Uniform-in-Time Convergence of Numerical Schemes for Richards' and Stefan's Models

Jérôme Droniou, Robert Eymard and Cindy Guichard

Abstract We prove that all Gradient Schemes—which include Finite Element, Mixed Finite Element, Finite Volume methods—converge uniformly in time when applied to a family of nonlinear parabolic equations which contains Richards and Stefan's models. We also provide numerical results to confirm our theoretical analysis.

1 Introduction

Let us consider the following generic nonlinear parabolic model

$$\begin{aligned} \partial_t \beta(\overline{u}) - \Delta\zeta(\overline{u}) &= f \text{ in } \Omega \times (0,T), \\ \beta(\overline{u})(x,0) &= \beta(u_{\text{ini}})(x) \text{ in } \Omega, \\ \zeta(\overline{u}) &= 0 \text{ on } \partial\Omega \times (0,T), \end{aligned} \tag{1}$$

where β, ζ are non-decreasing. This model includes both Richards' model (with $\zeta(s) = s$), which describes the flow of water in an underground medium, and Stefan's model (with $\beta(s) = s$), which arises in the study of the heat diffusion in a melting medium. The numerical approximation of both Richards' and Stefan's models has

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been extensively studied in the literature (see the fundamental work on the Stefan problem [13, 14] for a review of some numerical approximations, and see [12] for the Richards problem), but the convergence analysis of the considered schemes received a much reduced coverage and consists mostly in establishing space-time averaged (e.g. in $L^2(\Omega \times (0,T))$ results (in the case of finite volume schemes, see for example [6, 9]). Yet, the quantity of interest is often not $\overline{u}$ on $\Omega \times (0,T)$ but $\overline{u}$ at a given time, for example $t = T$. Existing numerical analysis results therefore do not ensure that this quantity of interest is really properly approximated by numerical methods.

The usual way to obtain pointwise-in-time approximation results for numerical schemes is to prove estimates in $L^\infty(0,T;L^2(\Omega))$ on $u - \overline{u}$, where u is the approximated solution. Establishing such error estimates is however only feasible when uniqueness of the solution $\overline{u}$ to (1) can be proved (which is the case for Richards' and Stefan's problem, but not for more complex non-linear parabolic problems) and requires moreover some regularity assumptions on $\overline{u}$. These assumptions clearly fail for (1) for which, because of the possible plateaux of β and ζ, the solution can develop jumps in its gradient.

The purpose of this article is to prove that, using Discrete Functional Analysis techniques (i.e. the translation to numerical analysis of nonlinear analysis techniques), one can establish an $L^\infty(0,T;L^2(\Omega))$ convergence result for numerical approximations of (1) without having to assume non-physical regularity assumptions on the data. Note that, although Richards' and Stefan's models are formally equivalent when β and ζ are strictly increasing (consider $\beta = \zeta^{-1}$ to pass from one model to the other), they change nature when these functions are allowed to have plateaux. Richards' model can degenerate to an ODE (if $\zeta = 0$ on the range of u_{ini}) and Stefan's model can become a non-transient elliptic equation (if $\beta = 0$). The technique we develop in this paper is however generic enough to work directly on (1), as well as on a vast number of numerical methods.

That being said, we nevertheless require a particular numerical framework to work in, in order to write precise equations and estimates. The framework we choose is that of Gradient Schemes, which has the double benefit of covering a vast number of numerical methods—Finite Element schemes, Mimetic Finite Difference schemes, Finite Volume schemes, etc.—and of having already been studied for many models—elliptic, parabolic, linear or non-linear, possibly degenerate, etc. We refer the reader to [3–5, 8, 10] for more details.

The paper is organised as follows. In the next section, we present the assumption and the notion of weak solution for (1). Section 3 presents the Gradient Schemes for (1). In Sect. 4, we state our uniform convergence result and give a short proof of it, based on the space-time averaged convergence results available in the literature. Finally, Sect. 5 provides some numerical results to illustrate our uniform-in-time convergence theorem.

Note that more complete proofs, as well as an entirely unified convergence analysis (not relying on previous convergence results) of Gradient Schemes for a more general and more non-linear model than (1), can be found in [2].

