An Asymptotic-Preserving Scheme for Systems of Conservation Laws with Source Terms on 2D Unstructured Meshes

C. Berthon, G. Moebs and R. Turpault

Abstract A finite volumes numerical scheme is here proposed for hyperbolic systems of conservation laws with source terms which degenerate into parabolic systems in large times when the source terms become stiff. In this framework, it is crucial that the numerical schemes are asymptotic-preserving i.e. that they degenerate accordingly. Here, an asymptotic-preserving numerical scheme is designed for any system within the aforementioned class on 2D unstructured meshes. This scheme is proved to be consistent and stable under a suitable CFL condition. Moreover, we show that it is also possible to prove that it preserves the set of (physically) admissible states under a geometrical property on the mesh. Finally, numerical examples are given to illustrate its behavior.

1 Introduction

The objective of this paper is to build a suitable numerical scheme for hyperbolic systems of conservation laws which can be written under the following form:

$$\partial_t \mathbf{U} + \operatorname{div}(\mathbf{F}(\mathbf{U})) = \gamma(\mathbf{U})(\mathbf{R}(\mathbf{U}) - \mathbf{U}),\ (t, x) \in \mathbb{R}_+ \times \mathbb{R}^2. \tag{1}$$

Here, the Jacobian of the flux $\mathbf{F}$ is assumed to be diagonalizable in $\mathbb{R}$. The set of admissible states is denoted $\mathscr{A}$. Moreover, $\mathbf{R}$ is a smooth function of $\mathbf{U}$ such that for all $\mathbf{U} \in \mathscr{A}$, $\mathbf{R}(\mathbf{U}) \in \mathscr{A}$. Finally, $\gamma(\mathbf{U})$ is a positive real function which represents the stiffness of the source term. The system (1) is assumed to fulfill the properties required in [3] so that it degenerates in long time and when the source term becomes stiff, more precisely when $\gamma(\mathbf{U})t \to \infty$, into a parabolic system. There are

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numerous examples of such systems, including the M_1 model for radiative transfer (see [17]) which is used here as when an illustration is required:

$$\mathbf{U} = \begin{pmatrix} E \\ Fx \\ Fy \\ T \end{pmatrix}, \quad \mathbf{F}(\mathbf{U}) = \begin{pmatrix} Fx & Fy \\ c^2 Pxx & c^2 Pxy \\ c^2 Pyx & c^2 Pyy \\ 0 & 0 \end{pmatrix}, \quad \mathbf{R}(\mathbf{U}) = \begin{pmatrix} \frac{\sigma(\mathbf{U}) a T^4 + \sigma_1(\mathbf{U})}{\sigma_m(\mathbf{U})} \\ \frac{\sigma_1(\mathbf{U}) Fx}{\sigma_m(\mathbf{U})} \\ \frac{\sigma_1(\mathbf{U}) Fy}{\sigma_m(\mathbf{U})} \\ \frac{\sigma(\mathbf{U}) E + \sigma_2(\mathbf{U}) \rho C_v T}{\rho C_v \sigma_m(\mathbf{U})} \end{pmatrix}, \tag{2}$$

where $\gamma(\mathbf{U}) = c\sigma_m(\mathbf{U})$ and

$$P = E\Big(\frac{1-\chi}{2} I_d + \frac{3\chi - 1}{2} \frac{F \otimes F}{\|F\|^2}\Big), \chi = \chi\Big(\xi = \frac{\|F\|}{cE}\Big) = \frac{3 + 4\xi^2}{5 + 2\sqrt{4 - 3\xi^2}}, \tag{3}$$

$$\sigma_m(\mathbf{U}) = \sigma(\mathbf{U}) \max\big(1, \frac{aT^3}{\rho C_v}\big), \sigma_1(\mathbf{U}) = \sigma_m(\mathbf{U}) - \sigma(\mathbf{U}), \sigma_2(\mathbf{U}) = \sigma_m(\mathbf{U}) - \sigma(\mathbf{U}) \frac{aT^3}{\rho C_v}. \tag{4}$$

The set of admissible states is:

$$\mathscr{A} = \big\{\mathbf{U} = (E, F_x, F_y, T)^\top \in \mathbb{R}^4 \,/\, E > 0,\ T > 0, \|F\| \leq cE\big\}. \tag{5}$$

