Modified Finite Volume Nodal Scheme for Euler Equations with Gravity and Friction

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Abstract In this work we present a new finite volume scheme valid on unstructured meshes for the Euler equation with gravity and friction indeed the classical Godunov type schemes are not adapted to treat the hyperbolic systems with source terms. The new method is based on a finite volume nodal scheme modified to capture correctly the behavior induced by the source terms.

1 Introduction

In many physical applications appear hyperbolic systems with source terms which model the balance between the convective effects, acoustic effects and the external forces. A classical example of this type of problem is the Euler equations with friction and gravity given by

$$\begin{cases} \partial_t \rho + \dfrac{1}{\varepsilon}\operatorname{div}(\rho \mathbf{u}) = 0 \\ \partial_t \rho \mathbf{u} + \dfrac{1}{\varepsilon}\operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \dfrac{1}{\varepsilon}\nabla p = \dfrac{1}{\varepsilon}\rho \mathbf{g} - \dfrac{\sigma}{\varepsilon^2}\rho \mathbf{u} \\ \partial_t \rho e + \dfrac{1}{\varepsilon}\operatorname{div}(\rho \mathbf{u} e) + \dfrac{1}{\varepsilon}\operatorname{div}(p\mathbf{u}) = \dfrac{1}{\varepsilon}\rho(\mathbf{g}, \mathbf{u}) - \dfrac{\sigma}{\varepsilon^2}\rho ||\mathbf{u}||^2 \end{cases} \tag{1}$$

with $\mathbf{g}$ a vector of gravity and ε a small parameter which comes from to a rescaling of the time and σ. The limit ε tend to zero correspond to the limit in long time for very large σ. This model is used for the astrophysics applications (for example atmospheric phenomena) and is an interesting model to begin the study of more complicated multi-fluid and multi-phases flows [5, 6]. At the numerical level, it is known that the classical Godunov and splitting schemes are not efficient to capture

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the behavior induced by the balance between source terms and hyperbolic part. Since some years, specific numerical methods have been designed, in particular the asymptotic preserving schemes (which capture the asymptotic limit independently of the relaxation parameter ε) and well-balanced schemes which discretize the steady states with a high accuracy. Our aim is to extend this type of method on unstructured meshes to the Euler equations. Firstly we recall some properties at the analytical level.

Proposition 1 *The system (1) satisfies the following properties:*

- *The density and the energy are non negative*
- *The entropy inequality* $\partial_t(\rho S) + \operatorname{div}(\rho \mathbf{u} S) \geq 0$ *is satisfied for weak solutions*
- *When* ε *tends to zero the system tends to*

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0 \\ \partial_t \rho e + \operatorname{div}(\rho \mathbf{u} e) + p \operatorname{div} \mathbf{u} = 0 \\ \mathbf{u} = \dfrac{1}{\sigma}(\mathbf{g} - \dfrac{1}{\rho}\nabla p) \end{cases} \tag{2}$$

- *The solutions of* $\mathbf{u} = \mathbf{0}$ *and* $\nabla p = \rho \mathbf{g}$ *are steady states (hydrostatic equilibrium) of the system (1)*

Proof The first property is a classical property of the Euler equations. The second and fourth one are discussed in [5].
We give a proof of the asymptotic limit. To obtain this result, we use a Hilbert expansion. Each variable is decomposed on the following form $\rho = \rho^0 + \varepsilon\rho^1 + \varepsilon^2\rho^2 + o(\varepsilon^2)$. Next we plug these definitions in the model.
The terms homogeneous to $\frac{1}{\varepsilon^2}$ are $-\rho^0\mathbf{u}^0 = \mathbf{0}$ and $-\rho^0||\mathbf{u}^0||^2 = 0$. Since ρ is strictly positive we obtain that $\mathbf{u}^0 = \mathbf{0}$.
The term homogeneous to $\frac{1}{\varepsilon}$ is $\nabla p^0 = \rho^0\mathbf{g} - \sigma\rho^0\mathbf{u}^1$.
To finish we give the terms homogeneous to $\frac{1}{\varepsilon^0}$, using $\mathbf{u}^0 = \mathbf{0}$ we have:

$$\begin{aligned} &\partial_t \rho^0 + \operatorname{div}(\rho^0 \mathbf{u}^1) = 0, \\ &\partial_t \rho^0 e^0 + \operatorname{div}(\rho^0 \mathbf{u}^1 e^0) + \operatorname{div}(p^0 \mathbf{u}^1) = \rho^0(\mathbf{g}, \mathbf{u}^1) - \sigma\rho^0||\mathbf{u}_1||^2. \end{aligned} \tag{3}$$

