

Flexibility Properties in Complex Analysis and Affine Algebraic Geometry

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Abstract In the last decades affine algebraic varieties and Stein manifolds with big (infinite-dimensional) automorphism groups have been intensively studied. Several notions expressing that the automorphisms group is big have been proposed. All of them imply that the manifold in question is an Oka–Forstnerič manifold. This important notion has also recently merged from the intensive studies around the homotopy principle in Complex Analysis. This homotopy principle, which goes back to the 1930s, has had an enormous impact on the development of the area of Several Complex Variables and the number of its applications is constantly growing. In this overview chapter we present three classes of properties: (1) density property, (2) flexibility, and (3) Oka–Forstnerič. For each class we give the relevant definitions, its most significant features and explain the known implications between all these properties. Many difficult mathematical problems could be solved by applying the developed theory, we indicate some of the most spectacular ones.

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1 Introduction

This is a survey of recent developments in Complex Analysis and Affine Algebraic Geometry which emphasize on objects described as elliptic, in the opposite to hyperbolic in the sense of Kobayashi or more general in the sense of Eisenman (all Eisenman measures on these objects vanish identically.)

Here is the scheme of properties we are going to discuss, together with the known implications between them. Although the properties do not require the manifolds to be Stein or affine algebraic, some of the implications do. We therefore assume the manifold to be Stein in the upper row and to be a smooth affine algebraic variety in the lower row.

$$\begin{array}{ccc}
 \text{density property (DP)} & \implies & \text{holomorphic flexible} \implies \text{Oka–Forstneric} \\
 \uparrow & & \uparrow \\
 \text{algebraic density property (ADP)} & & \text{algebraic flexible}
 \end{array} \tag{1}$$

In each of the following three sections we present one class of properties together with main features. We hope that researchers from the algebraic and from the holomorphic side can join their efforts to enlarge the class of examples and to find out which of the reverse implications between these properties hold.

In the last section we briefly recall that in the presence of a volume form there is a similar property to (algebraic) density property, called (algebraic) volume density property, which if replacing DP and ADP in Scheme (1) by these properties (AVDP, VDP) gives another scheme with the same implications true. Also we elaborate on the reverse implications in our Scheme (1).

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2 Density Property

2.1 Definition and Main Features

Considering a question promoted by Walter Rudin, Andersén and Lempert in 1989 [1, 3] proved a remarkable fact about the affine n -space $n \geq 2$, namely that the group generated by shears (maps of the form $(z_1, \dots, z_n) \mapsto (z_1, \dots, z_{n-1}, z_n + f(z_1, \dots, z_{n-1}))$ where $f \in \mathcal{O}(\mathbb{C}^{n-1})$ is a holomorphic function and any linear conjugate of such a map) and overshears (maps of the form $(z_1, \dots, z_n) \mapsto (z_1, \dots, z_{n-1}, z_n g(z_1, \dots, z_{n-1}))$ where $g \in \mathcal{O}^*(\mathbb{C}^{n-1})$ is a nowhere vanishing holomorphic function and any linear conjugate of such a map) are dense in

holomorphic automorphism group of \mathbb{C}^n , endowed with compact-open topology. The main importance of their work was not the mentioned result but the proof itself which implies, as observed by Forstnerič and Rosay in [17] for $X = \mathbb{C}^n$, the remarkable Andersén–Lempert theorem, see below. The natural generalization from \mathbb{C}^n to arbitrary manifolds X was made by Varolin [40] who introduced the following important property of a complex manifold:

Definition 1. A complex manifold X has the density property if in the compact-open topology the Lie algebra generated by completely integrable holomorphic vector fields on X is dense in the Lie algebra of all holomorphic vector fields on X .

Here a holomorphic vector field Θ on a complex manifold X is called completely integrable if the ODE

$$\begin{aligned} \frac{d}{dt}\varphi(x, t) &= \Theta(\varphi(x, t)) \\ \varphi(x, 0) &= x \end{aligned}$$

has a solution $\varphi(x, t)$ defined for all complex times $t \in \mathbb{C}$ and all starting points $x \in X$. It gives a complex one-parameter subgroup in the holomorphic automorphism group $\text{Aut}_{\text{hol}}(X)$.

The density property is a precise way of saying that the automorphism group of a manifold is big, in particular for a Stein manifold this is underlined by the main result of the theory (see [17] for \mathbb{C}^n , [40], a detailed proof can be found in the Appendix of [36] or in [14]).

Theorem 2 (Andersén–Lempert Theorem). *Let X be a Stein manifold with the density property and let Ω be an open subset of X . Suppose that $\Phi : [0, 1] \times \Omega \rightarrow X$ is a C^1 -smooth map such that*

- (1) $\Phi_t : \Omega \rightarrow X$ is holomorphic and injective for every $t \in [0, 1]$,
- (2) $\Phi_0 : \Omega \rightarrow X$ is the natural embedding of Ω into X , and
- (3) $\Phi_t(\Omega)$ is a Runge subset¹ of X for every $t \in [0, 1]$.

Then for each $\epsilon > 0$ and every compact subset $K \subset \Omega$ there is a continuous family, $\alpha : [0, 1] \rightarrow \text{Aut}_{\text{hol}}(X)$ of holomorphic automorphisms of X such that

$$\alpha_0 = id \text{ and } |\alpha_t - \Phi_t|_K < \epsilon \text{ for every } t \in [0, 1]$$

Philosophically one can think of the density property as a tool for realizing local movements by global maps (automorphisms). In some sense it is a substitute

¹Recall that an open subset U of X is Runge if any holomorphic function on U can be approximated by global holomorphic functions on X in the compact-open topology. Actually, for X Stein by Cartan’s Theorem A this definition implies more: for any coherent sheaf on X its section over U can be approximated by global sections.

for cutoff functions which in the differentiable category are used for globalizing local movements. In the holomorphic category we of course lose control on automorphism outside the compact set K . This makes constructions more complicate but still constructing sequences of automorphisms by iterated use of the Andersén–Lempert theorem has led to remarkable constructions.

Let us further remark that the implications of the density property for manifolds which are not Stein have not been explored very much yet. If the manifold is compact all (holomorphic) vector fields are completely integrable, the density property trivially hold and thus cannot give any further information on the manifold.

