Chapter 4

Lobachevsky geometry and nonlinear equations of mathematical physics

In this chapter we present a geometric approach to the interpretation of nonlinear partial differential equations which connects them with special coordinate nets on the Lobachevsky plane Λ^2. We introduce the class of Lobachevsky differential equations (Λ^2-class), which admit the aforementioned interpretation. The development of this geometric approach to nonlinear equations of contemporary mathematical physics enables us to apply in their study the rather well developed apparatus and methods of non-Euclidean hyperbolic geometry. Many known nonlinear equations, in particular, the sine-Gordon, Korteweg-de Vries, Burgers, Liouville, and other equations, which form the Λ^2-class, are generated by their own coordinate nets on the Lobachevsky plane Λ^2. This allows us to study the equations by means of net (intrinsic-geometrical) methods on the basis of Lobachevsky geometry. Overall, Chapter 4 is devoted to the application of geometric methods of hyperbolic geometry to the constructive investigation of equations of Λ^2-class.

4.1 The Lobachevsky class of equations of mathematical physics

In this section we introduce the notion of the Lobachevsky class of differential equations, which enables us to give to many nonlinear equations of contemporary mathematical physics a universal “net-type” geometric interpretation, based on Lobachevsky's non-Euclidean hyperbolic geometry [77, 79, 183–185]. Such an approach opens avenues for the application of tools and methods of non-Euclidean geometry to the study of partial differential equations of various types.

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4.1.1 The Gauss formula as a generalized differential equation

Let us consider in the parameter (x,t)-plane the quadratic differential form

$$ds^2 = E[u(x,t)]dx^2 + 2F[u(x,t)]dxdt + G[u(x,t)]dt^2, \tag{4.1.1}$$

whose coefficients,

$$E = E[u(x,t)], \quad F = F[u(x,t)], \quad G = G[u(x,t)], \tag{4.1.2}$$

depend on some unknown function $u(x,t)$ and its partial derivatives with respect to x and t.

Let us calculate the "curvature of the quadratic form" (4.1.1), using the Gauss formula (2.3.28):

$$\begin{aligned} K = &-\frac{1}{4W^2[u]} \cdot \det \begin{bmatrix} E[u] & (E[u])_x & (E[u])_t \\ F[u] & (F[u])_x & (F[u])_t \\ G[u] & (G[u])_x & (G[u])_t \end{bmatrix} \\ &- \frac{1}{2\sqrt{W[u]}} \left\{ \frac{\partial}{\partial t} \left(\frac{(E[u])_t - (F[u])_x}{\sqrt{W[u]}} \right) - \frac{\partial}{\partial x} \left(\frac{(F[u])_t - (G[u])_x}{\sqrt{W[u]}} \right) \right\}, \end{aligned} \tag{4.1.3}$$

where $W[u] = E[u] \cdot G[u] - F^2[u]$.

The right-hand side of (4.1.3) is the familiar (for the *given form of the coefficients* (4.1.2)) expression of the curvature K in terms of the coefficients $E[u]$, $F[u]$, $G[u]$ and their partial derivatives with respect to x and t (of order up to and including two).

If we assume that the curvature is an *a priori given* function $K = K(x,t)$, then the resulting relation (4.1.3) can be interpreted as a differential equation for $u(x,t)$:

$$\mathcal{F}[u(x,t)] = 0. \tag{4.1.4}$$

And conversely, if $u(x,t)$ is a solution of the differential equation (4.1.4), the quadratic form (4.1.1) defines in the parameter (x,t)-plane a metric with the square of the linear element given by (4.1.1) and with the given curvature $K(x,t)$. Thus, one can say that the metric (4.1.1) (or the differential form (4.1.1)) with its a priori prescribed curvature $K(x,t)$) *generates* (via (4.1.3)) the differential equation (4.1.4) for the function $u(x,t)$.

The equations generated in the aforementioned sense for the a priori choice of the constant negative curvature $K(x,t) \equiv -1$ (the case of the Lobachevsky plane Λ^2) will be called Λ^2-*equations*. The class of differential equations formed by the Λ^2-equation will be referred to as the *Lobachevsky class* (or the Λ^2-*class*).

In the more general case, when the curvature function $K = K(x,t)$ is arbitrary, we will say that the corresponding differential equation (an equation generated by a metric of variable curvature) belongs to the *G-class* (*the Gauss class*); such equations will be referred to as *G-equations*.

Let us clarify the geometric interpretation of equations introduced above on a number of examples of known nonlinear equations of mathematical physics.

Example 1. Consider the quadratic form (Chebyshev net metric):

$$ds^2 = dx^2 + 2\cos u(x,t)dxdt + dt^2. \tag{4.1.5}$$

In this case the coefficients are

$$E[u] = 1, \quad F[u] = \cos u(x,t), \quad G[u] = 1.$$

Calculating the curvature $K(x,t)$ of the form (4.1.5) by the Gauss formula (4.1.3) we get

$$\begin{aligned} K(x,t) = &-\frac{1}{4\sin^4 u} \det \begin{bmatrix} 1 & 0 & 0 \\ \cos u & -u_x \sin u & -u_t \sin u \\ 1 & 0 & 0 \end{bmatrix} \\ &- \frac{1}{2\sin u} \left\{ \frac{\partial}{\partial t} \left[\frac{u_x \sin u}{\sin u} \right] + \frac{\partial}{\partial x} \left[\frac{u_t \sin u}{\sin u} \right] \right\}, \end{aligned}$$

and so we arrive at the following G-equation:

$$u_{xt} = -K(x,t)\sin u(x,t) \tag{4.1.6}$$

(the *Chebyshev equation*).

Equation (4.1.6) is the already familiar to us (see § 2.5) equation that "governs" the variation of the net angle of the Chebyshev net of lines for the given curvature $K(x,t)$.

When $K \equiv -1$, (4.1.6) becomes the sine-Gordon equation[1]

$$u_{xt} = \sin u. \tag{4.1.7}$$

Example 2. Let us take a metric of the form

$$ds^2 = \eta^2 dx^2 + 2\eta \left(\frac{1}{2}\,\eta u^2 + \eta^3 \right) dxdt + \left[\eta^2 u_x^2 + \left(\frac{1}{2}\,\eta u^2 + \eta^3 \right)^2 \right] dt^2, \tag{4.1.8}$$

where $\eta = \text{const}$. In this case

$$E[u] = \eta^2, \quad F[u] = \eta \left(\frac{1}{2}\,\eta u^2 + \eta^3 \right),$$
$$G[u] = \eta^2 u_x^2 + \left(\frac{1}{2}\,\eta u^2 + \eta^3 \right)^2.$$

Setting $K \equiv -1$ (i.e., working in the *Lobachevsky plane* Λ^2), the Gauss formula (4.1.3) yields the Λ^2-equation

$$u_t = \frac{3}{2}\,u^2 u_x + u_{xxx} \tag{4.1.9}$$

[1] In this chapter, following the mathematical physics traditions, we write the sought-for solution of the differential equation in question as $u = u(x,t)$, where x and t are the independent variables.

(the *modified Korteweg-de Vries equation*).

Hence, the modified Korteweg-de Vries equation (MKdV) (4.1.9) is also defined by a coordinate net on the Lobachevsky plane (given by the form (4.1.8) of the metric). It is is natural to call such a net an MKdV-net.

Example 3. For the metric

$$ds^2 = \frac{e^u}{2}(dx^2 + dt^2), \tag{4.1.10}$$

with the coefficients

$$E[u] = \frac{e^u}{2}, \quad F[u] = 0, \quad G[u] = \frac{e^u}{2}$$

we obtain for $K \equiv -1$ the equation

$$\Delta_2 u = e^u, \quad \Delta_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial t^2} \tag{4.1.11}$$

(the *elliptic Liouville equation*).

If $u(x,t)$ is a solution of equation (4.1.11), then in accordance with (4.1.1), on the Lobachevsky plane there arises a net $\{(x,t)\}$ (the *Liouville net*) with the linear element (4.1.10), namely, the *isothermal* coordinate net.

The examples given above show how differential equations can be generated by metrics of a special form. As we will see later, many "concrete" nonlinear equations of mathematical physics belong to the Λ^2-class, i.e., are generated by pseudospherical metrics (metrics of curvature $K \equiv -1$). In general, the condition that the curvature of the generating metric is constant, $K \equiv$ const, is important, since in this case the curvature acquires the special meaning of an *invariant*, i.e., *it is preserved by transformations* generated by nets on two-dimensional smooth manifolds $\mathcal{M}_2$, connected with the realization of geometric algorithms for the integration of equations.

We should remark also that the geometric interpretation of equations introduced above, together with its clear geometric content is *universal*, since it "exhaust" all possible types in the standard classification of differential equations (as this was demonstrated on examples of hyperbolic, parabolic and elliptic equations, respectively).

It is also important to note that the *nonlinearity* in the "geometrically" derived equations of mathematical physics is *primarily a result of the nontriviality of the curvature* of the generating metric, as well as of the nonlinearity of its discriminant W.

The membership of equations in the Λ^2-class assumes that they possess certain general properties of geometric origin, the discussion of which we begin in the next subsection.

To finish the present subsection, we make an observation connected with the theory of nets [127]: Giving on the two-dimensional manifold $\mathcal{M}_2$ a metric of the type (4.1.1),

$$ds^2 = g_{ij}[u]dx^i dx^j, \quad g_{ij}[u] = \begin{pmatrix} E[u] & F[u] \\ F[u] & G[u] \end{pmatrix}, \tag{4.1.12}$$

is equivalent to giving on $\mathcal{M}_2$ a smooth tensor field (g_{ij}) of type $\binom{0}{2}$ that has the symmetry property

$$g_{ij} = g_{ji}$$

and is positive definite.

Every nondegenerate symmetric tensor g_{ij} gives rise to a net of lines on $\mathcal{M}_2$, the directing pseudovectors (tangent vectors to the one-parameter families of lines) of which, v_j and w_j, are solutions of the equation[2] (see [127])

$$g_{ij}x^i x^j = 0.$$

The specification of two fields of independent vectors v_j and w_j defines on $\mathcal{M}_2$ a two-parametric net of coorodinate lines $\{(x,t)\}$, $x \equiv x^1$, $t \equiv x^2$.

Therefore, it is totally correct to assert that a differential equation of the type (4.1.4) is generated not only by the metric (4.1.1) corresponding to it, but also by its "geometric preimage", the coordinate net on the two-dimensional smooth manifold $\mathcal{M}_2$ (and, in particular, on the Lobachevsky plane Λ^2).

4.1.2 Local equivalence of solutions of Λ^2-equations

Membership of equations in the Λ^2-class assumes that they have a general intrinsic-geometrical nature. In this subsection we give a theorem on the transformation of local solutions of Λ^2-equations which establishes their *local equivalence* [77, 79, 185].

Theorem 4.1.1 (Local equivalence of Λ^2-equations)**.** *Suppose two different analytic differential equations belong to the Λ^2-class. Then from a local analytic solution of one of these equations one can always construct a local analytic solution of the other, and conversely.*

In the case where one of the Λ^2-equations in Theorem 4.1.1 is the sine-Gordon equation, the content of this the theorem is concretized in Theorem 4.1.2.