Using the equation $\nabla p^0 = \rho^0\mathbf{g} - \sigma\rho^0\mathbf{u}^1$ we obtain

$$\begin{aligned} &\partial_t \rho^0 + \operatorname{div}(\rho^0 \mathbf{u}^1) = 0, \\ &\partial_t \rho^0 e^0 + \operatorname{div}(\rho^0 \mathbf{u}^1 e^0) + \operatorname{div}(p^0 \mathbf{u}^1) - (\mathbf{u}^1, \nabla p^0) = 0. \end{aligned} \tag{4}$$

with $\mathbf{u}^1 = \frac{1}{\sigma}(\mathbf{g} - \frac{1}{\rho}\nabla p_0)$. To conclude we use $\operatorname{div}(p\mathbf{u}) = (\mathbf{u}, \nabla p) + p \operatorname{div} \mathbf{u}$. □

Now we propose to design a scheme which captures and preserves these properties at the discrete level. To capture the diffusion limit system (2), we use asymptotic preserving (AP) methods.

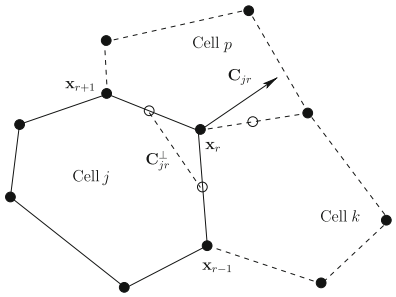

Fig. 1 Notations for nodal scheme. The corner quantity $\mathbf{C}_{jr}$ is equal to the orthogonal vector to the half of the vector that starts at $\mathbf{x}_{r-1}$ and finishes at $\mathbf{x}_{r+1}$. The center of the cell is an arbitrary point inside the cell

For a relaxation model as the Euler equations with the friction terms which depends of ε, the classical schemes like Godunov-type schemes admit a consistency error homogeneous to $O(\frac{\Delta x}{\varepsilon})$ and a CFL condition constrained by ε. However, for the AP schemes the consistency error and the CFL condition are independent of ε [1, 2, 7]. Whereas the well-balanced methods are schemes which discretize exactly or with a high accuracy the steady states [7, 8]. The idea to obtain good discretization is to plug the source terms in the fluxes to capture correctly the effects of these terms.

2 Derivation of the Scheme and Asymptotic Properties

Some asymptotic preserving and well-balanced schemes for Euler equations have been designed in 1D [5, 6]. However the situation is more complicated in 2D. Indeed in [2] we show that the classical extension of the AP scheme for linear hyperbolic systems with diffusion limit does not converge on unstructured meshes. In fact the limit diffusion scheme called 5-points scheme is not consistent on unstructured meshes. To treat this problem we have proposed new scheme based on a nodal formulation (these schemes localize the fluxes at the corner) for different models [2, 3]. Now extend these methods to solve the Euler equations. We use a modified Lagrange+remap scheme (nodal scheme for the Lagrangian part defined in [4] and a nodal advection scheme for the remap part).

Let us consider an unstructured mesh in dimension 2. The mesh is defined by a finite number of vertices $\mathbf{x}_r$ and cells Ω_j. We denote $\mathbf{x}_j$ the center of the cell chosen inside Ω_j. We also define the geometric quantity $\mathbf{C}_{jr} = \nabla_{\mathbf{x}_r} \Omega_j$ (Fig. 1).

Definition 1 The classical Lagrange+remap scheme (LP scheme) is

$$\begin{cases} |\,\Omega_j\,|\,\partial_t \rho_j + \frac{1}{\varepsilon}\left(\sum_{R_+}(\mathbf{C}_{jr},\mathbf{u}_r)\rho_j + \sum_{R_-}(\mathbf{C}_{jr},\mathbf{u}_r)\rho_{k(r)}\right) = 0 \\ |\,\Omega_j\,|\,\partial_t \rho_j\mathbf{u}_j + \frac{1}{\varepsilon}\left(\sum_{R_+}(\mathbf{C}_{jr},\mathbf{u}_r)(\rho\mathbf{u})_j + \sum_{R_-}(\mathbf{C}_{jr},\mathbf{u}_r)(\rho\mathbf{u})_{k(r)} + \sum_r \mathbf{G}_{jr}\right) = 0 \\ |\,\Omega_j\,|\,\partial_t \rho_j e_j + \frac{1}{\varepsilon}\left(\sum_{R_+}(\mathbf{C}_{jr},\mathbf{u}_r)(\rho e)_j + \sum_{R_-}(\mathbf{C}_{jr},\mathbf{u}_r)(\rho e)_{k(r)} + \sum_r(\mathbf{G}_{jr},\mathbf{u}_r)\right) = 0 \end{cases} \tag{5}$$