Remark 3. Andersén and Lempert proved that every algebraic vector field on \mathbb{C}^n is a finite sum of algebraic shear fields (fields of form $p(z_1, \dots, z_{n-1}) \frac{\partial}{\partial z_n}$ for a polynomial $p \in \mathbb{C}[\mathbb{C}^{n-1}]$ and their linear conjugates, i.e., fields whose one-parameter subgroups consist of shears) and overshear fields (fields of form $p(z_1, \dots, z_{n-1}) z_n \frac{\partial}{\partial z_n}$ for a polynomial $p \in \mathbb{C}[\mathbb{C}^{n-1}]$ and their linear conjugates, i.e., fields whose one-parameter subgroups consist of overshears). Together with the fact that any holomorphic automorphism of \mathbb{C}^n can be joined to the identity by a smooth path, this shows how the Andersén–Lempert theorem implies that the group generated by shears and overshears is dense in the holomorphic automorphism group of \mathbb{C}^n .

The algebraic density property can be viewed as a tool to prove the density property, whereas the ways of proving it are purely algebraic work.

Definition 4. An affine algebraic manifold X has the algebraic density property if the Lie algebra $\text{Lie}_{\text{alg}}(X)$ generated by completely integrable algebraic vector fields on it coincides with the Lie algebra $\text{VF}_{\text{alg}}(X)$ of all algebraic vector fields on it.

An algebraic vector field is an algebraic section of the tangent bundle, for example on \mathbb{C}^n it can be written as $\sum_{i=1}^n p_i(z_1, \dots, z_n) \frac{\partial}{\partial z_i}$ with polynomials $p_i \in \mathbb{C}[\mathbb{C}^n]$. If it is completely integrable, its flow gives a one-parameter subgroup in the holomorphic automorphism group not necessarily in the algebraic automorphism group. For example, a polynomial shear field of the form $p(z_1, \dots, z_{n-1}) z_n \frac{\partial}{\partial z_n}$ has the flow map $\gamma(t, z) = (z_1, \dots, z_{n-1}, \exp(tp(z_1, \dots, z_{n-1}))z_n)$. This is the reason that algebraic density property is in the intersection of affine algebraic geometry and complex analysis. It is an algebraic notion, proven using algebraic methods but has implications for the holomorphic automorphism group.

2.2 Applications and Examples

A first application we like to mention is to the notoriously difficult question whether every open Riemann surface can be properly holomorphically embedded into \mathbb{C}^2 . This is the only dimension for which the conjecture of Forster [12], saying that every Stein manifold of dimension n can be properly holomorphically embedded into \mathbb{C}^N for $N = \lfloor \frac{n}{2} \rfloor + 1$, is still unsolved. The conjectured dimension is sharp

by examples of Forster [12] and has been proven by Eliashberg, Gromov [10] and Schürmann [37] for all dimensions $n \geq 2$. Their methods of proof fail in dimension $n = 1$. But Fornaess Wold invented a clever combination of a use of shears (nice projection property) and Theorem 2 which led to many new embedding theorems for open Riemann surfaces. As an example we like to mention the following two recent results of Forstnerič and Fornaess Wold [18, 19] the first of them being the most general one for open subsets of the complex line:

Theorem 5. *Every domain in the Riemann sphere with at least one and at most countably many boundary components, none of which are points, admits a proper holomorphic embedding into \mathbb{C}^2 .*

Theorem 6. *If $\bar{\Sigma}$ is a (possibly reducible) compact complex curve in \mathbb{C}^2 with boundary $\partial\Sigma$ of class C^r for some $r > 1$, then the inclusion map $i : \Sigma = \bar{\Sigma} \setminus \Sigma \rightarrow \mathbb{C}^2$ can be approximated, uniformly on compacts in Σ , by proper holomorphic embeddings $\Sigma \rightarrow \mathbb{C}^2$.*

Many versions of embeddings with interpolation are also known and proven using the same methods invented by Fornaess Wold in [42].

Another application is to construct non-straightenable holomorphic embeddings of \mathbb{C}^k into \mathbb{C}^n for all pairs of dimensions $0 < k < n$, a fact which is contrary to the situation in affine algebraic geometry, namely contrary to the famous Abhyankar-Moh-Suzuki theorem for $k = 1, n = 2$ and also to work of Kaliman [26] or $2k + 1 < n$, whereas straightenability for the other dimension pairs is still unknown in algebraic geometry. The most recent and quite striking result in this direction says that there are even holomorphic families of pairwise non-equivalent holomorphic embeddings (referring to holomorphic automorphisms of the source and target in the definition below). Here non-straightenable for an embedding \mathbb{C}^k into \mathbb{C}^n means to be not equivalent to the standard embedding.

Definition 7. Two embeddings $\Phi, \Psi: X \hookrightarrow \mathbb{C}^n$ are *equivalent* if there exist automorphisms $\varphi \in \text{Aut}(\mathbb{C}^n)$ and $\psi \in \text{Aut}(X)$ such that $\varphi \circ \Phi = \Psi \circ \psi$.

Theorem 8. *see [33]. Let n, l be natural numbers with $n \geq l + 2$. There exist, for $k = n - l - 1$, a family of holomorphic embeddings of \mathbb{C}^l into \mathbb{C}^n parametrized by \mathbb{C}^k , such that for different parameters $w_1 \neq w_2 \in \mathbb{C}^k$ the embeddings $\psi_{w_1}, \psi_{w_2} : \mathbb{C}^l \hookrightarrow \mathbb{C}^n$ are non-equivalent.*

We would like to mention a nice application of Theorem 8 to actions of compact (or equivalently complex reductive, see [31]) groups on \mathbb{C}^n . It was a long-standing problem, whether all holomorphic actions of such groups on affine space are linear after a change of variables (see for example the overview article [24]). The first counterexamples to that (Holomorphic Linearization) problem were constructed by Derksen and the first author in [9]. The method from [9] is holomorphic in a parameter and therefore applied to our parametrized situation leads to the following result ([33])

Theorem 9. *For any $n \geq 5$ there is a holomorphic family of \mathbb{C}^* -actions on \mathbb{C}^n parametrized by \mathbb{C}^{n-4}*

$$\mathbb{C}^{n-4} \times \mathbb{C}^* \times \mathbb{C}^n \rightarrow \mathbb{C}^n, \quad (w, \theta, z) \mapsto \theta_w(z)$$

so that for different parameters $w_1 \neq w_2 \in \mathbb{C}^{n-4}$ there is no equivariant isomorphism between the actions θ_{w_1} and θ_{w_2} .

