_0^T e^{2\delta V_s}\left|X_s - X'_s\right|^2 L_s dQ_s\right)^{p/2} \\
&\le \frac{C_p}{\left(\delta-1\right)^{p/2}}\left\|\left(X, U\right) - \left(X', U'\right)\right\|_{\delta V}^p \\
&\le \frac{1}{2^p}\left\|\left(X, U\right) - \left(X', U'\right)\right\|_{\delta V}^p
\end{aligned}$$

for $\delta \ge 1 + 4C_p^{2/p}$. Hence

$$\left\|\Gamma\left(X, U\right) - \Gamma\left(X', U'\right)\right\|_{\delta V} \le \frac{1}{2}\left\|\left(X, U\right) - \left(X', U'\right)\right\|_{\delta V}$$

and the result follows. ∎

Theorem 5.21. *Let* $p > 1$, $n_p = 1 \wedge (p-1)$ *and the assumptions* (**BSDE-A0**), (**BSDE-LH**$_\Phi$) *be satisfied. Let*

$$V_t = V_t^{(p)} \overset{def}{=} \int_0^t \left(L_s dQ_s + \frac{1}{n_p}\left(\ell_s\right)^2 ds\right).$$

Assume also that there exists $\delta > \frac{p}{p-1}$ such that for $q = \frac{p\delta}{p+\delta}$ and $n_q = 1 \wedge (q-1)$,

$$\begin{aligned}
&(i) \quad \mathbb{E}\, e^{pV_T} |\eta|^p + \mathbb{E}\left(\int_0^T e^{V_s} |\Phi(s,0,0)|\, dQ_s\right)^p < \infty, \\
&(ii) \quad \mathbb{E}\left(\int_0^T L_s dQ_s\right)^\delta + \mathbb{E}\left(\int_0^T (\ell_s)^2\, ds\right)^{\delta/2} < \infty, \\
&(iii) \quad \mathbb{E}\, \exp\left[\delta\left(\frac{1}{n_q} - \frac{1}{n_p}\right)\int_0^T (\ell_s)^2\, ds\right] < \infty.
\end{aligned} \tag{5.46}$$

Then the BSDE (5.40) has a unique solution $(Y, Z) \in S_m^0 \times \Lambda_{m\times k}^0$ such that

$$\mathbb{E}\left(\sup_{t\in[0,T]} e^{pV_t} |Y_t|^p\right) < \infty. \tag{5.47}$$

Moreover there exists a positive constant C_p depending only on p such that for all $t \in [0,T]$

$$\begin{aligned}
&\mathbb{E}^{\mathcal{F}_t} \sup_{s\in[t,T]} \left|e^{V_s} Y_s\right|^p + \mathbb{E}^{\mathcal{F}_t}\left(\int_t^T e^{2V_s} |Z_s|^2\, ds\right)^{p/2} \\
&\quad \le C_p\, \mathbb{E}^{\mathcal{F}_t}\left[\left|e^{V_T}\eta\right|^p + \left(\int_t^T e^{V_s} |\Phi(s,0,0)|\, dQ_s\right)^p\right].
\end{aligned} \tag{5.48}$$

Remark 5.22. We remark that $q = \frac{p\delta}{p+\delta}$ defined in Theorem 5.21 satisfies $1 < q < p$. If $q \ge 2$ then $n_q = n_p = 1$ and the condition (5.46-*iii*) is clearly satisfied.

Proof of Theorem 5.21. Uniqueness follows from Theorem 5.10.

Existence. *Let $t \in [0,T]$* and

$$\begin{aligned}
\beta_t &= t + Q_t + \int_0^t L_s dQ_s + \int_0^t (\ell_s)^2\, ds + \int_0^t |\Phi(s,0,0)|\, dQ_s, \\
\gamma_t &= \beta_t + |\Phi(t,0,0)| + L_t + \ell_t.
\end{aligned}$$

Define, for $n \in \mathbb{N}^*$,

$$\begin{aligned}
L_t^n &= L_t \mathbf{1}_{[0,n]}(\gamma_t), \\
\ell_t^n &= \ell_t \mathbf{1}_{[0,n]}(\gamma_t), \\
\eta_n &= \eta \mathbf{1}_{[0,n]}(\beta_T + |\eta|), \\
\Phi_n(t,y,z) &= \Phi\big(t, y\mathbf{1}_{[0,n]}(\gamma_t), z\mathbf{1}_{[0,n]}(\gamma_t)\big) - \Phi(t,0,0)\ \mathbf{1}_{(n,\infty)}(\gamma_t).
\end{aligned}$$

By Lemma 5.20 we infer that the approximating BSDE

$$Y_t^n = \eta_n + \int_t^T \Phi_n\left(s, Y_s^n, Z_s^n\right) dQ_s - \int_t^T Z_s^n dB_s \tag{5.49}$$

has a unique solution $(Y^n, Z^n) \in S_m^q[0,T] \times \Lambda_{m\times k}^q(0,T)$, for all $q \geq 2$.

Let

$$V_t^n \overset{def}{=} \int_0^t \left(L_s^n dQ_s + \frac{1}{n_p}\left(\ell_s^n\right)^2 ds \right).$$

We have for all $n, i \in \mathbb{N}$

$$0 \leq V_t^n \leq V_t^{n+i} \leq (n+i)^2 + \frac{1}{n_p}(n+i)^3.$$

Therefore

$$\mathbb{E} \sup_{t\in[0,T]} e^{pV_t^{n+i}} |Y_t^n|^p \leq C_{n,i,p} \left(\mathbb{E} \sup_{t\in[0,T]} |Y_t^n|^{2p} \right)^{1/2} < \infty,$$

and since

$$\begin{aligned}
&\left\langle Y_t^n, \Phi_n\left(t, Y_t^n, Z_t^n\right) dQ_t \right\rangle \\
&\leq |Y_t^n| \left|\Phi(t,0,0)\right| \mathbf{1}_{[0,n]}(\gamma_t)\, dQ_t + |Y_t^n|^2 dV_t^n + \frac{n_p}{4}|Z_t^n|^2 dt \\
&\leq |Y_t^n| \left|\Phi(t,0,0)\right| \mathbf{1}_{[0,n]}(\gamma_t)\, dQ_t + |Y_t^n|^2 dV_t^{n+i} + \frac{n_p}{4}|Z_t^n|^2 dt,
\end{aligned}$$

we obtain, by Proposition 5.2-A, that

$$\begin{aligned}
&\mathbb{E}^{\mathcal{F}_t} \sup_{s\in[t,T]} e^{pV_s^{n+i}} |Y_s^n|^p + \mathbb{E}^{\mathcal{F}_t}\left(\int_t^T e^{2V_s^{n+i}} |Z_s^n|^2 ds \right)^{p/2} \\
&\leq C_q\, \mathbb{E}^{\mathcal{F}_t}\left[e^{pV_T^{n+i}} |\eta_n|^p + \left(\int_t^T e^{V_s^{n+i}} \left|\Phi_n(s,0,0)\right| dQ_s \right)^p \right] \\
&\leq C_q\, \mathbb{E}^{\mathcal{F}_t}\left[e^{pV_T^{n+i}} |\eta|^p + \left(\int_t^T e^{V_s^{n+i}} \left|\Phi(s,0,0)\right| dQ_s \right)^p \right].
\end{aligned}$$

By Beppo Levi's monotone convergence Theorem 1.9 it follows by letting $i \to \infty$ that for all $t \in [0,T]$,

$$\begin{aligned}
&\mathbb{E}^{\mathcal{F}_t} \sup_{s\in[t,T]} e^{pV_s} |Y_s^n|^p + \mathbb{E}^{\mathcal{F}_t}\left(\int_t^T e^{2V_s} |Z_s^n|^2 ds \right)^{p/2} \\
&\leq C_p\, \mathbb{E}^{\mathcal{F}_t}\left[e^{pV_T} |\eta|^p + \left(\int_t^T e^{V_s} \left|\Phi(s,0,0)\right| dQ_s \right)^p \right].
\end{aligned} \tag{5.50}$$

Consequently by (5.46−i) for all $n \in \mathbb{N}^*$,

$$\mathbb{E} \sup_{s\in[0,T]} e^{pV_s} |Y_s^n|^p + \mathbb{E} \left(\int_0^T e^{2V_s} |Z_s^n|^2 ds \right)^{p/2} \leq C < \infty.$$

Let $\delta > \frac{p}{p-1}$, $q = \frac{p\delta}{p+\delta} \in (1,p)$, $n_q = 1 \wedge (q-1)$ and $n_p = 1 \wedge (p-1)$ satisfy (5.46−ii, iii). If we define

$$\Delta_t = \left(\frac{1}{n_q} - \frac{1}{n_p} \right) \int_0^t (\ell_s)^2 ds \quad \text{and}$$

$$V_t^{(q)} = \int_0^t \left[L_s dQ_s + \frac{1}{n_q} (\ell_s)^2 ds \right] = V_t + \Delta_t,$$

we have for all $n \in \mathbb{N}^*$

$$\begin{aligned}
&\mathbb{E} \sup_{s\in[0,T]} e^{qV_s^{(q)}} |Y_s^n|^q \\
&\quad \leq \mathbb{E}\Big[e^{q\Delta_T} \sup_{s\in[0,T]} e^{qV_s} |Y_s^n|^q \Big] \\
&\quad \leq \Big[\mathbb{E} e^{\frac{pq}{p-q}\Delta_T} \Big]^{(p-q)/p} \Big[\mathbb{E} \sup_{s\in[0,T]} e^{pV_s} |Y_s^n|^p \Big]^{q/p} \\
&\quad = \Big[\mathbb{E} \exp(\delta \Delta_T) \Big]^{(p-q)/p} \left(\mathbb{E} \sup_{s\in[0,T]} e^{pV_s} |Y_s^n|^p \right)^{q/p} \\
&\quad < \infty.
\end{aligned}$$

Hence for all $n, i \in \mathbb{N}^*$

$$\mathbb{E} \sup_{s\in[0,T]} e^{qV_s^{(q)}} \left| Y_s^n - Y_s^{n+i} \right|^q < \infty.$$

Since

$$\begin{aligned}
&\left\langle Y_s^n - Y_s^{n+i}, \Phi_n\left(t, Y_s^n, Z_s^n\right) - \Phi_{n+i}\left(s, Y_s^{n+i}, Z_s^{n+i}\right) \right\rangle dQ_s \\
&\leq \Big\langle Y_s^n - Y_s^{n+i}, \Phi\left(s, Y_s^n \mathbf{1}_{[0,n]}(\gamma_s), Z_s^n \mathbf{1}_{[0,n]}(\gamma_s)\right) \\
&\qquad - \Phi\left(s, Y_s^{n+i} \mathbf{1}_{[0,n+i]}(\gamma_s), Z_s^{n+i} \mathbf{1}_{[0,n+i]}(\gamma_s)\right) \Big\rangle dQ_s \\
&\qquad - \left\langle Y_s^n - Y_s^{n+i}, \Phi(t,0,0) \right\rangle \left[\mathbf{1}_{(n,\infty)}(\gamma_s) - \mathbf{1}_{(n+i,\infty)}(\gamma_s) \right] dQ_s
\end{aligned}$$

$$\leq \left|Y_s^n - Y_s^{n+i}\right| \Big[|\Phi(t,0,0)| \ \mathbf{1}_{(n,\infty)}(\gamma_s)\, dQ_s$$

$$+ \left(L_s\, |Y_s^n|\, dQ_s + \ell_s\, |Z_s^n|\, ds\right) \left|\mathbf{1}_{[0,n]}(\gamma_s) - \mathbf{1}_{[0,n+i]}(\gamma_s)\right| \Big]$$

$$+ \left|Y_s^n - Y_s^{n+i}\right|^2 \left(L_s dQ_s + \frac{1}{n_q}\ell_s^2 ds\right) + \frac{n_q}{4}\left|Z_s^n - Z_s^{n+i}\right|^2 ds,$$

by Proposition 5.2-A, we infer that

$$\mathbb{E} \sup_{s\in[0,T]} e^{qV_s^{(q)}} \left|Y_s^n - Y_s^{n+i}\right|^q + \mathbb{E}\left(\int_0^T e^{2V_s^{(q)}} \left|Z_s^n - Z_s^{n+i}\right|^2 ds\right)^{q/2}$$

$$\leq C_q\, \mathbb{E}\, e^{qV_T^{(q)}}\, |\eta_n - \eta_{n+i}|^q$$

$$+ C_q\, \mathbb{E}\left[\int_0^T \mathbf{1}_{(n,\infty)}(\gamma_s)\, e^{V_s^{(q)}} \left(|\Phi(t,0,0)|\, dQ_s + L_s\, |Y_s^n|\, dQ_s + \ell_s\, |Z_s^n|\, ds\right)\right]^q$$

$$\leq C_q\, \mathbb{E}\left[e^{qV_T^{(q)}}\, |\eta|^q\, \mathbf{1}_{(n,\infty)}(\beta_T + |\eta|)\right]$$

$$+ C_q'\, \mathbb{E}\left(\int_0^T e^{V_s^{(q)}}\, |\Phi(t,0,0)|\ \mathbf{1}_{(n,\infty)}(\gamma_s)\, dQ_s\right)^q$$

$$+ C_q'\, \mathbb{E}\left[\left(\int_0^T L_s \mathbf{1}_{(n,\infty)}(\gamma_s)\, ds\right)^q \sup_{s\in[0,T]} e^{qV_s^{(q)}}\, |Y_s^n|^q\right]$$

$$+ C_q' \mathbb{E}\left[\left(\int_0^T \ell_s^2 \mathbf{1}_{(n,\infty)}(\gamma_s)\, ds\right)^{q/2} \left(\int_0^T e^{2V_s^{(q)}}\, |Z_s^n|^2\, ds\right)^{q/2}\right].$$

Note that

(a)

$$\mathbb{E}\left[e^{qV_T^{(q)}}\, |\eta|^q\, \mathbf{1}_{(n,\infty)}(\beta_T + |\eta|)\right] + \mathbb{E}\left(\int_0^T e^{V_s^{(q)}}\, |\Phi(t,0,0)|\ \mathbf{1}_{(n,\infty)}(\gamma_s)\, dQ_s\right)^q$$

$$\leq \left[\mathbb{E}\exp(\delta\Delta_T)\right]^{(p-q)/p} \left(\mathbb{E}\ \left[e^{pV_T}\, |\eta|^p\, \mathbf{1}_{(n,\infty)}(\beta_T + |\eta|)\right]\right)^{q/p}$$

$$+ \left[\mathbb{E}\exp(\delta\Delta_T)\right]^{(p-q)/p} \left[\mathbb{E}\left(\int_0^T e^{V_s}\, |\Phi(t,0,0)|\ \mathbf{1}_{(n,\infty)}(\gamma_s)\, dQ_s\right)^p\right]^{q/p}$$

$$\to 0, \quad \text{as } n \to \infty;$$

(b)

$$
\begin{aligned}
&\mathbb{E}\left[\left(\int_0^T L_s \mathbf{1}_{(n,\infty)}(\gamma_s)\, ds\right)^q \sup_{s\in[0,T]} e^{qV_s}\, |Y_s^n|^q\right] \\
&\leq \left[\mathbb{E}\left(\int_0^T L_s \mathbf{1}_{(n,\infty)}(\gamma_s)\, ds\right)^{\frac{qp}{p-q}}\right]^{(p-q)/p} \left(\mathbb{E} \sup_{s\in[0,T]} e^{pV_s}\, |Y_s^n|^p\right)^{q/p} \\
&= \left[\mathbb{E}\left(\int_0^T L_s \mathbf{1}_{(n,\infty)}(\gamma_s)\, ds\right)^{\delta}\right]^{p/(p+\delta)} \left(\mathbb{E} \sup_{s\in[0,T]} e^{pV_s}\, |Y_s^n|^p\right)^{\delta/(p+\delta)} \\
&\to 0, \quad \text{as } n \to \infty;
\end{aligned}
$$

(c)

$$
\begin{aligned}
&\mathbb{E}\left[\left(\int_0^T \ell_s^2 \mathbf{1}_{(n,\infty)}(\gamma_s)\, ds\right)^{q/2} \left(\int_0^T e^{2V_s}\, |Z_s^n|^2\, ds\right)^{q/2}\right] \\
&\leq \left[\mathbb{E}\left(\int_0^T \ell_s^2 \mathbf{1}_{(n,\infty)}(\gamma_s)\, ds\right)^{\delta/2}\right]^{p/(p+\delta)} \left[\mathbb{E}\left(\int_0^T e^{2V_s}\, |Z_s^n|^2\, ds\right)^{p/2}\right]^{\delta/(p+\delta)} \\
&\to 0, \quad \text{as } n \to \infty.
\end{aligned}
$$

Taking into account (a), (b) and (c) we deduce that there exists a pair $(Y, Z) \in S_m^0\,[0,T] \times \Lambda_{m\times k}^0\,(0,T)$ such that for $q = \frac{p\delta}{p+\delta}$

$$
\lim_{n\to\infty} \mathbb{E}\left(\sup_{s\in[0,T]} e^{qV_s}\, |Y_s^n - Y_s|^q\right) + \mathbb{E}\left[\left(\int_0^T e^{2V_s}\, |Z_s^n - Z_s|^2\, ds\right)^{q/2}\right] = 0.
$$

Now the inequality (5.48) clearly follows from (5.50) by Fatou's Lemma.

Finally passing to the limit in (5.49) we deduce using Lemma 5.16 that (Y, Z) is a solution of BSDE (5.40). ■

5.3.3.3 BSDEs with Locally Lipschitz Coefficients

For a (forward) SDE, it is not hard to deduce from existence and uniqueness under global Lipschitz conditions an existence and uniqueness result under local Lipschitz conditions, at least until a possible explosion time. The reason is that one just needs to follow each path of the solution.

For BSDEs, the situation is dramatically different. Indeed, in a sense, solving a BSDE amounts to combining the flow of a backward ODE with the operation of taking continuously in time the conditional expectation, given the current σ-algebra

$\mathcal{F}_t$. A backward stochastic differential equation is not solved by following each individual path of the solution. Consequently one cannot a priori deduce an existence and uniqueness result under local Lipschitz conditions from the same result under global Lipschitz conditions. However, this is in fact possible, because we have an a priori bound on the solution. This is what we shall explain in this section.

We consider the BSDE

$$Y_t = \eta + \int_t^T F\,(s, Y_s, Z_s)\,ds - \int_t^T Z_s dB_s,\ t \in [0, T]\,,\ a.s. \tag{5.51}$$

Assume that

▲ $\eta : \Omega \to \mathbb{R}^m$ *is an* $\mathcal{F}_T$*-measurable random vector;*
▲ $F : \Omega \times [0, T] \times \mathbb{R}^m \to \mathbb{R}^m$ *satisfies*

(BSDE-LL):

▲ *for all* $y \in \mathbb{R}^m$, $z \in \mathbb{R}^{m\times k}$, *the function* $F\,(\cdot, \cdot, y, z) : [0, T] \to \mathbb{R}^m$ *is* $\mathcal{P}$-m.s.p.
▲ *there exist measurable functions* $\ell, \kappa, \rho : [0, T] \to \mathbb{R}_+$ and $L : [0, T] \times \mathbb{R}_+ \to \mathbb{R}_+$ *satisfying:*

- L *is continuous and increasing in the second variable,*
- $$\int_0^T \left[L\,(t, q) + \ell^2\,(t) + \kappa\,(t) + \rho\,(t)\right] dt < \infty, \quad \text{for all } q \in \mathbb{R}_+,$$
- *for all* $y, y' \in \mathbb{R}^m$, $z, z' \in \mathbb{R}^{m\times k}$, dt*-a.e.*

$$\begin{aligned} &(i) \quad |F(t, y', z) - F(t, y, z)| \le L\,(t, |y| \vee |y'|)\,|y' - y|,\\ &(ii) \quad |F(t, y, z') - F(t, y, z)| \le \ell\,(t)\,|z' - z|,\\ &(iii) \quad |F(t, y, 0)| \le \rho\,(t) + \kappa\,(t)\,|y|\,. \end{aligned} \tag{5.52}$$

Let $p > 1$ and $n_p = 1 \wedge (p - 1)$. Define

$$\tilde{V}\,(t) = \int_0^t \left(\kappa\,(s) + \frac{1}{n_p}\ell^2\,(s)\right) ds.$$

Observe that for all $Y, Y' \in S_m^0\,[0, T]$ satisfying $|Y'| \le |Y|$ and all $Z \in \Lambda^0_{m\times k}\,(0, T)$,

$$\begin{aligned} \left\langle Y_t, \Phi\left(t, Y_t', Z_t\right) dt\right\rangle &\le \left\langle Y_t, \Phi\left(t, Y_t', 0\right) dt\right\rangle + |Y_t|\,\ell\,(t)\,|Z_t|\,dQ_t\\ &\le |Y_t|\,\rho\,(t)\,dt + |Y_t|^2\,d\tilde{V}_t + \frac{n_p}{4}\,|Z_t|^2\,dt. \end{aligned} \tag{5.53}$$

Note that if $(Y,Z) \in S_m^p[0,T] \times \Lambda^0_{m\times k}(0,T)$ is a solution of (5.51), then by Proposition 5.2-A and the inequality (5.53) there exists a $C_p > 0$ such that, $\mathbb{P}$-$a.s.$, for all $t \in [0,T]$

$$\mathbb{E}^{\mathcal{F}_t} \sup_{r\in[t,T]} \left|e^{\tilde{V}_r} Y_r\right|^p + \mathbb{E}^{\mathcal{F}_t} \left(\int_t^T e^{2\tilde{V}_r} |Z_r|^2 dr\right)^{p/2}$$
$$\leq C_p\, \mathbb{E}^{\mathcal{F}_t} \left[\left|e^{\tilde{V}(T)}\eta\right|^p + \left(\int_t^T e^{\tilde{V}(r)}\rho(r)\,dr\right)^p\right],$$

which yields $\mathbb{P}$-$a.s.$, for all $t \in [0,T]$:

$$|Y_t| \leq (C_p)^{1/p} e^{\tilde{V}(T)} \left[\left(\mathbb{E}^{\mathcal{F}_t}\ |\eta|^p\right)^{1/p} + \int_0^T \rho(r)\,dr\right] \overset{def}{=} R_t. \tag{5.54}$$

Define the continuous stochastic processes

$$\beta_t = e^{\tilde{V}(T)} \left[\left(\mathbb{E}^{\mathcal{F}_t}\ |\eta|^p\right)^{1/p} + \int_0^T \rho(s)\,ds\right] \tag{5.55}$$

and

$$\Gamma_t(\lambda) = \int_0^t L\left(s, \lambda + \lambda\left(\mathbb{E}^{\mathcal{F}_s}\ |\eta|^p\right)^{1/p}\right) ds, \quad \lambda \geq 1. \tag{5.56}$$

Theorem 5.23. *Let $p > 1$ and the assumption* (**BSDE-LL**) *be satisfied. If there exists a $\delta > \frac{p}{p-1}$ such that for all $\lambda \geq 1$*

$$\mathbb{E}\left[(\Gamma_T(\lambda))^\delta\right] + \mathbb{E}\ \left|e^{\Gamma_T(\lambda)}\eta\right|^p + \mathbb{E}\Big(\int_0^T e^{\Gamma_t(\lambda)} |\Phi(t,0,0)|\,dt\Big)^p < \infty, \tag{5.57}$$

then the BSDE (5.51) has a unique solution $(Y,Z) \in S_m^0[0,T] \times \Lambda^0_{m\times k}(0,T)$ such that for all $\lambda \geq 1$,

$$\mathbb{E} \sup_{s\in[0,T]} e^{p\lambda\Gamma_s(\lambda)} |Y_s|^p + \mathbb{E}\left(\int_0^T e^{2\lambda\Gamma_s(\lambda)} |Z_s|^2 ds\right)^{p/2} < \infty. \tag{5.58}$$

In particular $(Y,Z) \in S_m^p[0,T] \times \Lambda^p_{m\times k}(0,T)$ and (5.54) holds; if η is a bounded random variable then there exists a constant $C > 0$ such that $\mathbb{P}$-a.s. $\omega \in \Omega$

$$|Y_t(\omega)| \leq C, \quad \forall t \in [0,T].$$

Proof. Consider the projection operator $\pi : \Omega \times [0,T] \times \mathbb{R}^m \to \mathbb{R}^m$,

$$\pi_t(\omega, y) = \pi(\omega, t, y) = \begin{cases} y, & \text{if } |y| \le R_t(\omega), \\ \dfrac{y}{|y|} R_t(\omega) & \text{if } |y| > R_t(\omega). \end{cases}$$

Note that for all $y, y' \in \mathbb{R}^m$, $\pi(\cdot,\cdot,y)$ is a $\mathcal{P}$-m.c.s.p., $|\pi_t(y)| \le R_t$ and

$$\left|\pi_t(y) - \pi_t(y')\right| \le \left|y - y'\right|.$$

The function $\tilde{\Phi}(s,y,z) \overset{def}{=} \Phi(s, \pi_s(y), z)$ is globally Lipschitz with respect to (y,z):

$$\begin{aligned} \left|\tilde{\Phi}(s,y,z) - \tilde{\Phi}(s,y',z)\right| &= \left|\Phi(s,\pi_s(y),z) - \Phi(s,\pi_s(y'),z)\right| \\ &\le L_s\left(|\pi_s(y)| \vee \left|\pi_s(y')\right|\right) |\pi_s(y) - \pi_s(y')| \\ &\le L(s,R_s)\left|y - y'\right| \\ &\le [\kappa(s) + L(s,R_s)]\left|y - y'\right|, \end{aligned}$$

and

$$\begin{aligned} \left|\tilde{\Phi}(s,y,z) - \tilde{\Phi}(s,y,z')\right| &= \left|\Phi(s,\pi_s(y),z) - \Phi(s,\pi_s(y),z')\right| \\ &\le \ell(s)\,|z - z'|. \end{aligned}$$

Then by Theorem 5.21 the BSDE

$$Y_t = \eta + \int_t^T \tilde{\Phi}(s, Y_s, Z_s)\, dQ_s - \int_t^T Z_s dB_s, \; t \in [0,T], \tag{5.59}$$

has a unique solution $(Y,Z) \in S_m^0[0,T] \times \Lambda_{m\times k}^0(0,T)$ satisfying (5.58). Since by (5.53)

$$\begin{aligned} \left\langle Y_t, \tilde{\Phi}(t,Y_t,Z_t)\, dQ_t \right\rangle &= \langle Y_t, \Phi(t, \pi_t(Y_t), Z_t)\, dQ_t \rangle \\ &\le |Y_t|\,\rho_t dQ_t + |Y_t|^2 \kappa_t dQ_t + \frac{n_p}{4} |Z_t|^2 dt \end{aligned}$$

we infer by 5.54 that $|Y_t| \le R_t$ and consequently $\tilde{\Phi}(t,Y_t,Z_t) = \Phi(t,Y_t,Z_t)$, that is (Y,Z) is a solution of the Eq. (5.51). The solution is unique since any solution (Y,Z) of (5.51) satisfies $|Y_t| \le R_t$ and consequently it is a solution of (5.59). ∎

5.3.4 BSDEs with Monotone Coefficients

5.3.4.1 The First BSDE: Monotone Coefficient $\Phi(s, Y_s)\, dQ_s$

We first consider the BSDE

$$Y_t = \eta + \int_t^T \Phi(s, Y_s)\, dQ_s - \int_t^T Z_s dB_s, \quad t \in [0, T],\ a.s. \tag{5.60}$$

We assume that

$$(\mathbf{BSDE\text{-}MH0}_\Phi): \tag{5.61}$$

- ▲ $\eta : \Omega \to R^m$ *is an* $\mathcal{F}_T$*-measurable random vector;*
- ▲ Q is a $\mathcal{P}$-m.i.c.s.p. *such that* $Q_0 = 0$;
- ▲ $\Phi : \Omega \times [0, \infty[\times \mathbb{R}^m \to \mathbb{R}^m$ *satisfies:*

(a) $\forall\ y \in \mathbb{R}^m,\ \Phi(\cdot, \cdot, y) : \Omega \times [0, T] \to \mathbb{R}^m$ *is* $\mathcal{P}$*-measurable;*
(b) *the mapping* $y \longrightarrow \Phi(t, y) : \mathbb{R}^m \to \mathbb{R}^m$ *is continuous;*
(c) *there exist a* $\mathcal{P}$*-m.s.p.* $\mu : \Omega \times [0, T] \to \mathbb{R}$ *such that*

$$\int_0^T |\mu_t| dQ_t < \infty, \quad \mathbb{P}\text{-a.s.},$$

and for all $y, y' \in \mathbb{R}^m,\ d\mathbb{P} \otimes dQ_t$*-a.e.*

$$\langle y' - y, \Phi(t, y') - \Phi(t, y) \rangle \le \mu_t \left| y' - y \right|^2; \tag{5.62}$$

(d) *for all* $\rho \ge 0$

$$\int_0^T \Phi_\rho^{\#}(s)\, dQ_s < \infty, \quad a.s.$$

□

where

$$\Phi_\rho^{\#}(t) \overset{def}{=} \sup_{|y| \le \rho} |\Phi(t, y)|.$$

We recall the notations

$$S_m^{1+}\left([0, T]; e^{\bar{\mu}}\right) = \bigcup\nolimits_{p>1} S_m^p\left([0, T]; e^{\bar{\mu}}\right)$$

and

$$\bar{\mu}_t = \int_0^t \mu_s dQ_s, \quad \hat{\mu}_t = \int_0^t \mu_s^+ dQ_s.$$

Proposition 5.24. *Let $p \geq 1$ and the assumptions (5.61-**BSDE-MH0**$_\Phi$) be satisfied. If for all $\rho > 0$*

$$\mathbb{E}\ \left|e^{\bar{\mu}_T}\eta\right|^p + \mathbb{E}\ \left(\int_0^T e^{\hat{\mu}_s}\Phi_\rho^{\#}(s)\, dQ_s\right)^p < \infty \tag{5.63}$$

then the BSDE (5.60) has a unique solution $(Y, Z) \in S_m^1\left([0,T]; e^{\bar{\mu}}\right) \times \Lambda_{m\times k}^0\left(0,T; e^{\bar{\mu}}\right)$. Moreover

$$\begin{aligned}
(j)\quad & |Y_t| \leq \mathbb{E}^{\mathcal{F}_t}\ \left|e^{\bar{\mu}_T-\bar{\mu}_t}\eta\right| + \mathbb{E}^{\mathcal{F}_t}\int_t^T e^{\bar{\mu}_s-\bar{\mu}_t}\,|\Phi(s,0)|\, dQ_s,\\
& \qquad\qquad\qquad\qquad\qquad \forall t \in [0,T],\ \mathbb{P}\text{-a.s.},\\
(jj)\quad & \sup_{s\in[0,T]}\left(\mathbb{E}\, e^{\bar{\mu}_s}|Y_s|\right)^q + \mathbb{E}\sup_{s\in[0,T]}\left|e^{\bar{\mu}_s}Y_s\right|^q + \mathbb{E}\left(\int_0^T e^{2\bar{\mu}_s}|Z_s|^2 ds\right)^{q/2},\\
& \leq C_q\left(\mathbb{E}\left(e^{\bar{\mu}_T}|\eta|\right)\right)^q + \left(\mathbb{E}\int_0^T e^{\bar{\mu}_s}\,|\Phi(s,0)|\, dQ_s\right)^q, \quad \forall\, q \in (0,1),
\end{aligned} \tag{5.64}$$

and for $p > 1$ there exists a positive C_p (depending only on p) such that, $\mathbb{P}$-a.s., for all $t \in [0,T]$:

$$\begin{aligned}
&\mathbb{E}^{\mathcal{F}_t}\sup_{s\in[t,T]}\left|e^{\bar{\mu}_s}Y_s\right|^p + \mathbb{E}^{\mathcal{F}_t}\left(\int_t^T e^{2\bar{\mu}_s}|Z_s|^2 ds\right)^{p/2}\\
&\leq C_p\left[\mathbb{E}^{\mathcal{F}_t}\left|e^{\bar{\mu}_T}\eta\right|^p + \mathbb{E}^{\mathcal{F}_t}\left(\int_t^T e^{\bar{\mu}_s}\,|\Phi(s,0)|\, dQ_s\right)^p\right].
\end{aligned} \tag{5.65}$$

Remark 5.25. If $(\bar{\mu}_t)_{t\geq 0}$ is a deterministic process then the assumption (5.63) is equivalent to

$$\mathbb{E}\left(|\eta|^p\right) + \mathbb{E}\left(\int_0^T \Phi_\rho^{\#}(s) dQ_s\right)^p < \infty,$$

and the inequality (5.65) yields: for all $t \in [0,T]$

$$|Y_t| \leq e^{2\|\bar{\mu}\|_T}\left[\mathbb{E}^{\mathcal{F}_t}\ |\eta| + \mathbb{E}^{\mathcal{F}_t}\int_t^T |\Phi(s,0)|\, dQ_s\right].$$

Proof of Proposition 5.24. (I) Uniqueness follows from Theorem 5.10 and Theorem 5.11. If $(Y,Z) \in S_m^1\left([0,T]\,; e^{\bar{\mu}}\right) \times \Lambda^0_{m\times k}\left(0,T; e^{\bar{\mu}}\right)$ is a solution, then by Proposition 5.2 and

$$\langle Y_s, \Phi\left(s, Y_s\right) dQ_s\rangle \leq \left|\Phi\left(s,0\right)\right| \left|Y_s\right| dQ_s + \mu_s \left|Y_s\right|^2 dQ_s$$

the inequalities (5.64-j,jj) follow.

To prove the existence of the solution we write the equation in the form, $\mathbb{P}$-a.s.

$$Y_t = \eta + \int_t^T \left[F\left(s, Y_s\right) + \mu_s Y_s\right] dQ_s - \int_t^T Z_s dB_s, \quad t \in [0,T]\,, \tag{5.66}$$

where

$$F\left(s,y\right) = \Phi\left(s,y\right) - \mu_s y.$$

We remark that $\hat{\mu}_t - \hat{\mu}_s = \int_s^t \mu_r^+ dQ_r \geq \int_s^t \mu_r dQ_r = \bar{\mu}_t - \bar{\mu}_s$.

$(II\text{-}a)$ *Existence in the case: there exist* $b, c > 0$ *such that for all* $t \in [0,T]$

$$|\eta| + \left|\Phi\left(t,0\right)\right| + \left|e^{\hat{\mu}_T}\eta\right| + \int_0^T e^{\hat{\mu}_s}\left|\Phi\left(s,0\right)\right| dQ_s \leq b, \ a.s., \tag{5.67}$$

and

$$Q_t + |\bar{\mu}_t| + |\mu_t| + \Phi_b^{\#}(t) \leq c, \ a.s. \tag{5.68}$$

Step 1. Yosida approximation of $-F$.

Since $y \mapsto -F\left(t,y\right) = \mu_t y - \Phi(t,y) : \mathbb{R}^m \to \mathbb{R}^m$ is a monotone continuous operator (hence also a maximal monotone operator), it follows that for every $(\omega,t,y) \in \Omega \times [0,T] \times \mathbb{R}^m$ and $\varepsilon > 0$ there exists a unique $F_\varepsilon = F_\varepsilon\left(\omega,t,y\right) \in \mathbb{R}^m$ such that

$$F(\omega, t, y + \varepsilon F_\varepsilon) = F_\varepsilon.$$

From Annex B, Propositions 6.7 and 6.8, recall that $F_\varepsilon\left(\cdot,\cdot,y\right) : \Omega\times[0,T] \to \mathbb{R}^m$ is $\mathcal{P}$-m.s.p. for every $y \in \mathbb{R}^m$ and
$\forall\ \varepsilon, \delta > 0,\ \forall\ t \in [0,T], \forall\ y, y' \in \mathbb{R}^m, \quad a.s.$

$$\begin{aligned}
&(a) \quad \left\langle F_\varepsilon\left(t,y\right) - F_\varepsilon\left(t,y'\right), y - y'\right\rangle \leq 0,\\
&(b) \quad \left|F_\varepsilon\left(t,y\right) - F_\varepsilon\left(t,y'\right)\right| \leq \frac{2}{\varepsilon}|y - y'|,\\
&(c) \quad \left|F_\varepsilon\left(t,y\right)\right| \leq \left|F\left(t,y\right)\right|, \qquad \lim_{\varepsilon\to 0} F_\varepsilon\left(t,y\right) = F\left(t,y\right),
\end{aligned}$$

and

$$\left\langle y - y', F_\varepsilon(t, y) - F_\delta(t, y') \right\rangle \le (\varepsilon + \delta) \left\langle F_\varepsilon(t, y), F_\delta(t, y') \right\rangle.$$

Moreover, if $|y| \le b$ then

$$|F_\varepsilon(t, y)| \le |\mu_t| b + \Phi_b^\#(t). \tag{5.69}$$

Step 2. Approximating equation.
Let $0 < \varepsilon \le 1$. Since $y \longmapsto F_\varepsilon(r, y) + \mu_r y$ is a Lipschitz function with the Lipschitz constants $L_t = \dfrac{2}{\varepsilon} + c$ and $\ell_t = 0$ we infer by Theorem 5.21 that the approximating equation

$$Y_t^\varepsilon = \eta + \int_t^T \left[F_\varepsilon\left(r, Y_r^\varepsilon\right) + \mu_r Y_r^\varepsilon \right] dQ_r - \int_t^T Z_r^\varepsilon dB_r \tag{5.70}$$

has a unique solution $(Y^\varepsilon, Z^\varepsilon) \in S_m^q[0, T] \times \Lambda_{m \times k}^q(0, T)$ for all $q > 1$.
Step 3. Boundedness of $(Y^\varepsilon, Z^\varepsilon)_{0<\varepsilon\le 1}$.
We denote by C, C' generic constants independent of $\varepsilon, \delta \in]0, 1]$. Since

$$\mathbb{E} \sup_{t\in[0,T]} e^{q\bar{\mu}_t} |Y_t^\varepsilon|^q \le e^{qc} \mathbb{E} \sup_{t\in[0,T]} |Y_t^\varepsilon|^q < \infty$$

and

$$\begin{aligned} \left\langle Y_s^\varepsilon, \left[F_\varepsilon\left(s, Y_s^\varepsilon\right) + \mu_s Y_s^\varepsilon \right] dQ_s \right\rangle &\le |F_\varepsilon(s, 0)| \, |Y_s^\varepsilon| \, dQ_s + \mu_s |Y_s^\varepsilon|^2 dQ_s \\ &\le |\Phi(s, 0)| \, |Y_s^\varepsilon| \, dQ_s + \mu_s |Y_s^\varepsilon|^2 dQ_s \\ &\le |\Phi(s, 0)| \, |Y_s^\varepsilon| \, dQ_s + \mu_s^+ |Y_s^\varepsilon|^2 dQ_s \end{aligned}$$

we deduce, by Proposition 5.2, that for $p > 1$ and for all $t \in [0, T]$:

$$\begin{aligned} &\mathbb{E}^{\mathcal{F}_t} \sup_{s\in[t,T]} \left| e^{\hat{\mu}_s} Y_s^\varepsilon \right|^p + \mathbb{E}^{\mathcal{F}_t} \left(\int_t^T e^{\hat{\mu}_s} |Z_s^\varepsilon|^2 ds \right)^{p/2} \\ &\quad \le C_p \, \mathbb{E}^{\mathcal{F}_t} \left[\left| e^{\hat{\mu}_T} \eta \right|^p + \left(\int_t^T e^{\hat{\mu}_s} |\Phi(s, 0)| \, dQ_s \right)^p \right] \le C_p b^p, \end{aligned} \tag{5.71}$$

and (5.65) with (Y, Z) replaced by $(Y^\varepsilon, Z^\varepsilon)$.
By Corollary 6.81, for $p = 1$, we have $\mathbb{P}$-$a.s.$, for all $t \in [0, T]$:

$$|Y_t^\varepsilon| \le e^{\hat{\mu}_t} |Y_t^\varepsilon| \le \mathbb{E}^{\mathcal{F}_t} \left| e^{\hat{\mu}_T} \eta \right| + \mathbb{E}^{\mathcal{F}_t} \int_t^T e^{\hat{\mu}_s} |\Phi(s, 0)| \, dQ_s \le b. \tag{5.72}$$

Now from (5.69) we deduce

$$\begin{aligned}\left|F_{\varepsilon}\left(r, Y_r^{\varepsilon}\right)+\mu_r Y_r^{\varepsilon}\right| &\leq\left|F_{\varepsilon}\left(r, Y_r^{\varepsilon}\right)\right|+\left|\mu_r Y_r^{\varepsilon}\right| \\ &\leq \Phi_b^{\#}(t)+2\left|\mu_r\right| b \\ &\leq c+2 c b .\end{aligned}$$

Step 4. $(Y^{\varepsilon}, Z^{\varepsilon})$ *is a Cauchy sequence in* $S_m^q[0, T] \times \Lambda_{m \times k}^q(0, T)$, $q>0$.
Let $0<\varepsilon, \delta \leq 1$. We have

$$Y_t^{\varepsilon}-Y_t^{\delta}=\int_t^T d K_s^{\varepsilon, \delta}-\int_t^T\left(Z_s^{\varepsilon}-Z_s^{\delta}\right) d B_s,$$

where

$$K_t^{\varepsilon, \delta}=\int_0^t\left(F_{\varepsilon}(s, Y_s^{\varepsilon})+\mu_s Y_s^{\varepsilon}-F_{\delta}(s, Y_s^{\delta})-\mu_s Y_s^{\delta})\right) d Q_s .$$

Note that

$$\begin{aligned}&\left\langle Y_s^{\varepsilon}-Y_s^{\delta}, d K_s^{\varepsilon, \delta}\right\rangle \\ &=\left\langle Y_s^{\varepsilon}-Y_s^{\delta}, F_{\varepsilon}(s, Y_s^{\varepsilon})-F_{\delta}(s, Y_s^{\delta})\right\rangle d Q_s+\mu_s\left|Y_s^{\varepsilon}-Y_s^{\delta}\right|^2 d Q_s \\ &\leq(\varepsilon+\delta)\left\langle F_{\varepsilon}(s, Y_s^{\varepsilon}), F_{\delta}(s, Y_s^{\delta})\right\rangle d Q_s+\mu_s\left|Y_s^{\varepsilon}-Y_s^{\delta}\right|^2 d Q_s \\ &\leq(\varepsilon+\delta) c^2 d Q_s+c\left|Y_s^{\varepsilon}-Y_s^{\delta}\right|^2 d Q_s .\end{aligned}$$

Since $0 \leq Q_t \leq c$ and for every $q>1$

$$\mathbb{E} \sup_{t \in[0, T]} e^{q c Q_t}\left|Y_t^{\varepsilon}-Y_s^{\delta}\right|^q<\infty,$$

we infer from Proposition 5.2 with $D=N=0$, $\lambda=0$, that for $q \geq 2$,

$$\mathbb{E} \sup_{s \in[0, T]}\left|Y_s^{\varepsilon}-Y_s^{\delta}\right|^q+\mathbb{E}\left(\int_0^T\left|Z_s^{\varepsilon}-Z_s^{\delta}\right|^2 d s\right)^{q / 2} \leq C\,(\varepsilon+\delta)^{q / 2} .$$

For $0<q<2$ we have

$$\begin{aligned}&\mathbb{E} \sup_{s \in[0, T]}\left|Y_s^{\varepsilon}-Y_s^{\delta}\right|^q+\mathbb{E}\left(\int_0^T\left|Z_s^{\varepsilon}-Z_s^{\delta}\right|^2 d s\right)^{q / 2} \\ &\quad \leq\left(\mathbb{E} \sup_{s \in[0, T]}\left|Y_s^{\varepsilon}-Y_s^{\delta}\right|^{q+2}\right)^{q /(q+2)}+\left[\mathbb{E}\left(\int_0^T\left|Z_s^{\varepsilon}-Z_s^{\delta}\right|^2 d s\right)^{\frac{q+2}{2}}\right]^{q /(q+2)} \\ &\quad \leq C^{\prime}\,(\varepsilon+\delta)^{q / 2} .\end{aligned}$$

Hence there exists $(Y, Z) \in \bigcap_{q>0} S_m^q [0, T] \times \Lambda_{m\times k}^q (0, T)$ such that

$$\mathbb{E} \sup_{s\in[0,T]} |Y_s^\varepsilon - Y_s|^q + \mathbb{E}\left(\int_0^T |Z_s^\varepsilon - Z_s|^2 ds\right)^{q/2} \le C\ \varepsilon^{q/2}.$$

Note that

$$F_\varepsilon\left(r, Y_r^\varepsilon\right) + \mu_r Y_r^\varepsilon = \Phi\left(r, Y_r^\varepsilon + \varepsilon F_\varepsilon\left(r, Y_r^\varepsilon\right)\right) - \varepsilon\mu_r F_\varepsilon\left(r, Y_r^\varepsilon\right)$$

and $\left|F_\varepsilon\left(r, Y_r^\varepsilon\right)\right| + |\mu_r Y_r^\varepsilon| \le C$.

Passing to the limit as $\varepsilon \to 0_+$ in the approximating equation (5.70), we infer, by Lebesgue's dominated convergence theorem, that (Y, Z) is a solution of the BSDE (5.60). Moreover passing to the limit on a subsequence, by Fatou's Lemma we clearly infer that (Y, Z) satisfies (5.65), (5.64) and

$$\begin{aligned} &(j) \quad \mathbb{E} \sup_{s\in[0,T]} \left|e^{\hat{\mu}_s} Y_s\right|^p + \mathbb{E}\left(\int_0^T e^{\hat{\mu}_s} |Z_s|^2 ds\right)^{p/2} \le C_p b^p,\ \text{if } p > 1, \\ &(jj) \quad |Y_t| \le \left|e^{\hat{\mu}_t} Y_t\right| \le b, \quad \text{for all } t \in [0, T], \mathbb{P}\text{-a.s.,} \end{aligned} \tag{5.73}$$

since the same inequalities hold for $(Y^\varepsilon, Z^\varepsilon)$.

(*II-b*) *Existence under the assumption (5.67), but without (5.68).*

Let

$$\tau_n = \inf\{t \in [0, T] : Q_t \ge n\} \quad \text{and} \quad Q_t^n = Q_{t\wedge\tau_n}.$$

Let $\zeta_t = Q_t + |\bar{\mu}_t| + |\mu_t| + \Phi_b^\#(t)$ and $\hat{\mu}_t = \int_0^t \mu_s^+ dQ_s$.

Since

$$\begin{aligned} \left\langle u - v, \Phi(r, u)\, \mathbf{1}_{\zeta_r<n} - \Phi(r, v)\, \mathbf{1}_{\zeta_r<n}\right\rangle &\le \mu_r \mathbf{1}_{\zeta_r<n} |u - v|^2 \\ &\le \mu_r^+ |u - v|^2 \end{aligned}$$

by the step (*II-a*) the BSDE

$$\begin{aligned} Y_t^n &= \eta + \int_t^T \Phi\left(r, Y_r^n\right) \mathbf{1}_{\zeta_r<n} dQ_r - \int_t^T Z_r^n dB_r \\ &= \eta + \int_t^T \Phi\left(r, Y_r^n\right) \mathbf{1}_{\zeta_r<n} dQ_r^n - \int_t^T Z_r^n dB_r,\ t \in [0, T] \end{aligned}$$

has a unique solution $(Y^n, Z^n) \in \bigcap_{q>0} S_m^q [0, T] \times \Lambda_{m\times k}^q (0, T)$. The solution (Y^n, Z^n) satisfies (5.65), (5.64) and (5.73) with (Y, Z) replaced by (Y^n, Z^n). Note that from (5.73) written for (Y^n, Z^n) we have for $p \ge 1$,

$$\mathbb{E} \sup_{s\in[0,T]} \left| e^{\hat{\mu}_s} \left(Y_s^n - Y_s^{n+i} \right) \right|^p < \infty.$$

Since

$$\begin{aligned}
&\left\langle Y_t^n - Y_t^{n+i}, \left[\Phi\left(t, Y_t^n\right) \mathbf{1}_{\zeta_t < n} - \Phi\left(t, Y_t^{n+i}\right) \mathbf{1}_{\zeta_t < n+i} \right] dQ_t \right\rangle \\
&\quad \leq \left\langle Y_t^n - Y_t^{n+i}, \Phi\left(t, Y_t^n\right) \left(\mathbf{1}_{\zeta_t < n} - \mathbf{1}_{\zeta_t < n+i} \right) \right\rangle dQ_t + \mu_t \mathbf{1}_{\zeta_t < n+i} \left| Y_t^n - Y_t^{n+i} \right|^2 dQ_t \\
&\quad \leq \left| Y_t^n - Y_t^{n+i} \right| \mathbf{1}_{\zeta_t \geq n} \Phi_b^{\#}(t)\, dQ_t + \mu_t^+ \left| Y_t^n - Y_t^{n+i} \right| dQ_t,
\end{aligned}$$

we conclude by Corollary 6.81, that for $q = p$ if $p > 1$ and $q \in (0, 1)$ if $p = 1$ there exists a constant C_q such that $\mathbb{P}$-$a.s.$,

$$\begin{aligned}
&\mathbb{E} \sup_{s\in[0,T]} e^{q\hat{\mu}_s} \left| Y_s^n - Y_s^{n+i} \right|^q + \mathbb{E} \left(\int_0^T e^{2\hat{\mu}_s} \left| Z_s^n - Z_s^{n+i} \right|^2 ds \right)^{q/2} \\
&\leq C_q \Big[\left(\mathbb{E} \int_0^T e^{\hat{\mu}_s} \mathbf{1}_{\zeta_s \geq n} \Phi_b^{\#}(s)\, dQ_s \right)^q \mathbf{1}_{p=1} + \mathbb{E} \left(\int_0^T . e^{\hat{\mu}_s} \mathbf{1}_{\zeta_s \geq n} \Phi_b^{\#}(s)\, dQ_s \right)^q \mathbf{1}_{p>1} \Big].
\end{aligned}$$

Taking into account (5.63) we deduce that there exists a pair $(Y, Z) \in S_m^q \left([0, T]; e^{\hat{\mu}} \right) \times \Lambda_{m\times k}^q \left(0, T; e^{\hat{\mu}} \right)$ such that as $n \to \infty$

$$\lim_{n\to\infty} \mathbb{E} \sup_{s\in[0,T]} e^{q\hat{\mu}_s} \left| Y_s^n - Y_s \right|^q + \mathbb{E} \left(\int_0^T e^{2\hat{\mu}_s} \left| Z_s^n - Z_s \right|^2 ds \right)^{q/2} = 0.$$

Now using Lemma 5.16 we infer that (Y, Z) is a solution of the BSDE (5.76). We deduce by Fatou's Lemma from the inequalities (5.65), (5.64) and (5.73) written for (Y^n, Z^n) that the same inequalities hold for the limit (Y, Z).

(*II-c*) *Existence without the two assumptions (5.67) and (5.68).*

Let

$$\beta_t \overset{def}{=} Q_t + \int_0^t |\mu_s|\, dQ_s + \int_0^t |\Phi(s, 0)|\, dQ_s.$$

Define, for $n \in \mathbb{N}^*$,

$$\begin{aligned}
\eta_n &= \eta\, \mathbf{1}_{[0,n]} \left(\beta_T + |\eta| \right), \\
\Phi_n(t, y) &= \Phi(t, y) - \Phi(t, 0)\ \mathbf{1}_{[n,\infty[} \left(\beta_t + |\Phi(t, 0)| \right).
\end{aligned}$$

The condition (5.67) is satisfied:

$$\begin{aligned}
&|\eta_n| + |\Phi_n(t, 0)| + \left| e^{\hat{\mu}_T - \hat{\mu}_t} \eta_n \right| + \int_t^T e^{\hat{\mu}_s - \hat{\mu}_t} |\Phi_n(s, 0)|\, dQ_s \\
&\leq b_n = n + n + e^n n + e^n n T
\end{aligned}$$

and consequently by part (*II-b*) of this proof there exists a unique pair $(Y^n, Z^n) \in S_m^p\left([0,T]\,; e^{\hat{\mu}}\right) \times \Lambda_{m\times k}^p\left(0,T; e^{\hat{\mu}}\right)$ such that

$$Y_t^n = \eta_n + \int_t^T \Phi_n\left(s, Y_s^n\right) dQ_s - \int_t^T Z_s^n dB_s,\ t \in [0,T]\,,\ a.s. \tag{5.74}$$

and the inequalities (5.65), (5.64) for (Y^n, Z^n) in the place of (Y,Z) hold. Since

$$\begin{aligned}&\left\langle Y_s^n - Y_s^{n+i}, \Phi_n\left(s, Y_s^n\right) dQ_s - \Phi_{n+i}\left(s, Y_s^{n+i}\right) dQ_s\right\rangle \\ &\le \left|Y_s^n - Y_s^{n+i}\right| |\Phi(s,0)|\, \mathbf{1}_{[n,\infty[}\left(\beta_s + |\Phi(s,0)|\right) dQ_s + \mu_s \left|Y_s^n - Y_s^{n+i}\right|^2 dQ_s\end{aligned}$$

we deduce from Corollary 6.81 that in the case $p > 1$ we have

$$\begin{aligned}&\mathbb{E} \sup_{s\in[0,T]} e^{p\bar{\mu}_s} \left|Y_s^n - Y_s^{n+i}\right|^p + \mathbb{E}\left(\int_0^T e^{2\bar{\mu}_s}\left|Z_s^n - Z_s^{n+i}\right|^2 ds\right)^{p/2} \\ &\le C_p \mathbb{E}\left(e^{p\bar{\mu}_T}\left|\eta_n - \eta_{n+i}\right|^p\right) + C_p \mathbb{E}\left(\int_0^T e^{\bar{\mu}_s} \mathbf{1}_{\beta_s + |\Phi(s,0)| \ge n} |\Phi(s,0)|\, dQ_s\right)^p\end{aligned}$$

and in the case $p = 1$

$$\begin{aligned}&\sup_{s\in[0,T]}\left(\mathbb{E}\, e^{\bar{\mu}_s}\left|Y_s^n - Y_s^{n+i}\right|\right)^q + \mathbb{E} \sup_{s\in[0,T]} e^{q\bar{\mu}_s}\left|Y_s^n - Y_s^{n+i}\right|^q \\ &+ \mathbb{E}\left(\int_0^T e^{2\bar{\mu}_s}\left|Z_s^n - Z_s^{n+i}\right|^2 ds\right)^{q/2} \\ &\le C_q \left(\mathbb{E} e^{\bar{\mu}_T}\left|\eta_n - \eta_{n+i}\right|\right)^q + C_q\left(\mathbb{E}\int_0^T e^{\bar{\mu}_s}\mathbf{1}_{\beta_s + |\Phi(s,0)|\ge n}|\Phi(s,0)|\, dQ_s\right)^q\end{aligned}$$

for all $0 < q < 1$.

Hence for every $p \ge 1$ there exists $(Y,Z) \in S_m^q\left([0,T]\,; e^{\bar{\mu}}\right) \times \Lambda_{m\times k}^q\left(0,T; e^{\bar{\mu}}\right)$ (with $q = p$ if $p > 1$, and $0 < q < 1$ if $p = 1$) such that

$$\lim_{n\to\infty} \mathbb{E} \sup_{s\in[0,T]} e^{q\bar{\mu}_s}\left|Y_s^n - Y_s\right|^p + \mathbb{E}\left(\int_0^T e^{2\bar{\mu}_s}\left|Z_s^n - Z_s\right|^2 ds\right)^{q/2} = 0.$$

Using Fatou's Lemma, the inequalities (5.65) and (5.64) follow from the same inequalities written for (Y^n, Z^n). By Lemma 5.16 we infer that (Y,Z) is a solution of the BSDE (5.60). ■

Corollary 5.26. *Let* $p \ge 1$. *If in Proposition 5.24 we replace the assumption (5.63) by*

$$\mathbb{E}\, e^{p\bar{\mu}_T} |\eta|^p + \mathbb{E}\left(\int_0^T \sup_{|y|\leq\rho} \left|e^{\bar{\mu}_s}\Phi\left(s, e^{-\bar{\mu}_s} y\right) - \mu_s y\right| dQ_s\right)^p < \infty, \ \forall\rho \geq 0, \tag{5.75}$$

then the same conclusions follow.