2 Assumptions and Weak Solution for (1)

The notion of solution of (1) is that of a weak one in the following sense

$$
\left\{
\begin{array}{l}
\overline{u} \in L^2(\Omega \times (0,T))\,,\ \zeta(\overline{u}) \in L^2(0,T;H_0^1(\Omega))\,,\ \partial_t\beta(\overline{u}) \in L^2(0,T;H^{-1}(\Omega)),\\
\beta(\overline{u}) \in C([0,T];L^2(\Omega)\text{-w}),\\
\beta(\overline{u})(\cdot,0) = \beta(u_{\mathrm{ini}}) \text{ in } L^2(\Omega),\\
\displaystyle\int_0^T \langle \partial_t\beta(\overline{u})(\cdot,t), \overline{v}(\cdot,t)\rangle_{H^{-1},H_0^1}\mathrm{d}t + \int_0^T\int_\Omega \nabla\zeta(\overline{u})(x,t)\cdot\nabla\overline{v}(x,t)\mathrm{d}x\mathrm{d}t\\
\quad\displaystyle = \int_0^T\int_\Omega f(x,t)\overline{v}(x,t)\mathrm{d}x\mathrm{d}t\,, \qquad \forall \overline{v} \in L^2(0;T;H_0^1(\Omega))
\end{array}
\right.
\tag{2}
$$

where $C([0,T],L^2(\Omega)\text{-w})$ is the set of functions $[0,T] \to L^2(\Omega)$ which are continuous for the weak topology of $L^2(\Omega)$. We assume throughout this paper that

$$
\begin{array}{l}
\beta,\zeta : \mathbb{R} \mapsto \mathbb{R} \text{ are non-decreasing and Lipschitz-continuous,}\\
\beta(0) = \zeta(0) = 0 \text{ and } \exists A, B > 0 \text{ such that, for all } s \in \mathbb{R},\ |\zeta(s)| \geq A|s| - B,
\end{array}
\tag{3a}
$$

$$
\beta = \mathrm{Id} \text{ or } \zeta = \mathrm{Id} \quad (\textit{we let } \gamma = \zeta \textit{ if } \beta = \mathrm{Id} \textit{ and } \gamma = \beta \textit{ if } \zeta = \mathrm{Id})
\tag{3b}
$$

$$
\begin{array}{l}
\Omega \text{ is an open bounded subset of } \mathbb{R}^d,\ d \in \mathbb{N}^\star,\\
u_{\mathrm{ini}} \in L^2(\Omega), \quad f \in L^2(\Omega \times (0,T)).
\end{array}
\tag{3c}
$$

Under these assumptions, the weak continuity of $\beta(\overline{u}) : [0,T] \mapsto L^2(\Omega)$ is actually a consequence of the other regularity properties on $\overline{u}$, $\zeta(\overline{u})$, $\beta(\overline{u})$ and of the equation, see [2].

3 Gradient Scheme

The presentation of Gradient Schemes given here is minimal, we refer the reader to [3, 4, 7] for more details. A gradient scheme can be viewed as a general formulation of several discretisations of (1) which are based on approximations of the weak formulation (2). These approximations are based on some discrete spaces and mappings, the set of which we call a gradient discretisation.

Definition 1 We say that $\mathscr{D} = (X_{\mathscr{D},0}, \Pi_{\mathscr{D}}, \nabla_{\mathscr{D}}, \mathscr{I}_{\mathscr{D}}, (t^{(n)})_{n=0,\ldots,N})$ is a gradient discretisation for (1) if

1. $X_{\mathscr{D},0}$ is a finite dimensional real vector space (set of unknowns),
2. the linear mapping $\Pi_{\mathscr{D}} : X_{\mathscr{D},0} \to L^\infty(\Omega)$ is a piecewise constant reconstruction operator, that is there exists a set I of degrees of freedom such that $X_{\mathscr{D},0} = \mathbb{R}^I$ and there exists a family $(\Omega_i)_{i\in I}$ of disjoint subsets of Ω such that $\overline{\Omega} = \bigcup_{i\in I}\overline{\Omega_i}$ and, for all $u = (u_i)_{i\in I} \in X_{\mathscr{D},0}$ and all $i \in I$, $\Pi_{\mathscr{D}}u = u_i$ on Ω_i,
3. the linear mapping $\nabla_{\mathscr{D}} : X_{\mathscr{D},0} \to L^2(\Omega)^d$ gives a reconstructed discrete gradient.

4. $\mathcal{I}_{\mathcal{D}} : L^2(\Omega) \to X_{\mathcal{D},0}$ is a linear interpolation operator,
5. $t^{(0)} = 0 < t^{(1)} < t^{(2)} < \ldots < t^{(N)} = T$.