The linearization problem for holomorphic \mathbb{C}^* -actions on \mathbb{C}^n is thus solved to the positive for $n = 2$ by Suzuki [39] and still open for $n = 3$. For $n = 4$ there are uncountably many actions (non-linearizable ones among them) [8] and for $n \geq 5$ Theorem 9 implies that there are families. Moreover, there are families including a linear action as a single member of the family;

Theorem 10. *For any $n \geq 5$ there is a holomorphic family of \mathbb{C}^* -actions on \mathbb{C}^n parametrized by \mathbb{C}*

$$\mathbb{C} \times \mathbb{C}^* \times \mathbb{C}^n \rightarrow \mathbb{C}^n \quad (w, \theta, z) \mapsto \theta_w(z)$$

so that for different parameters $w_1 \neq w_2 \in \mathbb{C}$ there is no equivariant isomorphism between the actions θ_{w_1} and θ_{w_2} . Moreover, the action θ_0 is linear.

Open Problem: Suppose X is a Stein manifold with density property and $Y \subset X$ is a closed submanifold. Is there always another proper holomorphic embedding $\varphi : Y \hookrightarrow X$ which is not equivalent to the inclusion $i : Y \hookrightarrow X$?

We should remark that an affirmative answer to this question is stated in [41], but the author apparently had another (weaker) notion of equivalence in mind.

Here comes the essentially complete list of examples of Stein manifolds known to have the density property:

List of examples of Stein manifolds known to have the density property:

1. $X = G/R$ where G is linear algebraic and R a reductive subgroup has ADP and thus DP (defined on p. 2), except for $X = \mathbb{C}$ and $X = (\mathbb{C}^*)^n$. (this includes all examples known from the work of Andersén–Lempert and Varolin and Varolin–Toth and Kaliman–Kutzschebauch, the final result is proven by Donzelli–Dvorsky–Kaliman [7]);
2. The manifolds X given as a submanifold in \mathbb{C}^{n+2} with coordinates $u \in \mathbb{C}, v \in \mathbb{C}, z \in \mathbb{C}^n$ by the equation $uv = p(z)$, where the zero fiber of the polynomial $p \in \mathbb{C}[\mathbb{C}^n]$ is smooth (otherwise X is not smooth), have ADP [28].
3. The only known non-algebraic example with DP are the manifolds X given as a submanifold in \mathbb{C}^{n+2} with coordinates $u \in \mathbb{C}, v \in \mathbb{C}, z \in \mathbb{C}^n$ by the equation $uv = f(z)$, where the zero fiber of the holomorphic function $f \in \mathcal{O}(\mathbb{C}^n)$ is smooth (otherwise X is not smooth) [28].
4. Danilov–Gizatullin surfaces have ADP [6].

A variant of density property for (normal) singular varieties (considering vector fields vanishing on a subvariety in particular on the singular locus $\text{Sing}(X)$) was introduced in [34]. A version of Andersén–Lempert theorem holds in this situation which allows to approximate local movements taking place in the smooth part of X by automorphisms fixing the singular locus. It is proven in [34] that normal affine toric varieties have this property. Another version of this generalization considering holomorphic automorphisms of \mathbb{C}^n fixing a codimension two subvariety can be found in [27]. For more information on the density property we refer to the overview article [29].

3 Flexibility

3.1 Definition and Main Features

The notion of flexibility is the most recent among the described properties. It was defined in [4]. First the algebraic version:

Definition 11. Let X be a reduced algebraic variety defined over \mathbb{C} (any algebraically closed field would do). We let $\text{SAut}(X)$ denote the subgroup of $\text{Aut}_{\text{alg}}(X)$ generated by all algebraic one-parameter unipotent subgroups of $\text{Aut}_{\text{alg}}(X)$, i.e., algebraic subgroups isomorphic to the additive group \mathbb{G}_a (usually denoted \mathbb{C}^+ in complex analysis). The group $\text{SAut}(X)$ is called the *special automorphism group* of X ; this is a normal subgroup of $\text{Aut}_{\text{alg}}(X)$.

Definition 12. We say that a point $x \in X_{\text{reg}}$ is *algebraically flexible* if the tangent space $T_x X$ is spanned by the tangent vectors to the orbits $H \cdot x$ of one-parameter unipotent subgroups $H \subseteq \text{Aut}_{\text{alg}}(X)$. A variety X is called *algebraically flexible* if every point $x \in X_{\text{reg}}$ is.

Clearly, X is algebraically flexible if one point of X_{reg} is and the group $\text{Aut}_{\text{alg}}(X)$ acts transitively on X_{reg} .

The main feature of algebraic flexibility is the following result from [4] (whose proof mainly relies on the Rosenlicht theorem);

Theorem 13. *For an irreducible affine variety X of dimension ≥ 2 , the following conditions are equivalent.*

- (1) *The group $\text{SAut}(X)$ acts transitively on X_{reg} .*
- (2) *The group $\text{SAut}(X)$ acts infinitely transitively on X_{reg} .*
- (3) *X is an algebraically flexible variety.*

The paper [4] also contains versions of simultaneous transitivity (where the space X_{reg} is stratified by orbits of $\text{SAut}(X)$) and versions with jet-interpolation. Moreover, it was recently remarked that the theorem holds for quasi-affine varieties, see Theorem 1.11. in [11].

The holomorphic version of this notion is much less explored, it is obviously implied by the algebraic version in case X is an algebraic variety.

Definition 14. We say that a point $x \in X_{\text{reg}}$ is *holomorphically flexible* if the tangent space $T_x X$ is spanned by the tangent vectors of completely integrable holomorphic vector fields, i.e., holomorphic one-parameter subgroups in $\text{Aut}_{\text{hol}}(X)$. A complex manifold X is called *holomorphically flexible* if every point $x \in X_{\text{reg}}$ is.

Clearly, X is holomorphically flexible if one point of X_{reg} is and the group $\text{Aut}_{\text{hol}}(X)$ acts transitively on X_{reg} .

In the holomorphic category it is still open whether an analogue of Theorem 13 holds.

