Stochastic Modeling for Heterogeneous Two-Phase Flow

M. Köppel, I. Kröker and C. Rohde

Abstract The simulation of multiphase flow problems in porous media often requires techniques for uncertainty quantification to represent parameter values that are not known exactly. The use of the stochastic Galerkin approach becomes very complex in view of the highly nonlinear flow equations. On the other hand collocation-like methods suffer from low convergence rates. To overcome these difficulties we present a hybrid stochastic Galerkin finite volume method (HSG-FV) that is in particular well-suited for parallel computations. The new approach is applied to specific two-phase flow problems including the example of a porous medium with a spatially random change in mobility. We emphasize in particular the issue of parallel scalability of the overall method.

1 Introduction

We consider the influence of stochastic effects on a two-phase flow model, that governs the infiltration of a wetting fluid into a porous medium which is initially filled by a nonwetting fluid. Let us assume that both fluids are immiscible and incompressible, and let us neglect gravitational forces. The fractional flow formulation of the capillarity-free case for some domain $D \subset \mathbb{R}^2$ and time $T > 0$ leads to the following problem [5]:

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$$\mathbf{v} = -\mathbf{K}\lambda(S) \nabla p \quad \text{and} \quad \text{div}(\mathbf{v}) = q \qquad \text{in } D \times (0, T), \tag{1}$$

$$\phi S_t + \text{div}\,(\mathbf{v} f(\mathbf{x}, S)) - q = 0 \qquad \text{in } D \times (0, T). \tag{2}$$

The unknowns are the saturation of the wetting fluid $S = S(\mathbf{x}, t) \in [0,1]$, the global pressure $p = p(\mathbf{x}, t) \in \mathbb{R}$, and the total velocity field $\mathbf{v} = \mathbf{v}(\mathbf{x}, t) \in \mathbb{R}^2$. The total mobility $\lambda = \lambda(S)$ and the fractional flow function $f = f(\mathbf{x}, S)$ are given nonlinear functions of the saturation and additionally of space for the flux. Furthermore $\mathbf{K} = \mathbf{K}(\mathbf{x})$ stands for the intrinsic permeability, $\phi = \phi(\mathbf{x})$ for the porosity, and $q = q(\mathbf{x}, t)$ for a source or sink. Appropriate initial and boundary conditions have to be added.

Uncertainty can effect solutions of (1), (2) through e.g. given parameter functions, initial and boundary data. In this case the unknowns depend also on corresponding random variables. Let us first assume that the velocity field $\mathbf{v}$ is given and it remains to solve the hyperbolic transport equation for the saturation. Under generic conditions a representation in the form of a polynomial chaos expansion (PCE) exists. Restriction of a (stochastically) weak formulation to a finite number of modes leads to the stochastic Galerkin method. Combined with a finite volume discretization in space the PCE approach yields a coupled deterministic system to be solved. The degree of coupling increases with the non-linearity of the considered equations and with the order of polynomial expansion. This fact increases the computational effort and significantly reduces the scalability in parallelisation. We have suggested a hybrid stochastic Galerkin finite volume method (HSG-FV) in [2], that extends the methods presented in [8, 11], for general transport equations and will develop it here for the two-phase problem. Together with a review on the stochastic setting the new method is formulated in Sect. 2. We stress that the HSG-FV relies on an adaptive combination of PCE with a multi-element decomposition of the stochastic domain. It leads to a deterministic system that is significantly weaker coupled than the pure PCE approach. Therefore, the HSG-FV method allows for more efficient parallelization. In Sect. 3 we apply the HSG-FV to the two-phase flow problem (1), (2) with a nonlinear continuous flux function, present the finite volume method and numerical examples. Moreover the computational effort of the HSG-FV method is discussed at the end of the section. At last we briefly present the application of the HSG-FV method to the two-phase flow problem in a heterogeneous porous medium with randomly disturbed discontinuous flux function in Sect. 4.

