On Aposteriori Error Analysis of DG Schemes Approximating Hyperbolic Conservation Laws

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Abstract This contribution is concerned with aposteriori error analysis of discontinuous Galerkin (dG) schemes approximating hyperbolic conservation laws. In the scalar case the aposteriori analysis is based on the L^1 contraction property and the doubling of variables technique. In the system case the appropriate stability framework is in L^2, based on relative entropies. It is only applicable if one of the solutions, which are compared to each other, is Lipschitz. For dG schemes approximating hyperbolic conservation laws neither the entropy solution nor the numerical solution need to be Lipschitz. We explain how this obstacle can be overcome using a reconstruction approach which leads to an aposteriori error estimate.

1 Introduction

We investigate numerical approximations of systems of hyperbolic conservation laws. The problem has the general form

$$\mathbf{u}_t + \operatorname{div}(\mathbf{f}(\mathbf{u})) = \mathbf{0}, \tag{1}$$

where $\mathbf{u}(t, x) \in U$, for some state space $U \subset \mathbb{R}^d$ and we assume the flux function satisfies $\mathbf{f} \in C^2(U, \mathbb{R}^d)$. We study semi-discretisations of (1) by the discontinuous Galerkin (dG) method and derive an aposteriori error estimate. The discretisation of

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(1) by finite volume and dG schemes is standard, as it is well known that solutions may develop discontinuities in finite time. However, the aposteriori analysis has been only developed for special cases. In [9] aposteriori error estimates (in L^1) are derived in the scalar case. These arguments were generalized to fully applicable Runge–Kutta dG schemes in the scalar case in [4]. As pointed out in [12] these estimates are based on exploiting the L^1-contraction property of scalar hyperbolic conservation laws and the doubling of variables technique. The work [8], which establishes aposteriori error estimates for Friedrichs systems, is in the same spirit, but replaces the L^1 contraction framework by the relative entropy technique, which dates back to [2, 5].

A different approach is the construction of localized aposteriori error estimates via adjoint problems for space-time dG schemes in [7]. Nodal super-convergence of dG schemes was investigated in a sequence of works by Adjerid and coworkers, see [1] and references therein.

All the estimates mentioned before restrict themselves to one of the following two cases:

1. Equation (1) is required to be a scalar equation or a Friedrichs system.
2. Only continuous solutions $\mathbf{u}$ of (1) are considered.

In case of the estimates using adjoint problems the latter restriction is introduced via the stability assumptions on the solutions of the adjoint problems.

The main difficulty in constructing error estimates in the spirit of [4, 8, 9] for (multidimensional) systems of hyperbolic conservation laws without assuming (Lipschitz) continuity of solutions is encapsulated by the following: The appropriate stability theory for this class of PDE is the relative entropy technique. It has certain features (in contrast to the L^1-contraction stability theory available for scalar conservation laws, see [3, Chap. 6.2]) which make its use for constructing aposteriori error estimates more difficult:

1. It cannot be used to compare two discontinuous solutions but it can only compare a Lipschitz continuous solution to another (possibly discontinuous) one. At the same time the numerical solution obtained from a finite volume or dG scheme will be discontinuous and the exact (entropy) solution might also be discontinuous, even for smooth initial data.
2. It leads to an L^2-stability framework which is difficult to use with measure valued residuals, which may not belong to L^2.

We will sketch how to overcome these difficulties for dG spatial discretisations in one space dimension by a reconstruction technique. The details of our arguments, in particular the proofs of Lemma 1 and Theorem 1, can be found in [6]. Our error estimate is expected to be of optimal order (determined by the order of the dG scheme and the regularity of the solution) in the case the entropy solution is Lipschitz continuous. In the case the entropy solution is not Lipschitz the error estimate is not expected to converge, see Remark 7, but in that case uniqueness of the entropy solution cannot be guaranteed anyway.

