Convergence of Finite Volume Scheme for Degenerate Parabolic Problem with Zero Flux Boundary Condition

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Abstract This note is devoted to the study of the finite volume methods used in the discretization of degenerate parabolic-hyperbolic equation with zero-flux boundary condition. The notion of an entropy-process solution, successfully used for the Dirichlet problem, is insufficient to obtain a uniqueness and convergence result because of a lack of regularity of solutions on the boundary. We infer the uniqueness of an entropy-process solution using the tool of the nonlinear semigroup theory by passing to the new abstract notion of integral-process solution. Then, we prove that numerical solution converges to the unique entropy solution as the mesh size tends to 0.

1 Introduction

Our goal is to study convergence of a finite volume scheme for a degenerate parabolic equation with zero-flux boundary condition in a regular bounded domain $\Omega \in \mathbb{R}^\ell$ arising, e.g., in sedimentation and traffic models:

$$\begin{cases} u_t + \operatorname{div} f(u) - \Delta\phi(u) = 0 & \text{in} \quad Q = (0,T)\times\Omega, \\ u(0,x) = u_0(x) & \text{in} \quad \Omega, \\ (f(u) - \nabla\phi(u)).\eta = 0 & \text{on} \quad \Sigma = (0,T)\times\partial\Omega. \end{cases} \tag{P}$$

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Here ϕ is a non-decreasing Lipschitz continuous function, moreover, there exists $u_c \in [0, u_{\max}]$ with $u_{\max} > 0$ such that $\phi|_{[0,u_c]} \equiv 0$ but $\phi'|_{[u_c,u_{\max}]} > 0$. The case $u_c = u_{max}$ was understood in [7]. In the range $[0, u_c]$ of values of u, (P) degenerates into a hyperbolic problem, and admissibility criteria of Kruzhkov type are needed to single out the unique and physically motivated weak solution (see, e.g., [7, 13]). We require that the flux function f is Lipschitz, genuinely nonlinear on $[0, u_c]$; moreover, $[0, u_{\max}]$ is an invariant domain for the evolution of (P) due to assumption

$$f(0) = f(u_{\max}) = 0, \quad u_0 \in L^\infty(\Omega; [0, u_{\max}]) \tag{H1}$$

(the latter means the space of measurable on Ω functions with values in $[0, u_{\max}]$). In the work [4], inspired by [7] we proposed a new entropy formulation of (P) saying that $u \in L^\infty(Q; [0, u_{\max}])$ is an entropy solution of (P) if $u \in C([0, T]; L^1(\Omega))$ with $u(0) = u_0$, $\phi(u) \in L^2(0, T; H^1(\Omega))$ and $\forall k \in [0, u_{\max}]$

$$|u - k|_t + \operatorname{div}\left(sign(u-k)\big[f(u) - f(k) - \nabla\phi(u)\big]\right) \leq |f(k).\eta|\, d\mathscr{H}^\ell \tag{1}$$

in $\mathscr{D}'((0, T) \times \overline{\Omega})$, where η is the exterior unit normal vector to the boundary $\Sigma = (0, T) \times \partial\Omega$ and the last term is taken with respect to the Hausdorff measure $\mathscr{H}^\ell$ on Σ. Contrary to the Dirichlet case (cf. [9]) where the boundary condition is relaxed, (1) implies that zero-flux condition in (P) holds in the weak sense.

Existence of an entropy solution to (P) can be obtained by standard vanishing viscosity method, relying in particular on the *strong compactness* arguments derived from genuine nonlinearity of $f|_{[0,u_c]}$ and non-degeneracy of $\phi|_{[u_c,u_{max}]}$, see [12]. But in order to prove uniqueness, one faces a serious difficulty (not relevant in the case $u_c = u_{max}$, [7]) related to the lack of regularity of the flux $\mathscr{F}[u] := f(u) - \nabla\phi(u)$ and specifically, to the weak sense in which the normal component $\mathscr{F}[u].\eta$ of the flux annulates on Σ. Techniques of nonlinear semigroup theory (see, e.g., [5, 6]) can be used to circumvent this regularity problem in some cases (see [3, 4]) and to prove well-posedness for (P) in the sense (1). Let us present the key arguments: indeed, they are also important for study of convergence of the Finite Volume scheme for (P), which is the goal of this note. The standard doubling of variables method based upon formulation (1) readily leads to the uniqueness and L^1 contraction property