2 Hybrid Stochastic Galerkin Representation

Polynomial Chaos Let $\theta = \theta(\omega)$ be a random variable on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$, which satisfies $\theta \in L^2(\Omega)$. We assume that the distribution of θ is known and the probability density function (PDF) ρ is given. In this case the expectation of the random variable θ can be computed by $\mathbb{E}[\theta] := \int_\Omega \theta(\omega)\, \mathrm{d}\mathbb{P}(\omega) = \int \theta \, \mathrm{d}\rho(\theta)$. Then there exists a family $\left\{\phi_p(\theta)\right\}_{p\in\mathbb{N}_0}$ of $L^2(\Omega)$-orthonormal polynomials with respect to the PDF ρ. This means that $\left\{\phi_p(\theta)\right\}_{p\in\mathbb{N}_0}$ satisfies

$$\langle \phi_p(\theta),\, \phi_q(\theta) \rangle_{L^2(\Omega)} := \int_{\mathsf{I}} \phi_p(\theta)\phi_q(\theta)\,\mathrm{d}\rho(\theta) = \delta_{pq} \quad \text{for } p, q \in \mathbb{N}_0.$$

Here δ_{pq} denotes the Kronecker delta and I is the support of ϕ_p, for $p \in \mathbb{N}_0$. The choice of the polynomial basis depends on the PDF ρ. For example the Hermite polynomials could be used for the stochastic discretization of the Gauss distributed random variables, and Legendre polynomials allow the discretization of uniformly distributed random variables. Let $w = w(\mathbf{x}, t, \theta(\omega))$, $(\mathbf{x}, t) \in D \times [0, T]$, $\omega \in \Omega$ be a second order random field. Then w can be represented by the infinite series

$$w(\mathbf{x}, t, \theta(\omega)) = \sum_{p=0}^{\infty} w^p(\mathbf{x}, t)\phi_p(\theta(\omega)), \quad (\mathbf{x}, t, \omega) \in D \times [0, T] \times \Omega.$$

The coefficients $w^p = w^p(\mathbf{x}, t)$, $(\mathbf{x}, t) \in D \times [0, T]$ are defined by $w^p := \langle w,\, \phi_p \rangle_{L^2(\Omega)}$ for $p \in \mathbb{N}_0$. The expectation of the random field w is given by w^0, and the variance is given by the series $\sum_{p=1}^{\infty}(w^p)^2$. The truncation up to polynomial order $N_\mathrm{o} \in \mathbb{N}$ yields a finite sum

$$\Pi^{N_\mathrm{o}}\,[w]\,(\mathbf{x}, t, \theta(\omega)) := \sum_{p=0}^{N_\mathrm{o}} w^p(\mathbf{x}, t)\phi_p(\theta(\omega)), \quad (\mathbf{x}, t, \omega) \in D \times [0, T] \times \Omega. \tag{3}$$

The Cameron-Martin theorem [3] shows the convergence of (3). For more explanations we refer to [4, 12].

Extension to the Hybrid stochastic Galerkin discretization For the sake of brevity we assume that θ is uniformly distributed on the interval [0,1], ($\theta \sim \mathscr{U}(0, 1)$). The main idea of the presented method is the dyadical decomposition of the stochastic domain [0,1] and the appropriate rescaling of the polynomial basis $\{\phi_p\}_{p\in\mathbb{N}_0}$. Due to $\theta \sim \mathscr{U}(0, 1)$ we consider orthonormal Legendre polynomials. For $N_\mathrm{o} \in \mathbb{N}_0$ and $N_\mathrm{r} \in \mathbb{N}_0$ we define the *stochastic element* by $\mathsf{I}_l^{N_\mathrm{r}} := [2^{-N_\mathrm{r}}l, 2^{-N_\mathrm{r}}(l+1)]$, for $l = 0, \ldots, 2^{N_\mathrm{r}} - 1$, and a space of the piecewise polynomials $S^{N_\mathrm{o}, N_\mathrm{r}} := \left\{ w : [0, 1] \to \mathbb{R} \mid w_{|\mathsf{I}_l^{N_\mathrm{r}}} \in \mathbb{Q}_{N_\mathrm{o}}[\theta], \forall l \in \{0, \ldots, 2^{N_\mathrm{r}} - 1\} \right\}$, where $\mathbb{Q}_{N_\mathrm{o}}[\theta]$ denotes the space of real polynomials with degree $\le N_\mathrm{o}$. The basis of $S^{N_\mathrm{o}, N_\mathrm{r}}$ is spanned by the polynomials $\phi_{p,l}^{N_\mathrm{r}}$ defined by

