Discrete Relative Entropy for the Compressible Stokes System

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Abstract In this paper, we propose a discretization for the nonsteady compressible Stokes Problem. This scheme is based on Crouzeix-Raviart approximation spaces. The discretization of the momentum balance is obtained by the usual finite element technique. The discrete mass balance is obtained by a finite volume scheme, with an upwinding of the density. The time discretization will be implicit in time. We prove the existence of a discrete solution. We prove that our scheme satisfies a discrete version of the relative entropy. As a consequence, we obtain an error estimate for this system. This preliminary work will be used in order to obtain a error estimate for the compressible Navier-Stokes system and has to the author's knowledge not been studied previously.

1 Introduction

Let Ω an open bounded domain with lipschitz boundary subset of $\mathbb{R}^d$, $d = 2, 3$. We consider the following system

$$\partial_t \varrho + \operatorname{div}(\varrho \boldsymbol{u}) = 0,\ t \in (0, T),\ x \in \Omega \tag{1}$$

$$\partial_t \boldsymbol{u} - \mu \Delta \boldsymbol{u} - (\mu + \lambda)\nabla \operatorname{div} \boldsymbol{u} + \nabla_x p(\varrho) = \boldsymbol{0},\ t \in (0, T),\ x \in \Omega \tag{2}$$

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supplemented with the following initial conditions and boundary condition

$$\varrho(0,x)=\rho_0(x),\ \boldsymbol{u}(0,x)=\boldsymbol{u}_0,\ \boldsymbol{u}_{|\partial\Omega}=0. \tag{3}$$

We suppose that the pressure satisfies $p\in C(\mathbb{R}_+)\cap C^2(\mathbb{R}_+^*)$, $p(0)=0$ and $\lim_{+\infty}\frac{p'(\rho)}{\rho^{\gamma-1}}=p_\infty>0$ for $\gamma\geq 2$. Moreover if $\gamma\in[\frac{6}{5},2[$ we suppose also that $\liminf_0\frac{p'(\rho)}{\rho^{\alpha-1}}=p_0>0$, with $\alpha\leq 0$.

2 Weak Solutions, Relative Entropies

In this part, we give the definition of (finite energy) weak solutions for our system. We give the definition of the relative entropy. In the following we denote $\mathcal{H}(\varrho)=\rho\int_1^\rho\frac{p(t)}{t^2}\,\mathrm{dt}$. Let us denote $C_c^\infty([0,T]\times\Omega,\mathbb{R}^3)$ the space of all smooth functions on $[0,T]\times\Omega$ compactly supported in $[0,T]\times\Omega$.

Definition 1 Let $(\varrho_0,\boldsymbol{u}_0)\in L^\gamma(\Omega)\times H_0^1(\Omega)$ such that $\varrho_0\geq 0$ a.e in Ω. We shall say that $(\varrho,\boldsymbol{u})$ is a finite energy weak solution to the problem (1)–(3) emanating from the initial data $(\varrho_0,\boldsymbol{u}_0)$ if

$$\varrho\in L^\infty(0,T;L^\gamma(\Omega))\cap C_w([0,T],L^\gamma(\Omega)),\ \rho\geq 0\ p.p \text{ in } (0,T)\times\Omega,$$

$$\boldsymbol{u}\in L^2(0,T;H_0^1(\Omega))\cap C_w([0,T],L^2(\Omega))$$

and :

– The continuity equation (1) is satisfied in the following weak sense

$$\int_\Omega\varrho(\tau,\cdot)\varphi(\tau,\cdot)\,\mathrm{dx}-\int_\Omega\varrho_0\varphi(0,\cdot)=\int_0^\tau\int_\Omega\varrho(t,x)\partial_t\varphi(t,x)\,\mathrm{dx\,dt}+\int_0^\tau\int_\Omega\varrho\boldsymbol{u}\cdot\nabla_x\varphi\,\mathrm{dx\,dt}, \tag{4}$$

$\forall\tau\in[0,T],\ \forall\varphi\in C^\infty([0,T]\times\overline{\Omega})$.

– The momentum equation (2) is satisfied in the following weak sense

$$\int_\Omega\boldsymbol{u}\cdot\psi(\tau,x)\,\mathrm{dx}-\int_\Omega\mathbf{u}_0\cdot\psi(0,\cdot)\,\mathrm{dx}=\int_0^\tau\int_\Omega\boldsymbol{u}\cdot\partial_t\psi+p(\varrho)\operatorname{div}_x\psi-\mu\nabla_x\boldsymbol{u}:\nabla_x\psi-(\mu+\lambda)\operatorname{div}_x\boldsymbol{u}\operatorname{div}_x\psi\,\mathrm{dx\,dt}, \tag{5}$$

$\forall\tau\in[0,T],\ \forall\psi\in C_c^\infty([0,T]\times\Omega,\mathbb{R}^3)$.