$$\forall t \in [0, T] \quad \|u(t, \cdot) - \hat{u}(t, \cdot)\|_{L^1} \leq \|u_0 - \hat{u}_0\|_{L^1} \tag{2}$$

if we compare two solutions u, $\hat{u}$ such that the strong (in the sense of L^1 convergence, see [11, 13]) trace of the normal flux $\mathscr{F}[u].\eta$ at the boundary exists. In the sequel, we call such solutions *trace-regular*. Every entropy solution is a trace-regular in the case of the pure hyperbolic problem (case $u_c = u_{\max}$, see [7, 11, 13]). The idea of symmetry breaking in the doubling of variables (see [3]) permits an extension of (2) to a kind of weak-strong comparison principle where u is a general solution and $\hat{u}$ is a trace-regular solution. When a sufficiently large family of trace-regular solutions is available, uniqueness of a general solution and principle (2) may follow by density arguments. A closely related technique consists in exploiting the above weak-strong comparison arguments using the idea of integral solution and somewhat stronger regularity properties of *stationary solutions*. E.g., for the pure parabolic one ($u_c = 0$, see [3]) every entropy solution of the stationary problem

$$\hat{u} + \operatorname{div} f(\hat{u}) - \Delta\phi(\hat{u}) = g \text{ in } \Omega, \quad (f(\hat{u}) - \nabla\phi(\hat{u})).\eta = 0 \text{ on } \partial\Omega \tag{S}$$

with $g \in L^\infty(\Omega)$ is trace-regular if $f \circ \phi^{-1} \in \mathbf{C}^{0,\gamma}, \gamma > 0$ (see [3]). This observation, in conjunction with the use of integral solutions [6] of abstract evolution problem

$$u' + Au \ni h, \quad u(0) = u_0 \tag{3}$$

for suitably defined operator $A = A_{f,\phi}$ (problem (S) taking the form $(\text{Id} + A_{f,\phi})\, u \ni g$) permits to get uniqueness of entropy solution in [3], for the parabolic case $u_c = 0$. Let us stress that the question of uniqueness for (P) with $u_c \notin \{0, u_{max}\}$ and $\ell > 1$ remains open. The one-dimensional hyperbolic-parabolic case ($\ell = 1$, $\Omega = (a, b)$ with arbitrary $u_c \in [0, u_{\max}]$) has been treated by the authors in [4], using the above abstract approach along with the elementary observation that yields trace-regularity:

$$\big(f(\hat{u}) - \phi(\hat{u})_x\big)_x = g - u \in L^\infty((a, b)) \;\;\Rightarrow\;\; \mathscr{F}[u] = \big(f(\hat{u}) - \phi(\hat{u})_x\big) \in \mathbf{C}([a, b]).$$

Another essential aspect of the study of (P) is to justify convergence of numerical approximations. The difference with the existence proof is that, for numerical approximations, the use of *strong compactness* arguments is very technical, and *weak compactness* methods are often preferred. Such study relying on *nonlinear weak-∗ compactness* technique of [8, 9] is our goal in this note. We study a finite volume scheme discretization in the spirit of [9] for (P) on a family of admissible meshes $(\mathscr{O}_h)_h$ with implicit time stepping. According to the standard weak compactness estimates, as for the Dirichlet problem [9] approximate solutions $u^h := u_{\mathscr{O}_h,\delta t_h}$ converge up to a subsequence, as the discretization size h goes to zero, towards an *entropy-process solution* ν. This notion closely related to Young measures' techniques (see [8] and references therein) incorporates dependence on an additional variable $\alpha \in [0, 1]$ which may represent oscillations in the family $(u^h)_h$. It remains to prove the uniqueness of an entropy-process solution which implies the independence of $\nu(t, x, \alpha)$ on α so that $u(t, x) \equiv \nu(t, x, \alpha)$ is an entropy solution of (P). As for the proof of uniqueness of an entropy solution discussed above, we face the major difficulty due to the lack of regularity of $\mathscr{F}[u].\eta$. Hence, we found it useful to define the new notion of *integral-process solution* in the framework of abstract problem (3). Following the pattern of the uniqueness proofs in [3, 4], we compare an entropy-process solution of (P) and a trace regular solution of (S), then we prove that an entropy-process solution of (P) is an integral-process solution of (3) defined for an appropriate m-accretive operator $A_{f,\phi}$. The convergence result holds due to the fact that the integral-process solution coincides with the unique integral solution of (3); and the latter one coincides with the unique entropy solution of (P) in the sense (1).