$$\phi_{i,l}^{N_\mathrm{r}}(\xi) = \begin{cases} 2^{N_\mathrm{r}/2}\phi_i(2^{N_\mathrm{r}}\xi - l), & \xi \in \mathsf{I}_l^{N_\mathrm{r}}, \\ 0, & \text{else}, \end{cases} \quad i = 0, \ldots, N_\mathrm{o}, \quad l = 0, \ldots, 2^{N_\mathrm{r}} - 1.$$

The polynomials $\phi_{0,0}^{N_\mathrm{r}}, \ldots, \phi_{N_\mathrm{o}, 2^{N_\mathrm{r}}-1}^{N_\mathrm{r}}$ satisfy the orthogonality relation

$$\left\langle \phi_{i,k}^{N_\mathrm{r}},\, \phi_{j,l}^{N_\mathrm{r}} \right\rangle_{L^2(\Omega)} = \delta_{ij}\delta_{kl}, \tag{4}$$

and their support is given by the appropriate stochastic element $\text{supp}(\phi_{i,l}^{N_r}) = \mathsf{I}_l^{N_r}$. The projection $\Pi^{N_o,N_r} : L^2(\Omega) \to S^{N_o,N_r}$ of a second order random field $w(\mathbf{x}, t, \cdot) \in L^2(\Omega)$ is defined by $\Pi^{N_o,N_r}[w](\mathbf{x}, t, \theta) := \sum_{l=0}^{2^{N_r}-1} \sum_{i=0}^{N_o} w_{i,l}^{N_r}(\mathbf{x}, t)\phi_{i,l}^{N_r}(\theta)$, where the coefficients $w_{i,l}^{N_r}$ are defined by $w_{i,l}^{N_r}(\mathbf{x}, t) := \left\langle w(\mathbf{x}, t, \cdot), \phi_{i,l}^{N_r} \right\rangle_{L^2(\Omega)}$, for $0 \le p \le N_o$ and $0 \le l \le 2^{N_r} - 1$. The convergence of $\Pi^{N_o,N_r}[u]$ for $N_r, N_o \to \infty$ is discussed in [1]. The expectation and variance of the projection $\Pi^{N_o,N_r}[w]$ can be computed by the following formulae:

$$\mathbb{E}[\Pi^{N_o,N_r}[w](\mathbf{x}, t)] = \sum_{l=0}^{2^{N_r}-1} \sum_{p=0}^{N_o} w_{p,l}^{N_r}(\mathbf{x}, t) \left\langle \phi_{p,l}^{N_r}, \phi_{0,0}^{0} \right\rangle_{L^2(\Omega)}, \tag{5}$$

$$\begin{aligned} \text{Var}[\Pi^{N_o,N_r}[w](\mathbf{x}, t)] = &\sum_{l=0}^{2^{N_r}-1} \sum_{p=0}^{N_o} \sum_{q=0}^{N_o} w_{p,l}^{N_r}(\mathbf{x}, t) w_{q,l}^{N_r}(\mathbf{x}, t) \left\langle \phi_{p,l}^{N_r}\phi_{q,l}^{N_r}, \phi_{0,0}^{0} \right\rangle_{L^2(\Omega)} \\ &- \left(\mathbb{E}[\Pi^{N_o,N_r}[w](\mathbf{x}, t)]\right)^2. \end{aligned} \tag{6}$$

