Numerical Solution of Fluid-Structure Interaction by the Space-Time Discontinuous Galerkin Method

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Abstract This paper is devoted to the numerical solution of the interaction of compressible viscous flow with elastic structures. The flow in a time-dependent domain is described by the compressible Navier-Stokes equations written in the ALE formulation and the deformation of elastic structures is described by the dynamic linear elasticity system. For each individual problem we employ the discretization by the space-time discontinuous Galerkin finite element method (ST-DGM). The flow and elasticity problems are coupled via transmission conditions. The developed method is tested by numerical experiments.

1 Formulation of the Problem

1.1 Flow Problem

We are concerned with the problem of compressible flow in a time-dependent bounded domain $\Omega_t \subset I\!R^2$ with $t \in [0, T]$. The boundary of Ω_t is formed by

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J. Fuhrmann et al. (eds.), *Finite Volumes for Complex Applications VII - Elliptic, Parabolic and Hyperbolic Problems*, Springer Proceedings in Mathematics & Statistics 78,
DOI: 10.1007/978-3-319-05591-6_56,

three disjoint parts: $\partial\Omega_t = \Gamma_I \cup \Gamma_O \cup \Gamma_{W_t}$, where Γ_I is the inlet, Γ_O is the outlet and Γ_{W_t} represents impermeable time-dependent walls.

The time dependence of the domain Ω_t is taken into account with the aid of the *Arbitrary Lagrangian-Eulerian* (ALE) method (see, e.g., [4]). It is based on a regular one-to-one ALE mapping of the reference configuration Ω_0 onto the current configuration Ω_t : $\mathcal{A}_t : \bar{\Omega}_0 \longrightarrow \bar{\Omega}_t$, i.e. $\boldsymbol{X} \in \bar{\Omega}_0 \longmapsto \boldsymbol{x} = \boldsymbol{x}(\boldsymbol{X}, t) = \mathcal{A}_t(\boldsymbol{X}) \in \bar{\Omega}_t$. Further, we define the domain velocity $\tilde{z}(\boldsymbol{X}, t) = \frac{\partial}{\partial t}\mathcal{A}_t(\boldsymbol{X}),\ t \in [0, T],\ \boldsymbol{X} \in \Omega_0,\ z(\boldsymbol{x}, t) = \tilde{z}(\mathcal{A}_t^{-1}(\boldsymbol{x}), t),\ t \in [0, T],\ \boldsymbol{x} \in \Omega_t$ and the ALE derivative of the state vector function $\boldsymbol{w} = \boldsymbol{w}(\boldsymbol{x}, t)$ defined for $\boldsymbol{x} \in \Omega_t$ and $t \in [0, T]$: $\frac{D^{\mathcal{A}}}{Dt}\boldsymbol{w}(\boldsymbol{x}, t) = \frac{\partial \tilde{\boldsymbol{w}}}{\partial t}(\boldsymbol{X}, t),\ \tilde{\boldsymbol{w}}(\boldsymbol{X}, t) = \boldsymbol{w}(\mathcal{A}_t(\boldsymbol{X}), t),\ \boldsymbol{X} \in \Omega_0,\ \boldsymbol{x} = \mathcal{A}_t(\boldsymbol{X})$. Then the continuity equation, the Navier-Stokes equations and the energy equation can be written in the ALE form

$$\frac{D^{\mathcal{A}}\boldsymbol{w}}{Dt} + \sum_{s=1}^{2} \frac{\partial \boldsymbol{g}_s(\boldsymbol{w})}{\partial x_s} + \boldsymbol{w}\mathrm{div}z = \sum_{s=1}^{2} \frac{\partial \boldsymbol{R}_s(\boldsymbol{w}, \nabla \boldsymbol{w})}{\partial x_s}, \tag{1}$$