The remainder of this note is organized as follows. In Sect. 2 we present our scheme. In Sect. 3 we present the standard steps of convergence arguments for the problem (P), obtained as for Dirichlet problem [9]. In Sect. 4, we achieve the convergence result using classical and new tools of the nonlinear semigroup theory. In Remark 1, we sketch a convergence argument for Finite Volume schemes based upon a direct use of integral-process solutions, bypassing the entropy-process ones.

2 Description of the Finite Volume Scheme for (P)

Let us begin with considering an admissible mesh $\mathcal{O}$ of Ω (see [8, 9]) for space discretization and using the conventional notation present in the main literature. Because we consider the zero-flux boundary condition, we don't need to distinguish between interior and exterior control volumes K, only inner interfaces σ between volumes are needed in order to formulate the scheme. For $K \in \mathcal{O}$ and $\sigma \in \varepsilon_K$, we denote by $\tau_{K,\sigma}$ the transmissivity coefficient. For the approximation of the convective term, we consider the numerical convection fluxes $F_{K,\sigma} : \mathbb{R}^2 \longrightarrow \mathbb{R}$ that are consistent with f, monotone, Lipschitz regular, and conservative (see [8, 9]).

The values of the discrete unknowns u_K^{n+1} for all control volume $K \in \mathcal{O}$, and $n \in \mathbb{N}$ are defined thanks to the following relations: first we initialize the scheme by

$$u_K^0 = \frac{1}{m(K)} \int_K u_0(x)dx \quad \forall K \in \mathcal{O}, \tag{4}$$

then, we use the implicit scheme for the discretization of problem (P):
$\forall n > 0, \ \forall K \in \mathcal{O}$,

$$m(K)\frac{u_K^{n+1}-u_K^n}{\delta t} + \sum_{\sigma \in \varepsilon_K} \Big(F_{K,\sigma}(u_K^{n+1}, u_{K,\sigma}^{n+1}) - \tau_{K,\sigma}\big(\phi(u_{K,\sigma}^{n+1})-\phi(u_K^{n+1})\big)\Big) = 0. \tag{5}$$

If the scheme has a solution $(u_K^n)_{K,n}$, we will say that the approximate solution to (P) is the piecewise constant function $u_{\mathcal{O},\delta t}(t,x)$ defined by:

$$u_{\mathcal{O},\delta t}(t,x) = u_K^{n+1} \text{ for } x \in K \text{ and } t \in (n\delta t, (n+1)\delta t]. \tag{6}$$

A weakly consistent discrete gradient $\nabla_{\mathcal{O}}\phi(u_{\mathcal{O},\delta t})$ is defined "per diamond"; we refer to [10] for details. Let us stress that the zero-flux boundary condition is included in the scheme, since the flux terms on $\partial K \cap \partial \Omega$ are set to be zero in Eq. (5).

3 Analysis of the Approximate Solution: Classical Arguments

Following the guidelines of [8, 9], we can justify uniqueness of discrete solutions, obtain several uniform estimates (confinement of values of $u_{\mathscr{O},\delta t}$ in $[0, u_{max}]$, weak BV estimate for $u_{\mathscr{O},\delta t}$, discrete $L^2(0,T;H^1(\Omega))$ estimate of $\phi(u_{\mathscr{O},\delta t})$), and derive existence of $u_{\mathscr{O},\delta t}$. We refer to the PhD thesis [10] of the second author for details, with a particular emphasis on the treatment of boundary volumes. It follows that the discrete solution $u_{\mathscr{O},\delta t}$ satisfies the approximate continuous entropy formulation.