Together with the orthogonality relation (4) of $\phi_{q,l}^{N_r}$ for $q = 0, \ldots, N_o$, $l = 0, \ldots, 2^{N_r} - 1$ and the fact $\phi_{0,0}^{0} \equiv 1$ for $\mathscr{U}(0, 1)$ we obtain

$$\text{Var}[\Pi^{N_o,N_r}[w](\mathbf{x}, t)] = \sum_{l=0}^{2^{N_r}-1} \sum_{p=0}^{N_o} \left(w_{p,l}^{N_r}(\mathbf{x}, t)\right)^2 - \left(\mathbb{E}[\Pi^{N_o,N_r}[w](\mathbf{x}, t)]\right)^2.$$

3 Hybrid Stochastic Galerkin for the Two-Phase Flow Problem with Continuous Flux Function

In the deterministic case the continuous fractional flux function is defined as equivalent to $f(\mathbf{x}, S) \equiv f_w(S)$. $f(\mathbf{x}, S) \equiv f_w(S)$. The fractional flux of the wetting phase $f_w : [0, 1] \to \mathbb{R}$ is given by $f_w(S) = f_w(S, S^e) := \frac{\lambda_w(S,S^e)}{\lambda_w(S,S^e)+\lambda_o(S,S^e)}$. Here the mean mobility λ is given by $\lambda(S, S^e) = \lambda_o(S, S^e) + \lambda_w(S, S^e)$, where λ_w denotes the total mobility of the wetting phase and λ_o the total mobility of the non-wetting phase. The effective saturation S^e is defined by $S^e(S) := (S - S_{wc})/(1 - S_{or} - S_{wc})$, with the connate saturation $S_{wc} \in$ [0,1] and the irreducible saturation $S_{or} \in$ [0,1]. If the condition

$$\lambda(S, S^e) = const \tag{7}$$

is fulfilled, then the total velocity field $\mathbf{v}$ does not depend on the change of the saturation S. We use this property of $\mathbf{v}$ to stress the influence of the random disturbance.

In this section we consider the application of the HSG discretization to the two-phase flow problem with a given randomly disturbed velocity field. For this sake we replace $\mathbf{v} = (v^x, v^y)$ in (1) by $\mathbf{v}_s$ given by $\mathbf{v}_s = (v^x + c\theta, v^y)$, for $c \in \mathbb{R}$ and $\theta \sim \mathscr{U}(0, 1)$. Further we replace $\mathbf{v}_s$ and S in the Eq. (2) by their HSG representations $\Pi^{N_\mathrm{o},N_\mathrm{r}}[\mathbf{v}_s]$ and $\Pi^{N_\mathrm{o},N_\mathrm{r}}[S]$ and obtain

$$\mathbf{v} = -\mathbf{K}\lambda(S) \bigtriangledown p \quad \text{and} \quad \operatorname{div}(\mathbf{v}) = q, \tag{8}$$

$$\Pi^{N_\mathrm{o},N_\mathrm{r}}[S]_t + \operatorname{div}\left(\Pi^{N_\mathrm{o},N_\mathrm{r}}[\mathbf{v}_s]\, f\left(\Pi^{N_\mathrm{o},N_\mathrm{r}}[S]\right)\right) - q = 0, \tag{9}$$

$$S(\cdot, 0) = S_0. \tag{10}$$

We test the Eq. (9) with $\phi_{p,l}^{N_\mathrm{r}}$ for $p = 0, \dots, N_\mathrm{o}$ and $l = 0, \dots, 2^{N_\mathrm{r}} - 1$, that is

$$\int_\Omega \left(\Pi^{N_\mathrm{o},N_\mathrm{r}}[S]_t + \operatorname{div}\left(\Pi^{N_\mathrm{o},N_\mathrm{r}}[\mathbf{v}_s]\, f\left(\Pi^{N_\mathrm{o},N_\mathrm{r}}[S]\right)\right) - q\right) \phi_{p,l}^{N_\mathrm{r}}\, \mathrm{d}\mathbb{P}(\omega),$$