– The following energy inequality is satisfied

$$\int_{\Omega} \frac{1}{2}|\boldsymbol{u}|^2 + \mathcal{H}(\varrho)\,\mathrm{dx} + \int_0^{\tau}\int_{\Omega} \mu||\nabla_x \boldsymbol{u}||^2 + (\mu+\lambda)(\mathrm{div}_x\,\boldsymbol{u})^2\,\mathrm{dx}\,\mathrm{dt} \leq \int_{\Omega} \frac{1}{2}|\boldsymbol{u}_0|^2 + \mathcal{H}(\varrho_0)\,\mathrm{dx}, \tag{6}$$

$a.e$ $\tau \in [0, T]$.

2.1 Relative Entropy Inequality, Weak-Strong Uniqueness

The method of relative entropy has been successfully applied to partial differential equations of different types. Relative entropies are non-negative quantities that provide a kind of distance between two solutions of the same problem, one of which typically enjoys some extra regularity properties (see [2] for more details)

Definition 2 We define the relative entropy of $(\rho, \boldsymbol{u})$ with respect to $(r, \boldsymbol{U})$ by

$$\mathcal{E}([\varrho, \boldsymbol{u}], [r, \boldsymbol{U}]) = \int_{\Omega} \frac{1}{2}|\boldsymbol{u} - \boldsymbol{U}|^2 + E(\varrho, r)\,\mathrm{dx} \tag{7}$$

where $E(\rho, r) = \mathcal{H}(\rho) - \mathcal{H}'(r)(\rho - r) - \mathcal{H}(r)$. We also define a remainder, denoted by $\mathcal{R}$, as

$$\mathcal{R} = \int_{\Omega} \nabla_x \boldsymbol{U} : \nabla_x(\boldsymbol{U} - \boldsymbol{u})\,\mathrm{dx} + \int_{\Omega} (r - \varrho)\partial_t \mathcal{H}'(r) + \nabla_x \mathcal{H}'(r) \cdot (r\boldsymbol{U} - \varrho\boldsymbol{u})\,\mathrm{dx} - \int_{\Omega} \mathrm{div}_x\,\boldsymbol{U}(p(\varrho) - p(r))\,\mathrm{dx} + \int_{\Omega} \partial_t \boldsymbol{U} \cdot (\boldsymbol{U} - \boldsymbol{u})\,\mathrm{dx}\,. \tag{8}$$

Theorem 1 *Let $(\rho, \boldsymbol{u})$ be a weak solution of (1)–(3) in the sense of the definition 1 emanating from the initial condition $(\rho_0, \boldsymbol{u}_0)$. Then $(\rho, \boldsymbol{u})$ satisfy the relative energy inequality:*

$$\mathcal{E}([\varrho, \boldsymbol{u}], [r, \boldsymbol{U}])(\tau) + \int_0^{\tau}\int_{\Omega} \mu||\nabla_x(\boldsymbol{u} - \boldsymbol{U})||^2 + (\mu+\lambda)(\mathrm{div}_x(\boldsymbol{u} - \boldsymbol{U}))^2\,\mathrm{dx}\,\mathrm{dt} \leq \mathcal{E}([\varrho_0, \boldsymbol{u}_0], [r(0), \boldsymbol{U}(0)]) + \int_0^{\tau} \mathcal{R}([\varrho, \boldsymbol{u}], [r, \boldsymbol{U}])(t)\,\mathrm{dt} \tag{9}$$

$a.e$ $\tau \in [0, T]$, where $r \in C^{\infty}([0, T] \times \overline{\Omega}, \mathbb{R}_+^*)$ and $\boldsymbol{U} \in C^{\infty}([0, T] \times \Omega, \mathbb{R}^3)$.

Proof See [2].

Remark 1 For the choice of $r = \overline{\rho}$ and $\boldsymbol{U} = 0$, the relative energy inequality (9) reduces to the standard energy inequality.
Moreover, the relative energy inequality can be used to show that suitable weak solutions comply with the weak-strong uniqueness principle, meaning, a weak and strong solution emanating from the same initial data coincide as long as the latter exists. This can be seen by taking the strong solution as the test functions r, U in the relative entropy inequality (see [2]).