Open Problem: Are the three equivalences from Theorem 13 true for an irreducible Stein space X ? More precisely, if an irreducible Stein space X is holomorphically flexible, does the holomorphic automorphism group $\text{Aut}_{\text{hol}}(X)$ act infinitely transitively on X_{reg} ?

It is clear that holomorphic flexibility of X implies that $\text{Aut}_{\text{hol}}(X)$ acts transitively on X_{reg} , i.e., the implication (3) \Rightarrow (1) is true. Indeed, let $\theta_i, i = 1, 2, \dots, n$ be completely integrable holomorphic vector fields which span the tangent space $T_x X$ at some point $x \in X_{\text{reg}}$, where $n = \dim X$. If $\psi^i : \mathbb{C} \times X \rightarrow X, (t, x) \mapsto \psi_t^i(x)$ denote the corresponding one-parameter subgroups, then the map $\mathbb{C}^n \rightarrow X, (t_1, t_2, \dots, t_n) \mapsto \psi_{t_n}^n \circ \psi_{t_{n-1}}^{n-1} \circ \dots \circ \psi_{t_1}^1(x)$ is of full rank at $t = 0$ and thus by the Inverse Function Theorem a local biholomorphism from a neighborhood of 0 to a neighborhood of x . Thus the $\text{Aut}_{\text{hol}}(X)$ -orbit through any point of X_{reg} is open. If all orbits are open, each orbit is also closed, being the complement of all other orbits. Since X_{reg} is connected, this implies that it consists of one orbit.

The inverse implication (1) \Rightarrow (3) is also true. For the proof we appeal to the Hermann–Nagano Theorem which states that if \mathfrak{g} is a Lie algebra of holomorphic vector fields on a manifold X , then the orbit $R_{\mathfrak{g}}(x)$ (which is the union of all points z over any collection of finitely many fields $v_1, \dots, v_N \in \mathfrak{g}$ and over all times (t_1, \dots, t_N) for which the expression $z = \psi_{t_N}^N \circ \psi_{t_{N-1}}^{N-1} \circ \dots \circ \psi_{t_1}^1(x)$ is defined) is a locally closed submanifold and its tangent space at any point $y \in R_{\mathfrak{g}}(x)$ is $T_y R_{\mathfrak{g}}(x) = \text{span}_{v \in \mathfrak{g}} v(y)$. We consider the Lie algebra \mathfrak{g} generated by completely integrable holomorphic vector fields. Since by the assumption the orbit is X_{reg} we conclude that Lie combinations of completely integrable holomorphic vector fields span the tangent space at each point in X_{reg} . Now suppose at some point x_0 the completely integrable fields do not generate $T_{x_0} X_{\text{reg}}$, i.e., there is a proper linear subspace W of $T_{x_0} X_{\text{reg}}$, such that $v(x_0) \in W$ for all completely integrable holomorphic fields v . Any Lie combination of completely integrable holomorphic fields is a limit (in the compact open topology) of sums of completely integrable holomorphic fields due to the formula $\{v, w\} = \lim_{t \rightarrow 0} \frac{\phi_t^*(w) - w}{t}$ for the Lie bracket ($\phi_t^*(w)$ is a completely integrable field pulled back by an automorphism, thus completely integrable!). Therefore all Lie combinations of completely integrable fields evaluated at x_0 are contained in $W \subset T_{x_0} X_{\text{reg}}$, a contradiction.

In order to prove the remaining implication (3) \Rightarrow (2) one would like to find suitable functions $f \in \text{Ker}\theta$ for a completely integrable holomorphic vector field θ , vanishing at one point and not vanishing at some other point of X . In general these functions may not exist, an orbit of θ can be dense in X .

At this point it is worth mentioning that for a Stein manifold DP implies all three conditions from Theorem 13. For flexibility this is lemma 26 below, infinite transitivity (with jet-interpolation) is proved by Varolin in [41].

Also the generalized form of DP for Stein spaces defined in [34] implies all three conditions from Theorem 13.

3.2 Examples

Examples of algebraically flexible varieties are homogeneous spaces of semisimple Lie groups (or extensions of semisimple Lie groups by unipotent radicals), toric varieties without non-constant invertible regular functions, cones over flag varieties, and cones over Del Pezzo surfaces of degree at least 4, normal hypersurfaces of the form $uv = p(\bar{x})$ in $\mathbb{C}_{u,v,\bar{x}}^{n+2}$. Moreover, algebraic subsets of codimension at least 2 can be removed as recently shown by Flenner, Kaliman, and Zaidenberg in [11]

Theorem 15. *Let X be a smooth quasi-affine variety of dimension ≥ 2 and $Y \subset X$ a closed subscheme of codimension ≥ 2 . If X is flexible, then so is $X \setminus Y$.*

4 Oka–Forstnerič Manifolds

4.1 Historical Introduction to Oka Theory and Motivational Examples

The notion of Oka–Forstnerič manifolds is quite new (it was introduced by Forstnerič in [13], who called them Oka manifolds following a suggestion of Lárusson who underlined the importance of such a notion already in [35]) but the development merging into this important notion, called Oka theory, has a long history. It started with Oka’s theorem from 1939 that the second (multiplicative) Cousin problem on a domain of holomorphy is solvable with holomorphic functions if it is solvable with continuous functions. This implies that a holomorphic line bundle on such a domain is holomorphically trivial if it is topologically trivial.

Let us recall how the generalizations of the classical one variable results of Mittag–Leffler (construct meromorphic functions with prescribed main parts) and Weierstrass (construct meromorphic functions with prescribed zeros and poles) are generalized to several complex variables.

Let us recall the first (additive) Cousin problem which arises from the Mittag–Leffler problem, generalizing the famous Mittag–Leffler theorem from one variable to several variables: Given data $\{(U_i, m_i)\}$, where U_i is an open cover of a complex space X and $m_i \in \mathcal{M}(U_i)$ is a meromorphic function on U_i such that every difference $f_{ij} = m_i|_{U_{ij}} - m_j|_{U_{ij}}$ is holomorphic on $U_{ij} = U_i \cap U_j$, find a global meromorphic function $m \in \mathcal{M}(X)$ on X such that $m|_{U_i} - m_i$ is holomorphic on U_i for all i .