where $\boldsymbol{w} = (\rho, \rho v_1, \rho v_2, E)^T \in I\!R^4$, $\boldsymbol{g}_s(\boldsymbol{w}) = \boldsymbol{f}(\boldsymbol{w})_s - z_s\boldsymbol{w},\ \boldsymbol{f}_s = (\rho v_s, \rho v_1 v_s + \delta_{1s} p, \rho v_2 v_s + \ \delta_{2s} p, (E + p)v_s)^T,\quad \boldsymbol{R}_s(\boldsymbol{w}, \nabla\boldsymbol{w}) = (0, \tau_{s1}^V, \tau_{s2}^V, \tau_{s1}^V v_1 + \tau_{s2}^V v_2 + k\frac{\partial\theta}{\partial x_s})^T,\ s = 1, 2,\ \tau_{ij}^V = \lambda\delta_{ij}\mathrm{div}\boldsymbol{v} + 2\mu d_{ij}(\boldsymbol{v}), d_{ij}(\boldsymbol{v}) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right),\ i, j = 1, 2$. We have $\boldsymbol{R}_s(\boldsymbol{w}, \nabla\boldsymbol{w}) = \sum_{k=1}^{2} \mathbb{K}_{s,k}(\boldsymbol{w})\frac{\partial \boldsymbol{w}}{\partial x_k}$, where $\mathbb{K}_{s,k}(\boldsymbol{w})$ are 4×4 matrices depending on $\boldsymbol{w}$, and $\boldsymbol{f}_s(\boldsymbol{w}) = \mathbb{A}(\boldsymbol{w})\boldsymbol{w}$ with $\mathbb{A}(\boldsymbol{w}) = D\boldsymbol{f}_s(\boldsymbol{w})/D\boldsymbol{w}$.

The following notation is used: ρ—fluid density, p—pressure, E—total energy, $\boldsymbol{v} = (v_1, v_2)$—velocity vector, θ—absolute temperature, $c_v > 0$—specific heat at constant volume, $\gamma > 1$—Poisson adiabatic constant, $\mu > 0, \lambda = -2\mu/3$—viscosity coefficients, $k > 0$—heat conduction coefficient, τ_{ij}^V—components of the viscous part of the stress tensor. System (1) is completed by the thermodynamical relations $p = (\gamma - 1)\left(E - \rho\frac{|\boldsymbol{v}|^2}{2}\right),\quad \theta = \frac{1}{c_v}\left(\frac{E}{\rho} - \frac{|\boldsymbol{v}|^2}{2}\right)$ and equipped with the initial condition $\boldsymbol{w}(\boldsymbol{x}, 0) = \boldsymbol{w}^0(\boldsymbol{x}),\ \boldsymbol{x} \in \Omega_0$ and the boundary conditions:
$\rho = \rho_D,\quad \boldsymbol{v} = \boldsymbol{v}_D,\quad \sum_{j=1}^{2}\left(\sum_{i=1}^{2} \tau_{ij}^V n_i\right) v_j + k\frac{\partial\theta}{\partial \boldsymbol{n}} = 0$ on the inlet Γ_I,
$\boldsymbol{v} = z_D(t) =$ velocity of a moving wall, $\quad \frac{\partial\theta}{\partial \boldsymbol{n}} = 0$, on the moving wall Γ_{W_t},
$\sum_{j=1}^{2} \tau_{ij}^V n_j = 0,\quad \frac{\partial\theta}{\partial \boldsymbol{n}} = 0,\quad i = 1, 2$, on the outlet Γ_O,
with prescribed data $\rho_D,\ \boldsymbol{v}_D,\ z_D$. By $\boldsymbol{n}$ we denote the unit outer normal.