For any function $\chi : \mathbb{R} \mapsto \mathbb{R}$ and any $u \in X_{\mathcal{D},0}$, we denote by $\chi(u) \in X_{\mathcal{D},0}$ the element defined by $(\chi(u))_i = \chi(u_i)$ for any $i \in I$. Since $\Pi_{\mathcal{D}}$ is a piecewise constant reconstruction, we then have $\Pi_{\mathcal{D}}\chi(u) = \chi(\Pi_{\mathcal{D}}u)$. It is also customary to use the notations $\Pi_{\mathcal{D}}$ and $\nabla_{\mathcal{D}}$ for space-time dependent functions. We will also need a notation for the jump-in-time of piecewise constant functions in time. Hence, if $(v^{(n)})_{n=0,\ldots,N} \subset X_{\mathcal{D},0}$, we set

$$\begin{aligned}
&\text{for a.e. } x \in \Omega,\ \Pi_{\mathcal{D}}v(x,0) = \Pi_{\mathcal{D}}v^{(0)}(x) \text{ and } \forall n = 0,\ldots,N-1\,,\ \forall t \in (t^{(n)}, t^{(n+1)}] \ : \\
&\quad \Pi_{\mathcal{D}}v(x,t) = \Pi_{\mathcal{D}}v^{(n+1)}(x)\,,\ \nabla_{\mathcal{D}}v(x,t) = \nabla_{\mathcal{D}}v^{(n+1)}(x) \\
&\quad \text{and } \delta_{\mathcal{D}}v(t) = \delta_{\mathcal{D}}^{(n+\frac{1}{2})}v := \frac{v^{(n+1)} - v^{(n)}}{t^{(n+1)} - t^{(n)}}.
\end{aligned}$$

With these notations, the gradient scheme corresponding to a given gradient discretisation $\mathcal{D}$ is obtained by replacing the continuous functions and gradients in (2) with their discrete counterpart and using an implicit-in-time discretisation. It is therefore written: find $(u^{(n)})_{n=0,\ldots,N} \subset X_{\mathcal{D},0}$ such that

$$\left\{\begin{aligned}
&u^{(0)} = \mathcal{I}_{\mathcal{D}}u_{\rm ini} \text{ and, for all } v = (v^{(n)})_{n=0,\ldots,N} \subset X_{\mathcal{D},0}, \\
&\int_0^T \int_\Omega \Big[\Pi_{\mathcal{D}}\delta_{\mathcal{D}}\beta(u)(x,t)\Pi_{\mathcal{D}}v(x,t) + \nabla_{\mathcal{D}}\zeta(u)(x,t)\cdot\nabla_{\mathcal{D}}v(x,t)\Big]\mathrm{d}x\mathrm{d}t \\
&\quad = \int_0^T \int_\Omega f(x,t)\Pi_{\mathcal{D}}v(x,t)\mathrm{d}x\mathrm{d}t.
\end{aligned}\right. \tag{4}$$

As mentioned in the introduction, gradient schemes cover a wide number of well-known numerical methods [3]. Their convergence analysis is moreover based on a few (four, to be precise) properties that a gradient discretisation must satisfy: *coercivity*, *consistency*, *limit-conformity* and *compactness*. As we will not directly make much use of these properties but only of the following initial convergence result, we just refer the reader to [3, 4] for their precise definition.

Theorem 1 ([5, 8]) *Under Assumption* (3a)–(3c)*, there exists a unique solution to the gradient scheme* (4)*. Moreover, if* $(\mathcal{D}_m)_{m\in\mathbb{N}}$ *is a coercive, consistent, limit-conforming and compact sequence of gradient discretisations, if* $(u_m)_{m\in\mathbb{N}}$ *are the solutions to the corresponding gradient schemes and if* $\overline{u}$ *is the solution to* (2) *then, as* $m \to \infty$, $\Pi_{\mathcal{D}_m}u_m \to \overline{u}$ *weakly in* $L^2(\Omega \times (0,T))$, $\Pi_{\mathcal{D}_m}\gamma(u_m) \to \gamma(\overline{u})$ *in* $L^2(\Omega \times (0,T))$ *and* $\nabla_{\mathcal{D}_m}\zeta(u_m) \to \nabla\zeta(\overline{u})$ *in* $L^2(\Omega \times (0,T))^d$.