For solving this Mittag–Leffler problem one first solves the associated additive Cousin problem, defined as follows: The collection $f_{ij} \in \mathcal{O}(U_{ij})$ defines a 1-cocycle on the cover U_i with values in the sheaf \mathcal{O} of holomorphic functions, meaning that for each triple i, j, k of indexes we have

$$f_{ij} + f_{jk} + f_{ki} = 0 \quad \text{on} \quad U_{ijk} = U_i \cap U_j \cap U_k.$$

Given such a 1-cocycle $\{f_{ij}\}$, the Cousin I problem asks for a collection of holomorphic functions $f_j \in \mathcal{O}(U_j)$ (a 0-cochain) such that

$$f_i - f_j = f_{ij} \quad \text{on} \quad U_{ij}.$$

One expresses this by saying the cocycle splits or it is a 1-coboundary. From the solution to the additive Cousin problem one obtains by setting $m|_{U_i} = m_i - f_i$ a well-defined (since $m_i - m_j = f_{ij} = f_i - f_j$ on U_{ij}) global meromorphic function $m \in \mathcal{M}(X)$ solving the Mittag–Leffler problem.

The vanishing of the first Čech cohomology group $H^1(X, \mathcal{O})$ with coefficients in the sheaf \mathcal{O} means that every 1-cocycle splits on a refinement of the covering. In other words $H^1(X, \mathcal{O}) = 0$ implies that every 1-cocycle becomes a 1-coboundary on a refinement, so every Mittag–Leffler problem is solvable, in particular by Cartan’s Theorem B this is true for any Stein manifold.

The second (multiplicative) Cousin Problem arises from the problem of finding meromorphic functions with prescribed zeros and poles, solved by Weierstrass in one variable. Given data $\{(U_i, m_i)\}$, where U_i is an open cover of a complex space X and $m_i \in \mathcal{M}^*(U_i)$ is an invertible (i.e., not vanishing identically on any connected component) meromorphic function on U_i such that for any pair of indexes the quotient $f_{ij} := g_i g_j^{-1}$ is a nowhere vanishing holomorphic function $f_{ij} \in \mathcal{O}^*(U_{ij})$. Our data defines a divisor D on X and the problem is to find a global meromorphic function $m \in \mathcal{M}(X)$ defining this divisor, meaning, such a function that $m m_i^{-1}$ is a nowhere vanishing holomorphic function on U_i for every i . A solution is obtained by solving the second Cousin problem: Given a collection f_{ij} of nowhere vanishing holomorphic functions $f_{ij} : U_{ij} \rightarrow \mathbb{C}^*$ satisfying the 1-cocycle condition

$$f_{ii} = 1 \quad f_{ij} f_{ji} = 1 \quad f_{ij} f_{jk} f_{ki} = 1$$

on U_i, U_{ij}, U_{ijk} respectively, find nowhere vanishing holomorphic functions $f_j : U_j \rightarrow \mathbb{C}^*$ such that

$$f_i = f_{ij} f_j \quad \text{on} \quad U_{ij}.$$

If such f_i exist then $g_i f_i^{-1} = g_j f_j^{-1}$ on U_{ij} which defines a solution, a meromorphic function $m \in \mathcal{M}(X)$ representing our divisor.

The following cohomological formulation and proof of Oka’s Theorem are standard, see e.g. [14] Theorem 5.2.2..

Theorem 16. *If X is a complex space satisfying $H^1(X, \mathcal{O}) = 0$, then the homomorphism $H^1(X, \mathcal{O}^*) \rightarrow H^1(X, \mathbb{C}^*)$ induced by the sheaf inclusion $\mathcal{O}^* \hookrightarrow \mathbb{C}^*$ is injective. In particular if a multiplicative Cousin problem is solvable by continuous functions, then it is solvable by holomorphic functions. If in addition we have $H^2(X, \mathcal{O}) = 0$, then the above map is an isomorphism.*

Proof. Consider the exponential sheaf sequence (where $\sigma(f) = e^{2\pi i f}$).

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathcal{O} & \xrightarrow{\sigma} & \mathcal{O}^* & \longrightarrow & 1 \\
 & & \downarrow id & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & \mathbb{C} & \longrightarrow & \mathbb{C} & \xrightarrow{\sigma} & \mathbb{C}^* & \longrightarrow & 1
 \end{array}$$

Since due to partition of unity $H^1(X, \mathbb{C}) = H^2(X, \mathbb{C}) = 0$ the relevant portion of long exact cohomology sequence is:

$$\begin{array}{ccccccccc}
 H^1(X, \mathbb{Z}) & \longrightarrow & H^1(X, \mathcal{O}) & \longrightarrow & H^1(X, \mathcal{O}^*) & \xrightarrow{c_1} & H^2(X, \mathbb{Z}) & \longrightarrow & H^2(X, \mathcal{O}) \\
 & & \downarrow & & \downarrow & & \parallel & & \downarrow \\
 0 & \longrightarrow & H^1(X, \mathbb{C}^*) & \xrightarrow{c_1} & H^2(X, \mathbb{Z}) & \longrightarrow & 0 & & 0
 \end{array}$$

The map in the bottom row is an isomorphism $H^1(X, \mathbb{C}^*) \cong H^2(X, \mathbb{Z})$. If $H^1(X, \mathcal{O}) = 0$ the (1-st Chern class) map c_1 in the first row is injective $0 \rightarrow H^1(X, \mathcal{O}^*) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \cong H^1(X, \mathbb{C}^*)$. If in addition $H^2(X, \mathcal{O}) = 0$ this map is an isomorphism. \square

By Oka’s theorem on a complex space with $H^1(X, \mathcal{O}) = H^2(X, \mathcal{O}) = 0$ (by Theorem B this holds in particular on a Stein space) the natural map from equivalence classes of holomorphic line bundles into equivalence classes of continuous (complex) line bundles is an isomorphism. For higher rank vector bundles this cohomological proof fails due to non-commutativity of the relevant cohomology groups. Nevertheless, Grauert was able to prove the corresponding statement in higher dimensions. The following theorem is the holomorphic counterpart of Quillen’s and Suslin’s result that projective modules over affine space are free.

Theorem 17. *For a Stein space X the natural map $\text{Vect}_{\text{hol}}^r(X) \rightarrow \text{Vect}_{\text{top}}^r(X)$ of equivalence classes of rank r complex vector bundles is a bijection for every $r \in \mathbb{N}$.*

This theorem follows from the following result, named Grauert's Oka principle by H. Cartan, obtained by Grauert [20], Grauert and Kerner [21], and Ramspott [36] (see Theorem 5.3.2. in [14]).