Theorem 4.1.2. *Suppose an analytic equation of type* (4.1.4) *belongs to the Λ^2-class. Then for any local analytic solution $u(x,t)$ of this equation one can always construct a local analytic solution $z(\widetilde{x},\widetilde{t}\,)$ of the sine-Gordon equation*

$$z_{\widetilde{x}\widetilde{t}} = \sin z(\widetilde{x},\widetilde{t}\,), \quad z = z(\widetilde{x},\widetilde{t}\,)$$

by means of the formula

$$\cos z = \left[\frac{\partial f_1}{\partial\widetilde{x}}\,\frac{\partial f_1}{\partial\widetilde{t}}\;E[u(x,t)] + \left(\frac{\partial f_1}{\partial\widetilde{x}}\,\frac{\partial f_2}{\partial\widetilde{t}} + \frac{\partial f_1}{\partial\widetilde{t}}\,\frac{\partial f_2}{\partial\widetilde{x}}\right)F[u(x,t)] + \frac{\partial f_2}{\partial\widetilde{x}}\,\frac{\partial f_2}{\partial\widetilde{t}}\,G[u(x,t)]\right]\Bigg|_{\substack{x = f_1(\widetilde{x},\widetilde{t}) \\ t = f_2(\widetilde{x},\widetilde{t})}}, \tag{4.1.13}$$

[2] This equation gives the pseudovectors of the net, i.e., specifies the ratios x^1/x^2.

where $E[u], F[u], G[u]$ *are the coefficients of the pseudospherical metric that generates equation* (4.1.4)*.*

The functions f_1 *and* f_2 *appearing in* (4.1.13) *satisfy the system*

$$\begin{aligned} &\frac{\partial^2 f_1}{\partial\widetilde{x}\partial\widetilde{t}} + \Gamma^1_{\alpha\beta}\frac{\partial f_\alpha}{\partial\widetilde{x}}\frac{\partial f_\beta}{\partial\widetilde{t}} = 0, \\ &\frac{\partial^2 f_2}{\partial\widetilde{x}\partial\widetilde{t}} + \Gamma^2_{\alpha\beta}\frac{\partial f_\alpha}{\partial\widetilde{x}}\frac{\partial f_\beta}{\partial\widetilde{t}} = 0, \end{aligned} \qquad (\alpha,\,\beta = 1,2), \tag{4.1.14}$$

where $\Gamma^1_{\alpha\beta}$*,* $\Gamma^2_{\alpha\beta}$ *are the Christoffel symbols of the pseudospherical metric that generates the* Λ^2*-equation* (4.1.4)*, written in the variables* $x \equiv f_1$*,* $t \equiv f_2$ *(i.e.,* $\Gamma^\gamma_{\alpha\beta} = \Gamma^\gamma_{\alpha\beta}(f_1, f_2)$*,* $\alpha,\beta,\gamma = 1,2$*).*

Remark. The transformations established in theorems 4.1.1 and 4.1.2 are connected exclusively with a change of the independent variables and geometrically correspond to passing from one coordinate net to another in the plane Λ^2.

The proof of theorems 4.1.1 and 4.1.2 is prepared by § 2.5, which treats in detail the properties of Chebyshev nets and the conditions for passing to these nets in a regular domain on a surface, as well as by the methodology of Λ^2-equations introduced in Subsection 4.1.1. Hence, without repeating the arguments that we already used in the construction of Chebyshev nets, in the proof of the theorems given here the main attention is paid to the specifics of the corresponding algorithm in the case we are interested in, when the original given two-dimensional net is the net associated with a metric that generates a Λ^2-equation.

Proof of Theorem 4.1.2. Consider an Λ^2-equation of the type (4.1.4), as in the formulation of Theorem 4.1.2. Then this equation is generated by its corresponding metric

$$(ds^2)_1 = E[u]dx^2 + 2F[u]dxdt + G[u]dt^2, \quad K \equiv -1. \tag{4.1.15}$$

Let us determine whether it is possible to reduce the metric $(ds^2)_1$ to the Chebyshev metric

$$(ds^2)_2 = d\widetilde{x}^2 + 2\cos z(\widetilde{x},\widetilde{t})d\widetilde{x}d\widetilde{t} + d\widetilde{t}^2, \quad K \equiv -1, \tag{4.1.16}$$

i.e., whether it is possible to pass from the existing net $T(x,t)$ that generates equation (4.1.4) to the Chebyshev net $\mathrm{Cheb}(\widetilde{x},\widetilde{t})$.

Suppose that such a transition

$$T((x,t);(ds^2)_1) \longmapsto \mathrm{Cheb}((\widetilde{x},\widetilde{t});(ds^2)_2) \tag{4.1.17}$$

is effected on the plane Λ^2 by means of the transformation

$$x = x(\widetilde{x},\widetilde{t}), \quad t = t(\widetilde{x},\widetilde{t}), \tag{4.1.18}$$

and its correctness is guaranteed by the condition

$$\frac{D(x,t)}{D(\widetilde{x},\widetilde{t})} = \frac{\partial x}{\partial\widetilde{x}}\frac{\partial t}{\partial\widetilde{t}} - \frac{\partial x}{\partial\widetilde{t}}\frac{\partial t}{\partial\widetilde{x}} \neq 0. \tag{4.1.19}$$

Let us determine the conditions on the transformation (4.1.18), (4.1.19), under which it maps the net $T(x,t)$ into the Chebyshev net $\mathrm{Cheb}(\widetilde{x},\widetilde{t})$. In § 2.5 it was established that a criterion for a net to be a Chebyshev net is the vanishing of the corresponding two Christoffel symbol (see (2.5.16)), i.e., for the net $\mathrm{Cheb}(\widetilde{x},\widetilde{t})$ it holds that

$$\widetilde{\Gamma}^1_{12}=0,\quad \widetilde{\Gamma}^2_{12}=0. \tag{4.1.20}$$

As we have shown, conditions of the type (4.1.20) lead to the Servant-Bianchi system (2.5.22). Let us write this system for our case (for agreement with the notation of § 2.5, we re-denote (x,t) by (x^1,x^2) and $(\widetilde{x},\widetilde{t})$ by (y^1,y^2); also, $(x,t)\equiv(v_1,v_2)$ and $(\widetilde{x},\widetilde{t})\equiv(u_1,u_2)$, see (2.5.22)):

$$\begin{aligned}\Gamma^1_{\alpha\beta}\frac{\partial x^\alpha}{\partial y^2}\frac{\partial x^\beta}{\partial y^1}+\frac{\partial^2 x^1}{\partial y^2\partial y^1}&=0,\\ \Gamma^2_{\alpha\beta}\frac{\partial x^\alpha}{\partial y^2}\frac{\partial x^\beta}{\partial y^1}+\frac{\partial^2 x^2}{\partial y^2\partial y^1}&=0.\end{aligned} \tag{4.1.21}$$

The existence of a solution

$$x_1=f_1(y_1,y_2),\quad x_2=f_2(y_1,y_2) \tag{4.1.22}$$

of the system (4.1.21) means that it is possible to reduce the metric $(ds^2)_1$ (4.1.15) to the form $(ds^2)_2$ (4.1.16). In general, equations (4.1.21) establish the existence of a (virtual, in a certain sense) Chebyshev net on an arbitrary two-dimensional smooth manifold $\mathcal{M}_2$ and the degree of arbitrariness with which such a set is determined.

Now let us address the question of the unique determinacy of the transition (4.1.22) to a Chebyshev net.

Let x_1°, x_2° be some fixed values of the variables x_1, x_2 (and, accordingly, of some selected point $A(x_1^\circ,x_2^\circ)\in\mathcal{M}_2$ (or, in particular, $A(x_1^\circ,x_2^\circ)\in\Lambda^2$). Let us pick arbitrary values y_1°,y_2° that correspond in the new variables to x_1°, x_2° (coordinates of the Chebyshev net $\mathrm{Cheb}(y_1,y_2)$). In other words, in agreement with (4.1.22), we require that

$$x_1^\circ=f_1(y_1^\circ,y_2^\circ),\quad x_2^\circ=f_2(y_1^\circ,y_2^\circ). \tag{4.1.23}$$

Let $g_1(y_1)$ and $g_2(y_1)$ denote the functions that the sought-for functions $f_1(y_1,y_2)$ and $f_2(y_1,y_2)$ become when we set $y_2=y_2^\circ$:

$$f_1(y_1,y_2^\circ)=g_1(y_1),\quad f_2(y_1,y_2^\circ)=g_2(y_1).^3 \tag{4.1.24}$$

By (4.1.23), the functions g_1 and g_2 satisfy the conditions

$$g_1(y_1^\circ)=x_1^\circ,\quad g_2(y_1^\circ)=x_2^\circ. \tag{4.1.25}$$

In much the same way, let us introduce the functions $h_1(y_2)$ and $h_2(y_2)$:

$$f_1(y_1^\circ,y_2)=h_1(y_2),\quad f_2(y_1^\circ,y_2)=h_2(y_2), \tag{4.1.26}$$

[3] Obviously, the functions g_1 and g_2 can be given in a sufficiently arbitrary manner.

$$h_1(y_2^\circ) = x_1^\circ, \quad h_2(y_2^\circ) = x_2^\circ. \tag{4.1.27}$$

The freedom in the choice of the functions $g_1(y_1)$, $g_2(y_1)$, $h_1(y_2)$, $h_2(y_2)$ is restricted only by the natural condition

$$\frac{dg_1}{dy_1}\frac{dh_2}{dy_2} - \frac{dg_2}{dy_1}\frac{dh_1}{dy_2} \neq 0, \tag{4.1.28}$$

the geometric meaning of which will be made clear below.

Further, the substitution

$$y_1 = w_1 + w_2, \quad y_2 = w_1 - w_2 \tag{4.1.29}$$

brings (4.1.21) to the form of a *normal system* of second-order partial differential equations (a system solved with respect to the highest-order derivatives):

$$\begin{aligned} \frac{\partial^2 x_1}{\partial w_1^2} &= P[w_1, w_2], \\ \frac{\partial^2 x_2}{\partial w_2^2} &= Q[w_1, w_2]. \end{aligned} \tag{4.1.30}$$

Thanks to assumption, made in the theorems 4.1.1 and 4.1.2, that the functions $u(x,t)$ (the sought-for solutions of an equation of type (4.1.4)) are *analytic*, the Christoffel symbols $\Gamma^1_{\alpha\beta}$, $\Gamma^2_{\alpha\beta}$, as well as the resulting "right-hand sides" in (4.1.30), that is, the functions $P[w_1, w_2]$ and $Q[w_1, w_2]$, will also be analytic functions.

Thus, the system (4.1.30) with the initial data (4.1.23)–(4.1.27) (written in the variables w_1 and w_2) satisfies the conditions of the *Cauchy-Kovalevskaya theorem for a normal system of differential equations* [46]. By the Cauchy-Kovalevskaya theorem, the posed problem (4.1.30), (4.1.23)–(4.1.27) is always uniquely locally solvable, i.e., has a unique solution in a neighborhood of the chosen point (w_1°, w_2°):

$$y_1^\circ = w_1^\circ + w_2^\circ, \quad y_2^\circ = w_1^\circ - w_2^\circ.$$

Turning now to the variables y_1 and y_2, we conclude that in some neighborhood ω_A of the point $A(x_1^\circ, x_2^\circ) \in \Lambda^2$ there exists a unique solution (4.1.22) of the system (4.1.21) with the given initial conditions (4.1.23)–(4.1.27).

The arguments above can be interpreted geometrically as follows: the equations

$$x_1 = g_1(y_1), \quad x_2 = g_2(y_1)$$

define on Λ^2 a line that passes through the point $A(x_1^\circ, x_2^\circ)$ and represents in the new parametrization the line $y_2 = y_2^\circ$. Correspondingly, the equations

$$x_1 = h_1(y_2), \quad x_2 = h_2(y_2)$$

give the coordinate line $y_1 = y_1^\circ$ of the new net $\mathrm{Cheb}(y_1, y_2)$ that passes through the point A. Two such lines can be chosen arbitrarily, with the natural constraint that they must not be tangent to one another at the point A. (This requirement is ensured by fulfillment of condition (4.1.28).)