1.2 Elasticity Problem

We consider an elastic body $\Omega^b \subset I\!R^2$, which has a common boundary Γ_N^b with the reference domain Ω_0 occupied by the fluid at the initial time. Further, the boundary of Ω^b is formed by two disjoint parts $\partial\Omega^b = \Gamma_N^b \cup \Gamma_D^b,\ \Gamma_N^b \subset \Gamma_{W_0}$ and Γ_D^b is a fixed part of the boundary. Using the notation of the displacement of the body $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{X}, t),\ \boldsymbol{X} \in \Omega^b,\ t \in (0, T)$ we can write the equations describing the deformation of the elastic body Ω^b in the form

$$\rho^b \frac{\partial^2 \boldsymbol{u}}{\partial t^2} + c_M \rho^b \frac{\partial \boldsymbol{u}}{\partial t} - div\, \boldsymbol{\sigma}(\boldsymbol{u}) - c_K \frac{\partial}{\partial t} div\, \boldsymbol{\sigma}(\boldsymbol{u}) = \boldsymbol{f} \quad \text{in } \Omega^b \times (0,T), \tag{2}$$

$$\boldsymbol{u} = \boldsymbol{u}_D \quad \text{in } \Gamma_D^b \times (0,T), \quad \boldsymbol{\sigma}(\boldsymbol{u}) \cdot \boldsymbol{n} = \boldsymbol{g}_N \quad \text{in } \Gamma_N^b \times (0,T), \tag{3}$$

$$\boldsymbol{u}(x,0) = \boldsymbol{u}_0(x), \quad x \in \Omega^b, \quad \frac{\partial \boldsymbol{u}}{\partial t}(x,0) = z_0(x), \quad x \in \Omega^b. \tag{4}$$

Here $\boldsymbol{\sigma}(\boldsymbol{u}) = \{\sigma_{ij}\}_{i,j=1}^2$, $\sigma_{ij} = \lambda^b \text{div}\boldsymbol{u}\delta_{ij} + 2\mu^b e_{ij}^b(\boldsymbol{u})$ with $e_{ij}^b(\boldsymbol{u}) = (\partial u_i/\partial x_j + \partial u_j/\partial x_i)/2$. Further, $\boldsymbol{f} : \Omega^b \times (0,T) \to \mathbb{R}^2$—outer volume force, $\boldsymbol{u}_D : \Gamma_D^b \times (0,T) \to \mathbb{R}^2$—boundary displacement, $\boldsymbol{g}_N : \Gamma_N^b \times (0,T) \to \mathbb{R}^2$—boundary normal stress, $\boldsymbol{u}_0 : \Omega^b \to \mathbb{R}^2$—initial displacement, $z_0 : \Omega^b \to \mathbb{R}^2$—initial deformation velocity and $\rho^b > 0$—material density are given functions. The expressions $c_M \rho^b \frac{\partial \boldsymbol{u}}{\partial t}$ and $c_K \frac{\partial}{\partial t} div\, \boldsymbol{\sigma}(\boldsymbol{u})$ represent the damping terms, with $c_M, c_K \geq 0$.

The flow and structural problems are coupled by the transmission conditions

$$\boldsymbol{v} = \frac{\partial \boldsymbol{u}}{\partial t}, \quad \sum_{j=1}^{2} \sigma_{ij}(\boldsymbol{X},t) n_j(\boldsymbol{X}) = -\sum_{j=1}^{2} \tau_{ij}^f(\boldsymbol{x},t) n_j(\boldsymbol{X}), \quad i = 1,2, \tag{5}$$

$$\boldsymbol{X} \in \Gamma_N^b, \; \boldsymbol{x} = \boldsymbol{X} + \boldsymbol{u}(\boldsymbol{X},t), \; \tau_{ij}^f = -p\,\delta_{ij} + \tau_{ij}^V.$$

2 Discrete Problem

2.1 Discretization of the Flow Problem

The problem will be discretized by the space-time discontinuous Galerkin method (ST-DGM). We construct a polygonal approximation Ω_{ht} of the domain Ω_t. By $\mathcal{T}_{ht}$ we denote a partition of the closure $\overline{\Omega}_{ht}$ of the domain Ω_t into a finite number of closed triangles K with mutually disjoint interiors such that $\overline{\Omega}_{ht} = \bigcup_{K \in \mathcal{T}_{ht}} K$.