When $\sigma_m(\mathbf{U})t \to \infty$, the M_1 model degenerates into the so-called *equilibrium diffusion equation*:

$$\partial_t\big(\rho C_v T + aT^4\big) - \operatorname{div}\Big(\frac{c}{3\sigma} \nabla aT^4\Big) = 0. \tag{6}$$

The main difficulty when designing a numerical scheme for such systems is to enforce the correct degeneracy in the diffusion limit. In other words, the limit of the scheme when $\gamma(\mathbf{U})t \to \infty$ shall be a consistent approximation of the limit diffusion equation. This property is generally not fulfilled by numerical schemes hence the design of *asymptotic-preserving* (AP) schemes has been an important issue during the last decade. For 1D applications, several asymptotic-preserving schemes were proposed in this context. The most explored way to do so is to use a modified HLL scheme and cleverly control the numerical diffusion in the spirit of the work of Gosse and Toscani for the telegraph equations [20]. This technique has been widely used for the M_1 model and Euler equations with friction (see for instance [8, 10]) and extended to general cases [5]. Other techniques have also been used, such as [1, 6, 11]. The situation is much more difficult for 2D applications however. While it is quite straightforward for Cartesian grids (see [2] for example), it is way more complex on unstructured grids. One of the reasons is that the classical two-point flux scheme (or FV4 [18]) which is the target of many AP schemes is not consistent anymore. The only exception is the MPFA-based AP scheme for Friedrich systems developed in [9]. Our

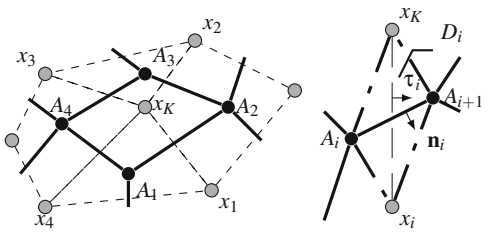

Fig. 1 Local mesh notations (l) one cell and the local notations (r) the diamond cell D_i associated with the ith interface of the cell $K \in \mathscr{M}$

goal is therefore to propose an AP finite volumes scheme for any system of the form (1). This scheme is a natural extension of the 1D scheme proposed in [5] based on the diamond scheme [12]. It is consistent and stable under a natural unrestrictive CFL condition. Moreover, it is also possible to enforce the preservation of the set of admissible states provided a geometrical property is satisfied by the mesh.

Notations. Since we intent to provide a finite volumes scheme which may be used in either *cell-centered* or *vertex-centered* contexts we call (*primary*) mesh $\mathscr{M}$ the set of all control volumes effectively used in the scheme. The *secondary* mesh is a set control volumes defined around the nodes of the *primary* mesh. Hence, the *primary* mesh is the primal mesh in the context of cell-centered schemes and the dual mesh in the context of vertex-centered schemes and the *secondary* mesh is the dual mesh in the context of cell-centered schemes and the primal mesh in the context of vertex-centered schemes. The notations used throughout this paper are summarized on Fig. 1:

- N_K is the number of nodes (and interfaces) of the cell $K \in \mathscr{M}$.
- x_K is the centroid of the cell K.
- The nodes of the cell K are locally denoted $\{A_i\}_{i=1\ldots N_K}$.
- The neighboring cells $(L_i)_i$ of the cell K are locally numbered from 1 to N_K such that $K \cap L_i = [A_i A_{i+1}]$. Their centroids are locally denoted $\{x_i\}_{i=1\ldots N_K}$.
- $d_i^K := \|x_K x_i\|$ and $e_i := \|A_i A_{i+1}\|$ is the length of the i^{th} interface of the cell K.
- $r^K := |K|/p_k$ where p_k is the perimeter of K.

2 Definition of the Scheme and Properties

The scheme proposed here is a direct generalization of the 1D scheme [5] where a Rusanov-type flux is selected for the hyperbolic part. As it was pointed in the introduction, the main difficulty is to select a scheme to degenerate into in the diffusive limit. The classical two-point finite volume scheme (a.k.a FV4 [18]) is not consistent with the diffusion equation on general meshes. The target scheme in the diffusive limit must therefore properly take into account the whole gradient. For the sake of consistency and simplicity, we choose to use the same gradient discretization in the hyperbolic part. Here, we adopt the *diamond scheme* strategy [12] but others could

be considered (see [7, 13–15, 19, 21] and references therein). With the diamond scheme to approximate the gradients, the resulting scheme is obtained:

$$\mathbf{U}_K^{n+1} = \mathbf{U}_i^n + \frac{\Delta t}{|K|}\sum_{i=1}^{N_K} e_i \alpha_{K,i}^n \mathscr{F}_{K,i}^n \cdot \mathbf{n}_i + \frac{\Delta t}{|K|}\sum_{i=1}^{N_K} e_i \alpha_{K,i}^n \mathbf{F}(\mathbf{U}_K^n)\cdot \mathbf{n}_i + \frac{\Delta t}{|K|}\sum_{i=1}^{N_K} e_i (1-\alpha_{K,i}^n) b_i^K (\mathbf{R}(\mathbf{U}_K^n) - \mathbf{U}_K^n), \tag{7}$$

$$\mathscr{F}_{K,i}^n = \frac{\mathbf{F}(\mathbf{U}_K^n) + \mathbf{F}(\mathbf{U}_i^n)}{2} - \frac{b_i^K \theta_i^K}{2}\nabla_i^K \mathbf{U}_K^n \cdot \mathbf{n}_i,\ \alpha_i^K = \frac{b_i^K}{b_i^K + \gamma_i^K r^K}, \tag{8}$$

$$\nabla_i^K \mathbf{U}_K^n \cdot \mathbf{n}_i = \frac{\mathbf{U}_i^n - \mathbf{U}_K^n}{2|D_i|} e_i + \frac{\mathbf{U}_{A_{i+1}}^n - \mathbf{U}_{A_i}^n}{2|D_i|} d_i^K \mathbf{n}_i \cdot \boldsymbol{\tau}_i, \tag{9}$$

where $\theta_i^K > 0$ is a parameter to be precised later and $\mathbf{U}_{A_i}^n$ is the value of the solution at the node A_i (see Fig. 1). This value is obtained as a mean value of the solution in the cells which share A_i as a node (see [12]).

Theorem 1 *Assume that $\theta_i^K \to 0$ when $r^K \to 0$, then the scheme (7)–(8) is consistent with (1).*

The proof of this theorem and the following can be found in [4]. In some applications, it is important to preserve the set of admissible states $\mathscr{A}$. It is all the more difficult since most finite volumes schemes for parabolic problems, including the diamond scheme, do not preserve the maximum principle. Only a few examples ensure this property (for example [16, 22]). Interestingly, it is sometimes possible to recover the maximum principle for our scheme under some geometric condition on the mesh.

Definition 1 The mesh is said to be δ-admissible if $\exists \delta > 0$ such that:

$$\forall K \in \mathscr{M},\ \forall i \in [1, N_K],\ 1 + \frac{e_{i-1}\overline{d^K}_{i-1}}{e_i^2}\frac{|D_i|}{3|D_{i-1}|} - \frac{e_{i+1}\overline{d^K}_{i+1}}{e_i^2}\frac{|D_i|}{3|D_{i+1}|} > \delta,$$
$$\overline{d^K}_i = d_i^K \mathbf{n}_i \cdot \boldsymbol{\tau}_i.$$

With this definition, an admissible mesh is δ-admissible for all $\delta \leq 1$ since all $\overline{d^K}_i$ are then equal to 0. This condition turned out to be satisfied by most of the meshes generated with reasonable constraints on the angles we tested. Equipped with this definition, we can obtain the following result.

Theorem 2 *Assume that the mesh is δ-admissible and that the secondary mesh is made of triangles. Let us also assume that α_i^K is constant inside each cell $K \in \mathscr{M}$ ($\alpha_i^K = \alpha^K$) and let us set $\theta_i^K = \dfrac{2|D_i|}{\delta e_i}$. Then, the scheme (7)–(8) preserves the set of admissible states $\mathscr{A}$ as soon as the following CFL condition holds:*

$$\max_{K\in\mathscr{A}, i\leq N_K} \{b_i^K \theta_i^K \delta_i^K\}\frac{\Delta t}{|K|} p_K \leq \frac{1}{2}. \tag{10}$$

Once again, a proof of this theorem is provided in [4]. There are two essential bricks here: the ability to write the 2D scheme as a convex combination of 1D schemes and the possibility to express the approximate gradient as $\sum_i \omega_{K,i}(U_K - U_i)$ with $\omega_{K,i} \geq 0$. Several comments have to be done concerning this theorem:

- The choice of θ_i^K tends to 0 when $r^K \to 0$ as it was requested for the sake of consistency.
- A similar theorem may be obtained on more general meshes. However, the geometrical condition quickly becomes cumbersome. On the other hand, other expressions of the discrete gradient such as [16] may also be used to ensure the property without any restriction on the mesh at the cost of a strongly nonlinear scheme.
- The main restriction is to consider α_i^K that are constant per cell. It is sometimes a severe limitation when the AP procedure defined in the following is applied.
- Other choices of θ_i^K allow to recover the same result e.g. $\theta_i^K = \max_{i \leq N_K} \dfrac{2|D_i|}{2\delta}$.