Thus, the solution (4.1.22) of the system (4.1.21) with the initial conditions (4.1.23)–(4.1.27), exists in some neighborhood ω_A and gives the transformation $T(x,t) \to \mathrm{Cheb}(\widetilde{x},\widetilde{t})$, which leads to the Chebyshev net of coordinate lines on Λ^2 (and, in general, on $\mathcal{M}_2$). This result has the following geometric explanation: if through the point $A \in \mathcal{M}_2$ ($A \in \Lambda^2$) one draws two intersecting (but not tangent to one another) lines l_1 and l_2, then in a sufficiently small neighborhood ω_A of A there exists a uniquely determined Chebyshev net in which l_1 and l_2 are included.

Substitution of the already obtained solution (4.1.22) in the metric (4.1.15) (keeping in mind the transformations performed above) reduces it to the form (4.1.16). Comparing the coefficients of the metric (4.1.15) that we reduced to the form (4.1.16) with the coefficients of the (original) metric (4.1.16) itself, we obtain the formula (4.1.13) for the construction of solutions of the sine-Gordon equation. Theorem 4.1.2 is proven. □

Let us make a number of comments.

Comment 4.1.1. The arbitrariness in the choice of the initial data (4.1.22)–(4.1.27) (with condition (4.1.28) in force) enables us to construct an infinite family $\{z\}$ of solutions of the sine-Gordon equation for each given solution u of the given Λ^2-equation of the type (4.1.14). Now choosing the same "base" generators for the net Cheb in the formulation of the problems for two different Λ^2-equations,

$$\mathcal{F}_1[u_1] = 0, \quad \mathcal{F}_2[u_2] = 0$$

performing the transitions

$$T_1 \longmapsto \mathrm{Cheb}, \quad T_2 \longmapsto \mathrm{Cheb},$$

and then applying Theorem 4.1.2, we arrive to a solution z of the sine-Gordon equation

$$z = \Omega_1[u_1] = \Omega_2[u_2],$$

that is shared by the two Λ^2-equations.

In view of the analyticity of the solutions u_1 and u_2 (for the corresponding Λ^2-equations), the relations obtained above imply their local equivalence, which is precisely what Theorem 4.1.1 establishes.

Comment 4.1.2. The method that we used in the proof of Theorem 4.1.1, of passing to the Chebyshev net (choosing the Chebyshev net as a *universal connecting* object) has a general character and, generally speaking, is not related to the curvature of the manifold $\mathcal{M}_2$ under consideration. Hence, if in the case of an arbitrary curvature $K = K(x,t)$ we argue in much the same way as in the proof of Theorem 4.1.2, we can obtain an analog of the transformations (4.1.13), (4.1.14) for the variable-curvature case. However, in this last case the curvature K no longer retains the meaning of an invariant of the transformation, and consequently in the formulation of Theorem 4.1.3 we need to "replace" the sine-Gordon equation by the Chebyshev equation.

Theorem 4.1.3. *For each local analytic solution $u(x,t)$ of any analytic equation generated by a metric of the type* (4.1.1) *of curvature $K(x,t)$ (G-equation), one can always construct a local analytic solution of the Chebyshev equation*

$$z_{\widetilde{x}\widetilde{t}} = -K \cdot \sin z(\widetilde{x}, \widetilde{t})$$

by means of relations (4.1.13), (4.1.14), *with the function z in them understood as a solution of the Chebyshev equation with the coefficient $K = K(f_1(\widetilde{x}, \widetilde{t}), f_2(\widetilde{x}, \widetilde{t}))$.*

Comment 4.1.3. The transformation established above for the solutions of the Λ^2- and G-equations has a local character. This is due, on the one hand, to the local character of the Cauchy-Kovalevskaya theorem applied, and on the other, to the problem of choosing a local Chebyshev net that is completely included in the global Chebyshev set "on the entire" $\mathcal{M}_2$.

The search for a possible transformation of nonlocal solutions should be connected to the search for a universal geometric object, defined "globally" on $\mathcal{M}_2$, or on the entire surface S that realizes the isometric immersion $\mathcal{M}_2 \overset{\text{isom}}{\longmapsto} \mathbb{E}^3$. In the case of pseudospherical surfaces as such an object it is appropriate to take the net of asymptotic lines (which is a Chebyshev net), given on entire surface S.

To construct a net of asymptotic lines on S we need to consider the problem of isometric immersion of of the generating metric of the form (4.1.1) in the space $\mathbb{E}^3$. Namely, given the coefficients $E[u], F[u], G[u]$, the task is to find the coefficients $L[u]$, $M[u]$, and $N[u]$ of the second fundamental form of the surface. This in turn is connected with the integration of the system of fundamental equations of the theory of surfaces in $\mathbb{E}^3$ (the Peterson-Codazzi and Gauss equations):

$$\begin{aligned} (L[u])_t + \Gamma^1_{11} M[u] + \Gamma^2_{11} N[u] &= (M[u])_x + \Gamma^1_{12} L[u] + \Gamma^2_{12} M[u], \\ (M[u])_t + \Gamma^1_{12} M[u] + \Gamma^2_{12} N[u] &= (N[u])_x + \Gamma^1_{22} L[u] + \Gamma^2_{22} M[u], \\ \frac{L[u]N[u] - M^2[u]}{E[u]G[u] - F^2[u]} &= K(x,t). \end{aligned} \tag{4.1.31}$$

The vanishing condition for the second fundamental form $\mathrm{II}(u,v)$ of the surface,

$$\mathrm{II}(u,v) = L[u]dx^2 + 2M[u]dxdt + N[u]dt^2 = 0$$

yields in a unique manner the transition from the variables (x,t) in the Λ^2-equation to the asymptotic Chebyshev coordinate set (x_a, t_a) on S determined by the sine-Gordon equation. Therefore, in this case one can talk about obtaining a "global analogue" of the transformation (4.1.18), which enables us to make the transition to the "global" Chebyshev net $\mathrm{Cheb}(x_a, t_a)$ of asymptotic lines on the entire surface S. Finding an exact solution of the system (4.1.31) is equivalent to obtaining a "*global*" *analogue of the substitution* (4.1.18), thanks to which the transformation (4.1.13), (4.1.14) acquires a "global" character.

Comment 4.1.4 (On correctness criteria for the application of approximate methods for constructing of solutions of the Λ^2- and G-equations)**.** In general, the construction of an exact nonlocal solution of the problem (4.1.13), (4.1.14), (4.1.23)–(4.1.27) has a transcendental character. For this reason we resort to possible criteria for verifying the correctness of the results obtained by the application of numerical methods.

Let $z^* = z^*(y_1, y_2)$ be an approximate solution of the Chebyshev equation (or of the sine-Gordon equation, respectively, when $K \equiv -1$). Given the function z^*, we extract its initial values

$$\begin{aligned} z^*(0, y_2) &= f_1^*(y_2), \\ z^*(y_1, 0) &= f_2^*(y_1), \\ f_1^*(0) &\simeq f_2^*(0). \end{aligned}$$

Next, from the initial data $f_1^*(y_2)$ and $f_2^*(y_1)$ we recover the "exact" solution $z(y_1, y_2)$ corresponding to them by means of successive approximations for the Chebyshev equation, written in the integral form (see § 3.6):

$$\begin{aligned} z_{m+1}(y_1, y_2) &= f_1^*(y_2) + f_2^*(y_1) - f_1^*(0) \\ &+ \int_0^{y_1}\int_0^{y_2} [-K(y_1, y_2)] \sin z_m(y_1, y_2) dy_1 dy_2. \end{aligned} \tag{4.1.32}$$

Under the assumption that the curvature is bounded, i.e.,

$$|K(y_1, y_2)| \le K_0, \quad K_0 = \text{const} > 0,$$

and choosing as the initial iteration in (4.1.32) $z_0 \equiv 0$, it is not hard to estimate the modulus of the difference of two successive approximations as

$$|z_{m+1} - z_m| \le (K_0)^m \frac{(y_1 y_2)^m}{(m!)^2},$$

which established the convergence of the sequence $\{z_m\}$:

$$\{z_m(y_1, y_2)\} \to z, \quad m \to \infty.$$

The coincidence, within the limits of the admissible accuracy ("residual") δ, of the solutions z and z^*:

$$z \simeq z^* + \delta,$$

represent the *correctness criterion* for the numerical algorithm that is being implemented.

In addition to this, one can use for verification the relations obtained simultaneously with formula (4.1.13) and stipulated by the intrinsic geometry of the Chebyshev net:

$$\left.\left(E[z^*] \cdot (f_{1_{y_1}})^2 + 2F[z^*] \cdot f_{1_{y_1}} f_{2_{y_1}} + G[z^*] \cdot (f_{2_{y_1}})^2\right)\right|_{\substack{x = f_1(y_1, y_2), \\ t = f_2(y_1, y_2)}} = 1,$$

$$\left.\left(E[z^*] \cdot (f_{1_{y_2}})^2 + 2F[z^*] \cdot f_{1_{y_2}} f_{2_{y_2}} + G[z^*] \cdot (f_{2_{y_2}})^2\right)\right|_{\substack{x = f_1(y_1, y_2) \\ t = f_2(y_1, y_2)}} = 1.$$

4.2 The generalized third-order Λ^2-equation. A method for recovering the structure of generating metrics

The recipe introduced in § 4.1 for generating a differential equation (Λ^2-equation) of the type (4.1.4) from a two-dimensional pseudospherical metric of the form (4.1.1) by means of the Gauss formula (4.1.3) presumes that it yields a "final" Λ^2-equation whose order is two units higher that the order of the metric one starts with. (By the order of the metric (4.1.1) we will mean the largest order of the derivatives of the unknown function $u(x,t)$ appearing in the coefficients $E[u(x,t)]$, $F[u(x,t)]$, and $G[u(x,t)]$ of the metric).

In this section we obtain a *generalized third-order* Λ^2*-equation* (generated by a corresponding pseudospherical metric (4.1.1) of *first order*). This equation will include as partial realizations all possible Λ^2-equations of order up to and including three (among them, for example, the nonlinear evolution equations of mathematical physics that we considered earlier, as well as other equations). Moreover, the obtained generalized equation will serve as a "support" in the elaboration of algorithms for recovering generating pseudospherical metrics for the nonlinear equations under investigation. Overall, the method proposed here offers a fundamentally new "geometric" way of "priming" the method of the inverse scattering transform (setting the "primer" problem of the form (3.9.3), (3.9.4)) *based on the obtained metric that generates* the equation.

4.2.1 The generalized third-order Λ^2-equation

Let us turn now to the direct derivation of the generalized third-order Λ^2-equation. We assume that the coefficients of the quadratic differential form (4.1.1) are of the form

$$E = E(u, u_x), \quad F = F(u, u_x), \quad G = G(u, u_x), \tag{4.2.1}$$

and insert them in the Gauss formula (4.1.3).