Proof. We remark that (Y, Z) solves the BSDE (5.60) if and only if $(\tilde{Y}_t, \tilde{Z}_t) := (e^{\bar{\mu}_t} Y_t, e^{\bar{\mu}_t} Z_t)$ is solution of the BSDE

$$\tilde{Y}_t = \tilde{\eta} + \int_t^T \tilde{\Phi}\left(s, \tilde{Y}_s\right) dQ_s - \int_t^T \tilde{Z}_s dB_s, \ \ t \in [0, T], \ a.s.$$

with

$$\begin{aligned}\tilde{\eta} &= e^{\bar{\mu}_T}\eta, \\ \tilde{\Phi}(t, y) &= -\mu_t y + e^{\bar{\mu}_t}\Phi(t, e^{-\bar{\mu}_t} y).\end{aligned}$$

Note that $\tilde{\eta}$ and $\tilde{\Phi}$ satisfy the same assumptions (5.61-**BSDE-MH0**$_{\Phi}$) as η and Φ, respectively, but with (5.62) replaced by

$$\left\langle y' - y, \tilde{\Phi}(t, y') - \tilde{\Phi}(t, y)\right\rangle \leq 0, \ \ \mathbb{P}\text{-a.s.}$$

and consequently the corresponding $\bar{\mu}$ and $\hat{\mu}$ for $\tilde{\Phi}$ are equal to 0. Therefore the condition (5.63) for $(\tilde{\eta}, \tilde{\Phi})$ means precisely (5.75). ■

5.3.4.2 The Second BSDE: Monotone Coefficient $F\,(t, Y_t, Z_t)\,dt$

In this subsection we study the BSDE

$$Y_t = \eta + \int_t^T F\,(s, Y_s, Z_s)\, ds - \int_t^T Z_s dB_s, \ \ a.s., \ \ t \in [0, T]. \tag{5.76}$$

We shall assume:

$$(\textbf{BSDE-MH}_F): \tag{5.77}$$

- ♦ *$\eta : \Omega \to \mathbb{R}^m$ is an $\mathcal{F}_T$-measurable random vector;*
- ♦ *the function $F\,(\cdot, \cdot, y, z) : \Omega \times [0, T] \to \mathbb{R}^m$ is $\mathcal{P}$-measurable for every $(y, z) \in \mathbb{R}^m \times \mathbb{R}^{m\times k}$;*
- ♦ *there exist some deterministic functions $\mu \in L^1\,(0, T; \mathbb{R})$ and $\ell \in L^2\,(0, T; \mathbb{R})$ such that*

$$\left|\begin{array}{ll} (I) & \text{for all } y, y' \in \mathbb{R}^m,\ z, z' \in \mathbb{R}^{m\times k},\ d\mathbb{P}\otimes dt\text{-}a.e.: \\ & \quad \textit{Continuity:} \\ & (C_y) \quad y \longrightarrow F(t,y,z) : \mathbb{R}^m \to \mathbb{R}^m \textit{ is continuous;} \\ & \quad \textit{Monotonicity condition:} \\ & (M_y) \quad \langle y'-y, F(t,y',z) - F(t,y,z)\rangle \le \mu(t)\,|y'-y|^2; \\ & \quad \textit{Lipschitz condition:} \\ & (L_z) \quad |F(t,y,z') - F(t,y,z)| \le \ell(t)\ |z'-z|; \\ (II) & \textit{Boundedness condition:} \\ & (B_F) \int_0^T F_\rho^{\#}(t)\,dt < \infty, \quad a.s., \quad \forall\, \rho \ge 0, \end{array}\right. \tag{5.78}$$

□

where

$$F_\rho^{\#}(t) = \sup\{|F(t,y,0)| : |y| \le \rho\}.$$

Theorem 5.27. *Let $p > 1$ and the assumptions (5.77-**BSDE-MH**$_F$) be satisfied. If for all $\rho \ge 0$:*

$$\mathbb{E}\ |\eta|^p + \mathbb{E}\left(\int_0^T F_\rho^{\#}(t)dt\right)^p < \infty,$$

then the BSDE (5.76):

$$Y_t = \eta + \int_t^T F(s, Y_s, Z_s)\,ds - \int_t^T Z_s dB_s, \quad a.s.$$

has a unique solution $(Y,Z) \in S_m^p[0,T] \times \Lambda_{m\times k}^p(0,T)$. Moreover, uniqueness holds in $S_m^{1+}[0,T] \times \Lambda_{m\times k}^0(0,T)$, where

$$S_m^{1+}[0,T] \stackrel{def}{=} \bigcup_{p>1} S_m^p[0,T].$$

Proof. The uniqueness is proved in Corollary 5.13. Let us prove existence.

We use again a contraction argument, which is slightly different from that in the proof of Theorem 5.17.

Note that a solution of the Eq. (5.76) is a fixed point of the mapping $\Gamma : S_m^p[0,T] \times \Lambda_{m\times k}^p(0,T) \to S_m^p[0,T] \times \Lambda_{m\times k}^p(0,T)$ defined by

$$(Y,Z) = \Gamma(X,U),$$

where

$$Y_t = \eta + \int_t^T F(r, Y_r, U_r)\,dr - \int_t^T Z_r dB_r,\ a.s.\ t \in [0,T].$$

By Proposition 5.24 and Remark 5.25, with $\Phi(\omega, t, y) = F(\omega, t, y, U_t(\omega))$, the mapping Γ is well defined since for all $\rho \geq 0$

$$\begin{aligned}
&\mathbb{E}\left(\int_0^T \Phi_\rho^{\#}(t)\,dt\right)^p \\
&\leq \mathbb{E}\left(\int_0^T \sup_{|y|\leq\rho} |F(t, y, 0)|\,dt + \int_0^T \ell(t)\,|U_t|\,dt\right)^p \\
&\leq 2^{p-1}\mathbb{E}\left(\int_0^T F_\rho^{\#}(t)\,dt\right)^p + 2^{p-1}\left(\int_0^T \ell^2(t)\,dt\right)^{p/2} \mathbb{E}\left(\int_0^T |U_t|^2\,dt\right)^{p/2} \\
&< \infty.
\end{aligned}$$

Let $M \in \mathbb{N}^*$ and $0 = T_0 < T_1 < \cdots < T_M = T$, with $T_i = \dfrac{iT}{M}$. Since the function $t \longmapsto \int_0^t \ell^2(r)\,dr : [0, T] \to \mathbb{R}_+$ is uniformly continuous, we see that

$$\boldsymbol{\alpha}\left(\tfrac{T}{M}\right) \stackrel{def}{=} \sup_{0<s-t<\frac{T}{M}} \int_t^s \ell^2(r)\,dr \to 0, \quad \text{as } M \to \infty.$$

First, we show that the Eq. (5.76) has a unique solution on $[T_{M-1}, T]$ in the Banach space $S_m^p[T_{M-1}, T] \times \Lambda_{m\times k}^p(T_{M-1}, T)$.

Let

$$\bar{\mu}(t) = \int_0^t \mu(r)\,dr.$$

To this end it is sufficient to prove that Γ is a strict contraction on the space $S_m^p[T_{M-1}, T] \times \Lambda_{m\times k}^p(T_{M-1}, T)$ with respect to the (equivalent) norm $\|\!|(Y, Z)|\!\|_M$

$$\|\!|(Y, Z)|\!\|_M^p \stackrel{def}{=} \mathbb{E}\left[\left(\sup_{r\in[T_{M-1},T]} e^{p\bar{\mu}(r)}|Y_r|^p\right) + \left(\int_{T_{M-1}}^T e^{2\bar{\mu}(r)}|Z_r|^2\,dr\right)^{p/2}\right],$$

for M large enough.

Let $(X, U), (X', U') \in S_m^p[0, T] \times \Lambda_{m\times k}^p(0, T)$. Then

$$Y_t - Y_t' = \int_t^T dK_r - \int_t^T \left(Z_r - Z_r'\right) dB_r, \quad t \in [T_{M-1}, T],$$

where

$$K_t = \int_0^t \left[F(r, Y_r, U_r) - F\left(r, Y_r', U_r'\right)\right] dr.$$

Since

$$\begin{aligned}\langle Y_r - Y_r', dK_r\rangle &\leq \left\langle Y_r - Y_r', F\left(r, Y_r', U_r\right) - F\left(r, Y_r', U_r'\right)\right\rangle dr + \mu\left(r\right)\left|Y_r - Y_r'\right|^2 dr \\ &\leq \ell\left(r\right)\left|U_r - U_r'\right|\left|Y_r - Y_r'\right| dr + \left|Y_r - Y_r'\right|^2 d\bar{\mu}\left(r\right)\end{aligned}$$

and

$$\mathbb{E}\left(\sup_{r\in[0,T]} e^{p\bar{\mu}(r)}\left|Y_r - Y_r'\right|^p\right) < \infty,$$

we have, by Proposition 5.2 with $[t, T]$ replaced by $[T_{M-1}, T]$ and $D = R = 0$, $\lambda = 0$,

$$\begin{aligned}&\mathbb{E}\left(\sup_{r\in[T_{M-1},T]} e^{p\bar{\mu}(r)}\left|Y_r - Y_r'\right|^p\right) + \mathbb{E}\left(\int_{T_{M-1}}^T e^{2\bar{\mu}(r)}\left|Z_r - Z_r'\right|^2 dr\right)^{p/2} \\ &\quad\leq C_p\mathbb{E}\left(\int_{T_{M-1}}^T e^{\bar{\mu}(r)}\ell\left(r\right)\left|U_r - U_r'\right| dr\right)^p \\ &\quad\leq C_p\left(\int_{T_{M-1}}^T \ell^2\left(r\right) dr\right)^{p/2}\mathbb{E}\left(\int_{T_{M-1}}^T e^{2\bar{\mu}(r)}\left|U_r - U_r'\right|^2 dr\right)^{p/2} \\ &\quad\leq C_p\left[\boldsymbol{\alpha}\left(\tfrac{T}{M}\right)\right]^{p/2}\left\|\left(X, U\right) - \left(X', U'\right)\right\|_M^p.\end{aligned}$$

Let $M_0 \in \mathbb{N}^*$ be such that

$$C_p\left[\boldsymbol{\alpha}\left(\tfrac{T}{M_0}\right)\right]^{p/2} \leq \frac{1}{2^p}.$$

Then

$$\left\|\Gamma\left(X, U\right) - \Gamma\left(X', U'\right)\right\|_{M_0} \leq \frac{1}{2}\left\|\left(X, U\right) - \left(X', U'\right)\right\|_{M_0}.$$

Hence the Eq. (5.76) has a unique solution in the space $S_m^p\left[T_{M_0-1}, T\right] \times \Lambda_{m\times k}^p\left(T_{M_0-1}, T\right)$. The next step is to solve the equation on the interval $\left[T_{M_0-2}, T_{M_0-1}\right]$ with the final value $Y\left(T_{M_0-1}\right)$. Repeating the same arguments, the proof is completed in M_0 steps. ■

Corollary 5.28. *Consider the BSDE:* $\forall\ t \in [0, T]$, $\mathbb{P}$-*a.s.*

$$Y_t = \eta + S_T - S_t + \int_t^T F\left(s, Y_s, Z_s\right) ds - \int_t^T Z_s dB_s. \tag{5.79}$$

If $p > 1$, $S \in S_m^p[0, T]$, $\eta \in L^p\left(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^m\right)$, *F satisfies the assumptions* ($\mathbf{MH}_F$), *and for all* $\rho \geq 0$

$$\mathbb{E}\left(\int_0^T \sup_{|y|\leq\rho} |F(t, y - S_t, 0)|\, dt\right)^p < \infty$$

then the Eq. (5.79) has a unique solution $(Y, Z) \in S_m^p[0, T] \times \Lambda_{m\times k}^p(0, T)$.

Proof. By the substitutions $\hat{Y}_t = Y_t + S_t$, $\hat{\eta} = \eta + S_T$ and $\hat{F}(t, y, z) = F(t, y - S_t, z)$ the Eq. (5.34) is transformed into

$$\hat{Y}_t = \hat{\eta} + \int_t^T \hat{F}\left(s, \hat{Y}_s, Z_s\right) ds - \int_t^T Z_s dB_s,$$

which satisfies the assumptions of Theorem 5.27. ■

5.3.4.3 The Third BSDE: Monotone Coefficient $\Phi(s, Y_s, Z_s)\, dQ_s$

We now generalize Theorem 5.27 to the case of the general BSDE (5.12) which we recall here:

$$Y_t = \eta + \int_t^T \Phi(s, Y_s, Z_s)\, dQ_s - \int_t^T Z_s dB_s,\ t \in [0, T],\ a.s. \tag{5.80}$$

The assumptions will be those from the beginning of Sect. 5.3.1.

Let $p, a > 1$ and $n_p = 1 \wedge (p - 1)$. Define

$$\bar{\mu}_t = \int_0^t \mu_s dQ_s \quad \text{and} \quad V_t = V_t^{(a,p)} = \int_0^t \mu_s dQ_s + \frac{a}{2n_p}\int_0^t (\ell_s)^2\, ds.$$

We say that $Y \in S_m^p\left([0, T]; e^{\bar{\mu}}\right)$ if $Y \in S_m^0[0, T]$ and

$$\mathbb{E} \sup_{s\in[0,T]} e^{p\bar{\mu}_s} |Y_s|^p < \infty.$$

In the same manner $Z \in \Lambda_{m\times k}^p\left(0, T; e^{\bar{\mu}}\right)$ if $Z \in \Lambda_{m\times k}^0(0, T)$ and

$$\mathbb{E}\left(\int_0^T e^{2\bar{\mu}_s} |Z_s|^2\, ds\right)^{p/2} < \infty.$$

We first prove the following:

Lemma 5.29. *Let* $p > 1$ *and the assumptions (5.13-**BSDE-H**$_\Phi$) (i.e. (5.14) and (5.15)) from Sect. 5.3.1 be satisfied. Moreover assume*

$$\begin{aligned} &(i) \quad \ell \in L^2(0, T) \text{ is a positive deterministic process,} \\ &(ii) \quad \mathbb{E}\left|e^{\bar{\mu}_T}\eta\right|^p + \mathbb{E}\left(\int_0^T e^{\bar{\mu}_s} |\Phi(s, 0, 0)|\, dQ_s\right)^p < \infty. \end{aligned} \tag{5.81}$$

If in addition

(h_1) $\mathbb{E}\left(\int_0^T \sup_{|y|\le\rho} \left|e^{\bar{\mu}_t}\Phi\left(s, e^{-\bar{\mu}_t}y, 0\right) - \mu_t y\right| dQ_t\right)^p < \infty$, *for all* $\rho \ge 0$, *or*

(h_2) $\mu \ge 0$ *and* $\mathbb{E}\left(\int_0^T e^{\bar{\mu}_t} \sup_{|y|\le\rho} |\Phi(t, y, 0)|\, dQ_s\right)^p < \infty$, *for all* $\rho \ge 0$,

then the BSDE (5.80) has a unique solution

$$(Y, Z) \in S_m^p\left([0,T]; e^{\bar{\mu}}\right) \times \Lambda_{m\times k}^p\left(0,T; e^{\bar{\mu}}\right).$$

Proof. Uniqueness follows from Theorem 5.10. To prove the existence we shall use the Banach fixed point theorem. Let $\Gamma : S_m^p\left([0,T]; e^{\bar{\mu}}\right) \times \Lambda_{m\times k}^p\left(0,T; e^{\bar{\mu}}\right) \to S_m^p\left([0,T]; e^{\bar{\mu}}\right) \times \Lambda_{m\times k}^p\left(0,T; e^{\bar{\mu}}\right)$ be defined by $(Y,Z) = \Gamma(X,U)$, where

$$Y_t = \eta + \int_t^T \Phi(s, Y_s, U_s)\, dQ_s - \int_t^T Z_s dB_s. \tag{5.82}$$

(A) Γ *is well defined.*

Let $(X,U) \in S_m^p\left([0,T]; e^{\bar{\mu}}\right) \times \Lambda_{m\times k}^p\left(0,T; e^{\bar{\mu}}\right)$. The function $\tilde{\Phi}(\omega,t,y) = \Phi(\omega,t,y,U_t(\omega))$ is monotone

$$\left\langle y - y', \Phi(r,y,U_r) - \Phi\left(r,y',U_r\right)\right\rangle \le \mu_r \left|y - y'\right|^2.$$

Under (h_1) the assumptions of Corollary 5.26 are satisfied, because we have

$$\mathbb{E}\left(\int_0^T \sup_{|y|\le\rho} \left|e^{\bar{\mu}_t}\Phi(t, e^{\bar{\mu}_t}y, U_t) - \mu_t y\right| dQ_t\right)^p$$
$$\le \mathbb{E}\left(\int_0^T \sup_{|y|\le\rho} \left|e^{\bar{\mu}_t}\Phi(t, e^{\bar{\mu}_t}y, 0) - \mu_t y\right| dQ_t + \int_0^T e^{\bar{\mu}_t}\ell(t)\,|U_t|\, dt\right)^p$$

and

$$\mathbb{E}\left(\int_0^T e^{\bar{\mu}_t}|U_t|\,\ell(t)\, dt\right)^p$$
$$\le \left(\int_0^T \ell^2(t)\, dt\right)^{p/2} \mathbb{E}\left(\int_0^T e^{2\bar{\mu}_t}|U_t|^2\, dt\right)^{p/2}$$
$$< \infty.$$

Under (h_2) the assumptions of Proposition 5.24 are satisfied because for $\mu \ge 0$ we have $\hat{\mu}_t = \int_0^t \mu_s^+ ds = \int_0^t \mu_s ds = \bar{\mu}_t$ and

$$\mathbb{E}\left(\int_0^T e^{\hat{\mu}_t}\sup_{|y|\leq\rho}\Phi(t,y,U_t)dQ_t\right)^p$$
$$\leq \mathbb{E}\left(\int_0^T e^{\bar{\mu}_t}\sup_{|y|\leq\rho}\Phi(t,y,0)dQ_t+\int_0^T e^{\bar{\mu}_t}\ell(t)|U_t|\,dt\right)^p<\infty.$$

(B) Γ *is a strict contraction*. Let $M\in\mathbb{N}^*$ and $0{=}T_0<T_1<\cdots<T_M{=}T$, with $T_i=\dfrac{iT}{M}$. To prove the existence on $[T_{M-1},T]$ of the solution it is sufficient to prove that Γ is a strict contraction on the space $S_m^p\left([T_{M-1},T];e^{\bar{\mu}}\right)\times\Lambda^p_{m\times k}\left(T_{M-1},T;e^{\bar{\mu}}\right)$ with respect to the norm $\|\!|(Y,Z)\|\!|_M$

$$\|\!|(Y,Z)\|\!|_M^p\overset{def}{=}\mathbb{E}\left[\left(\sup_{r\in[T_{M-1},T]}e^{p\bar{\mu}_r}|Y_r|^p\right)+\left(\int_{T_{M-1}}^T e^{2\bar{\mu}_r}|Z_r|^2\,dr\right)^{p/2}\right],$$

for M large enough. The proof continues exactly as in Theorem 5.27. Iteratively the existence follows on every interval $[T_{i-1},T_i]$, for $i=M,M-1,\ldots,2,1$, and finally we get the existence on $[0,T]$. ■

Theorem 5.30. *Let the assumptions (5.13-**BSDE-H**$_\Phi$) (i.e. (5.14) and (5.15)) from Sect. 5.3.1 be satisfied. Let $p,a>1$ be fixed, $n_p=1\wedge(p-1)$*

$$\bar{\mu}_t=\int_0^t\mu_s dQ_s\quad\text{and}\quad V_t\overset{def}{=}V_t^{(a,p)}=\int_0^t\mu_s dQ_s+\frac{a}{2n_p}\int_0^t(\ell_s)^2\,ds.$$

Assume there exists a $\delta>\frac{p}{p-1}$ such that for $q=\frac{p\delta}{p+\delta}$

$$\begin{aligned}(i)\quad &\mathbb{E}\,e^{pV_T}|\eta|^p+\mathbb{E}\left(\int_0^T e^{V_s}|\Phi(s,0,0)|\,dQ_s\right)^p<\infty,\\(ii)\quad &\mathbb{E}\left(\int_0^T(\ell_s)^2\,ds\right)^{\delta/2}<\infty,\\(iii)\quad &\mathbb{E}\,\exp\left[\tfrac{\delta a}{2}\left(\tfrac{1}{n_q}-\tfrac{1}{n_p}\right)\int_0^T(\ell_s)^2\,ds\right]<\infty.\end{aligned}\tag{5.83}$$

If in addition

(h_1) $\mathbb{E}\left(\int_0^T\sup_{|y|\leq\rho}\left|e^{\bar{\mu}_t}\Phi\left(s,e^{-\bar{\mu}_t}y,0\right)-\mu_t y\right|dQ_t\right)^p<\infty$, *for all $\rho\geq0$, or*

(h_2) $\mu\geq0$ *and* $\mathbb{E}\left(\int_0^T e^{\bar{\mu}_t}\sup_{|y|\leq\rho}|\Phi(t,y,0)|\,dQ_s\right)^p<\infty$, *for all $\rho\geq0$,*

then the BSDE (5.80) has a unique solution $(Y,Z)\in S_m^0([0,T])\times\Lambda^0_{m\times k}(0,T)$ such that

$$\mathbb{E} \sup_{t\in[0,T]} e^{pV_s} |Y_s|^p + \mathbb{E} \left(\int_0^T e^{2V_s} |Z_s|^2 ds \right)^{p/2} < \infty.$$

Moreover, for all $t \in [0, T]$*:*

$$\begin{aligned}
&\mathbb{E}^{\mathcal{F}_t} \sup_{s\geq t} e^{pV_s} |Y_s|^p + \mathbb{E}^{\mathcal{F}_t} \left(\int_t^T e^{2V_s} |Z_s|^2 ds \right)^{p/2} \\
&\quad \leq C_p \, \mathbb{E}^{\mathcal{F}_t} \left[e^{pV_T} |\eta|^p + \left(\int_t^T e^{V_s} |\Phi(s,0,0)| \, dQ_s \right)^p \right].
\end{aligned} \tag{5.84}$$

Proof. Uniqueness follows from Theorem 5.10.

Existence. By Lemma 5.29 we infer that the approximating BSDE

$$Y_t^n = \eta + \int_t^T \Phi\left(s, Y_s^n, Z_s^n \mathbf{1}_{[0,n]}(\ell_s)\right) dQ_s - \int_t^T Z_s^n dB_s \tag{5.85}$$

has a unique solution $(Y^n, Z^n) \in S_m^p\left([0,T]; e^{\bar{\mu}}\right) \times \Lambda_{m\times k}^p\left(0, T; e^{\bar{\mu}}\right)$.

Let $\ell_s^n = \ell_s \mathbf{1}_{[0,n]}(\ell_s)$ and

$$V_t^n \overset{def}{=} \int_0^t \left(\mu_s dQ_s + \frac{a}{2n_p} \left(\ell_s^n\right)^2 ds \right).$$

We have for all $n, i \in \mathbb{N}$

$$\bar{\mu}_t \leq V_t^n \leq V_t^{n+i} \leq \bar{\mu}_t + \frac{a}{2n_p} (n+i)^2 T.$$

Therefore

$$\mathbb{E} \sup_{t\in[0,T]} e^{pV_t^{n+i}} |Y_t^n|^p \leq C_{n,i} \left(\mathbb{E} \sup_{t\in[0,T]} e^{2p\bar{\mu}_t} |Y_t^n|^{2p} \right)^{1/2} < \infty.$$

Since

$$\begin{aligned}
&\left\langle Y_t^n, \Phi\left(t, Y_t^n, Z_t^n \mathbf{1}_{[0,n]}(\ell_t)\right) dQ_t \right\rangle \\
&\leq |Y_t^n| \, |\Phi(t,0,0)| \, dQ_t + |Y_t^n|^2 dV_t^n + \frac{n_p}{2a} |Z_t^n|^2 dt \\
&\leq |Y_t^n| \, |\Phi(t,0,0)| \, dQ_t + |Y_t^n|^2 dV_t^{n+i} + \frac{n_p}{2a} |Z_t^n|^2 dt
\end{aligned}$$

we obtain, by Proposition 5.2-A, that

$$\begin{aligned}&\mathbb{E}^{\mathcal{F}_t}\sup_{s\in[t,T]}e^{pV_s^{n+i}}\left|Y_s^n\right|^p+\mathbb{E}^{\mathcal{F}_t}\left(\int_t^T e^{2V_s^{n+i}}\left|Z_s^n\right|^2 ds\right)^{p/2}\\&\quad\le C_q\,\mathbb{E}^{\mathcal{F}_t}\Big[e^{pV_T^{n+i}}\left|\eta\right|^p+\Big(\int_t^T e^{V_s^{n+i}}\left|\Phi\left(s,0,0\right)\right|dQ_s\Big)^p\Big].\end{aligned}$$

By Beppo Levi's monotone convergence Theorem 1.9 it follows for $i\to\infty$ that for all $t\in[0,T]$,

$$\begin{aligned}&\mathbb{E}^{\mathcal{F}_t}\sup_{s\ge t}e^{pV_s}\left|Y_s^n\right|^p+\mathbb{E}^{\mathcal{F}_t}\left(\int_t^T e^{2V_s}\left|Z_s^n\right|^2 ds\right)^{p/2}\\&\quad\le C_p\,\mathbb{E}^{\mathcal{F}_t}\Big[e^{pV_T}\left|\eta\right|^p+\Big(\int_t^T e^{V_s}\left|\Phi\left(s,0,0\right)\right|dQ_s\Big)^p\Big].\end{aligned}\tag{5.86}$$

Consequently by (5.83-i) for all $n\in\mathbb{N}^*$,

$$\mathbb{E}\sup_{s\in[0,T]}e^{pV_s}\left|Y_s^n\right|^p+\mathbb{E}\left(\int_0^T e^{2V_s}\left|Z_s^n\right|^2 ds\right)^{p/2}\le C<\infty.$$

Let $\delta>\frac{p}{p-1}$, $q=\frac{p\delta}{p+\delta}$, $n_q\overset{def}{=}1\wedge(q-1)$ and $n_p\overset{def}{=}1\wedge(p-1)$ satisfy (5.83-ii, iii). Clearly $1<q<p$ and $0<n_q\le n_p$. If we define

$$\Delta_t=\frac{a}{2}\left(\frac{1}{n_q}-\frac{1}{n_p}\right)\int_0^t(\ell_s)^2 ds\quad\text{and}$$
$$V_t^{(a,q)}=\int_0^t\left[\mu_s dQ_s+\frac{a}{2n_q}(\ell_s)^2 ds\right]=V_t+\Delta_t,$$

we have, for all $n\in\mathbb{N}^*$,

$$\begin{aligned}&\mathbb{E}\sup_{s\in[0,T]}e^{qV_s^{(a,q)}}\left|Y_s^n\right|^q+\mathbb{E}\left(\int_0^T e^{2V_s^{(a,q)}}\left|Z_s^n\right|^2 ds\right)^{q/2}\\&\le\mathbb{E}\left(e^{q\Delta_T}\sup_{s\in[0,T]}e^{qV_s}\left|Y_s^n\right|^q\right)+\mathbb{E}\Big[e^{q\Delta_T}\left(\int_0^T e^{2V_s}\left|Z_s^n\right|^2 ds\right)^{q/2}\Big]\\&\le\left(\mathbb{E}e^{\delta\Delta_T}\right)^{\frac{p}{p+\delta}}\left[\left(\mathbb{E}\sup_{s\in[0,T]}e^{pV_s}\left|Y_s^n\right|^p\right)^{\frac{\delta}{p+\delta}}+\left(\mathbb{E}\left(\int_0^T e^{2V_s}\left|Z_s^n\right|^2 ds\right)^{p/2}\right)^{\frac{\delta}{p+\delta}}\right]\\&\le C<\infty.\end{aligned}$$

Hence for all $n, i \in \mathbb{N}^*$

$$\mathbb{E} \sup_{s\in[0,T]} e^{qV_s^{(a,q)}} \left|Y_s^n - Y_s^{n+i}\right|^q < \infty.$$

Since

$$\begin{aligned}
&\left\langle Y_s^n - Y_s^{n+i}, \Phi\left(s, Y_s^n, Z_s^n \mathbf{1}_{[0,n]}(\ell_s)\right) - \Phi\left(s, Y_s^{n+i}, Z_s^{n+i} \mathbf{1}_{[0,n+i]}(\ell_s)\right)\right\rangle dQ_s \\
&\leq \left|Y_s^n - Y_s^{n+i}\right|^2 \mu_s dQ_s + \left|Y_s^n - Y_s^{n+i}\right| \ell_s \left|Z_s^n \mathbf{1}_{[0,n]}(\ell_s) - Z_s^{n+i} \mathbf{1}_{[0,n+i]}(\ell_s)\right| ds \\
&\leq \left|Y_s^n - Y_s^{n+i}\right| \ell_s \left|Z_s^n\right| \left|\mathbf{1}_{[0,n]}(\ell_s) - \mathbf{1}_{[0,n+i]}(\ell_s)\right| ds \\
&\quad + \left|Y_s^n - Y_s^{n+i}\right|^2 \left(\mu_s dQ_s + \frac{a}{2n_q} \ell_s^2 ds\right) + \frac{n_q}{2a} \left|Z_s^n - Z_s^{n+i}\right|^2 ds,
\end{aligned}$$

by Proposition 5.2-A, we infer that

$$\begin{aligned}
&\mathbb{E}\left(\sup_{s\in[0,T]} e^{qV_s^{(a,q)}} \left|Y_s^n - Y_s^{n+i}\right|^q\right) + \mathbb{E}\left(\int_0^T e^{2V_s^{(a,q)}} \left|Z_s^n - Z_s^{n+i}\right|^2 ds\right)^{q/2} \\
&\leq C_q\, \mathbb{E}\left(\int_0^T \mathbf{1}_{(n,\infty)}(\ell_s)\, e^{V_s^{(a,q)}} \ell_s \left|Z_s^n\right| ds\right)^q \\
&\leq C_q \mathbb{E}\left[\left(\int_0^T \ell_s^2 \mathbf{1}_{(n,\infty)}(\ell_s)\, ds\right)^{q/2} \left(\int_0^T e^{2V_s^{(a,q)}} \left|Z_s^n\right|^2 ds\right)^{q/2}\right] \\
&\leq C_q \left[\mathbb{E}\left(\int_0^T \ell_s^2 \mathbf{1}_{(n,\infty)}(\ell_s)\, ds\right)^{\delta/2}\right]^{\frac{p}{p+\delta}} \left[\mathbb{E}\left(\int_0^T e^{2V_s^{(a,q)}} \left|Z_s^n\right|^2 ds\right)^{p/2}\right]^{\frac{\delta}{p+\delta}} \\
&\to 0,
\end{aligned}$$

as $n \to \infty$.

We deduce that there exists a pair $(Y, Z) \in S_m^0[0,T] \times \Lambda_{m\times k}^0(0,T)$ such that for $q = \frac{p\delta}{p+\delta}$

$$\lim_{n\to\infty} \mathbb{E}\left(\sup_{s\geq 0} e^{qV_s^{(a,q)}} \left|Y_s^n - Y_s\right|^q\right) + \mathbb{E}\left[\left(\int_0^T e^{2V_s^{(a,q)}} \left|Z_s^n - Z_s\right|^2 ds\right)^{q/2}\right] = 0.$$

Now the inequality (5.84) clearly follows from (5.86) by Fatou's Lemma.

Finally passing to the limit in (5.85) we deduce via Lemma 5.16 that (Y, Z) is a solution of BSDE (5.80). ■

5.3.5 Linear BSDEs

Let $m = 1$ and consider the BSDE

$$Y_t = \eta + \int_t^T [(a_s Y_s + b_s)\, dQ_s + \langle c_s, Z_s\rangle\, ds] - \int_t^T \langle Z_s, dB_s\rangle\,, \tag{5.87}$$

where

- η is an $\mathcal{F}_T$-measurable random variable;
- Q is a $\mathcal{P}$-m.i.c.s.p. such that $Q_0 = 0$;
- $(a_t)_{t\geq 0}$, $(b_t)_{t\geq 0}$ are $\mathbb{R}$-valued $\mathcal{P}$-m.s.p. and $(c_t)_{t\geq 0}$ is an $\mathbb{R}^k$-valued $\mathcal{P}$-m.s.p.;
- for some $p > 1$ and for all $\lambda \geq 0$,

$$\begin{aligned} &(j) \quad \mathbb{E}\left[\left(1+|\eta|^p\right)\exp\left(\lambda V_T\right)\right] < \infty,\\ &(jj) \quad \mathbb{E}\Big(\int_0^T |b_s| \exp\left(\lambda V_s\right) dQ_s\Big)^p < \infty, \end{aligned} \tag{5.88}$$

  where

$$V_t = \int_0^t |a_s|\, dQ_s + \frac{1}{n_p}\int_0^t |c_s|^2\, ds.$$

  By Theorem 5.21 the BSDE (5.87) has a unique solution satisfying

$$\begin{aligned} &\mathbb{E}\sup_{s\in[0,T]}\left|e^{V_s}Y_s\right|^p + \mathbb{E}\left(\int_0^T e^{2V_s}|Z_s|^2\, ds\right)^{p/2}\\ &\quad \leq C_p\, \mathbb{E}\Big[\left|e^{V_T}\eta\right|^p + \Big(\int_0^T e^{V_s}|b_s|\, dQ_s\Big)^p\Big]. \end{aligned}$$

Let

$$\Gamma_t = \exp\Big[\int_0^t \Big(a_r dQ_r - \frac{1}{2}|c_r|^2\, dr\Big) + \int_0^t \langle c_r, dB_r\rangle\Big].$$

Then

$$\begin{aligned} d\Gamma_t &= \Gamma_t a_t dQ_t + \Gamma_t \langle c_t, dB_t\rangle\,,\\ d\Gamma_t^{-1} &= \Gamma_t^{-1}\Big(-a_t dQ_t + |c_t|^2\, dt\Big) - \Gamma_t^{-1}\langle c_t, dB_t\rangle\,. \end{aligned}$$

Since for all $\delta > 0$, $\mathbb{E}\left[\exp\left(\delta\Lambda_T\right)\right] < \infty$ we have $\mathbb{E}\sup_{s\in[0,T]}|\Gamma_s|^\delta < \infty$ for all $\delta > 0$. Consequently there exists $1 < q < p$ such that

$$\mathbb{E}\left|\Gamma_T \eta + \int_0^T \Gamma_s b_s dQ_s\right|^q < \infty. \tag{5.89}$$

By the representation Theorem 2.42 there exists a unique stochastic process $R \in \Lambda^q_{1\times k}(0,T)$ such that

$$\Gamma_T \eta + \int_0^T \Gamma_s b_s dQ_s = \mathbb{E}\left(\Gamma_T \eta + \int_0^T \Gamma_s b_s dQ_s\right) + \int_0^T \langle R_s, dB_s\rangle .$$

Proposition 5.31. *Let the assumption (5.88) be satisfied. Then the solution of the BSDE (5.87) is given by*

$$\begin{aligned} (a) \quad Y_t &= \Gamma_t^{-1}\, \mathbb{E}^{\mathcal{F}_t}\left[\Gamma_T \eta + \int_t^T \Gamma_s b_s dQ_s\right], \\ (b) \quad Z_t &= \Gamma_t^{-1} R_t - c_t Y_t . \end{aligned} \tag{5.90}$$

Proof. It is sufficient to verify that (Y, Z) given by (5.90) is a solution of (5.87). We have

$$\begin{aligned} Y_t &= \Gamma_t^{-1}\, \mathbb{E}^{\mathcal{F}_t}\left[\Gamma_T \eta + \int_t^T \Gamma_s b_s dQ_s\right] \\ &= \Gamma_t^{-1}\left[\mathbb{E}\left(\Gamma_T \eta + \int_0^T \Gamma_s b_s dQ_s\right) + \int_0^t \langle R_s, dB_s\rangle - \int_0^t \Gamma_s b_s dQ_s\right] \\ &= \Gamma_t^{-1}\left[\mathbb{E}\left(\Gamma_T \eta + \int_0^T \Gamma_s b_s dQ_s\right) + \int_0^t \langle \Gamma_s Y_s c_s + \Gamma_s Z_s, dB_s\rangle - \int_0^t \Gamma_s b_s dQ_s\right]. \end{aligned}$$

Consequently, from Itô's formula,

$$\begin{aligned} dY_t &= \left[\Gamma_t^{-1}\left(-a_t\, dQ_t + |c_t|^2\, dt\right) - \Gamma_t^{-1}\langle c_t, dB_t\rangle\right]\Gamma_t Y_t \\ &\quad + \Gamma_t^{-1}\left[\langle \Gamma_t Y_t c_t + \Gamma_t Z_t, dB_t\rangle - \Gamma_t b_t\, dQ_t\right] - \Gamma_t^{-1}\langle c_t, c_t \Gamma_t Y_t + \Gamma_t Z_t\rangle\, dt \\ &= \left[-a_t Y_t\, dQ_t - b_t\, dQ_t - \langle c_t, Z_t\rangle\, dt\right] + \langle Z_t, dB_t\rangle . \end{aligned}$$

Since, moreover, $Y_T = \eta$, we conclude that (Y, Z) is a solution of the BSDE (5.87). ∎

5.3.6 Comparison Results

In this section we again restrict ourselves to the case $m = 1$.

5.3.6.1 Lipschitz Case

Let $(Y, Z) \in S^0[0,T] \times \Lambda_k^0(0,T)$ be a solution of the BSDE

$$Y_t = \eta + \int_t^T \Phi(s, Y_s, Z_s)\, dQ_s - \int_t^T \langle Z_s, dB_s\rangle \tag{5.91}$$

and $(\tilde{Y}, \tilde{Z}) \in S^0[0,T] \times \Lambda_k^0(0,T)$ a solution of the BSDE

$$\tilde{Y}_t = \tilde{\eta} + \int_t^T \tilde{\Phi}(s, \tilde{Y}_s, \tilde{Z}_s)\, dQ_s - \int_t^T \langle \tilde{Z}_s, dB_s\rangle. \tag{5.92}$$

Assume that the functions $\Phi, \tilde{\Phi} : \Omega \times [0,\infty[\times \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}$ are $(\mathcal{P}, \mathbb{R} \times \mathbb{R}^k)$-Carathéodory functions ($\mathcal{P}$-m.s.p. with respect to (ω, t) and continuous with respect to $(x, z) \in \mathbb{R} \times \mathbb{R}^k$) such that

$$\int_0^T |\Phi(s, Y_s, Z_s)|\, dQ_s + \int_0^T \left|\tilde{\Phi}(s, \tilde{Y}_s, \tilde{Z}_s)\right| dQ_s < \infty, \quad \text{a.s.} \tag{5.93}$$

We give a comparison result in the case when one of the two functions Φ and $\tilde{\Phi}$ satisfies some Lipschitz conditions.

Let $p > 1$. Without loss of generality we assume that Φ satisfies the assumptions of Theorem 5.21. Then the Eq. (5.91) has a unique solution (Y, Z) satisfying

$$\begin{aligned}\mathbb{E} \sup_{s\in[0,T]} \left|e^{V_s} Y_s\right|^p + \mathbb{E}\left(\int_0^T e^{2V_s} |Z_s|^2\, ds\right)^{p/2} \\ \leq C_p\, \mathbb{E}\left[\left|e^{V_T}\eta\right|^p + \left(\int_0^T e^{V_s} |\Phi(s,0,0)|\, dQ_s\right)^p\right].\end{aligned}$$

Proposition 5.32. *Let $p > 1$ and the assumptions of Theorem 5.21 be satisfied. Assume that $(\tilde{Y}, \tilde{Z})$ is a solution of the BSDE (5.92) and for all $\delta \geq 0$,*

$$\mathbb{E}\left(|\eta - \tilde{\eta}| \exp(\delta V_T) + \int_0^T \left|\Phi(s, \tilde{Y}_s, \tilde{Z}_s) - \tilde{\Phi}(s, \tilde{Y}_s, \tilde{Z}_s)\right| \exp(\delta V_s)\, dQ_s\right)^p < \infty.$$

If

$$\begin{array}{ll}(i) & \eta \geq \tilde{\eta}, \quad \mathbb{P}\text{-a.s. and} \\ (ii) & \Phi(t, \tilde{Y}_t, \tilde{Z}_t) \geq \tilde{\Phi}(t, \tilde{Y}_t, \tilde{Z}_t),\ d\mathbb{P} \otimes dQ_t\text{-a.e. on } \Omega \times \mathbb{R}_+.\end{array}$$

(a) Then $\mathbb{P}$-a.s. $\omega \in \Omega$, $Y_t(\omega) \geq \tilde{Y}_t(\omega)$, for all $t \in [0,T]$.
(b) If moreover there exists a $t_0 \in [0,T[$ such that $\mathbb{P}$-a.s.

$$(\eta - \tilde{\eta}) + \int_{t_0}^{T} \left[\Phi(s, \tilde{Y}_s, \tilde{Z}_s) - \tilde{\Phi}(s, \tilde{Y}_s, \tilde{Z}_s)\right] dQ_s > 0$$

then $Y_{t_0} > \tilde{Y}_{t_0}$ $\mathbb{P}$*-a.s. In particular if* $\eta > \tilde{\eta}$*,* $\mathbb{P}$*-a.s., then* $Y_t(\omega) > \tilde{Y}_t(\omega)$*, for all* $t \in [0, T]$*,* $\mathbb{P}$*-a.s.* $\omega \in \Omega$*.*

Proof. Observe that $Y_t - \tilde{Y}_t$ can be written in the form

$$Y_t - \tilde{Y}_t = (\eta - \tilde{\eta}) + \int_t^T \left\{\left[a_s\left(Y_s - \tilde{Y}_s\right) + b_s\right] dQ_s + \left\langle c_s, Z_s - \tilde{Z}_s\right\rangle ds\right\} - \int_t^T \left(Z_s - \tilde{Z}_s\right) dB_s,$$

where

$$a_s = \begin{cases} \dfrac{1}{Y_s - \tilde{Y}_s}\left[\Phi(s, Y_s, Z_s) - \Phi(s, \tilde{Y}_s, Z_s)\right], & \text{if } Y_s - \tilde{Y}_s \neq 0, \\ 0, & \text{if } Y_s - \tilde{Y}_s = 0, \end{cases}$$

$b_s = \Phi(s, \tilde{Y}_s, \tilde{Z}_s) - \tilde{\Phi}(s, \tilde{Y}_s, \tilde{Z}_s)$, and

$$c_s = \begin{cases} \dfrac{Z_s - \tilde{Z}_s}{\alpha_s \left|Z_s - \tilde{Z}_s\right|^2}\left[\Phi(s, \tilde{Y}_s, Z_s) - \Phi(s, \tilde{Y}_s, \tilde{Z}_s)\right], & \text{if } \alpha_s\left(Z_s - \tilde{Z}_s\right) \neq 0, \\ 0, & \text{if } \alpha_s\left(Z_s - \tilde{Z}_s\right) = 0, \end{cases}$$

(recall that α is a $\mathcal{P}$-m.s.p. such that $\alpha_s dQ_s = ds$).

From $|a_s| \leq L_s$, $|c_s| \leq \ell_s$, the assumption of the Proposition, and the argument of the preceding section, we deduce that

$$\sup_{s\in[0,T]} \left[\left|Y_s - \tilde{Y}_s\right|^p \exp(\delta V_s)\right] + \mathbb{E}\left(\int_0^T \left|Z_s - \tilde{Z}_s\right|^2 \exp(\delta V_s)\, ds\right)^{p/2} < \infty,$$

for all $\delta \geq 0$. Hence by Proposition 5.31

$$Y_t - \tilde{Y}_t = \Gamma_t^{-1}\, \mathbb{E}^{\mathcal{F}_t}\left[\Gamma_T(\eta - \tilde{\eta}) + \int_t^T \Gamma_s b_s dQ_s\right],$$

which clearly yields the conclusions of Proposition 5.32. ■

5.3.6.2 Monotone Case

We now give a comparison result for the solutions of the Eqs. (5.91) and (5.92) in the case when one of the two functions Φ and $\tilde{\Phi}$ satisfies a monotonicity condition. To be precise we assume without loss of generality that Φ satisfies the assumptions (5.13-**BSDE-H**$_\Phi$). Let

$$V_t = \int_0^t \left(\mu_r^+ dQ_r + \frac{a}{2n_p} (\ell_r)^2 dr \right).$$

Then for $a, p > 1$ and $n_p = (p-1) \wedge 1$,

$$\begin{aligned}
&\left(\tilde{Y}_r - Y_r\right)^+ \left[\Phi\left(r, \tilde{Y}_r, \tilde{Z}_r\right) - \Phi\left(r, Y_r, Z_r\right)\right] dQ_r \\
&\leq \left[\mu_r^+ \left(\left(Y_r - \tilde{Y}_r\right)^+\right)^2 + \ell_r \alpha_r \left(\tilde{Y}_r - Y_r\right)^+ \left|Z_r - \tilde{Z}_r\right|\right] dQ_r \\
&\leq \left[\left(Y_r - \tilde{Y}_r\right)^+\right]^2 dV_r + \frac{n_p}{2a} \mathbf{1}_{\tilde{Y}_r - Y_r > 0} \left|Z_r - \tilde{Z}_r\right|^2 dr.
\end{aligned}$$

Proposition 5.33. *Let the assumptions (5.13-**BSDE-H**$_\Phi$) be satisfied. Let $(Y, Z) \in S^0[0,T] \times \Lambda_k^0(0,T)$ be a solution of (5.91) and $\left(\tilde{Y}, \tilde{Z}\right) \in S^0[0,T] \times \Lambda_k^0(0,T)$ be a solution of (5.92), such that (5.93) and the condition*

$$\mathbb{E} \left\| \left(\tilde{Y} - Y\right)^+ e^V \right\|_T^p < \infty$$

are satisfied. Assume that $\mathbb{P}$-a.s.:

(*i*) $\eta \geq \tilde{\eta}$,
(*ii*) $\Phi(t, y, z) \geq \tilde{\Phi}(t, y, z)$, *for all* $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^k$.

Then $\mathbb{P}$-a.s., $Y_t(\omega) \geq \tilde{Y}_t(\omega)$, for all $t \in [0, T]$.

Proof. Recall from Proposition 2.33 that if

$$dX_t = dK_t + \langle G_t, dB_t \rangle,$$

then

$$dX_t^+ = \theta(X_t) dK_t + \theta(X_t) \langle G_t, dB_t \rangle + dP_t,$$

where

$$\theta(x) = \begin{cases} 0, & \text{if } x < 0, \\ \frac{1}{2}, & \text{if } x = 0, \\ 1, & \text{if } x > 0, \end{cases}$$

and $\{P_t : t \geq 0\}$, $P_0 = 0$, is an increasing continuous stochastic process defined by (2.33).

We have

$$d\left(\tilde{Y}_t - Y_t\right) = -\left[\tilde{\Phi}\left(t, \tilde{Y}_t, \tilde{Z}_t\right) - \Phi\left(t, Y_t, Z_t\right)\right] dQ_t + \left\langle \tilde{Z}_t - Z_t, dB_t \right\rangle,$$

and therefore

$$\left(\tilde{Y}_t - Y_t\right)^+ = (\tilde{\eta} - \eta)^+ + \int_t^T dK_r - \int_t^T \theta\left(\tilde{Y}_r - Y_r\right)\left\langle \tilde{Z}_r - Z_r, dB_r \right\rangle,$$

with

$$dK_r = \theta\left(\tilde{Y}_r - Y_r\right)\left[\left(\tilde{\Phi}\left(r, \tilde{Y}_r, \tilde{Z}_r\right) - \Phi\left(r, Y_r, Z_r\right)\right) dQ_r\right] - dP_r,$$

and (see (2.33))

$$P_t = \lim_{\varepsilon \to 0_+} \frac{1}{2\varepsilon} \int_0^t \rho\left(\frac{\tilde{Y}_s - Y_s}{\varepsilon}\right) \left|\tilde{Z}_s - Z_s\right|^2 ds.$$

Since for $a, p > 1$ and $n_p = (p-1) \wedge 1$,

$$\begin{aligned} \left(\tilde{Y}_r - Y_r\right)^+ dK_r &\le \left(\tilde{Y}_r - Y_r\right)^+ \left[\Phi\left(r, \tilde{Y}_r, \tilde{Z}_r\right) - \Phi\left(r, Y_r, Z_r\right)\right] dQ_r \\ &\le \left[\left(Y_r - \tilde{Y}_r\right)^+\right]^2 dV_r + \frac{n_p}{2a} \theta\left(\tilde{Y}_r - Y_r\right) \left|Z_r - \tilde{Z}_r\right|^2 dr, \end{aligned}$$

we obtain, by the inequality (6.107) from Proposition 6.80, that for all $0 \le t \le T$:

$$e^{pV_t} \left[\left(\tilde{Y}_t - Y_t\right)^+\right]^p \le \mathbb{E}^{\mathcal{F}_t} e^{pV_T} \left[(\tilde{\eta} - \eta)^+\right]^p = 0, \quad \mathbb{P}\text{-a.s.}$$

Consequently for all $0 \le t \le T$:

$$Y_t \ge \tilde{Y}_t, \quad \mathbb{P}\text{-a.s.}$$

■

We now give a strict comparison result in the case of monotone coefficients. Namely, we consider a solution $(Y, Z) \in S^0[0, T] \times \Lambda_k^0(0, T)$ of the BSDE

$$Y_t = \eta + \int_t^T \Phi\left(s, Y_s, Z_s\right) dQ_s - \int_t^T \left\langle Z_s, dB_s \right\rangle, \quad a.s., \quad t \in [0, T], \tag{5.94}$$

and a solution $\left(\tilde{Y}, \tilde{Z}\right) \in S^0[0, T] \times \Lambda_k^0(0, T)$ of the BSDE

$$\tilde{Y}_t = \tilde{\eta} + \int_t^T \tilde{\Phi}\left(s, \tilde{Y}_s, \tilde{Z}_s\right) dQ_s - \int_t^T \left\langle \tilde{Z}_s, dB_s \right\rangle, \tag{5.95}$$

where $\Phi, \tilde{\Phi} : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}$ satisfy:
(**CR1**)

- $\Phi(\cdot,t,y,z), \tilde{\Phi}(\cdot,t,y,z)$ *are* $\mathcal{F}_t$*-measurable for all* $(t,y,z) \in [0,T] \times \mathbb{R} \times \mathbb{R}^k$,
- $\Phi(\omega,\cdot,\cdot,\cdot), \tilde{\Phi}(\omega,\cdot,\cdot,\cdot)$ *are continuous* $\mathbb{P}$-a.s. $\omega \in \Omega$,
- Φ *satisfies the assumption* (5.13-**BSDE-H**$_\Phi$).

We have

$$Y_t - \tilde{Y}_t = (\eta - \tilde{\eta}) + \int_t^T \left[b_s dQ_s + \langle c_s, Z_s - \tilde{Z}_s \rangle ds\right] - \int_t^T \left(Z_s - \tilde{Z}_s\right) dB_s,$$

with $b_s = \Phi(s, Y_s, \tilde{Z}_s) - \tilde{\Phi}(s, \tilde{Y}_s, \tilde{Z}_s)$ and

$$c_s = \begin{cases} \dfrac{Z_s - \tilde{Z}_s}{\alpha_s \left|Z_s - \tilde{Z}_s\right|^2} \left[\Phi(s, Y_s, Z_s) - \Phi\left(s, Y_s, \tilde{Z}_s\right)\right], & \text{if } \alpha_s \left(Z_s - \tilde{Z}_s\right) \neq 0, \\ 0, & \text{if } \alpha_s \left(Z_s - \tilde{Z}_s\right) = 0, \end{cases}$$

(recall that α is a $\mathcal{P}$-m.s.p. such that $\alpha_s dQ_s = ds$). Note that $|c_s| \leq \ell_s$.

Assume that
(**CR2**)

- η *and* $\tilde{\eta}$ *are* $\mathcal{F}_T$*-measurable random variables;*
- *for some* $p > 1$ *and for all* $\delta \geq 0$,

$$\begin{aligned} (j) \quad & \mathbb{E}\left[\left(1 + |\eta - \tilde{\eta}|^p\right) \exp\left(\delta \int_0^T (\ell_r)^2 dr\right)\right] < \infty, \\ (jj) \quad & \mathbb{E}\left[\int_0^T \left|\Phi(s, Y_s, \tilde{Z}_s) - \tilde{\Phi}(s, \tilde{Y}_s, \tilde{Z}_s)\right| \exp\left(\delta \int_0^s (\ell_r)^2 dr\right) dQ_s\right]^p < \infty. \end{aligned}$$

Then by Proposition 5.31, for all $\delta \geq 0$

$$\mathbb{E} \sup_{s \in [0,T]} \left[\left|Y_s - \tilde{Y}_s\right|^p e^{\delta \int_0^s (\ell_r)^2 dr}\right] + \mathbb{E} \left(\int_0^T \left|Z_s - \tilde{Z}_s\right|^2 e^{\delta \int_0^s (\ell_r)^2 dr} ds\right)^{p/2} < \infty,$$

and for any stopping times $0 \leq \theta \leq \sigma \leq T$

$$\Gamma_\theta \left(Y_\theta - \tilde{Y}_\theta\right) = \mathbb{E}^{\mathcal{F}_\theta}\left[\Gamma_\sigma \left(Y_\sigma - \tilde{Y}_\sigma\right) + \int_\theta^\sigma \Gamma_s \left(\Phi(s, Y_s, \tilde{Z}_s) - \tilde{\Phi}(s, \tilde{Y}_s, \tilde{Z}_s)\right) dQ_s\right], \tag{5.96}$$

where

$$\Gamma_t = \exp\left[\int_0^t \langle c_r, dB_r \rangle - \int_0^t \frac{1}{2} |c_r|^2 dr\right].$$

Proposition 5.34. *Let* $(Y, Z) \in S^0[0,T] \times \Lambda_k^0(0,T)$ *be a solution for (5.94) and* $(\tilde{Y}, \tilde{Z}) \in S^0[0,T] \times \Lambda_k^0(0,T)$ *be a solution for (5.95), such that*

$$\mathbb{E}\left\| (\tilde{Y} - Y)^+ \exp\left(\int_0^{\cdot} \mu_r^+ dQ_r\right)\right\|_T^p < \infty.$$

Assume that the assumptions (**CR1**), (**CR2**) *are satisfied and* $\tilde{Z}$ *is a continuous stochastic process.*
If $0 \le t_0 < T$, $A \in \mathcal{F}_{t_0}$ *and*

$$\begin{array}{ll} (i) & \eta \ge \tilde{\eta}, \quad \mathbb{P}\text{-a.s.}, \\ (ii) & \Phi(t,y,z) \ge \tilde{\Phi}(t,y,z),\ \forall\ (t,y,z) \in [0,T]\times\mathbb{R}\times\mathbb{R}^k, \quad \mathbb{P}\text{-a.s.}, \\ (iii) & \Phi\left(\omega, t_0, Y_{t_0}, \tilde{Z}_{t_0}\right) > \tilde{\Phi}\left(\omega, t_0, Y_{t_0}, \tilde{Z}_{t_0}\right),\ \mathbb{P}\text{-a.s. } \omega \in A, \\ (iv) & Q_{t_0} < Q_t, \quad \text{for } t_0 < t \le T, \quad \mathbb{P}\text{-a.s.}, \end{array}$$

then

$$\begin{array}{ll} (j) & Y_t(\omega) \ge \tilde{Y}_t(\omega),\ \forall\ t \in [0,T], \quad \mathbb{P}\text{-a.s. } \omega \in \Omega, \quad \text{and} \\ (jj) & Y_{t_0}(\omega) > \tilde{Y}_{t_0}(\omega), \quad \mathbb{P}\text{-a.s. } \omega \in A. \end{array}$$

Proof. By Theorem 5.33 we have

$$\mathbb{P}\text{-a.s.}, \quad Y_t(\omega) \ge \tilde{Y}_t(\omega), \quad \text{for all } t \in [0,T].$$

Assume that $\mathbb{P}\left(\{Y_{t_0} = \tilde{Y}_{t_0}\} \cap A\right) > 0$. Let the stopping time

$$\begin{aligned}\tau = \inf\Big\{ s \in [t_0, T] : & \Gamma_s\left[\Phi(s, Y_s, \tilde{Z}_s) - \tilde{\Phi}(s, \tilde{Y}_s, \tilde{Z}_s)\right] \\ & \le \frac{1}{2}\Gamma_{t_0}\left[\Phi(\omega, t_0, Y_{t_0}, \tilde{Z}_{t_0}) - \tilde{\Phi}(\omega, t_0, \tilde{Y}_{t_0}, \tilde{Z}_{t_0})\right]\Big\},\end{aligned}$$

if the set under inf is non-empty and $\tau = T$ if the set is empty. Clearly $\tau > t_0$ a.s. on $\{Y_{t_0} = \tilde{Y}_{t_0}\}$. Setting in (5.96) $\theta = t_0$ and $\sigma = \tau$ we obtain

$$\begin{aligned} 0 &\ge \mathbb{E}^{\mathcal{F}_{t_0}}\left(\mathbf{1}_{\{Y_{t_0}=\tilde{Y}_{t_0}\}\cap A}\int_{t_0}^{\tau}\Gamma_s\left(\Phi(s, Y_s, \tilde{Z}_s) - \tilde{\Phi}(s, \tilde{Y}_s, \tilde{Z}_s)\right)dQ_s\right) \\ &\ge \frac{1}{2}\mathbf{1}_{\{Y_{t_0}=\tilde{Y}_{t_0}\}\cap A}\Gamma_{t_0}\left[\Phi(\omega, t_0, Y_{t_0}, \tilde{Z}_{t_0}) - \tilde{\Phi}(\omega, t_0, Y_{t_0}, \tilde{Z}_{t_0})\right]\mathbb{E}^{\mathcal{F}_{t_0}}\left(Q_\tau - Q_{t_0}\right) \\ &> 0, \text{ a.s. on } \{Y_{t_0} = \tilde{Y}_{t_0}\} \cap A, \end{aligned}$$

which is a contradiction. Hence $\mathbb{P}\left(\{Y_{t_0} = \tilde{Y}_{t_0}\} \cap A\right) = 0$ and the conclusion (*jj*) follows. ∎

Unlike in the Lipschitz case $\eta > \tilde{\eta}$ does not imply that $Y_t > \tilde{Y}_t$ for all $t \in [0, T]$, as the following example will show. Let $F(x) = \tilde{F}(x) = -\sqrt{x^+}$.

Clearly

$$(Y_t, Z_t) = \left(t^2, 0\right), \quad t \geq 0,$$

is the unique solution of the BSDE

$$Y_t = 1 + \int_t^1 \left(-2\sqrt{Y_s^+}\right) ds - \int_t^1 Z_s \, dB_s, \quad t \in [0, 1],$$

and $(\tilde{Y}_t, \tilde{Z}_t) = (0, 0)$, $t \geq 0$, is the unique solution of

$$\tilde{Y}_t = 0 + \int_t^1 \left(-2\sqrt{\tilde{Y}_s^+}\right) ds - \int_t^1 Z_s \, dB_s, \quad t \in [0, 1].$$

We have $Y_1 = 1 > 0 = \tilde{Y}_1$, but $Y_0 = \tilde{Y}_0$.

5.4 Semilinear Parabolic PDEs

We need to put our BSDE into a Markovian framework: the final condition η and the coefficient F of the BSDE will be functionals of B as “explicit” functions of the solution of a forward SDE driven by $\{B_t\}$.