To avoid any difficulties introduced by boundary conditions, we consider the following version of (1):

$$\mathbf{u}_t + (\mathbf{f}(\mathbf{u}))_x = \mathbf{0} \text{ in } (0,\infty) \times S^1, \tag{2}$$

for some initial data $\mathbf{u}_0 \in L^\infty(S^1, U)$, where S^1 denotes the periodic unit interval with the endpoints being identified with each other. We will assume that (2) is endowed with (at least) one convex entropy/entropy flux pair (η, q), i.e. $q, \eta \in C^1(U, \mathbb{R})$, η is strictly convex and

$$\mathrm{D}\eta\,\mathrm{D}\mathbf{f} = \mathrm{D}q, \tag{3}$$

where D denotes the Jacobian/gradient. For systems of hyperbolic conservation laws there is (usually) only one (physical) entropy/entropy flux pair, while in the scalar case every convex function is an entropy. Equation (3) gives rise to the additional conservation law

$$\eta(\mathbf{u})_t + q(\mathbf{u})_x = 0 \text{ in } (0,\infty) \times S^1, \tag{4}$$

for every strong solution $\mathbf{u}$ of (2). This is crucial for defining entropy solutions:

Definition 1 (*entropy solution*) A function $\mathbf{u} \in L^\infty([0,\infty) \times S^1, U)$ is called an *entropy solution* of the initial boundary value problem (2), with respect to the entropy/entropy-flux pair (η, q), if

$$\int_0^\infty \int_{S^1} \mathbf{u}\cdot\boldsymbol{\phi}_t + \mathbf{f}(\mathbf{u})\cdot\boldsymbol{\phi}_x \,\mathrm{d}x\,\mathrm{d}t + \int_{S^1} \mathbf{u}_0\cdot\boldsymbol{\phi}(0,\cdot)\,\mathrm{d}x = 0 \,\forall\, \boldsymbol{\phi} \in C_c^\infty([0,\infty), \mathbb{R}^d) \tag{5}$$

and

$$\int_0^\infty \int_{S^1} \eta(\mathbf{u})\phi_t + q(\mathbf{u})\phi_x \,\mathrm{d}x\,\mathrm{d}t + \int_{S^1} \eta(\mathbf{u}_0)\phi(0,\cdot)\,\mathrm{d}x \geq 0 \quad \forall \phi \in C_c^\infty([0,\infty) \times S^1, [0,\infty)). \tag{6}$$

We consider approximations of (2) by a class of semi-discrete dG schemes using Godunov type numerical fluxes in Sect. 2. In Sect. 3 we will introduce an explicitly computable reconstruction $\hat{\mathbf{u}}$ of the numerical solution $\mathbf{u}_h$. The reconstruction $\hat{\mathbf{u}}$ is continuous and satisfies a perturbed version of (2) with residuals in L^2. We state an aposteriori estimate, based on the relative entropy framework, of the error between the exact solution $\mathbf{u}$ and the reconstruction $\hat{\mathbf{u}}$ in Sect. 4. This implies an explicitly computable estimate for the difference of the entropy solution $\mathbf{u}$ and the numerical solution $\mathbf{u}_h$, see Theorem 1. We will compare this result to the result from [4] in the scalar case.

2 Semi-Discrete Discontinuous Galerkin Schemes

Before we state the dG schemes under consideration, let us fix some notation. We will discretise (2) in space using a consistent dG scheme. Let $I := [0, 1]$ be the unit interval and choose $0 = x_0 < x_1 < \cdots < x_N = 1$. By $I_n = [x_n, x_{n+1}]$ we denote the n–th sub-interval and by $h_n := x_{n+1} - x_n$ its size. Let $\mathbb{P}_p(I)$ be the space of polynomials of degree less than or equal to p on I, then we denote

$$\mathbb{V}_p := \left\{ \mathbf{g} : I \to \mathbb{R}^d \ : \ (g_i)|_{I_n} \in \mathbb{P}_p(I_n) \text{ for } i = 1, \dots, d, \ n = 0, \dots, N-1 \right\}, \tag{7}$$

where $\mathbf{g} = (g_1, \dots, g_d)^T$, is the usual space of piecewise p–th degree polynomials for vector valued functions over I. In addition, we define jump operators such that

$$[\mathbf{g}]_n := \mathbf{g}(x_n^-) - \mathbf{g}(x_n^+) := \lim_{s \searrow 0} \mathbf{g}(x_n - s) - \lim_{s \searrow 0} \mathbf{g}(x_n + s). \tag{8}$$