3 The Numerical Scheme

Now suppose that Ω is a bounded open set of $\mathbb{R}^d$, polygonal if $d = 2$ and polyhedral if $d = 3$. Let $\mathcal{T}$ be a decomposition of the domain Ω in simplices, which we call hereafter a triangulation of Ω, regardless of the space dimension. By $\mathcal{E}(K)$, we denote the set of the edges (d = 2) or faces (d = 3) σ of the elements $K \in \mathcal{T}$; for short, each edge or face will be called an edge hereafter. The set of all edges of the mesh is denoted by $\mathcal{E}$; the set of edges included in the boundary of Ω is denoted by $\mathcal{E}_{\text{ext}}$ and the set of internal edges (i.e $\mathcal{E} \setminus \mathcal{E}_{\text{ext}}$) is denoted by $\mathcal{E}_{\text{int}}$. The decomposition $\mathcal{T}$ is assumed to be regular in the usual sense of the finite element literature, and, in particular, $\mathcal{T}$ satisfies the following properties: $\overline{\Omega} = \cup_{K\in\mathcal{T}} \overline{K}$; if $K, L \in \mathcal{T}$, then $\overline{K} \cap \overline{L} = \emptyset$, $\overline{K} \cap \overline{L}$ is a vertex or $\overline{K} \cap \overline{L}$ is a common edge of K and L, which is denoted by $K|L$. For $K \in \mathcal{T}$ and $\sigma \in \mathcal{E}(K)$, we define $D_{K,\sigma}$ as the cone with basis σ and with vertex the mass center of K. For each internal edge of the mesh $\sigma = K|L$, $\boldsymbol{n}_{KL}$ stands for the unit normal vector of σ, oriented form K to L (so that $\boldsymbol{n}_{KL} = -\boldsymbol{n}_{LK}$). By $|K|$ and $|\sigma|$ we denote the (d and $d-1$ dimensional) measure, respectively, of an element K and of an edge σ, and h_K and h_σ stand for the diameter of K and σ, respectively. We measure the regularity of the mesh through the parameter θ defined by:

$$\theta = \inf\{\frac{\xi_K}{h_K}, K \in \mathcal{T}\} \tag{10}$$

where ξ_K stands for the diameter of the largest ball included in K. The space discretization relies on the Crouzeix-Raviart element. The reference element is the unit d-simplex and the discrete functional space is the space P_1 of affine polynomials. The degrees of freedom are determined by the following set of edge functionals:

$$\{F_\sigma, \sigma \in \mathcal{E}(K)\},\ F_\sigma(v) = \frac{1}{|\sigma|}\int_\sigma v \,\mathrm{d}\gamma\ .$$

The mapping from the reference element to the actual one is the standard affine mapping. Finally, the continuity of the average value of a discrete function v across each edge of the mesh, $F_\sigma(v)$, is required, thus the discrete space V_h is defined as follows:

$$V_h = \{v \in L^2(\Omega),\ \forall K \in \mathcal{T},\ v_{|K} \in \mathbb{P}_1(K) \text{ and } \forall \sigma \in \mathcal{E}_{\text{int}}, \sigma = K|L, F_\sigma(v_{|K}) = F_\sigma(v_{|L}), \forall \sigma \in \mathcal{E}_{\text{ext}}, F_\sigma(v) = 0\}.$$

The space of approximation for the velocity is the space $\boldsymbol{W}_h$ of vector-valued functions each component of which belongs to V_h : $\boldsymbol{W}_h = (V_h)^d$. The pressure and the density are approximated by the space L_h of piecewise constant functions:

$$L_h = \{q \in L^2(\Omega), q_{|K} = \text{constant}, \forall K \in \mathcal{T}\}.$$

We will also denote $L_h^+ = \{q \in L_h,\ q_K \geq 0,\ \forall K \in \mathcal{T}\}$ and $L_h^{++} = \{q \in L_h,\ q_K > 0,\ \forall K \in \mathcal{T}\}$.