Let $f : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ be continuous and globally monotone in x, uniformly with respect to t, $g : [0, T] \times \mathbb{R}^d \to \mathbb{R}^{d \times d}$ be continuous and globally Lipschitz in x uniformly with respect to t. Let $\{X_s^{t,x} \,;\, t \leq s \leq T\}$ denote the solution of the SDE

$$X_s^{t,x} = x + \int_t^s f(r, X_r^{t,x}) \, dr + \int_t^s g(r, X_r^{t,x}) dB_r, \; t \leq s \leq T, \tag{5.97}$$

and consider the backward SDE

$$Y_s^{t,x} = \kappa(X_T^{t,x}) + \int_s^T F(r, X_r^{t,x}, Y_r^{t,x}, \; Z_r^{t,x}) \, dr - \int_s^T Z_r^{t,x} \, dB_r, \; t \leq s \leq T, \tag{5.98}$$

where $\kappa : \mathbb{R}^d \to \mathbb{R}^m$ and $F : [0, T] \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \to \mathbb{R}^m$ are continuous and such that for some $K, \mu, p > 0$,

$$|\kappa(x)| \leq K(1 + |x|^p),$$

$$\sup_{|y| \leq \rho} |F(t, x, y, 0)| \leq \gamma(\rho, x),$$

$$\big\langle y - y', F(t,x,y,z) - F(t,x,y',z)\big\rangle \le \mu(t,x)|y - y'|^2,$$
$$|F(t,x,y,z) - F(t,x,y,z')| \le \ell(t,x)\|z - z'\|,$$

where for each $\rho > 0$, there exists a $K_\rho > 0$ such that $\gamma(\rho, x) \le K_\rho(1 + |x|^p)$ and one of the two following conditions hold:

- $|f(t,x)| + |g(t,x)| \le K(1 + |x|)$ and $|\mu(t,x)| + \ell^2(t,x) \le K$;
- $|f(t,x)| + |\mu(t,x)| + \ell^2(t,x) \le K(1 + |x|)$ and $|g(t,x)| \le K$.

In the case $m > 1$ we reinforce one of the above conditions into

$$|F(t,x,y,z) - F(t,x,y',z)| \le \ell(t,x)|y - y'].$$

This is necessary for our uniqueness proof of the viscosity solution of systems of PDEs, see Theorem 6.106 in Annex D.

Finally the following additional assumption is needed again for the uniqueness of viscosity solutions

$$|F(t,x,r,p) - F(t,y,r,p)| \le \mathbf{m}_R(|x - y|(1 + |p|)),$$

for all $x, y \in \mathbb{R}^d$ such that $|x| \le R$, $|y| \le R$, $r \in \mathbb{R}^m$, $p \in \mathbb{R}^d$, where for each $R > 0$, $\mathbf{m}_R \in C(\mathbb{R}_+)$ is increasing and $\mathbf{m}_R(0) = 0$.

Remark 5.35. (i) Clearly, for each $t \le s \le T$, $Y_s^{t,x}$ is $\mathcal{F}_s^t = g\{B_r - B_t,\ t \le r \le s\} \vee \mathcal{N}$ measurable, where $\mathcal{N}$ is the class of the $\mathbb{P}$-null sets of $\mathcal{F}$. Hence $Y_t^{t,x}$ is a.s. constant (i.e. deterministic).

(ii) It is not hard to see, using uniqueness for BSDEs, that $Y_{t+h}^{t,x} = Y_{t+h}^{t+h, X_{t+h}^{t,x}}$, $h > 0$.

We shall denote by

$$\mathcal{A}_t = \frac{1}{2}\sum_{i,j}(gg^*)_{ij}(t,x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_i f_i(t,x)\frac{\partial}{\partial x_i}$$

the infinitesimal generator of the Markov process $\{X_s^{t,x};\ t \le s \le T\}$.

5.4.1 Parabolic Systems in the Whole Space

We first consider the following system of backward semilinear parabolic PDEs

$$\begin{cases} \dfrac{\partial u_i}{\partial t}(t,x) + \mathcal{A}_t u_i(t,x) + F_i(t,x,u(t,x),(\nabla u g)(t,x)) = 0, \\ \qquad\qquad\qquad\qquad (t,x) \in [0,T] \times \mathbb{R}^d, \quad 0 \le i \le m; \\ u(T,x) = \kappa(x), \quad x \in \mathbb{R}^d; \end{cases} \tag{5.99}$$

where $F \in C([0,T] \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{m\times d}; \mathbb{R}^m)$, and $\kappa \in C(\mathbb{R}^d, \mathbb{R}^m)$ grows at most polynomially at infinity.

We can first establish the following:

Theorem 5.36. *Let $u \in C^{1,2}([0,T] \times \mathbb{R}^d; \mathbb{R}^m)$ be a classical solution of (5.99). Then for each $(t,x) \in [0,T] \times \mathbb{R}^d$, $\{(u(s,X_s^{t,x}), (\nabla u g)(s, X_s^{t,x})); t \le s \le T\}$ is the solution of the BSDE (5.98). In particular, $u(t,x) = Y_t^{t,x}$.*

Proof. The result follows by applying Itô's formula to $u(s, X_s^{t,x})$. ∎

We now want to connect (5.97)–(5.98) with (5.99) in the other direction, i.e. prove that (5.97)–(5.98) provides a solution of (5.99). In order to avoid restrictive assumptions on the coefficients in (5.97)–(5.98), we will consider (5.99) in the viscosity sense. This imposes just one restriction. Indeed for the notion of viscosity solution of the system of PDEs (5.99) to make sense, we need to make the following restriction: *for $0 \le i \le k$, the i-th coordinate of F depends only on the i-th row of the matrix z*. Then the first line in (5.99) reads

$$\frac{\partial u_i}{\partial t}(t,x) + \mathcal{A}_t u_i(t,x) + F_i(t,x,u(t,x),(\nabla u_i g)(t,x)) = 0,$$

which we rewrite in the form

$$-\frac{\partial u_i}{\partial t}(t,x) + \Phi_i(t,x,u(t,x), Du_i(t,x), D^2 u_i(t,x)) = 0,$$

where

$$\Phi : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^d \times \mathbb{S}^d \to \mathbb{R}^m$$

is defined by

$$\Phi_i(t,x,r,p,X) = -\frac{1}{2}\mathbf{Tr}[(gg^*)(t,x)X] - \langle f, p\rangle - F_i(t,x,r,pg(t,x)),$$

for all $1 \le i \le m$, $(t,x) \in [0,T] \times \mathbb{R}^d$, $r \in \mathbb{R}^m$, $p \in \mathbb{R}^d$, $X \in \mathbb{S}^d$.

We add the following assumptions. For each $\rho > 0$, there exists a K_ρ such that for some $p > 1$, all $(t,x) \in [0,T] \times \mathbb{R}^d$, $\rho > 0$,

$$\sup_{\{|y| \le \rho\}} |F(t,x,y,0)| \le K_\rho(1 + |x|^p),$$

and there exists a $K > 0$ such that for all $(t,x) \in [0,T] \times \mathbb{R}^d$, $y, y' \in \mathbb{R}^m$, $z, z' \in \mathbb{R}^d$,

$$|F(t,x,y,z) - F(t,x,y',z)| + |F(t,x,y,z) - F(t,x,y,z')| \le K(|y-y'| + |z-z'|).$$

The definition of the viscosity solution of a system of elliptic PDEs is given in Definition 6.94 in Annex D. The adaptation to systems of parabolic PDEs is obvious.

We now establish the main result of this section.

Theorem 5.37. *Under the above assumptions, $u(t,x) \overset{def}{=} Y_t^{t,x}$ is a continuous function of (t,x) and it is the unique viscosity solution of (5.99) which grows at most polynomially at infinity.*

Proof. Uniqueness follows from Theorem 6.106 in Annex D.

The continuity follows from the mean-square continuity of $\{Y_s^{t,x},\ x \in \mathbb{R}^d,\ 0 \le t \le s \le T\}$, which in turn follows from the continuity of $X_\cdot^{t,x}$ with respect to t, x and Theorem 5.10. The polynomial growth follows from classical moment estimates for $X_\cdot^{t,x}$, the assumptions on the growth of f and g, and Proposition 5.7.

To prove that u is a viscosity sub-solution, take any $1 \le i \le k$, $\varphi \in C^{1,2}([0,T]\times\mathbb{R}^d)$ and $(t,x) \in [0,T) \times \mathbb{R}^d$ such that $u_i - \varphi$ has a local maximum at (t,x). We assume without loss of generality that

$$u_i(t,x) = \varphi(t,x).$$

We suppose that

$$-\frac{\partial\varphi}{\partial t}(t,x) + \Phi_i(t,x,u(t,x),D\varphi(t,x),D^2\varphi(t,x)) > 0,$$

and we will find a contradiction.

Let $0 < \alpha \le T - t$ be such that for all $t \le s \le t+\alpha$, $|y-x| \le \alpha$,

$$u_i(s,y) \le \varphi(s,y),$$

$$-\frac{\partial\varphi}{\partial t}(s,y) + \Phi_i(s,y,u(s,y),D\varphi(s,y),D^2\varphi(s,y)) > 0,$$

and define

$$\tau = \inf\{s \ge t;\ |X_s^{t,x} - x| \ge \alpha\} \wedge (t+\alpha).$$

Let now

$$(\overline{Y}_s, \overline{Z}_s) = ((Y_{s\wedge\tau}^{t,x})^i, \mathbf{1}_{[0,\tau]}(s)(Z_s^{t,x})^i),\ t \le s \le t+\alpha.$$

It follows from the statement in Remark 5.35(ii) that

$$Y_{t+h}^{t,x} = u(t+h, X_{t+h}^{t,x}).$$

We hence have that (first approximating τ by a sequence of stopping times taking at most finitely many values)

$$Y_\tau^{t,x} = u(\tau, X_\tau^{t,x}).$$

Consequently $(\overline{Y}, \overline{Z})$ solves the one-dimensional BSDE

$$\overline{Y}_s = u_i(\tau, X_\tau^{t,x}) + \int_s^{t+\alpha} \mathbf{1}_{[0,\tau]}(r) F_i(r, X_r^{t,x}, u(r, X_r^{t,x}), \overline{Z}_r) dr \\ - \int_s^{t+\alpha} \overline{Z}_r dB_r, \ t \le s \le t + \alpha.$$

On the other hand, from Itô's formula,

$$(\hat{Y}_s, \hat{Z}_s) = (\varphi(s, X_{s\wedge\tau}^{t,x}), \mathbf{1}_{[0,\tau]}(s)(\nabla\varphi g)(s, X_s^{t,x})), \ t \le s \le t + \alpha$$

solves the one-dimensional BSDE, for all $s \in [t, t+\alpha]$

$$\hat{Y}_s = \varphi(\tau, X_\tau^{t,x}) - \int_s^{t+\alpha} \mathbf{1}_{[0,\tau]}(r)(\frac{\partial\varphi}{\partial r} + \mathcal{A}\varphi)(r, X_r^{t,x}) dr - \int_s^{t+\alpha} \hat{Z}_r dB_r.$$

From $u_i(\tau, X_\tau^{t,x}) \le \varphi(\tau, X_\tau^{t,x})$ and the choices of α and τ, we deduce from Proposition 5.34 that $\overline{Y}_t < \hat{Y}_t$, i.e. $u_i(t,x) < \varphi(t,x)$, which contradicts our standing assumption. ■

Remark 5.38. Suppose that $k = 1$ and F has the special form:

$$F(t, x, r, z) = c(t, x)r + h(t, x).$$

In that case, the BSDE is linear:

$$Y_s^{t,x} = \kappa(X_T^{t,x}) + \int_s^T [c(r, X_r^{t,x}) Y_s^{t,x} + h(r, X_r^{t,x})]\, dr - \int_s^T Z_r^{t,x}\, dB_r,$$

hence it has an explicit solution (see Proposition 5.31):

$$Y_s^{t,x} = \kappa(X_T^{t,x}) e^{\int_s^T c(r, X_r^{t,x})\, dr} + \int_s^T h(r, X_r^{t,x}) e^{\int_s^r c(\alpha, X_\alpha^{t,x})\, d\alpha}\, dr \\ - \int_s^T e^{\int_s^r c(\alpha, X_\alpha^{t,x}) d\alpha} Z_r^{t,x}\, dB_r.$$

Now $Y_t^{t,x} = \mathbb{E}(Y_t^{t,x})$, so that

$$Y_t^{t,x} = \mathbb{E}\left[\kappa(X_T^{t,x}) e^{\int_t^T c(s, X_s^{t,x})\, ds} + \int_t^T h(s, X_s^{t,x}) e^{\int_t^s c(r, X_r^{t,x}) dr} ds\right],$$

which is the well-known Feynman–Kac formula.

Clearly, Theorem 5.37 can be considered as a nonlinear extension of the Feynman–Kac formula.

Remark 5.39. We have proved that a certain function of (t, x), defined via the solution of a probabilistic problem, is the solution of a system of backward parabolic partial differential equations. Suppose that b, g and f do not depend on t, and let

$$v(t, x) = u(T - t, x),\ (t, x) \in [0, T] \times \mathbb{R}^d.$$

Then v solves the system of forward parabolic PDEs:

$$\frac{\partial v_i}{\partial t}(t, x) = \mathcal{A} v_i(t, x) + F_i(x, v(t, x), (\nabla v_i g)(t, x)),\ 1 \le i \le m, t > 0, x \in \mathbb{R}^d;$$
$$v(0, x) = \kappa(x),\ x \in \mathbb{R}^d.$$

On the other hand, we have that

$$v(t, x) = Y_{T-t}^{T-t,x} = \bar{Y}_0^{t,x},$$

where $\{(\bar{Y}_s^{t,x}, \bar{Z}_s^{t,x});\ 0 \le s \le t\}$, solves the BSDE

$$\bar{Y}_s^{t,x} = \kappa(X_t^x) + \int_s^t F(X_r^x, \bar{Y}_r^{t,x}, \bar{Z}_r^{t,x})dr$$
$$- \int_s^t \bar{Z}_r^{t,x} dB_r, 0 \le s \le t.$$

So we have a probabilistic representation for a system of forward parabolic PDEs, which is valid on $\mathbb{R}_+ \times \mathbb{R}^d$.

5.4.2 *Parabolic Dirichlet Problem*

We now combine the situation of the preceding subsection with that of Sect. 3.8.3, and we consider the following system of parabolic semilinear PDEs with Dirichlet boundary condition

$$\begin{cases} -\dfrac{\partial u_i}{\partial t}(t, x) + \Phi_i(t, x, u(t, x), Du_i(t, x), D^2 u_i(t, x)) = 0, \\ \qquad\qquad (t, x) \in [0, T] \times D, \quad 0 \le i \le m; \\ u(T, x) = \kappa(x), \quad x \in \overline{D}; \\ u(t, x) = \chi(t, x), \quad (t, x) \in [0, T] \times \partial D; \end{cases} \tag{5.100}$$

where in addition to the situation in the previous subsection, we give ourselves a function $\chi \in C([0, T] \times \partial D)$. We assume that

$$\chi(T, x) = \kappa(x),\ \forall x \in \partial D.$$

Now, together with the SDE (5.97), for each $(t, x) \in [0, T] \times \overline{D}$ we consider the BSDE, for all $s \in [t, T]$

$$Y_s^{t,x} = \chi(\tau_{t,x} \wedge T, X_{\tau_{t,x}\wedge T}^{t,x}) + \int_s^T \mathbf{1}_{\{r<\tau_{t,x}\}} F(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) dr - \int_s^T Z_r^{t,x} dB_r. \tag{5.101}$$

Again Itô's formula allows us to establish the following:

Theorem 5.40. *Suppose that the above conditions on the coefficients and the domain D are satisfied. Let $u \in C^{1,2}([0, T] \times D; \mathbb{R}^m) \cap C([0, T] \times \overline{D}; \mathbb{R}^m)$ be a classical solution of (5.100). Then for each $(t, x) \in [0, T] \times D$,*

$$\{(u(s \wedge \tau_{t,x}, X_{s\wedge\tau_{t,x}}^{t,x}, \mathbf{1}_{\{s<\tau_{t,x}\}}(\nabla u g)(s, X_s^{t,x})),\ t \le s \le T\}$$

is the solution of the BSDE (5.101).

We now wish to prove that $u(t, x) := Y_t^{t,x}$ is a viscosity solution of (5.100). From the discussion in Sect. 3.8.3, we deduce that the condition (3.111) is necessary for u to be continuous.

We now prove the following:

Theorem 5.41. *Under the above conditions, including those of Theorem 5.37 and (3.111), $u(t, x) := Y_t^{t,x}$ is a continuous function from $[0, T] \times \overline{D}$ into $\mathbb{R}^m$, and it is the unique viscosity solution of (5.100).*

Proof. Uniqueness follows from the arguments developed in Annex D. The continuity of u follows from Proposition 3.45 and the argument at the beginning of the proof of Theorem 5.37.

Let us prove that u is a viscosity sub-solution. Let $1 \le i \le k$, $\varphi \in C^{1,2}([0, T] \times \mathbb{R}^d)$ and $(t, x) \in [0, T) \times \overline{D}$ be such that $u_i - \varphi$ has a local maximum at (t, x), and $u_i(t, x) = \varphi(t, x)$. If $x \in D$, then the argument in the proof of Theorem 5.37 (with $\alpha < d(x, \partial D)$) establishes the required inequality. The same is true if $(t, x) \in [0, T] \times \partial D \backslash \Lambda$ (this time choosing $\alpha < d((t, x), \Lambda)$). Finally if $(t, x) \in \Lambda$, then $\tau_{t,x} = t$, a.s., hence $u(t, x) = \chi(t, x)$. The result follows. ∎

5.4.3 Parabolic Neumann Problem

We use again the notations from Sect. 5.4.1, and we add a nonlinear Neumann condition on the boundary of the bounded open connected subset D of $\mathbb{R}^d$, whose boundary ∂D is assumed to be of class C^2.

Let

$$G \in C([0, T] \times \partial D \times \mathbb{R}^m; \mathbb{R}^m)$$

be such that for some $\rho > 0$,

$$|G(t,x,y) - G(t,x,y')| \leq K|y - y'|, \tag{5.102}$$

for all $(t,x) \in [0,T] \times \partial D$, $y, y' \in \mathbb{R}^m$.

We now consider the following system of semilinear parabolic PDEs with nonlinear Neumann boundary condition:

$$\begin{cases} -\dfrac{\partial u_i}{\partial t}(t,x) + \Phi_i(t,x,u(t,x), Du_i(t,x), D^2 u_i(t,x)) = 0, \\ \qquad\qquad (t,x) \in [0,T] \times D, \quad 0 \leq i \leq m; \\ u(T,x) = \kappa(x), \quad x \in \overline{D}; \\ \dfrac{\partial u_i}{\partial n}(t,x) - G_i(t,x,u(t,x)) = 0,\ 1 \leq i \leq m,\ (t,x) \in [0,T] \times \partial D. \end{cases} \tag{5.103}$$

Let $X^{t,x}$ be the process solution of the reflected stochastic differential equation, for all $s \in [t, T,]$, $\mathbb{P}$-$a.s.$

$$\begin{cases} X_s^{t,x} + K_s^{t,x} = x + \displaystyle\int_t^s f(r, X_r^{t,x})dr + \int_t^s g(r, X_r^{t,x})dB_r, \\ X_s^{t,x} \in \overline{D}, \\ K_s^{t,x} = \displaystyle\int_t^s n(X_r^{t,x}) \mathbf{1}_{\partial D}\left(X_r^{t,x}\right) d \updownarrow K^{t,x} \updownarrow_r . \end{cases} \tag{5.104}$$

To each $(t,x) \in [0,T] \times \overline{D}$ we associate the BSDE

$$Y_s^{t,x} = \kappa(X_T^{t,x}) + \int_s^T F(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr \tag{5.105}$$

$$+ \int_s^T G(r, X_r^{t,x}, Y_r^{t,x}) d \updownarrow K^{t,x} \updownarrow_r - \int_s^T Z_r^{t,x} dB_r,\ t \leq s \leq T.$$

Itô's formula again allows us to establish the following:

Theorem 5.42. *Under the above assumptions on the coefficients and the domain D, if $u \in C^{1,2}([0,T] \times D; \mathbb{R}^m) \cap C^{0,1}([0,T] \times \overline{D}; \mathbb{R}^m)$ is a classical solution of (5.103), then for each $(t,x) \in [0,T] \times \mathbb{R}^d$, $\{(u(s, X_s^{t,x}), (\nabla u g)(s, X_s^{t,x})); t \leq s \leq T\}$ is the solution of the BSDE (5.104).*

Inspired by [59] we now prove:

Theorem 5.43. *Under the above conditions, including those of Theorem 5.37, if in addition either G does not depend upon its third argument, or else the additional assumptions from Proposition 5.83 below are satisfied, then $u(t,x) := Y_t^{t,x}$ is a continuous function from $[0,T] \times \overline{D}$ into $\mathbb{R}^m$, and it is the unique viscosity solution of (5.103).*

Proof. Uniqueness follows from a combination of the arguments in the proofs of Theorems 6.106 and 6.112. If G does not depend upon its third argument, then the continuity of u follows from Corollary 4.56 combined with the argument at the beginning of the proof of Theorem 5.37. In the other case, we refer to [46] for the proof of the continuity of u. We now prove that u is a viscosity sub-solution. Let $1 \le i \le k$, $\varphi \in C^{1,2}([0,T] \times \mathbb{R}^d)$ and $(t,x) \in [0,T) \times \overline{D}$ be such that $u_i - \varphi$ has a local maximum at (t,x), and $u_i(t,x) = \varphi(t,x)$. The case where $x \in D$ is treated as in the proof of Theorem 5.37. Suppose now that $x \in \partial D$. As usual we argue by contradiction. Suppose that for some $\alpha > 0$, all $(s,y) \in B((t,x),\alpha) \cap \overline{D}$ satisfy

$$\begin{cases} -\dfrac{\partial \varphi}{\partial t}(s,y) + \Phi_i(s,y,u(s,y),D\varphi(s,y),D^2\varphi(s,y)) > 0, \\ \qquad\qquad \dfrac{\partial \varphi}{\partial n}(s,y) - G_i(s,y,u(s,y)) > 0, \text{ if } y \in \partial D. \end{cases}$$

The contradiction can now be established as in the proof of Theorem 5.37, making use of the strict comparison result from Proposition 5.34. ■

5.5 BSDEs with a Subdifferential Coefficient

5.5.1 Uniqueness

We extend the estimates and the uniqueness result in the case of the multivalued BSDE

$$\begin{cases} -dY_t + \partial\varphi\,(Y_t)\,dt + \partial\psi\,(Y_t)\,dA_t \\ \qquad\qquad \ni F\,(t,Y_t,Z_t)\,dt + G\,(t,Y_t)\,dA_t - Z_t dB_t, \quad 0 \le t < T, \\ Y_T = \eta, \end{cases} \tag{5.106}$$

where again $T > 0$ is a fixed deterministic time and $\partial\varphi$ and $\partial\psi$ are subdifferential operators attached to the convex lower semicontinuous functions $\varphi, \psi : \mathbb{R}^m \to]-\infty, +\infty]$.

Such multivalued backward stochastic differential equations are also called *backward stochastic variational inequalities (BSVI).*

It is natural here to assume there exists a $u_0 \in \mathbb{R}^m$ such that $\partial\varphi\,(u_0) \neq \emptyset$ and $\partial\psi\,(u_0) \neq \emptyset$.

If $Q_t\,(\omega) \overset{def}{=} t + A_t\,(\omega)$ and $\{\alpha_t : t \in [0,T]\}$ is a real positive $\mathcal{P}$-m.s.p. (given by the Radon–Nikodym representation theorem) such that $0 \le \alpha \le 1$ and

$$dt = \alpha_t dQ_t \quad \text{and} \quad dA_t = (1-\alpha_t)\,dQ_t,$$

then the Eq. (5.106) becomes

$$\begin{cases} -dY_t + \partial_y \Psi\,(t,Y_t)\,dQ_t \ni \Phi\,(t,Y_t,Z_t)\,dQ_t - Z_t dB_t, \quad 0 \le t < T, \\ Y_T = \eta, \end{cases} \tag{5.107}$$

where

$$\Phi(\omega,t,y,z) \stackrel{def}{=} \alpha_t(\omega)\, F(\omega,t,y,z) + (1-\alpha_t(\omega))\, G(\omega,y),$$

$$\Psi(\omega,t,y) \stackrel{def}{=} \alpha_t(\omega)\,\varphi(y) + (1-\alpha_t(\omega))\,\psi(y),$$

(we use the convention $0\cdot\infty = 0$ and write $\partial\Psi$ for $\partial_y\Psi$).

We also remark that if $u_0 \in \mathrm{Dom}(\partial\varphi)\bigcap \mathrm{Dom}(\partial\psi)$, $\hat{u}_{01} \in \partial\varphi(u_0)$ and $\hat{u}_{02} \in \partial\psi(u_0)$, then

$$\hat{u}_t(\omega) = \alpha_t(\omega)\,\hat{u}_{01} + (1-\alpha_t(\omega))\,\hat{u}_{02} \in \partial_y\Psi(\omega,t,u_0).$$

We shall assume that the following assumptions hold:

$$\left(\mathbf{BSVI\text{-}H}_{\eta,\Psi,\Phi}\right): \tag{5.108}$$

(i) *$\eta : \Omega \to \mathbb{R}^m$ is an $\mathcal{F}_T$-measurable random vector;*
(ii) Q is a *$\mathcal{P}$-m.i.c.s.p. such that $Q_0 = 0$;*
(iii) *$(\omega,t) \longmapsto \alpha_t(\omega) : \Omega\times[0,T] \to [0,1]$ is $\mathcal{P}$-m.s.p. such that $\alpha_t dQ_t = dt$;*
(iv) *$\Phi : \Omega\times[0,T]\times\mathbb{R}^m\times\mathbb{R}^{m\times k} \to \mathbb{R}^m$ satisfies the assumptions (5.13-**BSDE-H$_\Phi$**);*
(v) *$\Psi : \Omega\times[0,T]\times\mathbb{R}^m \to]-\infty,+\infty]$ satisfies*

- ▲ $\Psi(\cdot,\cdot,y)$ is *$\mathcal{P}$-m.s.p. for all $y\in\mathbb{R}^m$,*
- ▲ *$y \longmapsto \Psi(\omega,t,y) : \mathbb{R}^m \to]-\infty,+\infty]$ is a proper convex l.s.c. function,*
- ▲ *$\exists\, u_0 \in \mathbb{R}^m$ and an $\mathbb{R}^m$-valued $\mathcal{P}$-m.s.p. $(\hat{u}_t)_{t\in[0,T]}$* such that

$$(u_0,\hat{u}_t) \in \partial_y\Psi(\omega,t,\cdot), \quad d\mathbb{P}\otimes dt\text{-}a.e.\ (\omega,t)\in\Omega\times[0,T].$$

□

Definition 5.44. A pair $(Y,Z) \in S_m^0[0,T]\times\Lambda^0_{m\times k}(0,T)$ of stochastic processes is a solution of the backward stochastic variational inequality (5.107) if there exist $K\in S_m^0[0,T]$, $K_0 = 0$, such that

$$\begin{aligned}
&(a)\quad \updownarrow K\updownarrow_T + \int_0^T |\Psi(t,Y_t)|\,dQ_t + \int_0^T |\Phi(t,Y_t,Z_t)|\,dQ_t < \infty, \text{ a.s.,}\\
&(b)\quad dK_t \in \partial_y\Psi(t,Y_t)\,dQ_t, \text{ a.s. that is: } \mathbb{P}\text{-a.s.,}\\
&\qquad \int_t^s \langle y(r)-Y_r, dK_r\rangle + \int_t^s \Psi(r,Y_r)dQ_r \le \int_t^s \Psi(r,y(r))dQ_r,\\
&\qquad\qquad \forall y\in C([0,T];\mathbb{R}^m),\ \forall\, 0\le t\le s\le T,
\end{aligned}$$

and $\mathbb{P}$-a.s., for all $t\in[0,T]$:

$$Y_t + K_T - K_t = \eta + \int_t^T \Phi(s,Y_s,Z_s)\,dQ_s - \int_t^T Z_s dB_s, \text{ a.s.} \tag{5.109}$$

(we also say that the triple (Y,Z,K) is a solution of the Eq. (5.107)).

Remark 5.45. If K is absolutely continuous with respect to dQ_t, i.e. there exists a progressively measurable stochastic process U such that

$$\int_0^T |U_t|\, dQ_t < \infty, \text{ a.s. and } K_t = \int_0^t U_s dQ_s, \text{ for all } t \in [0,T],$$

then $dK_t \in \partial\Psi(t,Y_t)\, dQ_t$ means, $\mathbb{P}$-a.s. $\omega \in \Omega$,

$$U_t \in \partial\Psi_y(t,Y_t), \quad dQ_t\text{-a.e.}$$

In this case we also say that the triple (Y,Z,U) is a solution of the Eq. (5.107).

If $dK_t \in \partial\Psi_y(t,Y_t)\, dQ_t$, $d\tilde{K}_t \in \partial\Psi_y(t,\tilde{Y}_t)dQ_t$ and

$$\int_0^T |\Psi(t,Y_t)|\, dQ_t + \int_0^T |\Psi(t,\tilde{Y}_t)| dQ_t < \infty, \text{ a.s.},$$

then, using the subdifferential inequalities

$$\begin{aligned}
&\int_t^s \langle \tilde{Y}_r - Y_r, dK_r\rangle + \int_t^s \Psi(r,Y_r)dQ_r \le \int_t^s \Psi(r,\tilde{Y}_r)dQ_r,\\
&\int_t^s \langle Y_r - \tilde{Y}_r, d\tilde{K}_r\rangle + \int_t^s \Psi(r,\tilde{Y}_r)dQ_r \le \int_t^s \Psi(r,Y_r)dQ_r,
\end{aligned}$$

we infer that, for all $0 \le t \le s \le T$

$$\int_t^s \langle Y_r - \tilde{Y}_r, dK_r - d\tilde{K}_r\rangle \ge 0, \; a.s. \tag{5.110}$$

Let $a, p > 1$ and

$$V_t = V_t^{a,p} \stackrel{def}{=} \int_0^t \left[\mu_s dQ_s + \frac{a}{2n_p}(\ell_s)^2 ds\right] \quad \text{and} \quad \bar{\mu}_t = \int_0^t \mu_s dQ_s.$$

Recall the notations

$$S_m^p\left([0,T]; e^{\bar{\mu}}\right) = \left\{Y \in S_m^0([0,T]) : e^{\bar{\mu}}Y \in S_m^p([0,T])\right\}$$

and

$$S_m^{1+}\left([0,T]; e^{\bar{\mu}}\right) = \bigcup_{p>1} S_m^p\left([0,T]; e^{\bar{\mu}}\right).$$

Note that if μ is a determinist process then $S_m^p\left([0,T]; e^{\bar{\mu}}\right) = S_m^p([0,T])$.

Proposition 5.46. *Let the assumptions* $\left(\mathbf{BSVI\text{-}H}_{\eta,\Psi,\Phi}\right)$ *be satisfied. Then for every* $a, p > 1$ *there exists a constant* $C_{a,p}$ *such that for all solutions* $(Y,Z) \in S_m^0[0,T] \times \Lambda_{m\times k}^0(0,T)$ *of the BSDE (5.107) satisfying*

$$\mathbb{E} \sup_{s\in[0,T]} e^{pV_s} |Y_s - u_0|^p < \infty,$$

the following inequality holds $\mathbb{P}$*-a.s., for all* $t \in [0, T]$*:*

$$\begin{aligned}
&\mathbb{E}^{\mathcal{F}_t} \sup_{s\in[t,T]} e^{pV_s} |Y_s - u_0|^p + \mathbb{E}^{\mathcal{F}_t} \left(\int_t^T e^{2V_s} |Z_s|^2 ds\right)^{p/2} \\
&\qquad\qquad + \mathbb{E}^{\mathcal{F}_t} \left(\int_t^T e^{2V_s} |\Psi(s, Y_s) - \Psi(s, u_0)| \, dQ_s\right)^{p/2} \\
&\le C_{a,p} \left[\mathbb{E}^{\mathcal{F}_t} e^{pV_T} |\eta - u_0|^p + \mathbb{E}^{\mathcal{F}_t} \left(\int_t^T e^{V_s} [|\hat{u}_s| + |\Phi(s, u_0, 0)|] \, dQ_s\right)^p\right].
\end{aligned} \tag{5.111}$$

Proof. We have

$$Y_t - u_0 = \eta - u_0 + \int_t^T [\Phi(s, Y_s, Z_s) \, dQ_s - dK_s] - \int_t^T Z_s dB_s.$$

Note that

$$\begin{aligned}
&\langle Y_t - u_0, \Phi(t, Y_t, Z_t)\rangle \, dQ_t \\
&\quad = \langle Y_t - u_0, (\Phi(t, Y_t, Z_t) - \Phi(t, u_0, Z_t))\rangle \, dQ_t \\
&\qquad + \langle Y_t - u_0, \Phi(t, u_0, Z_t) - \Phi(t, u_0, 0)\rangle \, dQ_t + \langle Y_t - u_0, \Phi(t, u_0, 0)\rangle \, dQ_t \\
&\quad \le |Y_t - u_0|^2 \mu_t dQ_t + |Y_t - u_0| \, |Z_t| \, \ell_t dt + |Y_t - u_0| \, |\Phi(t, u_0, 0)| \, dQ_t \\
&\quad \le |Y_t - u_0| \, |\Phi(t, u_0, 0)| \, dQ_t + |Y_t - u_0|^2 \, dV_t + \frac{n_p}{2a} |Z_t|^2 \, dt,
\end{aligned}$$

where $n_p = (p - 1) \wedge 1$.

From the subdifferential inequalities we have

$$\begin{aligned}
&|\Psi(t, Y_t) - \Psi(t, u_0)| \le \Psi(t, Y_t) - \Psi(t, u_0) + 2 |\hat{u}_t| \, |Y_t - u_0| , \quad \text{and} \\
&[\Psi(t, Y_t) - \Psi(t, u_0)] \, dQ_t \le \langle Y_t - u_0, dK_t\rangle ,
\end{aligned}$$

so

$$|\Psi(t, Y_t) - \Psi(t, u_0)| \, dQ_t \le \langle Y_t - u_0, dK_t\rangle + 2 |\hat{u}_t| \, |Y_t - u_0| \, dQ_t.$$

Hence

$$\begin{aligned}
&|\Psi(t, Y_t) - \Psi(t, u_0)| \, dQ_t + \langle Y_t - u_0, \Phi(t, Y_t, Z_t) \, dQ_t - dK_t\rangle \\
&\le |Y_t - u_0| \, [2 |\hat{u}_t| + |\Phi(t, u_0, 0)|] \, dQ_t + |Y_t - u_0|^2 \, dV_t + \frac{n_p}{2a} |Z_t|^2 \, dt.
\end{aligned}$$

Now (5.111) follows from Proposition 5.2. ■

Corollary 5.47. *Let $p = 1$. Let the assumptions $\left(\mathbf{BSVI\text{-}H}_{\eta,\Psi,\Phi}\right)$ be satisfied and $\Phi(t,y,z) \equiv \Phi(t,y)$ for all $t \in [0,T]$, $y \in \mathbb{R}^m$ and $z \in \mathbb{R}^{m\times k}$ (Φ is independent of z; $\ell_t \equiv 0$ and $V_t = \bar{\mu}_t = \int_0^t \mu_s dQ_s$). Let*

$$dN_t = [|\hat{u}_t| + |\Phi(t,u_0,0)|]\,dQ_t.$$

If $(Y,Z) \in S_m^0[0,T] \times \Lambda_{m\times k}^0(0,T)$ is a solution of the BSDE (5.107) satisfying

$$\mathbb{E}\sup_{s\in[0,T]} e^{\bar{\mu}_s}|Y_s - u_0| < \infty,$$

then the following inequality holds $\mathbb{P}$-a.s., for all $t \in [0,T]$:

$$e^{\bar{\mu}_t}|Y_t - u_0| \le \mathbb{E}^{\mathcal{F}_t} e^{\bar{\mu}_T}|\eta - u_0| + \mathbb{E}^{\mathcal{F}_t}\int_t^T e^{\bar{\mu}_s} dN_s.$$

Moreover for every $q \in (0,1)$ there exists a constant C_q such that

$$\begin{aligned}
&\sup_{s\in[0,T]}\left(\mathbb{E}\left(e^{\bar{\mu}_s}|Y_s|\right)\right)^q + \mathbb{E}\sup_{s\in[0,T]} e^{q\bar{\mu}_s}|Y_s|^q \\
&\quad + \mathbb{E}\left(\int_0^T e^{2\bar{\mu}_s}|Z_s|^2 ds\right)^{q/2} + \mathbb{E}\left(\int_0^T e^{2\bar{\mu}_s}|\Psi(s,Y_s) - \Psi(s,u_0)|\,dQ_s\right)^{q/2} \\
&\le C_q\left[\left(\mathbb{E}\left(e^{\bar{\mu}_T}|\eta - u_0|\right)\right)^q + \left(\mathbb{E}\int_0^T e^{\bar{\mu}_s} dN_s\right)^q\right].
\end{aligned}$$

Proof. From the proof of Proposition 5.46 we have

$$\begin{aligned}
&|\Psi(t,Y_t) - \Psi(t,u_0)|\,dQ_t + \langle Y_t - u_0, \Phi(t,Y_t,Z_t)\,dQ_t - dK_t\rangle \\
&\le |Y_t - u_0|\,[2\,|\hat{u}_t| + |\Phi(t,u_0,0)|]\,dQ_t + |Y_t - u_0|^2\, d\bar{\mu}_t
\end{aligned}$$

and the conclusions follow by Corollary 6.81. ■

Remark 5.48. A consequence of (5.111) is the following. Denoting

$$\Theta = e^{V_T}|\eta - u_0| + \int_0^T e^{V_s}[|\hat{u}_s| + |\Phi(s,u_0,0)|]\,dQ_s,$$

then for all $t \in [0,T]$:

$$|Y_t| \le |u_0| + C_{a,p}^{1/p} e^{-V_t}\left(\mathbb{E}^{\mathcal{F}_t}\Theta^p\right)^{1/p}, \quad a.s. \tag{5.112}$$

Corollary 5.49. *Let $p \ge 2$, $r_0 > 0$ and*

$$\Psi_{u_0,r_0}^{\#}(t) \stackrel{def}{=} \sup\{\Psi(t,u_0 + r_0 v) : |v| \le 1\}.$$

Then

$$r_0^{p/2}\mathbb{E}\left(\int_0^T e^{2V_s} d\updownarrow K\updownarrow_s\right)^{p/2} \leq C_{a,p}^{(r_0)}\Big[\mathbb{E}\, e^{pV_T}\left|\eta-u_0\right|^p + \mathbb{E}\left(\int_0^T e^{2V_s}\left[\Psi^{\#}_{u_0,r_0}(s)-\Psi(s,u_0)\right]dQ_s\right)^{p/2} + \mathbb{E}\left(\int_0^T e^{V_s}\left[\left|\hat{u}_s\right|+\left|\Phi(s,u_0,0)\right|\right]dQ_s\right)^p\Big]. \tag{5.113}$$

Proof. Let $v \in C([0,T];\mathbb{R}^m)$ be arbitrary. From the subdifferential inequality

$$\langle u_0 + r_0 v(t) - Y_t, dK_t\rangle + \Psi(t,Y_t)dQ_t \leq \Psi(t, u_0 + r_0 v(t))\, dQ_t,$$

we deduce

$$r_0 d\updownarrow K\updownarrow_t + \Psi(t,Y_t)dQ_t \leq \langle Y_t - u_0, dK_t\rangle + \Psi^{\#}_{u_0,r_0}(t)\, dQ_t.$$

Since

$$\langle Y_t - u_0, \hat{u}_t\rangle + \Psi(t,u_0) \leq \Psi(t,Y_t),$$

we see that

$$r_0 d\updownarrow K\updownarrow_t \leq \langle Y_t - u_0, dK_t\rangle + \left|\hat{u}_t\right|\left|Y_t-u_0\right| dQ_t + \left[\Psi^{\#}_{u_0,r_0}(t) - \Psi(t,u_0)\right]dQ_t.$$

Therefore

$$\begin{aligned} r_0 d\updownarrow K\updownarrow_t &+ \langle Y_t - u_0, \Phi(t,Y_t,Z_t)\, dQ_t - dK_t\rangle \\ &\leq \left[\Psi^{\#}_{u_0,r_0}(t) - \Psi(t,u_0)\right]dQ_t + \left|Y_t-u_0\right|\left[\left|\hat{u}_t\right| + \left|\Phi(t,u_0,0)\right|\right]dQ_t \\ &+ \left|Y_t-u_0\right|^2 dV_t + \frac{n_p}{2a}\left|Z_t\right|^2 dt. \end{aligned}$$

(5.113) now follows by Proposition 5.2. ■

Proposition 5.50 (Uniqueness). *Let* $a, p > 1$. *Let the assumptions (5.108-**BSVI-H**$_{\eta,\Psi,\Phi}$) be satisfied. If* $(Y,Z), (\hat{Y},\hat{Z}) \in S_m^0[0,T]\times\Lambda^0_{m\times k}(0,T)$ *are two solutions of the BSDE (5.107) corresponding respectively to* η *and* $\hat{\eta}$ *such that*

$$\mathbb{E}\sup_{s\in[0,T]} e^{pV_s}\left|Y_s - \hat{Y}_s\right|^p < \infty,$$

then $\mathbb{P}$-*a.s., for all* $t \in [0,T]$*:*

$$e^{pV_t}\left|Y_t - \hat{Y}_t\right|^p \leq \mathbb{E}^{\mathcal{F}_t}\left(e^{pV_T}\left|\eta-\hat{\eta}\right|^p\right) \tag{5.114}$$

and there exists a constant $C_{a,p}$ such that, $\mathbb{P}$-a.s., for all $t \in [0,T]$:

$$\mathbb{E}^{\mathcal{F}_t} \sup_{s\in[t,T]} e^{pV_s} \left|Y_s - \hat{Y}_s\right|^p + \mathbb{E}^{\mathcal{F}_t} \left(\int_t^T e^{2V_s} \left|Z_s - \hat{Z}_s\right|^2 ds\right)^{p/2} \\ \leq C_{a,p}\ \mathbb{E}^{\mathcal{F}_t}\ e^{pV_T} \left|\eta - \hat{\eta}\right|^p . \tag{5.115}$$

Uniqueness in the space $S_m^p\left([0,T]\,; e^V\right) \times \Lambda^0_{m\times k}(0,T)$ follows. Moreover, if $(\ell_t)_{t\in[0,T]}$ is a deterministic process, uniqueness of the solution (Y,Z) of the BSDE (5.107) holds in $S_m^{1+}\left([0,T]\,; e^{\bar{\mu}}\right) \times \Lambda^0_{m\times k}(0,T)$.

Proof. Let (Y,Z), $(\hat{Y},\hat{Z}) \in S_m^0([0,T]\,0) \times \Lambda^0_{m\times k}(0,T)$ be two solutions corresponding to η and $\hat{\eta}$ respectively. Then

$$Y_t - \hat{Y}_t = \eta - \hat{\eta} + \int_t^T dL_s - \int_t^T \left(Z_s - \hat{Z}_s\right) dB_s,$$

where

$$L_t = \int_0^t \left[\left(\Phi(s,Y_s,Z_s) - \Phi\left(s,\hat{Y}_s,\hat{Z}_s\right)\right) dQ_s - \left(dK_s - d\hat{K}_s\right)\right].$$

Since by (5.110) $\left\langle Y_s - \hat{Y}_s, dK_s - d\hat{K}_s\right\rangle \geq 0$, we have for all $a > 1$:

$$\left\langle Y_t - \hat{Y}_t, dL_t\right\rangle \leq \left|Y_t - \hat{Y}_t\right|^2 \mu_t dQ_t + \left|Y_t - \hat{Y}_t\right| \left|Z_t - \hat{Z}_t\right| \ell_t dt \\ \leq \left|Y_t - \hat{Y}_t\right|^2 \left[\mu_t dQ_t + \frac{a}{2n_p}(\ell_t)^2 dt\right] + \frac{n_p}{2a}\left|Z_t - \hat{Z}_t\right|^2 dt,$$

where $n_p = (p-1) \wedge 1$. (5.114) and (5.115) follow from Proposition 5.2 and, consequently, uniqueness follows, too.

Let now $(\ell_t)_{t\in[0,T]}$ be a deterministic process. If (Y,Z), $(\hat{Y},\hat{Z}) \in S_m^{1+}([0,T]\,; e^{\bar{\mu}}) \times \Lambda^0_{m\times k}(0,T)$, then there exists a $p > 1$ such that $Y, \hat{Y} \in S_m^p\left([0,T]\,; e^{\bar{\mu}}\right)$ and the uniqueness follows from the first step. ■

Proposition 5.51 (Uniqueness). *Let $p = 1$. Let the assumptions (5.108-**BSVI-$H_{\eta,\Psi,\Phi}$**) be satisfied and Φ be independent of $z \in \mathbb{R}^{m\times k}$ ($\ell_t \equiv 0$ and $V_t = \bar{\mu}_t = \int_0^t \mu_s dQ_s$). If $(Y,Z), (\hat{Y},\hat{Z}) \in S_m^0[0,T] \times \Lambda^0_{m\times k}(0,T)$ are two solutions of the BSDE (5.107) corresponding respectively to η and $\hat{\eta}$ such that*

$$\mathbb{E} \sup_{s\in[0,T]} e^{\bar{\mu}_s} \left|Y_s - \hat{Y}_s\right| < \infty,$$

then $\mathbb{P}$-a.s., for all $t \in [0,T]$:

$$e^{\bar{\mu}_t} \left|Y_t - \hat{Y}_t\right| \leq \mathbb{E}^{\mathcal{F}_t}\left(e^{\bar{\mu}_T}\left|\eta - \hat{\eta}\right|\right)$$

and for every $q \in (0,1)$ *there exists a constant* C_q *such that*

$$\begin{aligned}\sup_{s\in[0,T]}\left(\mathbb{E}\left(e^{\bar{\mu}_s}\left|Y_s-\hat{Y}_s\right|\right)\right)^q+\mathbb{E}\sup_{s\in[0,T]}e^{q\bar{\mu}_s}\left|Y_s-\hat{Y}_s\right|^q\\+\mathbb{E}\left(\int_0^T e^{2\bar{\mu}_s}\left|Z_s-\hat{Z}_s\right|^2ds\right)^{q/2}\\\leq C_q\left(\mathbb{E}\left(e^{\bar{\mu}_T}\left|\eta-\hat{\eta}\right|\right)\right)^q.\end{aligned}$$

Proof. Following the proof of Proposition 5.50 we now have

$$\left\langle Y_t-\hat{Y}_t,dL_t\right\rangle\leq\left|Y_t-\hat{Y}_t\right|^2\mu_t dQ_t$$

and the conclusions follow by Corollary 6.81. ■

5.5.2 Existence

We consider the following backward stochastic variational inequality (BSVI)

$$\begin{cases}-dY_t+\partial\varphi\left(Y_t\right)dt\ni F\left(t,Y_t,Z_t\right)dt-Z_tdB_t,\ 0\leq t<T,\\Y_T=\eta,\end{cases}\tag{5.116}$$

and we suppose that the following assumptions hold:

(A$_1$) $\eta:\Omega\to\mathbb{R}^m$ *is an* $\mathcal{F}_T$*-measurable random vector.*
(A$_2$) $F:\Omega\times[0,T]\times\mathbb{R}^m\times\mathbb{R}^{m\times k}\to\mathbb{R}^m$ *satisfies the assumptions* (5.77-**BSDE-MH**$_F$) *(from Sect. 5.3.4).*
(A$_3$) $\varphi:\mathbb{R}^m\to(-\infty,+\infty]$ *is a proper, convex l.s.c. function.*

Recall that the subdifferential of φ is given by

$$\partial\varphi\left(y\right)=\left\{\hat{y}\in\mathbb{R}^m:\langle\hat{y},v-y\rangle+\varphi\left(y\right)\leq\varphi\left(v\right),\ \forall\ v\in\mathbb{R}^m\right\},$$

and by $(y,\hat{y})\in\partial\varphi$ we understand that $y\in\mathrm{Dom}\left(\partial\varphi\right)$ and $\hat{y}\in\partial\varphi\left(y\right)$.
We define

$$\begin{aligned}&\mathrm{Dom}\left(\varphi\right)=\left\{y\in\mathbb{R}^m:\varphi\left(y\right)<\infty\right\},\\&\mathrm{Dom}\left(\partial\varphi\right)=\left\{y\in\mathbb{R}^m:\partial\varphi\left(y\right)\neq\emptyset\right\}\subset\mathrm{Dom}\left(\varphi\right).\end{aligned}$$

Let $\varepsilon>0$ and denote the Moreau regularization of φ by

$$\varphi_\varepsilon\left(y\right)\stackrel{def}{=}\inf\left\{\frac{1}{2\varepsilon}\left|y-v\right|^2+\varphi\left(v\right):v\in\mathbb{R}^m\right\}=\frac{1}{2\varepsilon}\left|y-J_\varepsilon\left(y\right)\right|^2+\varphi\left(J_\varepsilon\left(y\right)\right),\tag{5.117}$$

where $J_\varepsilon(y) = (I_{m\times m} + \varepsilon\partial\varphi)^{-1}(y)$. Note that φ_ε is a C^1 convex function and J_ε is a 1-Lipschitz function.

We mention some properties (see Annex B: Convex Functions): for all $x, y \in \mathbb{R}^m$

$$\begin{aligned}
&(a) \quad \nabla\varphi_\varepsilon(y) = \partial\varphi_\varepsilon(y) = \frac{y - J_\varepsilon(y)}{\varepsilon} \in \partial\varphi(J_\varepsilon y),\\
&(b) \quad |\nabla\varphi_\varepsilon(x) - \nabla\varphi_\varepsilon(y)| \le \frac{1}{\varepsilon}|x-y|,\\
&(c) \quad \langle\nabla\varphi_\varepsilon(x) - \nabla\varphi_\varepsilon(y), x-y\rangle \ge 0,\\
&(d) \quad \langle\nabla\varphi_\varepsilon(x) - \nabla\varphi_\delta(y), x-y\rangle \ge -(\varepsilon+\delta)\langle\nabla\varphi_\varepsilon(x), \nabla\varphi_\delta(y)\rangle.
\end{aligned} \tag{5.118}$$

Throughout this subsection we fix a pair $(u_0, \hat{u}_0) \in \partial\varphi$. Then by (6.26) from Annex B we have

$$\begin{cases} (j) \quad |\nabla\varphi_\varepsilon(u_0)| \le |\hat{u}_0|, \\ (jj) \quad \dfrac{|y - J_\varepsilon(y)|^2}{2\varepsilon} \le \varphi_\varepsilon(y) - \varphi(u_0) + |\hat{u}_0|\,|y-u_0| + \varepsilon|\hat{u}_0|^2. \end{cases} \tag{5.119}$$

We will make the following assumption:

(A_4) *There exist $p \ge 2$, a positive stochastic process $\beta \in L^1(\Omega\times(0,T))$, a positive function $b \in L^1(0,T)$ and real numbers $\kappa \ge 0$, $\lambda \in]0,1[$ such that*

$$\begin{gathered}
\textit{for all } (u,\hat{u}) \in \partial\varphi \textit{ and } z \in \mathbb{R}^{m\times k}: \\
\langle\hat{u}, F(t,u,z)\rangle \le \lambda|\hat{u}|^2 + \beta_t + b(t)|u|^p + \kappa|z|^2 \\
d\mathbb{P}\otimes dt\text{-a.e., } (\omega,t) \in \Omega\times[0,T].
\end{gathered} \tag{5.120}$$

We note that if $\langle\hat{u}, F(t,u,z)\rangle \le 0$ for all $(u,\hat{u}) \in \partial\varphi$, then the condition (5.120) is satisfied with $\beta_t = b(t) = \kappa = 0$. If for example $\varphi = I_{\overline{D}}$ (the convex indicator of the closed convex set $\overline{D}$) and $\mathbf{n}_y$ denotes any unit outward normal vector to $\overline{D}$ at $y \in \mathrm{Bd}\left(\overline{D}\right)$, then the condition $\left\langle\mathbf{n}_y, F(t,y,z)\right\rangle \le 0$ for all $y \in \mathrm{Bd}\left(\overline{D}\right)$ yields (5.120) with $\beta_t = b(t) = \kappa = 0$ (for example). In this last case by Itô's formula for $\psi\left(\tilde{Y}\right) = \left[\mathrm{dist}_{\overline{D}}\left(\tilde{Y}\right)\right]^2$, where

$$\begin{cases} -d\tilde{Y}_t = F\left(t, \tilde{Y}_t, \tilde{Z}_t\right)dt - \tilde{Z}_t dB_t, \quad 0 \le t < T, \\ \tilde{Y}_T = \eta, \end{cases}$$

and by the uniqueness of the triple (Y,Z,U) satisfying (5.107) we infer that $(Y,Z,U) = \left(\tilde{Y}, \tilde{Z}, 0\right)$.

Theorem 5.52 (Existence - Uniqueness). *Let $p \ge 2$ and assumptions (A_1–A_4) be satisfied with this p. Suppose moreover that, for all $\rho \ge 0$,*

$$\mathbb{E}|\eta|^p + \mathbb{E}\varphi^+(\eta) + \mathbb{E}\left(\int_0^T F_\rho^\#(s)ds\right)^p < \infty.$$

Then there exists a unique pair $(Y,Z) \in S_m^p[0,T] \times \Lambda_{m\times k}^p(0,T)$ *and a unique stochastic process* $U \in \Lambda_m^2(0,T)$ *such that*

$$\begin{aligned}
&(a) \quad \int_0^T |F(t,Y_t,Z_t)|\,dt < \infty, \ \mathbb{P}\text{-}a.s.,\\
&(b) \quad Y_t(\omega) \in \mathrm{Dom}(\partial\varphi), \ d\mathbb{P}\otimes dt\text{-}a.e. \ (\omega,t)\in\Omega\times[0,T],\\
&(c) \quad U_t(\omega) \in \partial\varphi(Y_t(\omega)), \ d\mathbb{P}\otimes dt\text{-}a.e. \ (\omega,t)\in\Omega\times[0,T],
\end{aligned}$$

and for all $t \in [0,T]$*:*

$$Y_t + \int_t^T U_s ds = \eta + \int_t^T F(s,Y_s,Z_s)\,ds - \int_t^T Z_s dB_s, \ a.s. \tag{5.121}$$

Moreover, uniqueness holds in $S_m^{1+}[0,T] \times \Lambda_{m\times k}^0(0,T)$*, where*

$$S_m^{1+}[0,T] \overset{def}{=} \bigcup_{p>1} S_m^p[0,T].$$

Proof. Let (Y,Z), $(\tilde{Y},\tilde{Z}) \in S_m^{1+}[0,T] \times \Lambda_{m\times k}^0(0,T)$ be two solutions. Then $Y,\tilde{Y} \in S_m^p[0,T]$, for some p. Uniqueness follows from Proposition 5.50.

The proof of the existence will be split into several steps.

Step 1. Approximating problem.

For $\varepsilon \in (0,1]$ consider the approximating equation: $\mathbb{P}$-$a.s.$, for all $t \in [0,T]$,

$$Y_t^\varepsilon + \int_t^T \nabla\varphi_\varepsilon\left(Y_s^\varepsilon\right) ds = \eta + \int_t^T F\left(s,Y_s^\varepsilon,Z_s^\varepsilon\right) ds - \int_t^T Z_s^\varepsilon dB_s, \tag{5.122}$$

where $\nabla\varphi_\varepsilon$ is the gradient of the Moreau regularization φ_ε of φ. It follows (*without assumption* (A_4)) from Theorem 5.27 that Eq. (5.122) has a unique solution $(Y^\varepsilon, Z^\varepsilon) \in S_m^p[0,T] \times \Lambda_{m\times k}^p(0,T)$.

Step 2. Boundedness of Y^ε *and* Z^ε*.*

Let $(u_0,\hat{u}_0) \in \partial\varphi$, $a > 1$ and

$$V(t) = V_t^{a,p} \overset{def}{=} \int_0^t \left[\mu(s) + \frac{a}{2n_p}\ell^2(s)\right] ds = \int_0^t \left[\mu(s) + \frac{a}{2}\ell^2(s)\right] ds$$

($p \geq 2$ yields $n_p = 1 \wedge (p-1) = 1$).

Let $(u_0,\hat{u}_0) \in \partial\varphi$ be fixed. From Proposition 5.46 with Ψ replaced by φ_ε and dQ_s by ds, there exists a constant $C_{a,p}$ (depending only on a and p) such that the following inequality holds $\mathbb{P}$-$a.s.$, for all $t \in [0,T]$:

$$\begin{aligned}&\mathbb{E}^{\mathcal{F}_t} \sup_{s\in[t,T]} e^{pV_s}\left|Y_s^\varepsilon-u_0\right|^p+\mathbb{E}^{\mathcal{F}_t}\left(\int_t^T e^{2V_s}\left|\varphi_\varepsilon(Y_s^\varepsilon)-\varphi_\varepsilon\left(u_0\right)\right|ds\right)^{p/2}\\&\qquad\qquad+\mathbb{E}^{\mathcal{F}_t}\left(\int_t^T e^{2V_s}\left|Z_s^\varepsilon\right|^2ds\right)^{p/2}\\&\leq C_{a,p}\left[\mathbb{E}^{\mathcal{F}_t}e^{pV_T}\left|\eta-u_0\right|^p+\mathbb{E}^{\mathcal{F}_t}\left(\int_t^T e^{V_s}\left[\left|\nabla\varphi_\varepsilon\left(u_0\right)\right|+\left|F\left(s,u_0,0\right)\right|\right]ds\right)^p\right].\end{aligned}\tag{5.123}$$

Note that $|\nabla\varphi_\varepsilon(u_0)|\leq|\hat{u}_0|$ and $|\varphi_\varepsilon(u_0)|\leq\varphi(u_0)+|\hat{u}_0|^2$. Hence there exists a constant C independent of ε such that

$$\begin{aligned}&(a)\quad \mathbb{E}\left\|Y^\varepsilon\right\|_T^2\leq\left(\mathbb{E}\left\|Y^\varepsilon\right\|_T^p\right)^{2/p}\leq C,\\&(b)\quad \mathbb{E}\int_0^T\left|Z_s^\varepsilon\right|^2ds\leq\left[\mathbb{E}\Big(\int_0^T\left|Z_s^\varepsilon\right|^2ds\Big)^{p/2}\right]^{2/p}\leq C,\\&(c)\quad \mathbb{E}\int_0^T\left|\varphi_\varepsilon(Y_s^\varepsilon)\right|ds\leq\left[\mathbb{E}\Big(\int_0^T\left|\varphi_\varepsilon(Y_s^\varepsilon)\right|ds\Big)^{p/2}\right]^{2/p}\leq C.\end{aligned}\tag{5.124}$$

Throughout the proof we shall fix $a=2$ and therefore

$$V_t=\int_0^t\left[\mu\left(s\right)+\ell^2\left(s\right)\right]ds.$$

Step 3. Boundedness of $\nabla\varphi_\varepsilon(Y^\varepsilon)$.