We will examine the following class of semi-discrete numerical schemes where $\mathbf{u}_h \in C^1([0, T), \mathbb{V}_p)$ is determined such that

$$0 = \sum_{n=0}^{N-1} \int_{I_n} ((\mathbf{u}_h)_t \cdot \boldsymbol{\phi} - \mathbf{f}(\mathbf{u}_h) \cdot \boldsymbol{\phi}_x) \, \mathrm{d}\, x + \sum_{n=0}^{N-1} \mathbf{F}(\mathbf{u}_h(x_n^-), \mathbf{u}_h(x_n^+)) \cdot [\boldsymbol{\phi}]_n \ \forall \boldsymbol{\phi} \in \mathbb{V}_p. \tag{9}$$

In the sequel we will assume that (9) has a solution and, in particular, that $\mathbf{u}_h$ takes values in U. We also set

$$[\mathbf{u}_h]_0 := \mathbf{u}_h(x_N^-) - \mathbf{u}_h(x_0^+) \tag{10}$$

to account for the periodic boundary conditions. In (9) $\mathbf{F} : U^2 \subset \mathbb{R}^{2d} \to \mathbb{R}^d$ is a numerical flux function. We restrict our attention to a certain class of numerical flux functions. We impose that there exists a function

$$\mathbf{w} : U \times U \to U \text{ such that } \mathbf{F}(\mathbf{u}, \mathbf{v}) = \mathbf{f}(\mathbf{w}(\mathbf{u}, \mathbf{v})) \tag{11}$$

and that there exists a constant $L > 0$ such that $\mathbf{w}$ satisfies

$$|\mathbf{w}(\mathbf{u}, \mathbf{v}) - \mathbf{u}| \leq L|\mathbf{u} - \mathbf{v}|, \qquad |\mathbf{w}(\mathbf{u}, \mathbf{v}) - \mathbf{v}| \leq L|\mathbf{u} - \mathbf{v}| \quad \forall \, \mathbf{u}, \mathbf{v} \in \mathbb{R}^d. \tag{12}$$

Remark 1 The restriction of the flux functions, in general, restricts our analysis to fluxes of Godunov type. Still, fluxes of Roe or Osher-Solomon type fall into this framework in some situations. We need this restriction in order to define the reconstructions in Sect. 3. If we do not have this restriction we may still define reconstructions but the error estimate will no longer be of optimal order for smooth solutions of (2).

3 Reconstructions

In order to derive error estimates for the scheme (9) we introduce reconstructions, which are similar to those used for dG schemes in time in [11], denoted by $\hat{\mathbf{u}}$ and $\hat{\mathbf{f}}$. For brevity we will omit the time dependency of all quantities in this section.

Definition 2 (*Reconstruction of* u_h) The reconstruction $\hat{\mathbf{u}}$ is the unique element of $\mathbb{V}_{p+1}$ such that

$$\sum_{n=0}^{N-1} \int_{I_n} \hat{\mathbf{u}} \cdot \boldsymbol{\phi} \, \mathrm{d}\, x = \sum_{n=0}^{N-1} \int_{I_n} \mathbf{u}_h \cdot \boldsymbol{\phi} \, \mathrm{d}\, x \; \forall \, \boldsymbol{\phi} \in \mathbb{V}_{p-1} \tag{13}$$

and

$$\begin{aligned} \hat{\mathbf{u}}(x_n^+) &= \mathbf{w}(\mathbf{u}_h(x_n^-), \mathbf{u}_h(x_n^+)) \\ \hat{\mathbf{u}}(x_{n+1}^-) &= \mathbf{w}(\mathbf{u}_h(x_{n+1}^-), \mathbf{u}_h(x_{n+1}^+)) \end{aligned} \quad \forall \, n \in \{0, \ldots, N-1\} \tag{14}$$

recalling that $\mathbf{u}_h(x_0^-) := \mathbf{u}_h(x_N^-)$, and $\mathbf{u}_h(x_N^+) := \mathbf{u}_h(x_0^+)$.