It is well-know that this discretization is nonconforming in $H^1(\Omega)^d$. We then define, for $1 \leq i \leq d$ and $u \in V_h$, $\partial_{h,i} u$ as the function of $L^2(\Omega)$ which is equal to the derivative of u with respect to the ith space variable almost everywhere. This notation allows us to define the discrete gradient, denoted by ∇_h for both scalar and vector-valued discrete functions and the discrete divergence of vector-valued discrete functions, denoted by div_h. We denote $||\cdot||_{1,b}$ the broken Sobolev H^1 semi-norm, which is defined for scalar as well as for vector-valued functions by

$$||v||_{1,b}^2 = \sum_{K \in \mathcal{T}} \int_K |\nabla v|^2 \,\text{dx} = \int_\Omega |\nabla_h v|^2 \,\text{dx}\,.$$

We denote by $\{u_{i,\sigma}, \sigma \in \mathcal{E}_{\text{int}}, 1 \leq i \leq d\}$ the set of velocity degrees of freedom We denote by φ_σ the usual Crouzeix-Raviart shape function associated to $\sigma \in \mathcal{E}_{\text{int}}$, i.e. the scalar function of V_h such that $F_\sigma(\varphi_\sigma) = 1$ and $F_{\sigma'}(\varphi_\sigma) = 0, \forall \sigma' \neq \sigma$.

Similarly, each degree of freedom for the density is associated to a cell K, and the set of density degrees of freedom is denoted by $\{\rho_K, K \in \mathcal{T}\}$. We define by r_h the following interpolation operator $r_h : H_0^1(\Omega) \to V_h$ by

$$r_h(v) = \sum_{\sigma \in \mathcal{E}_{\text{int}}} F_\sigma(v)\varphi_\sigma.$$

This operator naturally extends to vector-valued functions and we keep the same notation r_h for both the scalar and vector case.

Let us consider a partition $0 = t^0 < t^1 < ... < t^N = T$ of the time interval $[0, T]$, which, for the sake of simplicity, we suppose uniform. Let Δt be the constant time step $\Delta t = t^n - t^{n-1}$ for $n = 1, ..., N$. Let $(\rho^0, \boldsymbol{u}^0) \in L_h \times \boldsymbol{W}_h$.

Following [6] we consider the following numerical scheme :

Find $(\varrho^n)_{1 \leq n \leq N} \subset L_h$, $(\boldsymbol{u}^n)_{1 \leq n \leq N} \subset \mathbf{W}_h$ *such that* $\forall n = 1, ..., N$

$$|K|\frac{\varrho_K^n - \varrho_K^{n-1}}{\Delta t} + \sum_{\sigma \in \mathcal{E}(K), \sigma = K|L} |\sigma|\Big(\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{KL}\Big)^+ \rho_K^n - |\sigma|\Big(\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{KL}\Big)^- \rho_L^n = 0, \forall K \in \mathcal{T} \tag{11}$$

$$\frac{|D_\sigma|}{\Delta t}(u_{i,\sigma}^n - u_{i,\sigma}^{n-1}) + \mu \sum_{K\in\mathcal{T}} \int_K \nabla u_i^n \cdot \nabla \varphi_\sigma \,\mathrm{dx} + (\mu+\lambda) \sum_{K\in\mathcal{T}} \int_K \mathrm{div}(\boldsymbol{u}^n)\,\mathrm{div}(\varphi_\sigma \boldsymbol{e}_i)\,\mathrm{dx}$$
$$- \sum_{K\in\mathcal{T}} \int_K p_K^n \,\mathrm{div}(\varphi_\sigma \boldsymbol{e}_i)\,\mathrm{dx} = 0, \forall \sigma \in \mathcal{E}_{int}, 1 \le i \le d \tag{12}$$

with $p_K^n = p(\rho_K^n)$, $a^+ = \max(a,0)$, $a^- = -\min(a,0)$.

As usual, to the discrete unknowns, we associate piecewise constant functions on time intervals and on primal or dual meshes, so the density $\rho_{\Delta t,h}$, the pressure $p_{\Delta t,h}$ and the velocity $\boldsymbol{u}_{\Delta t,h}$ are defined almost everywhere on $(0,T)\times\Omega$ by

$$\varrho_{\Delta t,h}(t,x) = \sum_{n=1}^{N}\sum_{K\in\mathcal{T}} \varrho_K^n \mathbf{1}_{(t^{n-1},t^n)}\mathbf{1}_K, \quad \rho_{\Delta t,h}(t,x) = \sum_{n=1}^{N}\sum_{K\in\mathcal{T}} \rho_K^n \mathbf{1}_{(t^{n-1},t^n)}\mathbf{1}_K,$$

$$\boldsymbol{u}_{\Delta t,h}(t,x) = \sum_{n=1}^{N}\sum_{K\in\mathcal{T}} \boldsymbol{u}_\sigma^n \mathbf{1}_{(t^{n-1},t^n)}\mathbf{1}_{D_\sigma}.$$

3.1 Existence, Positivity and Stabilities Properties

Theorem 2 (Existence and positivity) *Let $(\rho^0, \boldsymbol{u}^0) \in L_h^{++} \times \mathbf{W}_h$. Then the problem (11), (12) admits at least a solution $(\varrho^n)_{1\le n\le N} \subset L_h^{++}$, $(\boldsymbol{u}^n)_{1\le n\le N} \subset \mathbf{W}_h$.*

Proof See [5].