and obtain the system

$$\partial_t S_\alpha^{N_\mathrm{r}} + \operatorname{div}\left\langle \Pi^{N_\mathrm{o},N_\mathrm{r}}[\mathbf{v}_s]\, f\left(\Pi^{N_\mathrm{o},N_\mathrm{r}}[S]\right),\, \phi_\alpha^{N_\mathrm{r}}\right\rangle_{L^2(\Omega)} - \left\langle q,\, \phi_\alpha^{N_\mathrm{r}}\right\rangle_{L^2(\Omega)} = 0, \tag{11}$$

with initial values

$$S_\alpha^{N_\mathrm{r}}(\cdot, 0) = \left\langle S_0,\, \phi_\alpha^{N_\mathrm{r}}\right\rangle_{L^2(\Omega)} \tag{12}$$

for the multi-index $\alpha = (p, l)$, $p = 0, \dots, N_\mathrm{o}$ and $l = 0, \dots, 2^{N_\mathrm{r}} - 1$. The HSG system (11) is symmetric hyperbolic [2].

Finite volume method For the computation of the numerical solution of the hyperbolic system (11), (12) the semi-discrete central-upwind scheme introduced by Kurganov and Petrova in [7] is applied. This central-upwind method allows to work with larger systems with a minimum of requirements on the eigenvalues. Together with the HSG discretization we obtain the following numerical scheme on the triangulation $\mathscr{T} = \bigcup T_j$ of $D = (-1, 1) \times (-1, 1)$, consisting of triangular cells T_j

$$\begin{aligned} \frac{d}{dt}\bar{\mathbf{S}}_j := &- \frac{1}{|T_j|} \sum_{k=1}^{3} h_{jk} \left(\frac{a_{jk}^{in} \mathbf{F}(\tilde{\mathbf{S}}_{jk}, M_j(k), t) + a_{jk}^{out} \mathbf{F}(\tilde{\mathbf{S}}_j, M_j(k), t)}{a_{jk}^{in} + a_{jk}^{out}} \right) \cdot \mathbf{n}_{jk} \\ &+ \frac{1}{|T_j|} \sum_{k=1}^{3} h_{jk} \frac{a_{jk}^{in} a_{jk}^{out}}{a_{jk}^{in} + a_{jk}^{out}} \left[\tilde{\mathbf{S}}_{jk}(M_j(k)) - \tilde{\mathbf{S}}_j(M_j(k)) \right] + \mathbf{q}_j \end{aligned}$$

for $\alpha = 0, \dots, P = (N_\mathrm{o} + 1)2^{N_\mathrm{r}} - 1$. Here $\bar{\mathbf{S}}_j = \left(\bar{S}^0, \dots, \bar{S}^P\right)$ is the cell average on the triangle $T_j \in \mathscr{T}$. The flux vector is given by $\mathbf{F} = (F^0, \dots, F^P)^T$, where

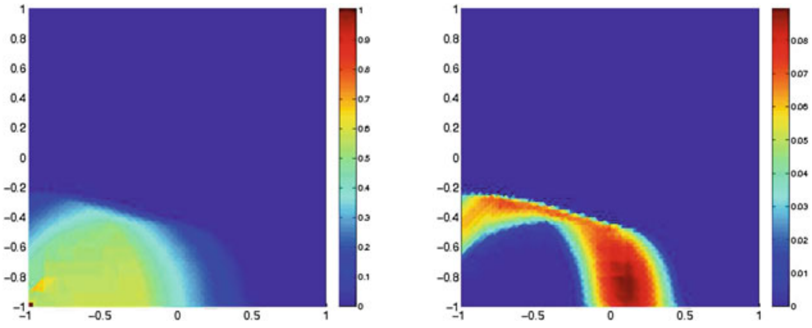


Fig. 1 **a** Expectation, **b** Variance for the problem (8)–(10) with a randomly perturbed velocity field and non-linear flux function. Computed with $N_{\mathrm{r}} = 5$, $N_{\mathrm{o}} = 4$, $T = 6$, spatial adaptivity with maximal refinement level 4

$$F^{\alpha}(\mathbf{S}, \mathbf{x}, t) := \left\langle f\left(\sum_{\beta=0}^{P} S^{\beta}(\mathbf{x}, t)\phi_{\beta}^{N_{\mathrm{r}}}\right) \Pi^{N_{\mathrm{o}}, N_{\mathrm{r}}}[\mathbf{v}_s](\mathbf{x}, t),\ \phi_{\alpha}^{N_{\mathrm{r}}}\right\rangle_{L^2(\Omega)} \qquad \text{for } \alpha = 0, \ldots, P.$$