Using the following stochastic subdifferential inequality given by Lemma 2.38

$$\varphi_\varepsilon(Y_t^\varepsilon)+\int_t^T\left\langle\nabla\varphi_\varepsilon(Y_s^\varepsilon),\,dY_s^\varepsilon\right\rangle\leq\varphi_\varepsilon(Y_T^\varepsilon)=\varphi_\varepsilon(\eta)\leq\varphi(\eta),$$

we deduce that, for all $t\in[0,T]$,

$$\begin{aligned}\varphi_\varepsilon(Y_t^\varepsilon)+\int_t^T\left|\nabla\varphi_\varepsilon(Y_s^\varepsilon)\right|^2ds\leq\varphi(\eta)+\int_t^T\left\langle\nabla\varphi_\varepsilon(Y_s^\varepsilon),F\left(s,Y_s^\varepsilon,Z_s^\varepsilon\right)\right\rangle ds\\-\int_t^T\left\langle\nabla\varphi_\varepsilon(Y_s^\varepsilon),\,Z_s^\varepsilon dB_s\right\rangle.\end{aligned}\tag{5.125}$$

Since $|\nabla\varphi_\varepsilon(y)|\leq|\nabla\varphi_\varepsilon(y)-\nabla\varphi_\varepsilon(u_0)|+|\nabla\varphi_\varepsilon(u_0)|\leq\dfrac{1}{\varepsilon}|y-u_0|+|\hat{u}_0|$ and

$$\begin{aligned}&\mathbb{E}\left(\int_0^T\left|\nabla\varphi_\varepsilon\left(Y_s^\varepsilon\right)\right|^2\left|Z_s^\varepsilon\right|^2ds\right)^{1/2}\\&\leq\frac{1}{\varepsilon}\mathbb{E}\left[\sup_{s\in[0,T]}\left|\nabla\varphi_\varepsilon\left(Y_s^\varepsilon\right)\right|\Big(\int_0^T\left|Z_s^\varepsilon\right|^2ds\Big)^{1/2}\right]\end{aligned}$$

$$\leq \left[\frac{2}{\varepsilon^2}\mathbb{E}\sup_{s\in[0,T]}|Y_s^\varepsilon - u_0|^2 + 2|\hat{u}_0|^2\right] + \mathbb{E}\left(\int_0^T |Z_s^\varepsilon|^2 ds\right)$$
$$< \infty,$$

we have

$$\mathbb{E}\int_t^T \langle \nabla\varphi_\varepsilon(Y_s^\varepsilon), Z_s^\varepsilon dB_s\rangle = 0.$$

Under assumption (A_4), since $\nabla\varphi_\varepsilon(Y_s^\varepsilon) \in \partial\varphi\left(J_\varepsilon\left(Y_s^\varepsilon\right)\right)$, it follows that

$$\begin{aligned}
&\left\langle \nabla\varphi_\varepsilon(Y_s^\varepsilon), F\left(s, Y_s^\varepsilon, Z_s^\varepsilon\right)\right\rangle \\
&= \frac{1}{\varepsilon}\left\langle Y_s^\varepsilon - J_\varepsilon\left(Y_s^\varepsilon\right), F\left(s, Y_s^\varepsilon, Z_s^\varepsilon\right) - F\left(s, J_\varepsilon\left(Y_s^\varepsilon\right), Z_s^\varepsilon\right)\right\rangle \\
&\qquad\qquad + \left\langle \nabla\varphi_\varepsilon(Y_s^\varepsilon), F\left(s, J_\varepsilon\left(Y_s^\varepsilon\right), Z_s^\varepsilon\right)\right\rangle \\
&\leq \frac{1}{\varepsilon}\mu^+(s)\left|Y_s^\varepsilon - J_\varepsilon\left(Y_s^\varepsilon\right)\right|^2 + \lambda\left|\nabla\varphi_\varepsilon(Y_s^\varepsilon)\right|^2 + \beta_s + b(s)\left|J_\varepsilon\left(Y_s^\varepsilon\right)\right|^p + \kappa\left|Z_s^\varepsilon\right|^2.
\end{aligned} \tag{5.126}$$

Using here the inequalities (5.119), then from (5.125) we infer that for all $t \in [0, T]$,

$$\begin{aligned}
\mathbb{E}\varphi_\varepsilon(Y_t^\varepsilon) + (1-\lambda)\,\mathbb{E}\int_t^T |\nabla\varphi_\varepsilon(Y_s^\varepsilon)|^2 ds &\leq \mathbb{E}\varphi(\eta) + 2\int_t^T \mu^+(s)\,\mathbb{E}\varphi_\varepsilon(Y_s^\varepsilon)ds \\
&\quad + C\,\mathbb{E}\int_t^T \left(\left[1 + \beta_s + b(s)\left(1 + |Y_s^\varepsilon - u_0|^p\right) + \kappa\,|Z_s^\varepsilon|^2\right]\right) ds
\end{aligned}$$

which yields, via estimates (5.124) and the backward Gronwall inequality (Corollary 6.62), that there exists a constant $C > 0$ independent of $\varepsilon \in (0, 1]$ such that

$$\begin{aligned}
&(a) \quad \mathbb{E}\varphi_\varepsilon(Y_t^\varepsilon) + \mathbb{E}\int_0^T |\nabla\varphi_\varepsilon(Y_s^\varepsilon)|^2 ds \leq C, \\
&(b) \quad \mathbb{E}\left|Y_t^\varepsilon - J_\varepsilon\left(Y_t^\varepsilon\right)\right|^2 \leq C\varepsilon.
\end{aligned} \tag{5.127}$$

Step 4. Cauchy sequence and convergence.

Let $\varepsilon, \delta \in (0, 1]$.
We can write

$$Y_t^\varepsilon - Y_t^\delta = \int_t^T dK_s^{\varepsilon,\delta} - \int_t^T Z_s^\varepsilon dB_s,$$

where

$$K_t^{\varepsilon,\delta} = \int_0^t \left[F\left(s, Y_s^\varepsilon, Z_s^\varepsilon\right) - F\left(s, Y_s^\delta, Z_s^\delta\right) - \nabla\varphi_\varepsilon\left(Y_s^\varepsilon\right) + \nabla\varphi_\delta\left(Y_s^\delta\right)\right] ds.$$

Then

$$\langle Y_t^\varepsilon - Y_t^\delta, dK_t^{\varepsilon,\delta}\rangle \leq (\varepsilon+\delta)\langle \nabla\varphi_\varepsilon(Y_t^\varepsilon), \nabla\varphi_\delta(Y_t^\delta)\rangle dt + |Y_t^\varepsilon - Y_t^\delta|^2 dV_t + \frac{1}{4}|Z_t^\varepsilon - Z_t^\delta|^2 dt,$$

and by Proposition 5.2, with $a = p = 2$,

$$\begin{aligned}
&\mathbb{E}\sup_{s\in[0,T]}\left|Y_s^\varepsilon - Y_s^\delta\right|^2 + \mathbb{E}\int_0^T \left|Z_s^\varepsilon - Z_s^\delta\right|^2 ds \\
&\quad\leq C\,\mathbb{E}\int_0^T (\varepsilon+\delta)\left\langle \nabla\varphi_\varepsilon(Y_s^\varepsilon), \nabla\varphi_\delta(Y_s^\delta)\right\rangle ds \\
&\quad\leq \frac{1}{2}C(\varepsilon+\delta)\left[\mathbb{E}\int_0^T |\nabla\varphi_\varepsilon(Y_s^\varepsilon)|^2 ds + \mathbb{E}\int_0^T \left|\nabla\varphi_\delta(Y_s^\delta)\right|^2 ds\right] \\
&\quad\leq C'(\varepsilon+\delta).
\end{aligned}$$

Hence there exist $(Y,Z,U) \in S_m^2[0,T] \times \Lambda_{m\times k}^2(0,T) \times \Lambda_m^2(0,T)$ and a sequence $\varepsilon_n \searrow 0$ such that

$$\begin{aligned}
&Y^{\varepsilon_n} \to Y, \text{ in } S_m^2[0,T] \text{ and a.s. in } C([0,T];\mathbb{R}^m), \\
&Z^{\varepsilon_n} \to Z, \text{ in } \Lambda_{m\times k}^2(0,T) \text{ and a.s. in } L^2\left(0,T;\mathbb{R}^{m\times k}\right), \\
&\nabla\varphi_\varepsilon(Y^\varepsilon) \rightharpoonup U, \text{ weakly in } \Lambda_m^2(0,T), \\
&J_{\varepsilon_n}(Y^{\varepsilon_n}) \to Y, \text{ in } \Lambda_m^2(0,T) \text{ and a.s. in } L^2(0,T;\mathbb{R}^m).
\end{aligned}$$

Passing to the limit in (5.122) we conclude that

$$Y_t + \int_t^T U_s ds = \eta + \int_t^T F(s, Y_s, Z_s)\, ds - \int_t^T Z_s dB_s, \text{ a.s.}$$

Since $\nabla\varphi_\varepsilon(Y_s^\varepsilon) \in \partial\varphi\left(J_\varepsilon\left(Y_s^\varepsilon\right)\right)$ it follows that for all $A \in \mathcal{F}$, $0 \leq s \leq t \leq T$ and $v \in S_m^2[0,T]$,

$$\mathbb{E}\int_s^t \mathbf{1}_A \langle \nabla\varphi_\varepsilon(Y_r^\varepsilon), v_r - Y_r^\varepsilon\rangle\, dr + \mathbb{E}\int_s^t \mathbf{1}_A\varphi(J_\varepsilon\left(Y_r^\varepsilon\right))dr \leq \mathbb{E}\int_s^t \mathbf{1}_A\varphi(v_r)dr.$$

Passing to lim inf for $\varepsilon = \varepsilon_n \searrow 0$ in the above inequality we obtain that $U_s \in \partial\varphi(Y_s)$. Hence $(Y,Z,U) \in S_m^p[0,T] \times \Lambda_{m\times k}^p(0,T) \times \Lambda_m^2(0,T)$ and (Y,Z,K), with $K_t = \displaystyle\int_0^t U_s ds$, is the solution of BSVI (5.116). The proof is complete. ∎

Remark 5.53. The existence Theorem 5.52 is well adapted to the Hilbert space setting, since we do not impose an assumption of the form

$$\operatorname{int}(\operatorname{Dom}(\varphi)) \neq \emptyset,$$

which is very restrictive for infinite dimensional spaces. In the context of Hilbert spaces Theorem 5.52 holds in the same form (see [57] where some examples of partial differential backward stochastic variational inequalities are given too).

Let $(u_0, \hat{u}_0) \in \partial\varphi$ be fixed. From the inequality (5.123) we have for $a = p = 2$

$$e^{2V(t)} |Y_t^\varepsilon - u_0|^2 \le C_{a,p} \Big[\mathbb{E}^{\mathcal{F}_t} e^{2V((T)} |\eta - u_0|^2 + \mathbb{E}^{\mathcal{F}_t} \Big(\int_t^T e^{V(s)} [|\hat{u}_0| + |F(s, u_0, 0)|] \, ds \Big)^2 \Big],$$

and consequently if $|\eta| + \int_0^T |F(s, u_0, 0)| \, ds \le M_0$, then a.s. for all $t \in [0, T]$,

$$|Y_t^\varepsilon| \le R_0 = |u_0| + C \, e^{2\|V\|_T} (|u_0| + T \, |\hat{u}_0| + M_0) \,. \tag{5.128}$$

Corollary 5.54. *If in Theorem 5.52 we replace the assumption (A$_4$) by*

(A$_5$) There exist $M_0, L > 0$ such that:

$$\begin{aligned} &(i) \quad 0 \le \ell_t \le L, \; a.e., \; t \in [0, T] \,, \\ &(ii) \quad |\eta| + \int_0^T |F(s, u_0, 0)| \, ds \le M_0, \; a.s., \omega \in \Omega \,, \\ &(iii) \quad \exists R_0 > 0 \textit{ sufficient large such that} \\ &\qquad\qquad \mathbb{E} \int_0^T \left(F_{R_0}^{\#}(s)\right)^2 ds < \infty, \end{aligned}$$

(in the proof R_0 is defined by (5.128)) the conclusions of Theorem 5.52 hold.

Proof. Let R_0 be defined by (5.128). The proof follows the same steps and computations as in Theorem 5.52 with the modification of Step 3: the estimate (5.126) now takes the following form (considering (5.128)),

$$\begin{aligned} \left\langle \nabla\varphi_\varepsilon(Y_s^\varepsilon), F\left(s, Y_s^\varepsilon, Z_s^\varepsilon\right)\right\rangle &\le |\nabla\varphi_\varepsilon(Y_s^\varepsilon)| \left|F\left(s, Y_s^\varepsilon, 0\right)\right| + |\nabla\varphi_\varepsilon(Y_s^\varepsilon)| \, L \, |Z_s^\varepsilon| \\ &\le \frac{1}{2} |\nabla\varphi_\varepsilon(Y_s^\varepsilon)|^2 + \left(F_{R_0}^{\#}(s)\right)^2 + L^2 \, |Z_s^\varepsilon|^2 \,. \end{aligned}$$

Using this inequality in (5.125) we directly obtain (5.127). ■

Remark 5.55. We note that if $F(\omega, t, y, z) = F(y, z)$, then the assumption (A$_5$) becomes $|\eta| \le M_0$, a.s., $\omega \in \Omega$.

Remark 5.56. In the particular case where φ is the convex indicator of a convex subset $D \subset \mathbb{R}^m$, the BSDE (5.116) is a reflected BSDE. As first noted in [34], the process K which maintains the solution inside D is absolutely continuous, unlike in the case of forward SDEs. The intuitive reason for this is that K does not need to fight against the martingale term. The situation is probably quite different in the case of nonconvex sets, but reflecting BSDEs at the boundary of nonconvex sets remains an open problem. The theory of reflected BSDEs was initiated in [25], where reflection in $\mathbb{R}$ above a given continuous adapted process was considered.

5.6 BSDEs with Random Final Time

5.6.1 BSDEs with a Monotone Coefficient

Let us now discuss the existence and uniqueness of a solution to an equation which we would like to write as

$$Y_t = \eta + \int_t^\infty \Phi(s, Y_s, Z_s)\, dQ_s - \int_t^\infty Z_s dB_s, \quad a.s., \ \forall\, t \geq 0. \tag{5.129}$$

In most cases the above integrals will not make sense. For this reason we shall give below a weaker formulation of the above BSDE.

We formulate the following assumptions:

$$\textbf{(BSDE-H}_\infty) \tag{5.130}$$

(i) $p, a > 1, n_p = 1 \wedge (p-1)$,

(ii) *$\eta \in L^p(\Omega, \mathcal{F}_\infty, \mathbb{P}; \mathbb{R}^m)$ and $(\xi, \zeta) \in S_d^p \times \Lambda_{d\times k}^p(0,\infty)$ is the unique pair such that*

$$\xi_t = \eta - \int_t^\infty \zeta_s dB_s, \quad t \geq 0, \ a.s.,$$

(in particular $(\xi_t)_{t\geq 0}$ is given by $\xi_t = \mathbb{E}^{\mathcal{F}_t}\eta$).

(iii) $(\omega, t) \longmapsto Q_t(\omega) : \Omega \times [0,\infty[\to \mathbb{R}$ is a $\mathcal{P}$-m.i.c.s.p. such that $Q_0 = 0$.

- ♦ $\forall\, y \in \mathbb{R}^m$, $z \in \mathbb{R}^{m\times k}$, *the function* $\Phi(\cdot,\cdot,y,z) : \Omega \times [0,\infty[\to \mathbb{R}^m$ *is* $\mathcal{P}$*-measurable*;
- ♦ *there exist* $\ell \in L^2_{loc}(\mathbb{R}_+; \mathbb{R}_+)$ *(a deterministic function) and two* $\mathcal{P}$*-m.s.p* $\mu, \alpha : \Omega \times [0,\infty[\to \mathbb{R}$, $\alpha \geq 0$, *such that* $\alpha_t dQ_t = dt$ *and*

$$\int_0^T |\mu_t| dQ_t < \infty, \quad \textit{for all } T > 0, \mathbb{P}\text{-a.s.};$$

♦ *for all* $y, y' \in \mathbb{R}^m$ and $z, z' \in \mathbb{R}^{m\times k}$, $d\mathbb{P}\otimes dQ_t$*-a.e.*:

$$\begin{array}{ll} & \textit{Continuity:} \\ (C_y) & y \longrightarrow \Phi(t,y,z) : \mathbb{R}^m \to \mathbb{R}^m \textit{ is continuous;} \\ & \textit{Monotonicity condition:} \\ (M_y) & \langle y'-y, \Phi(t,y',z)-\Phi(t,y,z)\rangle \le \mu_t |y'-y|^2 ; \\ & \textit{Lipschitz condition:} \\ (L_z) & |\Phi(t,y,z')-\Phi(t,y,z)| \le \alpha_t\, \ell(t)\, |z'-z|; \\ & \textit{Boundedness condition:} \\ (B_y) & \displaystyle\int_0^T \Phi^{\#}_\rho(s)\, dQ_s < \infty, \quad \forall\, \rho, T \ge 0, \end{array} \tag{5.131}$$

□

where $\Phi^{\#}_\rho(t) = \sup\{|\Phi(t,y,0)| : |y| \le \rho\}$.
Define

$$\bar{\mu}_t = \int_0^t \mu_s dQ_s$$

and

$$S^p_m\left(e^{\bar{\mu}}\right) = \left\{ Y \in S^0_m : \mathbb{E} \sup_{s\in[0,T]} \left|e^{\bar{\mu}_s} Y_s\right|^p < \infty \text{ for all } T \ge 0 \right\}.$$

Note that

$$\bar{\mu}_t \le \bar{\mu}^+_t \le \left\|\bar{\mu}^+\right\|_t = \sup_{s\in[0,t]} \left(\int_0^s \mu_r dQ_r\right)^+ \le \int_0^t \mu^+_r dQ_r .$$

Finally we recall the usual notation

$$V_t = V^{a,p}_t \overset{def}{=} \int_0^t \mu_s dQ_s + \frac{a}{2n_p}\int_0^t \ell^2(s)\, ds. \tag{5.132}$$

Theorem 5.57. *Let* $p, a > 1$ *and* V *be defined by (5.132). Let the assumptions* (**BSDE-H**$_\infty$) *be satisfied and*

$$\begin{array}{ll} (i) & \mathbb{E}\left(\sup_{t\in[0,T]} e^{p\bar{\mu}_t}|\eta|^p\right) < \infty < \infty, \textit{ for all } T \ge 0, \\ (ii) & \mathbb{E}\left(\displaystyle\int_0^\infty e^{V_t}\,|\Phi(t,\xi_t,\zeta_t)|\, dQ_t\right)^p < \infty. \end{array}$$

If, moreover, for all $\rho \geq 0$

$$(h_1) \quad \mathbb{E}\left(\int_0^T \sup_{|y|\leq\rho} \left|e^{\bar{\mu}_t}\Phi\left(s, e^{-\bar{\mu}_t}y, 0\right) - \mu_t y\right| dQ_t\right)^p < \infty, \text{ or}$$

$$(h_2) \quad \mu \geq 0 \text{ and } \mathbb{E}\left(\int_0^T e^{\bar{\mu}_t} \sup_{|y|\leq\rho} |\Phi(t, y, 0)|\, dQ_s\right)^p < \infty,$$

then there exists a unique solution $(Y_t, Z_t)_{t\geq 0} \in S_m^0 \times \Lambda_{m\times k}^0$ *of the BSDE (5.129) in the sense that (here* $\forall 0 \leq t \leq T$ *means for all* t *and all* T *such that* $0 \leq t \leq T$*)*

$$\begin{cases} (j) & Y_t = Y_T + \int_t^T \Phi(s, Y_s, Z_s)\, dQ_s - \int_t^T Z_s dB_s, \quad a.s., \ \forall\, 0 \leq t \leq T, \\ (jj) & \mathbb{E} \sup_{0\leq t\leq T} e^{pV_s} |Y_s|^p < \infty, \quad \text{for all } T \geq 0, \\ (jjj) & \lim_{T\to\infty} \mathbb{E}\, e^{pV_T} |Y_T - \xi_T|^p = 0. \end{cases} \tag{5.133}$$

Moreover

$$\mathbb{E}\left(\int_0^T e^{2V_s} |Z_s|^2 ds\right)^{p/2} < \infty, \quad \text{for all } T \geq 0,$$

and there exists a constant $C_{a,p}$ *depending only on* (a, p) *such that for all* $t \geq 0$*,*

$$\begin{aligned} &\mathbb{E}^{\mathcal{F}_t} \sup_{s\geq t} \left|e^{V_s}(Y_s - \xi_s)\right|^p + \mathbb{E}^{\mathcal{F}_t}\left(\int_t^\infty e^{2V_s} |Z_s - \zeta_s|^2 ds\right)^{p/2} \\ &\qquad \leq C_{a,p}\mathbb{E}^{\mathcal{F}_t}\left(\int_t^\infty e^{V_s} |\Phi(s, \xi_s, \zeta_s)|\, dQ_s\right)^p, \quad a.s. \end{aligned} \tag{5.134}$$

Proof. Uniqueness. If (Y, Z) and $\left(\hat{Y}, \hat{Z}\right)$ are two solutions of (5.133) in the space $S_m^p\left(e^{\bar{\mu}}\right) \times \Lambda_{m\times k}^0 = S_m^p\left(e^V\right) \times \Lambda_{m\times k}^0$. Then from (5.24) there exists a positive constant $C_{a,p}$ depending only on (a, p), such that

$$\begin{aligned} &\mathbb{E}\left(\sup_{t\in[0,T]} e^{pV_t}\left|Y_t - \hat{Y}_t\right|^p\right) + \mathbb{E}\left(\int_0^T e^{2V_s}\left|Z_s - \hat{Z}_s\right|^2 ds\right)^{p/2} \\ &\qquad \leq C_{a,p}\mathbb{E}\, e^{pV_T}\left|Y_T - \hat{Y}_T\right|^p \to 0, \quad \text{as } T \to \infty, \end{aligned}$$

where we have used $((5.133)(jj))$. Uniqueness follows.

Existence. Note that

$$\xi_t = \mathbb{E}^{\mathcal{F}_n}\eta - \int_t^n \zeta_s dB_s, \quad t \in [0, n], \ a.s.,$$

and since $\mathbb{E}\left(\sup_{t\in[0,T]} e^{p\bar{\mu}_t}|\eta|^p\right) < \infty$, by Corollary 6.83

$$\mathbb{E}\sup_{t\in[0,T]} e^{p\bar{\mu}_t}|\xi_t|^p + \mathbb{E}\left(\int_0^T e^{2\bar{\mu}_t}|\zeta_t|^2\,dt\right)^{p/2} \le C_p\,\mathbb{E}\left(\sup_{t\in[0,T]} e^{p\bar{\mu}_t}|\eta|^p\right) < \infty. \tag{5.135}$$

Hence

$$\begin{aligned}(\xi,\zeta) &\in S_m^p\left([0,T];e^{\bar{\mu}}\right)\times\Lambda^p_{m\times k}\left(0,T;e^{\bar{\mu}}\right)\\ &\equiv S_m^p\left([0,T];e^{V}\right)\times\Lambda^p_{m\times k}\left(0,T;e^{V}\right).\end{aligned}$$

For any fixed $n\in\mathbb{N}^*$, we consider the approximating equation

$$Y_t^n = \mathbb{E}^{\mathcal{F}_n}\eta + \int_t^n \Phi\left(s,Y_s^n,Z_s^n\right)dQ_s - \int_t^n Z_s^n dB_s,\quad t\in[0,n],\ a.s.$$

By Lemma 5.29, this equation has a unique solution $(Y^n,Z^n)\in S_m^p\left([0,n];e^{\bar{\mu}}\right)\times \Lambda^p_{m\times k}\left(0,n;e^{\bar{\mu}}\right)$. We set $Y_s^n=\xi_s$ and $Z_s^n=\zeta_s$ for $s>n$.

Since the approximating equation can be written in the form: $\mathbb{P}$-a.s., for all $t\in[0,n]$,

$$Y_t^n-\xi_t = \int_t^n \Phi\left(s,\xi_s+\left(Y_s^n-\xi_s\right),\zeta_s+\left(Z_s^n-\zeta_s\right)\right)dQ_s - \int_t^n\left(Z_s^n-\zeta_s\right)dB_s,$$

we deduce from (5.19) that $\mathbb{P}$-a.s., for all $0\le t\le n$,

$$\begin{aligned}\mathbb{E}^{\mathcal{F}_t}\sup_{s\ge t} e^{pV_s}|Y_s^n-\xi_s|^p &+ \mathbb{E}^{\mathcal{F}_t}\left(\int_t^\infty e^{2V_s}|Z_s^n-\zeta_s|^2\,ds\right)^{p/2}\\ &\le C_{a,p}\mathbb{E}^{\mathcal{F}_t}\left(\int_t^n e^{V_s}|\Phi(s,\xi_s,\zeta_s)|\,dQ_s\right)^p\\ &\le C_{a,p}\mathbb{E}^{\mathcal{F}_t}\left(\int_t^\infty e^{V_s}|\Phi(s,\xi_s,\zeta_s)|\,dQ_s\right)^p\end{aligned}\tag{5.136}$$

where $C_{a,p}$ is a constant depending only upon (a,p). In particular for $i\in\mathbb{N}^*$:

$$\begin{aligned}&\mathbb{E}\sup_{s\ge n}\left\|e^{V_s}\left(Y_s^{n+i}-\xi_s\right)\right\|^p + \mathbb{E}\left(\int_n^\infty e^{2V_s}\left|Z_s^{n+i}-\zeta_s\right|^2 ds\right)^{p/2}\\ &\quad\le C_{a,p}\mathbb{E}\left(\int_n^\infty e^{V_s}|\Phi(s,\xi_s,\zeta_s)|\,dQ_s\right)^p \to 0,\quad \text{as } n\to\infty.\end{aligned}\tag{5.137}$$

Note that by uniqueness

$$Y_t^{n+i} = Y_n^{n+i} + \int_t^n \Phi\left(s,Y_s^{n+i},Z_s^{n+i}\right)dQ_s - \int_t^n Z_s^{n+i}dB_s,\ t\in[0,n],\quad a.s.$$

Using the inequality (5.24) in this context we infer that

$$\mathbb{E}\left(\sup_{t\in[0,n]} e^{pV_t}\left|Y_t^{n+i}-Y_t^n\right|^p\right)+\mathbb{E}\left(\int_0^n e^{2V_s}\left|Z_s^{n+i}-Z_s^n\right|^2 ds\right)^{p/2} \\ \leq C_{a,p}\mathbb{E}\, e^{pVn}\left|Y_n^{n+i}-\xi_n\right|^p \to 0, \quad \text{as } n\to\infty.$$

Hence

$$\begin{aligned}
&\mathbb{E}\left(\sup_{s\geq 0} e^{pV_s}\left|Y_s^{n+i}-Y_s^n\right|^p\right)+\mathbb{E}\left(\int_0^\infty e^{2V_s}\left|Z_s^{n+i}-Z_s^n\right|^2 ds\right)^{p/2}\\
&\leq \mathbb{E}\left(\sup_{s\in[0,n]} e^{pV_s}\left|Y_s^{n+i}-Y_s^n\right|^p\right)+2^{p/2}\mathbb{E}\left(\int_0^n e^{2V_s}\left|Z_s^{n+i}-Z_s^n\right|^2 ds\right)^{p/2}\\
&+\mathbb{E}\left(\sup_{s>n} e^{pV_s}\left|Y_s^{n+i}-\xi_s\right|^p\right)+2^{p/2}\mathbb{E}\left(\int_n^\infty e^{2V_s}\left|Z_s^{n+i}-\zeta_s\right|^2 ds\right)^{p/2}\\
&\to 0, \quad \text{as} \quad n\to\infty.
\end{aligned}$$

This shows there exist progressively measurable stochastic processes $Y:\mathbb{R}_+\times\Omega\to\mathbb{R}^m$ and $Z:\mathbb{R}_+\times\Omega\to\mathbb{R}^{m\times k}$ such that

$$\lim_{n\to\infty}\mathbb{E}\left(\sup_{s\geq 0} e^{pV_s}\left|Y_s^n-Y_s\right|^p\right)+\mathbb{E}\left(\int_0^\infty e^{2V_s}\left|Z_s^n-Z_s\right|^2 ds\right)^{p/2}=0. \quad (5.138)$$

From (5.136) we deduce by letting $n\to\infty$ that for all $t\geq 0$, $\mathbb{P}$-a.s.,

$$\mathbb{E}^{\mathcal{F}_t}\sup_{s\geq t}\left|e^{V_s}\left(Y_s-\xi_s\right)\right|^p+\mathbb{E}^{\mathcal{F}_t}\left(\int_t^\infty e^{2V_s}\left|Z_s-\zeta_s\right|^2 ds\right)^{p/2} \\ \leq C_{a,p}\mathbb{E}^{\mathcal{F}_t}\left(\int_t^\infty e^{V_s}\left|\Phi\left(s,\xi_s,\zeta_s\right)\right| dQ_s\right)^p. \quad (5.139)$$

Since $(\xi,\zeta)\in S_m^p\left([0,T];e^V\right)\times\Lambda_{m\times k}^p\left(0,T;e^V\right)$, for all $T>0$, it clearly follows from (5.139) that

$$\mathbb{E}\sup_{s\in[0,T]}\left|e^{V_s}Y_s\right|^p+\mathbb{E}\left(\int_0^T e^{2V_s}\left|Z_s\right|^2 ds\right)^{p/2}<\infty.$$

Let $0\leq t\leq T\leq n$. Now by Lemma 5.5 we can pass to the limit in

$$Y_t^n=Y_T^n+\int_t^T\Phi\left(s,Y_s^n,Z_s^n\right)dQ_s-\int_t^T Z_s^n dB_s, \quad a.s.,\ t\in[0,T] \quad (5.140)$$

and taking into account (5.139) we deduce that (Y, Z) satisfies (5.133). The proof is complete.

Remark 5.58. If, moreover, there exists a constant b such that $\sup_{t\geq 0} V_t \leq b$, $\mathbb{P}$-a.s., then the conditions (5.143-(*jjj*)) can be replaced by the stronger statement than (5.133):

$$(jjj') \qquad \lim_{T\to\infty} \mathbb{E}\, e^{pV_T} |Y_T - \eta|^p = 0. \tag{5.141}$$

Indeed using the backward Burkholder–Davis–Gundy inequality (2.51) we have

$$c_p \mathbb{E}\left(\int_t^\infty |\zeta_r|^2 \, dr\right)^{p/2} \leq \mathbb{E} \sup_{s\geq t} |\eta - \xi_s|^p \leq C_p \mathbb{E}\left(\int_t^\infty |\zeta_r|^2 \, dr\right)^{p/2}.$$

□

Let $\tau : \Omega \to [0, \infty]$ be a stopping time and $\eta \in L^p(\Omega, \mathcal{F}_\tau, \mathbb{P}; \mathbb{R}^m)$, $p > 1$. We now consider the BSDE

$$Y_t = \eta + \int_{t\wedge\tau}^{\tau} \Phi(s, Y_s, Z_s)\, dQ_s - \int_{t\wedge\tau}^{\tau} Z_s dB_s, \quad a.s., \ \forall\, t \geq 0, \tag{5.142}$$

in the sense which will be made precise in the next theorem. Plainly the BSDE (5.142) a particular case of Eq. (5.129) where Φ is of the form $\mathbf{1}_{[0,\tau]}\Phi$, since by Lemma 2.43 $Z_t = 0$ for all $t > \tau$.

Recall that the unique pair $(\xi, \zeta) \in S_d^p \times \Lambda_{d\times k}^p(0, \infty)$ such that

$$\xi_t = \eta - \int_t^\infty \zeta_s dB_s, \quad t \geq 0, \ a.s.,$$

satisfies $\xi_t = \mathbb{E}^{\mathcal{F}_{t\wedge\tau}}\eta$ and $\zeta_t = \mathbf{1}_{[0,\tau]}(t)\, \zeta_t$.

Define

$$V_t \overset{def}{=} \int_0^{t\wedge\tau} \mu_s dQ_s + \frac{a}{2n_p}\int_0^{t\wedge\tau} \ell^2(s)\, ds \quad \text{and} \quad \bar{\mu}_t = \int_0^{t\wedge\tau} \mu_s dQ_s.$$

We deduce from Theorem 5.57:

Corollary 5.59. *Let $a, p > 1$ and $\tau : \Omega \to [0, \infty]$ be a stopping time. Let the assumptions* (**BSDE-H$_\infty$**) *with $\Phi(s, y, z) = \mathbf{1}_{[0,\tau]}(s)\, \Phi(s, y, z)$ be satisfied and $\eta \in L^p(\Omega, \mathcal{F}_\tau, \mathbb{P}; \mathbb{R}^m)$. Assume moreover*

$$(i) \qquad \mathbb{E}\left(e^{p\|\bar{\mu}^+\|_{T\wedge\tau}} |\eta|^p\right) < \infty, \text{ for all } T \geq 0,$$

$$(ii) \qquad \mathbb{E}\left(\int_0^\tau e^{V_t} |\Phi(t, \xi_t, \zeta_t)|\, dQ_t\right)^p < \infty,$$

and for all $\rho \geq 0$

(h_1) $\mathbb{E}\left(\int_0^{T\wedge\tau} \sup_{|y|\leq\rho} \left| e^{\bar{\mu}_t} \Phi\left(s, e^{-\bar{\mu}_t} y, 0\right) - \mu_t y \right| dQ_t\right)^p < \infty$, *or*

(h_1) $\mu \geq 0$ *and* $\mathbb{E}\left(\int_0^{T\wedge\tau} e^{\bar{\mu}_t} \sup_{|y|\leq\rho} |\Phi(t, y, 0)| \, dQ_s\right)^p < \infty.$

(A) *Then there exists a unique solution* $(Y_t, Z_t)_{t\geq 0} \in S_m^0 \times \Lambda_{m\times k}^0$, $(Y_t, Z_t) = (\eta, 0)$ *if* $t > \tau$, *of the BSDE (5.142) in the sense that*

$$\begin{cases} (j) \quad Y_t = Y_T + \int_t^T \Phi(s, Y_s, Z_s) \, dQ_s - \int_t^T Z_s dB_s, \quad a.s., \\ \qquad\qquad\qquad\qquad\qquad\qquad \textit{for all } 0 \leq t \leq T, \\ (jj) \ \mathbb{E} \sup_{s\in[0,T]} e^{pV_s} |Y_s|^p < \infty, \quad \textit{for all } T \geq 0, \\ (jjj) \ \lim_{T\to\infty} \mathbb{E}\, e^{pV_{T\wedge\tau}} |Y_{T\wedge\tau} - \xi_{T\wedge\tau}|^p = 0. \end{cases} \tag{5.143}$$

Moreover

$$\mathbb{E}\left(\int_0^\tau e^{2V_s} |Z_s|^2 \, ds\right)^{p/2} < \infty$$

and there exists a constant $C_{a,p}$ *depending only on* (a, p) *such that for all* $t \geq 0$,

$$\begin{aligned} \mathbb{E} \sup_{t\wedge\tau\leq s\leq\tau} \left| e^{V_s} (Y_s - \xi_s) \right|^p + \mathbb{E}\left(\int_{t\wedge\tau}^\tau e^{2V_s} |Z_s - \zeta_s|^2 \, ds\right)^{p/2} \\ \leq C_{a,p} \mathbb{E}\left(\int_{t\wedge\tau}^\tau e^{V_s} |\Phi(s, \xi_s, \zeta_s)| \, dQ_s\right)^p. \end{aligned} \tag{5.144}$$

(B) *If, moreover, there exists a constant* b *such that* $\sup_{0\leq t\leq\tau} V_t \leq b$, $\mathbb{P}$*-a.s., then the conditions (5.143-(jjj)) can be replaced by*

$$(jjj') \qquad \lim_{T\to\infty} \mathbb{E}\, e^{pV_{T\wedge\tau}} |Y_{T\wedge\tau} - \eta|^p = 0. \tag{5.145}$$

□

5.6.2 BSVIs with Random Final Time

In this section we are interested in the following generalized backward stochastic variational inequality (BSVI for short):

$$\begin{cases} Y_t + \int_{t\wedge\tau}^{\tau} dK_s = \eta + \int_{t\wedge\tau}^{\tau} \left[F\left(s, Y_s, Z_s\right) ds + G\left(s, Y_s\right) dA_s\right] \\ \qquad\qquad\qquad\qquad\qquad\qquad - \int_{t\wedge\tau}^{\tau} Z_s dB_s,\ t \geq 0, \\ dK_t \in \partial\varphi\left(Y_t\right) dt + \partial\psi\left(Y_t\right) dA_t \quad \text{on } \mathbb{R}_+, \end{cases} \tag{5.146}$$

where $\partial\varphi$, $\partial\psi$ are the subdifferentials of the convex lower semicontinuous functions φ, ψ, $\{A_t : t \geq 0\}$ is a progressively measurable increasing continuous stochastic process, and τ is a stopping time.

In fact we will define and prove the existence of the solution for an equivalent form of (5.146):

$$\begin{cases} Y_t + \int_t^{\infty} dK_s = \eta + \int_t^{\infty} \Phi\left(s, Y_s, Z_s\right) dQ_s - \int_t^{\infty} Z_s dB_s, \quad t \geq 0, \\ dK_t \in \partial_y \Psi\left(t, Y_t\right) dQ_t, \ \text{on } \mathbb{R}_+, \end{cases} \tag{5.147}$$

with Q, Φ and Ψ adequately defined.

We mention that the presence of the process A is justified by the possible applications of Eq. (5.146) in obtaining a probabilistic interpretation for the solution of PDEs with Neumann boundary conditions; since τ is a stopping time the BSVI (5.146) can be used for elliptic PDEs.

Because (5.146) is quite a complicated equation, in order to simplify the presentation we shall restrict ourselves to $p = 2$. The case $p \geq 2$ can be found in [47].

We begin to give the main assumptions for this section.

(A_1) *The random variable* $\tau : \Omega \to [0, \infty]$ *is a stopping time.*

(A_2) *The random variable* $\eta : \Omega \to \mathbb{R}^m$ *is* $\mathcal{F}_\tau$*-measurable,* $\mathbb{E}\left|\eta\right|^2 < \infty$ *and the stochastic process* $(\xi, \zeta) \in S_m^2 \times \Lambda_{m\times k}^2 (0, \infty)$ *is the unique pair associated to* η *given by the martingale representation formula (Corollary 2.44)*

$$\begin{cases} \xi_t = \eta - \int_t^{\infty} \zeta_s dB_s, \ t \geq 0, \ a.s., \\ \xi_t = \mathbb{E}^{\mathcal{F}_t} \eta = \mathbb{E}^{\mathcal{F}_{t\wedge\tau}} \eta \quad \text{and} \quad \zeta_t = \mathbf{1}_{[0,\tau]}(t)\, \zeta_t. \end{cases}$$

(A_3) *The process* $\{A_t : t \geq 0\}$ *is a progressively measurable increasing continuous stochastic process such that* $A_0 = 0$,

$$Q_t(\omega) = t + A_t(\omega),$$

and $\{\alpha_t : t \geq 0\}$ *is a real positive p.m.s.p. (given by the Radon–Nikodym representation theorem) such that* $\alpha \in [0, 1]$ *and*

$$dt = \alpha_t dQ_t \quad \text{and} \quad dA_t = (1 - \alpha_t)\, dQ_t.$$

(A$_4$) *The functions* $F : \Omega \times \mathbb{R}_+ \times \mathbb{R}^m \times \mathbb{R}^{m\times k} \to \mathbb{R}^m$ *and* $G : \Omega \times \mathbb{R}_+ \times \mathbb{R}^m \to \mathbb{R}^m$ *are such that*

$$\begin{cases} F(\cdot,\cdot,y,z), \ G(\cdot,\cdot,y) \ \text{are p.m.s.p., for all } (y,z) \in \mathbb{R}^m \times \mathbb{R}^{m\times k}, \\ F(\omega,t,\cdot,\cdot), \ G(\omega,t,\cdot) \ \text{are continuous functions, } d\mathbb{P}\otimes dt\text{-a.e.,} \end{cases}$$

and $\mathbb{P}$*-a.s.,*

$$\int_0^T F_\rho^\#(s)\,ds + \int_0^T G_\rho^\#(s)\,dA_s < \infty, \quad \forall\, \rho, T \ge 0,$$

where

$$F_\rho^\#(\omega,s) := \sup_{|y|\le\rho} |F(\omega,s,y,0)|, \quad G_\rho^\#(\omega,s) := \sup_{|y|\le\rho} |G(\omega,s,y)|.$$

(A$_5$) *Assume that there exist three progressively measurable positive stochastic processes* $\mu, \nu, \ell : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ *such that*

$$\int_0^T \left(\mu_s + (\ell_s)^2\right) ds + \int_0^T \nu_s dA_s < \infty, \ \text{for all } T > 0, \ \mathbb{P}\text{-a.s.,}$$

and $\mathbb{P}$-a.s. $\omega \in \Omega$, *for all* $t \in [0,\tau(\omega)]$, $y, y' \in \mathbb{R}^m$, $z, z' \in \mathbb{R}^{m\times k}$,

$$\begin{array}{ll} (i) & \langle y'-y, F(t,y',z) - F(t,y,z)\rangle \le \mu_t \ |y'-y|^2, \\ (ii) & \langle y'-y, G(t,y') - G(t,y)\rangle \le \nu_t \ |y'-y|^2, \\ (iii) & |F(t,y,z') - F(t,y,z)| \le \ell_t \ |z'-z|. \end{array} \tag{5.148}$$

Let us introduce the functions

$$\begin{aligned} &H(\omega,t,y,z) := \mathbf{1}_{[0,\tau(\omega)]}(t)\left[\alpha_t(\omega) F(\omega,t,y,z) + (1-\alpha_t(\omega))\, G(\omega,t,y)\right], \\ &\bar{\mu}_t := \int_0^t \mathbf{1}_{[0,\tau]}(s)\,\mu_s ds, \quad \bar{\nu}_t := \int_0^t \mathbf{1}_{[0,\tau]}(s)\,\nu_s dA_s, \\ &\sigma_t := \mathbf{1}_{[0,\tau]}(t)\left[\mu_t\alpha_t + \nu_t(1-\alpha_t)\right], \quad \bar{\sigma}_t := \int_0^t \mathbf{1}_{[0,\tau]}(s)\,\sigma_s dQ_s = \bar{\mu}_t + \bar{\nu}_t, \\ &\lambda_t = \mathbf{1}_{[0,\tau]}(t)\,\alpha_t\ell_t, \quad \hat{\lambda}_t = \int_0^t \mathbf{1}_{[0,\tau]}(s)\,(\ell_s)^2\alpha_s dQ_s = \int_0^{t\wedge\tau} (\ell_s)^2\,ds. \end{aligned} \tag{5.149}$$

The relations (5.148) yield

$$\begin{array}{ll} (a) & \langle y'-y, H(t,y',z) - H(t,y,z)\rangle \le \sigma_t\,|y'-y|^2, \\ (b) & |H(t,y,z') - H(t,y,z)| \le \boldsymbol{\lambda}_t \ |z'-z|. \end{array} \tag{5.150}$$

Let

$$V_t = \int_0^t \mathbf{1}_{[0,\tau]}(s)\left[\left(\mu_s + (\ell_s)^2\right)\alpha_s + \nu_s(1-\alpha_s)\right]dQ_s = \bar{\sigma}_t + \hat{\lambda}_t. \tag{5.151}$$

Concerning φ and ψ we shall assume:

(A_6) *$\varphi, \psi : \mathbb{R}^m \to [0, +\infty]$ are proper convex lower semicontinuous (l.s.c.) functions, $\partial\varphi$ and $\partial\psi$ are the subdifferentials of φ and ψ, respectively, and there exists a $u_0 \in \mathbb{R}^m$ such that $0 \in \partial\varphi(u_0) \cap \partial\psi(u_0)$ (which is equivalent to $\varphi(u_0) \le \varphi(y)$ and $\psi(u_0) \le \psi(y)$* for all $y \in \mathbb{R}^m$). *Define*

$$\Psi(\omega, t, y) = \mathbf{1}_{[0,\tau(\omega)]}(t) \left[\alpha_t(\omega)\varphi(y) + (1 - \alpha_t(\omega))\psi(y)\right].$$

(A_7) *If $\mathbb{P}(\tau > N) > 0$, for all $N \in \mathbb{N}^*$, then for every $\bar{\eta} \in \overline{\eta, u_0} = conv\{\eta, u_0\}$ there exist two progressively measurable stochastic processes $\bar{\xi}^{(1)}, \bar{\xi}^{(2)}$ such that $\bar{\xi}_t^{(1)} \in \partial\varphi\left(\mathbb{E}^{\mathcal{F}_t}\bar{\eta}\right)$, $\bar{\xi}_t^{(2)} \in \partial\psi\left(\mathbb{E}^{\mathcal{F}_t}\bar{\eta}\right)$ a.e. $t \ge 0$ and*

$$\mathbb{E}\left(\int_0^\tau e^{V_s}|\bar{\xi}_s^{(1)}|ds\right)^2 + \mathbb{E}\left(\int_0^\tau e^{V_s}|\bar{\xi}_s^{(2)}|dA_s\right)^2 < \infty;$$

if $\bar{\xi}_s = \mathbf{1}_{[0,\tau]}(s)[\bar{\xi}_s^{(1)}\alpha_s + \bar{\xi}_s^{(2)}(1 - \alpha_s)]$, then $\bar{\xi}_s(\omega) \in \partial\Psi\left(\omega, s, \mathbb{E}^{\mathcal{F}_s}\bar{\eta}\right)$ and

$$\mathbb{E}\left(\int_0^\tau e^{V_s}\left|\bar{\xi}_s\right|dQ_s\right)^2 < \infty$$

(in the case $\bar{\eta} = \eta$ we define $\xi_t = \mathbb{E}^{\mathcal{F}_t}\eta$ and in the place of $(\bar{\xi}^{(1)}, \bar{\xi}^{(2)}, \bar{\xi})$ we shall use the notation $(\hat{\xi}^{(1)}, \hat{\xi}^{(2)}, \hat{\xi})$).

Remark 5.60. In place of the assumption (A_7) we can consider two particular cases:

(A_7') $\eta : \Omega \to \mathbb{O}$, where $\mathbb{O}$ is the closed convex set defined by

$$\mathbb{O} \overset{def}{=} \{y \in \mathbb{R}^m : \varphi(y) = \varphi(u_0) \text{ and } \psi(y) = \psi(u_0)\};$$

or

(A_7'') there exist $r_0 > 0$ and $v_0 \in \text{Dom}(\varphi) \cap \text{Dom}(\psi)$ such that

$$\begin{aligned} &(i) \quad \eta : \Omega \to \overline{B(v_0, r_0)} \subset \text{int}(\text{Dom}(\varphi)) \cap \text{int}(\text{Dom}(\psi)), \\ &(ii) \quad \mathbb{E}\left(e^{V_\tau}(\tau + A_\tau)\right) < \infty. \end{aligned}$$

Indeed:

(A_7') $\Rightarrow$(A_7): since $\bar{\xi}_t = \mathbb{E}^{\mathcal{F}_t}\bar{\eta} \in \mathbb{O}$ for all $t \ge 0$ we can set $\bar{\xi}^{(1)} = \bar{\xi}^{(2)} = \bar{\xi} = 0$ for every $\bar{\eta} \in \overline{\eta, u_0}$;
(A_7'') $\Rightarrow$ (A_7): by Proposition 6.2-(d) there exists an $M_0 > 0$ such that $\partial\varphi(u) \subset B(0, M_0)$ and $\partial\psi(u) \subset B(0, M_0)$ for all $u \in \overline{B(v_0, r_0)}$ and consequently $\left|\bar{\xi}_t^{(1)}\right| + \left|\bar{\xi}_t^{(2)}\right| \le 2M_0$ for all $\bar{\eta} \in \overline{\eta, u_0}$ because $\mathbb{E}^{\mathcal{F}_t}\bar{\eta} \in \overline{B(v_0, r_0)}$ for all $t \ge 0$. Observe that from $\left(A_7''\text{-ii}\right)$ it follows that $\mathbb{P}(\tau = \infty) = 0$.

Definition 5.61. By the notation $dK_t \in \partial\psi(Y_t)\,dA_t$ we shall understand that:

- K is an $\mathbb{R}^m$-valued locally bounded variation stochastic process;
- Y is an $\mathbb{R}^m$-valued continuous stochastic process such that $\int_0^T \psi(Y_t)\,dA_t < \infty$, a.s. $\forall\, T \geq 0$; and
- $\mathbb{P}$-a.s., for all $0 \leq t \leq s$

$$\int_t^s \langle y(r) - Y_r, dK_r\rangle + \int_t^s \psi(Y_r)\,dA_r \leq \int_t^s \varphi(y(r))\,dA_r,\ \forall y \in C(\mathbb{R}_+;\mathbb{R}^m),$$

(we have an analogous definition for $dK_t \in \partial\varphi(Y_t)\,dt$).

Remark 5.62. The condition $0 \in \partial\varphi(u_0) \cap \partial\psi(u_0)$ does not restrict the generality of the problem because from $\mathrm{Dom}(\partial\varphi) \cap \mathrm{Dom}(\partial\psi) \neq \emptyset$ it follows that *there exists* $u_0 \in \mathrm{Dom}(\partial\varphi) \cap \mathrm{Dom}(\partial\psi)$ and $\hat{u}_{01} \in \partial\varphi(u_0)$, $\hat{u}_{02} \in \partial\psi(u_0)$; in this case equation (5.146) is equivalent to

$$\begin{cases} Y_t + \displaystyle\int_{t\wedge\tau}^{\tau} d\hat{K}_s = \eta + \int_{t\wedge\tau}^{\tau}\left[\hat{F}(s,Y_s,Z_s)\,ds + \hat{G}(s,Y_s)\,dA_s\right] \\ \qquad\qquad\qquad\qquad\qquad - \displaystyle\int_{t\wedge\tau}^{\tau} Z_s dB_s, \quad t \geq 0, \\ d\hat{K}_t \in \partial\hat{\varphi}(Y_t)\,dt + \partial\hat{\psi}(Y_t)\,dA_t,\ \text{on } \mathbb{R}_+, \end{cases}$$

where $\hat{F}(s,y,z) = F(t,y,z) - \hat{u}_{01}$, $\hat{G}(s,y,z) = G(t,y) - \hat{u}_{02}$, $\hat{\varphi}(y) = \varphi(y) - \langle \hat{u}_{01}, y - u_0\rangle$, $\hat{\psi}(y) = \psi(y) - \langle \hat{u}_{02}, y - u_0\rangle$, $\partial\hat{\varphi}(y) = \partial\varphi(y) - \hat{u}_{01}$, $\partial\hat{\psi}(y) = \partial\psi(y) - \hat{u}_{02}$ and $d\hat{K}_t = dK_t - \hat{u}_{01}dt - \hat{u}_{02}dA_t$.

Let $\varepsilon > 0$ and define the Moreau–Yosida regularization of φ by

$$\varphi_\varepsilon(y) := \inf\left\{\frac{1}{2\varepsilon}|y - v|^2 + \varphi(v) : v \in \mathbb{R}^m\right\},$$

which is a C^1 convex function and $\nabla\varphi_\varepsilon(x) = \partial\varphi_\varepsilon(x) \in \partial\varphi(J_\varepsilon x)$, where $J_\varepsilon x = x - \varepsilon\nabla\varphi_\varepsilon(x)$. (For further properties see Annex B, Section "Convex Functions".) Since $0 \in \partial\varphi(u_0)$ we deduce that $\varphi(u_0) = \varphi_\varepsilon(u_0) \leq \varphi_\varepsilon(u) \leq \varphi(u)$, $J_\varepsilon(u_0) = u_0$ and $\nabla\varphi_\varepsilon(u_0) = 0$.

We introduce the *compatibility conditions* between φ, ψ and F, G:

(A_8) *There exists a* $c > 0$ *and two progressively measurable stochastic processes* $f, g\colon \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ *satisfying*

$$\mathbb{E}\int_0^\tau e^{2V_s}|f_s|^2\,ds + \mathbb{E}\int_0^\tau e^{2V_s}|g_s|^2\,dA_s < \infty,$$

such that for all $\varepsilon > 0$, $t \geq 0$, $y \in \mathbb{R}^m$, $z \in \mathbb{R}^{m\times k}$, $\mathbb{P}$-a.s.

$$\begin{array}{ll}(i) & \langle \nabla\varphi_\varepsilon(y), \nabla\psi_\varepsilon(y)\rangle \geq 0, \\ (ii) & \langle \nabla\varphi_\varepsilon(y), G(t,y)\rangle \leq c\,|\nabla\psi_\varepsilon(y)|\,[|G(t,y)| + g_t], \\ (iii) & \langle \nabla\psi_\varepsilon(y), F(t,y,z)\rangle \leq c\,|\nabla\varphi_\varepsilon(y)|\,[|F(t,y,z)| + f_t],\end{array} \tag{5.152}$$

and

$$\begin{array}{ll}(iv) & \langle \nabla\varphi_\varepsilon(y), -G(t,u_0)\rangle \leq c\,|\nabla\psi_\varepsilon(y)|\,[|G(t,u_0)| + g_t], \\ (v) & \langle \nabla\psi_\varepsilon(y), -F(t,u_0,0)\rangle \leq c\,|\nabla\varphi_\varepsilon(y)|\,[|F(t,u_0,0)| + f_t].\end{array} \tag{5.153}$$

Example 5.63. (e_1) If $\varphi = \psi$ then the compatibility assumptions (5.152) and (5.153) are clearly satisfied.

(e_2) Let $m = 1$. Since $\nabla\varphi_\varepsilon$ and $\nabla\psi_\varepsilon$ are increasing monotone functions on $\mathbb{R}$, we see that, if $G(t,u_0) = F(t,u_0,0) = 0$ and

$$(y - u_0)\,G(t,y) \leq 0 \quad \text{and} \quad (y - u_0)\,F(t,y,z) \leq 0, \;\; \forall\, t, y, z,$$

then the compatibility assumptions (5.152) and (5.153) are satisfied.

(e_3) Let $m = 1$. If $\varphi, \psi : \mathbb{R} \to (-\infty, +\infty]$ are the convex indicator functions

$$\varphi(y) = \begin{cases} 0, & \text{if } y \in [a,b], \\ +\infty, & \text{if } y \notin [a,b], \end{cases} \quad \text{and} \quad \psi(y) = \begin{cases} 0, & \text{if } y \in [c,d], \\ +\infty, & \text{if } y \notin [c,d], \end{cases}$$

where $-\infty \leq a \leq b \leq +\infty$ and $-\infty \leq c \leq d \leq +\infty$ are such that $[a,b] \cap [c,d] \neq \emptyset$ (see the assumption (A_6)), then

$$\nabla\varphi_\varepsilon(y) = \frac{1}{\varepsilon}\left[(y-b)^+ - (a-y)^+\right], \quad \text{and}$$
$$\nabla\psi_\varepsilon(y) = \frac{1}{\varepsilon}\left[(y-d)^+ - (c-y)^+\right].$$

The assumption (A_8-i) is clearly fulfilled; the next *compatibility assumptions* (A_8-*ii*, *iii*, *iv*, *v*) *are satisfied if for example* $G(t,u_0) = F(t,u_0,0) = 0$ and

$$G(t,y) \geq 0, \textit{ for } y \leq a, \quad G(t,y) \leq 0, \textit{ for } y \geq b,$$

and, respectively,

$$F(t,y,z) \geq 0, \textit{ for } y \leq c, \quad F(t,y,z) \leq 0, \textit{ for } y \geq d.$$

We complete the assumptions with some general boundedness conditions

(A_9) *For all* $\rho > 0$

$$\begin{aligned}
&(i)\ \mathbb{E}\, e^{2V_\tau}\Big(|\eta - u_0|^2 + \varphi(\eta) - \varphi(u_0) + \psi(\eta) - \psi(u_0)\Big) < \infty,\\
&(ii)\ \mathbb{E}\left(\int_0^\tau e^{V_s} F_\rho^\#(s)\, ds\right)^2 + \mathbb{E}\left(\int_0^\tau e^{V_s} G_\rho^\#(s)\, dA_s\right)^2 < \infty,\\
&(iii)\ \mathbb{E}\left[\int_0^\tau e^{2V_s}\left|F_\rho^\#(s)\right|^2 ds + \int_0^\tau e^{2V_s}\left|G_\rho^\#(s)\right|^2 dA_s\right] < \infty,\\
&(iv)\ \mathbb{E}\left(\int_0^T e^{V_s} dQ_s\right)^2 < \infty, \text{ for all } T \geq 0;
\end{aligned} \tag{5.154}$$

and some special boundedness conditions

(A_{10}) *There exist* $L, b > 0$ *such that for all* $0 \leq t \leq \tau$, $\mathbb{P}$-a.s.

$$\begin{aligned}
&(a)\quad \ell_t + \int_0^\tau (\ell_s)^2\, ds \leq L,\\
&(b)\ e^{V_\tau}|\eta - u_0| + |H(t, u_0, 0)| + \int_0^\tau e^{V_s}|H(s, u_0, 0)|\, dQ_s \leq b,
\end{aligned} \tag{5.155}$$

where again H is defined by

$$H(t, y, z) = \alpha_t F(t, y, z) + (1 - \alpha_t) G(t, y).$$

We also recall the definition of

$$\Psi(t, y) = \alpha_t \varphi(y) + (1 - \alpha_t)\psi(y).$$

Since $V \geq 0$, we remark that under (A_{10}) we have

$$|\eta - u_0| \leq \left|e^{V_\tau}(\eta - u_0)\right| \leq b$$

and for all $t \geq 0$,

$$|\xi_t - u_0| \leq e^{V_t}|\xi_t - u_0| = \mathbb{E}^{\mathcal{F}_t}\left(e^{V_t}|\xi_t - u_0|\right) \leq b.$$

Therefore by Proposition 6.80-A, for all $q > 0$

$$\mathbb{E}\left(\int_0^\tau e^{2V_s}|\zeta_s|^2\, ds\right)^{q/2} \leq C_{b,p}. \tag{5.156}$$

Using the definition of Q, H and Ψ we can rewrite (5.146) in the form

$$\begin{cases}
Y_t + \int_t^\infty dK_s = \eta + \int_t^\infty H(s, Y_s, Z_s)\, dQ_s - \int_t^\infty Z_s dB_s,\ t \geq 0,\\
dK_s \in \partial_y \Psi(s, Y_s)\, dQ_s \text{ on } \mathbb{R}_+.
\end{cases} \tag{5.157}$$

Definition 5.64. We call $(Y_t, Z_t)_{t\geq 0}$ a solution of (5.157) if

(d_1) $(Y, Z) \in S_m^0 \times \Lambda_{m\times k}^0$,
(d_2) $(Y_t, Z_t) = (\xi_t, \zeta_t) = (\eta, 0)$, if $t > \tau$,
(d_3) $\mathbb{P}$-a.s., for all $T \geq 0$,

$$\int_0^T [|F(s, Y_s, Z_s)| + |\varphi(Y_s)|]\, ds + \int_0^T [|G(s, Y_s)| + |\psi(Y_s)|]\, dA_s < \infty,$$

(d_4) there exists a $K \in S_m^0$ such that $\mathbb{P}$-a.s.

$$\begin{array}{ll} (i) & \updownarrow K \updownarrow_T < \infty, \quad \forall T \geq 0, \\ (ii) & dK_t \in \partial_y \Psi(t, Y_t)\, dQ_t, \end{array}$$

(d_5) $e^{2V_T} |Y_T - \xi_T|^2 + \int_T^\infty e^{2V_s} |Z_s - \zeta_s|^2 ds \xrightarrow{prob.} 0$, as $T \to \infty$,
(d_6) $\mathbb{P}$-a.s., for all $0 \leq t \leq T$,

$$Y_t + K_T - K_t = Y_T + \int_t^T H(s, Y_s, Z_s)\, dQ_s - \int_t^T Z_s dB_s \tag{5.158}$$

(we also say that the triplet (Y, Z, K) is a solution of (5.157)).