Theorem 1 *Let $u_{\mathscr{O},\delta t}$ be the approximate solution of the problem* (P) *defined by* (4),(5),(6). *Then the following approximate entropy inequalities hold:*
for all $k \in [0, u_{\max}]$, for all $\xi \in \mathscr{C}^\infty([0,T)\times\mathbb{R}^\ell)$, $\xi \geq 0$,

$$\int_0^T\int_\Omega \Big\{|u_{\mathscr{O},\delta t}-k|\xi_t + sign(u_{\mathscr{O},\delta t}-k)\Big[f(u_{\mathscr{O},\delta t})-f(k)-\nabla_{\mathscr{O}}\phi(u_{\mathscr{O},\delta t})\Big].\nabla\xi\Big\}\,dxdt$$
$$+\int_0^T\int_{\partial\Omega}|f(k).\eta(x)|\,\xi(t,x)d\mathscr{H}^{\ell-1}(x)dt+\int_\Omega|u_0-k|\xi(0,x)dx \geq -\upsilon_{\mathscr{O},\delta t}(\xi), \tag{7}$$

where $\forall\xi \in \mathscr{C}^\infty([0,T)\times\mathbb{R}^\ell)$, $\upsilon_{\mathscr{O},\delta t}(\xi)\to 0$ when $h\to 0$.

In order to pass to the limit in (7) using only the L^∞ bound on $u_{\mathscr{O},\delta t}$, one can adapt the notion of an entropy-process solution to problem (P) in the entropy sense (1).

Definition 1 Let $\mu \in L^\infty(Q\times(0,1))$. The function $\mu=\mu(t,x,\alpha)$ is called an entropy-process solution to the problem (P) if $\forall k\in[0,u_{\max}]$, $\forall\xi\in\mathscr{C}^\infty([0,T)\times\mathbb{R}^\ell)$, with $\xi\geq 0$, the following inequalities hold:

$$\int_0^T\int_\Omega\int_0^1\Big\{|\mu-k|\xi_t+sign(\mu-k)\Big[f(\mu)-f(k)\Big].\nabla\xi\Big\}\,dxdtd\alpha$$
$$-\int_0^T\int_\Omega\nabla|\phi(u)-\phi(k)|.\nabla\xi dxdt+\int_0^T\int_{\partial\Omega}|f(k).\eta(x)|\,\xi(t,x)d\mathscr{H}^{\ell-1}(x)dt$$
$$+\int_\Omega|u_0-k|\xi(0,x)dx\geq 0,\quad \text{where } u(t,x):=\int_0^1\mu(t,x,\alpha)d\alpha.$$

From Theorem 1 we derive the following result which, however, will not be conclusive. In the sequel, we will upgrade (or circumvent, see Remark 1) this claim.

Proposition 1 *Let $u_{\mathscr{O},\delta t}$ be the approximate solution of the problem* (P) *defined by* (4), (5). *There exists an entropy-process solution μ of* (P) *in the sense of Definition 1 and a subsequence of $(u_{\mathscr{O},\delta t})_{\mathscr{O},\delta t}$, such that:*
- *The sequence $(u_{\mathscr{O},\delta t})_{\mathscr{O},\delta t}$ converges to μ in the nonlinear weak-$*$ sense.*

- *Moreover,* $(\phi(u_{\mathcal{O},\delta t}))_{\mathcal{O},\delta t}$ *converges strongly in* $L^2(Q)$ *to* $\phi(u)$*,* $u = \int_0^1 \mu(t,x,\alpha)\, d\alpha$*, and* $(\nabla_{\mathcal{O}}\phi(u_{\mathcal{O},\delta t}))_{\mathcal{O},\delta t} \rightharpoonup \nabla\phi(u)$ *in* $(L^2(Q))^\ell$ *weakly, as* $h, \delta t \to 0$.