By $\mathcal{F}_h, \mathcal{F}_h^B, \mathcal{F}_h^I$ we denote the systems of all faces of all elements $K \in \mathcal{T}_{ht}$, boundary faces and inner faces, respectively. Further, we introduce the set of "Dirichlet" boundary faces $\mathcal{F}_h^D = \{\Gamma \in \mathcal{F}_h^B$; a Dirichlet condition is prescribed on $\Gamma\}$. Each face Γ is associated with a unit normal $\boldsymbol{n}_\Gamma$, which has the same orientation as the outer normal on $\Gamma \in \mathcal{F}_h^B$. We set h_Γ = length of $\Gamma \in \mathcal{F}_h$.

We introduce the space of piecewise polynomial functions $\boldsymbol{S}_{ht}^r = \{v; v|_K \in P_r(K)\ \forall K \in \mathcal{T}_{ht}\}^4$, where $r > 0$ is an integer and $P_r(K)$ denotes the space of all polynomials on K of degree $\leq r$. A function $\varphi \in \boldsymbol{S}_{ht}^r$ is, in general, discontinuous on interfaces $\Gamma \in \mathcal{F}_h^I$. By $\varphi_\Gamma^{(L)}$ and $\varphi_\Gamma^{(R)}$ we denote the values of $\varphi \in \boldsymbol{S}_{ht}^r$ on Γ from the side of the element $K_\Gamma^{(L)}$ and $K_\Gamma^{(R)}$ adjacent to Γ lying in the opposite direction to $\boldsymbol{n}_\Gamma$ and in the direction of $\boldsymbol{n}_\Gamma$, respectively. Then we set $\langle\varphi\rangle_\Gamma = (\varphi_\Gamma^{(R)} + \varphi_\Gamma^{(L)})/2$ and $[\varphi]_\Gamma = \varphi_\Gamma^{(L)} - \varphi_\Gamma^{(R)}$.

The discrete problem is derived in the following way: We multiply system (1) by a test function $\varphi_h \in \boldsymbol{S}_{ht}^r$, integrate over $K \in \mathcal{T}_{ht}$, apply Green's theorem, sum over all elements $K \in \mathcal{T}_{ht}$, use the concept of the numerical flux and introduce suitable terms mutually vanishing for a regular exact solution and linearize the resulting forms (see, e.g. [1, 3]). In this way we get the following forms:

$$\hat{a}_h(\overline{\boldsymbol{w}}_h, \boldsymbol{w}_h, \varphi_h, t) = \sum_{K \in \mathcal{T}_{ht}} \int_K \sum_{s=1}^{2} \sum_{k=1}^{2} \mathbb{K}_{s,k}(\overline{\boldsymbol{w}}_h) \frac{\partial \boldsymbol{w}_h}{\partial x_k} \cdot \frac{\partial \varphi_h}{\partial x_s} \,\mathrm{d}\boldsymbol{x} \tag{6}$$

$$- \sum_{\Gamma \in \mathcal{F}_{ht}^I} \int_\Gamma \sum_{s=1}^{2} \left\langle \sum_{k=1}^{2} \mathbb{K}_{s,k}(\overline{\boldsymbol{w}}_h) \frac{\partial \boldsymbol{w}_h}{\partial x_k} \right\rangle (\boldsymbol{n}_\Gamma)_s \cdot [\varphi_h] \,\mathrm{d}S$$

$$- \sum_{\Gamma \in \mathcal{F}_{ht}^D} \int_\Gamma \sum_{s=1}^{2} \sum_{k=1}^{2} \mathbb{K}_{s,k}(\overline{\boldsymbol{w}}_h) \frac{\partial \boldsymbol{w}_h}{\partial x_k} (\boldsymbol{n}_\Gamma)_s \cdot \varphi_h \,\mathrm{d}S$$

$$- \Theta \sum_{\Gamma \in \mathcal{F}_{ht}^I} \int_\Gamma \sum_{s=1}^{2} \left\langle \sum_{k=1}^{2} \mathbb{K}_{k,s}^T(\overline{\boldsymbol{w}}_h) \frac{\partial \varphi_h}{\partial x_k} \right\rangle (\boldsymbol{n}_\Gamma)_s \cdot [\boldsymbol{w}_h] \,\mathrm{d}S$$