For coefficients of the form (4.2.1) the determinant appearing in formula (4.1.3) (in the first right-hand side term) takes on the form

$$\begin{aligned}
&\det\begin{pmatrix} E[u] & (E[u])_x & (E[u])_t \\ F[u] & (F[u])_x & (F[u])_t \\ G[u] & (G[u])_x & (G[u])_t \end{pmatrix} \\
&= \det\begin{pmatrix} E & (E_u u_x + E_{u_x} u_{xx}) & (E_u u_t + E_{u_x} u_{xt}) \\ F & (F_u u_x + F_{u_x} u_{xx}) & (F_u u_t + F_{u_x} u_{xt}) \\ G & (G_u u_x + G_{u_x} u_{xx}) & (G_u u_t + G_{u_x} u_{xt}) \end{pmatrix} \\
&= \det\begin{pmatrix} E & E_u & E_{u_x} \\ F & F_u & F_{u_x} \\ G & G_u & G_{u_x} \end{pmatrix} \cdot \det\begin{pmatrix} u_x & u_t \\ u_{xx} & u_{xt} \end{pmatrix}.
\end{aligned} \tag{4.2.2}$$

In our case, for the coefficients (4.2.1), the second term in the right-hand side of (4.1.3) becomes

$$\frac{1}{2\sqrt{W}}\left\{\frac{\partial}{\partial t}\left(\frac{(E[u])_t-(F[u])_x}{\sqrt{W[u]}}\right)-\frac{\partial}{\partial x}\left(\frac{(F[u])_t-(G[u])_x}{\sqrt{W[u]}}\right)\right\}$$
$$=\frac{1}{4W^2}\left\{2W(E_{tt}-G_{xx})+(F_t-G_x)W_x-(E_t-F_x)W_t\right\}. \tag{4.2.3}$$

The "components" figuring in relations (4.2.2) and (4.2.3) are given by

$$\begin{aligned}
&\text{(a)}\quad E_x=E_u u_x+E_{u_x}u_{xx}, \qquad E_t=E_u u_t+E_{u_x}u_{xt},\\
&\text{(b)}\quad G_x=G_u u_x+G_{u_x}u_{xx}, \qquad G_t=G_u u_t+G_{u_x}u_{xt},\\
&\text{(c)}\quad F_x=F_u u_x+F_{u_x}u_{xx}, \qquad F_t=F_u u_t+F_{u_x}u_{xt},
\end{aligned} \tag{4.2.4}$$

$$\begin{aligned}
&\text{(a)}\quad E_{tt}=E_{uu}u_t^2+2E_{uu_x}u_t u_{xt}+E_u u_{tt}+E_{u_x u_x}u_{xt}^2+E_{u_x}u_{xtt},\\
&\text{(b)}\quad G_{xx}=G_{uu}u_x^2+2G_{uu_x}u_x u_{xx}+G_u u_{xx}+G_{u_x u_x}u_{xx}^2+G_{u_x}u_{xxx},
\end{aligned} \tag{4.2.5}$$

$$\begin{aligned}
&\text{(a)}\quad E_tW_x=E_uW_u u_x u_t+E_uW_{u_x}u_t u_{xx}+E_{u_x}W_u u_x u_{xt}\\
&\qquad\qquad +E_{u_x}W_{u_x}u_{xt}u_{xx},\\
&\text{(b)}\quad G_xW_x=G_uW_u u_x^2+G_uW_{u_x}u_x u_{xx}+G_{u_x}W_u u_x u_{xx}\\
&\qquad\qquad +G_{u_x}W_{u_x}u_{xx}^2,\\
&\text{(c)}\quad E_tW_t=E_uW_u u_t^2+E_uW_{u_x}u_t u_{xt}+E_{u_x}W_u u_t u_{xt}\\
&\qquad\qquad +E_{u_x}W_{u_x}u_{xx}u_{xt},\\
&\text{(d)}\quad F_xW_t=F_uW_u u_x u_t+F_uW_{u_x}u_x u_{xt}+F_{u_x}W_u u_t u_{xx}\\
&\qquad\qquad +F_{u_x}W_{u_x}u_{xt}u_{xx}.
\end{aligned} \tag{4.2.6}$$

Substitution of expressions (4.2.4)–(4.2.6) in relations (4.2.2) and (4.2.3) (i.e., essentially, in the Gauss formula (4.1.3)) allow us to *interpret the Gauss formula* as a partial differential *equation* for the unknown function $u(x,t)$, which appears in the generating metric of the form (4.2.1). Hence, we arrive at a generalized Gauss equation of the third order, generated by a first-order metric of arbitrary Gaussian curvature $K(x,t)$:

$$\sum_{\alpha,\beta,\gamma=1}^{2}a_{\alpha\beta\gamma}u_{\alpha\beta\gamma}+\sum_{\alpha,\beta,\gamma,\delta=1}^{2}a_{\alpha\beta,\gamma\delta}u_{\alpha\beta}u_{\gamma\delta}+\sum_{\alpha,\beta,\gamma=1}^{2}b_{\alpha,\beta\gamma}u_\alpha u_{\beta\gamma}$$
$$+\sum_{\alpha,\beta=1}^{2}c_{\alpha,\beta}u_\alpha u_\beta+\sum_{\alpha,\beta=1}^{2}d_{\alpha\beta}u_{\alpha\beta}=-4K(x,t)\cdot W^2 \tag{4.2.7}$$

(*generalized third-order G-equation*).

Each of the indices α, β, γ, and δ in (4.2.7) can take only two values: 1 or 2. An index attached to the function $u(x,t)$ denotes the derivative with respect to the corresponding variable "x" $\equiv$ "1", "t" $\equiv$ "2"; for example, $u_1\equiv u_x$, $u_{12}\equiv u_{xt}$, and so on. All nontrivial (non-zero) coefficients of the generalized equation (4.2.7) are given below in Table 4.2.1.

In the expressions listed in the table we use the notation

$$D \equiv \begin{vmatrix} E & E_u & E_{u_x} \\ F & F_u & F_{u_x} \\ G & G_u & G_{u_x} \end{vmatrix}.$$

Table 4.2.1

u_{xxx}	a_{111}	$2WG_{u_x}$
u_{xxt}	a_{112}	$-4WF_{u_x}$
u_{xtt}	a_{122}	$2WE_{u_x}$
u_{xx}^2	$a_{11,11}$	$2WG_{u_xu_x} - W_{u_x}G_{u_x}$
$u_{xx}u_{xt}$	$a_{11,12}$	$2(W_{u_x}F_{u_x} - 2WF_{u_xu_x})$
u_{xt}^2	$a_{12,12}$	$2WE_{u_xu_x} - W_{u_x}E_{u_x}$
u_xu_{xx}	$b_{1,11}$	$4WG_{uu_x} - G_uW_{u_x} - W_uG_{u_x}$
u_xu_{xt}	$b_{1,12}$	$D + F_{u_x}W_u + W_{u_x}F_u - 4WF_{uu_x}$
u_tu_{xx}	$b_{2,11}$	$F_uW_{u_x} + W_uF_{u_x} - D - 4WF_{uu_x}$
u_tu_{xt}	$b_{2,12}$	$4WE_{uu_x} - E_uW_{u_x} - W_uE_{u_x}$
u_x^2	$c_{1,1}$	$2WG_{uu} - W_uG_u$
u_xu_t	$c_{1,2}$	$2(W_uF_u - 2WF_{uu})$
u_t^2	$c_{2,2}$	$2WE_{uu} - W_uE_u$
u_{xx}	d_{11}	$2WG_u$
u_{xt}	d_{12}	$-4WF_u$
u_{tt}	b_{22}	$2WE_u$

The obtained equation (4.2.7) with the functional coefficients given in Table 4.2.1 is the generalized third-order Gauss equation (G-equation). In the geometrically characteristic case $K(x,t) \equiv -1$ (Lobachevsky plane), equation (4.2.7) becomes the generalized third-order Λ^2-equation; below we will focus on precisely this last equation.

4.2.2 The method of structural reconstruction of the generating metrics for Λ^2-equations

Let us formulate a *general algorithm of structural reconstruction* of the generating Λ^2 metric for nonlinear $(1+1)$-equations[4] and exemplify it in detail to construct a pseudospherical metric for the modified Korteweg-de Vries equation.

The study of the problem of deriving, for a given differential equation, a *geometric interpretation* (namely, given the equation, find the corresponding Λ^2-metric that generated it) is connected with subjecting equation (4.2.7) to additional constraints, which characterize the structure of the equation under study. Derivatives of the type $\{u_{0,n}\}$, defined in the sought-for metric for all solutions of the equation under study, are taken with respect to the independent variables.

[4]In a $(1+1)$-equation the unknown function depends on *one* space variable x and *one* time variable t.

This enables us to associate to each such term containing $u_{0,n}$ the components with the corresponding terms of the initial equation. This leads to a *system of relations for the coefficients of the sought-for metric*. The derivatives of the form $u_{m,0}$, $m = 1, 2$, are replaced by expressions determined by the form of the equation under study (for instance, $u_t = \mathcal{F}[u]$ or $u_{xt} = \mathcal{F}[u]$).

As promised, we will next implement in detail the method of reconstruction of a generating pseudospherical metric in the case of the modified Korteweg-de Vries equation.

Example. *Construction of a generating Λ^2-metric for the modified Korteweg-de Vries equation* (MKdV equation). We consider the MKdV equation, well known in mathematical physics:

$$u_t = \frac{3}{2}\,u^2 u_x + u_{xxx}. \tag{4.2.8}$$

Under the assumption that the pseudospherical metric that generates equation (4.2.8) is a first-order metric with the coefficients (4.2.1),

$$ds^2 = E(u, u_x)dx^2 + 2F(u, u_x)dxdt + G(u, u_x)dt^2,$$

let us find under what (detailed) conditions on the coefficients (4.2.1) of this metric the resulting generalized equation (4.2.7) is precisely the MKdV equation.

Here it is natural to interpret the equation (4.2.8) itself as a *constraint* on the unknown function $u = u(x, t)$ and its derivatives.

To begin with, let us write several differential consequences of equation (4.2.8) that will be needed later in order to perform certain manipulations in the generalized equation (4.2.7):

$$\begin{aligned}
u_t &= \frac{3}{2}u^2 u_x + u_{xxx},\\
u_{xt} &= 3uu_x^2 + \frac{3}{2}u^2 u_{xx} + u_{xxxx},\\
u_{xxt} &= 3u_x^3 + 9uu_x u_{xx} + \frac{3}{2}u^2 u_{xxx} + u_{xxxxx},\\
u_{xxxt} &= 18u_x^2 u_{xx} + 9uu_{xx}^2 + 12uu_x u_{xxx} + \frac{3}{2}u^2 u_{xxxx} + u_{xxxxxx},\\
u_{tt} &= 9u^3 u_x^2 + \frac{9}{4}u^4 u_{xx} + 18u_x^2 u_{xx} + 9uu_{xx}^2\\
&\quad + 15uu_x u_{xxx} + 3u^2 u_{xxxx} + u_{xxxxxx},\\
u_{xtt} &= 27u^2 u_x^3 + 27u^3 u_x u_{xx} + 45u_x u_{xx}^2 + \frac{9}{4}u^4 u_{xxx}\\
&\quad + 33u_x^2 u_{xxx} + 33uu_{xx}u_{xxx} + 21uu_x u_{xxxx} + u_{xxxxxxx}.
\end{aligned} \tag{4.2.9}$$