Remark 5.65. If there exists a constant C such that $\sup_{t\in[0,\tau]} |V_t(\omega)| \leq C$, $\mathbb{P}$-a.s. $\omega \in \Omega$, then the condition (d_5) from Definition 5.64 is equivalent to

$$|Y_T - \eta|^2 + \int_T^\infty |Z_s|^2 ds \xrightarrow{prob.} 0, \text{ as } T \to \infty. \tag{5.159}$$

In the rest of this book, a constant depending upon $p > 0$ is denoted by C_p; in this section since we are only considering the case $p = 2$ we will denote the corresponding constant by C_2.

We now give the main result.

Theorem 5.66. *Let the assumptions $(A_1$–$A_{10})$ be satisfied. Then the backward stochastic variational inequality (5.157) has a unique solution $(Y, Z, K) \in S_m^0 \times \Lambda_{m\times k}^0 \times S_m^0$ such that*

$$\begin{array}{ll} (j) & \mathbb{E} \sup\limits_{s\geq 0} e^{2V_s} |Y_s - u_0|^2 + \mathbb{E} \int_0^\infty e^{2V_s} |Z_s|^2 ds < \infty, \\ (jj) & \lim\limits_{T\to\infty} \mathbb{E}\Big[e^{2V_T} |Y_T - \xi_T|^2 + \int_T^\infty e^{2V_s} |Z_s - \zeta_s|^2 ds \Big] = 0. \end{array} \tag{5.160}$$

Moreover there exists $U^{(1)}, U^{(2)} \in \Lambda_m^0$, with $U^{(1)} \in \partial\varphi(Y_t)\, d\mathbb{P} \otimes dt$ a.e. and $U_t^{(2)} \in \partial\psi(Y_t)\, d\mathbb{P} \otimes dA_t$ a.e., so that with $U_t = \mathbf{1}_{[0,\tau]}(t)[\alpha_t U_t^{(1)} + (1 - \alpha_t) U_t^{(2)}]$, $dK_t = U_t dQ_t \in \partial_y \Psi(t, Y_t) dQ_t$,

$$U_t = \mathbf{1}_{[0,\tau]}(t)\left[\alpha_t U_t^{(1)} + (1-\alpha_t)\,U_t^{(2)}\right]$$

and for all $0 \le t \le T$,

$$Y_t + \int_t^T U_s dQ_s = Y_T + \int_t^T H(s, Y_s, Z_s)\,dQ_s - \int_t^T Z_s dB_s.$$

The solution also satisfies for some positive constants:

(A) *for all* $t \ge 0$ *and all* $q > 0$,

$$\begin{aligned} &(j) \quad |Y_t - u_0| \le e^{V_t}|Y_t - u_0| \le C_b, \\ &(jj) \quad \mathbb{E}\left(\int_0^\infty e^{2V_r}|Z_r|^2\,dr\right)^{q/2} \le C_{q,b}; \end{aligned} \tag{5.161}$$

(B) *for all* $t \ge 0$,

$$\begin{aligned} &\mathbb{E}^{\mathcal{F}_t}\sup_{s\ge t}\left|e^{V_s}(Y_s - u_0)\right|^2 + \mathbb{E}^{\mathcal{F}_t}\left(\int_t^\infty e^{2V_s}|Z_s|^2\,ds\right) \\ &\quad + \mathbb{E}^{\mathcal{F}_t}\int_t^\infty e^{2V_s}\mathbf{1}_{[0,\tau]}(s)\left[|\varphi(Y_s) - \varphi(u_0)|\,ds + |\psi(Y_s) - \psi(u_0)|\,dA_s\right] \\ &\le C_2\,\mathbb{E}^{\mathcal{F}_t}\left[e^{2V_\tau}|\eta - u_0|^2 + \left(\int_t^\tau e^{V_s}\left(|F(s, u_0, 0)|\,ds + |G(s, u_0)|\,dA_s\right)\right)^2\right]; \end{aligned} \tag{5.162}$$

(C) *for all* $t \ge 0$,

$$\begin{aligned} &\mathbb{E}\sup_{s\ge t} e^{2V_s}|Y_s - \xi_s|^2 + \mathbb{E}\int_t^\infty e^{2V_s}|Z_s - \zeta_s|^2 ds \\ &\qquad + \mathbb{E}\int_t^\infty e^{2V_s}|\Psi(s, Y_s) - \Psi(s, \xi_s)|\,dQ_s \\ &\le C_2\mathbb{E}\left(\int_t^\infty \mathbf{1}_{[0,\tau]}(s)\,e^{V_s}\left[|\hat{\xi}_s|dQ_s + (|F(s, \xi_s, 0)| + \ell_s|\zeta_s|)\,dsx\right.\right. \\ &\qquad \left.\left. + |G(s, \xi_s)|\,dA_s\right]\right)^2; \end{aligned} \tag{5.163}$$

(D) *for all* $t \ge 0$

$$\begin{aligned} &\mathbb{E}\left[e^{2V_t}(\varphi(Y_t) - \varphi(u_0) + \psi(Y_t) - \psi(u_0))\right] \\ &\quad + \frac{1}{2}\mathbb{E}\int_t^\infty \mathbf{1}_{[0,\tau]}(s)\,e^{2V_s}\left(\left|U_s^{(1)}\right|^2 ds + \left|U_s^{(2)}\right|^2 dA_s\right) \\ &\le \mathbb{E}\left[e^{2V_\tau}(\varphi(\eta) - \varphi(u_0) + \psi(\eta) - \psi(u_0))\right] \\ &\quad + (1+c)^2\,\mathbb{E}\int_t^\infty \mathbf{1}_{[0,\tau]}(s)\,e^{2V_s}\left(|F(s, Y_s, Z_s)|^2 + |f_s|^2\right)ds \\ &\quad + (1+c)^2\,\mathbb{E}\int_t^\infty \mathbf{1}_{[0,\tau]}(s)\,e^{2V_s}\left(|G(s, Y_s)|^2 + |g_s|^2\right)dA_s. \end{aligned} \tag{5.164}$$

Proof. Uniqueness. If (Y, Z, K), (Y', Z', K') are two solutions, in the sense of Definition 5.64, that satisfy (5.160), then

$$\mathbb{E} \sup_{t\in[0,T]} e^{2V_s}|Y_s - Y_s'|^2 < \infty.$$

Applying the monotonicity and Lipschitz property of the function H and taking into account that

$$\left\langle Y_s - Y_s', dK_s - dK_s' \right\rangle \geq 0$$

for $dK_s \in \partial_y \Psi(s, Y_s)dQ_s$ and $dK_s' \in \partial_y \Psi(s, Y_s')dQ_s$, then

$$\begin{aligned}
&\langle Y_s - Y_s', \left[H\left(s, Y_s, Z_s\right) - H(s, Y_s', Z_s')\right] dQ_s - dK_s + dK_s' \rangle \\
&\leq |Y_s - Y_s'|^2 dV_s + \frac{1}{4}|Z_s - Z_s'|^2 ds.
\end{aligned}$$

Using Corollary 6.82 from Annex C, it follows that

$$\begin{aligned}
&\mathbb{E} \sup_{s\in[0,T]} e^{2V_s} \left|Y_s - Y_s'\right|^2 + \mathbb{E} \int_0^T e^{2V_s} \left|Z_s - Z_s'\right|^2 ds \\
&\leq C_2\, \mathbb{E} \left(e^{2V_T} \left|Y_T - Y_T'\right|^2\right) \xrightarrow[T\to\infty]{} 0,
\end{aligned}$$

which yields the uniqueness.

The proof of the existence will be split into several steps.

A. *Approximating problem.* Let $n \in \mathbb{N}^*$ and $\varepsilon = \dfrac{1}{n}$.

Let

$$\begin{aligned}
\Psi^n\left(\omega, t, y\right) &= \mathbf{1}_{[0,n\wedge\tau(\omega)]}\left(t\right)\left[\alpha_t\left(\omega\right)\varphi_\varepsilon\left(y\right) + \left(1 - \alpha_t\left(\omega\right)\right)\psi_\varepsilon\left(y\right)\right] \\
\nabla_y \Psi^n\left(\omega, t, y\right) &= \mathbf{1}_{[0,n\wedge\tau(\omega)]}\left(t\right)\left[\alpha_t\left(\omega\right)\nabla_y\varphi_\varepsilon\left(y\right) + \left(1 - \alpha_t\left(\omega\right)\right)\nabla_y\psi_\varepsilon\left(y\right)\right] \\
H_n\left(\omega, t, y, z\right) &= \mathbf{1}_{[0,n]}\left(t\right) H\left(\omega, t, y, z\right) \\
&= \mathbf{1}_{[0,n\wedge\tau(\omega)]}\left(t\right)\left[\alpha_t\left(\omega\right) F\left(\omega, t, y, z\right) + \left(1 - \alpha_t\left(\omega\right)\right) G\left(\omega, t, y\right)\right]
\end{aligned}$$

and

$$\Phi_n\left(\omega, t, y, z\right) = H_n\left(\omega, t, y, z\right) - \nabla_y \Psi^n\left(\omega, t, y\right).$$

We note that

$$\begin{aligned}
&\left|\Phi_n\left(t, u_0, 0\right)\right| dQ_t \\
&= \mathbf{1}_{[0,n\wedge\tau]}\left(t\right)\left[\left|H\left(t, u_0, 0\right)\right| dQ_t + \left|\nabla_y \Psi^n\left(t, u_0\right)\right| dQ_t\right]
\end{aligned}$$

$$= \mathbf{1}_{[0,n\wedge\tau]}(t)\,|H(t,u_0,0)|\,dQ_t$$
$$\leq \mathbf{1}_{[0,n\wedge\tau]}(t)\,[|F(t,u_0,0)|\,dt + |G(t,u_0)|\,dA_t]$$

and

$$\left|\nabla_y\Psi^n(s,y) - \nabla_y\Psi^n(s,y')\right| \leq n\mathbf{1}_{[0,n\wedge\tau]}(t)\left|y-y'\right|.$$

We consider the approximating stochastic equation: for all $t \geq 0$,

$$Y_t^n + \int_t^\infty \nabla_y\Psi^n(s,Y_s^n)dQ_s = \eta + \int_t^\infty H_n(s,Y_s^n,Z_s^n)dQ_s - \int_t^\infty Z_s^n dB_s, \quad (5.165)$$

or equivalently

$$\begin{cases} Y_t^n - u_0 = \left(\mathbb{E}^{\mathcal{F}_n}\eta - u_0\right) + \displaystyle\int_t^n \Phi_n(s,u_0+\left(Y_s^n-u_0\right),Z_s^n)dQ_s \\ \qquad\qquad\qquad\qquad\qquad\qquad - \displaystyle\int_t^n Z_s^n dB_s, \quad \forall\, t\in[0,n], \\ (Y_t^n,Z_t^n) = (\xi_t,\zeta_t), \ \forall\, t>n. \end{cases} \quad (5.166)$$

To show the existence of a solution (Y^n,Z^n) of (5.166) we intend to use Lemma 5.29-(h_2).

Since $\langle y'-y, \nabla\varphi_\varepsilon(y') - \nabla\varphi_\varepsilon(y)\rangle \geq 0$ (and similarly for $\nabla\psi_\varepsilon$) we notice that Φ_n satisfies the inequalities

$$\begin{aligned} &(a) \quad \langle y'-y, \Phi_n(t,y',z) - \Phi_n(t,y,z)\rangle \\ &\qquad\quad \leq \mathbf{1}_{[0,n\wedge\tau]}(t)\,\left[\mu_t\alpha_t + \nu_t(1-\alpha_t)\right]|y'-y|^2 \leq \sigma_t\,|y'-y|^2 \\ &(b) \quad |\Phi_n(t,y,z') - \Phi_n(t,y,z)| \leq \mathbf{1}_{[0,n\wedge\tau]}(t)\ \ell_t\alpha_t\ |z'-z| \leq \alpha_t L\ |z'-z|. \end{aligned} \quad (5.167)$$

Consequently the corresponding assumptions (5.13-**BSDE-H**$_\Phi$) for Φ_n are satisfied.

We have

$$\mathbb{E}\left(e^{2\bar{\sigma}_n}\left|\mathbb{E}^{\mathcal{F}_n}\eta - u_0\right|^2\right) \leq \mathbb{E}\left(e^{2\bar{\sigma}_n}\,|\eta-u_0|^2\right) \leq b^2 < \infty.$$

For the assumption (h_2) from Lemma 5.29 we have for all $\rho > 0$,

$$\begin{aligned} &\mathbb{E}\left(\int_0^n e^{\bar{\sigma}_s}\sup_{|y|\leq\rho}|\Phi_n(s,y,0)|\,dQ_s\right)^2 \\ &\leq \mathbb{E}\left(\int_0^{n\wedge\tau} e^{\bar{\sigma}_s}\left|F_\rho^\#(s)\right|ds + \int_0^{n\wedge\tau} e^{\bar{\sigma}_s}\left|G_\rho^\#(s)\right|dA_s\right. \\ &\qquad\qquad \left. + \int_0^{n\wedge\tau} e^{\bar{\sigma}_s}\sup_{|y|\leq\rho}\left|\nabla_y\Psi^n(s,y)\right|dQ_s\right)^2 \end{aligned}$$

$$\leq 3\mathbb{E}\left(\int_0^{n\wedge\tau} e^{\bar{\sigma}_s}\left|F_\rho^\#(s)\right| ds\right)^2 + 3\mathbb{E}\left(\int_0^{n\wedge\tau} e^{\bar{\sigma}_s}\left|G_\rho^\#(s)\right| dA_s\right)^2$$
$$+ 3\mathbb{E}\left(\int_0^{n\wedge\tau} e^{\bar{\sigma}_s} n\left(\rho + |u_0|\right) dQ_s\right)^2 < \infty$$

because $\nabla_y \Psi^n(s, u_0) = 0$ and

$$\sup_{|y|\leq\rho}\left|\nabla_y \Psi^n(s, y)\right| = \sup_{|y|\leq\rho}\left|\nabla_y \Psi^n(s, y) - \nabla_y \Psi^n(s, u_0)\right| \leq n \sup_{|y|\leq\rho}|y - u_0|.$$

By Lemma 5.29-(h_2) Eq. (5.166) has a unique solution $(Y^n, Z^n) \in S_m^0 \times \Lambda_{m\times k}^0$ such that

$$\mathbb{E}\sup_{s\in[0,n]}\left|e^{V_s}\left(Y_s^n - u_0\right)\right|^2 + \mathbb{E}\int_0^n e^{2V_s}|Z_s^n|^2 ds < \infty.$$

Consequently for all $T > n$,

$$\begin{aligned}
&\mathbb{E}\sup_{s\in[0,T]}\left|e^{V_s}\left(Y_s^n - u_0\right)\right|^2 \\
&\quad\leq \mathbb{E}\sup_{s\in[0,n]}\left|e^{V_s}\left(Y_s^n - u_0\right)\right|^2 + \mathbb{E}\sup_{s\in[n,T]}\left|e^{V_s}\left(\mathbb{E}^{\mathcal{F}_s}\eta - u_0\right)\right|^2 \\
&\quad\leq \mathbb{E}\sup_{s\in[0,n]}\left|e^{V_s}\left(Y_s^n - u_0\right)\right|^2 + b^2 < \infty.
\end{aligned}$$

Now we remark that

$$\begin{aligned}
|\Psi^n(s, y) - \Psi^n(s, u_0)|\, dQ_s &= \left(\Psi^n(s, y) - \Psi^n(s, u_0)\right) dQ_s \\
&\leq \left\langle y - u_0, \nabla_y \Psi^n(s, y)\right\rangle,
\end{aligned}$$

and therefore

$$\begin{aligned}
&|\Psi^n(s, Y_s^n) - \Psi^n(s, u_0)|\, dQ_s + \left\langle Y_s^n - u_0, \Phi_n\left(s, Y_s^n, Z_s^n\right) dQ_s\right\rangle \\
&\leq |Y_s^n - u_0|\, |H(s, u_0, 0)|\, dQ_s + |Y_s^n - u_0|^2\, dV_s + \frac{1}{4}|Z_s^n|^2 ds.
\end{aligned}$$

By Proposition 6.80 we have for all $q, T > 0$

$$\begin{aligned}
&\mathbb{E}\left(\int_0^T e^{2V_s}|\Psi^n(s, Y_s^n) - \Psi^n(s, u_0)|\, dQ_s\right)^{q/2} + \mathbb{E}\left(\int_0^T e^{2V_s}|Z_s^n|^2 ds\right)^{q/2} \\
&\leq C_q\mathbb{E}\left[\sup_{s\in[0,T]}\left|e^{V_s}\left(Y_s^n - u_0\right)\right|^q + \left(\int_0^T e^{V_s}|H(s, u_0, 0)|\, dQ_s\right)^q\right],
\end{aligned} \tag{5.168}$$

and

$$\begin{aligned}
&\mathbb{E}^{\mathcal{F}_t} \sup_{s\in[t,T]} \left|e^{V_s}\left(Y_s^n-u_0\right)\right|^2+\mathbb{E}^{\mathcal{F}_t}\left(\int_t^T e^{2V_s}\left|Z_s^n\right|^2 ds\right)\\
&\qquad +\mathbb{E}^{\mathcal{F}_t}\left(\int_t^T e^{2V_s}\left|\Psi^n(s,Y_s^n)-\Psi^n(s,u_0)\right| dQ_s\right)\\
&\le C_2\,\mathbb{E}^{\mathcal{F}_t}\left[\left|e^{V_T}\left(Y_T^n-u_0\right)\right|^2+\left(\int_t^T e^{V_s}\left|H(s,u_0,0)\right| dQ_s\right)^2\right].
\end{aligned} \tag{5.169}$$

B. *Boundedness of* Y^n *and* Z^n.

If $n\le T$, then

$$\mathbb{E}^{\mathcal{F}_t}\left|e^{V_T}\left(Y_T^n-u_0\right)\right|^2=\mathbb{E}^{\mathcal{F}_t}\left|e^{V_T}\mathbb{E}^{\mathcal{F}_T}(\eta-u_0)\right|^2\le b^2.$$

Passing to the limit as $T\to\infty$ in (5.169) we infer (by the Beppo Levi monotone convergence theorem) that for all $t\ge 0$

$$\begin{aligned}
&\mathbb{E}^{\mathcal{F}_t} \sup_{s\ge t} \left|e^{V_s}\left(Y_s^n-u_0\right)\right|^2+\mathbb{E}^{\mathcal{F}_t}\left(\int_t^\infty e^{2V_s}\left|Z_s^n\right|^2 ds\right)\\
&\qquad +\mathbb{E}\left(\int_0^\infty e^{2V_s}\left|\Psi^n(s,Y_s^n)-\Psi^n(s,u_0)\right| dQ_s\right)\\
&\le C_2\,\mathbb{E}^{\mathcal{F}_t}\left[\left|e^{V_\tau}(\eta-u_0)\right|^2+\left(\int_t^\infty e^{V_s}\left|H(s,u_0,0)\right| dQ_s\right)^2\right].
\end{aligned} \tag{5.170}$$

In particular, using the assumption (5.155), we deduce that for all $t\ge 0$

$$|Y_t^n-u_0|\le e^{V_t}\,|Y_t^n-u_0|\le C_2 b^2 \overset{def}{=} R_0. \tag{5.171}$$

Moreover from (5.168) for all $q>0$

$$\mathbb{E}\left(\int_0^\infty e^{2V_r}\left|Z_r^n\right|^2 dr\right)^{q/2}\le C_{q,b}. \tag{5.172}$$

C. *Estimates on* $|Y_t^n-\xi_t|$ *and* $|Z_t^n-\zeta_t|$ *for large* $t\ge 0$.

If there exists an $N_0>0$ such that $\tau(\omega)\le N_0$, $\mathbb{P}$-a.s. $\omega\in\Omega$, then $Y_t^n=\xi_t=\eta$ and $Z_t^n=\zeta_t=0$ for all $t\ge N_0$.

We next consider the case where $\mathbb{P}(\tau>N)>0$ for all $N\in\mathbb{N}^*$.

Since

$$\xi_t=\xi_n-\int_t^n \zeta_s dB_s,\ \forall\, t\in[0,n]\,,$$

we infer, from (5.166), that (Y^n, Z^n) satisfies for all $t \in [0, n]$ the equality

$$Y_t^n - \xi_t = \int_t^n \Phi_n\left(s, \xi_s + \left(Y_s^n - \xi_s\right), \zeta_s + (Z_s^n - \zeta_s)\right) dQ_s - \int_t^n (Z_s^n - \zeta_s) dB_s.$$

We have

$$\Psi^n(t, u_0) \le \Psi^n(t, \xi_t) \le \Psi(t, \xi_t) = \Psi(t, \mathbb{E}^{\mathcal{F}_t}\eta) \le \mathbb{E}^{\mathcal{F}_t}\Psi(t, \eta).$$

From $\left\langle \nabla_y \Psi^n(t, \xi_t), y - \xi_t \right\rangle \le \Psi^n(t, y) - \Psi^n(t, \xi_t) \le \langle y - \xi_t, \nabla_y \Psi^n(t, y)\rangle$ we infer that

$$\begin{aligned} |\Psi^n(t, y) - \Psi^n(t, \xi_t)| &\le \Psi^n(t, y) - \Psi^n(t, \xi_t) + 2\left|\nabla_y \Psi^n(t, \xi_t)\right| |y - \xi_t| \\ &\le \langle y - \xi_t, \nabla_y \Psi^n(t, y)\rangle + 2\left|\nabla_y \Psi^n(t, \xi_t)\right| |y - \xi_t| \\ &\le \langle y - \xi_t, \nabla_y \Psi^n(t, y)\rangle + 2|\hat{\xi}_t|\,|y - \xi_t| \end{aligned}$$

where $\hat{\xi}_t \in \partial_y \Psi(t, \xi_t)$ is given by the assumption (A_7). Using the inequality (5.167)) it follows that, as signed measures on $\mathbb{R}_+$,

$$\begin{aligned} &\left|\Psi^n\left(t, Y_t^n\right) - \Psi^n(t, \xi_t)\right| dQ_t + \langle Y_t^n - \xi_t, \Phi_n(t, Y_t^n, Z_t^n)\rangle\, dQ_t \\ &\quad \le |Y_t^n - \xi_t| \left[2|\hat{\xi}_t| + |H(t, \xi_t, \zeta_t)|\right] dQ_t + |Y_t^n - \xi_t|^2 dV_t + \frac{1}{4}|Z_t^n - \zeta_t|^2 dt. \end{aligned} \tag{5.173}$$

Since

$$\begin{aligned} \mathbb{E} \sup_{t\in[0,T]} e^{2V_t} |Y_t^n - \xi_t|^2 &\le 2\mathbb{E}\left[\sup_{t\in[0,T]} e^{2V_t} |Y_t^n - u_0|^2 + \sup_{t\in[0,T]} e^{2V_t} |\xi_t - u_0|^2\right] \\ &\le 2R_0^2 + \mathbb{E} \sup_{t\in[0,T]} e^{2V_t} \left|\mathbb{E}^{\mathcal{F}_t}\eta - u_0\right|^2 \le 2R_0^2 + \mathbb{E} \sup_{t\in[0,T]} \mathbb{E}^{\mathcal{F}_t}\left(e^{2V_t} |\eta - u_0|^2\right) \\ &\le 2R_0^2 + b^2 \end{aligned}$$

by Proposition 5.2 we deduce that for $0 \le t \le T$,

$$\begin{aligned} &\mathbb{E} \sup_{s\in[t,T]} e^{2V_s} |Y_s^n - \xi_s|^2 + \mathbb{E}\int_t^T e^{2V_s}|Z_s^n - \zeta_s|^2 ds \\ &\qquad + \mathbb{E}\int_t^T e^{2V_s}\left|\Psi^n\left(s, Y_s^n\right) - \Psi^n(s, \xi_s)\right| dQ_s \\ &\le C_p\left[\mathbb{E}\left(e^{pV_T}\left|Y_T^n - \xi_T\right|^2\right) + \mathbb{E}\left(\int_t^T e^{V_s}\left[|\hat{\xi}_s| + |H(s, \xi_s, \zeta_s)|\right] dQ_s\right)^2\right]. \end{aligned} \tag{5.174}$$

Recall that $(Y_s^n, Z_s^n) = (\xi_s, \zeta_s),\ \forall\, s > n$. Passing to the limit $T \to \infty$ in (5.174), we obtain by the Beppo Levi monotone convergence theorem

$$\begin{aligned}
&\mathbb{E}\sup_{s\geq t} e^{pV_s}\left|Y_s^n-\xi_s\right|^2+\mathbb{E}\int_t^\infty e^{2V_s}|Z_s^n-\zeta_s|^2ds\\
&\qquad\qquad+\mathbb{E}\int_t^\infty e^{2V_s}\left|\Psi^n\left(s,Y_s^n\right)-\Psi^n\left(s,\xi_s\right)\right|dQ_s\\
&\leq C_2\mathbb{E}\left(\int_t^\infty e^{V_s}\left[|\hat{\xi}_s|+|H(s,\xi_s,\zeta_s)|\right]dQ_s\right)^2.
\end{aligned}\tag{5.175}$$

D. *Boundedness of* $\nabla\varphi_\varepsilon(Y_t^n)$ *and* $\nabla\psi_\varepsilon(Y_t^n)$.

By the stochastic subdifferential inequality from Lemma 2.38 and Remark 2.39, for all $0\leq t\leq T$

$$\begin{aligned}
&e^{2V_t}\left[\varphi_\varepsilon(Y_t^n)-\varphi_\varepsilon\left(u_0\right)\right]\leq e^{2V_T}\left[\varphi_\varepsilon(Y_T^n)-\varphi_\varepsilon\left(u_0\right)\right]\\
&\quad+\int_t^T e^{2V_s}\left\langle\nabla\varphi_\varepsilon(Y_s^n),\Phi_n(s,Y_s^n,Z_s^n)\right\rangle dQ_s-\int_t^T e^{2V_s}\left\langle\nabla\varphi_\varepsilon(Y_s^n),Z_s^ndB_s\right\rangle
\end{aligned}$$

(and a similar inequality for ψ_ε). We infer that

$$\begin{aligned}
&e^{2V_t}\left[\varphi_\varepsilon(Y_t^n)-\varphi_\varepsilon\left(u_0\right)+\psi_\varepsilon(Y_t^n)-\varphi_\varepsilon\left(u_0\right)\right]+\int_t^T\mathbf{1}_{s\leq n\wedge\tau}e^{2V_s}\Big[\alpha_s|\nabla\varphi_\varepsilon(Y_s^n)|^2\\
&\qquad+\left\langle\nabla\varphi_\varepsilon(Y_s^n),\nabla\psi_\varepsilon(Y_s^n)\right\rangle+(1-\alpha_s)\left|\nabla\psi_\varepsilon(Y_s^n)\right|^2\Big]dQ_s\\
&\quad\leq e^{2V_T}\left[\varphi_\varepsilon(Y_T^n)-\varphi_\varepsilon\left(u_0\right)+\psi_\varepsilon(Y_T^n)-\varphi_\varepsilon\left(u_0\right)\right]\\
&\qquad+\int_t^T\mathbf{1}_{[0,n]}\left(s\right)e^{2V_s}\left\langle\nabla\varphi_\varepsilon(Y_s^n)+\nabla\psi_\varepsilon(Y_s^n),H(s,Y_s^n,Z_s^n)\right\rangle dQ_s\\
&\qquad-\int_t^T e^{2V_s}\left\langle\nabla\varphi_\varepsilon(Y_s^n)+\nabla\psi_\varepsilon(Y_s^n),Z_s^ndB_s\right\rangle.
\end{aligned}\tag{5.176}$$

Using the definition of the function $H(t,y,z)$ given in (5.149), the compatibility assumptions (5.152) yield

$$\begin{aligned}
&\left\langle\nabla\varphi_\varepsilon(y),H(t,y,z)\right\rangle=\mathbf{1}_{[0,\tau]}\left(t\right)\left\langle\nabla\varphi_\varepsilon(y),\alpha_tF(t,y,z)+(1-\alpha_t)\,G(t,y)\right\rangle\\
&\leq\mathbf{1}_{[0,\tau]}\left(t\right)\Big[\alpha_t|\nabla\varphi_\varepsilon(y)||F(t,y,z)|+c\left(1-\alpha_t\right)|\nabla\psi_\varepsilon(y)|\left(|G(t,y)|+g_s\right)\Big]
\end{aligned}\tag{5.177}$$

and respectively

$$\begin{aligned}
&\left\langle\nabla\psi_\varepsilon(y),H(t,y,z)\right\rangle=\mathbf{1}_{[0,\tau]}\left(t\right)\left\langle\nabla\psi_\varepsilon(y),\alpha_tF(t,y,z)+(1-\alpha_t)\,G(t,y)\right\rangle\\
&\leq\mathbf{1}_{[0,\tau]}\left(t\right)\Big[c\;\alpha_t|\nabla\varphi_\varepsilon(y)|\left(|F(t,y,z)|+f_s\right)+(1-\alpha_t)\,|G(t,y)||\nabla\psi_\varepsilon(y)|\Big].
\end{aligned}\tag{5.178}$$

Recall that $\varphi\left(y\right)\geq\varphi_\varepsilon\left(u_0\right)=\varphi\left(u_0\right)$ and $\psi\left(y\right)\geq\psi_\varepsilon\left(u_0\right)=\psi\left(u_0\right)$. From (5.152), (5.176–5.178) and the inequality $a\left(x+y\right)\leq\frac{1}{2}a^2+x^2+y^2$ we obtain

$$
\begin{aligned}
& e^{2V_t}\left(\varphi_\varepsilon(Y_t^n)-\varphi(u_0)+\psi_\varepsilon(Y_t^n)-\psi(u_0)\right) \\
& +\frac{1}{2}\int_t^T \mathbf{1}_{[0,n\wedge\tau]}(s)\, e^{2V_s}\Big[|\nabla\varphi_\varepsilon(Y_s^n)|^2 ds+|\nabla\psi_\varepsilon(Y_s^n)|^2 dA_s\Big] \\
& \leq e^{2V_T}\left[\varphi_\varepsilon(Y_T^n)-\varphi(u_0)+\psi_\varepsilon(Y_T^n)-\psi(u_0)\right] \\
& \quad +(1+c)^2\int_t^T \mathbf{1}_{[0,n\wedge\tau]}(s)\, e^{2V_s}\left(|F(s,Y_s^n,Z_s^n)|^2+|f_s|^2\right) ds \\
& \quad +(1+c)^2\int_t^T \mathbf{1}_{[0,n\wedge\tau]}(s)\, e^{2V_s}\left(|G(s,Y_s^n)|^2+|g_s|^2\right) dA_s \\
& \quad -\int_t^T e^{2V_s}\left\langle \nabla\varphi_\varepsilon(Y_s^n)+\nabla\psi_\varepsilon(Y_s^n), Z_s^n dB_s\right\rangle .
\end{aligned}
\tag{5.179}
$$

The stochastic integral from this last inequality has the property

$$
\mathbb{E}^{\mathcal{F}_t}\int_t^T e^{2V_r}\left\langle \nabla\varphi_\varepsilon(Y_r^n)+\nabla\psi_\varepsilon(Y_r^n), Z_r^n dB_r\right\rangle = 0,
$$

because by $\nabla\varphi_\varepsilon(u_0)=\nabla\psi_\varepsilon(u_0)=0$ we have

$$
|\nabla\varphi_\varepsilon(Y_s^n)+\nabla\psi_\varepsilon(Y_r^n)| \leq 2n\,|Y_s^n-u_0|
$$

and by (5.171) and (5.172)

$$
\begin{aligned}
& \mathbb{E}\left(\int_t^T e^{4V_s}\,|\nabla\varphi_\varepsilon(Y_s^n)+\nabla\psi_\varepsilon(Y_s^n)|^2\,|Z_s^n|^2 ds\right)^{1/2} \\
& \leq 2nR_0\,\mathbb{E}\left(\int_0^T e^{2V_s}|Z_s^n|^2 ds\right)^{1/2} < \infty .
\end{aligned}
$$

Let $T \geq n$. By Jensen's inequality it follows that

$$
\begin{aligned}
\mathbb{E}\left[e^{2V_T}\left(\varphi_\varepsilon(Y_T^n)+\psi_\varepsilon(Y_T^n)\right)\right] & \leq \mathbb{E}\left[e^{2V_T}\left(\varphi(\xi_T)+\psi(\xi_T)\right)\right] \\
& \leq \mathbb{E}\left[e^{2V_T}\left(\varphi(\eta)+\psi(\eta)\right)\right].
\end{aligned}
$$

Now from inequality (5.179) we infer by Beppo Levi's monotone convergence theorem for $T\to\infty$

$$
\begin{aligned}
& \mathbb{E}\left[e^{2V_t}\left(\varphi_\varepsilon(Y_t^n)-\varphi(u_0)+\psi_\varepsilon(Y_t^n)-\psi(u_0)\right)\right] \\
& \quad +\frac{1}{2}\mathbb{E}\int_t^\infty \mathbf{1}_{[0,n\wedge\tau]}(s)\, e^{2V_s}\Big[|\nabla\varphi_\varepsilon(Y_s^n)|^2 ds+|\nabla\psi_\varepsilon(Y_s^n)|^2 dA_s\Big] \\
& \leq \mathbb{E}\left[e^{2V_\tau}\left(\varphi(\eta)-\varphi(u_0)+\psi(\eta)-\psi(u_0)\right)\right] \\
& \quad +(1+c)^2\,\mathbb{E}\int_t^\infty \mathbf{1}_{[0,\tau]}(s)\, e^{2V_s}\left(|F(s,Y_s^n,Z_s^n)|^2+|f_s|^2\right) ds \\
& \quad +(1+c)^2\,\mathbb{E}\int_t^\infty \mathbf{1}_{[0,\tau]}(s)\, e^{2V_s}\left(|G(s,Y_s^n)|^2+|g_s|^2\right) dA_s .
\end{aligned}
\tag{5.180}
$$

By (5.171), (5.168) and the assumption (5.154(iii)) we deduce that there exists a constant C independent of n such that

$$\begin{aligned}&\mathbb{E}\int_t^{\infty}\mathbf{1}_{[0,\tau]}(s)\,e^{2V_s}\left(|F(s,Y_s^n,Z_s^n)|^2+|f_s|^2\right)ds\\&\leq\mathbb{E}\int_t^{\infty}\mathbf{1}_{[0,\tau]}(s)\,e^{2V_s}\Big[2\left|F^{\#}_{R_0+|u_0|}(s)\right|^2+2L^2|Z_s^n|^2+|f_s|^2\Big]ds\leq C\end{aligned}$$

and

$$\begin{aligned}&\mathbb{E}\int_t^{\infty}\mathbf{1}_{[0,\tau]}(s)\,e^{2V_s}\left(|G(s,Y_s^n)|^2+|g_s|^2\right)dA_s\\&\leq\mathbb{E}\int_t^{\infty}\mathbf{1}_{[0,\tau]}(s)\,e^{2V_s}\left[\left|G^{\#}_{R_0+|u_0|}(s)\right|^2+|g_s|^2\right]dA_s\leq C.\end{aligned}$$

Therefore from (5.180) we have

$$\mathbb{E}\left[e^{2V_t}\left(\varphi_\varepsilon(Y_t^n)-\varphi(u_0)+\psi_\varepsilon(Y_t^n)-\psi(u_0)\right)\right]\leq C,\ \text{ for all } t\geq 0 \tag{5.181}$$

and

$$\mathbb{E}\int_0^{\infty}\mathbf{1}_{[0,n\wedge\tau]}(r)\left[e^{2V_r}|\nabla\varphi_\varepsilon(Y_r^n)|^2dr+e^{2V_r}|\nabla\psi_\varepsilon(Y_r^n)|^2dA_r\right]\leq C. \tag{5.182}$$

Since

$$\varphi_\varepsilon(y)-\varphi(u_0)=\frac{\varepsilon}{2}|\nabla\varphi_\varepsilon(y)|^2+[\varphi(y-\varepsilon\nabla\varphi_\varepsilon(y))-\varphi(u_0)]$$

we see from (5.181) that, for all $t\geq 0$,

$$\mathbb{E}\left[e^{2V_t}\left(|\varepsilon\nabla\varphi_\varepsilon(Y_t^n)|^2+|\varepsilon\nabla\psi_\varepsilon(Y_t^n)|^2\right)\right]\leq 2C\varepsilon \tag{5.183}$$

(recall that $\varepsilon=1/n$).

E. *Cauchy sequences and convergence.*

Note that by assumption (A_{10}) and (5.156) we have $|\xi_s-u_0|\leq b$ and

$$\begin{aligned}&\mathbb{E}\left(\int_n^{\infty}\mathbf{1}_{[0,\tau]}(s)\,e^{V_s}\ell_s\,|\zeta_s|\,ds\right)^2\\&\leq\mathbb{E}\left[\left(\int_n^{\infty}\mathbf{1}_{[0,\tau]}(s)\,(\ell_s)^2\,ds\right)\left(\int_0^{\infty}\mathbf{1}_{[0,\tau]}(s)\,e^{2V_s}\,|\zeta_s|^2\,ds\right)\right]\to 0,\quad\text{as } n\to\infty.\end{aligned}$$

Hence by assumption (5.154ii)

$$\mathbb{E}\left(\int_n^\infty e^{V_s}|H(s,\xi_s,\zeta_s)|dQ_s\right)^2$$
$$\leq 3\mathbb{E}\left(\int_n^\infty \mathbf{1}_{[0,\tau]}(s)\,e^{V_s}F^{\#}_{b+|u_0|}(s)\,ds\right)^2 + 3\mathbb{E}\left(\int_n^\infty \mathbf{1}_{[0,\tau]}(s)\,e^{V_s}G^{\#}_{b+|u_0|}(s)\,dA_s\right)^2$$
$$+3\mathbb{E}\left(\int_n^\infty e^{V_s}\mathbf{1}_{[0,\tau]}(s)\,\ell_s\,|\zeta_s|\,ds\right)^2 \to 0, \quad \text{as } n\to\infty,$$

and by (5.175),

$$\begin{aligned}&\mathbb{E}\sup_{s\geq n} e^{2V_s}|Y_s^{n+i}-\xi_s|^2 + \mathbb{E}\int_n^\infty e^{2V_s}|Z_s^{n+i}-\zeta_s|^2 ds\\ &\quad +\mathbb{E}\int_n^\infty e^{2V_s}\left|\Psi^{n+i}\left(s,Y_s^{n+i}\right)-\Psi^{n+i}(s,\xi_s)\right|dQ_s\\ &\leq C_p\mathbb{E}\left(\int_n^\infty e^{V_s}\left[\left|\hat{\xi}_s\right|+|H(s,\xi_s,\zeta_s)|\right]dQ_s\right)^2 \longrightarrow 0,\ n\to\infty.\end{aligned} \tag{5.184}$$

By uniqueness it follows that, for all $t\in[0,n]$,

$$Y_t^{n+i}-Y_t^n = Y_n^{n+i}-\xi_n + \int_t^n dK_s^{n,i} - \int_t^n (Z_s^{n+i}-Z_s^n)dB_s,\ \ a.s.,$$

where on $[0,n]$

$$\begin{aligned}&dK_s^{n,i}\\ &=\left[H_{n+i}(s,Y_s^{n+i},Z_s^{n+i})-H_n(s,Y_s^n,Z_s^n)-\nabla_y\Psi^{n+i}(s,Y_s^{n+i})+\nabla_y\Psi^n(s,Y_s^n)\right]dQ_s\\ &=\left[H(s,Y_s^{n+i},Z_s^{n+i})-H(s,Y_s^n,Z_s^n)-\nabla_y\Psi(s,Y_s^{n+i})+\nabla_y\Psi(s,Y_s^n)\right]dQ_s.\end{aligned}$$

By (6.28, with $a=0$, $\varepsilon=1/n$ and $\delta=1/(n+i)$)

$$\begin{aligned}&-\left\langle Y_s^{n+i}-Y_s^n,\left(\nabla_y\Psi(s,Y_s^{n+i})-\nabla_y\Psi(s,Y_s^n)\right)dQ_s\right\rangle\\ &\leq(\varepsilon+\delta)\mathbf{1}_{[0,\tau]}(s)\left(\left\langle\nabla\varphi_\varepsilon(Y_s^{n+i}),\nabla\varphi_\delta(Y_s^n)\right\rangle ds+\left\langle\nabla\psi_\varepsilon(Y_s^{n+i}),\nabla\psi_\delta(Y_s^n)\right\rangle dA_s\right),\end{aligned}$$

and using (5.150) we have on $[0,n]$

$$\begin{aligned}&\left\langle Y_s^{n+i}-Y_s^n,dK_s^{n,i}\right\rangle\\ &\leq\frac{\varepsilon+\delta}{2}\mathbf{1}_{[0,\tau]}(s)\Big[\left(|\nabla\varphi_\varepsilon(Y_s^n)|^2+|\nabla\varphi_\delta(Y_s^{n+i})|^2\right)ds\\ &\qquad+\left(|\nabla\psi_\varepsilon(Y_s^n)|^2+|\nabla\psi_\delta(Y_s^{n+i})|^2\right)dA_s\Big]\\ &\qquad+|Y_s^{n+i}-Y_s^n|^2dV_s+\frac{1}{4}|Z_s^{n+i}-Z_s^n|^2ds.\end{aligned}$$

Since by (5.171),

$$\mathbb{E}\sup_{s\in[0,n]} e^{2V_s}|Y_s^{n+i}-Y_s^n|^2 \le 2\mathbb{E}\sup_{s\in[0,n]} e^{2V_s}\left[\left|Y_s^{n+i}-u_0\right|^2+|Y_s^n-u_0|^2\right]$$
$$\le 2R_0^2<\infty,$$

we obtain by Proposition 5.2 that

$$\begin{aligned}&\mathbb{E}\sup_{s\in[0,n]} e^{2V_s}|Y_s^{n+i}-Y_s^n|^2+\mathbb{E}\int_0^n e^{2V_s}|Z_s^{n+i}-Z_s^n|^2ds\\&\le C\,\mathbb{E}e^{2V_n}|Y_n^{n+i}-\xi_n|^2\\&\quad+(\varepsilon+\delta)\,C\,\mathbb{E}\int_0^{n\wedge\tau} e^{2V_s}\left(|\nabla\varphi_\varepsilon(Y_s^n)|^2+|\nabla\varphi_\delta(Y_s^{n+i})|^2\right)ds\\&\quad+(\varepsilon+\delta)\,C\,\mathbb{E}\int_0^{n\wedge\tau} e^{2V_s}\left(|\nabla\psi_\varepsilon(Y_s^n)|^2+|\nabla\psi_\delta(Y_s^{n+i})|^2\right)dA_s.\end{aligned}\tag{5.185}$$

The estimates (5.182) and (5.184) give us

$$\mathbb{E}\sup_{s\in[0,n]} e^{2V_s}|Y_s^{n+i}-Y_s^n|^2+\mathbb{E}\int_0^n e^{2V_s}|Z_s^{n+i}-Z_s^n|^2ds$$
$$\le\mathbb{E}\sup_{s\ge n} e^{2V_s}|Y_s^{n+i}-\xi_s|^2+\frac{C}{n}\longrightarrow 0,\ \text{as } n\to\infty.$$

Hence

$$\mathbb{E}\sup_{s\ge 0} e^{2V_s}|Y_s^{n+i}-Y_s^n|^2$$
$$\le\mathbb{E}\sup_{s\in[0,n]} e^{2V_s}|Y_s^{n+i}-Y_s^n|^2+\mathbb{E}\sup_{s\ge n} e^{2V_s}|Y_s^{n+i}-\xi_s|^2\longrightarrow 0,\ \text{as } n\to\infty$$

and

$$\mathbb{E}\int_0^\infty e^{2V_s}|Z_s^{n+i}-Z_s^n|^2ds$$
$$\le\mathbb{E}\int_0^n e^{2V_s}|Z_s^{n+i}-Z_s^n|^2ds+\mathbb{E}\int_n^\infty e^{2V_s}|Z_s^{n+i}-\zeta_s|^2ds\longrightarrow 0,\ \text{as } n\to\infty.$$

F. *Passage to the limit.*

Consequently there exists $(Y,Z)\in S_m^0\times\Lambda_{m\times k}^0$ such that

$$\mathbb{E}\sup_{s\ge 0} e^{2V_s}|Y_s^n-Y_s|^2+\mathbb{E}\int_0^\infty e^{2V_s}|Z_s^n-Z_s|^2ds\longrightarrow 0,\ \text{as } n\to\infty.$$

We have that $(Y_t,Z_t)=(\eta,0)$ for $t>\tau$, since $Y_t^n=\xi_t=\eta$ and $Z_t^n=\zeta_t=0$ for $t>\tau$.

Taking into account (5.183) and

$$
\begin{aligned}
&|\Psi^n(s,y)-\Psi^n(s,u_0)| \\
&= \mathbf{1}_{[0,n\wedge\tau]}(s)\left[\alpha_s\left(\varphi_\varepsilon(y)-\varphi_\varepsilon(u_0)\right)+(1-\alpha_s)\left(\psi_\varepsilon(y)-\psi_\varepsilon(u_0)\right)\right] \\
&\geq \mathbf{1}_{[0,n\wedge\tau]}(s)\,\alpha_s\left[\varphi\left(y-\varepsilon\nabla\varphi_\varepsilon(y)\right)-\varphi(u_0)\right] \\
&\qquad +\mathbf{1}_{[0,n\wedge\tau]}(s)\,(1-\alpha_s)\left|\psi\left(y-\varepsilon\nabla\psi_\varepsilon(y)\right)-\psi(u_0)\right|,
\end{aligned}
$$

the inequality (5.162) follows from (5.170) by Fatou's Lemma.

Also by Fatou's Lemma from (5.175) we obtain (5.163) and from (5.171) and (5.172) we deduce (5.161).

From (5.182) there exist two p.m.s.p. $U^{(1)}$ and $U^{(2)}$, such that along a subsequence still indexed by n, we have for $\varepsilon=\frac{1}{n}\to 0$

$$
\begin{aligned}
&e^V\nabla\varphi_\varepsilon(Y^n)\mathbf{1}_{[0,\tau\wedge n]}\rightharpoonup e^V U^{(1)}\mathbf{1}_{[0,\tau)}, \quad \text{weakly in } L^2\left(\Omega\times\mathbb{R}_+, d\mathbb{P}\otimes dt;\mathbb{R}^m\right), \\
&e^V\nabla\psi_\varepsilon(Y^n)\mathbf{1}_{[0,\tau\wedge n]}\rightharpoonup e^V U^{(2)}\mathbf{1}_{[0,\tau)}, \quad \text{weakly in } L^2\left(\Omega\times\mathbb{R}_+, d\mathbb{P}\otimes dA_t;\mathbb{R}^m\right).
\end{aligned}
$$

Using (5.183) and applying Fatou's Lemma we have

$$
\begin{aligned}
\mathbb{E}\Big(e^{2V_t}\left[\varphi(Y_t)-\varphi(u_0)\right]\Big) &\leq \liminf_{n\to+\infty}\mathbb{E}\Big(e^{2V_t}\left[\varphi\left(Y_t^n-\varepsilon\nabla\varphi_\varepsilon(Y_t^n)\right)-\varphi(u_0)\right]\Big) \\
&\leq \liminf_{n\to+\infty}\mathbb{E}\Big(e^{2V_t}\left[\varphi_\varepsilon\left(Y_t^n\right)-\varphi(u_0)\right]\Big),
\end{aligned}
$$

and similarly for ψ. Passing to $\liminf_{n\to+\infty}$ in (5.180) we obtain (5.164).

From (5.165) we have for all $0\leq t\leq T\leq n$, $\mathbb{P}$-a.s.

$$
Y_t^n+\int_t^T\nabla_y\Psi^n(s,Y_s^n)dQ_s = Y_T^n+\int_t^T H(s,Y_s^n,Z_s^n)dQ_s-\int_t^T Z_s^n dB_s,
$$

and passing to the limit we conclude that

$$
Y_t+\int_t^T U_s dQ_s = Y_T+\int_t^T H(s,Y_s,Z_s)dQ_s-\int_t^T Z_s dB_s, \text{ a.s.} \tag{5.186}
$$

with

$$
U_s=\mathbf{1}_{[0,\tau]}(s)\left[\alpha_s U_s^1+(1-\alpha_s)\,U_s^2\right], \text{ for } s\geq 0. \tag{5.187}
$$

By (5.118$-b$), we see that, for all $E\in\mathcal{F}$, $0\leq s\leq t$ and $X\in S_m^2$,

$$
\begin{aligned}
\mathbb{E}\int_s^t\left\langle e^{2V_r}\nabla\varphi_\varepsilon(Y_r^n), X_r-Y_r^n\right\rangle\mathbf{1}_E dr+\mathbb{E}\int_s^t e^{2V_r}\varphi(Y_r^n-\varepsilon\nabla\varphi_\varepsilon(Y_s^n))\mathbf{1}_E dr \\
\leq \mathbb{E}\int_s^t e^{2V_r}\varphi(X_r)\mathbf{1}_E dr.
\end{aligned}
$$

Passing to lim inf for $n \to \infty$ in the above inequality we obtain $U_s^{(1)} \in \partial\varphi(Y_s)$, $d\mathbb{P}\otimes ds$-a.e. and, with similar arguments, $U_s^{(2)} \in \partial\psi(Y_s)$, $d\mathbb{P}\otimes dA_s$-a.e. Summarizing the above conclusions we conclude that (Y, Z, U) is a solution of the BSVI (5.157).

We want to highlight the fact that the assumption (A_{10}-b) is too strong for many applications. The next two results are concerned with the existence of a solution for the backward stochastic variational inequality (5.146) recalled here for convenience:

$$\begin{cases} Y_t + \displaystyle\int_{t\wedge\tau}^{\tau} dK_s = \eta + \int_{t\wedge\tau}^{\tau} \left[F\left(s, Y_s, Z_s\right) ds + G\left(s, Y_s\right) dA_s\right] \\ \qquad\qquad\qquad\qquad\qquad - \displaystyle\int_{t\wedge\tau}^{\tau} Z_s dB_s, \quad \text{for } t \geq 0, \\ dK_t \in \partial\varphi\left(Y_t\right) dt + \partial\psi\left(Y_t\right) dA_t, \ \text{on } \mathbb{R}_+, \end{cases} \tag{5.188}$$

without the boundedness conditions from (A_{10}).

Consider the closed convex sets

$$\begin{aligned} \mathbb{O}_\varphi &= \left\{y \in \mathbb{R}^m : \varphi\left(y\right) = \varphi\left(u_0\right)\right\}, \\ \mathbb{O}_\psi &= \left\{y \in \mathbb{R}^m : \psi\left(y\right) = \psi\left(u_0\right)\right\}, \text{ and} \\ \mathbb{O} &= \mathbb{O}_\varphi \cap \mathbb{O}_\psi. \end{aligned}$$

Since every point of $\mathbb{O}$ is a minimum point for φ and ψ, it follows that $\nabla\varphi_\varepsilon(u) = \nabla\psi_\varepsilon(u) = 0$ for all $u \in \mathbb{O}$.

Theorem 5.67. *Let the assumptions* $(A_1), \ldots, (A_9)$ *be satisfied and assume that there exists a* $\delta_0 > 0$ *such that*

$$\overline{B\left(u_0, \delta_0\right)} \subset \operatorname{int}\left(\mathbb{O}\right). \tag{5.189}$$

Assume moreover there exists $\delta \in (0, \infty]$ *and* $q = 1 + \frac{\delta}{2+\delta}$ *(with* $q = 2$ *if* $\delta = \infty$*) such that*

$$\begin{aligned} &(i) \quad \text{for } 0 < \delta < \infty\text{: } \mathbb{E}\left(\int_0^\infty \left(\ell_s\right)^2 ds\right)^{1+\delta} < \infty, \\ &(i') \quad \text{for } \delta = \infty\text{: } \left(\ell_s\right)_{s\geq 0} \text{ is a deterministic process and } \int_0^\infty \left(\ell_s\right)^2 ds < \infty, \\ &(ii) \quad \lim_{t\to\infty} \mathbb{E}\left(\int_t^\infty \mathbf{1}_{[0,\tau]}\left(s\right) e^{V_s} \left|\left|F(s, \xi_s, 0)\right| ds + \left|G(s, \xi_s,)\right| dA_s\right)^q = 0. \end{aligned} \tag{5.190}$$

Then the BSVI (5.188) has a unique solution $(Y, Z, K) \in S_m^0 \times \Lambda_{m\times k}^0 \times S_m^0$ *in the sense of Definition 5.64 such that for* $q = 1 + \frac{\delta}{2+\delta}$ *and* $q = 2$ *if* $\delta = \infty$*,*

$$(j) \quad \mathbb{E} \sup_{s \geq 0} e^{2V_s} |Y_s - u_0|^2 + \mathbb{E} \int_0^\infty e^{2V_s} |Z_s|^2 ds < \infty,$$

$$(jj) \quad \lim_{T \to \infty} \left[\mathbb{E}\, e^{qV_T} |Y_T - \xi_T|^q + \mathbb{E} \left(\int_T^\infty e^{2V_s} |Z_s - \zeta_s|^2 ds \right)^{q/2} \right] = 0. \tag{5.191}$$

Moreover the inequalities (5.162) and (5.163) hold.

Proof. (***I***) ***Uniqueness***. The proof of uniqueness is similar to that given for Theorem 5.66 except that now by Corollary 6.82 from Annex C, we have

$$\mathbb{E} \sup_{s \in [0,T]} e^{qV_s} \left|Y_s - Y_s'\right|^q + \mathbb{E} \left(\int_0^T e^{2V_s} \left|Z_s - Z_s'\right|^2 ds \right)^{q/2}$$
$$\leq C_q\, \mathbb{E} \left(e^{qV_T} \left|Y_T - Y_T'\right|^q \right) \xrightarrow[T \to \infty]{} 0.$$

(***II***) ***Existence.*** *Step 1. Approximation of the problem's data to satisfy* (A_{10}). Let

$$\sigma_s = \mathbf{1}_{[0,\tau]}(s) \left[\mu_s \alpha_s + \nu_s (1 - \alpha_s)\right], \quad dQ_s = ds + dA_s,$$
$$\beta_t = Q_{t \wedge \tau} + \int_0^{t \wedge \tau} \sigma_s dQ_s + \int_0^{t \wedge \tau} |F(s, u_0, 0)|\, ds + \int_0^{t \wedge \tau} |G(s, u_0)|\, dA_s,$$
$$\gamma_t = t + \beta_t + \ell_t + |F(t, u_0, 0)| + |G(t, u_0)| \text{ and}$$
$$\lambda_t = t + \ell_t.$$

Define, for $n \in \mathbb{N}^*$,

$$\ell_t^n = \ell_t \mathbf{1}_{[0,n]}(\lambda_t),$$
$$\eta_n = (\eta - u_0)\, \mathbf{1}_{[0,n]}(\beta_\tau + |\eta - u_0|) + u_0 \in \overline{\eta, u_0},$$
$$\hat{F}_n(t, y, z) = F\left(t, y, z\mathbf{1}_{[0,n]}(\lambda_t)\right) - F(t, u_0, 0)\ \mathbf{1}_{(n,\infty)}(\gamma_t),$$
$$\hat{G}_n(t, y) = G(t, y) - G(t, u_0)\ \mathbf{1}_{(n,\infty)}(\gamma_t),$$
$$\hat{H}_n(t, y, z) = \left[\alpha_t \hat{F}_n(t, y, z) + (1 - \alpha_t)\, \hat{G}_n(t, y)\right] \mathbf{1}_{[0,\tau]}(t),$$

and

$$V_t^n = \int_0^{t \wedge \tau} \left[\mu_s ds + \left(\ell_s^n\right)^2 ds + \nu_s dA_s\right] = V_t - \int_0^{t \wedge \tau} (\ell_s)^2\, \mathbf{1}_{(n,\infty)}(\lambda_s)\, ds.$$

Let (ξ^n, ζ^n) be given by the martingale representation theorem (Corollary 2.44): for all $t \geq 0$, $\xi_t^n = \mathbb{E}^{\mathcal{F}_t} \eta_n$ and

$$\xi_t^n = \eta_n - \int_t^\infty \zeta_s^n dB_s,$$

or equivalently, for all $T > 0$

$$\xi_t^n = \mathbb{E}^{\mathcal{F}_T} \eta_n - \int_t^T \zeta_s^n dB_s, \quad t \in [0, T].$$

It is easy to verify that

$$\mathbb{E} \sup_{t \geq 0} e^{2V_t} |\eta_n - u_0|^2 \leq \mathbb{E} \sup_{t \geq 0} e^{2V_t} |\eta - u_0|^2 \leq \mathbb{E} \left(e^{2V_\tau} |\eta - u_0|^2 \right) < \infty,$$

$$\mathbb{E} \sup_{t \geq 0} e^{2V_t} |\eta_n - \eta|^2 \leq \mathbb{E} \left[e^{2V_\tau} |\eta - u_0|^2 \mathbf{1}_{\beta_T + |\eta - u_0| > n} \right].$$

Applying Corollary 6.83, first on $[t, T]$ and then letting $T \to \infty$, we infer that for all $t \geq 0$

$$\begin{aligned}
&(a)\ \mathbb{E}^{\mathcal{F}_t} \sup_{s \geq t} e^{2V_s} |\xi_s - u_0|^2 + \mathbb{E}^{\mathcal{F}_t} \int_t^\infty e^{2V_s} |\zeta_s|^2 ds \leq C_2\, \mathbb{E}^{\mathcal{F}_t} \left(e^{2V_\tau} |\eta - u_0|^2 \right), \\
&(b)\ \mathbb{E}^{\mathcal{F}_t} \sup_{s \geq t} e^{2V_s} |\xi_s^n - u_0|^2 + \mathbb{E}^{\mathcal{F}_t} \int_t^\infty e^{2V_s} |\zeta_s^n|^2 ds \leq C_2\, \mathbb{E}^{\mathcal{F}_t} \left(e^{2V_\tau} |\eta - u_0|^2 \right), \\
&(c)\ \mathbb{E}^{\mathcal{F}_t} \sup_{s \geq t} e^{2V_s} |\xi_s^n - \xi_s|^2 + \mathbb{E}^{\mathcal{F}_t} \int_t^\infty e^{2V_s} |\zeta_s^n - \zeta_s|^2 ds \\
&\qquad\qquad \leq C_2\, \mathbb{E}^{\mathcal{F}_t} \left[e^{2V_\tau} |\eta - u_0|^2 \mathbf{1}_{\beta_T + |\eta - u_0| > n} \right].
\end{aligned} \tag{5.192}$$

Since the assumptions $(A_1, \ldots, A_9)$ are satisfied by $(\eta, F, G, \varphi, \psi, V, \mu, \nu, \ell)$ it follows that the same assumptions are satisfied replacing $(\eta, F, G, \varphi, \psi, V, \mu, \nu, \ell)$ by $(\eta_n, \hat{F}_n, \hat{G}_n, \varphi, \psi, V^n, \mu, \nu, \ell^n)$.