Theorem 3 (Energy estimate) *Let $(\varrho_0, \boldsymbol{u}_0) \in L^\gamma(\Omega) \times H_0^1(\Omega, \mathbb{R}^3)$, such that $\varrho_0 > 0$ a.e $x \in \Omega$.*

Let $\varrho_K^0 = \frac{1}{|K|}\int_K \varrho_0 \,\mathrm{dx}$ and $\boldsymbol{u}^0 = r_h(\boldsymbol{u}_0)$.

Let $(\varrho^n, \boldsymbol{u}^n) \in L_h^{++} \times \mathbf{W}_h$, $n = 1, ..., N$ be a solution of (11), (12) emanating from the initial data $(\varrho^0, \boldsymbol{u}^0)$. Then we have the following balance discrete energy

$$\max_{n=0,\dots,N} \sum_{K\in\mathcal{T}} |K|\,\mathcal{H}(\varrho_K^n) + \max_{n=0,\dots,N} \sum_{i,\sigma\in\mathcal{E}_{\mathrm{int}}} \frac{1}{2}|D_\sigma|(u_{i,\sigma}^n)^2 + \mu\Delta t \sum_{n=0}^{N} ||\boldsymbol{u}^n||_{1,b}^2$$
$$+ (\mu+\lambda)\Delta t \sum_{k=0}^{N} ||\,\mathrm{div}_h\, \boldsymbol{u}^n||_{L^2(\Omega)}^2 \le c(d, \theta_0, \varrho_0, \boldsymbol{u}_0),$$

Proof See [5].

3.1.1 Discrete Relative Entropy Inequality

The following result is crucial for the rest of the article. It can be seen as a discrete balance version of (9).

Theorem 4 *Let* $(\varrho_0, \boldsymbol{u}_0) \in L^\gamma(\Omega) \times H_0^1(\Omega, \mathbb{R}^3)$, *such that* $\varrho_0(x) > 0$ *a.e* $x \in \Omega$ *and* $\mathcal{H}(\varrho_0) \in L^1(\Omega)$.

Let $\varrho_K^0 = \frac{1}{|K|}\int_K \varrho_0 \,\mathrm{dx}$ *and* $\boldsymbol{u}^0 = r_h(\boldsymbol{u}_0)$.

Let $(\varrho^n, \boldsymbol{u}^n) \in L_h \times \boldsymbol{W}_h, n = 1, ..., N$ *be a solution of (11), (12) emanating from the initial data* $(\rho^0, \boldsymbol{u}^0)$. *Let* $(r, \boldsymbol{U}) \in C^1([0,T] \times \overline{\Omega}) \cap C^2([0,T] \times \overline{\Omega}, \mathbb{R}^3)$ *such that* $r(t,x) > 0, \forall (t,x) \in [0,T] \times \overline{\Omega}$ *and* $\boldsymbol{U}(t)_{|\partial\Omega} = \boldsymbol{0}$. *Let* $\boldsymbol{U}_h^n = r_h(\boldsymbol{U}(t^n))$, $r_K^n = \frac{1}{|K|}\int_K r(t^n, x)\,\mathrm{dx}$ *Then we have the following inequality*