In the case of the MKdV equation and its consequences (4.2.9) considered here, the generalized Λ^2-equation (4.2.7) (for $K \equiv -1$) reduces to a differential equations that contains only derivatives of the unknown function $u(x, t)$ with respect to x of order up to and including 7:

$$4W^2 = 2WE_{u_x}\Big(27u^3 u_x u_{xx} + \frac{9}{4}u^4 u_{xxx} + 27u^2 u_x^3 + 21uu_x u_{xxxx}$$

$$
\begin{aligned}
&+\ 33u_x^2 u_{xxx} + 45u_{xx}^2 u_x + 33uu_{xx}u_{xxx} + u_{xxxxxxx}\Big) \\
&-\ 4WF_{u_x}\left(\frac{3}{2}u^2 u_{xxx} + 9uu_x u_{xx} + 3u_x^3 + u_{xxxxx}\right) \\
&+\ 2Wu_{xxx}G_{u_x} + u_x u_{xx}(4WG_{uu_x} - G_u W_{u_x} - W_u G_{u_x}) \\
&+\ u_x^2(2WG_{uu} - W_u G_u) + 2Wu_{xx}G_u + u_{xx}^2(2WG_{u_x u_x} - W_{u_x}G_{u_x}) \\
&+\ 2u_{xx}\left(\frac{3}{2}\,u^2 u_{xx} + 3uu_x^2 + u_{xxxx}\right)(W_{u_x}F_{u_x} - 2WF_{u_x u_x}) \\
&+\left(\frac{3}{2}u^2 u_x + u_{xxx}\right)\left(\frac{3}{2}u^2 u_{xx} + 3uu_x^2 + u_{xxxx}\right)(4WE_{uu_x} - E_u W_{u_x} - W_u E_{u_x}) \\
&+\left(\frac{3}{2}\,u^2 u_x + u_{xxx}\right)^2(2WE_{uu} - W_u E_u) \qquad (4.2.10)\\
&+\ u_{xx}\left(\frac{3}{2}u^2 u_x + u_{xxx}\right)(F_u W_{u_x} + W_u F_{u_x} - D - 4WF_{uu_x}) + 2WE_u \times \\
&\times\left(\frac{9}{4}u^4 u_{xx} + 9u^3 u_x^2 + 3u^2 u_{xxxx} + 15uu_x u_{xxx} + 18u_{xx}u_x^2 + 9uu_{xx}^2 + u_{xxxxxx}\right) \\
&+\left(\frac{3}{2}u^2 u_{xx} + 3uu_x^2 + u_{xxxx}\right)^2(2WE_{u_x u_x} - W_{u_x}E_{u_x}) \\
&+\ u_x\left(\frac{3}{2}u^2 u_{xx} + 3uu_x^2 + u_{xxxx}\right)(D + F_{u_x}W_u + W_{u_x}F_u - 4WF_{uu_x}) \\
&+ 2u_x\left(\frac{3}{2}u^2 u_x + u_{xxx}\right)(W_u F_u - 2WF_{uu}) - 4WF_u\left(\frac{3}{2}u^2 u_{xx} + 3uu_x^2 + u_{xxxx}\right).
\end{aligned}
$$

The next step in the implementation of the reconstruction algorithm consists in "ordering" expression (4.2.1) according to groups of terms in front of the derivatives $u_{xxxxxxx}$, $u_{xxxxxx}, \ldots$, $u_{xxx}, \ldots$ (in order of decrease of the order of differentiation). We note again that the indicated derivatives (defined on each solution u of the MKdV equation) acquire here the meaning of independent "variables".

The first ordered term, which includes the 7-th order derivative, has the form

$$2W \cdot E_{u_x} \cdot u_{xxxxxxx} + \cdots\,; \qquad (4.2.11)$$

Since relation (4.2.10) means that equation (4.2.7) holds identically on all solutions of the MKdV equation (with the constraint (4.2.9) accounted for in (4.2.7)), all "functional coefficients" in front of the derivatives of the unknown functions u in (4.2.10) must be equal to zero. An examination of the first three ordered terms, in front of the derivatives of u with respect to x of order 7, 6, and 5 in (4.2.10) leads, in conjunction with (4.2.11), to the system

$$
\begin{aligned}
2WE_{u_x} &= 0,\\
2WE_u &= 0,\\
-4WF_{u_x} &= 0.
\end{aligned}
\qquad (4.2.12)
$$

From (4.2.12) we obtain (under the natural assumption that $W \neq 0$):

$$E = \eta^2 = \text{const}, \quad F = F(u). \qquad (4.2.13)$$

Expression (4.2.13) is the first result on the path of finding the precise form of the coefficients of the generating metric. At the same time, it allows us to simplify considerably the form of equation (4.2.10), to

$$\begin{aligned} 4W^2 = {} & 2Wu_{xxx}G_{u_x} + u_xu_{xx}(4WG_{uu_x} - G_uW_{u_x} - W_uG_{u_x}) \\ & + u_{xx}\Big(\frac{3}{2}u^2u_x + u_{xxx}\Big)(F_uW_{u_x} - D) \\ & + u_x^2(2WG_{uu} - W_uG_u) + 2Wu_{xx}G_u + u_{xx}^2(2WGu_xu_x - W_{u_x}G_{u_x}) \\ & + u_x\Big(\frac{3}{2}u^2u_{xx} + 3uu_x^2 + u_{xxxx}\Big)(D + W_{u_x}F_u) \\ & + 2u_x\Big(\frac{3}{2}u^2u_x + u_{xxx}\Big)(W_uF_u - 2WF_{uu}) \\ & - 4WF_u\Big(\frac{3}{2}u^2u_{xx} + 3uu_x^2 + u_{xxxx}\Big); \end{aligned} \tag{4.2.14}$$

moreover,

$$D = \eta^2G_{u_x}F_u, \quad W_u = \eta^2G_u - 2FF_u, \quad W_{u_x} = \eta^2G_{u_x}.$$

Continuing the implementation of the algorithm, let us write the conditions expressing the "vanishing" of the coefficients in front of the derivatives u_{xxxx} and u_{xxx} in (4.2.14):

for u_{xxxx}:

$$2F_u \cdot (\eta^2G_{u_x} \cdot u_x - 2W) = 0, \tag{4.2.15}$$

for u_{xxx}:

$$\eta^2u_xG_uF_u - 2u_x(FF_u^2 + WF_{uu}) + WG_{u_x} = 0. \tag{4.2.16}$$

It is readily verified that the equality $F_u = 0$ cannot be a consequence of relation (4.2.15), since otherwise (recalling (4.2.13) and (4.2.16)) all coefficients of the generating metric of the type (4.2.1) would be constant.

Thus, (4.3.15) yields

$$W = \frac{1}{2}\,a^2G_{u_x}u_x. \tag{4.2.17}$$

Accordingly, equation (4.2.16) becomes

$$2\eta^2u_xG_uF_u - 4u_xFF_u^2 - 2\eta^2u_xG_{u_x}F_{uu} + \eta^2G_{u_x}^2 = 0. \tag{4.2.18}$$

At this iteration step of the algorithm, if one takes (4.2.17) and (4.2.18) into account, the generalized Λ^2-equation (equation (4.2.7) → (4.2.10) → (4.2.14)) can be simplified further to

$$\begin{aligned} 2\eta^2G_{u_x}^2u_x^2 = {} & u_{xx}\big(u_x(3u_xG_{u_x}G_{uu_x} - G_u(G_{u_x} + G_{u_xu_x}u_x)\big) \\ & + \frac{3}{2}u^2u_x(F_u(G_{u_x} + G_{u_xu_x}u_x) - 2G_{u_x}F_u) + 2u_xG_{u_x}G_u) \\ & - u_{xx}^2G_{u_x}(u_xG_{u_xu_x} - G_{u_x}) \\ & + u_x^3(2G_{u_x}G_{uu} - G_{u_xu}G_u + 3u^2(G_{u_xu}F_u - 2G_{u_x}F_{uu})). \end{aligned} \tag{4.2.19}$$

Continuing the algorithmic scheme, now already for equation (4.2.19), we write the "vanishing coefficients" in the remaining terms:

for u_{xx}:

$$G_{u_x}(u_x G_{u_x u_x} - G_{u_x}) = 0. \tag{4.2.20}$$

Since $W \neq 0$, relation (4.2.17) shows that in (4.2.20) we cannot have $G_{u_x} = 0$. Setting the expression inside the parentheses in (4.2.20) equal to zero, one can readily get that

$$G = \lambda(u)u_x^2 + f(u). \tag{4.2.21}$$

Moreover, for u_{xx}:

$$3u_x^2 G_{u_x} G_{uu_x} - u_x G_u G_{u_x} - u_x^2 G_u G_{u_x u_x} + \frac{3}{2}u^2 u_x F_u G_{u_x} + \frac{3}{2}u^2 u_x^2 F_u G_{u_x u_x} - 3u^2 u_x G_{u_x} F_u + 2u_x G_{u_x} G_u = 0. \tag{4.2.22}$$

Using (4.2.21), equation (4.2.21) can be simplified considerably to

$$\lambda\lambda_u u_x^4 = 0, \quad \lambda = \text{const},$$

and so

$$W = \lambda\eta^2 u_x^2, \quad G_{u_x} = 2\lambda u_x.$$

The results obtained to this point allow us to rewrite equation (4.2.16) in the compact form

$$g_1(u) \cdot u_x + g_2(u) \cdot u_x^2 = 0, \tag{4.2.23}$$

where

$$g_1(u) = 2\eta^2 G_u F_u - 4FF_u^2,$$
$$g_2(u) = -4\lambda\eta^2 F_{uu} + 4\lambda^2\eta^2.$$

Since the coefficients (4.2.23) must vanish: $g_1(u) = 0$ and $g_2(u) = 0$, it holds that

$$\begin{aligned} \eta^2 G_u &= 2FF_u \\ F_{uu} &= \lambda. \end{aligned} \tag{4.2.24}$$

The second equation in (4.2.24) immediately yields

$$F = F(u) = \frac{\lambda}{2}u^2 + C_1 u + C_2, \quad C_1, C_2 = \text{const}. \tag{4.2.25}$$

Integration of the first equation in (4.2.24) gives

$$G = \frac{1}{\eta^2}\,F^2 + C, \quad C = C(u_x). \tag{4.2.26}$$

From the calculation of the already obtained determinant of the metric,

$$W = EG - F^2 = \lambda\eta^2 u_x^2$$

we obtain, using the coefficients E, F, G given by the expressions (4.2.13), (4.2.25), (4.2.26),

$$C(u_x) = \lambda u_x^2, \quad \lambda = \eta^2 = \text{const}. \tag{4.2.27}$$

Substituting the coefficients E, F, G (4.2.25), (4.2.26), with relation (4.2.27) accounted for, in the generalized third-order Λ^2-equation (4.2.19), transformed to the form

$$2\lambda\eta^4 = 2\left(\lambda\Big(\frac{\lambda}{2}u^2 + C_1 u + C_2\Big) + (\lambda u + C_1)^2\right) - 3\lambda u^2\eta^2,$$

and subsequently comparing the coefficients of like powers of the function u, we obtain the exact values of the constants involved:

$$C_1 = 0, \quad C_2 = \eta^4.$$

Putting all together, we finally obtain the exact explicit representation for the coefficients of the sought-for generating metric:

$$E = \eta^2, \quad F = \eta^2\Big(\frac{u^2}{2} + \eta^2\Big), \quad G = \eta^2 u_x^2 + \eta^2\Big(\frac{u^2}{2} + \eta^2\Big)^2, \tag{4.2.28}$$

and consequently the pseudospherical metric itself that generates the modified Korteweg-de Vries equation (4.2.8):

$$ds^2 = \eta^2 dx^2 + 2\eta^2\Big(\frac{u^2}{2} + \eta^2\Big)dxdt + \left[\eta^2 u_x^2 + \eta^2\Big(\frac{u^2}{2} + \eta^2\Big)^2\right]dt^2. \tag{4.2.29}$$

Thus, we fully implemented the algorithm of the *method of structural reconstruction* of the generating pseudospherical metric for the modified Korteweg-de Vries equation. Overall, the question whether the proposed algorithm is applicable to a given nonlinear equation is directly connected with the *compatibility* (or *consistency*) *problem* , as well as with the *explicit solvability of the system of equations*, obtained on the basis of the generalized third-order Λ^2 equation, which expresses the vanishing of all the "functional coefficients" in the equation of the type (4.2.7) (in the equation (4.2.10) in each concrete case).