With respect to (A_{10}) we have

$$\ell_t^n + \int_0^\infty \left(\ell_s^n\right)^2 ds \leq n + \int_0^n n^2 ds = n + n^3$$

and

$$\begin{aligned}
&e^{V_\tau^n} |\eta_n - u_0| + \left| \hat{H}_n(t, u_0, 0) \right| + \int_0^\tau e^{V_s^n} \left| \hat{H}_n(s, u_0, 0) \right| dQ_s \\
&\leq e^{V_\tau} |\eta_n - u_0| + |H(t, u_0, 0)| \mathbf{1}_{[0,n]}(\gamma_t) + \int_0^\tau e^{V_s} |H(s, u_0, 0)| \mathbf{1}_{[0,n]}(\gamma_s) dQ_s \\
&\leq n + e^n n + e^n n^2 = b_n.
\end{aligned}$$

Hence $\left(\eta_n, \hat{F}_n, \hat{G}_n, \mu, \nu, \ell^n, V^n\right)$ satisfies (A_{10}).

Step 2. Approximating equation and estimates. By *Step 1* we are in the conditions of Theorem 5.66 and therefore the approximating equation

$$\begin{cases} Y_t^n + \int_t^\infty U_s^n dQ_s = \eta_n + \int_t^\infty \hat{H}_n\left(s, Y_s^n, Z_s^n\right) dQ_s - \int_t^\infty Z_s^n dB_s, \\ dK_s^n = U_s^n dQ_s \in \partial_y \Psi\left(s, Y_s^n\right) dQ_s \end{cases}$$

has a unique solution $(Y^n, Z^n, K^n) \in S_m^0 \times \Lambda_{m\times k}^0 \times S_m^0$, $\left(Y_s^n, Z_s^n\right) = \left(\xi_s^n, 0\right)$ for $s > \tau$ and $U_s^n = \alpha_s U_s^{1,n} + (1-\alpha_s)\, U_s^{2,n}$ such that

$$\begin{aligned} &(j) \quad \mathbb{E}\sup_{s\geq 0} e^{2V_s^n}\left|Y_s^n - u_0\right|^2 + \mathbb{E}\int_0^\infty e^{2V_s^n}\left|Z_s^n\right|^2 ds < \infty, \\ &(jj) \quad \lim_{T\to\infty} \mathbb{E}\Big[e^{2V_T^n}\left|Y_T^n - \xi_T^n\right|^2 + \int_T^\infty e^{2V_s^n}\left|Z_s^n - \zeta_s^n\right|^2 ds\Big] = 0. \end{aligned} \tag{5.193}$$

Moreover the inequalities (5.161), (5.162), (5.163) and (5.164) hold with (η, F, G, φ, ψ, V, μ, ν, ℓ, C_b, $C_{q,b}$) by (η_n, $\hat{F}_n$, $\hat{G}_n$, φ, ψ, V^n, μ, ν, ℓ^n, C_n, $C_{q,n}$).

Using in (5.193) $V_t^{n+i} - (n+i)^3 \leq V_t^n$ we get for all $i \in \mathbb{N}$

$$\begin{aligned} &(j') \quad \mathbb{E}\sup_{s\geq 0} e^{2V_s^{n+i}}\left|Y_s^n - u_0\right|^2 + \mathbb{E}\int_0^\infty e^{2V_s^{n+i}}\left|Z_s^n\right|^2 ds < \infty, \\ &(jj') \quad \lim_{T\to\infty} \mathbb{E}\Big[e^{2V_T^{n+i}}\left|Y_T^n - \xi_T^n\right|^2 + \int_T^\infty e^{2V_s^{n+i}}\left|Z_s^n - \zeta_s^n\right|^2 ds\Big] = 0. \end{aligned} \tag{5.194}$$

Since $\langle Y_s^n - u_0, U_s^n - 0\rangle \geq 0$ and

$$\begin{aligned} &|\Psi(s, Y_s^n) - \Psi(s, u_0)|\, dQ_s + \left\langle Y_s^n - u_0, \hat{H}_n\left(s, Y_s^n, Z_s^n\right) - U_s^n\right\rangle dQ_s \\ &\leq |Y_s^n - u_0|\left|\hat{H}_n\left(s, u_0, 0\right)\right| dQ_s + |Y_s^n - u_0|^2\, dV_s^n + \frac{1}{4}|Z_s^n|^2 ds \\ &\leq |Y_s^n - u_0|\, \mathbf{1}_{[0,\tau]}(s)\left(|F\left(s, u_0, 0\right)| ds + |G\left(s, u_0\right)| dA_s\right) + |Y_s^n - u_0|^2\, dV_s^{n+i} \\ &\quad + \frac{1}{4}|Z_s^n|^2 ds, \end{aligned}$$

we infer by Corollary 6.82 for $p = 2$ and $0 \leq t \leq T$,

$$\begin{aligned} &\mathbb{E}^{\mathcal{F}_t}\sup_{s\in[t,T]}\left|e^{V_s^{n+i}}\left(Y_s^n - u_0\right)\right|^2 + \mathbb{E}^{\mathcal{F}_t}\left(\int_t^T e^{2V_s^{n+i}}\left|Z_s^n\right|^2 ds\right) \\ &\qquad + \mathbb{E}^{\mathcal{F}_t}\left(\int_t^T e^{2V_s^{n+i}}\left|\Psi(s, Y_s^n) - \Psi(s, u_0)\right| dQ_s\right) \\ &\leq C_2\, \mathbb{E}^{\mathcal{F}_t}\Big[\left|e^{V_T^{n+i}}\left(Y_T^n - u_0\right)\right|^2 \\ &\qquad + \left(\int_t^T \mathbf{1}_{[0,\tau]}(s)\, e^{V_s^{n+i}}\left(|F\left(s, u_0, 0\right)| ds + |G\left(s, u_0\right)| dA_s\right)\right)^2\Big]. \end{aligned} \tag{5.195}$$

But by (5.192-b) and $V_T^{n+i} \leq V_T$ we have

$$\left[\mathbb{E}^{\mathcal{F}_t}\left(\left|e^{V_T^{n+i}}\left(Y_T^n - u_0\right)\right|^2\right)\right]^{1/2}$$
$$\leq \left[\mathbb{E}^{\mathcal{F}_t}\left(e^{2V_T^{n+i}}\,|Y_T^n - \xi_T^n|^2\right)\right]^{1/2} + \left[\mathbb{E}^{\mathcal{F}_t}\left(e^{2V_T}\,|\xi_T^n - u_0|^2\right)\right]^{1/2}$$
$$\leq \left[\mathbb{E}^{\mathcal{F}_t}\left(e^{2V_T^{n+i}}\,|Y_T^n - \xi_T^n|^2\right)\right]^{1/2} + \left[\mathbb{E}^{\mathcal{F}_t}\left(e^{2V_\tau}\,|\eta - u_0|^2\right)\right]^{1/2}.$$

Using Beppo Levi's monotone convergence theorem and (5.194-jj'') we can pass to the limit in (5.195), first $\limsup_{T\to\infty}$ and then $\lim_{i\to\infty}$. We obtain

$$\begin{aligned}
&\mathbb{E}^{\mathcal{F}_t}\sup_{s\geq t}\left|e^{V_s}\left(Y_s^n - u_0\right)\right|^2 + \mathbb{E}^{\mathcal{F}_t}\left(\int_t^\infty e^{2V_s}\,|Z_s^n|^2\,ds\right)\\
&\qquad + \mathbb{E}^{\mathcal{F}_t}\left(\int_t^\infty e^{2V_s}\,|\Psi(s,Y_s^n) - \Psi(s,u_0)|\,dQ_s\right)\\
&\leq C_2\,\mathbb{E}^{\mathcal{F}_t}\bigg[e^{2V_\tau}\,|\eta - u_0|^2\\
&\qquad + \left(\int_t^\infty \mathbf{1}_{[0,\tau]}(s)\,e^{V_s}\left(|F(s,u_0,0)|\,ds + |G(s,u_0)|\,dA_s\right)\right)^2\bigg],
\end{aligned} \tag{5.196}$$

and, in particular,

$$\mathbb{E}\sup_{s\geq 0} e^{2V_s}\,|Y_s^n - u_0|^2 < \infty. \tag{5.197}$$

Step 3. Cauchy sequence and convergences. Let

$$\mathcal{K}_t^n = \int_0^t \left[\hat{H}_n\left(s, Y_s^n, Z_s^n\right) - U_s^n\right] dQ_s.$$

We have for any $j > i > 1$

$$\begin{aligned}
&\left\langle Y_s^n - Y_s^{n+i}, d\left(\mathcal{K}_s^n - \mathcal{K}_s^{n+i}\right)\right\rangle\\
&\leq \left\langle Y_s^n - Y_s^{n+i}, H\left(s, Y_s^n, Z_s^n \mathbf{1}_{[0,n]}(\lambda_s)\right) - H\left(s, Y_s^{n+i}, Z_s^{n+i}\mathbf{1}_{[0,n+i]}(\lambda_s)\right)\right\rangle dQ_s\\
&\quad - \left\langle Y_s^n - Y_s^{n+i}, H(s,u_0,0)\right\rangle\ \left[\mathbf{1}_{[n,\infty[}(\gamma_s) - \mathbf{1}_{[n+i,\infty[}(\gamma_s)\right] dQ_s\\
&\leq \left|Y_s^n - Y_s^{n+i}\right|\Big[\,|H(s,u_0,0)|\ \mathbf{1}_{[n,\infty[}(\gamma_s)\,dQ_s\\
&\quad + \mathbf{1}_{[0,\tau]}(s)\,\ell_s\left|\mathbf{1}_{[0,n]}(\lambda_s) - \mathbf{1}_{[0,n+i]}(\lambda_s)\right|\,|Z_s^n|\,ds\Big]\\
&\quad + \left|Y_s^n - Y_s^{n+i}\right|^2\left(\sigma_s dQ_s + \mathbf{1}_{[0,\tau]}(s)\,\mathbf{1}_{[0,n+j]}(\lambda_s)\,(\ell_s)^2\,ds\right) + \frac{1}{4}\left|Z_s^n - Z_s^{n+i}\right|^2 ds,
\end{aligned}$$

then for all $T > 0$, by Proposition 5.2 with $q = 1 + \frac{\delta}{2+\delta} \in (1,2)$ and $q = 2$ if $\delta = \infty$,

$$\begin{aligned}
&\mathbb{E}\left(\sup_{s\in[0,T]} e^{qV_s^{n+j}}\left|Y_s^n - Y_s^{n+i}\right|^q\right) + \mathbb{E}\left(\int_0^T e^{2V_s^{n+j}}\left|Z_s^n - Z_s^{n+i}\right|^2 ds\right)^{q/2} \\
&\leq C_q\,\mathbb{E}\,e^{qV_T^{n+j}}\left|Y_T^n - Y_T^{n+i}\right|^q \\
&+C_q\,\mathbb{E}\left(\int_0^T \mathbf{1}_{(n,\infty)}(\gamma_s)\,e^{V_s^{n+j}}\Big[\left|H(t,u_0,0)\right|\,dQ_s + \mathbf{1}_{[0,\tau]}(s)\,\ell_s\left|Z_s^n\right|ds\Big]\right)^q .
\end{aligned}$$

But

$$\begin{aligned}
&\mathbb{E}\left(\int_0^T \mathbf{1}_{(n,\infty)}(\gamma_s)\,e^{V_s^{n+j}}\mathbf{1}_{[0,\tau]}(s)\,\ell_s\left|Z_s^n\right|ds\Big]\right)^q \\
&\leq \mathbb{E}\left[\left(\int_0^T \mathbf{1}_{[0,\tau]}(s)\,(\ell_s)^2\,\mathbf{1}_{(n,\infty)}(\lambda_s)\,ds\right)^{q/2}\left(\int_0^T e^{2V_s^{n+j}}\left|Z_s^n\right|^2 ds\right)^{q/2}\right] \\
&\leq \Lambda_{n,\delta}\times\left[\mathbb{E}\left(\int_0^T e^{2V_s}\left|Z_s^n\right|^2 ds\right)\right]^{\frac{q}{2}},
\end{aligned}$$

with

$$\Lambda_{n,\delta} = \begin{cases}\left[\mathbb{E}\left(\int_0^T \mathbf{1}_{[0,\tau]}(s)\,(\ell_s)^2\,\mathbf{1}_{(n,\infty)}(\lambda_s)\,ds\right)^{1+\delta}\right]^{\frac{1}{2+\delta}}, & \text{if } 0<\delta<\infty, \\ \int_0^\infty (\ell_s)^2\,\mathbf{1}_{(n,\infty)}(\lambda_s)\,ds, & \text{if } q = 2\ (\delta = \infty,\ \ell \text{ is deterministic});\end{cases}$$

the last inequality is obtained by Hölder's inequality since $\frac{1}{2+\delta} + \frac{1}{2/q} = 1$. Thus

$$\begin{aligned}
&\mathbb{E}\left(\sup_{s\in[0,T]} e^{qV_s^{n+j}}\left|Y_s^n - Y_s^{n+i}\right|^q\right) + \mathbb{E}\left(\int_0^T e^{2V_s^{n+j}}\left|Z_s^n - Z_s^{n+i}\right|^2 ds\right)^{q/2} \\
&\leq C_q\,\mathbb{E}\,e^{qV_T^{n+j}}\left|Y_T^n - Y_T^{n+i}\right|^q + C_q\,\mathbb{E}\left(\int_0^T e^{V_s}\Big[\left|H(t,u_0,0)\right|\,\mathbf{1}_{(n,\infty)}(\gamma_s)\,dQ_s\right)^q \\
&\qquad +C_q\,\Lambda_{n,\delta}\times\left[\mathbb{E}\left(\int_0^T e^{2V_s}\left|Z_s^n\right|^2 ds\right)\right]^{\frac{q}{2}}.
\end{aligned} \tag{5.198}$$

Here we have

$$\begin{aligned}
&\left(\mathbb{E}\, e^{2V_T^{n+j}}\left|Y_T^n - Y_T^{n+i}\right|^2\right)^{1/2} \\
&\quad \le \left(\mathbb{E}\, e^{2V_T^{n+j}}\left|Y_T^n - \xi_T^n\right|^2\right)^{1/2} + \left(\mathbb{E}\, e^{2V_T^{n+j}}\left|\xi_T^n - \xi_T^{n+i}\right|^2\right)^{1/2} \\
&\qquad + \left(\mathbb{E}\, e^{2V_T^{n+j}}\left|\xi_T^{n+i} - Y_T^{n+i}\right|^2\right)^{1/2} \\
&\quad \le \left(\mathbb{E}\, e^{2V_T^{n+j}}\left|Y_T^n - \xi_T^n\right|^2\right)^{1/2} + \left(\mathbb{E}\, e^{2V_T^{n+j}}\left|\xi_T^{n+i} - Y_T^{n+i}\right|^2\right)^{1/2} \\
&\qquad + \left[\mathbb{E}\left(e^{2V_\tau}\left|\eta - u_0\right|^2 \mathbf{1}_{\beta_\tau + |\eta - u_0| > n}\right)\right]^{1/2},
\end{aligned}$$

and as $T \to \infty$ we infer

$$\limsup_{T\to\infty} \mathbb{E}\, e^{2V_T^{n+j}}\left|Y_T^n - Y_T^{n+i}\right|^2 \le \mathbb{E}\left(e^{2V_\tau}\left|\eta - u_0\right|^2 \mathbf{1}_{\beta_\tau + |\eta - u_0| > n}\right).$$

Using (5.196) for the boundedness of $\mathbb{E}\int_0^\infty e^{2V_s}\left|Z_s^n\right|^2 ds$ we get from (5.198) as $T \to \infty$ and then passing to the limit as $j \to \infty$:

$$\begin{aligned}
&\mathbb{E}\left(\sup_{s\ge 0} e^{qV_s}\left|Y_s^n - Y_s^{n+i}\right|^q\right) + \mathbb{E}\left(\int_0^\infty e^{2V_s}\left|Z_s^n - Z_s^{n+i}\right|^2 ds\right)^{q/2} \\
&\quad \le C\left[\mathbb{E}\left(e^{2V_\tau}\left|\eta - u_0\right|^2 \mathbf{1}_{|\eta - u_0| > n}\right)\right]^{q/2} \\
&\qquad + C\,\mathbb{E}\left(\int_0^\tau e^{V_s}\Big[\left|H\left(t, u_0, 0\right)\right|\, \mathbf{1}_{(n,\infty)}\left(\gamma_s\right) dQ_s\right)^q + C\,\Lambda_{n,\delta},
\end{aligned}$$

which yields by (5.190) the existence of a pair $(Y, Z) \in S_m^0 \times \Lambda_{m\times k}^0$ such that

$$\lim_{n\to\infty}\left[\mathbb{E}\,\sup_{s\ge 0} e^{qV_s}\left|Y_s^n - Y_s\right|^q + \mathbb{E}\left(\int_0^\infty e^{2V_s}\left|Z_s^n - Z_s\right|^2 ds\right)^{q/2}\right] = 0. \tag{5.199}$$

Now by Fatou's lemma from (5.196) we obtain (5.162) and consequently (5.191-j).

To verify (5.191-jj), following the proof of Theorem 5.66, we have

$$\begin{aligned}
&\left|\Psi\left(s, Y_s^n\right) - \Psi\left(s, \xi_s\right)\right| dQ_s + \left\langle Y_s^n - \xi_s, \hat{H}_n\left(s, Y_s^n, Z_s^n\right) - U_s^n\right\rangle dQ_s \\
&\le \left|Y_s^n - \xi_s\right|\Big[|\hat{\xi}_s| + |\hat{H}_n(s, \xi_s, \zeta_s)|\Big] dQ_s \\
&\quad + \left|Y_s^n - \xi_s\right|^2 dV_s^{n+i} + \frac{1}{4}\left|Z_s^n - \zeta_s\right|^2 ds.
\end{aligned}$$

By (5.192-a) and (5.197) we have

$$\mathbb{E}\sup_{s\geq 0} e^{2V_s^{n+i}}\left|Y_s^n-\xi_s\right|^2 \leq \mathbb{E}\sup_{s\geq 0} e^{2V_s}\left|Y_s^n-\xi_s\right|^2<\infty.$$

Furthermore, using (5.194-jj'') and (5.192-c), we have

$$\limsup_{T\to\infty}\mathbb{E}\, e^{2V_T^{n+i}}\left|Y_T^n-\xi_T\right|^2 \leq \mathbb{E}\left(e^{2V_\tau}\left|\eta-u_0\right|^2\mathbf{1}_{\beta_\tau+|\eta-u_0|>n}\right).$$

In the same manner as above when we proved (5.196) we obtain, by Corollary 6.82 for $p\in(1,2]$, similar inequalities with (ξ_s,ζ_s) in place of $(u_0,0)$ and passing successively to the limit $T\to\infty$ and $i\to\infty$ we get that for all $t\geq 0$

$$\begin{aligned}&\mathbb{E}\sup_{s\geq t} e^{pV_s}\left|Y_s^n-\xi_s\right|^p+\mathbb{E}\left(\int_t^\infty e^{2V_s}|Z_s^n-\zeta_s|^2ds\right)^{p/2}\\ &\qquad\qquad+\mathbb{E}\left(\int_t^\infty e^{2V_s}\left|\Psi\left(s,Y_s^n\right)-\Psi\left(s,\xi_s\right)\right|dQ_s\right)^{p/2}\\ &\leq C_p\bigg[\mathbb{E}\left(e^{pV_\tau}\left|\eta-u_0\right|^p\mathbf{1}_{\beta_\tau+|\eta-u_0|>n}\right)\\ &\quad+\mathbb{E}\left(\int_t^\infty\mathbf{1}_{[0,\tau]}(s)\,e^{V_s}\big[|\hat{\xi}_s|+|\hat{H}_n(s,\xi_s,\zeta_s)|\big]dQ_s\right)^p\bigg].\end{aligned}\tag{5.200}$$

Since $|\hat{H}_n(s,\xi_s,\zeta_s)|\leq|H(s,\xi_s,0)|+\ell_s\,|\zeta_s|+|H(s,u_0,0)|\mathbf{1}_{(n,\infty)}(\gamma_s)$, from (5.200) with Fatou's Lemma applied to the left-hand side and the Lebesgue dominated convergence theorem applied to the right-hand side we infer by taking the limit as $n\to\infty$

$$\begin{aligned}&\mathbb{E}\sup_{s\geq t} e^{pV_s}\left|Y_s-\xi_s\right|^p+\mathbb{E}\left(\int_t^\infty e^{2V_s}|Z_s-\zeta_s|^2ds\right)^{p/2}\\ &\qquad\qquad+\mathbb{E}\left(\int_t^\infty e^{2V_s}\left|\Psi\left(s,Y_s\right)-\Psi\left(s,\xi_s\right)\right|dQ_s\right)^{p/2}\\ &\leq C_p\,\mathbb{E}\left(\int_t^\infty\mathbf{1}_{[0,\tau]}(s)\,e^{V_s}\left[|\hat{\xi}_s|dQ_s+|H(s,\xi_s,0)|\,dQ_s+\ell_s\,|\zeta_s|\,ds\right]\right)^p,\end{aligned}\tag{5.201}$$

which yields (5.163) if we choose $p=2$.

In the case $p=q=1+\frac{\delta}{2+\delta}\in(1,2)$, by Hölder's inequality, we have

$$\begin{aligned}&\mathbb{E}\left(\int_t^\infty\mathbf{1}_{[0,\tau]}(s)\,e^{V_s}\ell_s|\zeta_s|ds\right)^q\\ &\quad\leq\mathbb{E}\left[\left(\int_t^\infty\mathbf{1}_{[0,\tau]}(s)\,(\ell_s)^2\,ds\right)^{q/2}\left(\int_t^\infty e^{2V_s}|\zeta_s|^2ds\right)^{q/2}\right]\\ &\quad\leq\left[\mathbb{E}\left(\int_t^\infty\mathbf{1}_{[0,\tau]}(s)\,(\ell_s)^2\,ds\right)^{1+\delta}\right]^{\frac{1}{2+\delta}}\left[\mathbb{E}\int_t^\infty e^{2V_s}|\zeta_s|^2ds\right]^{\frac{q}{2}}.\end{aligned}$$

In the case $p = q = 2$, $\delta = \infty$ and ℓ is a deterministic process

$$\mathbb{E}\left(\int_t^\infty \mathbf{1}_{[0,\tau]}(s)\, e^{V_s}\ell_s|\zeta_s|ds\right)^2 \le \left(\int_t^\infty (\ell_s)^2\, ds\right)\mathbb{E}\left(\int_t^\infty e^{2V_s}|\zeta_s|^2 ds\right).$$

Using the assumptions (5.190) and (5.154-ii), from (5.201) we infer (5.191-*jj*).
Step 4. Estimates on the subdifferential term $dK_s^n = U_s^n dQ_s \in \partial_y\Psi\left(s, Y_s^n\right) dQ_s$. We now use the assumption (5.189) on the interior of Dom (φ). From the proof of Corollary 5.49 we have

$$\begin{aligned}
&\delta_0\, d \updownarrow K^n \updownarrow_t + \left\langle Y_t - u_0, \hat{H}_n\left(t, Y_t^n, Z_t^n\right) dQ_t - dK_t^n\right\rangle \\
&\quad\le \left[\Psi^{\#}_{u_0,\delta_0}(t) - \Psi(t,u_0)\right] dQ_t + |Y_t^n - u_0|\left[|\hat{u}_t| + \left|\hat{H}_n(t,u_0,0)\right|\right] dQ_t \\
&\quad + |Y_t^n - u_0|^2\, dV_t + \frac{1}{4}|Z_t^n|^2\, dt,
\end{aligned}$$

where

$$\begin{aligned}
\Psi^{\#}_{u_0,\delta_0}(t) &= \sup\left\{\mathbf{1}_{[0,\tau]}(t)\left[\alpha_t\varphi(u_0+\delta_0 v) + (1-\alpha_t)\,\psi(u_0+\delta_0 v)\right] : |v| \le 1\right\} \\
&= \sup\left\{\mathbf{1}_{[0,\tau]}(t)\,|\alpha_t\varphi(u_0) + (1-\alpha_t)\,\psi(u_0)|\right\} \\
&= \Psi(t,u_0),
\end{aligned}$$

and $\hat{u}_t = 0 \in \partial_y\Psi(\omega,t,u_0)$. Hence

$$\begin{aligned}
&\delta_0 d \updownarrow K^n \updownarrow_t + \left\langle Y_t^n - u_0, \hat{H}_n\left(t, Y_t^n, Z_t^n\right) dQ_t - dK_t^n\right\rangle \\
&\quad\le |Y_t^n - u_0|\,|H(t,u_0,0)|\, dQ_t + |Y_t^n - u_0|^2\, dV_t + \frac{1}{4}|Z_t^n|^2\, dt.
\end{aligned}$$

By Proposition 6.80-B we obtain

$$\begin{aligned}
&\delta_0\mathbb{E}\int_0^T e^{2V_s} d \updownarrow K^n \updownarrow_s \\
&\quad\le C_2\left[\mathbb{E}\, e^{2V_T}\,|Y_T^n - u_0|^2 + \mathbb{E}\left(\int_0^T e^{V_s}\,|H(s,u_0,0)|\, dQ_s\right)^2\right] \le C.
\end{aligned}$$

From the convergence (5.199) and the equality

$$Y_0^n + K_t^n = Y_t^n + \int_0^t \hat{H}_n\left(s, Y_s^n, Z_s^n\right) dQ_s - \int_0^t Z_s^n dB_s,\ \forall\, t \ge 0,$$

it follows, via Lemma 5.16, that there exists a $K \in S_m^0$ such that

$$\|K^n - K\|_T \xrightarrow{prob.} 0, \quad \text{as } n \to \infty.$$

As in Proposition 1.20 and Corollary 1.22 we obtain

$$\mathbb{E}\int_0^\tau e^{2V_s} d \updownarrow K \updownarrow_s \leq \liminf_{n\to+\infty} \mathbb{E}\int_0^\tau e^{2V_s} d \updownarrow K^n \updownarrow_s \leq C$$

and

$$dK_t \in \partial_y \Psi\left(t, Y_t\right) dQ_t \text{ on } \mathbb{R}_+.$$

Finally passing to the limit in

$$Y_t^n + K_T^n - K_t^n = Y_T^n + \int_t^T \hat{H}_n\left(s, Y_s^n, Z_s^n\right) dQ_s - \int_t^T Z_s^n dB_s,$$

we complete the proof. ■

Remark 5.68. In this last theorem, in contrast to the results in Theorem 5.66, we have not been able to show that the process K is absolutely continuous.

To end this section we discuss a particular case of BSVI (5.146) that we recall here for the convenience of the reader:

$$\begin{cases} Y_t + \displaystyle\int_{t\wedge\tau}^{\tau} dK_s = \eta + \int_{t\wedge\tau}^{\tau}\left[F\left(s, Y_s, Z_s\right) ds + G\left(s, Y_s\right) dA_s\right] \\ \qquad\qquad\qquad\qquad\qquad\qquad -\displaystyle\int_{t\wedge\tau}^{\tau} Z_s dB_s, \quad t \geq 0, \\ dK_t \in \partial\varphi\left(Y_t\right) dt + \partial\psi\left(Y_t\right) dA_t, \text{ on } \mathbb{R}_+, \end{cases} \tag{5.202}$$

where the assumptions $(\mathrm{A}_1), \ldots, (\mathrm{A}_{10})$ from the beginning of this section will to be replaced by

(L_1): $(\mathrm{A}_1) + (\mathrm{A}_2) + (\mathrm{A}_3)$ *are satisfied;*

(L_2): *the functions* $F : \Omega\times\mathbb{R}_+\times\mathbb{R}^m\times\mathbb{R}^{m\times k} \to \mathbb{R}^m$ *and* $G : \Omega\times\mathbb{R}_+\times\mathbb{R}^m \to \mathbb{R}^m$ *satisfy*

$$F\left(\cdot,\cdot,y,z\right),\ G\left(\cdot,\cdot,y\right) \text{ is p.m.s.p., for each } (y,z) \in \mathbb{R}^m \times \mathbb{R}^{m\times k},$$

and there exists an $L > 0$ *such that,* $\mathbb{P}$-a.s. $\omega \in \Omega$, *a.e.* $t \geq 0$, for all y, y', z, z'

$$\begin{array}{rl} (i) & \langle y'-y, F(t,y',z) - F(t,y,z)\rangle \leq \frac{L}{2}\left|y'-y\right|^2, \\ (ii) & \left|F(t,y,z') - F(t,y,z)\right| \leq \sqrt{\frac{L}{2}}\left|z'-z\right|, \\ (iii) & \left|F\left(\omega,t,y,0\right)\right| \leq \frac{L}{2}\left(1+|y|\right), \\ (iv) & \langle y'-y, G(t,y') - G(t,y)\rangle \leq L\left|y'-y\right|^2, \\ (v) & \left|G\left(\omega,t,y\right)\right| \leq L\left(1+|y|\right). \end{array} \tag{5.203}$$

We remark that in this case $\mu_t = \frac{L}{2}\mathbf{1}_{[0,\tau]}(t)$, $\ell_t = \sqrt{\frac{L}{2}}\mathbf{1}_{[0,\tau]}(t)$, $\nu_t = L\mathbf{1}_{[0,\tau]}(t)$ and

$$V_t = \int_0^{t\wedge\tau} \left[\mu_s ds + \nu_s dA_s + (\ell_s)^2\, ds\right] = LQ_{t\wedge\tau},$$

$$F_\rho^{\#}(s) = \sup_{|y|\le\rho} |F(t,y,0)| \le \frac{L}{2}(1+\rho),$$

$$G_\rho^{\#}(s) = \sup_{|y|\le\rho} |G(t,y)| \le L(1+\rho).$$

(L_3): $(A_6)+(A_7)+(A_8)$ *are satisfied;*
(L_5): *assume that*

$$\mathbb{E}\, e^{2LQ_\tau}\left(1+|\eta|^2+|\varphi(\eta)|+|\psi(\eta)|\right) < \infty. \tag{5.204}$$

We highlight that under $(L_1),\ldots,(L_5)$, the assumptions $(A_1),\ldots,(A_9)$ are satisfied. Also from (L_5) we have

$$\frac{(2L)^j}{j!}\,\mathbb{E}\left[(\tau + A_\tau)^j\right] \le \mathbb{E}\, e^{2LQ_\tau} < \infty, \quad \text{for all } j\in\mathbb{N}^*,$$

and consequently $\tau<\infty$, $\mathbb{P}$-a.s. Moreover it is not difficult to verify that by (L_2), (L_5) and (5.192-a) the condition (5.190) is satisfied for all $\delta\in(0,\infty)$ and for all $q = 1+\frac{\delta}{2+\delta}\in(1,2)$. Hence with the exception of assumption (5.189) on the interior of $\mathrm{Dom}(\varphi)$ all other assumptions of Theorem 5.67 are satisfied.

Theorem 5.69. *Under the assumptions $(L_1),\ldots,(L_5)$ the BSVI (5.202) has a unique solution $(Y,Z,K)\in S_m^0\times\Lambda_{m\times k}^0\times S_m^0$ in the sense of Definition 5.64 which satisfies for all $q\in(1,2)$:*

$$(j)\quad \mathbb{E}\sup_{s\ge0} e^{2LQ_s}|Y_s-u_0|^2 + \mathbb{E}\int_0^\infty e^{2LQ_s}|Z_s|^2\,ds < \infty,$$

$$(jj)\quad \lim_{T\to\infty}\left[\mathbb{E}\, e^{qLQ_T}|Y_T-\xi_T|^q + \mathbb{E}\left(\int_T^\infty e^{2LQ_s}|Z_s-\zeta_s|^2\,ds\right)^{q/2}\right] = 0.$$

Moreover there exist $U^{(1)},U^{(2)}\in\Lambda_m^0$, $U_t^{(1)}\in\partial\varphi(Y_t)$ and $U_t^{(2)}\in\partial\psi(Y_t)$, $d\mathbb{P}\otimes dt$-a.e. such that $dK_t = U_t dQ_t\in\partial_y\Psi(t,Y_t)\,dQ_t$, where

$$U_t = \mathbf{1}_{[0,\tau]}(t)\left[\alpha_t U_t^{(1)} + (1-\alpha_t)\,U_t^{(2)}\right]$$

and

$$Y_t + \int_t^T U_s dQ_s = Y_T + \int_t^T H(s, Y_s, Z_s) dQ_s - \int_t^T Z_s dB_s, \ a.s.$$

The inequalities (5.162), (5.163) and (5.164) hold with $V_t = LQ_t$.

Proof. The proof is similar to that of Theorem 5.67: the *Steps 1–3* are exactly the same. To pass to the limit in the approximating equation

$$\begin{cases} Y_t^n + \int_t^\infty U_s^n dQ_s = \eta_n + \int_t^\infty \hat{H}_n\left(s, Y_s^n, Z_s^n\right) dQ_s - \int_t^\infty Z_s^n dB_s, \\ dK_s^n = U_s^n dQ_s = U_s^{1,n} ds + U_s^{2,n} dA_s, \\ \qquad \text{with } U_s^{1,n} ds \in \partial\varphi\left(Y_s^n\right) ds, \ U_s^{2,n} dA_s \in \partial\psi\left(Y_s^n\right) dA_s, \end{cases} \tag{5.205}$$

we need a new argument for *Step 4* since now the interior condition (5.189) is not satisfied.
Step 4′. Estimates on subdifferential terms $U^{1,n}$ and $U^{2,n}$. By Theorem 5.66 we have

$$\begin{aligned} &\mathbb{E}\left[e^{2V_t^n}\left(\varphi(Y_t^n) - \varphi(u_0) + \psi(Y_t^n) - \psi(u_0)\right)\right] \\ &\quad + \frac{1}{2}\mathbb{E}\int_{t\wedge\tau}^{\tau} e^{2V_s^n}\left[\left|U_s^{1,n}\right|^2 ds + \left|U_s^{2,n}\right|^2 dA_s\right] \\ &\le \mathbb{E}\left[e^{2V_\tau^n}\left(\varphi(\eta_n) - \varphi(u_0) + \psi(\eta_n) - \psi(u_0)\right)\right] \\ &\quad + 4\mathbb{E}\int_t^\infty \mathbf{1}_{[0,\tau]}(s)\, e^{2V_s^n}\left[|\hat{F}_n(s, Y_s^n, Z_s^n)|^2 ds + |\hat{G}_n(s, Y_s^n)|^2 dA_s\right]. \end{aligned}$$

Note that $V_t^n = V_t - \theta_t^n$, where $\theta_t^n = \int_0^{t\wedge\tau} \frac{L}{2}\mathbf{1}_{(n,\infty)}(\lambda_s)\, ds$. Since

$$\begin{aligned} \varphi(\eta_n) - \varphi(u_0) &= (\varphi(\eta) - \varphi(u_0))\,\mathbf{1}_{[0,n]}(\beta_\tau + |\eta - u_0|), \\ \psi(\eta_n) - \psi(u_0) &= (\psi(\eta) - \psi(u_0))\,\mathbf{1}_{[0,n]}(\beta_\tau + |\eta - u_0|), \\ |\hat{F}_n(s, Y_s^n, Z_s^n)| &\le \frac{L}{2}\left(1 + |Y_s^n|\right) + \sqrt{\frac{L}{2}}\,|Z_s^n|\,\mathbf{1}_{[0,n]}(\lambda_s) + |F(s, u_0, 0)|\,\mathbf{1}_{[0,n]}(\gamma_s), \\ |\hat{G}_n(s, Y_s^n)| &\le L\left(1 + |Y_s^n|\right) + |G(s, u_0)|\,\mathbf{1}_{[0,n]}(\gamma_s), \end{aligned}$$

we obtain

$$\mathbb{E}\left[e^{2V_t^n}\left(\varphi(Y_t^n) - \varphi(u_0) + \psi(Y_t^n) - \psi(u_0)\right)\right] \le C,$$

and

$$\mathbb{E}\int_0^\infty \mathbf{1}_{[0,\tau]}(s)\, e^{2V_s^n}\left[\left|U_s^{1,n}\right|^2 ds + \left|U_s^{2,n}\right|^2 dA_s\right] \le C.$$

Consequently there exist two p.m.s.p. $U^{(1)}$ and $U^{(2)}$, such that along a subsequence still indexed by n,

$$\begin{aligned} e^V U^{1,n} e^{\theta^n} \mathbf{1}_{[0,\tau]} &\rightharpoonup e^V U^{(1)} \mathbf{1}_{[0,\tau]}, \quad \text{weakly in } L^2\left(\Omega \times \mathbb{R}_+, d\mathbb{P} \otimes dt; \mathbb{R}^m\right), \\ e^V U^{2,n} e^{\theta^n} \mathbf{1}_{[0,\tau]} &\rightharpoonup e^V U^{(2)} \mathbf{1}_{[0,\tau]}, \quad \text{weakly in } L^2\left(\Omega \times \mathbb{R}_+, d\mathbb{P} \otimes dA_t; \mathbb{R}^m\right). \end{aligned}$$

Passing to the limit in (5.205) the result follows in a standard manner (see the proof of Theorem 5.66). ■

Remark 5.70. If $\tau = T < \infty$ is a deterministic final time, then the assertions of Theorem 5.69 are also true with $q = 2$ (and $\delta = \infty$) by setting $\ell_s = \sqrt{\frac{L}{2}} \mathbf{1}_{[0,T]}(s)$.

5.6.3 Weak Variational Solutions

In this subsection we discuss again the existence and the uniqueness of a solution (Y, Z) of BSVI (5.146) that we recall here:

$$\begin{cases} Y_t + \displaystyle\int_{t\wedge\tau}^{\tau} dK_s = \eta + \int_{t\wedge\tau}^{\tau} \left[F(s, Y_s, Z_s)\, ds + G(s, Y_s)\, dA_s\right] - \int_{t\wedge\tau}^{\tau} Z_s dB_s,\ t \geq 0, \\ dK_t \in \partial\varphi(Y_t)\, dt + \partial\psi(Y_t)\, dA_t,\ \text{on } \mathbb{R}_+, \end{cases} \tag{5.206}$$

under the assumptions $(A_1), \ldots, (A_9)$ presented in Sect. 5.6.2. Adding the assumption (A_{10}) we have Theorem 5.66. Replacing the assumption (A_{10}) by (5.190) and the interior condition (5.189) we have Theorem 5.67. Furthermore if the stochastic processes $(\mu_t)_{t\geq 0}$, $(\nu_t)_{t\varepsilon 0}$, $(\ell_t)_{t\geq 0}$ are constants and some boundedness assumptions (5.203-*iii*, *v*) and (5.204) are satisfied then we can renounce assumption (A_{10}), and obtain the existence and uniqueness of a solution (Y, Z) for (5.206): see Theorem 5.69.

The aim of this subsection is to obtain existence and uniqueness under the assumptions $(A_1), \ldots, (A_9)$ and (5.190), i.e. to see what happens in Theorem 5.67 without the interior condition (5.189). It is not clear how we can obtain some estimates on the subdifferential term $dK_s^n = U_s^n dQ_s \in \partial_y \Psi\left(s, Y_s^n\right) dQ_s$, except for the particular case treated in Theorem 5.69. For this reason we shall give a weak variational formulation for the solution as in [47]. The stochastic variational formulation for forward SDEs was introduced by Răşcanu in [62].

Let us define the space $\mathcal{L}_m^p$, $p \geq 0$, of continuous semimartingales M of the form

$$M_t = \gamma - \int_0^t \Lambda_r dQ_r + \int_0^t \Theta_r dB_r,$$

where $\gamma \in \mathbb{R}^m$, Λ and Θ are two p.m.s.p. such that on every interval $[0, T] \subset \mathbb{R}_+$, $\Lambda \in L^p\left(\Omega; L^1\left(0, T; \mathbb{R}^m\right)\right)$, $\Theta \in L^p\left(\Omega; L^2\left(0, T; \mathbb{R}^{m\times k}\right)\right)$.

For an intuitive introduction, let $M \in \mathcal{L}_m^0$ and (Y, Z, K) be a solution of (5.157), in the sense of Definition 5.64. By Itô's formula for $\frac{1}{2}|M_t - Y_t|^2$ and the subdifferential inequality

$$\int_t^T \langle M_r - Y_r, dK_r\rangle + \int_t^T \Psi(r, Y_r)\, dQ_r \le \int_t^T \Psi(r, M_r)\, dQ_r$$

we obtain the inequality

$$\begin{aligned}
&\frac{1}{2}|M_t - Y_t|^2 + \frac{1}{2}\int_t^T |\Theta_r - Z_r|^2\, dr + \int_t^T \Psi(r, Y_r)\, dQ_r \\
&\le \frac{1}{2}|M_T - Y_T|^2 + \int_t^T \Psi(r, M_r)\, dQ_r + \int_t^T \langle M_r - Y_r, \Lambda_r - H(r, Y_r, Z_r)\rangle dQ_r \\
&\qquad - \int_t^T \langle M_r - Y_r, (\Theta_r - Z_r) dB_r\rangle.
\end{aligned}$$

Therefore, we propose the following weak formulation for the solution.

Definition 5.71. We call $(Y_t, Z_t)_{t\ge 0}$ a weak variational solution of (5.206) if $(Y, Z) \in S_m^0 \times \Lambda_{m\times k}^0$, $(Y_t, Z_t) = (\xi_t, \zeta_t) = (\eta, 0)$ for $t > \tau$ and

$$\begin{aligned}
(i)\quad & \int_0^T (|H(r, Y_r, Z_r)| + \Psi(r, Y_r))\, dQ_r < \infty, \ \mathbb{P}\text{-a.s., for all } T \ge 0, \\
(ii)\quad & \frac{1}{2}|M_t - Y_t|^2 + \frac{1}{2}\int_t^s |\Theta_r - Z_r|^2\, dr + \int_t^s \Psi(r, Y_r)\, dQ_r \\
& \le \frac{1}{2}|M_s - Y_s|^2 + \int_t^s \Psi(r, M_r)\, dQ_r \\
& + \int_t^s \langle M_r - Y_r, \Lambda_r - H(r, Y_r, Z_r)\rangle dQ_r - \int_t^s \langle M_r - Y_r, (\Theta_r - Z_r) dB_r\rangle, \\
& \qquad \forall\, 0 \le t \le s \le \tau,\ \forall\, M_\cdot = \gamma - \textstyle\int_0^\cdot \Lambda_r dQ_r + \int_0^\cdot \Theta_r dB_r \in \mathcal{L}_m^0, \\
(iii)\quad & e^{2V_T}|Y_T - \xi_T|^2 + \int_T^\infty e^{2V_s}|Z_s - \zeta_s|^2\, ds \xrightarrow{prob.} 0, \text{ as } T \to \infty.
\end{aligned} \tag{5.207}$$

Theorem 5.72. *Let the assumptions* $(A_1, \ldots, A_9)$ *and* (5.190-(i') *and* $(ii$*, with* $q = 2)$*) be satisfied. Then the BSVI (5.206) has a unique weak variational solution* $(Y, Z) \in S_m^0 \times \Lambda_{m\times k}^0$ *in the sense of Definition 5.71 such that*

$$\begin{aligned}
(j)\quad & \mathbb{E}\sup_{s\ge 0} e^{2V_s}|Y_s - u_0|^2 + \mathbb{E}\int_0^\infty e^{2V_s}|Z_s|^2\, ds < \infty, \\
(jj)\quad & \lim_{T\to\infty}\left[\mathbb{E}\, e^{2V_T}|Y_T - \xi_T|^2 + \mathbb{E}\int_T^\infty e^{2V_s}|Z_s - \zeta_s|^2\, ds\right] = 0.
\end{aligned} \tag{5.208}$$

Moreover the inequalities (5.162) and (5.163) hold.

Proof. Existence We remark that we are in the conditions of Theorem 5.67 without the interior condition (5.189). Therefore we start with the same approximating equation as in the proof of Theorem 5.67

$$\begin{cases} Y_t^n + \int_t^\infty U_s^n dQ_s = \eta_n + \int_t^\infty \hat{H}_n\left(s, Y_s^n, Z_s^n\right) dQ_s - \int_t^\infty Z_s^n dB_s, \\ dK_s^n = U_s^n dQ_s \in \partial_y \Psi\left(s, Y_s^n\right) dQ_s \equiv \mathbf{1}_{[0,\tau]}(s)\, \partial\varphi\left(Y_s^n\right) dQ_s \end{cases} \tag{5.209}$$

and we follow exactly the same *Steps 1–3* as there.

We obtain the existence of $(Y, Z) \in S_m^0 \times \Lambda_{m\times k}^0$ such that

$$\mathbb{E}\left[\sup_{s\geq 0} e^{2V_s}|Y_s^n - Y_s|^2 + \int_0^\infty e^{2V_s}|Z_s^n - Z_s|^2 ds\right] \longrightarrow 0, \text{ as } n\to\infty,$$

$(Y_t, Z_t) = (\eta, 0)$ for $t > \tau$ and (Y, Z) satisfies (5.208), the inequalities (5.162) and (5.163), and (5.207-*i*, *iii*).

Let $M_\cdot = \gamma - \int_0^\cdot \Lambda_r dQ_r + \int_0^\cdot \Theta_r dB_r \in \mathcal{L}_m^0$. By Itô's formula for $\frac{1}{2}|M_t - Y_t^n|^2$ we deduce that, for all $0 \le t \le s$,

$$\begin{aligned} \frac{1}{2}|M_t - Y_t^n|^2 + \frac{1}{2}\int_t^s |\Theta_r - Z_r^n|^2 dr + \int_t^s \Psi\left(r, Y_r^n\right) dQ_r \le \frac{1}{2}\mathbb{E}\,|M_s - Y_s^n|^2 \\ + \int_t^s \Psi\left(r, M_r^n\right) dQ_r + \int_t^s \langle M_r - Y_r^n, \Lambda_r - \hat{H}_n\left(r, Y_r^n, Z_r^n\right)\rangle dQ_r \\ - \int_t^s \langle M_r - Y_r^n, (\Theta_r - Z_r^n) dB_r\rangle. \end{aligned}$$

Passing to the lim inf it follows that the pair (Y, Z) satisfies the inequality (5.207-*ii*).

Uniqueness. In order to prove the uniqueness of the solution, let $\left(\hat{Y}, \hat{Z}\right) \in S_m^0 \times \Lambda_{m\times k}^0$ and $\left(\tilde{Y}, \tilde{Z}\right) \in S_m^0 \times \Lambda_{m\times k}^0$ be two weak variational solutions of (5.206) corresponding to $\hat{\eta}$ and $\tilde{\eta}$, respectively. Therefore for all $M_\cdot = \gamma - \int_0^\cdot \Lambda_r dQ_r + \int_0^\cdot \Theta_r dB_r \in \mathcal{L}_m^0$,

$$\begin{aligned} &\frac{1}{2}\left(|M_t - \hat{Y}_t|^2 + |M_t - \tilde{Y}_t|^2\right) + \frac{1}{2}\int_t^s \left(|\Theta_r - \hat{Z}_r|^2 + |\Theta_r - \tilde{Z}_r|^2\right) dr \\ &\quad + \int_t^s \left(\Psi(r, \hat{Y}_r) + \Psi(r, \tilde{Y}_r)\right) dQ_r \\ &\le \frac{1}{2}\left(|M_s - \hat{Y}_s|^2 + |M_s - \tilde{Y}_s|^2\right) + 2\int_t^s \Psi(r, M_r)\, dQ_r \\ &\quad + \int_t^s \left(\langle M_r - \hat{Y}_r, \Lambda_r - H(r, \hat{Y}_r, Z_r)\rangle + \langle M_r - \tilde{Y}_r, \Lambda_r - H(r, \tilde{Y}_r, \tilde{Z}_r)\rangle\right) dQ_r \\ &\quad - \int_t^s \left(\langle M_r - \hat{Y}_r, (\Theta_r - \hat{Z}_r) dB_r\rangle + \langle M_r - \tilde{Y}_r, (\Theta_r - \tilde{Z}_r) dB_r\rangle\right), \ \forall\, 0 \le t \le s. \end{aligned}$$

Let $Y=\frac{\hat{Y}+\tilde{Y}}{2}$, $Z=\frac{\hat{Z}+\tilde{Z}}{2}$ and $h_r=\dfrac{1}{2}\Big[H(r,\hat{Y}_r,\hat{Z}_r)+H(r,\tilde{Y}_r,\tilde{Z}_r)\Big]$.

From the convexity of φ we see that

$$2\varphi(Y_r)\le\varphi(\hat{Y}_r)+\varphi(\tilde{Y}_r),$$

and using the identity

$$2\left\langle\frac{u+v}{2},\frac{f+g}{2}\right\rangle+\frac{1}{2}\langle u-v,f-g\rangle=\langle u,f\rangle+\langle v,g\rangle,$$

we obtain

$$\begin{aligned}&\langle M_r-\hat{Y}_r,\Lambda_r-H(r,\hat{Y}_r,\hat{Z}_r)\rangle+\langle M_r-\tilde{Y}_r,\Lambda_r-H(r,\tilde{Y}_r,\tilde{Z}_r)\rangle\\&=2\langle M_r-Y_r,\Lambda_r-h_r\rangle+\frac{1}{2}\langle\hat{Y}_r-\tilde{Y}_r,H(r,\hat{Y}_r,\hat{Z}_r)-H(r,\tilde{Y}_r,\tilde{Z}_r)\rangle,\end{aligned}$$

and

$$\begin{aligned}&\int_t^s\langle M_r-\hat{Y}_r,(\Theta_r-\hat{Z}_r)dB_r\rangle+\int_t^s\langle M_r-\tilde{Y}_r,(\Theta_r-\tilde{Z}_r)dB_r\rangle\\&=2\int_t^s\langle M_r-Y_r,(\Theta_r-Z_r)\rangle\,dB_r+\frac{1}{2}\int_t^s\langle\hat{Y}_r-\tilde{Y}_r,(\hat{Z}_r-\tilde{Z}_r)dB_r\rangle.\end{aligned}$$

Therefore, since

$$\frac{1}{2}\left(|m-u|^2+|m-v|^2\right)=\left|m-\frac{u+v}{2}\right|^2+\frac{1}{4}|u-v|^2,$$

we have for all $M_\cdot=\gamma-\int_0^\cdot\Lambda_r dQ_r+\int_0^\cdot\Theta_r dB_r\in\mathcal{L}_m^0$

$$\begin{aligned}\left|\hat{Y}_t-\tilde{Y}_t\right|^2+\int_t^s\left|\hat{Z}_r-\tilde{Z}_r\right|^2dr&\le 8B_{t,s}(M)+\left|\hat{Y}_s-\tilde{Y}_s\right|^2\\&\quad+2\int_t^s\langle\hat{Y}_r-\tilde{Y}_r,H(r,\hat{Y}_r,\hat{Z}_r)-H(r,\tilde{Y}_r,\tilde{Z}_r)\rangle dQ_r\\&\quad-2\int_t^s\langle\hat{Y}_r-\tilde{Y}_r,(\hat{Z}_r-\tilde{Z}_r)dB_r\rangle,\quad\forall\,0\le t\le s,\end{aligned}\tag{5.210}$$

where

$$\begin{aligned}B_{t,s}(M)&=\frac{1}{2}|M_s-Y_s|^2+\int_t^s\Psi(r,M_r)\,dQ_r\\&+\int_t^s\langle M_r-Y_r,\Lambda_r-h_r\rangle dQ_r-\frac{1}{2}|M_t-Y_t|^2\\&-\frac{1}{2}\int_t^s|\Theta_r-Z_r|^2\,dr-\int_t^s\Psi(r,Y_r)\,dQ_r-\int_t^s\langle M_r-Y_r,(\Theta_r-Z_r)\,dB_r\rangle.\end{aligned}$$

Let

$$M_t^\varepsilon = e^{-\frac{Q_t}{Q_\varepsilon}} \left[Y_0 + \frac{1}{Q_\varepsilon} \int_0^t e^{\frac{Q_r}{Q_\varepsilon}} Y_r dQ_r \right]. \tag{5.211}$$

Clearly, $M^\varepsilon \in \mathcal{L}_m^0$ since $M_t^\varepsilon = M_0^\varepsilon + \int_0^t dM_r^\varepsilon = Y_0 + \int_0^t \frac{Y_r - M_r^\varepsilon}{Q_\varepsilon} dQ_r + \int_0^t 0 dB_r$. By Lemma 6.21 it follows that for all $0 \le t \le s \le T$

$$\begin{aligned}
&(a) \quad \lim_{\varepsilon \to 0_+} \left[\sup_{r \in [0,T]} |M_r^\varepsilon - Y_r| \right] = 0, \\
&(b) \quad \lim_{\varepsilon \to 0_+} \int_t^s \mathbf{1}_{[0,\tau]}(r)\, \varphi\big(M_r^\varepsilon\big) dr = \int_t^s \mathbf{1}_{[0,\tau]}(r)\, \varphi(Y_r) dr, \\
&(c) \quad \lim_{\varepsilon \to 0_+} \int_t^s \mathbf{1}_{[0,\tau]}(r)\, \psi\big(M_r^\varepsilon\big) dA_r = \int_t^s \mathbf{1}_{[0,\tau]}(r)\, \psi(Y_r) dA_r
\end{aligned}$$

and consequently

$$\limsup_{\varepsilon \to 0_+} B_{t,s}(M^\varepsilon) \le 0,$$

because $\Psi\left(r, M_r^\varepsilon\right) dQ_r = \mathbf{1}_{[0,\tau]}(r) \left[\varphi\big(M_r^\varepsilon\big) dr + \psi\big(M_r^\varepsilon\big) dA_r \right]$.

Using the inequality

$$\langle \hat{Y}_r - \tilde{Y}_r, H(r, \hat{Y}_r, \hat{Z}_r) - H(r, \tilde{Y}_r, \tilde{Z}_r) \rangle dQ_r \le |\hat{Y}_r - \tilde{Y}_r|^2 dV_r + \frac{1}{4} |\hat{Z}_r - \tilde{Z}_r|^2 dr$$

from (5.210) with $M = M^\varepsilon$, $\varepsilon \to 0_+$, we obtain that for all $0 \le t \le s$,

$$\begin{aligned}
\left|\hat{Y}_t - \tilde{Y}_t\right|^2 + \frac{1}{2} \int_t^s \left|\hat{Z}_r - \tilde{Z}_r\right|^2 dr \le \left|\hat{Y}_s - \tilde{Y}_s\right|^2 + 2 \int_t^s \left|\hat{Y}_r - \tilde{Y}_r\right|^2 dV_r \\
-2 \int_t^s \langle \hat{Y}_r - \tilde{Y}_r, (\hat{Z}_r - \tilde{Z}_r) dB_r \rangle,
\end{aligned}$$

which yields, by Proposition 6.69

$$\begin{aligned}
e^{2V_t} \left|\hat{Y}_t - \tilde{Y}_t\right|^2 + \frac{1}{2} \int_t^s e^{2V_r} \left|\hat{Z}_r - \tilde{Z}_r\right|^2 dr \le e^{2V_s} \left|\hat{Y}_s - \tilde{Y}_s\right|^2 \\
- 2 \int_t^s e^{2V_r} \langle \hat{Y}_r - \tilde{Y}_r, (\hat{Z}_r - \tilde{Z}_r) dB_r \rangle.
\end{aligned}$$

Taking the expectation and then passing to the limit as $s \to \infty$ uniqueness follows (see the properties of the solutions given in (5.208)).

5.7 Semilinear Elliptic PDEs

5.7.1 Elliptic Equations in the Whole Space

We will first consider elliptic PDEs in $\mathbb{R}^d$, and then in a bounded open subset of $\mathbb{R}^d$, with Dirichlet boundary condition.