with the fluxes defined by the problem

$$\begin{cases} \mathbf{G}_{jr} = p_j \mathbf{C}_{jr} + c_{jr}\widehat{\alpha}_{jr}(\mathbf{u}_j - \mathbf{u}_r) \\ \sum_j c_{jr}\widehat{\alpha}_{jr}\mathbf{u}_r = \sum_j p_j \mathbf{C}_{jr} + \sum_j c_{jr}\widehat{\alpha}_{jr}\mathbf{u}_j \end{cases} \tag{6}$$

The wave speed is defined by $c_{jr} = \rho_j c_j$. The expression of the flux $\mathbf{u}_r$ comes from a classical relation of the GLACE scheme: $\sum_j \mathbf{G}_{jr} = \mathbf{0}$. For the advection fluxes we define $\mathbf{u}_{jr} = (\mathbf{C}_{jr}, \mathbf{u}_r)$, $R_+ = (r/\mathbf{u}_{jr} > 0)$, $R_- = (r/\mathbf{u}_{jr} < 0)$ and $\rho_{k(r)} = \frac{\sum_{j/\mathbf{u}_{jr}>0} \mathbf{u}_{jr}\rho_j}{\sum_{j/\mathbf{u}_{jr}>0} \mathbf{u}_{jr}}$.

To obtain an AP scheme, we apply the Jin-Levermore procedure [9]. This method consists to incorporate the steady state of the system into the fluxes. The balance equation between source term and hyperbolic part is $\mathrm{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla p = \rho\mathbf{g} - \frac{\sigma}{\varepsilon}\rho\mathbf{u}$. But the proof of the asymptotic limit shows that $\mathrm{div}(\rho\mathbf{u}\otimes\mathbf{u})$ is negligible in the limit. Indeed the previous equation shows that $\mathbf{u} = O(\varepsilon)$, consequently the important relation for the diffusion regime is $\nabla p + O(\varepsilon^2) = \rho\mathbf{g} - \frac{\sigma}{\varepsilon}\rho\mathbf{u}$. To incorporate this relation into the fluxes we use a first order Taylor expansion $p(\mathbf{x}_j) = p(\mathbf{x}_r) + (\mathbf{x}_j - \mathbf{x}_r, \nabla p(\mathbf{x}_r))$. Using the relation between ∇p and the source term we obtain $p(\mathbf{x}_j) = p(\mathbf{x}_r) + \frac{\sigma}{\varepsilon}(\mathbf{x}_j - \mathbf{x}_r, \rho(\mathbf{x}_r)\mathbf{g} - \frac{\sigma}{\varepsilon}\rho(\mathbf{x}_r)\mathbf{u}(\mathbf{x}_r))$. Now we use the the discrete equivalent of the previous equation: $p_j = p_{jr} + (\mathbf{x}_j - \mathbf{x}_r, \rho_r\mathbf{g} - \frac{\sigma}{\varepsilon}\rho_r\mathbf{u}_r)$. If we consider that $\mathbf{G}_{jr}$ is homogeneous to $p_{jr}\mathbf{C}_{jr}$. We obtain $\mathbf{G}_{jr} \approx p_j\mathbf{C}_{jr} + \widehat{\beta}_{jr}\rho_r(\mathbf{g} - \frac{\sigma}{\varepsilon})\mathbf{u}_r$ with $\widehat{\beta}_{jr} = \mathbf{C}_{jr} \otimes (\mathbf{x}_r - \mathbf{x}_j)$ then we obtain the new fluxes, we plug the previous relation in the fluxes (6). To finish we use discretization localized to the interfaces of the cells for the source term. To justify this discretization we use the following identity $\sum_r \widehat{\beta}_{jr} = \mid \Omega_j \mid \widehat{I}_d$ introduced in [2].