Let us now formulate the general scheme of the algorithm of the *method of structural reconstruction* of the generating metric for a nonlinear third-order differential equation:

1. Reduce the Λ^2-equation (4.2.7), with the differential consequences of the equation under study accounted for, to a relation whose terms are arranged according to the order of the derivatives of the unknown function $u(x,t)$. (In the example considered above, that was equation (4.2.10).)

2. Derive the system of differential equations for the coefficients of the sought-for generating metric, $E(u,u_x)$, $F(u,u_x)$, $G(u,u_x)$, from the condition that all the "functional cofficients" in front of the terms with the derivatives of the unknown function u of different orders vanish.

3. Investigate of the compatibility of the aforementioned system of differential relations. Construct exact solutions of this system.

4.3 Orthogonal nets and the nonlinear equations they generate

As one can see from the discussion above (see § 4.2), given some nonlinear equation, the recovery of its generating Λ^2- or G-metric takes a rather large amount of work. For that reason, one of the approaches that allows one, to a certain extent, to "optimize" the problem of associating to Λ^2- and G-equations the Λ^2- and G-metrics that generate them, consists in cleverly describing those equations that are generated by two-dimensional metrics that have certain specific geometric properties, namely, metrics associated with certain classes of coordinate nets on two-dimensional smooth manifolds that have intuitive geometric features. As it turns out, such nets define a considerable number of nonlinear equations of current interest in mathematical physics.

A rich class of metrics that generate a sufficient number of well-known nonlinear equations is associated with the *orthogonal nets*. Such nets are given by the condition that the second coefficient of the metric of type (4.1.1) vanishes:

$$F[u(x,t)] \equiv 0, \quad (x,t) \in \mathbb{R}^2. \tag{4.3.1}$$

Accordingly, the metric itself, written in the orthogonal coordinate system, reads

$$ds^2 = E[u]dx^2 + G[u]dt^2. \tag{4.3.2}$$

Let us study the problem of finding the G-equations generated by metrics of the form (4.3.2) (the curvature $K(x,t)$ is assumed to be arbitrary).

Setting

$$E[u] = a^2[u], \quad G[u] = b^2[u],$$

(and then $W = a^2[u]b^2[u] > 0$), we rewrite the metric (4.3.2) as

$$ds^2 = a^2[u]dx^2 + b^2[u]dt^2. \tag{4.3.3}$$

Let us substitute (4.3.3) in the Gauss formula (4.1.3). This yields the equation

$$\left\{\left(\frac{a_u}{b}\,u_t\right)_t + \left(\frac{b_u}{a}\,u_x\right)_x\right\} = -2K(x,t)\cdot W^{1/2}. \tag{4.3.4}$$

It is convenient to recast (4.3.4) as

$$\left[\left(\frac{b_u}{a}\right)u_{xx} + \left(\frac{a_u}{b}\right)u_{tt}\right] + \left[\left(\frac{b_u}{a}\right)_u u_x^2 + \left(\frac{a_u}{b}\right)_u u_t^2\right] = -2K(x,t)\cdot W^{1/2}. \tag{4.3.5}$$

Equation (4.3.5) is the general G-equation generated by metrics of the form (4.3.3), written in an orthogonal net parametrization. Let us determine under what conditions on $a[u]$ and $b[u]$ the left-hand side of (4.3.5) expresses the action of one of the standard operators of mathematical physics: the Laplace operator, the wave operator, etc.

1) Equation (4.3.5) will be *elliptic* if, in particular, its left-hand side represents the Laplacian of the function u, which is the case whenever the following system of conditions are satisfied:

$$\begin{cases} \dfrac{a_u}{b} = \eta, \\ \dfrac{b_u}{a} = \eta, \end{cases} \qquad \eta = \text{const.} \tag{4.3.6}$$

Notice that fulfillment of conditions (4.3.6) automatically implies that the terms inside the second pair of brackets in the left-hand side of (4.3.5) vanish.

Integrating the system (4.3.6), we find for $a[u]$ and $b[u]$ the expressions

$$a[u] = A_1 \cdot e^{\eta u} + A_2 \cdot e^{-\eta u},$$
$$b[u] = A_1 \cdot e^{\eta u} - A_2 \cdot e^{-\eta u}, \quad A_1, A_2 = \text{const.}$$

Therefore, if conditions (4.3.6) are satisfied, then the metric (4.3.3) takes on the form

$$ds^2 = \left(A_1 \cdot e^{\eta u} + A_2 \cdot e^{-\eta u}\right)^2 dx^2 + \left(A_1 \cdot e^{\eta u} - A_2 \cdot e^{-\eta u}\right)^2 dt^2. \tag{4.3.7}$$

The metric (4.3.7) thus obtained, written in orthogonal coordinates, generates a general elliptic G-equation of the form

$$\Delta_2 u = -\frac{1}{\eta} \cdot K(x,t) \cdot \left(A_1^2 \cdot e^{2\eta u} - A_2^2 \cdot e^{-2\eta u}\right), \tag{4.3.8}$$

where $\Delta_2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial t^2}$ is the Laplace operator.

By suitably choosing the constants A_1 and A_2 appropriately we can obtain as particular cases of the general equation well-known nonlinear equations encountered in mathematical physics. Let us give such examples of metrics and the equations they generate.

a) $A_1 = \frac{1}{\sqrt{2}}, \;\; A_2 = 0, \;\; \eta = \frac{1}{2}.$

Generating metric:

$$ds^2 = \frac{e^u}{2}\, dx^2 + \frac{e^u}{2}\, dt^2. \tag{4.3.9}$$

G-equation generated – the *elliptic Liouville equation*:

$$\Delta_2 u = -K(x,t) \cdot e^u. \tag{4.3.10}$$

When $K \equiv -1$ (the case of the Lobachevsky plane Λ^2) we obtain an important subcase of equation (4.3.10):

$$\Delta_2 u = e^u. \tag{4.3.11}$$

b) $A_1 = A_2 = \frac{1}{2}$, $\eta = \frac{1}{2}$.
Generating metric:

$$ds^2 = \cosh^2 \frac{u}{2}\, dx^2 + \sinh^2 \frac{u}{2}\, dt^2. \tag{4.3.12}$$

The G-equation corresponding to the metric (4.3.12):

$$\Delta_2 u = -K(x,t) \cdot \sinh u, \tag{4.3.13}$$

and its "Λ^2-analogue", the elliptic sinh-Gordon equation:

$$\Delta_2 u = \sinh u. \tag{4.3.14}$$

2) Now let us study the *hyperbolic* G-equations, which are "included" in (4.3.5) and are generated by a metric of the general form (4.3.3). Indeed, if the conditions

$$\begin{cases} \dfrac{b_u}{a} = \eta, \\ \dfrac{a_u}{b} = -\eta, \end{cases} \qquad \eta = \text{const.} \tag{4.3.15}$$

are satisfied, then in the left-hand side of (4.3.5) one obtains the Laplace operator.
The system (4.3.15) has the solutions

$$\begin{aligned} a[u] &= C_1 \sin \eta u - C_2 \cos \eta u, \\ b[u] &= -C_2 \sin \eta u - C_1 \cos \eta u, \end{aligned} \qquad \eta = \text{const}, \quad C_1, C_2 = \text{const.} \tag{4.3.16}$$

Using (4.3.16), let us write the generating metric of general form (4.3.3) for the case at hand:

$$ds^2 = (C_1 \sin \eta u - C_2 \cos \eta u)^2\, dx^2 + (C_2 \sin \eta u + C_1 \cos \eta u)^2\, dt^2. \tag{4.3.17}$$

The metric (4.3.17) generates the general hyperbolic G-equation

$$u_{xx} - u_{tt} = -K \cdot [C_1 C_2 \cdot \cos(2\eta u) - (C_1^2 - C_2^2) \cdot \sin \eta u \cdot \cos \eta u]. \tag{4.3.18}$$

Upon choosing for the constant parameters in (4.3.18) the values

$$C_1 = 0, \quad C_2 = 1, \quad \eta = \frac{1}{2},$$

we obtain the classical Chebyshev equation (see § 2.5):

$$U_{xx} - U_{tt} = -K(x,t) \sin U, \quad U = 2u, \tag{4.3.19}$$

in the variables x, t, relative to an orthogonal coordinate system.
Let us give additional examples that demonstrates how orthogonal coordinate nets can be applied in the analysis of nonlinear equations.
Let us consider the metric (4.3.2) with coefficients of the form

$$E = E(u_x), \quad G = G(u), \quad \text{under the condition} \ \ K \equiv -1.$$

Here are two examples.

a) Taking the pseudospherical metric

$$ds^2 = u_x^2 dx^2 + \sinh^2 u \, dt^2 \tag{4.3.20}$$

as the generating metric yields as Λ^2-equation the hyperbolic cosh-Gordon equation

$$u_{xt} = \cosh u. \tag{4.3.21}$$

b) The pseudospherical metric

$$ds^2 = u_x^2 dx^2 + \cosh^2 u \, dt^2 \tag{4.3.22}$$

generates the hyperbolic Λ^2-equation called the sinh-Gordon equation,

$$u_{xt} = \sinh u. \tag{4.3.23}$$

The fact that is possible to associate nonlinear equations to orthogonal generating nets on the Lobachevsky plane Λ^2 enables one to propose geometric algorythms for their integration. Such methods are treated in the next section.

4.4 Net methods for constructing solutions of Λ^2-equations

The geometric interpretation of differential equations presented in this chapter assigns to each Λ^2-equation a pseudospherical metric that generates it (or a generating coordinate net on the Lobachevsky plane Λ^2). This geometric "view" allows one to pass from the investigation of the equations themselves to the analysis of their geometric preimages – the generating coordinates nets, and thus to enlist in the study of equations the tools of non-Euclidean differential geometry. In the realization of this approach it is expedient to use sufficiently well studied integrable Λ^2-equations (for example, the sine-Gordon equation) and the corresponding coordinate nets as *canonical* (supporting) information for constructing transformations that connect them with geometric objects (nets) that characterize other equations under study. A classical example of canonical (supporting) net is the "Chebyshev" net. As we will show below, an important role is played also by the semigeodesic net, used to construct transformations between solutions of elliptic equations.

It is important to emphasize that the transformations obtained connect solutions of various Λ^2-equations and arise "*at the level*" of the transformation of the *preimages* of the equations studied – the generating nets on the Lobachevsky plane Λ^2, and they do not "touch upon" the equations themselves. That is to say, the transformations obtained are the result of transformations between various generating nets on Λ^2 and the associated transformation of solutions, but not of transformations of the equations. Here the constant negative *curvature* $K \equiv -1$ of the generating pseudospherical metrics has the *meaning of an invariant* of the transformations performed. The diagram in Figure 4.4.1 explains the general algorithm and the sequence of links in of the net approach to the construction of solutions of Λ^2-equations.