Let $\{X_t^x;\ t \geq 0\}$ denote the solution of the forward SDE:

$$X_t^x = x + \int_0^t f(X_s^x)\,ds + \int_0^t g(X_s^x)\,dB_s,\ t \geq 0, \tag{5.212}$$

where $f : \mathbb{R}^d \to \mathbb{R}^d$ is continuous and globally monotone, $g : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ is globally Lipschitz, and consider the backward SDE

$$Y_t^x = Y_T^x + \int_t^T F(X_s^x, Y_s^x, Z_s^x)\,ds - \int_t^T Z_s^x\,dB_s,\ \text{for all } t, T \text{ s.t. } 0 \leq t \leq T, \tag{5.213}$$

where $F : \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k\times d} \to \mathbb{R}^k$ is continuous and such that for some K, K', $\mu < 0$, $p > 0$,

$$|F(x,y,z)| \leq K'(1 + |x|^p + |y| + |z|),$$
$$\langle y - y', F(x,y,z) - F(x,y',z)\rangle \leq \mu|y - y'|^2,$$
$$|F(x,y,z) - F(x,y,z')| \leq K\|z - z'\|.$$

We assume moreover that for some $\lambda > 2\mu + K^2$, and all $x \in \mathbb{R}^d$,

$$\mathbb{E}\int_0^\infty e^{\lambda t}|F(X_t^x,0,0)|^2 dt < \infty, \tag{5.214}$$

which essentially implies that $\lambda < 0$.

Under these assumptions, the BSDE (5.213) has a unique solution, in the sense of Theorem 5.27.

It is not hard to see, using uniqueness for BSDEs, that

$$Y_t^x = Y_0^{X_t^x}, \quad t > 0. \tag{5.215}$$

Denote by

$$\mathcal{A} = \frac{1}{2}\sum_{i,j}(gg^*)_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_i f_i(x)\frac{\partial}{\partial x_i}$$

the infinitesimal generator of the Markov process $\{X_t^x;\ t \geq 0\}$, and consider the following system of semilinear elliptic PDEs in $\mathbb{R}^d$

$$\mathcal{A}u_i(x) + F_i(x, u(x), (\nabla ug)(x)) = 0, x \in \mathbb{R}^d,\ 0 \leq i \leq k. \tag{5.216}$$

As in Sect. 5.4, one easily establishes the following:

Theorem 5.73. *Let $u \in C^2(\mathbb{R}^d; \mathbb{R}^m)$ be a classical solution of (5.216) such that for some $M, q > 0$,*

$$|u(x)| \leq M(1 + |x|^q),\ \forall x \in \mathbb{R}^d.$$

Then for each $x \in \mathbb{R}^d$, $\{(u(X_t^x), (\nabla ug)(X_t^x));\ t \geq 0\}$ is the solution of the BSDE (5.213). In particular $u(x) = Y_0^x$.

We now want to prove that (5.212)–(5.213) provide a viscosity solution to (5.216)

Again, for the notion of a viscosity solution of the system of PDEs we need (5.216) to make sense, therefore we need to make the following restriction: *for $0 \leq i \leq k$, the i-th coordinate of F depends only on the i-th row of the matrix z.*

Define the mapping

$$\Phi : \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^d \times \mathbb{S}^d \to \mathbb{R}^m$$

by

$$\Phi_i(x, r, p, X) = -\frac{1}{2}\mathbf{Tr}[g(x)g^*(x)X] - \langle f(x), p\rangle - F_i(x, r, pg(x)),\ 1 \leq i \leq m.$$

Then the system (5.216) reads

$$\Phi_i(x, u(x), Du_i(x), D^2u_i(x)) = 0,\ x \in \mathbb{R}^d,\ 0 \leq i \leq m.$$

All the assumptions from Theorem 5.37 are assumed to hold below (with of course f, g and F independent of the time variable t). The notion of a viscosity solution of (5.216) is defined by Definition 6.94 in Annex D.

We can now prove the following:

Theorem 5.74. *Under the above assumptions, $u(x) \stackrel{def}{=} Y_0^x$ is a continuous function which satisfies*

$$|Y_0^x| \leq c\sqrt{\mathbb{E}\int_0^\infty e^{\lambda t}|F(X_t^x, 0, 0)|^2 dt}, \tag{5.217}$$

for any $\lambda > 2\mu + K^2$, and it is a viscosity solution of (5.216).

Proof. The continuity follows from the mean-square continuity of $\{Y_\cdot^x,\ x \in \mathbb{R}^d\}$. The inequality (5.217) follows from (5.134) with $\eta = 0$ (hence $\xi = 0$ and $\zeta = 0$).

To prove that u is a viscosity sub-solution, we take any $1 \le i \le m$, $\varphi \in C^2(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$ such that $u_i - \varphi$ has a local maximum at x. We assume without loss of generality that

$$u_i(x) = \varphi(x).$$

We suppose that

$$\Phi_i(x, u(x), D\varphi_i(x), D^2\varphi_i(x)) > 0,$$

and we will find a contradiction.

Let $\alpha > 0$ be such that whenever $|y - x| \le \alpha$,

$$u_i(y) \le \varphi(y),$$
$$\Phi_i(y, u(y), D\varphi_i(y), D^2\varphi_i(y)) > 0,$$

and define, for some $T > 0$,

$$\tau = \inf\{t > 0;\ |X_t^x - x| \ge \alpha\} \wedge T.$$

Let now

$$(\overline{Y}_t, \overline{Z}_t) = ((Y_{t\wedge\tau}^x)^i, \mathbf{1}_{[0,\tau]}(t)(Z_t^x)^i),\ 0 \le t \le T.$$

$(\overline{Y}, \overline{Z})$ solves the one-dimensional BSDE

$$\overline{Y}_t = u_i(X_\tau^x) + \int_t^T \mathbf{1}_{[0,\tau]}(s) F_i(X_s^x, u(X_s^x), \overline{Z}_s)ds - \int_t^T \overline{Z}_s dB_s,\ 0 \le t \le T.$$

On the other hand, from Itô's formula,

$$(\hat{Y}_t, \hat{Z}_t) = (\varphi(X_{t\wedge\tau}^x), \mathbf{1}_{[0,\tau]}(t)(\nabla\varphi g)(X_t^x)),\ 0 \le t \le T$$

solves the BSDE

$$\hat{Y}_t = \varphi(X_\tau^x) - \int_t^T \mathbf{1}_{[0,\tau]}(s)\mathcal{A}\varphi(X_s^x)ds - \int_t^T \hat{Z}_s dB_s,\ 0 \le t \le T.$$

From $u_i \le \varphi$, and the choice of α and τ, we deduce with the help of Proposition 5.34 that $\overline{Y}_0 < \hat{Y}_0$, i.e. $u_i(x) < \varphi(x)$, which is a contradiction. ■

5.7.2 Elliptic Dirichlet Problem

We now give a similar result for a system of elliptic PDEs in an open bounded subset of $\mathbb{R}^d$, with Dirichlet boundary condition, following [20]. Let $D \subset \mathbb{R}^d$ be a bounded domain (i.e. D is an open bounded subset of $\mathbb{R}^d$), whose boundary ∂D is of class C^1. We are given a function $\chi \in C(\mathbb{R}^d)$ and we consider the system of elliptic PDEs

$$\begin{cases} \Phi_i(x, u(x), Du(x), D^2u(x)) = 0, \ 1 \le i \le m, \ x \in D; \\ u_i(x) = \chi_i(x), \qquad\qquad\qquad\quad 1 \le i \le m, \ x \in \partial D. \end{cases} \tag{5.218}$$

The process $\{X_t^x;\ t \ge 0\}$ is defined as in the preceding subsection. For each $x \in \overline{D}$, we define the stopping time

$$\tau_x = \inf\{t \ge 0;\ X_t^x \notin \overline{D}\}.$$

Let $\{(Y_t^x, Z_t^x);\ 0 \le t \le \tau_x\}$ be the solution, in the sense of Corollary 5.59, of the BSDE

$$Y_t^x = \chi(X_{\tau_x}^x) + \int_{t\wedge\tau_x}^{\tau_x} F(X_s^x, Y_s^x, Z_s^x)\, ds - \int_{t\wedge\tau_x}^{\tau_x} Z_s^x dB_s,\ t \ge 0. \tag{5.219}$$

Using Itô's formula, it is not hard to establish the following:

Theorem 5.75. *Let $u \in C^2(D;\mathbb{R}^m)\cap C^0(\overline{D};\mathbb{R}^m)$ be a classical solution of (5.218). Then for each $x \in \mathbb{R}^d$, $\{(u(X_t^x), (\nabla u g)(X_t^x));\ t \ge 0\}$ is the solution of the BSDE (5.219). In particular $u(x) = Y_0^x$.*

We now assume that $P(\tau_x < \infty) = 1$, for all $x \in \overline{D}$, that the set

$$\Lambda = \{x \in \partial D;\ P(\tau_x > 0) = 0\} \quad \text{is closed,} \tag{5.220}$$

and that for some $\lambda > 2\mu + K^2$, and all $x \in \overline{D}$,

$$\mathbb{E}e^{\lambda\tau_x} < \infty.$$

We again define $u(x) = Y_0^x$. Besides some arguments which we have already used, the continuity of u also relies on the following:

Proposition 5.76. *Under the condition (5.220), the mapping $x \to \tau_x$ is a.s. continuous on $\overline{D}$.*

Proof. Let $\{x_n,\ n \in \mathbb{N}\}$ be a sequence in $\overline{D}$ such that $x_n \to x$, as $n \to \infty$. We first show that

$$\limsup_{n\to\infty} \tau_{x_n} \le \tau_x \quad \text{a.s.} \tag{5.221}$$

Suppose that (5.221) is false. Then

$$P(\tau_x < \limsup_{n\to\infty} \tau_{x_n}) > 0. \tag{5.222}$$

For each $\varepsilon > 0$, let

$$\tau_x^\varepsilon = \inf\{t \geq 0;\ d(X_t^x, D) \geq \varepsilon\}.$$

From (5.222), there exists ε and T such that

$$P(\tau_x^\varepsilon < \limsup_{n\to\infty} \tau_{x_n} \leq T) > 0.$$

But since $X_\cdot^{x_n} \to X_\cdot^x$ uniformly on $[0, T]$ a.s., this implies that

$$P(\limsup_{n\to\infty} \tau_{x_n}^{\varepsilon/2} \leq \tau_x^\varepsilon < \limsup_{n\to\infty} \tau_{x_n} \leq T) > 0,$$

which would mean that for some n, X^{x_n} exits the $\varepsilon/2$-neighbourhood of D before exiting D, which is impossible.

We next prove that

$$\liminf_{n\to\infty} \tau_{x_n} \geq \tau_x \quad \text{a.s.} \tag{5.223}$$

For this part of the proof, we will need the assumption (5.220) that Λ is closed.

It suffices to prove that (5.223) holds a.s. on $\Omega_M = \{\tau_x \leq M\}$, with M arbitrary. From the result of the first step, for almost all $\omega \in \Omega_M$, there exists an $n(\omega)$ such that $n \geq n(\omega)$ implies $\tau_{x_n} \leq M + 1$. From the a.s. (on Ω_M) uniform convergence of $X_\cdot^{x_n} \to X_\cdot^x$ on the interval $[0, M + 1]$, $X_\cdot^x$ hits the set

$$\overline{\{X_{\tau_{x_n}}^{x_n};\ n \in \mathbb{N}\}} \subset \overline{\Lambda} = \Lambda$$

on the random interval $[0, \liminf_n \tau_{x_n}]$ a.s. on Ω_M. The result follows, since $X_\cdot^x$ exits $\overline{D}$ when it hits Λ. ■

We now prove the following:

Theorem 5.77. *Under the assumptions of Theorem 5.74, the above conditions on D and the condition (5.220), $u(x) \overset{def}{=} Y_0^x$ is continuous on $\overline{D}$ and it is a viscosity solution of the system of Eq. (5.218).*

Proof. We only prove that u is a sub-solution. Let $1 \leq i \leq m$, $\varphi \in C^2(\mathbb{R}^d)$ $u_i - \varphi$ have a local maximum at $x \in \overline{D}$, such that $u_i(x) = \varphi(x)$. If $x \in \Lambda$, then $\tau_x = 0$, and hence $u(x) = \chi(x)$. If however $x \in D \cup (\partial D \backslash \Lambda)$, the result follows by the same argument as in the proof of Theorem 5.74.

5.7.3 Elliptic Equations with Neumann Boundary Conditions

The data and assumptions are the same as in Sect. 5.4.3, except that we suppress the dependence of all coefficients upon the time variable. Moreover we also assume that all assumptions of Section 5.4.1 are satisfied.

Consider the following system of semilinear elliptic PDEs with nonlinear Neumann boundary condition

$$\begin{cases} \Phi_i(x, u(x), Du_i(x), D^2u_i(x)) = 0, \ x \in D, \quad 0 \le i \le m; \\ \dfrac{\partial u_i}{\partial n}(x) - G_i(x, u(x)) = 0, \ x \in \partial D, \ 1 \le i \le m. \end{cases} \tag{5.224}$$

Let X^x be the process solution of the reflected stochastic differential equation, for all $t \ge 0$, $\mathbb{P}$ a.s.

$$\begin{cases} X_t^x + K_t^x = x + \displaystyle\int_0^t f(r, X_r^x)dr + \int_0^t g(r, X_r^x)dB_r, \\ X_t^x \in \overline{D}, \ K_t^x = \displaystyle\int_0^t n(X_r^x)\mathbf{1}_{\partial D}\left(X_r^x\right) d \updownarrow K^x \updownarrow_r . \end{cases} \tag{5.225}$$

To each $x \in \overline{D}$ we associate the BSDE

$$Y_t^x = Y_T^x + \int_t^T F(r, X_r^x, Y_r^x, Z_r^x)dr + \int_t^T G(r, X_r^x, Y_r^x)d \updownarrow K^x \updownarrow_r \tag{5.226}$$

$$- \int_t^T Z_r^x dB_r, \ \text{for all pairs } 0 \le t < T.$$

Itô's formula again allows us to establish the following:

Theorem 5.78. *Let $u \in C^2(D;\mathbb{R}^m) \cap C^1(\overline{D};\mathbb{R}^m)$ be a classical solution of (5.224). Then for each $x \in \mathbb{R}^d$, $\{(u(X_t^x), (\nabla ug)(X_t^x));\ t \ge 0\}$ is the solution of the BSDE (5.226). In particular $u(x) = Y_0^x$.*

We now have:

Theorem 5.79. *Under the above conditions and those of Theorem 5.43, $u(x) := Y_0^x$ is a continuous function of x, and it is a viscosity solution of (5.224).*

The proof of this Theorem is easily done by combining the arguments in the proofs of Theorems 5.74 and 5.43.

5.8 Parabolic Variational Inequality

The aim of this section is to prove the existence of a viscosity solution for the following parabolic variational inequality (PVI) with a mixed nonlinear multivalued Neumann–Dirichlet boundary condition:

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} + \mathcal{A}_t u\left(t,x\right) + F\left(t,x,u(t,x),(\nabla u g)(t,x)\right) \in \partial\varphi\left(u(t,x)\right), \\ \hfill t \in (0,T)\,,\ x \in D, \\ -\dfrac{\partial u(t,x)}{\partial n} + G\left(t,x,u(t,x)\right) \in \partial\psi\left(u(t,x)\right),\ \ t \in (0,T)\,,\ x \in \mathrm{Bd}\,(D)\,, \\ u(T,x) = \kappa(x),\ \ x \in \overline{D}, \end{cases} \tag{5.227}$$

where the operator $\mathcal{A}_t$ is given by

$$\mathcal{A}_t v(x) = \frac{1}{2}\mathrm{Tr}\big[g(t,x)g^*(t,x)D^2 v(x)\big] + \big\langle f(t,x),\nabla v(x)\big\rangle,$$

and D is an open connected bounded subset of $\mathbb{R}^d$ of the form

$$D = \left\{x \in \mathbb{R}^d : \phi\left(x\right) < 0\right\},\ \ \mathrm{Bd}\,(D) = \left\{x \in \mathbb{R}^d : \phi\left(x\right) = 0\right\},$$

where $\phi \in C_b^3\left(\mathbb{R}^d\right)$, $|\nabla\phi\left(x\right)| = 1$, for all $x \in \mathrm{Bd}\,(D)$. The outward normal derivative of v at the point $x \in \mathrm{Bd}\,(D)$ is given by

$$\frac{\partial v\left(x\right)}{\partial n} = \sum_{j=1}^{d} \frac{\partial \phi\left(x\right)}{\partial x_j}\frac{\partial v\left(x\right)}{\partial x_j} = \left\langle \nabla\phi\left(x\right), \nabla v\left(x\right)\right\rangle.$$

The functions $f : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$, $g : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{d\times d}$, $F : \mathbb{R}_+ \times \overline{D} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, $G : \mathbb{R}_+ \times \mathrm{Bd}\,(D) \times \mathbb{R} \to \mathbb{R}$ and $\kappa : \overline{D} \to \mathbb{R}$ are continuous. Assume that for all $T > 0$, there exist $a, L > 0$ (which can depend on T) such that $\forall t \in [0,T]\,,\ \forall x, \tilde{x} \in \mathbb{R}^d$:

$$\left|f\left(t,x\right) - f\left(t,\tilde{x}\right)\right| + \left|g\left(t,x\right) - g\left(t,\tilde{x}\right)\right| \le a\left|x - \tilde{x}\right|, \tag{5.228}$$

and $\forall t \in [0,T]$, $\forall x \in \overline{D}$, $x' \in \mathrm{Bd}\,(D)$, $y, \tilde{y} \in \mathbb{R}, z, \tilde{z} \in \mathbb{R}^d$:

$$\begin{array}{rl} (i) & (y-\tilde{y})\left[F(t,x,y,z) - F(t,x,\tilde{y},z)\right] \le \frac{L}{2}|y-\tilde{y}|^2, \\ (ii) & |F(t,x,y,z) - F(t,x,y,\tilde{z})| \le \sqrt{\frac{L}{2}}|z-\tilde{z}|, \\ (iii) & |F(t,x,y,0)| \le \frac{L}{2}\left(1+|y|\right), \\ (iv) & (y-\tilde{y})\left[G(t,x',y) - G(t,x',\tilde{y})\right] \le L|y-\tilde{y}|^2, \\ (v) & |G(t,x',y)| \le L\left(1+|y|\right). \end{array} \tag{5.229}$$

We also assume that

$$\begin{array}{rl} (i) & \varphi, \psi : \mathbb{R} \to (-\infty,+\infty]\ \text{ are proper convex l.s.c. functions}, \\ (ii) & 0 = \varphi\left(0\right) \le \varphi\left(y\right)\ \text{and}\ 0 = \psi\left(0\right) \le \psi\left(y\right),\ \forall\ y \in \mathbb{R}, \\ (iii) & \kappa(x) \in \mathrm{int}\left(\mathrm{Dom}\left(\varphi\right)\right) \cap \mathrm{int}\left(\mathrm{Dom}\left(\psi\right)\right)\ \text{ for all } x \in \overline{D}, \end{array} \tag{5.230}$$

and the *compatibility conditions*:
for all $\varepsilon > 0$, $t \geq 0$, $x \in \mathrm{Bd}(D)$, $\tilde{x} \in \overline{D}$, $y \in \mathbb{R}$ and $z \in \mathbb{R}^d$

$$\begin{aligned} &(i)\ \nabla\varphi_\varepsilon(y)\, G(t,x,y) \leq |\nabla\psi_\varepsilon(y)|\, |G(t,x,y)|, \\ &(ii)\ \nabla\psi_\varepsilon(y)\, F(t,\tilde{x},y,z) \leq |\nabla\varphi_\varepsilon(y)|\, |F(t,\tilde{x},y,z)|, \end{aligned} \tag{5.231}$$

where $a^+ = \max\{0,a\}$ and $\nabla\varphi_\varepsilon(y)$, $\nabla\psi_\varepsilon(y)$ are the unique solutions U and V, respectively, of equations

$$\partial\varphi(y-\varepsilon U) \ni U \quad \text{and} \quad \partial\psi(y-\varepsilon V) \ni V$$

(for the Moreau–Yosida approximations $\nabla\varphi_\varepsilon$, $\nabla\psi_\varepsilon$ see section "Convex function" from Annex B and for the compatibility conditions see Example 5.63). We mention that in the one dimensional case (which is our case here)

$$\partial\varphi(y) = \left[\varphi'_-(y), \varphi'_+(y)\right] \quad \text{and} \quad \partial\psi(y) = \left[\psi'_-(y), \psi'_+(y)\right].$$

Since $\overline{D}$ is bounded and κ is continuous it follows from (5.230-*iii*) that there exists an $M_0 > 0$ such that

$$\sup_{x\in\overline{D}} |\kappa(x)| + \sup_{x\in\overline{D}} \varphi(\kappa(x)) + \sup_{x\in\overline{D}} \psi(\kappa(x)) \leq M_0.$$

Let $(t,x) \in [0,T] \times \overline{D}$ be arbitrarily fixed. Consider the stochastic basis $\left(\Omega, \mathcal{F}, \mathbb{P}, \left(\mathcal{F}_s^t\right)_{s\geq 0}\right)$, where the filtration is generated by a d-dimensional Brownian motion as follows: $\mathcal{F}_s^t = \mathcal{N}$ if $0 \leq s \leq t$ and

$$\mathcal{F}_s^t = \sigma\{B_r - B_t : t \leq r \leq s\} \vee \mathcal{N}, \quad \text{if } s > t.$$

From Theorem 4.54 and Theorem 4.47 we infer that there exists a unique pair $(X^{t,x}, A^{t,x}) : \Omega \times [0,\infty[\to \mathbb{R}^d \times \mathbb{R}^d$ of continuous progressively measurable stochastic processes such that, $\mathbb{P}$-a.s.:

$$\begin{cases} (j)\ X_s^{t,x} \in \overline{D} \text{ and } X_{s\wedge t}^{t,x} = x \text{ for all } s \geq 0, \\ (jj)\ 0 = A_u^{t,x} \leq A_s^{t,x} \leq A_v^{t,x} \text{ for all } 0 \leq u \leq t \leq s \leq v, \\ (jjj)\ X_s^{t,x} + \displaystyle\int_t^s \nabla\phi\left(X_r^{t,x}\right) dA_r^{t,x} = x + \int_t^s f\left(r, X_r^{t,x}\right) dr \\ \qquad\qquad\qquad\qquad\qquad + \displaystyle\int_t^s g\left(r, X_r^{t,x}\right) dB_r, \quad \forall\, s \geq t, \\ (jv)\ A_s^{t,x} = \displaystyle\int_t^s \mathbf{1}_{\mathrm{Bd}(\overline{D})}\left(X_r^{t,x}\right) dA_r^{t,x}, \quad \forall\, s \geq t. \end{cases} \tag{5.232}$$

Then by Proposition 4.55 and Corollary 4.56 we have for all $p \geq 1$, $\lambda > 0$ and $s \geq t$,

$$\begin{aligned} &(j)\quad \mathbb{E}\sup_{r\in[t,s]}|X_r^{t,x}-X^{t,y}|^p+\mathbb{E}\sup_{r\in[t,s]}\left|A_r^{t,x}-A_r^{t,y}\right|^p\leq Ce^{C(s-t)}\,|x-y|^p\,,\\ &(jj)\quad \mathbb{E}e^{\lambda A_s^{t,x}}\leq\exp\left(C\lambda+C\lambda t+\frac{C^2\lambda^2}{2}t\right), \end{aligned} \tag{5.233}$$

and for every $T>0$, $p\geq 1$ and continuous functions $h_1,h_2:[0,T]\times\overline{D}\to\mathbb{R}$, the mappings

$$(t,x)\mapsto\left(X_\cdot^{t,x},A_\cdot^{t,x}\right):[0,T]\times\overline{D}\to S_d^p\,[0,T]\times S_1^p\,[0,T]$$

and

$$(t,x)\mapsto\mathbb{E}\int_t^T h_1(s,X_s^{t,x})ds+\mathbb{E}\int_t^T h_2(s,X_s^{t,x})\,dA_s^{t,x}:[0,T]\times\overline{D}\to\mathbb{R}$$

are continuous.

We consider the backward stochastic variational inequality (BSVI):

$$\begin{cases} Y_s^{t,x}+\displaystyle\int_s^T dK_r^{t,x}=\kappa\left(X_T^{t,x}\right)+\int_s^T 1_{[t,T]}(r)\;F\left(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x}\right)dr,\\ \qquad+\displaystyle\int_s^T 1_{[t,T]}(r)\,G\left(r,X_r^{t,x},Y_r^{t,x}\right)dA_r^{t,x}-\int_s^T\langle Z_r^{t,x},dB_r\rangle\,,\;\forall\;s\in[0,T]\,,\\ Y_s^{t,x}=Y_t^{t,x},\;Z_s^{t,x}=0,\;K_s^{t,x}=U_s^{t,x}=V_s^{t,x}=0,\;\forall\;s\in[0,t]\,,\\ K_s^{t,x}=\displaystyle\int_0^s\left(U_r^{t,x}dr+V_r^{t,x}dA_r^{t,x}\right),\quad\forall\;s\in[0,T]\,,\\ U_s^{t,x}\in\partial\varphi\left(Y_s^{t,x}\right)\text{ and }V_s^{t,x}\in\partial\psi\left(Y_s^{t,x}\right)\quad a.e.\text{ on }\;\Omega\times[t,T]\,. \end{cases} \tag{5.234}$$

Note that the backward stochastic variational inequality (5.234) satisfies the assumptions of Theorem 5.69 and Remark 5.70 with $\tau=T$, $\eta=\kappa\left(X_T^{t,x}\right)$ satisfying (A_7''), $\mu_s=\frac{L}{2}\mathbf{1}_{[0,T]}(s)$, $\ell_s=\sqrt{\frac{L}{2}}\mathbf{1}_{[0,T]}(s)$, $\nu_s=L\mathbf{1}_{[0,T]}(s)$, $V_s=LQ_{s\wedge T}^{t,x}$, $u_0=0$, where

$$Q_s^{t,x}=s+A_s^{t,x}\quad\text{and}\quad\mathbb{E}\left(e^{\lambda Q_T^{t,x}}\right)<\infty,\text{ for all }\lambda>0.$$

Therefore (5.234) has a unique solution $(Y^{t,x},Z^{t,x},K^{t,x})$ of continuous progressively measurable stochastic processes such that

$$\mathbb{E}\sup_{r\in[t,T]}e^{2LQ_r^{t,x}}\left|Y_r^{t,x}\right|^2+\mathbb{E}\left(\int_t^T e^{2LQ_r^{t,x}}\left|Z_r^{t,x}\right|^2dr\right)<\infty,$$

and $dK_s^{t,x}=U_s^{t,x}ds+V_s^{t,x}dA_s^{t,x}$, where $U^{t,x},V^{t,x}$ are progressively measurable stochastic processes and $U_s^{t,x}\in\partial\varphi\left(Y_s^{t,x}\right)$, $V_s^{t,x}\in\partial\varphi\left(Y_s^{t,x}\right)$ $d\mathbb{P}\otimes dt-a.e.$ on $\Omega\times[t,T]$.

Moreover by (5.162) and (5.164) the solution satisfies for all $s \in [t, T]$:

$$\begin{aligned}
&\mathbb{E}^{\mathcal{F}_s} \sup_{r\in[s,T]} e^{2LQ_r^{t,x}} |Y_r^{t,x}|^2 + \mathbb{E}^{\mathcal{F}_s} \left(\int_s^T e^{2LQ_r^{t,x}} |Z_r^{t,x}|^2 \, dr \right) \\
&\quad + \mathbb{E}^{\mathcal{F}_s} \int_s^T e^{2LQ_r^{t,x}} \left[\varphi\left(Y_r^{t,x}\right) dr + \left|\psi\left(Y_r^{t,x}\right)\right| dA_r^{t,x} \right] \\
&\leq C_2 \mathbb{E}^{\mathcal{F}_s} \Big[e^{2LQ_T^{t,x}} \left|\kappa\left(X_T^{t,x}\right)\right|^2 \\
&\quad + \Big(\int_s^T e^{LQ_r^{t,x}} \left(\left|F\left(r, X_r^{t,x}, 0, 0\right)\right| dr + \left|G\left(r, X_r^{t,x}, 0,\right)\right| dA_r^{t,x} \right) \Big)^2 \Big] \\
&\leq C_2 \, \mathbb{E}^{\mathcal{F}_s} \left[e^{2LQ_T^{t,x}} M_0 + \left(e^{LQ_T^{t,x}} - e^{LQ_s^{t,x}} \right)^2 \right] \\
&\leq C_{M_0} \mathbb{E}^{\mathcal{F}_s} \left(e^{2LQ_T^{t,x}} \right),
\end{aligned} \tag{5.235}$$

and

$$\begin{aligned}
&\mathbb{E}\left[e^{2LQ_s^{t,x}} \varphi(Y_s^{t,x}) + \psi(Y_s^{t,x}) \right] + \frac{1}{2} \mathbb{E} \int_s^T e^{2LQ_r^{t,x}} \left(|U_r^{t,x}|^2 \, dr + |V_r^{t,x}|^2 \, dA_r \right) \\
&\leq \mathbb{E}\left[e^{2LQ_T^{t,x}} \left(\varphi(\kappa\left(X_T^{t,x}\right)) + \psi(\kappa\left(X_T^{t,x}\right)) \right) \right] \\
&\quad + 4\mathbb{E} \int_s^T e^{2LQ_r^{t,x}} \left(|F(r, Y_r^{t,x}, Z_r^{t,x})|^2 \right) dr + 4\mathbb{E} \int_s^T e^{2LQ_r^{t,x}} \left(|G(r, Y_r^{t,x})|^2 \right) dA_r \\
&\leq C_{M_0,L} \mathbb{E} e^{2LQ_T^{t,x}}.
\end{aligned} \tag{5.236}$$

We define

$$u(t, x) = Y_t^{t,x}, \quad (t, x) \in [0, T] \times \overline{D}, \tag{5.237}$$

which is a deterministic quantity since $Y_t^{t,x}$ is $\mathcal{F}_t^t \equiv \mathcal{N}$-measurable.

From the Markov property, we have

$$u(s, X_s^{t,x}) = Y_s^{t,x}.$$

By (5.236) we infer that

$$u(t, x) \in \text{Dom}\,(\varphi) \cap \text{Dom}\,(\psi) \;\text{ for all } (t, x) \in [0, T] \times \overline{D}. \tag{5.238}$$

In the sequel we shall prove that u defined by (5.237) is a viscosity solution of (5.227). Reversing the time by $\tilde{u}(t, x) = u(T - t, x)$, the PVI (5.227) becomes (6.137) and the uniqueness of the viscosity solution follows from Theorem 6.112.

We now give the definition of the viscosity solution of the PVI (5.227).

A triple $(p, q, X) \in \mathbb{R}\times\mathbb{R}^d \times\mathbb{S}^d$ is a parabolic super-jet to u at $(t, x) \in (0, T)\times\overline{D}$ if for all $(s, y) \in (0, T) \times \overline{D}$

$$u(s,y) \le u(t,x) + p(s-t) + \langle q, y-x\rangle + \tfrac{1}{2}\langle X(y-x), y-x\rangle \\ +o(|s-t| + |y-x|^2);$$

the set of parabolic super-jets is denoted $\mathcal{P}^{2,+}u(t,x)$. The set of parabolic sub-jets is defined by $\mathcal{P}^{2,-}_{\mathcal{O}}u = -\mathcal{P}^{2,+}_{\mathcal{O}}(-u)$.

Let

$$\Phi(t,x,y,q,X) = -\frac{1}{2}\mathrm{Tr}\left((gg^*)(t,x)X\right) - \langle f(t,x), q\rangle - F(t,x,y,qg(t,x)),$$
$$\Gamma(t,x,y,q) = \langle \nabla\phi(x), q\rangle - G(t,x,y).$$

We clearly have

$$\Phi\left(s,y,r,\nabla v(y), D^2v(y)\right) = -\mathcal{A}_s v(y) - F(s,y,r,\nabla v(y)g(s,y)). \tag{5.239}$$

Definition 5.80. Let $u : [0,T]\times\overline{D} \to \mathbb{R}$ be a continuous function, which satisfies $u(T,x) = \kappa(x)$, $\forall\, x \in \overline{D}$.

(a) u is a viscosity sub-solution of (5.227) if:

$$\left|\begin{array}{ll} u(t,x) \in \mathrm{Dom}(\varphi), & \forall (t,x) \in (0,T)\times\overline{D}, \\ u(t,x) \in \mathrm{Dom}(\psi), & \forall (t,x) \in (0,T)\times \mathrm{Bd}(D), \end{array}\right.$$

and for any $(t,x) \in (0,T)\times\overline{D}$ and any $(p,q,X) \in \mathcal{P}^{2,+}u(t,x)$:

$$\begin{cases} (d_1) & p + \Phi(t,x,u(t,x),q,X) + \varphi'_-(u(t,x)) \le 0 \ \text{ if } x \in D, \\ (d_2) & \min\Big\{p + \Phi(t,x,u(t,x),q,X) + \varphi'_-(u(t,x)), \\ & \quad \Gamma(t,x,u(t,x),q) + \psi'_-(u(t,x))\Big\} \le 0 \ \text{ if } x \in \mathrm{Bd}(D). \end{cases} \tag{5.240}$$

(b) The viscosity super-solution of (5.227) is defined in a similar manner as above, with $\mathcal{P}^{2,+}$ replaced by $\mathcal{P}^{2,-}$, the left derivative replaced by the right derivative, min by max, and the inequalities $\le$ by $\ge$.

(c) A continuous function $u : [0,\infty)\times\overline{D}$ is a viscosity solution of (6.137) if it is both a viscosity sub- and super-solution.

Theorem 5.81. *Let the assumptions (5.228), (5.229), (5.230) and (5.231) be satisfied. If u defined by (5.237) is continuous on* $[0,T]\times\overline{D}$*, then u is a viscosity solution of PVI (5.227).*

Proof. We show only that u is a viscosity sub-solution of (5.227) (the proof of the super-solution property is similar).

Let $(t,x) \in [0,T]\times\overline{D}$ and $(p,q,X) \in \mathcal{P}^{2,+}u(t,x)$.

Cases. $(t,x) \in [0,T] \times \mathrm{Bd}\,(D)$.

Aiming to deduce a contradiction we suppose that

$$\min\Big\{-p+\Phi\,(t,x,u(t,x),q,X)+\varphi'_-\big(u(t,x)\big),\Gamma(t,x,u(t,x),q)+\psi'_-\big(u(t,x)\big)\Big\}>0.$$

It follows by continuity of F, G, u, f, g, ϕ, Φ, Γ left continuity and nondecreasing monotonicity of φ'_- and ψ'_- that there exists $\varepsilon > 0$, $\delta > 0$ such that for all $(s,x') \in [0,T] \times \overline{D}$, $|s-t| \leq \delta$, $|x'-x| \leq \delta$,

$$-(p+\varepsilon)+\Phi\big(s,x',u(s,x'),q+(X+\varepsilon I)\left(x'-x\right),X+\varepsilon I\big)+\varphi'_-\big((u(s,x')\big)>0 \tag{5.241}$$

and

$$\Gamma(s,x',u(s,x'),q+(X+\varepsilon I)\left(x'-x\right))+\psi'_-\big(u(s,x')\big)>0. \tag{5.242}$$

Now since $(p,q,X) \in \mathcal{P}^{2,+}u(t,x)$ there exists $0 < \delta' \leq \delta$ such that for all $s \in [0,T]$, $s \neq t$, $x' \in \overline{D}$, $x' \neq x$, $|s-t| \leq \delta'$, $|x'-x| \leq \delta'$ we have

$$\begin{aligned} u(s,x') < \hat{u}(s,x') &\overset{def}{=} u(t,x)+(p+\varepsilon)\,(s-t)+\langle q,x'-x\rangle \\ &\quad+\frac{1}{2}\big\langle\,(X+\varepsilon I)\,(x'-x),x'-x\big\rangle. \end{aligned}$$

By (5.239) the condition (5.241) becomes

$$-\frac{\partial\hat{u}(r,x')}{\partial t}-\mathcal{A}_s\hat{u}(s,x')-F\,\big(s,x',u(s,x'),\nabla\hat{u}(s,x')g(s,x')\big)+\varphi'_-\big((u(s,x')\big)>0. \tag{5.243}$$

The condition (5.242) can be written as follows

$$\big\langle\nabla\hat{u}(s,x'),\nabla\phi(x')\big\rangle-G(s,x',u(s,x'))+\psi'_-\big(u(s,x')\big)>0. \tag{5.244}$$

Let

$$\theta \overset{def}{=} \big(t+\delta'\big)\wedge\inf\big\{s>t:\ |X_s^{t,x}-x|\geq\delta'\big\}\,.$$

We note that $(Y_s^{t,x},Z_s^{t,x})$, $t \leq s \leq \theta$, solves the BSDE

$$\begin{cases} Y_s^{t,x}=u\left(\theta,X_\theta^{t,x}\right)+\displaystyle\int_s^\theta\big(F(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})-U_r^{t,x}\big)dr-\int_s^\theta Z_r^{t,x}dB_r \\ \qquad\qquad\qquad\qquad\qquad+\displaystyle\int_s^\theta\big(G(r,X_r^{t,x},Y_r^{t,x})-V_r^{t,x}\big)dA_r^{t,x}, \\ U_s^{t,x}\in\partial\varphi\left(Y_s^{t,x}\right)\ \text{and}\ V_s^{t,x}\in\partial\psi\left(Y_s^{t,x}\right)\quad d\mathbb{P}\otimes dt\text{-}a.e. \end{cases}$$

Moreover, it follows from Itô's formula that

$$(\hat{Y}_s^{t,x}, \hat{Z}_s^{t,x}) = \big(\hat{u}(s, X_s^{t,x}), (\nabla \hat{u} g)\,(s, X_s^{t,x})\big),\ t \le s \le \theta$$

satisfies

$$\hat{Y}_s^{t,x} = \hat{u}(\theta, X_\theta^{t,x}) - \int_s^\theta \Big[\frac{\partial \hat{u}(r, X_r^{t,x})}{\partial t} + \mathcal{A}_r \hat{u}(r, X_r^{t,x})\Big] dr - \int_s^\theta \hat{Z}_r^{t,x} dB_r \\ + \int_s^\theta \big\langle \nabla_x \hat{u}\left(r, X_r^{t,x}\right), \nabla \phi(X_r^{t,x})\big\rangle\, dA_r^{t,x}.$$

Let $(\tilde{Y}_s^{t,x}, \tilde{Z}_s^{t,x}) = (\hat{Y}_s^{t,x} - Y_s^{t,x}, \hat{Z}_s^{t,x} - Z_s^{t,x})$. We have

$$\tilde{Y}_s^{t,x} = \big[\hat{u}(\theta, X_\theta^{t,x}) - u\left(\theta, X_\theta^{t,x}\right)\big] + \int_s^\theta \Big[- \frac{\partial \hat{u}(r, X_r^{t,x})}{\partial t} - \mathcal{A}_r \hat{u}(r, X_r^{t,x}) \\ - F(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) + U_r^{t,x}\Big] dr - \int_s^\theta \tilde{Z}_r^{t,x} dB_r \\ + \int_s^\theta \Big[\big\langle \nabla_x \hat{u}(r, X_r^{t,x}), \nabla \phi(X_r^{t,x})\big\rangle - G(r, X_r^{t,x}, Y_r^{t,x}) + V_r^{t,x}\Big] dA_r^{t,x}.$$

Let

$$\beta_s = \mathcal{A}_s \hat{u}(s, X_s^{t,x}) + F(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}) \ \text{ and}$$
$$\hat{\beta}_s = \mathcal{A}_s \hat{u}(s, X_s^{t,x}) + F(s, X_s^{t,x}, Y_s^{t,x}, \hat{Z}_s^{t,x}).$$

Since $|\hat{\beta}_s - \beta_s| \le \sqrt{\frac{L}{2}}\,|\hat{Z}_s^{t,x} - Z_s^{t,x}|$, there exists a bounded d-dimensional p.m.s.p. $\{\zeta_s; t \le s \le \theta\}$ such that $\hat{\beta}_s - \beta_s = \langle \zeta_s, \tilde{Z}_s^{t,x}\rangle$ and therefore

$$\tilde{Y}_s^{t,x} = \hat{u}(\theta, X_\theta^{t,x}) - u(\theta, X_\theta^{t,x}) + \int_s^\theta \Big[- \frac{\partial \hat{u}(r, X_r^{t,x})}{\partial t} + \langle \zeta_r, \tilde{Z}_r^{t,x}\rangle - \hat{\beta}_r + U_r^{t,x}\Big] dr \\ + \int_s^\theta \Big[\big\langle \nabla_x \hat{u}(r, X_r^{t,x}), \nabla \phi(X_r^{t,x})\big\rangle - g(r, X_r^{t,x}, Y_r^{t,x}) + V_r^{t,x}\Big] dA_r^{t,x} - \int_s^\theta \tilde{Z}_r^{t,x} dB_r.$$

Let

$$\Lambda_s = \exp\left(\int_t^s \langle \zeta_r, dB_r\rangle - \frac{1}{2}\int_t^s |\zeta_r|^2\, dr\right),\ t \le s \le \theta.$$

Then by Itô's formula,

$$\Lambda_s = 1 + \int_t^s \Lambda_r\ \langle \zeta_r, dB_r\rangle\,,\ t \le s \le \theta,$$

and so

$$d(\tilde{Y}_r^{t,x}\ \Lambda_r) = \Lambda_r\Big[\frac{\partial \hat{u}(r,X_r^{t,x})}{\partial t} + \hat{\beta}_r - U_r^{t,x}\Big]dr + \Lambda_r\langle \tilde{Z}_r^{t,x} + \tilde{Y}_r^{t,x}\zeta_r, dB_r\rangle$$
$$+ \Lambda_r\Big[-\big\langle \nabla_x \hat{u}(r,X_r^{t,x}), \nabla\phi(X_r^{t,x})\big\rangle + g(r,X_r^{t,x},Y_r^{t,x}) - V_r^{t,x}\Big]dA_r^{t,x}.$$

Then

$$\tilde{Y}_t^{t,x} = \mathbb{E}\Big[\Lambda_\theta\ \big(\hat{u}(\theta,X_\theta^{t,x}) - u(\theta,X_\theta^{t,x})\big)\Big] + \mathbb{E}\int_t^\theta \Lambda_r\Big[-\tfrac{\partial \hat{u}(r,X_r^{t,x})}{\partial t} - \hat{\beta}_r + U_r^{t,x}\Big]dr$$
$$+\mathbb{E}\int_t^\theta \Lambda_r\Big[\big\langle \nabla_x \hat{u}(r,X_r^{t,x}), \nabla\phi(X_r^{t,x})\big\rangle - g(r,X_r^{t,x},Y_r^{t,x}) + V_r^{t,x}\Big]dA_r^{t,x}. \tag{5.245}$$

Since $U_r^{t,x} \in \partial\varphi\left(Y_r^{t,x}\right)$ and $V_r^{t,x} \in \partial\psi\left(Y_r^{t,x}\right)$, we have

$$U_r^{t,x}dr \geq \varphi'_-\big(u(r,X_r^{t,x})\big)dr, \quad V_r^{t,x}dA_r^{t,x} \geq \psi'_-\big(u(r,X_r^{t,x})\big)dA_r^{t,x},$$

and therefore by (5.243) and (5.244)

$$-\frac{\partial \hat{u}(r,X_r^{t,x})}{\partial t} - \hat{\beta}_r + U_r^{t,x} > 0,$$

and

$$\Big[\big\langle \nabla_x \hat{u}(r,X_r^{t,x}), \nabla\phi(X_r^{t,x})\big\rangle - g(r,X_r^{t,x},Y_r^{t,x}) + V_r^{t,x}\Big]dA_r^{t,x} \geq 0.$$

Moreover, the choice of δ' and θ implies that $u(\theta,X_\theta^{t,x}) < \hat{u}(\theta,X_\theta^{t,x})$. Hence

$$0 = \hat{u}\,(t,x) - u\,(t,x) = \tilde{Y}_t^{t,x} \geq \mathbb{E}\Big[\Lambda_\theta\ \big(\hat{u}(\theta,X_\theta^{t,x}) - u(\theta,X_\theta^{t,x})\big)\Big] > 0,$$

which is a contradiction. It follows that (5.240-d_2) holds.

Cases. $(t,x) \in [0,T] \times D$.

The proof follows the same steps from *Case 5.8*, where we now choose δ and δ' such that $\overline{B\,(x,\delta')} \subset \overline{B\,(x,\delta)} \subset D$ and, by condition (5.232-*iv*), $A_r^{t,x} = 0$ for all $t \leq r \leq \theta$.

This proves that u is a viscosity sub-solution of PVI (5.227). Symmetric arguments show that u is also a super-solution; hence u is a viscosity solution of PVI (5.227).

Corollary 5.82. *We have*

$$u(t,x) \in \mathrm{Dom}\,(\partial\varphi)\,, \quad \forall (t,x) \in [0,T] \times D.$$

Proof. Let $(t, x) \in [0, T] \times D$ be fixed. We have two cases:

(1) $\mathrm{Dom}\,(\partial\varphi) = \mathrm{Dom}\,(\varphi)$, and so, from (5.238), $u(t, x) \in \mathrm{Dom}\,(\partial\varphi)$.
(2) $\mathrm{Dom}\,(\partial\varphi) \neq \mathrm{Dom}\,(\varphi)$. Let $b \in \mathrm{Dom}\,\varphi \setminus \mathrm{Dom}\,(\partial\varphi)$. Then $b = \sup(\mathrm{Dom}\,\varphi)$ or $b = \inf \mathrm{Dom}\,\varphi$. If $b = \sup(\mathrm{Dom}\,\varphi)$ and $u(t, x) = b$, then $(0, 0, 0) \in \mathcal{P}^{2,+}u(t, x)$ since

$$u(s, y) \leq u(t, x) + o\big(|s-t| + |y-x|^2\big)$$

and from (6.143-d_1) it follows that $\varphi'_{-}(b) = \varphi'_{-}\big(u(t, x)\big) < \infty$ and consequently $b \in \mathrm{Dom}\,(\partial\varphi)$; a contradiction which shows that $u(t, x) < b$. Similarly for $b = \inf(\mathrm{Dom}\,\varphi)$. ■

The problem now is to see when $(t, x) \longmapsto u(t, x) = Y_t^{tx} : [0, T] \times \overline{D} \to \mathbb{R}$ is continuous. A recent result of Maticiuc and Răşcanu [46] gives a sufficient condition for u to be continuous. The idea is to show that if $(t_n, x_n) \to (t, x)$ then $(Y_\cdot^{t_n, x_n})_{n\in\mathbb{N}^*}$ is tight in the Skorohod space $\mathbb{D}\,([0, T], \mathbb{R})$ of càdlàg functions endowed with the S-topology (introduced by Jakubowski in [41]). This topology makes the mapping $y \longmapsto \int_0^s G\,(r, y\,(r))\,dA_r$ continuous from $\mathbb{D}\,([0, T], \mathbb{R})$ into $\mathbb{R}$. The result is the following:

Proposition 5.83. *Let the assumptions (5.228), …, (5.231) be satisfied. If moreover there exists an $L_0 > 0$ such that*

$$\begin{array}{ll} (i) & F \text{ is independent of } z, \\ (ii) & g\,(t, x) \text{ is an invertible matrix, for all } (t, x) \in [0, T] \times \overline{D}, \\ (iii) & |G\,(t, x, y) - G\,(t', x', y')| \leq L_0\,(|t - t'| + |x - x'| + |y - y'|) \\ & \qquad \text{for all } t, t' \in [0, T],\ x, x' \in \mathrm{Bd}\,(\mathcal{D}),\ y, y' \in \mathbb{R} \end{array} \tag{5.246}$$

then the function

$$(t, x) \mapsto u\,(t, x) = Y_t^{t,x} : [0, T] \times \overline{\mathcal{D}} \to \mathbb{R}$$

is continuous.

Finally let f, g, F, G be independent of t and $(X_s^x,\ A_s^x,\ Y_s^{x;t},\ Z_s^{x;t},\ U_s^{x;t},\ V_s^{x;t})_{0\leq s\leq t}$ be defined by

$$\begin{cases} (j)\ X_s^x \in \overline{D} \text{ for all } s \geq 0, \\ (jj)\ 0 = A_0^x \leq A_u^x \leq A_s^x \text{ for all } 0 \leq u \leq s, \\ (jjj)\ X_s^x + \displaystyle\int_0^s \nabla\phi(X_r^x)dA_r^x = x + \int_0^s f(X_r^x)dr + \int_0^s g(X_r^x)dB_r, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall\ s \geq 0, \\ (jv)\ A_s^x = \displaystyle\int_0^s \mathbf{1}_{\mathrm{Bd}(\overline{D})}\left(X_r^x\right)dA_r^x, \quad \forall\ s \geq 0, \end{cases}$$

and

$$\begin{cases} Y_s^{x;t} + \int_s^t (U_r^{x;t} dr + V_r^{x;t} dA_r^x) = \kappa(X_t^x) + \int_s^t F(X_r^x, Y_r^{x;t}, Z_r^{x;t}) dr, \\ \qquad + \int_s^t G(X_r^x, Y_r^{x;t}) dA_r^x - \int_s^t \langle Z_r^{x;t}, dB_r \rangle, \ \forall\, s \in [0,t], \\ U_s^{x;t} \in \partial\varphi(Y_s^{x;t}) \text{ and } V_s^{x;t} \in \partial\psi(Y_s^{x;t}) \quad a.e. \text{ on } \Omega \times [0,t]. \end{cases}$$

Summarizing Theorem 5.81 and Theorem 6.112 we have:

Theorem 5.84. *Let the assumptions (5.228),…,(5.231) be satisfied. Assume there exists a continuous function* $\mathbf{m} : [0,\infty) \to [0,\infty)$, $\mathbf{m}(0) = 0$, *such that*

$$\begin{array}{ll} (i) & yG(x,y) \le 0, \quad \forall x \in \mathrm{Bd}(D) \text{ and } y \in \mathbb{R}, \\ (ii) & \left|F(x,y) - F(x',y)\right| \le \mathbf{m}(|x-x'|) \ \forall\, x, x' \in \overline{D} \text{ and } y \in \mathbb{R}. \end{array} \tag{5.247}$$

If $(t,x) \longmapsto u(t,x) \overset{def}{=} Y_0^{x;t} : [0,T] \times \overline{D} \to \mathbb{R}$ *is continuous (this is true in particular under the assumptions of Proposition 5.83), then u is the unique viscosity solution of the parabolic variational inequality*

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} - \mathcal{A}u(t,x) + \partial\varphi(u(t,x)) \ni F(x, u(t,x), (\nabla u g)(t,x)), \ t > 0, \ x \in D, \\ \dfrac{\partial u(t,x)}{\partial n} + \partial\psi(u(t,x)) \ni G(x, u(t,x)), \ t > 0, \ x \in \mathrm{Bd}(D), \\ u(0,x) = \kappa(x), \quad x \in \overline{D}, \end{cases}$$

where the operator $\mathcal{A}$ *is given by*

$$\mathcal{A}v(x) = \frac{1}{2}\mathrm{Tr}\big[g(x)g^*(x)D^2v(x)\big] + \big\langle f(x), \nabla v(x)\big\rangle.$$

5.9 Invariant Sets of BSDEs

Let $\{B_t : t \ge 0\}$ be a k-dimensional standard Brownian motion defined on some complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We denote by $\{\mathcal{F}_t : t \ge 0\}$ the natural filtration generated by $\{B_t, t \ge 0\}$ and augmented by the $\mathbb{P}$-null sets of $\mathcal{F}$.

Let $x \in \mathbb{R}^d$, $0 \le t \le \tilde{T} \le T$. Consider the SDE

$$\begin{cases} X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x}) dr + \int_t^s \sigma(r, X_r^{t,x}) dB_r, \ t \le s \le T, \\ X_s^{t,x} = x, \quad 0 \le s \le t, \end{cases} \tag{5.248}$$

and the BSDE

$$\begin{cases} Y_s^{t,x} = \kappa(X_{\tilde{T}}^{t,x}) + \int_s^{\tilde{T}} F(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr - \int_s^{\tilde{T}} Z_r^{t,x} dB_r, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t \le s \le \tilde{T}, \\ Y_s^{t,x} = \kappa(X_{\tilde{T}}^{t,x}), \quad \tilde{T} \le s \le T, \\ Y_s^{t,x} = Y_t^{t,x}, \quad 0 \le s \le t. \end{cases} \tag{5.249}$$

The aim of this section is to state necessary and sufficient conditions which guarantee that the solution of the BSDE (5.249) does not leave a given set

$$\mathcal{E} = \{E(t,x) \subset \mathbb{R}^m : (t,x) \in [0,T] \times \mathbb{R}^d\},$$

i.e., under which we have that for all $0 \le t \le \tilde{T} \le T$, $x \in \mathbb{R}^d$ and $\kappa(X_{\tilde{T}}^{t,x}) \in E(\tilde{T}, X_{\tilde{T}}^{t,x})$ $a.s.\ \omega \in \Omega$:

$$Y_s^{t,x} \in E(s, X_s^{t,x}) \quad a.s.\ \omega \in \Omega, \quad \forall s \in [t, \tilde{T}].$$

As a by-product, we will derive a result on the existence of constrained viscosity solutions to some PDEs. Together with the Eqs. (5.248) and (5.249), we consider the following system of semilinear parabolic PDEs

$$\begin{cases} \dfrac{\partial u_i(t,x)}{\partial t} + \mathcal{A}(t)u_i(t,x) + f_i(t,x,u(t,x),\sigma^*(t,x)\nabla_x u_i(t,x)) = 0, \\ u(T,x) = \kappa(x), \quad (t,x) \in [0,T] \times \mathbb{R}^d,\ 1 \le i \le n, \end{cases} \tag{5.250}$$

with the second order differential operator

$$\begin{aligned} \mathcal{A}(t)\varphi(x) &= \frac{1}{2}\mathrm{Tr}[\sigma\sigma^*(t,x)D_x^2\varphi(x)] + < b(t,x), \nabla_x\varphi(x) > \\ &= \frac{1}{2}\sum_{j,\ell=1}^m (\sigma\sigma^*)_{j\ell}(t,x)\frac{\partial^2\varphi(x)}{\partial x_j \partial x_\ell} + \sum_{j=1}^m b_j(t,x)\frac{\partial\varphi(x)}{\partial x_j}, \quad \varphi \in C^2\left(\mathbb{R}^d\right), \end{aligned}$$

where $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : [0,T] \times \mathbb{R}^d \to \mathbb{R}^{d\times k}$ and $f_i : [0,T] \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}$, $1 \le i \le n$.

We make the following standard assumptions:

(AV_1) We assume that the functions $b : [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$, $\sigma : [0,T]\times\mathbb{R}^d \to \mathbb{R}^{d\times k}$, $F : [0,T] \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{m\times k} \to \mathbb{R}^m$ and $f : [0,T] \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}^m$ are continuous and such that, for some constants $L > 0$ and $q \ge 2$,

$$|b(t,x) - b(t,\tilde{x})| + \|\sigma(t,x) - \sigma(t,\tilde{x})\| \le L\,|x - \tilde{x}|,$$

$$\begin{aligned} i)\ & |F(t,x,y,z)| \le L\left(1+|x|^q+|y|+\|z\|\right), \\ ii)\ & |F(t,x,y,z)-F(t,x,y,\tilde z)| \le L\,\|z-\tilde z\|, \\ iii)\ & \langle F(t,x,y,z)-F(t,x,\tilde y,z), y-\tilde y\rangle \le L\,|y-\tilde y|^2 \end{aligned}$$

and

$$\begin{aligned} j)\ & |f(t,x,y,u)| \le L\left(1+|x|^q+|y|+|u|\right), \\ jj)\ & |f(t,x,y,u)-f(t,x,y,\tilde u)| \le L\,|u-\tilde u|, \\ jjj)\ & \langle f(t,x,y,u)-f(t,x,\tilde y,u), y-\tilde y\rangle \le L\,|y-\tilde y|^2, \end{aligned}$$

for all $t \in [0,T]$, $x,\tilde x \in \mathbb{R}^d$, $y,\tilde y \in \mathbb{R}^m$, and $z,\tilde z \in \mathbb{R}^{m\times k}$, $u,\tilde u \in \mathbb{R}^k$.

(AV_2) We assume that $\kappa : \mathbb{R}^d \to \mathbb{R}^m$ is a Borel measurable function of at most polynomial growth, i.e., there are some $a > 0, q \ge 1$ such that, for all $x \in \mathbb{R}^d$,

$$|\kappa(x)| \le a(1+|x|^q),\ \forall x \in \mathbb{R}^d.$$

We shall now recall some basic properties of forward and backward stochastic differential equations.

Proposition 5.85. *Under the assumptions* (AV_1) *and* (AV_2)*:*

I. *Equations (1.1) and (1.2) have unique solutions* $X^{t,x} \in S_d^2[0,T]$ *and*

$$\left(Y^{t,x}, Z^{t,x}\right) \in S_m^2[0,T] \times \Lambda^2_{m\times k}[0,T]$$

with $Z_s^{t,x} = 0$ *for* $s \in [0,t] \cup [\tilde T, T]$ *and the solutions satisfy:*

II. *For all* $p \ge 2$*, there exist some constants* $C_p > 0$*,* $q \in \mathbb{N}^*$*, which don't depend on* $t,t' \in [0,T]$ *and* $x,x' \in \mathbb{R}^m$*, such that*

$$\begin{aligned} a)\ & \mathbb{E}\left(\sup_{s\in[0,T]}|X_s^{t,x}|^p\right) \le C_p(1+|x|^p), \\ b)\ & \mathbb{E}\left(\sup_{s\in[0,T]}|X_s^{t,x}-X_s^{t',x'}|^p\right) \le \\ & \le C_p(1+|x|^p+|x'|^{pq})(|t-t'|^{p/2}+|x-x'|^p), \end{aligned} \tag{5.251}$$

and

$$\begin{aligned} c)\ & \mathbb{E}\left(\sup_{s\in[0,T]}|Y_s^{t,x}|^p\right) \le C_p(1+|x|^{pq}), \\ d)\ & \mathbb{E}\left(\sup_{s\in[0,T]}|Y_s^{t,x}-Y_s^{t',x'}|^2\right) \le C_2[\mathbb{E}|\kappa(X_T^{t,x})-\kappa(X_T^{t',x'})|^2, \\ & +\mathbb{E}\int_0^T |1_{[t,T]}(r)F(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x}) \\ & \qquad -1_{[t',T]}(r)F(r,X_r^{t',x'},Y_r^{t,x},Z_r^{t,x})|^2 dr]. \end{aligned}$$

III. *There are some Borel measurable functions* $u : [0,T]\times\mathbb{R}^d \to \mathbb{R}^m$*, and* $v : [0,T]\times\mathbb{R}^d \to \mathbb{R}^{m\times d}$ *such that for all* $0 \le t \le s \le \tilde T \le T$

$$Y_s^{t,x} = u\left(s \wedge \tilde{T}, X_{s\wedge\tilde{T}}^{t,x}\right), \quad Z_s^{t,x} = (v\sigma)\left(s \wedge \tilde{T}, X_{s\wedge\tilde{T}}^{t,x}\right), \quad d\mathbb{P} \otimes ds - a.e.$$

(see [30]).