Definition 3 (*Reconstruction of* $f(u_h)$) The reconstruction $\hat{\mathbf{f}}$ is the unique element of $\mathbb{V}_{p+1}$ such that

$$\begin{aligned} \sum_{n=0}^{N-1} \int_{I_n} \hat{\mathbf{f}}_x \cdot \boldsymbol{\phi} \, \mathrm{d}\, x = &- \sum_{n=0}^{N-1} \int_{I_n} \mathbf{f}(\mathbf{u}_h) \cdot \boldsymbol{\phi}_x \, \mathrm{d}\, x \\ &+ \sum_{n=0}^{N-1} \mathbf{f}(\mathbf{w}(\mathbf{u}_h(x_n^-), \mathbf{u}_h(x_n^+))) \cdot [\boldsymbol{\phi}]_n \; \forall \, \boldsymbol{\phi} \in \mathbb{V}_p \end{aligned} \tag{15}$$

and

$$\hat{\mathbf{f}}(x_n^+) = \mathbf{f}(\mathbf{w}(\mathbf{u}_h(x_n^-), \mathbf{u}_h(x_n^+))) \; \forall \, n \in \{0, \ldots, N-1\}. \tag{16}$$

Lemma 1 (Properties of the reconstruction) *The reconstructions* $\hat{\mathbf{u}}$ *and* $\hat{\mathbf{f}}$ *are uniquely defined and continuous. Moreover, the reconstructions are explicitly and locally computable.*

Proof The proof of uniqueness and continuity of $\hat{\mathbf{u}}$ is straightforward. To assert the continuity of $\hat{\mathbf{f}}$, we use an analagous argument to that of [11, Lemma 2.1] by testing (15) with piecewise constant functions.

Using the specific reconstruction (15) and (9) we see that

$$0 = \sum_{n=0}^{N-1} \int_{I_n} ((\mathbf{u}_h)_t \cdot \boldsymbol{\phi} - \hat{\mathbf{f}}_x \cdot \boldsymbol{\phi}) \, \mathrm{d}\, x \; \forall \, \boldsymbol{\phi} \in \mathbb{V}_p. \tag{17}$$

As $(\mathbf{u}_h)_t$ and $\hat{\mathbf{f}}_x$ are piecewise polynomials of degree p this implies the *pointwise* equation

$$(\mathbf{u}_h)_t + \hat{\mathbf{f}}_x = \mathbf{0} \quad \text{a.e. in } S^1 \tag{18}$$

which is equivalent to

$$\hat{\mathbf{u}}_t + \mathbf{f}(\hat{\mathbf{u}})_x = \mathbf{R}_h := \hat{\mathbf{u}}_t - (\mathbf{u}_h)_t + \mathbf{f}(\hat{\mathbf{u}})_x - \hat{\mathbf{f}}_x. \tag{19}$$

Remark 2 Equation (19) shows that $\hat{\mathbf{u}}$ solves a perturbed version of (1). As $\mathbf{f}(\hat{\mathbf{u}})$ and $\hat{\mathbf{f}}$ are continuous and piecewise polynomial, $\mathbf{R}_h(t,\cdot) \in L^2(S^1)$ for all $t > 0$. In addition $\mathbf{R}_h$ is explicitly computable.

Remark 3 The reconstruction $\hat{\mathbf{u}}$ is Lipschitz continuous because it is piecewise polynomial and continuous. However, it must be noted that it is not clear whether the Lipschitz constant of $\hat{\mathbf{u}}$ is uniformly bounded if h goes to zero.

Remark 4 It might be expected that the x-derivatives appearing in the definition of $\mathbf{R}_h$ in (19) might lead to a suboptimal order of the error estimate. This is precisely the point at which we need assumption (11) in order to obtain an error estimate of optimal order. For details we refer to [6].