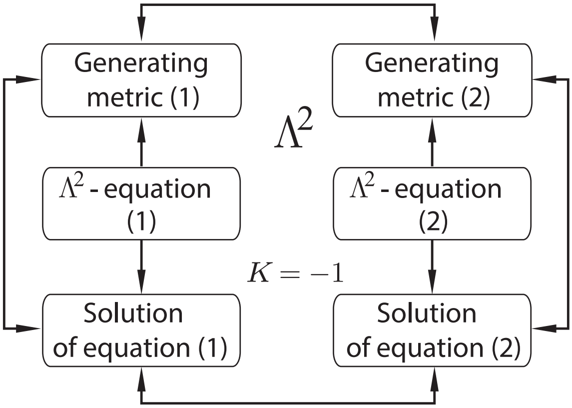

Figure 4.4.1

4.4.1 On mutual transformations of solutions of the Laplace equation and the elliptic Liouville equation

In this subsection we obtain exact explicit formulas for the construction of exact solutions of the elliptic Liouville equation [77, 90]

$$\Delta_2 u = e^u, \quad u = u(x,t) \tag{4.4.1}$$

from solutions of the Laplace equation

$$\Delta_2 v = 0, \quad v = v(x,t). \tag{4.4.2}$$

To construct solutions of the Λ^2-equation (4.1.1) we involve another (auxiliary) Λ^2-equation, namely

$$y_{\tau\tau} - y = 0, \quad y = y(\tau), \tag{4.4.3}$$

i.e., the ordinary differential equation generated by the pseudospherical metric

$$ds^2 = y^2(\tau)d\chi^2 + d\tau^2, \quad K(x,t) \equiv -1, \tag{4.4.4}$$

which plays the role of the supporting metric in our approach.

Recall that the Liouville equation (4.4.1) itself is generated by a Λ^2-metric of the form (see § 4.1)

$$ds^2 = \frac{e^u}{2}(dx^2 + dt^2). \tag{4.4.5}$$

The metric (4.4.5) generating the Liouville equation (4.4.1) is associated with the *isothermal* coordinate net on the Lobachevsky plane Λ^2, while the metric (4.4.4) that generates the Laplace equation (4.4.2) is associated with the *semigeodesic* coordinate net on Λ^2.

In the plane Λ^2, let us pass from the *semigeodesic* coordinate net $T^{\text{sg}}(\chi,\tau)$ to the *isothermal* net $T^{\text{is}}(x,t)$ (the Liouville net) via

$$\begin{aligned} w(x,t) &= \chi, \\ v(x,t) &= \int \frac{d\tau}{y(\tau)}. \end{aligned} \tag{4.4.6}$$

Substitution of (4.4.6) in the metric (4.4.4) reduces the latter to a metric (4.4.5), provided the following conditions are satisfied:

$$\begin{aligned} &v_x^2 + w_x^2 = v_t^2 + w_t^2, \\ &v_x v_t + w_x w_t = 0. \end{aligned} \tag{4.4.7}$$

Then the solution $u(x,t)$ of the Liouville equation (4.4.1) is given by the formula

$$u(x,t) = \ln\left[2y^2(\tau(x,t)) \cdot (v_x^2 + w_x^2)\right]. \tag{4.4.8}$$

It is easy to see that the system (4.4.7) connects two arbitrary *harmonically conjugate* functions $v(x,t)$ and $w(x,t)$, which satisfy the classical Cauchy-Riemann conditions [105]

$$\begin{aligned} v_x &= w_t, \\ v_t &= -w_x, \end{aligned} \tag{4.4.9}$$

and hence also the Laplace equation:

$$\begin{aligned} \Delta_2 v &= 0, \\ \Delta_2 w &= 0. \end{aligned} \tag{4.4.10}$$

Let us turn now to the construction of a solution $u(x,t)$ of equation (4.4.1) by means of formula (4.4.8). To this end, using the general solution

$$y(\tau) = C_1 e^{\tau} + C_2 e^{-\tau}, \quad C_1, C_2 = \text{const} \tag{4.4.11}$$

of equation (4.4.3), we write the metric (4.4.4):

$$ds^2 = (C_1 e^{\tau} + C_2 e^{-\tau})^2 d\chi^2 + d\tau^2 \tag{4.4.12}$$

Now let us substitute the solution (4.4.11) in the second relation in (4.4.6). This yields the representation

$$\tau = \tau(v(x,t)),$$

which is necessary for (4.4.8).

Depending on the signs of the constants C_1 and C_2 chosen in the solution (4.4.11), the second relation in (4.4.6) yields three possible variants:

$$\begin{aligned} &1) \quad y^2(\tau(v)) = \frac{1}{v^2}, \\ &2) \quad y^2(\tau(v)) = \frac{1}{\sinh^2 v}, \\ &3) \quad y^2(\tau(v)) = \frac{1}{\sin^2 v}. \end{aligned} \tag{4.4.13}$$

Formula (4.4.8) in conjunction with (4.4.13) yields three formulas for constructing solutions of the elliptic Liouville equation (4.4.1) from an arbitrary solution $v(x,t)$, $v(x,t) \not\equiv \text{const}$, of the Laplace equation (4.4.2) [77, 90]:

$$\begin{aligned} u(x,t) &= \ln\left[\frac{2(v_x^2+v_t^2)}{v^2}\right], \\ u(x,t) &= \ln\left[\frac{2(v_x^2+v_t^2)}{\sinh^2 v}\right], \\ u(x,t) &= \ln\left[\frac{2(v_x^2+v_t^2)}{\sin^2 v}\right]. \end{aligned} \tag{4.4.14}$$

It goes without saying that the validity of the geometrically derived transformations (4.4.14) can be verified by their direct substitution in the Liouville equation (4.4.1). To this end, the following assertion proves useful.

If $\overset{(k)}{v}(x,t) \not\equiv \text{const}$ *is a solution of the Laplace equation* (4.4.2), *then the function* $\overset{(k+1)}{v}(x,t)$, *defined as*

$$\overset{(k+1)}{v}(x,t) = \ln\Big(\overset{(k)}{v}_x{}^2 + \overset{(k)}{v}_t{}^2\Big), \tag{4.4.15}$$

is also a solution of the Laplace equation (4.4.2).

Formula (4.4.15) expresses a transformation (or self-transformation) for the Laplace equation that is analogous to the Bäcklund transformation. The transformation (4.4.15) is the natural result of applying the obtained transformation (4.4.14) to the Laplace and Liouville equations.

From the point of view of the theory of functions of a complex variables, the result obtained above implies that, *given any analytic function* $f(z) = v(x,t) + iw(x,t)$, *one can always construct* (*by means of formulas* (4.4.14)) *solutions of the three types of the elliptic Liouville equation.*

Let us give the "gradient" form of the solutions $u(x,t)$ in (4.4.14):

$$\begin{aligned} u(x,t) &= \ln\Big[2\big(\operatorname{grad}(\ln v)\big)^2\Big], \\ u(x,t) &= \ln\Big[2\Big(\operatorname{grad}\Big(\ln\Big(\tanh\frac{v}{2}\Big)\Big)\Big)^2\Big], \\ u(x,t) &= \ln\Big[2\Big(\operatorname{grad}\Big(\ln\Big(\tan\frac{v}{2}\Big)\Big)\Big)^2\Big]. \end{aligned} \tag{4.4.16}$$

For work connected with the study of the Liouville equation (4.4.1) we refer the reader also to [15].

4.4.2 On the equation $\Delta_2 u^* = e^{-u^*}$

Side by side with the Liouville equation (4.4.1), in applications [16, 33] one encounters also the equation of close form

$$\Delta_2 u^* = e^{-u^*}, \tag{4.4.17}$$

which is taken by the simple "reflection" $u^{**} = -u^*$ into the equation

$$\Delta_2 u^{**} = -e^{u^*}. \tag{4.4.18}$$

Like equation (4.4.1), equation (4.4.18) can be interpreted as a relation that generates a metric of the form (4.4.5), but in the case of an a priori given constant *positive* curvature $K \equiv +1$.[5]

The construction of solutions of equation (4.4.18) will be carried out by the general geometric algorithm discussed in Subsection 4.4.1. Namely, to construct the solution $u^{**}(x,t)$ of (4.4.18) we take as supporting metric the metric (4.4.4), but with prescribed constant positive curvature $K \equiv +1$. Then such a metric will generate, instead of (4.4.3), the related auxiliary equation

$$(y^{**})_{\tau\tau} + y^{**} = 0, \quad y^{**} = y^{**}(\tau). \tag{4.4.19}$$

Let us use the substitution (4.4.6) to pass from the metric (4.4.4) (the semi-geodesic net $T^{\text{sg}}(\chi,\tau)$, curvature $K \equiv +1$) to the metric (4.4.5) (respectively, the isothermal net $T^{\text{is}}(x,t)$, curvature $K \equiv +1$).

Starting from the general solution of the equation (4.4.19),

$$y^{**}(\tau) = C_1 \sin\tau + C_2 \cos\tau, \quad C_1, C_2 = \text{const}, \tag{4.4.20}$$

we make the transition

$$T^{\text{sg}}(\chi,\tau) \longmapsto T^{\text{is}}(x,t).$$

Note that the relations (4.4.7) retain their form also in the case of curvature $K \equiv +1$ (up to the transformation of $y(\tau)$ into $y^{**}(\tau)$). Moreover, the function $[y^{**}(\tau)]^2$ is defined in terms of the solution y^{**} of equation (4.4.17), via the second relation in (4.4.6), as

$$[y^{**}(\tau(v))]^2 = \frac{1}{\cosh^2 v}.$$

Substituting this expression in (4.4.8) we finally construct the solution $u^*(x,t)$ (or the solution $u^{**}(x,t)$) from the solution $v(x,t)$ of the Laplace equation as

$$u^*(x,t) = \ln\left[\frac{\cosh^2 v}{2(v_x^2 + v_t^2)}\right]. \tag{4.4.21}$$

We will next discuss some important related issues arising in the study of the equation of Liouville type (4.4.1), (4.4.17), (4.4.18) at hand and the derived transformations (4.4.14)–(4.4.16) and (4.4.21).

[5] The Gaussian curvature $K \equiv +1$ is an "indicator" of spherical geometry.

4.4.3 Some applications connected with equations of Liouville type

1) *Centrally-symmetric metrics*. The well-known theoretical physics problem[6] of finding centrally-symmetric forms of two-dimensional metrics of constant curvature is connected with the search for "radial" solutions $u(r)$, $r = \sqrt{x^2 + y^2}$, of the Liouville equation (4.4.1) (for $K \equiv \text{const} < 0$) and of equation (4.4.17) (for $K \equiv \text{const} > 0$). The transformations (4.4.14) and (4.4.21) established above indicate that the search for such metrics relies on finding fundamental solutions $v(r)$ of the Laplace equation (4.4.2). Therefore, one can assert that for $K \equiv \text{const} < 0$ there exists three forms of centrally-symmetric metrics, while for $K \equiv \text{const} > 0$ there is only one such metric. It is interesting to note that the Bäcklund self-transformation (4.4.15) for the Laplace equation is the identity transformation on the "radial" solutions $v(r)$ of this equation.

2) *On problems of combustion theory.* The mathematical modeling of a number of problems of combustion theory, such as thermal explosion, forced auto-ignition, and others (which consider the thermal action of the surrounding medium on the reaction domain Ω) is connected with the study of initial-boundary value problems for the heat balance equation [16, 33]

$$\frac{\partial \vartheta}{\partial t} = \frac{1}{\delta}\Delta_2 \vartheta + e^{\vartheta},$$

where the quantity ϑ represents the temperature field in Ω. In particular, the fundamental problem of stationary theory (for $\vartheta_t \equiv 0$), which is "governed" by the Liouville-type equation of

$$\Delta_2 \vartheta_{\mathrm{ST}} + \delta e^{\vartheta_{\mathrm{ST}}} = 0,$$

is the investigation of the critical conditions, under which the problem under study is no longer solvable in the natural class of regular functions, which from the physical point of view corresponds to a forced explosion or auto-ignition (i.e., to *a discontinuity* (jump) of the solution ϑ_{ST}).