Definition 2 The scheme LP-AP is

$$\begin{cases} \mid \Omega_j \mid \partial_t \rho_j + \frac{1}{\varepsilon}\left(\sum_{R_+} \mathbf{u}_{jr}\rho_j + \sum_{R_-} \mathbf{u}_{jr}\rho_{k(r)}\right) = 0 \\ \mid \Omega_j \mid \partial_t \rho_j\mathbf{u}_j + \frac{1}{\varepsilon}\left(\sum_{R_+} \mathbf{u}_{jr}(\rho\mathbf{u})_j + \sum_{R_-} \mathbf{u}_{jr}(\rho\mathbf{u})_{k(r)} + \sum_r \mathbf{G}_{jr}\right) \\ = \frac{1}{\varepsilon}\sum_r \rho_r\widehat{\beta}_{jr}\left(\mathbf{g} - \frac{\sigma}{\varepsilon}\mathbf{u}_r\right) \\ \mid \Omega_j \mid \partial_t \rho_j e_j + \frac{1}{\varepsilon}\left(\sum_{R_+} \mathbf{u}_{jr}(\rho e)_j + \sum_{R_-} \mathbf{u}_{jr}(\rho e)_{k(r)} + \sum_r (\mathbf{G}_{jr}, \mathbf{u}_r)\right) \\ = \frac{1}{\varepsilon}\left(\sum_r \rho_r\widehat{\beta}_{jr}\mathbf{g} - \frac{\sigma}{\varepsilon}\sum_r (\mathbf{u}_r, \widehat{\beta}_{jr}\mathbf{u}_r)\right) \end{cases} \tag{7}$$

with the fluxes

$$\begin{cases} \mathbf{G}_{jr} = p_j\mathbf{C}_{jr} + c_{jr}\widehat{\alpha}_{jr}(\mathbf{u}_j - \mathbf{u}_r) + \rho_r\widehat{\beta}_{jr}(\mathbf{g} - \frac{\sigma}{\varepsilon}\mathbf{u}_r) \\ \left(\sum_j c_{jr}\widehat{\alpha}_{jr} + \frac{\sigma}{\varepsilon}\rho_r\sum_j \widehat{\beta}_{jr}\right)\mathbf{u}_r = \sum_j p_j\mathbf{C}_{jr} + \sum_j c_{jr}\widehat{\alpha}_{jr}\mathbf{u}_j + \rho_r(\sum_j \widehat{\beta}_{jr})\mathbf{g} \end{cases} \tag{8}$$

Proposition 2 *If the local matrices are invertible and the density is positive then the scheme LP-AP tends formally to the following diffusion scheme*

$$\begin{cases} \mid \Omega_j \mid \partial_t \rho_j + \left(\sum_{R_+} \mathbf{u}_{jr} \rho_j + \sum_{R_-} \mathbf{u}_{jr} \rho_{k(r)} \right) = 0 \\ \mid \Omega_j \mid \partial_t \rho_j e_j + \left(\sum_{R_+} \mathbf{u}_{jr} (\rho e)_j + \sum_{R_-} \mathbf{u}_{jr} (\rho e)_{k(r)} + p_j \sum_r (\mathbf{C}_{jr}, \mathbf{u}_r) \right) = 0 \\ \left(\sum_j \sigma_r \rho_r \widehat{\beta}_{jr} \right) \mathbf{u}_r = \sum_j p_j \mathbf{C}_{jr} + \rho_r \left(\sum_j \widehat{\beta}_{jr} \right) \mathbf{g} \end{cases} \tag{9}$$

Proof To obtain this result, we plug the Hilbert expansion in the scheme (7)–(8). We begin by simplify the source terms with the last part of the fluxes $\sum_r \mathbf{G}_{jr}$ and $\sum_r (\mathbf{G}_{jr}, \mathbf{u}_r)$. After we plug these definitions in the model.
The term homogeneous to $\frac{1}{\varepsilon}$ is $\left(\sum_r \sigma_r \rho_r^0 \widehat{\beta}_{jr} \right) \mathbf{u}_r^0 = \mathbf{0}$. Since the density is positive and the matrix is invertible [2] then $\mathbf{u}_r^0 = \mathbf{0}$. The term in the second equation homogeneous to $\frac{1}{\varepsilon}$ is $\sum_r p_j^0 \mathbf{C}_{jr} + \sum_r c_{jr}^0 \widehat{\alpha}_{jr} (\mathbf{u}_j^0 - \mathbf{u}_r^0) = \mathbf{0}$. Using $\mathbf{u}_r = \mathbf{0}$ and since $\sum_r \mathbf{C}_{jr} = \mathbf{0}$ (property of nodal schemes) this term gives $\sum_r c_{jr}^0 \widehat{\alpha}_{jr} \mathbf{u}_j^0 = \mathbf{0}$. The matrix $\sum_j c_{jr} \widehat{\alpha}_{jr}$ is invertible [4] and $\rho_j > 0$ then $\mathbf{u}_j^0 = \mathbf{0}$. To finish we study the terms homogeneous to $\frac{1}{\varepsilon^0}$ using $\mathbf{u}_r^0 = \mathbf{0}$ and $\mathbf{u}_j^0 = \mathbf{0}$:

$$\begin{aligned} &\partial_t \mid \Omega_j \mid \rho_j^0 + \sum_{R_+} \mathbf{u}_{jr}^1 \rho_j^0 + \sum_{R_-} \mathbf{u}_{jr}^1 \rho_{k(r)}^0 = 0 \\ &\partial_t \mid \Omega_j \mid \rho_j^0 e_j^0 + \sum_r \mathbf{C}_{jr} (p_j^0, \mathbf{u}_r^1) + \sum_{R_+} \mathbf{u}_{jr}^1 \rho_j^0 e_j^0 + \sum_{R_-} \mathbf{u}_{jr}^1 (\rho e)_{k(r)}^0 = 0 \end{aligned} \tag{10}$$

and, since $\mathbf{u}_r^0 = \mathbf{0}$ and $\mathbf{u}_j^0 = \mathbf{0}$, we obtain

$$\sigma_r \rho_r^0 \Big(\sum_j \widehat{\beta}_{jr} \Big) \mathbf{u}_r^1 = \sum_j p_j^0 \mathbf{C}_{jr} + \Big(\sum_j \rho_r^0 \widehat{\beta}_{jr} \Big) \mathbf{g} \tag{11}$$

To finish we couple (11) and (10). □

3 Discretization of the Steady States

For some applications as gravitational flows in astrophysics it is very important to treat with a good accuracy the steady states and to initialize the computations with such steady states, otherwise spurious velocity in the hydrostatic equilibrium configuration ($\nabla p = \rho \mathbf{g}$ and $\mathbf{u} = \mathbf{0}$) may disrupt the simulation. For nearly steady flows numerical perturbations dominate the small physical perturbations. In this section we show that the AP scheme is a well-balanced scheme [8] and is more efficient to treat these configurations. For some equation as shallow water equations a well balanced scheme is a method which preserve exactly the steady states. However this definition is not adapted to study the Euler equations with gravity. Indeed the steady state for the shallow water equations the steady states are algebraic unlike the steady states of the Euler equations which are differential.

Definition 3 (*Well-balanced scheme*) We assume that the initial data $(\rho_j, \mathbf{u}_j, e_j)$ satisfy the discrete steady state at the interface ($\nabla_r p = \rho_r \mathbf{g}$ for Euler equations). A scheme is well-balanced if the scheme is exact for the discrete steady state.

For the Shallow water equations the discrete steady state is an exact discretization of the continuous steady states. This is not the case for the Euler equations. Consequently for the Euler equations the numerical error given by a well-balanced scheme come from only to the error between continuous and discrete steady state.

Lemma 1 *Assume the initial data is given by* $\mathbf{u}_j = \mathbf{0}$ *and* $\nabla_r p = \rho_r \mathbf{g}$ *which is equivalent to*

$$\nabla_r p = -\Big(\sum\nolimits_j \widehat{\beta}_{jr}\Big)^{-1} \sum\nolimits_j p_j \mathbf{C}_{jr} = \rho_r \mathbf{g}$$

with ρ_r *a mean of* ρ_j *around* r. *Then the scheme LP-AP is stationary for the hydrostatic equilibrium.*

Proof We write the nodal solver

$$\Big(\sum\nolimits_j c_{jr}\widehat{\alpha}_{jr} + \frac{\sigma_r}{\varepsilon}\rho_r \sum\nolimits_j \widehat{\beta}_{jr}\Big)\mathbf{u}_r = \sum\nolimits_j p_j \mathbf{C}_{jr} + \sum\nolimits_j \widehat{\alpha}_{jr} c_{jr}\mathbf{u}_j + \Big(\sum\nolimits_j \widehat{\beta}_{jr}\Big)\rho_r \mathbf{g}$$