Due to Remarks 2 and 3 the relative entropy framework can be used to estimate the difference between $\hat{\mathbf{u}}$ and the entropy solution $\mathbf{u}$ in terms of $\mathbf{R}_h$ and $\hat{\mathbf{u}}$ even if $\mathbf{u}$ is discontinuous. Once we obtained such an estimate we can estimate the error of the numerical scheme by

$$\|\mathbf{u} - \mathbf{u}_h\|_{L^\infty(0,T;L^2(S^1))} \leq \|\mathbf{u} - \hat{\mathbf{u}}\|_{L^\infty(0,T;L^2(S^1))} + \|\hat{\mathbf{u}} - \mathbf{u}_h\|_{L^\infty(0,T;L^2(S^1))}. \tag{20}$$

4 The Aposteriori Error Estimate

In the remainder of this paper we make the following assumption on the flux and the entropy which is standard in relative entropy arguments. We will assume that there are constants $0 < C_{\overline{\mathbf{f}}} < \infty$ and $0 < C_{\underline{\eta}} < C_{\overline{\eta}} < \infty$ such that

$$|\mathbf{v}^T \mathrm{H}[\mathbf{f}(\mathbf{u})]\mathbf{v}| \leq C_{\overline{\mathbf{f}}}|\mathbf{v}|^2, \quad C_{\underline{\eta}}|\mathbf{v}|^2 \leq \mathbf{v}^T \mathrm{H}[\eta(\mathbf{u})]\mathbf{v} \leq C_{\overline{\eta}}|\mathbf{v}|^2 \quad \forall\, \mathbf{v} \in \mathbb{R}^d, \mathbf{u} \in U, \tag{21}$$

where $|\cdot|$ is the Euclidean norm for vectors, and $\mathrm{H}[\cdot]$ denotes the Hessian of a function or vector field.

Using an analogous argument to [3, Theorem: 5.3.1] we infer

Theorem 1 (Aposteriori error estimate) *Let* $\mathbf{f} \in W_2^\infty(U, \mathbb{R}^d)$ *satisfy* (21). *Let* $\mathbf{u}$ *be an entropy solution of* (2) *with periodic boundary conditions. Then, for* $0 \leq t \leq T$ *the error between the numerical solution* $\mathbf{u}_h$, *given by* (9), *and* $\mathbf{u}$ *satisfies*

$$\begin{aligned}\|\mathbf{u}(t,\cdot)-\mathbf{u}_h(t,\cdot)\|^2_{L^2(S^1)} &\le \|\hat{\mathbf{u}}(t,\cdot)-\mathbf{u}_h(t,\cdot)\|^2_{L^2(S^1)}\\ &\quad + C_{\underline{\eta}}^{-1}\Big(\|\mathbf{R}_h\|^2_{L^2((0,t)\times S^1)} + C_{\overline{\eta}}\|\mathbf{u}_0-\hat{\mathbf{u}}_0\|^2_{L^2(S^1)}\Big)\\ &\quad\times \exp\Big(\int_0^t \frac{C_{\overline{\eta}}C_{\overline{\mathbf{f}}}\|\hat{\mathbf{u}}_x(s,\cdot)\|_{L^\infty(S^1)}+C^2_{\overline{\eta}}}{C_{\underline{\eta}}}\,\mathrm{d}\,s\Big),\end{aligned} \tag{22}$$

where $\hat{\mathbf{u}}$ *is the reconstruction of* $\mathbf{u}_h$ *given in Definition (2) and* $\mathbf{R}_h$ *is defined in* (19).

Remark 5 Note that all the terms on the right hand side of (22) are explicitly computable. Provided $\|\hat{\mathbf{u}}_x(s,\cdot)\|_{L^\infty(S^1)}$ is uniformly bounded in h the right hand side of (22) is expected to be of optimal order. This is expected in case of an at least Lipschitz continuous entropy solution. This is also confirmed by numerical experiments, see [6].