$$
\begin{aligned}
&\sum_{i,\sigma\in\mathcal{E}_{\text{int}}} \frac{1}{2}\frac{|D_\sigma|}{\Delta t}\Big((u_{i,\sigma}^n - U_{i,\sigma}^n)^2 - (u_{i,\sigma}^{n-1} - U_{i,\sigma}^{n-1})^2\Big) \\
&+ \sum_{K\in\mathcal{T}} \frac{|K|}{\Delta t}\Big(E(\varrho_K^n | r_K^n) - E(\varrho_K^{n-1} | r_K^{n-1})\Big) \\
&+ \mu ||\boldsymbol{u}^n - \boldsymbol{U}^n||_{1,b}^2 + (\mu + \lambda)||\operatorname{div}_h(\boldsymbol{u}^n - \boldsymbol{U}_h^n)||_{L^2(\Omega)}^2 \\
&\le \sum_{i,\sigma\in\mathcal{E}_{\text{int}}} \frac{|D_\sigma|}{\Delta t}(U_{i,\sigma}^n - u_{i,\sigma}^n)(U_{i,\sigma}^n - U_{i,\sigma}^{n-1}) + \mu \sum_{K\in\mathcal{T}} \int_K \nabla \boldsymbol{U}_h^n : \nabla(\boldsymbol{U}_h^n - \boldsymbol{u}^n)\,\mathrm{dx} \\
&+ (\mu+\lambda)\int_\Omega \operatorname{div}_h \boldsymbol{U}_h^n \operatorname{div}_h(\boldsymbol{U}_h^n - \boldsymbol{u}^n)\,\mathrm{dx} + \sum_{K\in\mathcal{T}} \operatorname{div}_K^{\text{up}}(\varrho^n \boldsymbol{u}^n)\mathcal{H}'(r_K^n) \\
&+ \sum_{K\in\mathcal{T}} \frac{|K|}{\Delta t}(r_K^n - \rho_K^n)(\mathcal{H}'(r_K^n) - \mathcal{H}'(r_K^{n-1})) - \int_\Omega p^n \operatorname{div} \boldsymbol{U}_h^n \,\mathrm{dx} + \mathcal{R}^{n,h}
\end{aligned} \tag{13}
$$

where $\Delta t \sum_{n=1}^N |\mathcal{R}^{n,h}| \le c(\varrho_0, \boldsymbol{u}_0, r, \boldsymbol{U})\Delta t$.

The following result is the main result of our article and it is a consequence of the previous. We give an error estimate for our system.

Theorem 5 *Let* $(\varrho_0, \boldsymbol{u}_0) \in L^\gamma(\Omega) \times H_0^1(\Omega, \mathbb{R}^3)$, *such that* $\varrho_0(x) > 0$ *a.e* $x \in \Omega$ *and* $\mathcal{H}(\varrho_0) \in L^1(\Omega)$.

Let $\varrho_K^0 = \frac{1}{|K|}\int_K \varrho_0 \,\mathrm{dx}$ *and* $\boldsymbol{u}^0 = r_h(\boldsymbol{u}_0)$.

Let $(\varrho^n, \boldsymbol{u}^n) \in L_h \times \boldsymbol{W}_h, n = 1, ..., N$ *be a solution of (11), (12) emanating from the initial data* $(\varrho^0, \boldsymbol{u}^0)$. *Let* $(r, \boldsymbol{U}) \in C^1([0,T] \times \overline{\Omega}) \cap C^2([0,T] \times \overline{\Omega}, \mathbb{R}^3)$ *be a strong solution of (1)–(3) such that* $\forall (t,x) \in [0,T] \times \overline{\Omega}, r(t,x) > 0$. *Let* $\boldsymbol{U}_h^n = r_h(\boldsymbol{U}(t^n))$, $r_K^n = \frac{1}{|K|}\int_K r(t^n, x)\,\mathrm{dx}$. *Then we have the following inequality*

$$
\begin{aligned}
&\sum_{i,\sigma\in\mathcal{E}_{\rm int}} \frac{1}{2}\frac{|D_\sigma|}{\Delta t}\Big((u^n_{i,\sigma}-U^n_{i,\sigma})^2-(u^{n-1}_{i,\sigma}-U^{n-1}_{i,\sigma})^2\Big)\\
&\quad+\sum_{K\in\mathcal{T}}\frac{|K|}{\Delta t}\Big(E(\varrho^n_K|r^n_K)-E(\varrho^{n-1}_K|r^{n-1}_K)\Big)\\
&\quad+\mu||\boldsymbol{u}^n-\boldsymbol{U}^n||^2_{1,b}+(\mu+\lambda)||\,\mathrm{div}_h(\boldsymbol{u}^n-\boldsymbol{U}^n_h)||^2_{L^2(\Omega)}\\
&\quad\le\sum_{K\in\mathcal{T}}(r^n_K-\varrho^n_K)\int_K\frac{\nabla p(r^n)}{r^n}\cdot(\boldsymbol{u}^n-\boldsymbol{U}^n_h)\,\mathrm{dx}\\
&\quad-\sum_{K\in\mathcal{T}}\int_K\Big(p^n-p'(r^n_K)(\varrho^n_K-r^n_K)-p(r^n_K)\Big)\,\mathrm{div}\,\boldsymbol{U}^n_h\,\mathrm{dx}\\
&\quad+\mathcal{R}^{n,h}
\end{aligned}
\tag{14}
$$