For the convenience of the reader we recall the definition of a viscosity solution corresponding to the PDE (5.250).

Definition 5.86. a) A lower semicontinuous function $u : [0,T] \times \mathbb{R}^d \to \mathbb{R}^m$ is a *viscosity super-solution* of (5.250), if, firstly, $u_i(T,x) \geq \kappa_i(x)$, for all $x \in \mathbb{R}^d$, $1 \leq i \leq n$, and secondly, for any $1 \leq i \leq n$, $\varphi \in C^{1,2}((0,T) \times \mathbb{R}^d)$ and $(t,x) \in [0,T] \times \mathbb{R}^d$ such that $u_i - \varphi$ achieves a local minimum at (t,x), it holds that

$$\frac{\partial}{\partial t}\varphi(t,x) + \mathcal{A}(t)\varphi(t,x) + f_i(t,x,u(t,x),(\sigma^*\nabla\varphi)(t,x)) \leq 0.$$

b) An upper semicontinuous function $u : [0,T] \times \mathbb{R}^d \to \mathbb{R}^m$ is a *viscosity sub-solution* of (5.250), if, firstly, $u_i(T,x) \leq \kappa_i(x)$, for all $x \in \mathbb{R}^d$, $1 \leq i \leq n$, and secondly, for any $1 \leq i \leq n$, $\varphi \in C^{1,2}((0,T) \times \mathbb{R}^d)$ and $(t,x) \in [0,T] \times \mathbb{R}^d$ such that $u_i - \varphi$ attains a local maximum at (t,x), we have that

$$\frac{\partial}{\partial t}\varphi(t,x) + \mathcal{A}(t)\varphi(t,x) + f_i(t,x,u(t,x),(\sigma^*\nabla\varphi)(t,x)) \geq 0.$$

c) Finally, a continuous function $u : [0,T] \times \mathbb{R}^d \to \mathbb{R}^m$ is a *viscosity solution* of (5.250) if it is both a viscosity super-solution and a viscosity sub-solution of this equation.

From Sect. 5.4.1 of this chapter we have:

Proposition 5.87. *We suppose that the function f satisfies hypothesis (AV_1) and that $\kappa : \mathbb{R}^d \to \mathbb{R}^m$ is a continuous function satisfying (AV_2). Let $X^{t,x}$ and $(Y^{t,x}, Z^{t,x})$ be the solutions to (5.248) and (5.249), respectively, where the driver F of BSDE (1.2) is of the form*

$$F(t,x,y,z) = (f_1(t,x,y,z^*e_1), \ldots, f_n(t,x,y,z^*e_n)),$$

and e_i denotes the unit vector pointing in the i-th coordinate direction of $\mathbb{R}^m$. Then $u(t,x) = Y_t^{t,x}$, $(t,x) \in [0,T] \times \mathbb{R}^d$, is a deterministic continuous function of at most polynomial growth. This function is a viscosity solution to (5.250). Moreover if, for each $R > 0$, there exists a continuous function $\alpha_R : \mathbb{R}_+ \to \mathbb{R}$, $\alpha_R(0) = 0$, such that, for all t, y, z, x, x' with $|x| \leq R$, $|x'| \leq R$,

$$|f(t,x,y,z) - f(t,x',y,z)| \leq \alpha_R(|x - x'|(1 + \|z\|)), \tag{5.252}$$

then u is the unique viscosity solution in the class $C_{pol}([0,T] \times \mathbb{R}^d, \mathbb{R}^m)$.

We now give the notion of the viability property for BSDEs and PDEs. We recall some notations. For any closed set $S \subset \mathbb{R}^d$ we denote by $x \to d_S(x) = \min\{|x-y : y \in S\}$ the distance function to S, and for $x \in \mathbb{R}^d$, we denote by $\Pi_S(x) := \{z \in S: d_S(x) = |x-z|\}$ the set of projections of x on S.

For all $t \in [0,T]$, $x \in \mathbb{R}^d$, let $E(t,x)$ be a non-empty and closed subset of $\mathbb{R}^m$. We consider the following set of *moving* constraints

$$\mathcal{E} = \{E(t,x) : (t,x) \in [0,T] \times \mathbb{R}^d\}.$$

Definition 5.88 (Viability for BSDEs). The moving set $E(t,x)$, $(t,x) \in [0,T] \times \mathbb{R}^d$, is viable (invariant) for the BSDE (5.249) (or Eq. (5.249) is said to be $\mathcal{E}$-viable on $[0,T]$) if, for all $(t,x) \in [0,T] \times \mathbb{R}^d$, $\tilde{T} \in [t,T]$, and all Borel measurable functions $\kappa : \mathbb{R}^d \to \mathbb{R}^m$ of at most polynomial growth, such that $\kappa(\tilde{x}) \in E(\tilde{T},\tilde{x})$, $\mathbb{P} \circ \left[X_{\tilde{T}}^{t,s}\right]^{-1}(d\tilde{x})$-a.s., it holds that the solution of (1.2) satisfies

$$Y_s^{t,x} \in E(s, X_s^{t,x}), \quad \forall s \in [t,\tilde{T}], \ \mathbb{P}\text{-a.s.}$$

Viability for PDEs: Equation (5.250) is said to be $\mathcal{E}$-viable ($\mathcal{E}$-invariant) on $[0,T]$ if, for all $\tilde{T} \in [0,T]$ and $\kappa \in C_{pol}(\mathbb{R}^d, \mathbb{R}^m)$ such that $\kappa(\tilde{x}) \in E(\tilde{T},\tilde{x})$, for all $\tilde{x} \in \mathbb{R}^d$, it holds that there exists a viscosity solution $u \in C_{pol}([0,\tilde{T}] \times \mathbb{R}^d, \mathbb{R}^m)$ of (5.250) with time horizon $\tilde{T}$ and terminal condition $u(\tilde{T},x) = \kappa(x)$, $x \in R^m$, such that

$$u(t,x) \in E(t,x), \ \forall (t,x) \in [0,\tilde{T}] \times \mathbb{R}^d.$$

From Proposition 5.87 we see immediately that:

Remark 5.89. If BSDE (5.249) is $\mathcal{E}$-viable then PDE (5.250) is also $\mathcal{E}$-viable.

Therefore the next result also concerns constrained the BSDEs and the PDEs.

Theorem 5.90 (Viability Criterion for BSDEs). *Assume that* (AV_1) *and* (AV_2) *are satisfied and moreover*

(i) *the function* $(t,x) \mapsto d^2_{E(t,x)}(y) : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ *is jointly upper semicontinuous,*

(ii) *there exist some constants* $M > 0$, $p \geq 1$ *such that*

$$d^2_{E(t,x)}(0) \leq M(1 + |x|^p), \ \forall (t,x) \in [0,T] \times \mathbb{R}^d.$$

Then the following assertions (c) *and* (cc) *are equivalent:*

(c) *Equation (5.249) is* $\mathcal{E}$*-viable on* $[0,T]$.

(cc) *For any sufficiently large* $C > 0$ *and for every* $z \in \mathbb{R}^{m \times d}$*, the function* $h(t,x,y) = d^2_{E(t,x)}(y)$ *is an upper semicontinuous viscosity sub-solution of the PDE*

$$\frac{\partial V(t,x,y)}{\partial t} + \mathcal{L}_z(t)V(t,x,y) + Cd^2_{E(t,x)}(y) = 0, \qquad (t,x,y) \in [0,T] \in \mathbb{R}^d \times \mathbb{R}^m. \tag{5.253}$$

In the above relation, $\mathcal{L}_z(t)$ denotes the following second order differential operator

$$\mathcal{L}_z(t)\varphi(x,y) = \frac{1}{2}\mathrm{Tr}[\Gamma_z\Gamma_z^*(t,x,y)D^2_{(x,y)}\varphi(x,y)] + \left\langle B_z(t,x,y), \nabla_{(x,y)}\varphi(x,y)\right\rangle, \tag{5.254}$$

where

$$\Gamma_z(t,x,y) = \begin{pmatrix} \sigma(t,x) \\ z\sigma(t,x) \end{pmatrix}, \quad B_z(t,x,y) = \begin{pmatrix} b(t,x) \\ -F(t,x,y,z\sigma(t,x)) \end{pmatrix}.$$

This theorem yields:

Corollary 5.91 (Viability Criterion for BSDEs). *We assume that the moving sets of Theorem 5.90 are independent of the spatial variable, $E(t,x) \equiv E(t)$, $(t,x) \in [0,T] \times \mathbb{R}^m$. Then the following assertions (j) and (jj) are equivalent:*

(j) *Equation (5.249) is $\mathcal{E}$-viable on* $[0,T]$.
(jj) *The function $h(t,y) = d^2_{E(t)}(y)$ is an upper semicontinuous viscosity subsolution of the PDE:*

$$\frac{\partial V(t,y)}{\partial t} + \mathcal{A}_z(t;x)V(t,y) + Cd^2_{E(t)}(y) = 0, \quad (t,y) \in [0,T] \times \mathbb{R}^m,$$

for all $x \in \mathbb{R}^d$, $z \in \mathbb{R}^{m\times d}$, where

$$\mathcal{A}_z(t;x)\psi(y) = \frac{1}{2}\mathrm{Tr}[z\sigma\sigma^*(t,x)z^*D^2_y\psi(y)] - \left\langle F(t,x,y,z\sigma(t,x)), \nabla_y\psi(y)\right\rangle,$$

and $C > 0$ is any sufficiently large constant.

Before proving the main results stated above, we shall present some clarifying examples. In the first example we find a criterion such that a family of moving balls has the viability property for a given BSDE.

Example 5.92 (Control Security Tube). We consider an arbitrary function $r \in C^1([0,T];\mathbb{R}_+)$ with $r(t) > 0$ for all $t \in [0,T]$, and we put

$$E(t) = \{y \in \mathbb{R}^m : |y| \le r(t)\}, \quad t \in [0,T].$$

Then the square-distance function is

$$d^2_{E(t)}(y) = h_0(t,y) = ((|y| - r(t))^+)^2,$$

and, for $|y| > r(t)$, the operator $\mathcal{A}_z(t)$ applied to h_0 at (t, y) takes the form

$$\mathcal{A}_z(t)h_0(t, y) = \frac{|y| - r(t)}{|y|} |z\sigma(t, x)|^2 + \frac{r(t)}{|y|^3} |(z\sigma(t, x))^* y|^2 - 2\frac{|y| - r(t)}{|y|} \langle F(t, x, y, z\sigma(t, x)), y \rangle .$$

Hence, the inequality in Corollary 5.91(jj) yields that, for all $(t, x, y, z) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{m\times d}$ with $|y| > r(t)$,

$$2\frac{|y| - r(t)}{|y|} \left[\langle F(t, x, y, z\sigma(t, x)), y\rangle + |y|\, r'(t)\right] \le \frac{|y| - r(t)}{|y|} \|z\sigma(t, x)\|^2 + \frac{r(t)}{|y|^3} |(z\sigma(t, x))^* y|^2 + C(|y| - r(t))^2 ,$$

from where we easily deduce the following necessary condition for the $\mathcal{E}$-viability of BSDE (5.249):

For all (t, x, y, z) with $|y| = r(t)$ and $(z\sigma(t, x))^* y = 0$,

$$2r(t)\, r'(t) + 2\langle F(t, x, y, z\sigma(t, x)), y\rangle \le \|z\sigma(t, x)\|^2 . \tag{5.255}$$

If the assumption $(AV_1\text{-}i)$ is replaced by

$$i')\qquad |F(t, x, y, z)| \le L(1 + |y|) ,$$

for all (t, x, y, z), then this condition is not only necessary but also sufficient as the following argument proves. We fix any $(t, x, y, z) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{m\times d}$ with $|y| > r(t)$, and for simplicity of notation we put $\overline{y} = |y|^{-1} r(t)\, y$ and, for $1 \le j \le m$

- $u_j = (z\sigma(t, x))_j = \sum_{i=1}^{d} z_{\cdot,i}\sigma_{i,j}(t, x)$,
- $\hat{u}_j = |y|^{-2} \langle u_j, y\rangle y, \quad u_j^{\perp} = u_j - \hat{u}_j$,
- $\hat{u} = (\hat{u}_1, \ldots, \hat{u}_m), \quad u^{\perp} = (u_1^{\perp}, \ldots, u_m^{\perp}) = u - \hat{u}$.

From the assumptions (AV_1) and $(AV_1\text{-}i')$ we get that, for some generic constant C which can change from line to line but does not depend on (t, x, y, z),

$$\begin{aligned}
&2\frac{|y| - r(t)}{|y|} \left(\langle F(t, x, y, u), y\rangle + |y|\, r'(t)\right) \\
&= 2\langle F(t, x, y, u), y - \overline{y}\rangle + 2(|y| - r(t))\, r'(t) \\
&\le 2\langle F(t, x, \overline{y}, u), y - \overline{y}\rangle + 2(|y| - r(t))\, r'(t) + C(|y| - r(t))^2 \\
&\le 2\langle F(t, x, \overline{y}, u^{\perp}), y - \overline{y}\rangle + 2(|y| - r(t))\, r'(t) + C(|y| - r(t))^2
\end{aligned}$$

$$
\begin{aligned}
&+ C\,(|y| - r\,(t))\,|\hat{u}| \\
&= 2\frac{|y| - r(t)}{|y|}\left(\left\langle F(t,x,\overline{y},u^{\perp}), y\right\rangle + C\,|y|\,|\hat{u}| + |y|\,r'(t)\right) \\
&+ C\,(|y| - r\,(t))^2 \\
&\le 2\frac{|y| - r(t)}{|y|}\left(\left\langle F(t,x,\overline{y},u^{\perp}), \overline{y}\right\rangle + |y|\,r'(t) + C\,(|y| - r\,(t))\right) \\
&+ C\,(|y| - r(t))\,\|\hat{u}\| + C\,(|y| - r\,(t))^2 .
\end{aligned}
$$

Thus, since $|y|\,r'(t) \le r\,(t)\,r'(t) + C\,(|y| - r\,(t))$, for all $(t,y) \in [0,T] \times \mathbb{R}^m$, we can deduce from (5.255) that

$$
\begin{aligned}
&2\frac{|y| - r(t)}{|y|}\left(\langle F(t,x,y,u), y\rangle + |y|\,r'(t)\right) \\
&\le \frac{|y| - r(t)}{|y|}\left\|u^{\perp}\right\|^2 + C\,(|y| - r(t))\,\|\hat{u}\| + C\,(|y| - r\,(t))^2 \\
&\le \frac{|y| - r(t)}{|y|}\|u\|^2 + \frac{r(t)}{|y|}\|\hat{u}\|^2 + C\,(|y| - r\,(t))^2 \\
&\le \frac{|y| - r(t)}{|y|}\|u\|^2 + \frac{r(t)}{|y|^3}|u^* y|^2 + C\,(|y| - r\,(t))^2 .
\end{aligned}
$$

This proves the sufficiency of (5.255).

The next example shows that, in the general case, there is no possibility of null-controllability of BSDEs; although we don't consider controlled equations, we can interpret the choice of the coefficients as controls.

Example 5.93. For any given $(t_0, y_0) \in \left]0,T\right[\times \mathbb{R}^m$, we introduce the family of moving constraints

$$
E\,(t) = \begin{cases} \mathbb{R}^m, & \text{if } t \neq t_0, \\ \{y_0\}\,, & \text{if } t = t_0. \end{cases}
$$

The associated square-distance function is of the form:

$$
h\,(t,y) = d^2_{E(t)}\,(y) = \begin{cases} 0, & \text{if } t \neq t_0, \\ |y - y_0|^2\,, & \text{if } t = t_0. \end{cases}
$$

This function is upper semicontinuous in $(t,y) \in [0,T] \times \mathbb{R}^m$, and if $t = t_0$, $y \neq y_0$, then, for every $a \in \mathbb{R}$, there is some $\varphi_a \in C^{1,2}\,([0,T] \times \mathbb{R}^m)$ with

$$
\left(\frac{\partial}{\partial t}, \nabla_y, D_y^2\right)\varphi_a\,(t_0, y) = (-a, 2\,(y - y_0)\,, 2I)
$$

such that $h - \varphi_a$ achieves a local maximum at (t_0, y). Since

$$\mathcal{A}_z(t_0; x)\,\varphi_a(t_0, y) = |z\sigma(t, x)|^2 - 2\langle F(t, x, y, z\sigma(t, x)), y - y_0\rangle$$

does not depend on $a \in \mathbb{R}$, we can choose $a > 0$ sufficiently large in order to guarantee that the inequality in Corollary 5.91(jj) is not satisfied. This shows that Eq. (5.249) cannot be $\mathcal{E}$-viable.

The proof of Theorem 5.90 reduces to that of the following two lemmas, see [15].

Lemma 5.94. *Under our standard assumptions we have the equivalence between the following statements:*

i) Equation (5.249) is $\mathcal{E}$-viable on $[0, T]$.
ii) There exists a $C > 0$ such that, for all $t, \tilde{T}$ with $0 \le t \le \tilde{T} \le T$, and for all $x \in \mathbb{R}^d$, the solution of BSDE (5.249) with time horizon $\tilde{T}$ and arbitrary Borel measurable terminal function $\kappa : \mathbb{R}^d \to \mathbb{R}^m$ of at most polynomial growth satisfies:

$$d^2_{E(t,x)}(Y_t^{t,x}) \le e^{C(\tilde{T}-t)}\mathbb{E}d^2_{E(\tilde{T}, X^{t,x}_{\tilde{T}})}(Y^{t,x}_{\tilde{T}}).$$

Lemma 5.95. *Let $Y^{t,x}$ be the solution of BSDE (5.249) with time horizon $\tilde{T}$ and arbitrary terminal function $\kappa \in C_{pol}\left([0, \tilde{T}] \times \mathbb{R}^d\right)$.*

Let C be a positive constant and $h : [0, T] \times \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}$ be an upper semicontinuous function of at most polynomial growth such that, for some positive constants $M, p > 0$,

$$|h(t, x, y') - h(t, x, y)| \le M|y - y'|(1 + |x|^p + |y|^p + |y'|^p) \tag{5.256}$$

for all $(t, x) \in [0, T] \times \mathbb{R}^d$ and all $y, y' \in \mathbb{R}^m$. Then the following assertions are equivalent:

i) For all $x \in \mathbb{R}^d$ and $t, \tilde{T}$ with $0 \le t \le \tilde{T} \le T$, it holds that

$$h(t, x, Y_t^{t,x}) \le e^{C(\tilde{T}-t)}\mathbb{E}h(\tilde{T}, X^{t,x}_{\tilde{T}}, Y^{t,x}_{\tilde{T}}).$$

ii) For every $z \in \mathbb{R}^{m\times d}$, the function h is a viscosity sub-solution of the equation

$$\frac{\partial V(t, x, y)}{\partial t} + \mathcal{L}_z(t)V(t, x, y) + Ch(t, x, y) = 0 \text{ on } [0, \tilde{T}]\times\mathbb{R}^d\times\mathbb{R}^m. \tag{5.257}$$

Recall that $\mathcal{L}_z(t)$ is defined in (5.254).

Proof of Lemma 5.94. We first remark that (ii) obviously implies (i). Thus, it only remains to show that (ii) can be deduced from (i). Let $\tilde{T} \in [0, T]$, $(t, x) \in [0, \tilde{T}] \times \mathbb{R}^d$. For simplicity of notation we put $u(t, x) = Y_t^{t,x}$, and we select a Borel measurable mapping $\hat{u} : [t, T] \times \mathbb{R}^d \to \mathbb{R}^m$ such that $\hat{u}(s, x') \in \prod_{E(s,x')}(u(s, x'))$, for all $(s, x') \in [t, T] \times \mathbb{R}^d$. Recall that $\prod_{E(s,x')}(z) = \{y \in E(s, x') : |z - y| = d_{E(s,x')}(z)\}$. Then, since Eq. (1.2) is $\mathcal{E}$-viable, the unique square integrable adapted solution $(\tilde{Y}^{t,x}, \tilde{Z}^{t,x})$ of the BSDE

$$\tilde{Y}_s^{t,x} = \hat{u}(\tilde{T}, X_{\tilde{T}}^{t,x}) + \int_s^{\tilde{T}} F(r, X_r^{t,x}, \tilde{Y}_r^{t,x}, \tilde{Z}_r^{t,x})dr - \int_s^{\tilde{T}} \tilde{Z}_r^{t,x} dW_r, \quad s \in [t, T],$$

is such that $\tilde{Y}_s^{t,x} \in E(s, X_s^{t,x})$, $t \le s \le T$, $\mathbb{P}$-$a.s.$

Consequently, $\mathbb{E}d^2_{E(s,X_s^{t,x})}(Y_s^{t,x}) \le \mathbb{E}|Y_s^{t,x} - \tilde{Y}_s^{t,x}|^2$, and a standard estimate of $\mathbb{E}|Y_s^{tx} - \tilde{Y}_s^{tx}|^2$ involving Itô's formula and Gronwall's formula, yields the desired result:

$$\begin{aligned}
&\mathbb{E}d^2_{E(s,X_s^{t,x})}(Y_s^{t,x}) \\
&\le \mathbb{E}|Y_s^{t,x} - \tilde{Y}_s^{t,x}|^2 \le e^{C(\tilde{T}-s)}\mathbb{E}|Y_{\tilde{T}}^{t,x} - \tilde{Y}_{\tilde{T}}^{t,x}|^2 \\
&= e^{C(\tilde{T}-s)}\mathbb{E}|Y_{\tilde{T}}^{t,x} - \hat{u}(\tilde{T}, X_{\tilde{T}}^{t,x})|^2 = e^{C(\tilde{T}-s)}\mathbb{E}d^2_{E(\tilde{T},X_{\tilde{T}}^{t,x})}(Y_{\tilde{T}}^{t,x}),
\end{aligned}$$

$0 \le t \le s \le \tilde{T} \le T$, $x \in \mathbb{R}^d$. This completes the proof of Lemma 5.94.

We now come to the proof of Lemma 5.95.

Proof of Lemma 5.95. We first show that, under the assumption (i), we have (ii). To this end we fix an arbitrary function $\varphi : [0, T] \times \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}$ of class $C_{pol}^{1,2,2}$ and a point $(t, x, y) \in (0, T) \times \mathbb{R}^d \times \mathbb{R}^m$ such that the mapping $h - \varphi$ achieves a global maximum at (t, x, y). For an arbitrary but fixed $z \in \mathbb{R}^{m\times d}$ we denote by $(Y^\varepsilon, Z^\varepsilon) \in S_m^2[t, t + \varepsilon] \times \Lambda^2_{m\times k}(t, t + \varepsilon)$ the unique solution of the BSDE

$$Y_s^\varepsilon = \kappa_\varepsilon(X_{t+\varepsilon}^{t,x}) + \int_s^{t+\varepsilon} F(r, X_r^{t,x}, Y_r^\varepsilon, Z_r^\varepsilon)dr - \int_s^{t+\varepsilon} Z_r^\varepsilon dW_r, \; t \le s \le t + \varepsilon,$$

where

$$\kappa_\varepsilon(x') = y + z(x' - x) - \varepsilon z b(t, x) - \varepsilon F(t, x, y, z\sigma(t, x)).$$

From the assumption made on h in assertion (i), we obtain

$$\begin{aligned}
&h(t, x, Y_t^\varepsilon) - h(t, x, y) \\
&\quad \le e^{C\varepsilon}[\mathbb{E}h(t + \varepsilon, X_{t+\varepsilon}^{t,x}, Y_{t+\varepsilon}^\varepsilon) - h(t, x, y))] + (e^{C\varepsilon} - 1)h(t, x, y) \\
&\quad \le e^{C\varepsilon}[\mathbb{E}\varphi(t + \varepsilon, X_{t+\varepsilon}^{t,x}, Y_{t+\varepsilon}^\varepsilon) - \varphi(t, x, y)] + (e^{C\varepsilon} - 1)h(t, x, y).
\end{aligned}$$

Then, with the help of a Taylor expansion of φ, we get

$$\begin{aligned}
&\frac{1}{\varepsilon}\left(h(t,x,Y_t^\varepsilon)-h(t,x,y)\right)\\
&\le e^{C\varepsilon}[\frac{\partial\varphi}{\partial t}(t,x,y)+\frac{1}{\varepsilon}\mathbb{E}\left\langle\nabla_{(x,y)}\varphi(t,x,y),\begin{pmatrix}X_{t+\varepsilon}^{t,x}-x\\ Y_{t+\varepsilon}^{\varepsilon}-y\end{pmatrix}\right\rangle+\\
&+\frac{1}{2\varepsilon}\mathbb{E}\left\langle D_{(x,y)}^2\varphi(t,x,y)\begin{pmatrix}X_{t+\varepsilon}^{t,x}-x\\ Y_{t+\varepsilon}^{\varepsilon}-y\end{pmatrix},\begin{pmatrix}X_{t+\varepsilon}^{t,x}-x\\ Y_{t+\varepsilon}^{\varepsilon}-y\end{pmatrix}\right\rangle+\\
&+\frac{1}{\varepsilon}\mathbb{E}\gamma^{t,x,y}(t+\varepsilon,X_{t+\varepsilon}^{t,x},Y_{t+\varepsilon}^{\varepsilon})]+\frac{e^{C\varepsilon}-1}{\varepsilon}h(t,x,y),
\end{aligned}\tag{5.258}$$

where,

$$\begin{aligned}
&\gamma^{t,x,y}(t',x',y')\\
&=\int_0^1\left(\frac{\partial}{\partial t}\varphi\left(t+\theta\left(t'-t\right),x',y'\right)-\frac{\partial}{\partial t}\varphi(t,x,y)\right)\left(t'-t\right)d\theta\\
&+\int_0^1\int_0^\theta\Big\langle\Big(D_{(x,y)}^2\varphi\left(t,x+\vartheta\left(x'-x\right),y+\vartheta\left(y'-y\right)\right)-D_{(x,y)}^2\varphi(t,x,y)\Big)\\
&\qquad\qquad\begin{pmatrix}x'-x\\ y'-y\end{pmatrix},\begin{pmatrix}x'-x\\ y'-y\end{pmatrix}\Big\rangle d\vartheta d\theta.
\end{aligned}$$

Note that

$$\begin{aligned}
\begin{pmatrix}X_{t+\varepsilon}^{t,x}-x\\ Y_{t+\varepsilon}^{\varepsilon}-y\end{pmatrix}&=\int_t^{t+\varepsilon}\begin{pmatrix}b(r,X_r^{t,x})\\ z(b(r,X_r^{t,x})-b(t,x))-F(t,x,y,z\sigma(t,x))\end{pmatrix}dr\\
&+\int_t^{t+\varepsilon}\begin{pmatrix}\sigma(r,X_r^{t,x})\\ z\sigma(r,X_r^{t,x})\end{pmatrix}dW_r.
\end{aligned}$$

Hence,

$$\lim_{\varepsilon\to0}\frac{1}{\varepsilon}\mathbb{E}\begin{pmatrix}X_{t+\varepsilon}^{t,x}-x\\ Y_{t+\varepsilon}^{\varepsilon}-y\end{pmatrix}=\begin{pmatrix}b(t,x)\\ -F(t,x,y,z\sigma(t,x))\end{pmatrix}$$

and

$$\begin{aligned}
&\lim_{\varepsilon\to0}\frac{1}{\varepsilon}\mathbb{E}\left\langle D_{(x,y)}^2\varphi(t,x,y)\begin{pmatrix}X_{t+\varepsilon}^{t,x}-x\\ Y_{t+\varepsilon}^{\varepsilon}-y\end{pmatrix},\begin{pmatrix}X_{t+\varepsilon}^{t,x}-x\\ Y_{t+\varepsilon}^{\varepsilon}-y\end{pmatrix}\right\rangle\\
&=\frac{1}{2}\mathrm{Tr}\left((\sigma,z\sigma)\,(\sigma,z\sigma)^*\,(t,x)\,D_{(x,y)}^2\varphi(t,x,y)\right).
\end{aligned}$$

Moreover, from the assumptions on h,

$$\frac{1}{\varepsilon}|h(t,x,Y_t^\varepsilon) - h(t,x,y)| \le \frac{M}{\varepsilon}|Y_t^\varepsilon - y|(1+|x|^p+|y|^p+|Y_t^\varepsilon|^p).$$

Therefore, applying the following auxiliary lemma, the proof of which will be given at the end of this section, we can take the limit as $\varepsilon \to 0$ in (5.258) and obtain assertion (ii).

Lemma 5.96. *Under the assumptions of Lemma 5.95, and with the notations introduced above, we have*

$$\begin{aligned} a)\ & \lim_{\varepsilon\searrow 0}\frac{1}{\varepsilon^2}|Y_t^\varepsilon - y|^2 = 0,\\ b)\ & \lim_{\varepsilon\searrow 0}\frac{1}{\varepsilon}\mathbb{E}|\gamma^{t,x,y}(\varepsilon, X_{t+\varepsilon}^{t,x}, Y_{t+\varepsilon}^\varepsilon)| = 0,\end{aligned}$$

for all $(t,x,y)\in[0,T]\times\mathbb{R}^d\times\mathbb{R}^m$.

We shall now prove the reverse implication: Under the assumption that (ii) holds we have to show the validity of (i). For this we first remark that, for any continuous function $\Phi : [0,T]\times\mathbb{R}^d\times\mathbb{R}^m\times\mathbb{R}\times\mathbb{R}^{m+n}\times\mathbb{S}^{m+n}\times\mathbb{R}^{m\times d}\to\mathbb{R}$ satisfying the standard assumptions of monotonicity with respect to the $\mathbb{R}^{m+n}$-variable and of degenerate ellipticity with respect to the $\mathbb{S}^{m+n}$-variable (see Annex D),

$$(\alpha)\ \left|\begin{array}{l} h(t,x,y) \text{ is a viscosity sub-solution of the PDE}\\ \Phi(t,x,y,\partial_t h(t,x,y),\nabla_{(x,y)}h(t,x,y),D^2_{(x,y)}h(t,x,y);z)=0,\\ \hfill \text{for all } z\in\mathbb{R}^{m\times d}\end{array}\right.$$

if and only if

$$(\beta)\ \left|\begin{array}{l} h(t,x,y) \text{ is a viscosity sub-solution of the PDE}\\ \Phi(t,x,y,\partial_t h(t,x,y),\nabla_{(x,y)}h(t,x,y),D^2_{(x,y)}h(t,x,y);g(t,x))=0,\\ \hfill \text{for all } g\in C_{pol}([0,T]\times\mathbb{R}^d;\mathbb{R}^{m\times d}).\end{array}\right.$$

Indeed, in order to see that (β) implies (α), it suffices to choose $g\in C_{pol}([0,T]\times\mathbb{R}^d;\mathbb{R}^{m\times d})$ with $g(t,x)=z\in\mathbb{R}^{m\times d}$. On the other hand, to get the necessity of (β) under (α), we remark that, for all test functions $\varphi\in C^{1,2,2}$ for which $h-\varphi$ achieves a local maximum at (t,x,y), and with the notation

$$(a,p,S)=\left(\frac{\partial}{\partial t}\varphi,\nabla_{(x,y)}\varphi,D^2_{(x,y)}\varphi\right)(t,x,y),$$

we have that $\Phi(t,x,y,a,p,S;z)\ge 0$, for all $z\in\mathbb{R}^{m\times d}$, and hence also for $z=g(t,x)$, where g runs over the set of functions belonging to $C_{pol}([0,T]\times\mathbb{R}^d;$

$\mathbb{R}^{m\times d}$). We now fix any $g \in C_{pol}([0,T]\times\mathbb{R}^d;\mathbb{R}^{m\times d})$ and consider the unique square integrable adapted solution $(X, \overline{Y}^{t,x,y})$ of the (forward) SDE

$$\begin{pmatrix} X_s^{t,x} \\ \overline{Y}_s^{t,x,y} \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \int_t^s \begin{pmatrix} b(r, X_r^{t,x}) \\ -F(r, X_r^{t,x}, \overline{Y}_r^{t,x,y}, g(r, X_r^{t,x})\sigma(r, X_r^{t,x})) \end{pmatrix} dr \\ + \int_t^s \begin{pmatrix} \sigma(r, X_r^{t,x}) \\ g(r, X_r^{t,x})\sigma(r, X_r^{t,x}) \end{pmatrix} dW_r, \quad s \in [t,T].$$

Of course, here the process $X^{t,x}$ is nothing else than the unique solution of SDE (5.248). Moreover, we denote by $\left(\tilde{Y}_{k,\cdot}^{t,x,y}, \tilde{Z}_{k,\cdot}^{t,x,y}\right) \in S_m^2\left[t,\tilde{T}\right] \times \Lambda_{m\times k}^2\left(t,\tilde{T}\right)$ the unique solution of the BSDE

$$\tilde{Y}_{k,s}^{t,x,y} = h_k(\tilde{T}, X_{\tilde{T}}^{t,x}, \overline{Y}_{\tilde{T}}^{t,x,y}) + C\int_s^{\tilde{T}} h_k(r, X_r^{t,x}, \overline{Y}_r^{t,x,y})dr \\ -\int_s^{\tilde{T}} \tilde{Z}_{k,r}^{t,x,y} dW_r, \quad s \in [t,\tilde{T}],$$

where $\tilde{T} \in [0,T]$ and $(h_k)_{k\geq 1} \subset C_{pol}([0,T]\times\mathbb{R}^d\times\mathbb{R}^m)$ is a monotonically decreasing sequence of continuous functions with at most polynomial growth, such that its pointwise limit is equal to h. Then the function

$$V_k(t,x,y) = \tilde{Y}_{k,t}^{t,x,y}, \quad (t,x,y) \in [0,\tilde{T}]\times\mathbb{R}^d\times\mathbb{R}^m,$$

is a continuous viscosity solution of the equation

$$\begin{cases} \dfrac{\partial V_k(t,x,y)}{\partial t} + \mathcal{L}_{g(t,x)}(t)V_k(t,x,y) + Ch_k(t,x,y) = 0, \\ V_k(\tilde{T},x,y) = h_k(\tilde{T},x,y), \quad (x,y) \in \mathbb{R}^d\times\mathbb{R}^m, \end{cases}$$

and it is the unique solution in the class of continuous functions of at most polynomial growth. We also note that, by the Markov property,

$$\tilde{Y}_{k,s}^{t,x,y} = V_k(t, X_s^{t,x}, \overline{Y}_s^{t,x,y}), s \in [t,\tilde{T}].$$

Since, due to assumption (*ii*), h is an upper semicontinuous viscosity sub-solution of at most polynomial growth of the above PDE, we know that h must be smaller than or equal to the viscosity solution V_k. Thus,

$$\begin{aligned} &\mathbb{E}h(s, X_s^{t,x}, \overline{Y}_s^{t,x,y}) \\ &\quad \leq \mathbb{E}V_k(s, X^{t,x}, \overline{Y}_s^{t,x,y}) \\ &\quad = \mathbb{E}h_k(\tilde{T}, X_{\tilde{T}}^{t,x}, \overline{Y}_{\tilde{T}}^{t,x,y}) + C\int_s^{\tilde{T}} \mathbb{E}h_k(r, X_r^{t,x}, \overline{Y}_r^{t,x,y})dr,\ s \in [t,\tilde{T}], \end{aligned}$$

then, by passing to the limit as $k \to \infty$ and applying Gronwall's inequality, we obtain the following estimate

$$\mathbb{E}h(s, X_s^{t,x}, \overline{Y}_s^{t,x,y}) \le e^{C(\tilde{T}-s)}\mathbb{E}h(\tilde{T}, X_{\tilde{T}}^{t,x}, \overline{Y}_{\tilde{T}}^{t,x,y}).$$

Setting $s = t$ and $y = u(t,x) = Y_t^{t,x}$ and using the assumption (5.256) we obtain for some positive constant C_1,

$$\begin{aligned}
&h(t,x,u(t,x)) \\
&\quad \le e^{C(\tilde{T}-t)}\mathbb{E}h(\tilde{T}, X_{\tilde{T}}^{t,x}, \overline{Y}_{\tilde{T}}^{t,x,y}) \\
&\quad \le e^{C(\tilde{T}-t)}\Big[\mathbb{E}h(\tilde{T}, X_{\tilde{T}}^{t,x}, Y_{\tilde{T}}^{t,x}) \\
&\qquad +M\ \mathbb{E}\left(|\overline{Y}_{\tilde{T}}^{t,x,y} - Y_{\tilde{T}}^{t,x}|(1+|X_{\tilde{T}}^{t,x}|^p + |\overline{Y}_{\tilde{T}}^{t,x,y}|^p + |Y_{\tilde{T}}^{t,x}|^p)\right)\Big] \\
&\le e^{C(\tilde{T}-t)}\Bigg[\mathbb{E}h(\tilde{T}, X_{\tilde{T}}^{t,x}, Y_{\tilde{T}}^{t,x}) \\
&\qquad +C_1(1+|x|^{pq}+|y|^p)\left(\mathbb{E}\int_t^{\tilde{T}} \left|Z_r^{t,x} - (g\sigma)(r, X_r^{t,x})\right|^2 dr\right)^{1/2}\Bigg]
\end{aligned}$$

for all $g \in C_{pol}([0,T]\times\mathbb{R}^d;\mathbb{R}^{m\times d})$. Since by a result from [30] (Theorem 4.1) there is a Borel measurable function $v : [0,\tilde{T}]\times\mathbb{R}^d \to \mathbb{R}^{m\times d}$ such that

$$Z_s^{t,x} = (v\sigma)\left(s, X_s^{t,x}\right),\ s \in [t,\tilde{T}],\ ds\, d\mathbb{P}-a.e.,$$

we deduce that by density (and Lebesgue's dominated convergence theorem)

$$h(t,x,u(t,x)) \le e^{C(\tilde{T}-t)}\mathbb{E}h(\tilde{T}, X_{\tilde{T}}^{t,x}, Y_{\tilde{T}}^{t,x}).$$

Since this result holds true for all $x \in \mathbb{R}^d$, $0 \le t \le \tilde{T} \le T$, we have proved (i).

Let us now prove Lemma 5.96.

Proof of Lemma 5.96. We first prove part a) of the lemma. Obviously, we have that

$$\begin{aligned}
Y_t^\varepsilon &= \kappa_\varepsilon(X_{t+\varepsilon}^{t,x}) + \int_t^{t+\varepsilon} F(r, X_r^{t,x}, Y_r^\varepsilon, Z_r^\varepsilon)dr - \int_t^{t+\varepsilon} Z_r^\varepsilon dW_r \\
&= y + \int_t^{t+\varepsilon} z\left(b\left(r, X_r^{t,x}\right) - b(t,x)\right) dr \\
&+\int_t^{t+\varepsilon} \left(F(r, X_r^{t,x}, Y_r^\varepsilon, Z_r^\varepsilon) - F(t,x,y,z\sigma(t,x)\right) dr \\
&-\int_t^{t+\varepsilon} \left[Z_r^\varepsilon - z\sigma(r, X_r^{t,x})\right] dW_r.
\end{aligned}$$

Thus for $0 < \varepsilon < \dfrac{1}{6L^2}$,

$$
\begin{aligned}
&|Y_t^\varepsilon - y|^2 + \mathbb{E}\int_t^{t+\varepsilon} \left|Z_r^\varepsilon - z\sigma(r, X_r^{t,x})\right|^2 dr \\
&\le 3\varepsilon\,|z|^2 \int_t^{t+\varepsilon} \mathbb{E}\left|b(r, X_r^{t,x}) - b(t,x)\right|^2 dr \\
&\quad +3\varepsilon\mathbb{E}\int_t^{t+\varepsilon} \left|F(r, X_r^{t,x}, Y_r^\varepsilon, Z_r^\varepsilon) - F(r, X_r^{t,x}, Y_r^\varepsilon, z\sigma(t, X_r^{t,x})\right|^2 dr \\
&\quad +3\varepsilon\mathbb{E}\int_t^{t+\varepsilon} \left|F(r, X_r^{t,x}, Y_r^\varepsilon, z\sigma(t, X_r^{t,x})) - F(t, x, y, z\sigma(t,x)\right|^2 dr \\
&\le 3\varepsilon\,|z|^2 \int_t^{t+\varepsilon} \mathbb{E}\left|b(r, X_r^{t,x}) - b(t,x)\right|^2 dr + \frac{1}{2}\mathbb{E}\int_t^{t+\varepsilon} \left|Z_r^\varepsilon - z\sigma(t, X_r^{t,x})\right|^2 dr \\
&\quad +3\varepsilon\mathbb{E}\int_t^{t+\varepsilon} \left|F(r, X_r^{t,x}, Y_r^\varepsilon, z\sigma(t, X_r^{t,x})) - F(t, x, y, z\sigma(t,x)\right|^2 dr,
\end{aligned}
$$

which yields

$$
\limsup_{\varepsilon\searrow 0} \frac{1}{\varepsilon^2}\mathbb{E}\,|Y_t^\varepsilon - y|^2 + \limsup_{\varepsilon\searrow 0} \frac{1}{2\varepsilon^2}\mathbb{E}\int_t^{t+\varepsilon} \left|Z_r^\varepsilon - z\sigma(r, X_r^{tx})\right|^2 dr \le 0.
$$

Finally, the proof of part b) of Lemma 5.96 uses the same argument as that of Lemma 4.82. The only difference is that the role of the diffusion process $X^{t,x}$ in the proof of Lemma 4.82 is now replaced by that of the pair $(X^{t,x}, Y^\varepsilon)$.

5.10 Exercises

Without further mention, $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\ge 0})$ will be a stochastic basis, $\{B_t : t \ge 0\}$ will be a k-dimensional Brownian motion with respect to this basis and $\mathcal{F}_t = \mathcal{F}_t^B$ for all $t \ge 0$.

Exercise 5.1. Consider the BSDE

$$
Y_t = \eta + \int_t^T \Phi\left(s, Y_s, Z_s\right) dQ_s - \int_t^T Z_s dB_s, \tag{5.259}
$$

under the assumptions (5.41). Let

$$
V_t = \int_0^t L_s dQ_s + \frac{1}{n_p}\int_0^t (\ell_s)^2\, ds.
$$

Show that if $p \geq 2$ and for all $\delta \geq 0$

$$\mathbb{E}\ \left|e^{\delta V_T}\eta\right|^p + \mathbb{E}\Big(\int_0^T e^{\delta V_t}\left|\Phi(t,0,0)\right| dQ_t\Big)^p < \infty,$$

then the BSDE (5.259) has a unique solution $(Y,Z) \in S_m^0[0,T] \times \Lambda_{m\times k}^0(0,T)$ such that

$$\mathbb{E}\sup_{s\in[0,T]} e^{\delta p V_s}\left|Y_s\right|^p + +\mathbb{E}\left(\int_0^T e^{2\delta V_s}\left|Z_s\right|^2 ds\right)^{p/2} < \infty, \text{ for all } \delta \geq 0.$$

Remark. Note that our assumptions hold in particular if both V_t has exponential moments of all orders (e.g. the tail of its law behaves like that of a Gaussian random variable) and $|\eta| + \mathbb{E}\int_0^T |\Phi(t,0,0)| dQ_t$ has a finite moment of some order $p > 1$.

Exercise 5.2 (g-Expectation). Consider the BSDE: $\mathbb{P}$-a.s., for all $t \in [0,T]$

$$Y_t = \eta + \int_t^T g(s,Y_s,Z_s)\,ds - \int_t^T \langle Z_s, dB_s\rangle\,, \tag{5.260}$$

where we assume:

(*i*) $\eta \in L^p(\Omega,\mathcal{F}_T,\mathbb{P};\mathbb{R})$, $p > 1$;
(*ii*) *for every* $(y,z) \in \mathbb{R}\times\mathbb{R}^k$, *the function* $g(\cdot,\cdot,y,z) : \Omega\times[0,T] \to \mathbb{R}$ *is* $\mathcal{P}$-*measurable*;
(*iii*) g satisfies the assumptions of Theorem 5.27 (F replaced by g) and $g(t,y,0) = 0$ for all $y \in \mathbb{R}$, a.e. $t \in [0,T]$.

Then by Theorem 5.17 the BSDE (5.260) has a unique solution $(Y,Z) \in S_1^p[0,T] \times \Lambda_k^p(0,T)$. Moreover if $\tau : \Omega \to [0,T]$ is a stopping time and $\eta \in L^p(\Omega,\mathcal{F}_\tau,\mathbb{P};\mathbb{R})$ then $(Y_t,Z_t) = (\eta,0)$ for all $t \geq \tau$.

Define the g-expectation of η by $\mathbf{E}_g(\eta) \overset{def}{=} Y_0$ and the conditional g-expectation of η with respect to $\mathcal{F}_t$ by $\mathbf{E}_g(\eta|\mathcal{F}_t) \overset{def}{=} Y_t$. Clearly $\mathbf{E}_0(\eta) = \mathbb{E}\eta$ and $\mathbf{E}_0(\eta|\mathcal{F}_t) = \mathbb{E}(\eta|\mathcal{F}_t)$.

Show that:

1. $\mathbf{E}_g(a) = a$, for all $a \in \mathbb{R}$.
2. $\eta_1 \leq \eta_2$, $\mathbb{P}$-a.s. $\implies$ $\mathbf{E}_g(\eta_1) \leq \mathbf{E}_g(\eta_2)$.
3. $\eta_1 \leq \eta_2$, $\mathbb{P}$-a.s. and $\mathbf{E}_g(\eta_1) = \mathbf{E}_g(\eta_2)$ $\implies$ $\eta_1 = \eta_2$, $\mathbb{P}$-a.s.
4. If $g(t,\cdot,\cdot) : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is a convex function, a.e. $t \in [0,T]$, then $\mathbf{E}_g : L^p(\Omega,\mathcal{F}_T,\mathbb{P}) \to \mathbb{R}$ is convex, too.
5. Let $U \in L^p(\Omega,\mathcal{F}_t,\mathbb{P})$. Then $\mathbf{E}_g(1_A\eta) = \mathbf{E}_g(1_A U)$, for all $A \in \mathcal{F}_t$, if and only if $U = Y_t$.
6. $\mathbf{E}_g(a|\mathcal{F}_t) = a$, for all $a \in \mathbb{R}$.
7. $\mathbf{E}_g(\eta|\mathcal{F}_t) = \eta$, for all $\eta \in L^p(\Omega,\mathcal{F}_t,\mathbb{P})$.

8. $\eta_1 \le \eta_2$, $\mathbb{P}$-a.s. $\implies$ $\mathbf{E}_g(\eta_1|\mathcal{F}_t) \le \mathbf{E}_g(\eta_2|\mathcal{F}_t)$, $\mathbb{P}$-a.s.
9. $\mathbf{E}_g(1_A\eta|\mathcal{F}_t) = 1_A\mathbf{E}_g(\eta|\mathcal{F}_t)$, for all $A \in \mathcal{F}_t$.

Exercise 5.3 (Peano Type Result). Consider the BSDE

$$Y_t = \eta + \int_t^T G(s, Y_s, Z_s)\, ds - \int_t^T \langle Z_s, dB_s\rangle,$$

where $\eta \in L^p(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$, $p \ge 2$, and $G : [0,T] \times \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}$ is a function such that

- $G(\cdot, x, z) : [0,T] \to \mathbb{R}$ is measurable for all $x \in \mathbb{R}$ and $z \in \mathbb{R}^k$,
- $G(t, \cdot, \cdot) : \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}$ is continuous for all $t \in [0,T]$,
- there exists an $L > 0$ such that for all $(t, y, z) \in [0,T] \times \mathbb{R} \times \mathbb{R}^k$,

$$|G(t, y, z)| \le L(1 + |y| + |z|).$$

Under these conditions we shall prove that the BSDE (5.260) has at least one solution $(Y, Z) \in S^p[0,T] \times \Lambda_k^p(0,T)$.

Let $0 < \varepsilon \le \varepsilon_0 = 1 \wedge (1/L)$ and $G_\varepsilon : [0,T] \times \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}$,

$$G_\varepsilon(t, y, z) = \inf\left\{G(t, u, v) + \frac{1}{\varepsilon}|y - u| + \frac{1}{\varepsilon}|z - v| : (u, v) \in \mathbb{R} \times \mathbb{R}^k\right\}.$$

Prove that:

1. For all $t \in [0,T]$, $y, y' \in \mathbb{R}$ and $z, z' \in \mathbb{R}^k$:
   (a) $|G_\varepsilon(t, y, z)| \le L(1 + |y| + |z|)$;
   (b) $|G_\varepsilon(t, y, z) - G_\varepsilon(t, y', z')| \le \frac{1}{\varepsilon}(|y - y'| + |z - z'|)$;
   (c) $yG_\varepsilon(t, y, z) \le L|y| + \left(L + L^2\right)|y|^2 + \frac{1}{4}|z|^2$;
   (d) $0 < \delta < \varepsilon \implies G_\delta(t, y, z) \ge G_\varepsilon(t, y, z)$;
   (e) if $\lim_{\varepsilon\to 0}(y_\varepsilon, z_\varepsilon) = (y, z)$, then $\lim_{\varepsilon\to 0} G_\varepsilon(t, y_\varepsilon, z_\varepsilon) = G(t, y, z)$.
2. The BSDEs

$$Y_t^\varepsilon = \eta + \int_t^T G_\varepsilon\left(s, Y_s^\varepsilon, Z_s^\varepsilon\right) ds - \int_t^T Z_s^\varepsilon dB_s,$$

$$U_t = \eta + \int_t^T L(1 + |U_s| + |V_s|)\, ds - \int_t^T Z_s dB_s$$

have unique solutions $(Y^\varepsilon, Z^\varepsilon)$, $(U, V) \in S^p[0,T] \times \Lambda_{m\times k}^p(0,T)$ and:

(a)

$$\mathbb{E}^{\mathcal{F}_t}\left(\sup_{s\in[t,T]}|Y_s^\varepsilon|^p\right)+\mathbb{E}^{\mathcal{F}_t}\left(\int_t^T|Z_s^\varepsilon|^2\,ds\right)^{p/2}$$
$$\le C_p\exp\left[\left(L+L^2\right)(T-t)\right]\left[\mathbb{E}^{\mathcal{F}_t}|\eta|^p+L^p(T-t)^p\right]$$

where C_p is a constant depending only on p.

(b) For all $0<\delta<\varepsilon\le\varepsilon_0=1\wedge(1/L)$, $\mathbb{P}$-a.s.,

$$Y_t^{\varepsilon_0}\le Y_t^\varepsilon\le Y_t^\delta\le U_t,\quad \text{for all } t\in[0,T],$$

and there exists a $Y\in S^p[0,T]$ such that

$$\lim_{\varepsilon\to 0}\mathbb{E}\left(\sup_{s\in[0,T]}|Y_s^\varepsilon-Y_s|^p\right)=0.$$

(c) There exists a $Z\in\Lambda^p_{m\times k}(0,T)$ such that

$$\lim_{\varepsilon\to 0}\mathbb{E}\left(\int_0^T|Z_s^\varepsilon-Z_s|^2\,ds\right)^{p/2}=0.$$

Exercise 5.4 (BSDE Reflected Above 0**).** Let $\xi\in L^2\left(\Omega,\mathcal{F}_T^B,\mathbb{P};\mathbb{R}\right)$, where $\{B_t,\ 0\le t\le T\}$ is a k-dimensional BM, and $F:\mathbb{R}\times\mathbb{R}^k\to\mathbb{R}$ be a Lipschitz continuous mapping. Consider for each $n\in\mathbb{N}$ the solution $\{(Y_t^n,Z_t^n),\ 0\le t\le T\}$ of the BSDE

$$Y_t^n=\xi+\int_t^T F(Y_s^n,Z_s^n)ds+n\int_t^T(Y_s^n)^-ds-\int_0^t\langle Z_s^n,dB_s\rangle,$$

and let $K_t^n=n\int_0^t(Y_s^n)^-ds$.

1. Show that $Y_t^{n+1}\ge Y_t^n$, $0\le t\le T$.
2. Show that

$$\sup_n\mathbb{E}\left(\sup_{0\le t\le T}|Y_t^n|^2\right)<\infty.$$

3. Deduce that there exists a progressively measurable process $\{Y_t,\ 0\le t\le T\}$ such that $Y_t^n\to Y_t$ a.s. for all $t\in[0,T]$, and

$$\mathbb{E}\left(\sup_{0\le t\le T}|Y_t|^2\right)<\infty.$$

4. Show that $Y_t^n \geq \tilde{Y}_t^n$, where $\{\tilde{Y}_t^n,\ 0 \leq t \leq T\}$ solves the BSDE

$$\tilde{Y}_t^n = \xi + \int_t^T F(\tilde{Y}_s^n, \tilde{Z}_s^n)ds - n\int_t^T \tilde{Y}_s^n ds - \int_0^t \langle \tilde{Z}_s^n, dB_s \rangle .$$

5. Identify $\lim_{n\to\infty} \tilde{Y}_t^n$ and deduce that $Y_t \geq 0$, $0 \leq t \leq T$, a.s., and (with the help of Dini's theorem) that $\sup_{0\leq t\leq T}(Y_t^n)^- \to 0$ in mean square.
6. Show that $\{Z_t^n,\ 0 \leq t \leq T\}$ is a Cauchy sequence in $\Lambda_k^2(0,T)$. Hint: check that

$$\int_t^T (Y_s^n - Y_s^p)(dK_s^n - dK_s^p) \leq \int_t^T \left[(Y_s^p)^- dK_s^n + (Y_s^n)^- dK_s^p\right] \to 0.$$

7. Deduce that K_t^n converges in probability to a progressively measurable increasing continuous stochastic process K_t.
8. Show that the just constructed triple $\{(X_t, Z_t, K_t),\ 0 \leq t \leq T\}$ is a unique progressively measurable solution of the reflected BSDE: for all $t \in [0,T]$, $\mathbb{P}$-a.s.

$$\begin{cases} (i) & Y \text{ is a continuous stochastic process, } Y_t \geq 0, \\ (ii) & K \text{ is c.i.s.p., } \displaystyle\int_0^T Y_s dK_s = 0, \\ (iii) & \mathbb{E}\displaystyle\int_0^T |Z_s|^2 dt < \infty, \\ (iv) & Y_t = \xi + \displaystyle\int_t^T F(Y_s, Z_s)ds + K_T - K_t - \int_t^T \langle Z_s, dB_s \rangle . \end{cases}$$

9. With the help of Tanaka's formula applied to $(Y_t)^+ = Y_t$, show that in the sense of inequality between measures,

$$0 \leq dK_t \leq \mathbf{1}_{\{Y_t=0\}} \left[F(Y_t, Z_t)\right]^- dt.$$

   Deduce that K is absolutely continuous.
10. Show that the points 2–9 constitute a particular case of Theorem 5.52.

Exercise 5.5. Let $\eta \in L^0(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$ be such that $0 \leq \eta \leq 1$, $\mathbb{P}$-a.s. Prove that the BSDE

$$Y_t = \eta + \int_t^T Y_s(1 - Y_s)\, ds - \int_t^T \langle Z_s, dB_s \rangle$$

has a unique solution $(Y, Z) \in S_1^2[0,T] \times \Lambda_k^2(0,T)$. Moreover

$$\mathbb{E}\left(\int_0^T |Z_s|^2 ds\right)^{p/2} < \infty, \quad \text{for all } p > 0,$$

$0 \leq Y_t \leq 1, \quad \mathbb{P} - a.s.$

Exercise 5.6. Let $\varepsilon > 0$, $\kappa : \mathbb{R} \to \mathbb{R}$ be a continuous bounded function and $g : \mathbb{R} \to \mathbb{R}$ be a bounded Lipschitz continuous function. Consider the PDEs

$$\begin{cases} u'_t(t,x) + \frac{1}{2}u''_{xx}(t,x) = 0, & (t,x) \in]0,T[\times\mathbb{R}, \\ u(T,x) = \kappa(x) & x \in \mathbb{R}, \end{cases} \tag{5.261}$$

and

$$\begin{cases} (u^\varepsilon)'_t(t,x) + \frac{1}{2}(u^\varepsilon)''_{xx}(t,x) + \sin\left(\frac{x}{\varepsilon}\right) g\left(u^\varepsilon(t,x), (u^\varepsilon)'_x(t,x)\right) = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad (t,x) \in]0,T[\times\mathbb{R}, \\ u(T,x) = \kappa(x), \qquad x \in \mathbb{R}. \end{cases} \tag{5.262}$$

1. Write the BSDEs in $(Y^{t,x}, Z^{t,x})$ and respectively in $(Y^{\varepsilon;t,x}, Z^{\varepsilon;t,x})$ such that $u(t,x) = Y_t^{t,x}$ and $u^\varepsilon(t,x) = Y_t^{\varepsilon;t,x}$ are viscosity solutions of the PDEs (5.261) and, respectively, (5.262). Are the corresponding viscosity solutions unique?
2. Prove that

$$\lim_{\varepsilon\to 0} u^\varepsilon(0,x) = u(0,x), \quad \text{for all } x \in \mathbb{R}.$$

Exercise 5.7. Let E be a non-empty closed subset of $\mathbb{R}^m$, $g : \mathbb{R}^k \to E$ be a bounded Borel measurable function and $F : \Omega \times [0,T] \to \mathbb{R}^m$ be a bounded progressively measurable stochastic process. Let $(Y,Z) \in S_m^1[0,T] \times \Lambda_{m\times k}^1(0,T)$ be such that

$$Y_t = g(B_T) + \int_t^T F_s ds - \int_t^T Z_s dB_s, \quad a.s., \quad t \in [0,T].$$

Show that $(i) \Rightarrow (ii)$, where:

(i) $\mathbb{P}$-a.s., $\{Y_t : t \in [0,T]\} \subset E$, for all bounded Borel measurable function $g : \mathbb{R}^k \to E$;
(ii) E is a convex set.