$$- \Theta \sum_{\Gamma \in \mathcal{F}_{ht}^D} \int_\Gamma \sum_{s=1}^{2} \sum_{k=1}^{2} \mathbb{K}_{k,s}^T(\overline{\boldsymbol{w}}_h) \frac{\partial \varphi_h}{\partial x_k} (\boldsymbol{n}_\Gamma)_s \cdot \boldsymbol{w}_h \,\mathrm{d}S,$$

$$d_h(\boldsymbol{w}_h, \varphi_h, t) = \sum_{K \in \mathcal{T}_{ht}} \int_K (\boldsymbol{w}_h \cdot \varphi_h) \,\mathrm{div} z \,\mathrm{d}x, \tag{7}$$

$$J_h(\boldsymbol{w}_h, \varphi_h, t) = \sum_{\Gamma \in \mathcal{F}_{ht}^I} \int_\Gamma \frac{\mu C_W}{h_\Gamma} [\boldsymbol{w}_h] \cdot [\varphi_h] \,\mathrm{d}S + \sum_{\Gamma \in \mathcal{F}_{ht}^D} \int_\Gamma \frac{\mu C_W}{h_\Gamma} \boldsymbol{w}_h \cdot \varphi_h \,\mathrm{d}S, \tag{8}$$

$$\ell_h(\boldsymbol{w}_h, \varphi_h, t) = \sum_{\Gamma \in \mathcal{F}_{ht}^D} \int_\Gamma \frac{\mu C_W}{h_\Gamma} \boldsymbol{w}_B \cdot \varphi_h \,\mathrm{d}S \tag{9}$$

$$- \Theta \sum_{\Gamma \in \mathcal{F}_{ht}^D} \int_\Gamma \sum_{k=1}^{2} \mathbb{K}_{k,s}^T(\overline{\boldsymbol{w}}_h) \frac{\partial \varphi_h}{\partial x_k} (\boldsymbol{n}_\Gamma)_s \cdot \boldsymbol{w}_B \,\mathrm{d}S,$$

$$\hat{b}_h(\overline{\boldsymbol{w}}_h, \boldsymbol{w}_h, \varphi_h, t) = \tag{10}$$

$$
\begin{aligned}
&- \sum_{K \in \mathcal{T}_{ht_{k+1}}} \int_K \sum_{s=1}^{2} ((\mathbb{A}_s(\overline{\boldsymbol{w}}_h(x)) - z_s(x)\mathbb{I})\boldsymbol{w}_h(x)) \cdot \frac{\partial \varphi_h(x)}{\partial x_s} dx \\
&+ \sum_{\Gamma \in \mathcal{F}_{ht}^I} \int_\Gamma \left(\mathbb{P}_g^+ \left(\langle \overline{\boldsymbol{w}}_h \rangle_\Gamma, \boldsymbol{n}_\Gamma\right) \boldsymbol{w}_h^{(L)} + \mathbb{P}_g^- \left(\langle \overline{\boldsymbol{w}}_h \rangle_\Gamma, \boldsymbol{n}_\Gamma\right) \boldsymbol{w}_h^{(R)}\right) \cdot [\varphi_h] \, \mathrm{d}S \\
&+ \sum_{\Gamma \in \mathcal{F}_{ht}^B} \int_\Gamma \left(\mathbb{P}_g^+ \left(\langle \overline{\boldsymbol{w}}_h \rangle_\Gamma, \boldsymbol{n}_\Gamma\right) \boldsymbol{w}_h^{(L)} + \mathbb{P}_g^- \left(\langle \overline{\boldsymbol{w}}_h \rangle_\Gamma, \boldsymbol{n}_\Gamma\right) \overline{\boldsymbol{w}}_h^{(R)}\right) \cdot \varphi_h \, \mathrm{d}S,
\end{aligned}
$$