The initial values are given by $\bar{S}_j^{\alpha,0} := \frac{1}{T_j}\int_{T_j} S_0 \left\langle \phi_{0,0}^{0},\ \phi_{\alpha}^{N_{\mathrm{r}}}\right\rangle_{L^2(\Omega)}$ for $\alpha = 0, \ldots, P$. For the triangle $T_j \in \mathscr{T}$, h_{jk} with $k = 1, 2, 3$ denotes the length of the k-th edge. The point $M_j(k)$ is the midpoint of the k-th edge and $\mathbf{n}_{jk}$ is the outer normal on the k-th edge, a_{jk}^{in} and a_{jk}^{out} are the so-called directional local speeds associated with the k-th edge. We use the Runge-Kutta method for the time discretization, the CFL-condition depends on the Jacobian of $\mathbf{F}$. For the computation of the reconstructions $\tilde{\mathbf{S}}_j$ and $\tilde{\mathbf{S}}_{jk}$ we refer to the work of Kurganov and Petrova [7].

Remark 1 The velocity field $\mathbf{v}$ is computed with the Taylor-Hood FEM approach, respective CG-solver, implemented in the FEM-toolbox *Alberta* [9]. The initial Delaunay triangulation is generated below the mesh generator *Triangle* [10]. We use an adaptive dynamic mesh refinement and coarsening, which uses discrete gradient heuristics and hierarchical refinement given by the bisection of the triangle on the longest edge. We perform our computation on the domain $D = (-1, 1) \times (-1, 1)$ with the initial edge-length 0.1 and max. refinement level 4.

Numerical experiments Let us apply the previously introduced numerical flux to (9), (10). We define mean mobility functions of the wetting λ_w and non-wetting λ_o phase by $\lambda_w(S, S^e) := \frac{(S^e(S))^2}{\mu_w(S^e(S)^2+(1-S^e(S))^2)}$ and $\lambda_o(S, S^e) := \frac{(1-S^e(S))^2}{\mu_o(S^e(S)^2+(1-S^e(S))^2)}$. Then the condition (7) is satisfied for $\mu_w = \mu_o = 0.3 \cdot 10^{-3}$. Therefore we can again use the velocity field $\mathbf{v} = (v^x, v^y)$ computed with the FEM framework *Alberta* at the first time-step during the computation. The randomly perturbed velocity field $\mathbf{v}_s$ is given by $\mathbf{v}_s = (v^x + c\theta, v^y)$, where $c = 0.1$ and $\theta \sim \mathscr{U}(0, 1)$. The irreducible and connate saturations are again given by $S_{or} = 0.3$ and $S_{wc} = 0.1$. Figure 1 shows the expectation and variance computed with (5), (6) at $T = 6$.

Table 1 (a) L^1-error for the problem (8)–(10) with a randomly perturbed velocity field and non-linear flux function, at $T = 6$

	(a)					(b)			
N_o	$N_r = 2$	$N_r = 3$	$N_r = 4$	$N_r = 5$	N_o	$N_r = 2$	$N_r = 3$	$N_r = 4$	$N_r = 5$
2	2.74e-2	1.89e-2	1.10e-2	1.11e-3	2	18.5	21.2	17.8	19.6
3	2.54e-2	1.79e-2	1.0e-2	1.01e-3	3	48.4	56.2	46.2	49.7
4	2.34e-2	1.74e-2	1.10e-2	1.1e-3	4	109.9	129.3	107.3	114.3

(b) Computation time (in hours) for the problem (8)–(10) with a randomly perturbed velocity field and nonlinear flux function, at $T = 6$ computed on 2^{N_r} CPU's