4 Uniform Convergence Result

Our main result is the following. As mentioned in the introduction, we only sketch its proof and refer the reader to [2] for the details.

Theorem 2 ([2]) Under the assumptions and notations of Theorem 1, $\Pi_{\mathcal{D}_m}\gamma(u_m) \to \gamma(\overline{u})$ strongly in $L^\infty(0,T;L^2(\Omega))$.

Proof The keys to this proof are the following integration-by-parts properties satisfied by the continuous and discrete solutions. Defining $\beta_r(s)$ = closest z to 0 such that $\beta(z) = s$ (pseudo-inverse of β) and $B(z) = \int_0^z \zeta(\beta_r(s))\mathrm{d}s$, we have, for any $T_0 \in (0,T]$,

$$\int_\Omega B(\beta(\overline{u})(x,T_0))\mathrm{d}x + \int_0^{T_0}\int_\Omega \nabla\zeta(\overline{u})(x,t)\cdot\nabla\zeta(\overline{u})(x,t)\mathrm{d}x\mathrm{d}t$$
$$= \int_\Omega B(\beta(u_{\mathrm{ini}}(x)))\mathrm{d}x + \int_0^{T_0}\int_\Omega f(x,t)\zeta(\overline{u})(x,t)\mathrm{d}x\mathrm{d}t \tag{5}$$

and

$$\int_\Omega B(\Pi_{\mathcal{D}_m}\beta(u_m)(x,T_0))\mathrm{d}x + \int_0^{T_0}\int_\Omega \nabla_{\mathcal{D}_m}\zeta(u_m)(x,t)\cdot\nabla_{\mathcal{D}_m}\zeta(u_m)(x,t)\mathrm{d}x\mathrm{d}t$$
$$\leq \int_\Omega B(\Pi_{\mathcal{D}_m}\beta(\mathcal{I}_{\mathcal{D}_m}u_{\mathrm{ini}})(x))\mathrm{d}x + \int_0^{t^{(k)}}\int_\Omega f(x,t)\Pi_{\mathcal{D}_m}\zeta(u_m)(x,t)\mathrm{d}x\mathrm{d}t, \tag{6}$$

where $k \in \{1,\ldots,N\}$ is such that $t^{(k-1)} < T_0 \leq t^{(k)}$. These formula are obtained by plugging respectively $\bar{v} = \zeta(\overline{u})$ and $v = u_m$ in (2) and (4). Properly justifying (5) is however not straightforward because of the lack of regularity of $\overline{u}$.

Let $T_0 \in [0,T]$ and $(T_m)_{m\in\mathbb{N}}$ which converges to T_0. We apply (6) to $T_0 = T_m$ and let $m \to \infty$. The consistency of $(\mathcal{D}_m)_{m\in\mathbb{N}}$ ensures that $\mathcal{I}_{\mathcal{D}_m}u_{\mathrm{ini}} \to u_{\mathrm{ini}}$ in $L^2(\Omega)$. Hence, using the strong convergence in $L^2(\Omega\times(0,T))^d$ of $\nabla_{\mathcal{D}_m}\zeta(u_m)$ to $\nabla\zeta(\overline{u})$ and (5), we find

$$\limsup_{m\to\infty}\int_\Omega B(\Pi_{\mathcal{D}_m}\beta(u_m)(x,T_m))\mathrm{d}x \leq \int_\Omega B(\beta(\overline{u})(x,T_0))\mathrm{d}x. \tag{7}$$

Using the scheme (4), we easily obtain, for any $\varphi \in L^2(\Omega)$, estimates on the variations of $t \mapsto \langle \Pi_{\mathcal{D}_m}\beta(u_m)(t),\varphi\rangle_{L^2(\Omega)}$ which show that $(\Pi_{\mathcal{D}_m}\beta(u_m))_{m\in\mathcal{D}_m}$ is relatively compact in $L^\infty(0,T;L^2(\Omega)\text{-w})$ and therefore converges uniformly in time for the weak topology of $L^2(\Omega)$. We deduce that $\Pi_{\mathcal{D}_m}\beta(u_m)(T_m) \to \beta(\bar{u})(T_0)$ weakly in $L^2(\Omega)$ and the convexity of B therefore ensures that

$$\int_\Omega B(\beta(\overline{u})(x,T_0))\mathrm{d}x \leq \liminf_{m\to\infty}\int_\Omega B(\Pi_{\mathcal{D}_m}\beta(u_m)(x,T_m))\mathrm{d}x. \tag{8}$$