Proof The proof is essentially the same as in main reference papers dealing with finite volume scheme for degenerate parabolic equations (see [2, 9]). □

4 Reduction of Entropy-Process Solution: Semigroup Arguments

In the context of the Dirichlet problem (see [8, 9]) there holds the uniqueness and reduction result stating that an entropy-process solution μ is α-independent, so that it reduces to an entropy solution. The lack of regularity of the fluxes at the boundary makes it difficult to prove the analogous result with zero-flux conditions. Here, we show how this difficulty can be bypassed, using classical tools and a new notion of *integral-process solution* in the abstract context of nonlinear semigroup theory [6].

4.1 Notion of Integral-Process Solution and Equivalence Result

Given a Banach space X and an accretive operator $A \subset X \times X$, $u \in C([0,T];X)$ is called integral solution (see Bénilan et al. [5, 6]) of the abstract evolution problem (3) if, $\|\cdot\|$ being the norm and $[u,v] := \lim_{\lambda\downarrow 0}\frac{\|u+\lambda v\|-\|u\|}{\lambda}$ the bracket on X, one has $u(0)=u_0$ and the following family of inequalities holds:

$$\forall(\hat{u},\hat{z}) \in A \quad \|u(t)-\hat{u}\| - \|u(s)-\hat{u}\| \le \int_s^t [u(\tau)-\hat{u}, h(\tau)-\hat{z}], \quad 0 \le s \le t \le T.$$

For m-accretive operators the classical in the nonlinear semigroup theory notion of mild solution coincides with the notion of integral solution, so that we have

Proposition 2 *Assume that* A *is* m*-accretive, with* $\overline{Dom(A)}^{\|\cdot\|_X} = X$*. Then for any* $h \in L^1((0,T);X)$*,* $u_0 \in X$ *there exists a unique integral solution of* (3).

We refer to [6] for the proof of uniqueness of an integral solution and to [5] for a generalization relevant to our case: continuity of $u : [0,T] \to X$ can be relaxed, cf. (9). We propose a variant of the above notion that we call *integral-process solution*. This notion is motivated by an application in the setting where X is a Lebesgue space on $\Omega \subset \mathbb{R}^\ell$ and ν is a *nonlinear weak-$*$ limit* (see [8]) of approximate solutions.

Definition 2 Let A be an accretive operator on X, $h \in L^1(0,T;X)$ and $u_0 \in X$. An X-valued function ν of $(t,\alpha) \in [0,T] \times [0,1]$ is an integral-process solution of abstract problem $u' + Au \ni h$ on $[0,T]$ with datum $\nu(0,\cdot,\alpha) \equiv u_0(\cdot)$, if for all $(\hat{u},\hat{z}) \in A$

$$\int_0^1 \Big(\|\nu(t,\alpha) - \hat{u}\| - \|\nu(s,\alpha) - \hat{u}\| \Big) d\alpha \le \int_0^1\!\!\int_s^t \Big[v(\tau,\alpha) - \hat{u}, h(\tau) - \hat{z} \Big] d\tau d\alpha \quad (8)$$

for $0 < s \le t \le T$ and the initial condition is satisfied in the sense

$$\text{ess-}\lim_{t\downarrow 0} \int_0^1 \|\nu(t,\alpha) - u_0\| d\alpha = 0. \quad (9)$$

The main fact concerning integral-process solutions is the following result [10].

Theorem 2 *Assume that A is m-accretive in X and $u_0 \in \overline{D(A)}$. Then ν is an integral-process solution of* (3) *if and only if ν is independent on α and for all α, $\nu(.,\alpha)$ coincides with the unique integral and mild solution $u(\cdot)$ of* (3).

4.2 Convergence of the Scheme

Let us define the operator $A_{f,\phi}$ on $L^1(\Omega; [0, u_{\max}]) \subset X = L^1(\Omega)$ endowed with $\|\cdot\|_1$:

$$(v, z) \in A_{f,\phi} = \Big\{ v \text{ such that } v \text{ is a } trace\ regular \text{ solution of (S), with } g = v + z \Big\}$$

(instead of $L^1(\Omega)$ we can work in $L^1(\Omega; [0, u_{\max}])$ due to the confinement principle for solutions of (S)). The main result of this paper is the following theorem.