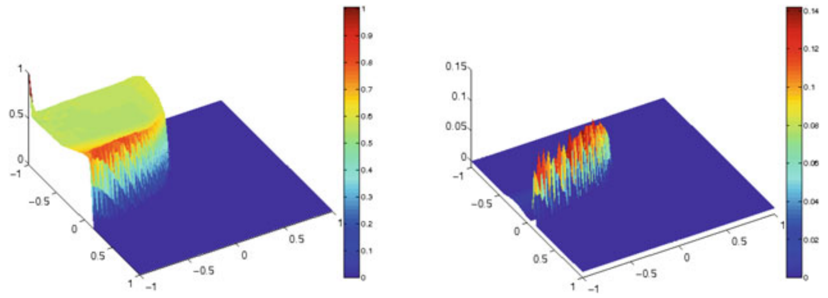
Up to our knowledge there is no analytical solution of the problem. A comparable simulation with the Monte Carlo finite volume (MC-FV) method in two space dimensions is not possible with nowadays computer power. Therefore we compare our numerical results with the most fine HSG-FV solution we could realize, that means $N_r = 6$ and $N_o = 4$. Due to the accuracy tests in one space dimension in comparison with a MC-solution, considered in [2], we can expect that this comparison represents the behaviour of the method correctly. Table 1 shows the L^1-error and computing times for $N_r = 1, \ldots, 5$ and $N_o = 1, \ldots, 4$. These results seem to indicate that the overall approach leads to convergence for increasing N_r and N_o. The computing times show, that the computational effort per node does not change significantly for increasing N_r and a constant number of stochastic elements $\mathbf{I}_l^{N_r}$ per node.

4 HSG for Two-Phase Flows in a Heterogeneous Porous Media

Now we focus on two-phase flow problems with a non-linear, spatially discontinuous flux function. A specific application is a heterogeneous porous medium characterised by two different materials. In the deterministic case the considered spatial domain D is decomposed such that $D = D^1 \cup D^2$. Within one subdomain D^i, $i = 1, 2$, the medium is supposed to be homogeneous. Hence, the descriptive parameters depend on the spatial position. By the introduction of a discontinuity function $\gamma : D \to [0, 1]$, in order to determine the location, and a uniformly distributed random variable θ, we define the randomly perturbed discontinuous fractional flux

$$f_w(\mathbf{x}, \gamma, S, \theta) := \gamma(x + c\theta, y) f^{1,w}(S) + (1 - \gamma(x + c\theta, y)) f^{2,w}(S), \quad \mathbf{x} \in D, \tag{13}$$

where $c \in \mathbb{R}$. The related HSG of the randomly perturbed problem (1), (2) is (non-strictly) hyperbolic (cf. [2, 6] for details). Figure 2 shows expectation and variance of the numerical solution of the problem (1), (2) with a randomly perturbed discontinuous flux (13) and deterministic velocity field $\mathbf{v}$ for $N_o = 3$ and $N_r = 3$ at $T = 15$. Computed with $S^1_{wc} = 0.1$, $S^1_{or} = 0.3$ and $S^2_{wc} = 0.4$, $S^2_{or} = 0.2$, $\mu_o = 3 \cdot 10^{-3}$, $\mu_w = 3 \cdot 10^{-3}$, $\theta \sim \mathscr{U}(0, 1)$ and coefficient $c = 0.4$.



(a) (b)

Fig. 2 **a** Expectation, **b** Variance for the quarter five-spot problem with random perturbed discontinuous flux. Computed with $N_{\mathrm{r}} = 3$, $N_{\mathrm{o}} = 3$ at $T = 15$

The numerical results show a realistic behaviour close to the discontinuity, in particular the variance shows the expected dependence of the uncertainty.

5 Outlook

In the future work we intend to develop an appropriate stochastic representation of the total velocity field **v** in the elliptic equation (1), and apply the developed numerical scheme to more general heterogeneous two-phase flow problems.

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