Combining (7) and (8) we find that

$$\lim_{m\to\infty}\int_\Omega B(\Pi_{\mathscr{D}_m}\beta(u_m)(x,T_m))\mathrm{d}x = \int_\Omega B(\beta(\overline{u})(x,T_0))\mathrm{d}x. \tag{9}$$

We also notice that, by weak convergence in $L^2(\Omega)$ of $\Pi_{\mathscr{D}_m}\beta(u_m)(T_m)$ to $\beta(\overline{u})(T_0)$ and the convexity of B,

$$\int_\Omega B(\beta(\overline{u})(x,T_0))\mathrm{d}x \le \liminf_{m\to\infty}\int_\Omega B\left(\frac{\Pi_{\mathscr{D}_m}\beta(u_m)(x,T_m)+\beta(\overline{u})(x,T_0)}{2}\right)\mathrm{d}x. \tag{10}$$

The definition of B ensures that, for all $s, s' \in \mathbb{R}$,

$$(\gamma(s)-\gamma(s'))^2 \le C_1\left[B(\beta(s))+B(\beta(s'))-2B\left(\frac{\beta(s)+\beta(s')}{2}\right)\right]$$

where C_1 only depends on the Lipschitz constants of β and ζ. We deduce that

$$\begin{aligned}
&\|\gamma(\Pi_{\mathscr{D}_m}u_m)(\cdot,T_m)-\gamma(\overline{u})(\cdot,T_0)\|^2_{L^2(\Omega)}\\
&\le C_1\int_\Omega \left[B(\beta(\Pi_{\mathscr{D}_m}u_m)(x,T_m))+B(\beta(\overline{u})(x,T_0))\right]\mathrm{d}x\\
&\quad -2C_1\int_\Omega B\left(\frac{\beta(\Pi_{\mathscr{D}_m}u_m)(x,T_m)+\beta(\overline{u})(x,T_0)}{2}\right)\mathrm{d}x.
\end{aligned}$$

Taking the lim sup as $m\to\infty$ of this relation and using (9) and (10), we find that $\Pi_{\mathscr{D}_m}\gamma(u_m(\cdot,T_m)) \to \gamma(\overline{u})(\cdot,T_0)$ in $L^2(\Omega)$ as $m\to\infty$. Since this is true for any sequence $T_m \to T_0$, and since we can prove that $\gamma(\overline{u})$ is continuous $[0,T]\mapsto L^2(\Omega)$, this proves that $\Pi_{\mathscr{D}_m}\gamma(u_m)\to\gamma(\overline{u})$ uniformly on $[0,T]$ for the topology of $L^2(\Omega)$.

5 Numerical Tests

In order to illustrate the uniform-in-time convergence properties, we first present the gradient scheme which has been selected for running the test cases. The gradient scheme is built on a conforming simplicial mesh of the polyhedral domain Ω (see [1] for the precise definitions of such a mesh). The degrees of freedom of any $u \in X_{\mathscr{D},0}$ are the values u_s for all interior vertices s of the mesh. Then $\Pi_{\mathscr{D}}u$ is taken piecewise constant in the regions K_s (see Fig. 1), whereas $\nabla_{\mathscr{D}}u$ is the gradient of the P^1 finite element function obtained from the values u_s.

For both following tests, the meshes used for the discretisation of the domain $\Omega = (0,1)^2$ come from from the FVCA5 2D benchmark on anisotropic diffusion problem [11]. These triangle meshes show no symmetry which could artificially increase the convergence rate, and all angles of triangles are acute. This family of meshes is built using the same pattern reproduced at different scales: the first (coarsest) mesh and the third mesh are shown in Fig. 2. We consider the two cases