$C_W > 0$ is a sufficiently large constant. We set $\Theta = 1$ or $\Theta = 0$ or $\Theta = -1$ and get the so-called symmetric version (SIPG) or incomplete version (IIPG) or nonsymmetric version (NIPG), respectively, of the discretization of viscous terms. The symbols $\mathbb{P}_g^+(\boldsymbol{w}, \boldsymbol{n})$ and $\mathbb{P}_g^-(\boldsymbol{w}, \boldsymbol{n})$ denote the "positive" and "negative" parts of the matrix $\mathbb{P}_g(\boldsymbol{w}, \boldsymbol{n}) = \sum_{s=1}^2 (\mathbb{A}_s(\boldsymbol{w}) - z_s\mathbb{I})n_s$ defined, e.g., in [2]. The boundary state $\boldsymbol{w}_B$ is defined on the basis of the prescribed Dirichlet boundary conditions and extrapolation.

For the space-time discretization we consider a partition $0 = t_0 < t_1 < \ldots < t_M = T$ of the time interval $[0, T]$ and denote $I_m = (t_{m-1}, t_m)$, $\tau_m = t_m - t_{m-1}$, for $m = 1, \ldots, M$. We define the space $\mathbf{S}_{h\tau}^{rq} = \{\phi\,;\, \phi|_{I_m} = \sum_{i=0}^q \zeta_i \phi_i$, where $\phi_i \in S_{ht}^r$, $\zeta_i \in P^q(I_m)\}^2$ with integers $r, q \geq 1$. $P^q(I_m)$ denotes the space of all polynomials in t on I_m of degree $\leq q$. For $\varphi \in \mathbf{S}_{h\tau}^{rq}$ we set $\varphi_m^\pm = \varphi(t_m^\pm) = \lim_{t \to t_m \pm} \varphi(t)$, $\{\varphi\}_m = \varphi_m^+ - \varphi_m^-$. The initial state $\boldsymbol{w}_{h\tau}(0-) \in \mathbf{S}_{h0}^p$ is defined as the $L^2(\Omega_{h0})$-projection of $\boldsymbol{w}^0$ on $\mathbf{S}_{h0}^r$. Moreover, we introduce the prolongation $\overline{\boldsymbol{w}}_{h\tau}(t)$ of $\boldsymbol{w}_{h\tau}|_{I_{m-1}}$ on the interval I_m. By $(\cdot, \cdot)_t$ we denote the $L^2(\Omega_{ht})$-scalar product.

Now the *space-time DG approximate solution* is defined as a function $\boldsymbol{w}_{h\tau} \in \mathbf{S}_{h\tau}^{rq}$ satisfying the following relation for $m = 1, \ldots, M$:

$$
\begin{aligned}
&\int_{I_m} \left(\left(\frac{D^{\mathcal{A}} \boldsymbol{w}_{h\tau}}{Dt}(t), \varphi_{h\tau} \right)_t + \hat{a}_h(\overline{\boldsymbol{w}}_{h\tau}, \boldsymbol{w}_{h\tau}, \varphi_{h\tau}, t) \right) \mathrm{d}t \qquad (11) \\
&+ \int_{I_m} \left(\hat{b}_h(\overline{\boldsymbol{w}}_{h\tau}, \boldsymbol{w}_{h\tau}, \varphi_{h\tau}, t) + \int_{I_m} J_h(\boldsymbol{w}_{h\tau}, \varphi_{h\tau}, t) \right) \mathrm{d}t \\
&+ (\{\boldsymbol{w}_{h\tau}\}_{m-1}, \varphi_{h\tau}(t_{m-1}+)) = \int_{I_m} \ell_h(\boldsymbol{w}_{hD}, \varphi_{h\tau}, t) \, \mathrm{d}t, \quad \forall \varphi_{h\tau} \in \mathbf{S}_{h\tau}^{rq}.
\end{aligned}
$$