Theorem 3 *Assume operator $A_{f,\phi}$ on $L^1(\Omega; [0, u_{\max}])$ is m-accretive densely defined, then any entropy-process-solution of* (P) *is its unique entropy solution. In particular, the scheme* (4),(5) *for discretization of* (P) *in the sense* (1) *is convergent:*

$$\forall p \in [1, +\infty) \quad u_{\mathcal{O},\delta t} \longrightarrow u \ \ in\ L^p(0, T \times \Omega) \ \ as \ \ \max(\delta t, h) \longrightarrow 0.$$

Proof First, in Proposition 1 we prove that the approximate solutions $u_{\mathcal{O},\delta t}$ converge towards an entropy-process solution μ. Then, with the technique of [3, 4] we compare the entropy-process solution μ and a trace-regular solution $\hat{u}$ of stationary problem (S). We find that μ is also an integral-process solution. By Theorem 2, μ is independent on the variable α. Therefore $\mu(\cdot, \alpha)$ coincides with the unique integral solution of the abstract evolution problem (3) governed by operator $A_{f,\phi}$; we know from the analysis of [3, 4] that it is also the unique entropy solution of (P). □

Theorem 3 is applicable in the following three cases where trace-regularity for the solutions of (S) can be justified, at least for a dense set of source terms.

Proposition 3 *Assume that $\ell \ge 1$, and $u_c = u_{\max}$* (i.e., (P) *is purely hyperbolic*). *Then $A_{f,\phi}$ is m-accretive densely defined on $L^1(\Omega; [0, u_{\max}])$.*

Proposition 4 *Assume that $\ell \ge 1$ and $u_c = 0$* (i.e. (P) *is non-degenerate parabolic*). *Then $A_{f,\phi}$ is m-accretive densely defined on $L^1(\Omega; [0, u_{\max}])$ if $f \circ \phi^{-1} \in \mathcal{C}^{0,\gamma}$, $\gamma > 0$.*

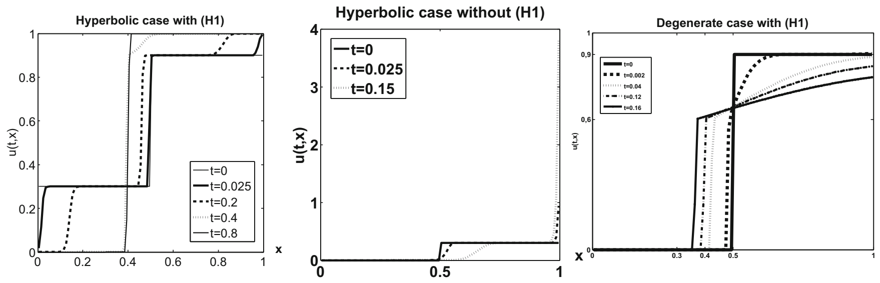

Fig. 1 **a** $f(u) = u(1-u)$, $\phi \equiv 0$, **b** $f(u) = \frac{u^2}{2}$, $\phi \equiv 0$, **c** $f(u) = u(1-u)$, $\phi(u) = (u-0.6)^+$

Proposition 5 *Assume that $\Omega = (a, b)$ (thus, $\ell = 1$). Then $A_{f,\phi}$ is m-accretive densely defined on $L^1(\Omega; [0, u_{\max}])$.*

Prop. 3 follows by the strong trace results of [11, 13] (cf. [7]), Prop. 4 is justified like in [3], while Prop. 5 was an ingredient of the uniqueness proof in [4].