Remark 6 In [6] it is shown that $\|\mathbf{R}_h\|^2_{L^2((0,t)\times S^1)}$ can be estimated without explicitly computing $\hat{\mathbf{u}}$. The estimate [6, Lemma 5.6] is rather technical but it shows that for every $t\in(0,T)$

$$\begin{aligned}\|\mathbf{R}_h\|_{L^2(S^1)} \lesssim C\sum_{n=0}^{N-1} h_n\Big(\big|[\mathbf{u}_h]_n\big|^2+\big|[\mathbf{u}_h]_{n+1}\big|^2\Big)\\ \times\Big(\frac{\big|[\mathbf{u}_h]_n\big|+\big|[\mathbf{u}_h]_{n+1}\big|}{h_n}+\|(\mathbf{u}_h)_x\|_{L^\infty(I_n)}\Big),\end{aligned} \tag{23}$$

where $C>0$ is a computable constant and the "$\lesssim$" in (23) should indicate that there are additional terms needed to estimate $\|\mathbf{R}_h\|_{L^2(S^1)}$ which are of the same order as the right hand side of (23).

Remark 7 Let us compare the estimate which is obtained by combining Theorem 1 and (23) to the estimate which is obtained if the arguments from [4] are applied to our scheme (9) in the scalar case:

$$\begin{aligned}\|u(t,\cdot)-u_h(t,\cdot)\|_{L^1(S^1)} &\le \|u(0,\cdot)-u_h(0,\cdot)\|_{L^1(S^1)}\\ &\quad+\sqrt{K_1\int_0^t\sum_n\big(h_nR_n+h_{n+\frac12}R_{n+\frac12}\big)}\\ &\quad+\sqrt{K_2\int_0^t\sum_n\big(\|\bar u_h-u_h\|_{L^\infty(I_n)}R_n+\big|\bar u_h(x_n^+)-u_h(x_n^+)\big|R_{n+\frac12}\big)},\end{aligned} \tag{24}$$

where $\bar u_h$ is the intervalwise mean of u_h, $h_{n+\frac12}=\frac12(h_n+h_{n+1})$ and

$$\begin{aligned}R_n(t) &:= \textstyle\int_{I_n}|(u_h)_t(t,x)+f(u_h)_x(t,x)|\,\mathrm{d}\,x\\ R_{n+\frac12} &:= \big|[u_h(t,\cdot)]_n+[u_h(t,\cdot)]_{n+1}\big|\end{aligned} \tag{25}$$

where K_1, K_2 are computable constants. In this comparison obviously the two estimators are rather different. The two most important differences making (24) preferable in the scalar case are the following:

1. The estimator in (24) is proportional to $\sqrt{t}$ while the estimator in (22) depends exponentially on time.
2. The estimator in (24) is expected to converge even for discontinuous entropy solutions. In contrast the estimator in (22) depends exponentially on $\|\hat{\mathbf{u}}_x\|_{L^\infty([0,t]\times S^1)}$. For discontinuous entropy solutions $\|\hat{\mathbf{u}}_x\|_{L^\infty([0,t]\times S^1)}$ will be of order h^{-1}. Thus, for discontinuous entropy solutions, the right hand of side (22) will (at best) behave like $h^{2p+2}e^{-1/h}$ which blows up for $h \to 0$. Therefore, while (22) indeed holds even for discontinuous $\mathbf{u}$, the estimator will not converge for $h \to 0$ and its practical use is limited in that case.

Both of these observations are consequences of the use of the relative entropy. We see that it provides a much weaker kind of stability that the L^1-contraction property does for scalar equations.

Remark 8 (Higher space dimensions and time-discretisation) There are two immediate directions for generalisation of the results stated here. The first direction is to derive aposteriori error estimates for fully discrete Runge-Kutta-discontinuous Galerkin schemes. We are optimistic that similar methods to those used in [10] will permit us to obtain such estimates. A special emphasis in this analysis should be put on considering explicit discretisations in time as they are most commonly used in practice.

The second direction is the generalisation to several space dimensions. The crucial issue there is to find appropriate reconstructions of the numerical solution as well as of the numerical fluxes. This is the subject of ongoing research. Once such reconstructions are determined the other arguments presented here can immediately be applied, as they are by no means restricted to the one dimensional case.

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