Theorem 18. *If X is a Stein space and $\pi : Z \rightarrow X$ is a holomorphic fiber bundle with a complex homogeneous fiber whose structure group is a complex Lie group acting transitively on the fiber, then the inclusion $\Gamma_{\text{hol}}(X, Z) \hookrightarrow \Gamma_{\text{cont}}(X, Z)$ of the space of global holomorphic sections into the space of global continuous sections is a weak homotopy equivalence. In particular every continuous section is homotopic to a holomorphic section.*

An equivariant version of Grauert's Oka principle with respect to an action of a reductive complex Lie group has been proven by Heinzner and Kutzschebauch [23]. This principle in particular implies that the method of constructing counterexamples to the linearization problem, found by Schwarz in the algebraic setting [38], does not work in the holomorphic category. Moreover, the above-mentioned Oka principle was recently used by Kutzschebauch, Lárusson, and Schwarz [32] to show among others a strong linearization result: A generic holomorphic action, which is locally over a common categorical quotient isomorphic to a linear action on \mathbb{C}^n , is in fact globally isomorphic to that linear action.

The next step in Oka theory was made by Gromov in his seminal paper [22], which marks the beginning of modern Oka theory. He introduced the notion of dominating spray and ellipticity (see the last section). The great improvement compared to Grauert's Oka principle is the fact that not the fiber together with the transition maps of the bundle, but only certain properties of the fiber totally independent of transition maps allow to derive the conclusions. In the above-cited classical works, the structure group was indeed assumed to be a complex Lie group. However, in modern Oka theory the structure group is completely irrelevant. Moreover, modern Oka theory allows to consider sections of stratified elliptic submersions, generalizing the case of locally trivial fiber bundles. The emphasis shifted from the cohomological to the homotopy theoretic aspect, focusing on those analytic properties of a complex manifold Y which ensure that every continuous map from a Stein space X to Y is homotopic to a holomorphic map, with natural additions concerning approximation and interpolation of maps that are motivated by the extension and approximation theorems for holomorphic functions on Stein spaces. The approximation and extension are needed for generalizing from maps $X \rightarrow Y$ (which can be considered as sections of the trivial bundle $X \times Y \rightarrow X$ with fiber Y) to sections of holomorphic submersions $Z \rightarrow X$ with Oka–Forstnerič fibers and moreover to stratified elliptic submersions.

4.2 Definition and Main Features

Definition 19. A complex manifold Y is an Oka–Forstnerič manifold if every holomorphic map $f : K \rightarrow Y$ from (a neighborhood of) a compact convex set

$K \subset \mathbb{C}^n$ (any dimension n) can be approximated uniformly on K by entire maps $\mathbb{C}^n \rightarrow Y$.

The property in this definition is also called Convex Approximation Property (CAP), if the dimension n is fixed we speak of $(CAP)_n$, thus (CAP) means $(CAP)_n$ for all n . By work of Forstnerič (CAP) is equivalent to any of 13 different Oka properties, one of them is mentioned in the following Theorem which includes all versions of the classical Oka–Grauert principle discussed in the Introduction. This theorem answers Gromov’s question whether Runge approximation on a certain class of compact sets in Euclidean spaces suffices to infer the Oka property. Since all these 13 Oka properties are equivalent characterizations of the same class of manifolds Forstnerič called them Oka manifolds. In order to honor his work on the equivalence of all the Oka properties the author finds the notation Oka–Forstnerič manifolds more appropriate.

Theorem 20. *Let $\pi : Z \rightarrow X$ be a holomorphic submersion of a complex space Z onto a reduced Stein space X . Assume that X is exhausted by a sequence of open subsets $U_1 \subset U_2 \subset \dots \cup_j U_j = X$ such that each restriction $Z|_{U_j} \rightarrow U_j$ is a stratified holomorphic fiber bundle whose fibers are Oka manifolds. Then sections $X \rightarrow Z$ satisfy the following*

Parametric Oka property (POP): *Given a compact $\mathcal{O}(X)$ -convex subset K of X , a closed complex subvariety A of X , compact sets $P_0 \subset P$ in a Euclidean space \mathbb{R}^m , and a continuous map $f : P \times X \rightarrow Z$ such that*

- (a) *for every $p \in P$, $f(p, \cdot) : X \rightarrow Z$ is a section of $Z \rightarrow X$ that is holomorphic on a neighborhood of K (independent of p) and such that $f(p, \cdot)|_A$ is holomorphic on A , and*
- (b) *$f(p, \cdot)$ is holomorphic on X for every $p \in P_0$*

there is a homotopy $f_t : P \times X \rightarrow Z$ ($t \in [0, 1]$), with $f_0 = f$, such that f_t enjoys properties (a) and (b) for all $t \in [0, 1]$, and also the following hold:

- (i) *$f_1(p, \cdot)$ is holomorphic on X for all $p \in P$*
- (ii) *f_t is uniformly close to f on $P \times K$ for all $t \in [0, 1]$*
- (iii) *$f_t = f$ on $(P_0 \times X) \cup (P \times A)$ for all $t \in [0, 1]$*

As a general reference for Oka theory we refer to the monograph [14] and the overview article [15].

4.3 Applications and Examples

The number of applications of the Oka theory is growing, we already indicated the classical Cousin problems and Grauert’s classification of holomorphic vector bundles over Stein spaces in the introduction. The only application we would like to mention is a recent solution to a problem posed by Gromov, called the Vaserstein problem. It is a natural question about the K_1 -group of the ring of holomorphic

functions or in simple terms it is asking whether (and when) in a matrix whose entries are holomorphic functions (parameters) the Gauss elimination process can be performed in a way holomorphically depending on the parameter. This is to our knowledge the only application where a stratified version of an Oka theorem is needed, i.e., no proof using a non-stratified version is known.

Theorem 21. (see [25]). *Let X be a finite dimensional reduced Stein space and $f: X \rightarrow SL_m(\mathbb{C})$ be a holomorphic mapping that is null-homotopic. Then there exist a natural number K and holomorphic mappings $G_1, \dots, G_K: X \rightarrow \mathbb{C}^{m(m-1)/2}$ such that f can be written as a product of upper and lower diagonal unipotent matrices*

$$f(x) = \begin{pmatrix} 1 & 0 \\ G_1(x) & 1 \end{pmatrix} \begin{pmatrix} 1 & G_2(x) \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & G_K(x) \\ 0 & 1 \end{pmatrix}$$

for every $x \in X$.