where $\Delta t\sum_{n=1}^N|\mathcal{R}^{n,h}|\le C(\theta_0,\varrho_0,\boldsymbol{u}_0)(h^{\epsilon(\gamma)}+\Delta t)$ *with* $\epsilon(\gamma)=\frac{1}{2}$ *for* $\gamma\ge\frac{3}{2}$ *and* $\epsilon(\gamma)=\frac{5}{2}-\frac{3}{\gamma}$ *for* $\gamma\in[\frac{6}{5},\frac{3}{2}]$, *and we obtain the following estimation* $||\boldsymbol{u}_{\delta t,h}-\boldsymbol{U}||^2_{L^\infty(0,T;L^2(\Omega))}+||\varrho_{\delta t,h}-r||^\gamma_{L^\infty(0,T,L^\gamma(\Omega))}\le C(\theta_0,\varrho_0,\boldsymbol{u}_0)(h^{\epsilon(\gamma)}+\Delta t)$.

Proof We begin with a algebraic inequality whose straightforward proof is left to the reader

Lemma 1 *Let* $0<a<b<\infty$. *Then there exists* $c=c(a,b)>0$ *such that for all* $\rho\in[0,\infty[$ *and* $r\in[a,b]$ *there holds*

$$
E(\rho|r)\ge c(a,b)\Big(1_{[\frac{a}{2},2b]}+\rho^\gamma 1_{\mathbb{R}_+\setminus[\frac{a}{2},2b]}+(\rho-r)^2 1_{\mathbb{R}_+\setminus[\frac{a}{2},2b]}\Big).
\tag{15}
$$

We return to (14). We set $a=\min_{[0,T]\times\overline{\Omega}}r$ and $b=\max_{[0,T]\times\overline{\Omega}}r$. We write

$$
\begin{aligned}
&\sum_{K\in\mathcal{T}}\int_K\Big(p^n-p'(r^n_K)(\rho^n_K-r^n_K)-p(r^n_K)\Big)\,\mathrm{div}\,\boldsymbol{U}^n_h\,\mathrm{dx}\\
&\quad=\sum_{K,\rho^n_K\in[a/2,2b]}\int_K\Big(p^n-p'(r^n_K)(\rho^n_K-r^n_K)-p(r^n_K)\Big)\,\mathrm{div}\,\boldsymbol{U}^n_h\,\mathrm{dx}\\
&\quad+\sum_{K,\rho^n_K\in\mathbb{R}_+\setminus[a/2,2b]}\int_K\Big(p^n-p'(r^n_K)(\rho^n_K-r^n_K)-p(r^n_K)\Big)\,\mathrm{div}\,\boldsymbol{U}^n_h\,\mathrm{dx}
\end{aligned}
$$

Now using the behavior of p as ρ goes to infinity and (15) we obtain

$$
|\sum_{K\in\mathcal{T}}\int_K\Big(p^n-p'(r^n_K)(\rho^n_K-r^n_K)-p(r^n_K)\Big)\,\mathrm{div}\,\boldsymbol{U}^n_h\,\mathrm{dx}\,|\le c(r,\boldsymbol{U})\sum_{K\in\mathcal{T}}|K|E(\rho^n_K|r^n_K)
$$

We write

$$\sum_{K\in\mathcal{T}}(r_K^n-\rho_K^n)\int_K\frac{\nabla p(r^n)}{r^n}\cdot(\boldsymbol{u}^n-\boldsymbol{U}_h^n)\,\mathrm{dx}$$
$$=\sum_{\rho_K^n<\frac{a}{2}}(r_K^n-\rho_K^n)\int_K\frac{\nabla p(r^n)}{r^n}\cdot(\boldsymbol{u}^n-\boldsymbol{U}_h^n)\,\mathrm{dx}$$
$$+\sum_{\rho_K^n\in[\frac{a}{2},2b]}(r_K^n-\rho_K^n)\int_K\frac{\nabla p(r^n)}{r^n}\cdot(\boldsymbol{u}^n-\boldsymbol{U}_h^n)\,\mathrm{dx}$$
$$+\sum_{\rho_K^n>2b}(r_K^n-\rho_K^n)\int_K\frac{\nabla p(r^n)}{r^n}\cdot(\boldsymbol{u}^n-\boldsymbol{U}_h^n)\,\mathrm{dx}\,.$$