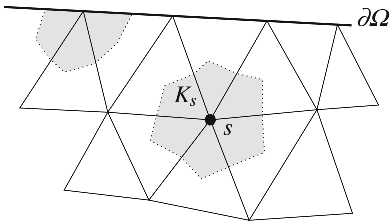

Fig. 1 Definition of K_s

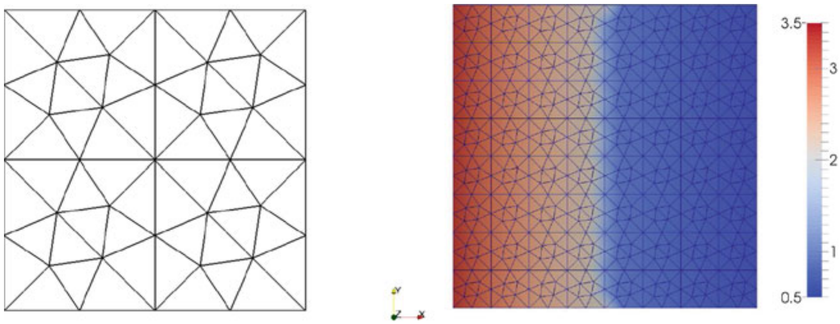

Fig. 2 First mesh and approximate solution u at time 0.5 for the Stefan problem on the third mesh

Table 1 Data and numerical results for Stefan and Richards problems

<table>
<tr><td colspan="2">test case</td><td colspan="2">Stefan</td><td colspan="2">Richards</td></tr>
<tr><td colspan="2">$\beta(u)$</td><td colspan="2">u</td><td colspan="2">$\begin{cases} 0 & \text{if } u \le 0 \\ u & \text{if } 0 \le u \end{cases}$</td></tr>
<tr><td colspan="2">$\zeta(u)$</td><td colspan="2">$\begin{cases} u & \text{if } u \le 1 \\ 1 & \text{if } 1 \le u \le 2 \\ u-1 & \text{if } 2 \le u \end{cases}$</td><td colspan="2">$u$</td></tr>
<tr><td colspan="2">analytical sol. $u(x_1,x_2,t)$</td><td colspan="2">$\begin{cases} 2\exp(t-x_1)\ (>2) & \text{if } x_1 < t \\ \exp(t-x_1)\ (<1) & \text{if } t < x_1 \end{cases}$</td><td colspan="2">$\begin{cases} \exp(t-x_1)-1\ (\ge 0) & \text{if } x_1 \le t \\ t-x_1\ (\le 0) & \text{if } t \le x_1 \end{cases}$</td></tr>
<tr><td>h</td><td>δt</td><td>error on $\zeta(u)$ in $L^\infty(0,T;L^2(\Omega))$</td><td>num. order</td><td>error on $\beta(u)$ in $L^\infty(0,T;L^2(\Omega))$</td><td>num. order</td></tr>
<tr><td>0.250</td><td>0.01024</td><td>0.362E-01</td><td>-</td><td>0.223E-02</td><td>-</td></tr>
<tr><td>0.125</td><td>0.00256</td><td>0.191E-01</td><td>0.920</td><td>0.891E-03</td><td>1.327</td></tr>
<tr><td>0.063</td><td>0.00064</td><td>0.895E-02</td><td>1.096</td><td>0.297E-03</td><td>1.584</td></tr>
<tr><td>0.031</td><td>0.00016</td><td>0.392E-02</td><td>1.192</td><td>0.101E-03</td><td>1.562</td></tr>
<tr><td>0.016</td><td>0.00004</td><td>0.175E-02</td><td>1.166</td><td>0.334E-04</td><td>1.590</td></tr>
</table>

of a Stefan problem and of a Richards problems, for which there exists an analytical solution with $f = 0$. These analytical solutions show the regularity properties of "natural" solutions on the time period [0, 1] during which a free boundary moves from $x_1 = 0$ to $x_1 = 1$. In the Stefan problem, this free boundary is the surface between two thermodynamical states of a material. In the Richards problem, it is the limit between a fully saturated zone and a partially saturated zone. These test cases are built on 1D solutions, using fully 2D meshes, hence providing realistic conditions (an example of a numerical solution is shown in the right part of Fig. 2). For both of them, the corresponding data and numerical results are given in Table 1. The convergence orders are computed from the values of h, and the constant time steps have been taken proportional to h^2. Note that in both cases, the proposed analytical solution is a strong solution for $x_1 < t$ and $x_1 > t$ and the Rankine-Hugoniot condition holds at the free boundary $x_1 = t$. It is therefore a weak solution to (2), extended to the case of non-homogeneous Dirichlet boundary conditions. These results confirm our uniform-in-time convergence result (Theorem 2). We also observe that in the Stefan case, where u is discontinuous and $\zeta(u)$ is only of class H^1, the convergence order remains close to 1 whereas in the Richards case, where u is of class C^1 in space, the convergence orders are greater.

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