Here the assumption null-homotopic means that the map is homotopic through continuous maps to a constant map (matrix), which since Grauert’s Oka principle, Theorem 18, is equivalent of being null-homotopic through holomorphic maps. This is an obvious necessary condition since multiplying all lower/upper diagonal matrices in the product by $t \in [0, 1]$ yields a homotopy to the (constant) identity matrix. It is a result of Vaserstein that null-homotopic is also sufficient in order to factorize the map as a product with continuous entries. Thus we have the Oka principle. For the existence of a holomorphic factorization there are only topological obstructions, it exists iff a topological factorization exists.

Now we come to examples of Oka–Forstnerič manifolds:

A Riemann surface is an Oka–Forstnerič manifold iff it is non-hyperbolic, i.e., one of $\mathbb{P}^1 \setminus \mathbb{C}$, \mathbb{C}^* , or a compact torus \mathbb{C}/Γ .

Oka–Forstnerič manifolds enjoy the following functorial properties, for elliptic manifolds (see Definition below) these properties are unknown.

- If $\pi : E \rightarrow B$ is a holomorphic covering map of complex manifolds then B is Oka–Forstnerič iff E is ([14] Proposition 5.5.2).
- If E and X are complex manifolds and $\pi : E \rightarrow X$ is a holomorphic fiber bundle whose fiber is an Oka–Forstnerič manifold, the X is an Oka–Forstnerič manifold iff E is ([14] Theorem 5.5.4).
- If a complex manifold Y is exhausted by open domains $D_1 \subset D_2 \subset \dots \subset \bigcup_{j=1}^\infty D_j = Y$ such that every D_j is an Oka–Forstnerič manifold, then Y is an Oka–Forstnerič manifold. In particular every long \mathbb{C}^n is an Oka–Forstnerič manifold. (A manifold is called a long \mathbb{C}^n if all D_j are biholomorphic to \mathbb{C}^n . If the inclusion $D_i \subset D_{i+1}$ is not a Runge pair, on those manifolds the ring of holomorphic functions may consist of constants only!!)

The main source of examples are the elliptic manifolds (see Definition 22 below), a notion invented by Gromov. This includes by our scheme (1) of implications all holomorphic flexible manifolds and all manifolds with the density property,

in particular complex Lie groups and homogeneous spaces of Lie groups, the manifolds from the classical theorems of Oka and Grauert. For a Stein manifold ellipticity is equivalent to being an Oka–Forstnerič manifold. For general manifolds this is an open question. One possible counterexample is the complement of the ball in \mathbb{C}^n , the set $\{z \in \mathbb{C}^n : |z_1|^2 + |z_2|^2 + \dots + |z_n|^2 > 1\}$. It was recently shown by Andrist and Wold [5] that it is not elliptic for $n \geq 3$, whereas it has two “nearby” properties implied by being an Oka–Forstnerič manifold, strongly dominable and CAP_{n-1} ([16]).

5 Proof of the Implications from Scheme (1): Ellipticity in the Sense of Gromov

First remark that the two bottom up arrows in Scheme (1) are obvious from the definitions. In order to prove the left–right arrows let’s define the notions introduced by Gromov [22] revolutionizing Oka theory (see also [14, Chap. 5]):

Definition 22. Let Y be a complex manifold.

- (1) A holomorphic spray on Y is a triple (E, π, s) consisting of a holomorphic vector bundle $\pi : E \rightarrow Y$ (a spray bundle) and a holomorphic map $s : E \rightarrow Y$ (a spray map) such that for each $y \in Y$ we have $s(0_y) = y$.
- (2) A spray (E, π, s) on Y is dominating on a subset $U \subset Y$ if the differential $d_{0_y}s : T_{0_y}E \rightarrow T_yY$ maps the vertical tangent space E_y of $T_{0_y}E$ surjectively onto T_yY for every $y \in U$, s is dominating if this holds for all $y \in Y$.
- (3) A complex manifold Y is elliptic if it admits a dominating holomorphic spray.

The main result of Gromov can now be formulated in the following way.

Theorem 23. *An elliptic manifold is an Oka–Forstnerič manifold.*

Of course Gromov proved the full Oka principle for elliptic manifolds. This proof can now be decomposed in two stages. The main (and the only) use of ellipticity is to prove a homotopy version of Runge (Oka–Weil) theorem, which in particular gives CAP (= Oka–Forstnerič) and the second stage is CAP implies Oka principle.

Gromov’s theorem proves our implication

$$\text{holomorphically flexible} \implies \text{Oka – Forstneric manifold}$$

using the following example of a spray given by Gromov and Lemma 25;

Example 24. Given completely integrable holomorphic vector fields $\theta_1, \theta_2, \dots, \theta_N$ on a complex manifold X such that at each point $x \in X$ they span the tangent space, $\text{span}(\theta_1(x), \theta_2(x), \dots, \theta_N(x)) = T_xX$. Let $\psi^i : \mathbb{C} \times X \rightarrow X, (t, x) \mapsto \psi^i_t(x)$ denote the corresponding one-parameter subgroups; Then the map $s : \mathbb{C}^N \times X \rightarrow X$

defined by $((t_1, t_2, \dots, t_n), x) \mapsto \psi_{t_n}^n \circ \psi_{t_{n-1}}^{n-1} \circ \dots \circ \psi_{t_1}^1(x)$ is of full rank at $t = 0$ for any x . It is therefore a dominating spray map from the trivial bundle $X \times \mathbb{C}^N \rightarrow X$.

Lemma 25. *If a Stein manifold X is holomorphically flexible, then there are finitely many completely integrable holomorphic fields which span the tangent space $T_x X$ at every point $x \in X$*

Proof. To prove that there are finitely many completely integrable holomorphic fields that span each tangent space, let us start with n fields $\theta_1, \dots, \theta_n$ which span the tangent space at some point x_0 and thus outside a proper analytic subset A . The set A may have countably many irreducible components A_1, A_2, A_3, \dots

It suffices now to find a holomorphic automorphism $\Phi \in \text{Aut}_{\text{hol}}(X)$ such that $\Phi(X \setminus A) \cap A_i \neq \emptyset$ for every $i = 1, 2, 3, \dots$. Indeed, for such an automorphism Φ the completely integrable holomorphic vector fields $\Phi_*(\theta_1), \dots, \Phi_*(\theta_n)$ span the tangent space at a general point in each A_i , i.e., together with the fields $\theta_1, \dots, \theta_n$ they span the tangent space at each point outside an analytic subset B of a smaller dimension than A . Then the induction by dimension implies the desired conclusion.