Using (15) and Poincare's inequality we obtain $\forall\delta>0$,

$$|\sum_{\rho_K^n<\frac{a}{2}}(r_K^n-\rho_K^n)\int_K\frac{\nabla p(r^n)}{r^n}\cdot(\boldsymbol{u}^n-\boldsymbol{U}_h^n)\,\mathrm{dx}\,|$$
$$\le c(r,\delta)\sum_{K\in\mathcal{T}}|K|E(\rho_K^n|r_K^n)+\delta||\boldsymbol{u}^n-\boldsymbol{U}_h^n||_{1,b}^2,$$
$$|\sum_{\rho_K^n\in[\frac{a}{2},2b]}(r_K^n-\rho_K^n)\int_K\frac{\nabla p(r^n)}{r^n}\cdot(\boldsymbol{u}^n-\boldsymbol{U}_h^n)\,\mathrm{dx}\,|$$
$$\le c(r,\delta)\sum_{K\in\mathcal{T}}|K|E(\rho_K^n|r_K^n)+\delta||\boldsymbol{u}^n-\boldsymbol{U}_h^n||_{1,b}^2.$$

Now we have

$$\sum_{\rho_K^n>2b}|K|(\rho_K^n)^\gamma\le c\sum_{K\in\mathcal{T}}|K|E(\rho_K^n|r_K^n),\qquad\sum_{\rho_K^n>2b}|K|(\rho_K^n)^{\gamma/2}\le c\sum_{K\in\mathcal{T}}|K|E(\rho_K^n|r_K^n)$$

Then,

$$|\sum_{\rho_K^n>2b}(r_K^n-\rho_K^n)\int_K\frac{\nabla p(r^n)}{r^n}\cdot(\boldsymbol{u}^n-\boldsymbol{U}_h^n)\,\mathrm{dx}\,|$$
$$\le c(r)\sum_{\rho_K^n>2b}\max(\rho_K^n,(\rho_K^n)^{\gamma/2})\int_K||\boldsymbol{u}^n-\boldsymbol{U}_h^n||\,\mathrm{dx}$$
$$c(r)\sum_{\rho_K^n>2b}\sqrt{|K|}(\rho_K^n)^{\gamma/2}||\boldsymbol{u}^n-\boldsymbol{U}_h^n||_{L^2(K)}$$
$$+c(r)\sum_{\rho_K^n>2b}|K|^{1/\gamma}\rho_K^n||\boldsymbol{u}^n-\boldsymbol{U}_h^n||_{L^{\gamma'}(K)}$$

$$
\begin{aligned}
&C(r,\delta)\sum_{K\in\mathcal{T}}|K|E(\rho_K^n|r_K^n)+\delta||\boldsymbol{u}^n-\boldsymbol{U}_h^n||_{1,b}^2\\
&+c(r,\delta)\sum_{K\in\mathcal{T}}|K|E(\rho_K^n|r_K^n)+\delta||\boldsymbol{u}^n-\boldsymbol{U}_h^n||_{L^{\gamma'}(\Omega)}^{\gamma'}\\
&\le C(r,\delta)\sum_{K\in\mathcal{T}}|K|E(\rho_K^n|r_K^n)+\delta||\boldsymbol{u}^n-\boldsymbol{U}_h^n||_{1,b}^2\\
&+c(r,\delta)\sum_{K\in\mathcal{T}}|K|E(\rho_K^n|r_K^n)+\delta||\boldsymbol{u}^n-\boldsymbol{U}_h^n||_{L^6(\Omega)}^6\\
&\le C(r,\delta)\sum_{K\in\mathcal{T}}|K|E(\rho_K^n|r_K^n)+\delta||\boldsymbol{u}^n-\boldsymbol{U}_h^n||_{1,b}^2
\end{aligned}
$$

since $\gamma \ge \frac{6}{5}$. We obtain finally

$$
\begin{aligned}
&\sum_{i,\sigma\in\mathcal{E}_{\rm int}}\frac{1}{2}\frac{|D_\sigma|}{\Delta t}\Big((u_{i,\sigma}^n-U_{i,\sigma}^n)^2-(u_{i,\sigma}^{n-1}-U_{i,\sigma}^{n-1})^2\Big)\\
&\quad+\sum_{K\in\mathcal{T}}\frac{|K|}{\Delta t}\Big(E(\rho_K^n|r_K^n)-E(\rho_K^{n-1}|r_K^{n-1})\Big)\\
&\quad\le c(r,\boldsymbol{U},\mu)\Big(\sum_{i,\sigma\in\mathcal{E}_{\rm int}}|D_\sigma|(u_{i,\sigma}^n-U_{i,\sigma}^n)^2+\sum_{K\in\mathcal{T}}|K|E(\rho_K^n|r_K^n)\Big)+\mathcal{R}^{n,h}.
\end{aligned}
$$

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