Using the definition of $\mathbf{u}_j$ and p_j, we obtain that the right hand side term is equal to zero. By uniqueness of the solution $\mathbf{u}_r = \mathbf{0}$. Since $\mathbf{u}_r = \mathbf{0}$ and $\mathbf{u}_j = \mathbf{0}$ then $\mathbf{G}_{jr} = p_j \mathbf{C}_{jr} + \rho_r \widehat{\beta}_{jr}\mathbf{g}$, $\partial_t \rho_j = 0$, $\partial_t \rho_j e_j = 0$ and $\partial_t \rho_j \mathbf{u}_j + \frac{1}{\varepsilon}\sum_r \mathbf{G}_{jr} = \frac{1}{\varepsilon}\sum_r \widehat{\beta}_{jr}\rho_r \mathbf{g}$. Next we use the property $\sum_r \mathbf{C}_{jr} = \mathbf{0}$ consequently we obtain that $\sum_r \mathbf{G}_{jr} = \rho_r \sum_r \widehat{\beta}_{jr}\mathbf{g}$ and we conclude that LP-AP scheme is a WB scheme. □

4 Numerical Results

Firstly we study the convergence of the LP and LP-AP schemes for two different steady states where the density is constant or linear. In the two cases we define $\mathbf{g} = (0, -1)$. The initial data for the first test case are defined by $\rho_j = 1, \mathbf{u}_j = \mathbf{0}$ and $e_j = \frac{1}{\gamma - 1}(\mathbf{x}_j, \mathbf{g}) + C$ with C a constant. The initial data for the second test case are defined by $\rho_j(t, \mathbf{x}) = y + b, \mathbf{u}_j = \mathbf{0}$ and $p_j(t, \mathbf{x}) = -(\frac{y^2}{2} + by)g$.

Now we propose two remarks about the numerical results given in Tables 1 and 2. For the constant density case, the AP scheme preserves exactly the steady state unlike the classical scheme which converges with the first order. For the non constant density case, we remark that the AP scheme is more accurate that the classical scheme. Indeed the AP scheme converge with the second order (Table 2).

We also validate the AP property. For this we consider a Sod problem with $\sigma = 1$ and $\varepsilon = 0.005$. We compare the classical scheme on fine grid (480×480 cells) and coarse grid (60×60 cells) and the AP scheme on coarse grid.

Table 1 L^2 error for the first test case (constant density)

Schemes	LP-AP			LP		
Meshes/cells	40	80	160	40	80	160
Cartesian	5.9×10^{-17}	1×10^{-16}	7.1×10^{-17}	0.00470	0.00239	0.00121
Random	1.1×10^{-16}	1.5×10^{-16}	3×10^{-16}	0.01519	0.00947	0.00526
Kershaw	1.4×10^{-16}	2.2×10^{-16}	3.2×10^{-16}	0.08503	0.050	0.02908

Table 2 L^2 error for the second test case (linear density)

Schemes	LP-AP			LP		
Meshes/cells	80	160	320	80	160	320
Cartesian	2.3×10^{-15}	9.4×10^{-15}	3.4×10^{-14}	0.0034068	0.0016984	0.0000848
Random	3.4×10^{-5}	1×10^{-5}	2.8×10^{-6}	0.00967	0.00529	0.002823
Kershaw	1.1×10^{-6}	1.8×10^{-7}	2.6×10^{-8}	0.03687	0.008363	0.00215

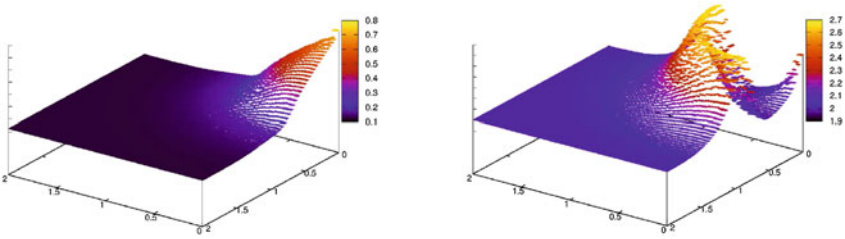

Fig. 2 Density (*left*) and energy (*right*) for the classical LP scheme. Coarse grid