Remark 1 Actually, the use of entropy-process solutions can be circumvented. Observe that the stationary problem (S) can be discretized with the scheme analogous to the time-implicit scheme used for the evolution problem (P). Consider the situation where strong compactness (and convergence to $\hat{u} \in \mathrm{Dom}(A_{f,\phi})$) can be proved for approximate solutions $\hat{u}_{\mathcal{O}}$ of (S) but only nonlinear weak-$*$ compactness for approximate solutions $u_{\mathcal{O},\delta t}$ of (P) is known (this occurs when $\ell = 1$, where compactness of $\hat{u}_{\mathcal{O}}(x_i)$, for all $x_i \in \mathbb{Q}$, is immediate: see the arguments developed in [1]). Then convergence of the stationary scheme is easily proved, moreover, one infers inequalities (8) for the limit $\nu(\cdot, \alpha)$ of $u_{\mathcal{O},\delta t}$. Then, the result of Theorem 2 proves convergence of the scheme for the evolution problem. In a future work, this argument will be applied to a large variety of one-dimensional degenerate parabolic conservation laws with boundary conditions or interface coupling conditions.

5 Numerical Experiments

We conclude with $1D$ numerical illustrations presented in Fig. 1a, c obtained with the explicit analogue of the scheme (4),(5) under the *ad hoc* CFL restrictions. On this occasion, we use the scheme to highlight the importance of hypothesis (H1). In the test of Fig. 1b assumption (H1) fails, and a boundary layer appears. If one refines the mesh one observes convergence of $u_{\mathcal{O}_h,\delta t_h}$ towards a function bounded by $\|u_0\|_\infty$ while the sequence $(u_{\mathcal{O}_h,\delta t_h})_h$ seems unbounded. However, the condition of zero flux imposed in (5) is relaxed in the limit, making formulation (1) inappropriate outside the framework (H1). Introduction of appropriate boundary formulation satisfied by the limit of the scheme, in absence of (H1), is postponed to future work.

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References

1. Andreianov, B.: In: H.H. Chen G.-Q., K. Karlsen (eds.) Hyperbolic Conservation Laws and Related Analysis with Applications, Springer Proceedings in Mathematics and Statistics, vol. 29, pp. 1–22
2. Andreianov, B., Bendahmane, M., Karlsen, K.: Discrete duality finite volume schemes for doubly nonlinear degenerate hyperbolic-parabolic equations. J. Hyperb. Diff. Eq. 7, 1–67 (2010)
3. Andreianov, B., Bouhsiss, F.: Uniqueness for an elliptic-parabolic problem with Neumann boundary condition. J. Evol. Eq. **4**, 273–295 (2004)
4. Andreianov, B.: Gazibo Karimou, M.: Entropy formulation of degenerate parabolic equation with zero-flux boundary condition. Z. Angew. Math. Phys. **64**(5), 1471–1491 (2013)
5. Barthélémy, L., Bénilan, P.: Subsolutions for abstract evolution equations. Potential Anal. **1**(1), 93–113 (1992)
6. Bénilan, P., Crandall, M.G., Pazy, A.: Nonlinear evolution equations in Banach spaces. (Preprint book)
7. Bürger, R., Frid, H., Karlsen, K.H.: On the well-posedness of entropy solutions to conservation laws with a zero-flux boundary condition. J. Math. Anal. Appl. **326**(1), 108–120 (2007)
8. Eymard, R., Gallouët, T., Herbin, R.: Finite volume methods. Handb. Numer. Anal. **7**, 713–1018 (2000)
9. Eymard, R., Gallouët, T., Herbin, R., Michel, A.: Convergence of a finite volume scheme for nonlinear degenerate parabolic equations. Numer. Math. **92**(1), 41–82 (2002)
10. Gazibo Karimou, M.: Etudes mathématiques et numériques des problèmes paraboliques avec des conditions aux limites. Thèse de Doctorat Besançon (2013)
11. Panov, E.Y.: Existence of strong traces for quasi-solutions of multidimensional conservation laws. J. Hyperb. Diff. Eq. **4**(4), 729–770 (2007)
12. Panov, E.Y.: On the strong pre-compactness property for entropy solutions of a degenerate elliptic equation with discontinuous flux. J. Differ. Eq. **247**(10), 2821–2870 (2009)
13. Vasseur, A.: Strong traces for solutions of multidimensional scalar conservation laws. Arch. Ration. Mech. Anal. **160**(3), 181–193 (2001)