The scheme (7)–(8) is not AP in general but a simple procedure may be used to recover this property. It consists in appliying the scheme to the system:

$$\partial_t \mathbf{U} + \operatorname{div}(\mathbf{F}(\mathbf{U})) = (\gamma + \bar{\gamma})\big(\bar{\mathbf{R}}(\mathbf{U}) - \mathbf{U}\big), \ \bar{\mathbf{R}}(\mathbf{U}) = \frac{\gamma \mathbf{R}(\mathbf{U}) + \bar{\gamma}\mathbf{U}}{\gamma + \bar{\gamma}}, \tag{11}$$

which is obviously equivalent to (1). Then, a formal Chapmann-Enskog expansion leads to the following scheme in the diffusion limit:

$$\mathbf{U}_K^{n+1} = \mathbf{U}_K^n - \frac{\Delta t}{|K|} \sum_{i=1}^{N_K} e_i \frac{b_i^K}{(\gamma_i^K + \overline{\gamma^K}_i) r^K} \Big[\mathscr{F}(\mathbf{U}_K^n) \cdot \mathbf{n}_i - \mathbf{F}(\mathbf{U}_K^n) \cdot \mathbf{n}_i \Big]_{|\mathbf{R}(\mathbf{U}_K^n) = \mathbf{U}_K^n}. \tag{12}$$

Now, $\overline{\gamma^K}_i$ may be chosen so that the scheme is AP. This procedure is illustrated in the case of the M_1 model for radiative transfer.

AP correction for the M_1 model. For the M_1 model (2), $b_i^K = c$ and the equilibrium gives $F_x = F_y = 0$ and $E = aT^4$. The first and fourth equations of (12)–(8) hence become:

$$(\rho C_v + aT^4)_K^{n+1} = (\rho C_v T + aT^4)_K^n + \frac{\Delta t}{|K|} \sum_{i=1}^{N_K} \frac{c^2 e_i}{2(\sigma_{m,i}^K + \bar{\sigma}_i^K) r^K} \nabla_i^K (aT^4)^n \cdot \mathbf{n}_i.$$

In order to ensure that this scheme is consistent with the equilibrium diffusion equation (6), the terms $\bar{\sigma}_i^K$ have to be chosen accordingly. For example, if we take:

$$(\sigma_{m,i}^K + \bar{\sigma}_i^K) = \sigma_{m,i}^K \frac{3c\theta_i^K}{2r^K} > 0. \tag{13}$$

then the limit scheme in the diffusion regime is:

$$(\rho C_v T + aT^4)_K^{n+1} = (\rho C_v T + aT^4)_K^n + \frac{\Delta t}{|K|} \sum_{i=1}^{N_K} \frac{ce_i}{3\sigma_i^K} \nabla_i^K (aT^4)_i^n \cdot \mathbf{n}_i,$$

which is consistent with the diffusion equation (6). In fact, as expected, it is nothing but the diamond scheme applied to (6). Moreover, if σ is a constant and $\theta_i^K = \theta^K$ then $\bar{\sigma}_i^K = \bar{\sigma}^K$ and Theorem 2 can be applied. In order to meet such a requirement, one may choose: $\theta_i^K = \max_{i \le N_K} \frac{2|D_i|}{2\delta}$.