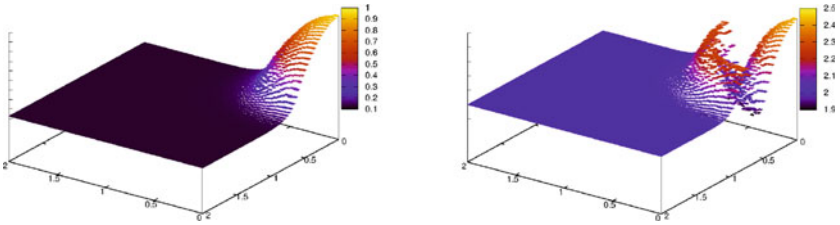

Fig. 3 Density (*left*) and energy (*right*) for the LP AP scheme. Coarse grid

We observe that the AP scheme (Fig. 3) on coarse grid capture correctly the behavior of the solution computed on the fine grid (Fig. 4) at least better than the classical scheme on coarse grid (Fig. 2) which is more diffusive (the numerical viscosity is homogeneous to $\frac{\Delta x}{\varepsilon}$).

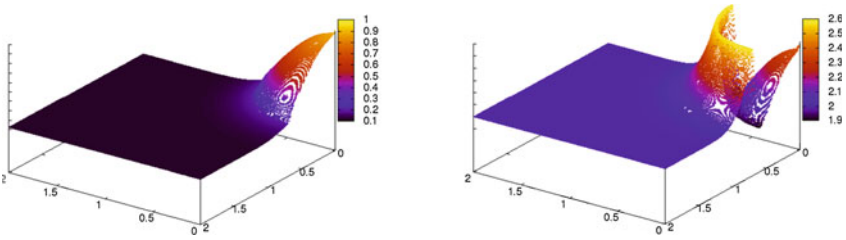

Fig. 4 Density (*left*) and energy (*right*) for the classical LP scheme. Fine grid

5 Conclusion

In this paper we study a modified Lagrange+remap scheme in 2D to capture the behavior induced by the source terms in the Euler equations. We obtain an AP scheme which captures theoretically the diffusion limit independently of the parameter ε. Moreover this scheme preserves experimentally the positivity of ρ and e. To finish, this scheme is well-balanced and converges with the second order for the hydrostatic equilibrium. This new scheme is more accurate than the classical one for these steady states. Contrary to the Shallow water equations where the steady states are algebraic, for the Euler equations the steady states are differential, consequently it is more difficult to obtain a WB scheme exact for all steady states. In the future it will be important to discuss the entropy property and semi-implicit time scheme with a CFL independent of ε.

References

1. Berthon, C., Turpault, R.: Asymptotic preserving hll schemes. Numer. Methods Partial. Diff. Eqn. **27**(6), 1396–1422 (2011)
2. Buet,C., Després, B., Franck, E.: Design of asymptotic preserving schemes for the hyperbolic heat equation on unstructured meshes. Numer. Math. **122**(2), 227–278 (2012)
3. Buet, C., Després, B., Franck, E.: Asymptotic preserving scheme with maximum principle for non linear radiative transfer model on unstructured meshes, C.R. Acad. Sci., Paris, Sér. I, Math., **350**(11–12), 633–638 (2012)
4. Carré, G., Del Pino, S., Després, B., Labourasse, E.: A Cell-centered lagrangian hydrodynamics scheme on general unstructured meshes in arbitrary dimension. JCP. **228**(14), 5160–518 (2009)
5. Chalons, C., Coquel, F., Godlewski, E., Raviart, P-A., Seguin N.: Godunov-type schemes for hyperbolic systems with parameter dependent source. The case of Euler system with friction. M3AS. **20**(11), 2109–2166 2010
6. Chalons C., Girardin M., Kokh S.: Large time step asymptotic preserving numerical schemes for the gas dynamics equations with source terms. SIAM J. Sci. Comput. **35**(6), A2874A2902
7. Gosse, L., Toscani, G.: An asymptotic-preserving well-balanced scheme for the hyperbolic heat equations C. R. Acad. Sci Paris, Ser. I **334**, 337–342 (2002)
8. Greenberg, J., Leroux, A.Y.A.: well balanced scheme for the numerical processing of source terms in hyperbolic equations SIAM J. Numer. Anal **33**(1), 1996
9. Jin S., Levermore D.: Numerical schemes for hyperbolic conservation laws with stiff relaxation terms. JCP. **126**,449–467 (1996)