In order to construct Φ consider a monotonically increasing sequence of compacts $K_1 \subset K_2 \subset \dots$ in X such that $\bigcup_i K_i = X$ and a closed imbedding $\iota : X \hookrightarrow \mathbb{C}^m$. For every continuous map $\varphi : X \rightarrow \mathbb{C}^m$ denote by $\|\varphi\|_i$ the standard norm of the restriction of φ to K_i . Let d be the metric on the space $\text{Aut}_{\text{hol}}(X)$ of holomorphic automorphisms of X given by the formula

$$d(\Phi, \Psi) = \sum_{i=1}^{\infty} 2^{-i} (\min(\|\Phi - \Psi\|_i, 1) + \min(\|\Phi^{-1} - \Psi^{-1}\|_i, 1)) \tag{4.1}$$

where automorphisms $\Phi^{\pm 1}, \Psi^{\pm 1} \in \text{Aut}_{\text{hol}}(X)$ are viewed as continuous maps from X to \mathbb{C}^m . This metric makes $\text{Aut}_{\text{hol}}(X)$ a complete metric space.

Set $Z_i = \{\Psi \in \text{Aut}_{\text{hol}}(X) : \Psi(A_i) \cap (X \setminus A) \neq \emptyset\}$. Note that Z_i is open in $\text{Aut}_{\text{hol}}(X)$ and let us show that it is also everywhere dense.

Since completely integrable holomorphic fields generate the tangent space at each point of X , we can choose such a field θ non-tangent to A_i . Then for every $\Psi \in \text{Aut}_{\text{hol}}(X)$ its composition with general elements of the flow induced by θ is in Z_i . That is, a perturbation of Ψ belongs to Z_i which proves that Z_i is everywhere dense in $\text{Aut}_{\text{hol}}(X)$. By the Baire category theorem the set $\bigcap_{i=1}^{\infty} Z_i$ is not empty which yields the existence of the desired automorphism. \square

Since the question whether holomorphic maps are approximable by morphisms is an important issue in algebraic geometry, we would like to remark at this point that there is an application of Oka-theory to this question. Clearly there is an obvious notion of algebraic spray, thus algebraic ellipticity. Also the proof of the above lemma generalizes showing that an algebraically flexible manifold is algebraically elliptic. These algebraically elliptic manifolds satisfy an algebraic version of CAP. However, in the algebraic category, simple examples show that algebraic CAP does not imply the full algebraic Oka principle, but only a weaker statement that being

approximable by algebraic morphisms is a homotopy invariant property (at least for maps from affine algebraic manifolds to algebraically elliptic manifolds). For a precise treatment of this question we refer to [14] chapter 7.10.

The implication $DP \implies$ holomorphically flexible is contained in the following Lemma.

Lemma 26. *If a Stein manifold X has the density property, the completely integrable holomorphic vector fields span the tangent space at each point $x \in X$.*

Proof. It follows from the density property that Lie combinations of completely integrable holomorphic vector fields span the tangent space $T_x X$ at any given point $x \in X$. Observe that every Lie bracket $[\nu, \mu]$ of completely integrable holomorphic vector fields can be approximated by a linear combination of such fields which follows immediately from the equality $[\nu, \mu] = \lim_{t \rightarrow 0} \frac{\phi_t^*(\nu) - \nu}{t}$ where ϕ_t is the flow generated by μ . Thus the completely integrable holomorphic vector fields span $T_x X$ at any $x \in X$. □

6 Concluding Remarks and Open Problems

There is also another property which has similar consequences as the density property for holomorphic automorphisms preserving a volume form.

Definition 27. Let a complex manifold X be equipped with a holomorphic volume form ω (i.e., ω is nowhere vanishing section of the canonical bundle). We say that X has the volume density property (VDP) with respect to ω if in the compact-open topology the Lie algebra generated by completely integrable holomorphic vector fields ν such that $\nu(\omega) = 0$ is dense in the Lie algebra of all holomorphic vector fields that annihilate ω (note that condition $\nu(\omega) = 0$ is equivalent to the fact that ν is of ω -divergence zero). If X is affine algebraic we say that X has the algebraic volume density property (AVDP) with respect to an algebraic volume form ω if the Lie algebra generated by completely integrable algebraic vector fields ν such that $\nu(\omega) = 0$, coincides with the Lie algebra of all algebraic vector fields that annihilate ω .

For Stein manifolds with the volume density property (VDP) an Andersén–Lempert theorem for volume preserving maps holds. The implication $(AVDP) \implies (VDP)$ holds but its proof is not trivial (see [30]). Also $(VDP) \implies$ **holomorphic flexibility** is true (see [29]). Thus we can have a scheme of implications like (1) with (DP) replaced by (VDP) and (ADP) replaced by (AVDP).

Volume density property and density property are not implied by each other, if X has density property it may not even admit a holomorphic volume form, if X has volume density property with respect to one volume form it may not have it with respect to another volume form and there is no reason to expect it has density property. For example, $(\mathbb{C}^*)^n$ for $n > 1$ has (algebraic) volume density property

with respect to the Haar form, it does not have algebraic density property [2] and it is expected not to have density property. It is a potential counterexample to the reverse of the left horizontal arrow in scheme (1).

Concerning the reverse implications in scheme (1): The variety $(\mathbb{C}^*)^n$, $n > 1$ is an obvious counterexample to the reverse of the right vertical arrow, the others are more delicate.

Open Problem: Which of the other three implications in scheme (1) are reversible for a Stein manifold (resp. smooth affine algebraic variety for the vertical arrow)?

The main problem here is that no method is known how to classify (meaning exclude the existence of any other than the obvious) completely integrable holomorphic vector fields on Stein manifolds with any of our flexibility properties. There is not even a classification of completely integrable holomorphic vector fields on \mathbb{C}^2 available.

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