3 Numerical Results

Validation tests are performed in this paragraph in order to illustrate the behavior of the scheme. A Riemann problem for the M_1 model for radiative transfer is considered on $[0, 5] \times [0, 1]$ with:

$$(E, F_x, F_y, T)^\top (0, x) = \begin{cases} (aT_L^4, cf_{x,L} aT_L^4, 0, T_L)^\top, \text{ if } x < 1, \\ (aT_R^4, 0, 0, T_R)^\top, \text{ otherwise.} \end{cases}$$

In the following, $T_L = 10000$, $T_R = 300$ and $f_{x,L} = 0$, $\rho C_v = 10^{-2}$ and $c = 3.10^8$. First, σ is set to 0 since the preservation of admissible states is expected to be more difficult than in the presence of the (regularizing) source-term. Two different meshes are used: a "coarse" one (5152 triangles) and a "fine" one (132006 triangles). Both of these meshes are $\delta-$admissible with optimal $\delta = \delta_1 = 1.095$ for the coarse grid and $\delta_2 = 5.59910^{-2}$ for the fine one. The reference solution is the exact solution of the corresponding 1D Riemann problem. Figure 2 shows the solutions along $x = \frac{1}{2}$. Here, the conservation of admissible states is enforced by using $\theta_i^K = \max_{i \le N_K} \frac{2|D_i|}{2\delta}$ where $\delta = \delta_1$ on the coarse mesh and δ_2 on the fine one. The solution computed on the coarse grid is comparable to a 1D Rusanov scheme with a similar number of cells. On the other hand, since $\delta_2 \ll \delta_1$, the numerical diffusion of the scheme is way larger on the fine mesh than on the coarse one. The approximation is hence better on the coarse grid in this case. Now if $\delta = \delta_1$ on the fine mesh, as shown in the right of Fig. 2, the quality of the approximation behaves as expected, i.e. the approximation is better on the fine grid.

Next, we fix $\sigma = 1000$ to investigate the AP property. The results showed on Fig. 3 are compared with a grid-converged 1D approximation of the equilibrium diffusion equation. The tests are performed with and without the asymptotic-preserving correction on the fine grid. We immediately see that with the AP correction, the scheme provides an approximation which is nearly indistinguishable from the reference solution. On the other hand, as expected, if the AP correction is turned off i.e. $\overline{\gamma^K} = 0$),

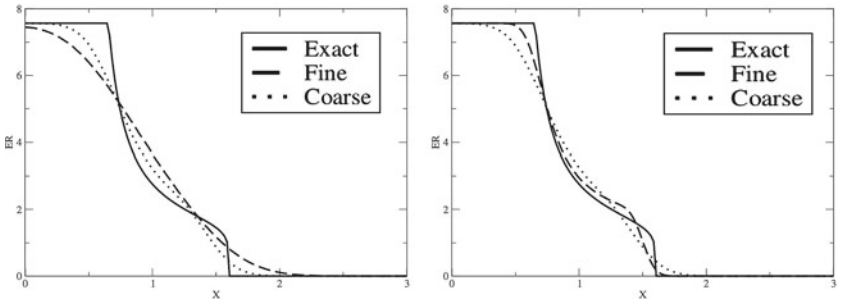

Fig. 2 Exact and computed E along $x = \frac{1}{2}$ with $\sigma = 0$ at $t = 2.10^{-9}$. (l) conservation of $\mathscr{A}$ enforced; (r) same value of δ for both meshes

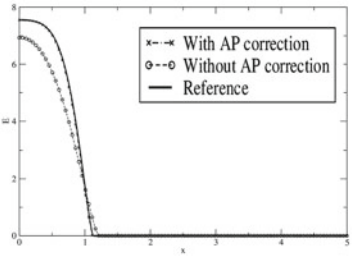

Fig. 3 Ref. and computed E with and w/o AP correction along $x = \frac{1}{2}$ with $\sigma = 1000$ at $t = 2.10^{-6}$

Fig. 4 Radiative flow in a channel (*top*) E (*bottom*) χ

there is a large discrepancy between the computed and the reference solution. Finally, a test-case involving the evolution of the radiation in a channel with multiple obstacles is performed. The entry condition on the left side of the channel models a beam of high energy ($F_L = cE_L = ca10000^4$) compared to the initial state of the domain ($F_0 = 0$, $E_0 = a10^4$), $\sigma = 1$ and 11 obstacles (with wall boundary conditions) are scattered in the channel. A vertex-centered approached is used on a mesh consisting of 15348 cells refined near the obstacles (see Fig. 4). Let us emphasize that this case is numerically very challenging and that it is all the more critical to preserve the set of admissible states here since very small numerical errors may yield

unadmissible values, which immediately cause the code to crash. Several values of θ_i^K were tested and even a value 5 % larger than the choice stated in the theorem produces unadmissible results. In this sense, it seems that the condition of Theorem 2 is optimal.

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