In this connection we remark that the relations (4.4.14)–(4.4.16) and (4.4.21) discussed above leave unchanged the domain Ω in which the problem for the Liouville-type equation (4.4.1), (4.4.17) and the corresponding problem for the Laplace equation (4.4.2) (with the corresponding nonlinear boundary conditions) are posed. For this reason, the possible singularities of the solution ϑ_{ST} come from the singularities of the right-hand sides in (4.4.14)–(4.4.16), (4.4.21). For example, the solution ϑ_{ST}, computed by means of the third formula in (4.4.14), is regular in the domain

$$\Omega_0\colon\ k\pi < v(x,t) < (k+1)\pi, \quad k \text{ an integer}.$$

That is to say, there are geometric constraints on the configuration of the domain $\Omega: \ \Omega = \Omega_0$ that must be satisfied in order for the evolution of the process to be regular. This agrees with the known results of physical

[6]Encountered, first of all, in the general theory of relativity.

investigations [16]. Moreover, the blow-up regime $|\theta_{\mathrm{ST}}| > M$, for all $M > 0$, corresponds exactly to the degeneration of the metric (4.4.5) that generates the Liouville-type equation when the discriminant $W[\theta]$ vanishes: $W[\theta] = 0$, and to the singularities that arise in the Liouville net on $\mathcal{M}_2$ $(K \equiv \pm 1)$.

3) *The multidimensional Liouville equation.* A formal generalization of the structure of the transformations (4.4.14), (4.4.21) allows us to guess a class of self-similar solutions (of a linear argument) for the multidimensional Liouville-type equation:

$$\Delta_n u = e^u, \tag{4.4.22}$$

$$\Delta_n \widetilde{u} = e^{-\widetilde{u}}, \tag{4.4.23}$$

where $\Delta_n = \dfrac{\partial^2}{\partial x_1^2} + \cdots + \dfrac{\partial^2}{\partial x_n^2}$, $\bar{x} = (x_1, \ldots, x_n)$.

The solutions of this class are given as follows:

for equation (4.4.22):

$$\begin{aligned} u(\bar{x}) &= \ln\left(\frac{2}{\alpha^2(\bar{x})}\right), \\ u(\bar{x}) &= \ln\left(\frac{2}{\sinh^2\alpha(\bar{x})}\right), \\ u(\bar{x}) &= \ln\left(\frac{2}{\sin^2\alpha(\bar{x})}\right); \end{aligned} \tag{4.4.24}$$

for equation (4.4.23):

$$\widetilde{u}(\bar{x}) = \ln\left(\frac{\cosh^2\alpha(\bar{x})}{2}\right), \tag{4.4.25}$$

where $\alpha(\bar{x}) = a_1x_1 + \cdots + a_nx_n$, $\sum\limits_{i=1}^{n} a_i^2 = 1$.

4.4.4 Example of "net-based" construction of "kink" type solutions of the sine-Gordon equation

Let us construct, applying the net method, a solution $u(x,t)$ of the sine-Gordon equation (4.1.7). The symmetry transformation

$$(x,t) \mapsto (x,-t)$$

takes (4.1.7) into an equation of the form

$$\bar{u}_{xt} = -\sin\bar{u}, \tag{4.4.26}$$

with

$$\bar{u}(x,t) = u(x,-t),$$
$$u(x,t) = \bar{u}(x,-t).$$

Equations (4.1.7) and (4.4.26) represent particular realizations of the Chebyshev equation (4.1.6) that is generated by the metric of the Chebyshev net. Specifically, equation (4.1.7) is generated by a pseudospherical metric of the form (4.1.5) (curvature $K \equiv -1$), while equation (4.4.26) is generated by a metric of the same form (4.1.5), but with an a priori prescribed constant positive curvature $K \equiv +1$.

To construct a solution $\bar{u}(x,t)$ of the equation (4.4.26) we turn to the auxiliary metric of curvature $K \equiv +1$, written in the semigeodesic coordinates (χ, τ):

$$ds^2 = (y^{**})^2(\tau) d\chi^2 + d\tau^2, \quad K(x,t) \equiv +1. \tag{4.4.27}$$

The metric (4.4.27) generates again equation (4.4.19), which has a general solution of the form

$$y^{**}(\tau) = A_1 \sin\tau + A_2 \cos\tau, \quad A_1, A_2 = \text{const}. \tag{4.4.28}$$

Setting $A_1 = 0$ and $A_2 = 1$ in (4.4.28), we select the particular solution

$$Y^{**}(\tau) = \cos\tau$$

and rewrite with it the metric (4.4.27):

$$ds^2 = \cos^2\tau d\chi^2 + d\tau^2. \tag{4.4.29}$$

The quadratic form (4.4.29) with curvature $K \equiv +1$ is reduced to a metric of the form (4.1.5), written in the coordinates of the Chebyshev net $\text{Cheb}(x,t)$ of the same curvature, by means of the substitution

$$\begin{aligned} x + t &= \chi, \\ x - t &= \int \frac{d\tau}{\sin\tau}. \end{aligned} \tag{4.4.30}$$

In this way we arrive at the metric

$$ds^2 = dx^2 + 2\cos 2\tau(x,t) dx dt + dt^2. \tag{4.4.31}$$

Comparing (4.4.31) with the classical Chebyshev metric (4.1.5), we find the solution $\bar{u}(x,t)$ of equation (4.4.26):

$$\bar{u}(x,t) = 2\tau(x,t). \tag{4.4.32}$$

The function $\tau(x,t)$ is calculated from the second relation in (4.4.30):

$$\tau(x,t) = 2\arctan e^{x-t}. \tag{4.4.33}$$

Correspondingly, turning to the original solution $u(x,t)$ of the sine-Gordon equation and using (4.4.32), (4.4.31), and (4.4.26), we obtain from (4.4.33) the expression

$$u(x,t) = 4\arctan e^{x+t}. \tag{4.4.34}$$

The solution (4.4.34) is a “kink”-type solution or one-soliton solution of the form (3.2.11) (of unit amplitude).

The examples given above show how the method of mutual transformation of nets on manifolds of constant curvature can be used to construct exact solutions of nonlinear differential equations.

4.5 Geometric generalizations of a series of model equations of mathematical physics

In this section we provide a list of G-equations that *generalize* a series of important – from the point of view of mathematical physics and applications – *nonlinear equations*, together with the metrics that generate them. Usually, partial differential equations are generalized by increasing the dimension of the differential operators they involve (Laplacians, d’Alembertians and so on), which essentially means that one considers physical models of higher dimensions. In our treatment here, the generalization of known $(1+1)$-equations will be done by means of introducing in the “process of generating” the equation (see § 4.1) an arbitrary curvature $K(x,t)$, which will be a priori prescribed for the generating metric. Such an approach allows us to preserve the form of the generating metric for the resulting G-equation (the same metric as for the original Λ^2-equation), and hence preserve the very type of the generating coordinate net on $\mathcal{M}_2$ associated with this equation. Overall, the approach relies on the application of unified methods of geometric investigation to the Λ^2-equation at hand (a nonlinear equation with constant coefficients), as well as to its generalization, the G-equation (a generalized analog with functional coefficients). On the other hand, the presence of an“additional” functional coefficient in the G-equation enables us, in the construction of the corresponding models, to exploit supplementary properties of the physical processes under study “governed” by that equation.

We next list a number of physically important *generalized equations* of contemporary mathematical physics and the metrics (of arbitrary curvature $K(x,t)$) that generate them. For each metric we indicated the type of the generating coordinate net – the unified geometric preimage of the Λ^2-equation and of the generalized G-equation corrresponding to it.

I. *Chebyshev equation* (*generalized sine-Gordon equation*):

$$u_{xt} = -K(x,t)\sin u(x,t),$$

generating metric:

$$ds^2 = dx^2 + 2\cos u(x,t)dxdt + dt^2$$

(*Chebyshev net*).

II. *Generalized Korteweg-de Vries equation* (KdV *G-equation*):

$$u_t = u_x + (1 + K(x,t) + 6u)u_x + u_{xxx},$$

generalized metric:

$$
\begin{aligned}
ds^2 &= [(1-u)^2 + \eta^2]dx^2 \\
&\quad + 2[(1-u)(-u_{xx} + \eta u_x - \eta^2 u - 2u^2 + \eta^2 + 2u) + \eta(\eta^3 + 2\eta u - 2u_x)]dxdt \\
&\quad + [(-u_{xx} + \eta u_x - \eta^2 u - 2u^2 + \eta^2 + 2u)^2 + (\eta^3 + 2\eta u - 2u_x)^2]dt^2, \quad \eta = \text{const.}
\end{aligned}
$$

III. *Generalized modified Korteweg-de Vries equation* (MKdV *G-equation*):

$$u_t = \Big(1 + K(x,t) + \frac{3}{2}\,u^2\Big)u_x + u_{xxx},$$

generating metric:

$$ds^2 = \eta^2 dx^2 + 2\eta\Big(\eta\frac{u^2}{2} + \eta^3\Big)dxdt + \Big[\eta^2 u_x^2 + \Big(\eta\frac{u^2}{2} + \eta^3\Big)^2\Big]dt^2, \quad \eta = \text{const}$$

(MKdV *net*).

IV. *Generalized Burgers equation* (*Burgers G-equation*):

$$u_t = (1 + K(x,t) + u)\cdot u_x + u_{xx},$$

generating metric:

$$
\begin{aligned}
ds^2 &= \Big(\frac{u^2}{4} + \eta^2\Big)dx^2 + 2\left[\eta^2\frac{u}{2} + \frac{u}{4}\Big(\frac{u^2}{2} + u_x\Big)\right]dxdt \\
&\quad + \left[\Big(\frac{u^2}{4} + \frac{u_x}{2}\Big)^2 + \eta^2\frac{u^2}{4}\right]dt^2, \quad \eta = \text{const}
\end{aligned}
$$

(*Burgers net*).

V. *Generalized Liouville equation* (*G-Liouville equation*):

a) elliptic:

$$\Delta_2 u = -K(x,t)\,e^u,$$

generating metric:

$$ds^2 = \frac{e^u}{2}\,(dx^2 + dt^2)$$

(*elliptic Liouville net – isothermal coordinate net*).

b) hyperbolic:

$$u_{xt} = -K(x,t)\,e^u,$$

generating metric:

$$ds^2 = (u_x^2 + \eta^2)dx^2 + 2\eta e^u dxdt + e^{2u}dt^2$$

(*hyperbolic Liouville set*).

VI. *Generalized* sinh-*Gordon equation* (sinh-*Gordon G-equation*):

a) elliptic:

$$\Delta_2 u = -K(x,t)\sinh u,$$

generating metric:

$$ds^2 = \cosh^2\frac{u}{2}\,dx^2 + \sinh^2\frac{u}{2}\,dt^2.$$

b) hyperbolic:

$$u_{xt} = -K(x,t)\sinh u,$$

generating metric:

$$ds^2 = (u_x^2 + \eta^2)dx^2 + 2\eta\cosh u\,dxdt + \cosh^2 u\,dt^2.$$

VII. *Generalized equation generated by a "semi-geodesic" metric*:

$$y_{xx} + K(x,t)y(x) = 0,$$

generating metric:

$$ds^2 = dx^2 + y^2(x)dt^2$$

(*semi-geodesic coordinate net*).

The geometric class of the equations listed above awaits addition of new model equations of mathematical physics together with the generating metrics recovered for them.