2.2 Discretization of the Elasticity Problem

The elasticity problem will also be discretized by the ST-DGM. To this end, the problem is reformulated as a couple of equations of the first order in time: find functions $\boldsymbol{u}$ and $\boldsymbol{z} : \Omega^b \times [0, T] \to \mathbb{R}^2$ such that

$$\rho^b \frac{\partial \boldsymbol{z}}{\partial t} + c\rho^b \boldsymbol{z} - div\, \boldsymbol{\sigma}(\boldsymbol{u}) = \boldsymbol{f} \quad \text{in } \Omega^b \times (0,T), \tag{12}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} - \boldsymbol{z} = 0 \quad \text{in } \Omega^b \times (0,T), \tag{13}$$

$$\boldsymbol{u} = \boldsymbol{u}_D \quad \text{in } \Gamma_D^b \times (0,T), \quad \boldsymbol{\sigma}(\boldsymbol{u}) \cdot \boldsymbol{n} = \boldsymbol{g}_N \quad \text{in } \Gamma_N^b \times (0,T), \tag{14}$$

$$\boldsymbol{u}(x,0) = \boldsymbol{u}_0(x), \quad \boldsymbol{z}(x,0) = z_0(x), \quad x \in \Omega^b. \tag{15}$$

Now we proceed in a similar way as in Sect. 2.1. By Ω_h^b we denote a polygonal approximation of the domain Ω^b. The sets $\Gamma_{Dh}^b, \Gamma_{Nh}^b \subset \partial\Omega_h^b$ will approximate Γ_D^b and Γ_N^b. Let $\mathcal{T}_h^b$ be a partition of the closure $\overline{\Omega}_h^b$ We define the finite dimensional space $\boldsymbol{S}_{hs}^b = \left\{v \in L^2(\Omega_h^b);\, v|_K \in P_s(K),\, K \in \mathcal{T}_h^b\right\}^2$, where $s > 0$ is an integer. By $\mathcal{F}_h^b, \mathcal{F}_h^{bD}, \mathcal{F}_h^{bN}, \mathcal{F}_h^{bI}$ we denote the system of all faces of all elements $K \in \mathcal{T}_h^b$, boundary Dirichlet, Neumann faces and inner faces. If we introduce the forms

$$\begin{aligned} a_h^b(\boldsymbol{u}, \boldsymbol{v}) = & \sum_{K \in \mathcal{T}_h^b} \int_K \boldsymbol{\sigma}(\boldsymbol{u}) : \boldsymbol{e}(\boldsymbol{v})\, \mathrm{d}x - \sum_{\Gamma \in \mathcal{F}_h^{bI}} \int_\Gamma (\langle \boldsymbol{\sigma}(\boldsymbol{u}) \rangle \cdot \boldsymbol{n}) \cdot [\boldsymbol{v}]\, \mathrm{d}S \\ & - \sum_{\Gamma \in \mathcal{F}_h^{bD}} \int_\Gamma (\boldsymbol{\sigma}(\boldsymbol{u}) \cdot \boldsymbol{n}) \cdot \boldsymbol{v}\, \mathrm{d}S - \Theta \sum_{\Gamma \in \mathcal{F}_h^{bI}} \int_\Gamma (\langle \boldsymbol{\sigma}(\boldsymbol{v}) \rangle \cdot \boldsymbol{n}) \cdot [\boldsymbol{u}]\, \mathrm{d}S \\ & - \Theta \sum_{\Gamma \in \mathcal{F}_h^{bD}} \int_\Gamma (\boldsymbol{\sigma}(\boldsymbol{v}) \cdot \boldsymbol{n}) \cdot \boldsymbol{u}\, \mathrm{d}S, \end{aligned} \tag{16}$$

$$J_h^b(\boldsymbol{u}, \boldsymbol{v}) = \sum_{\Gamma \in \mathcal{F}_h^{bI}} \int_\Gamma \frac{C_W^b}{h_\Gamma} [\boldsymbol{u}] \cdot [\boldsymbol{v}]\, \mathrm{d}S + \sum_{\Gamma \in \mathcal{F}_h^{bD}} \int_\Gamma \frac{C_W^b}{h_\Gamma} \boldsymbol{u} \cdot \boldsymbol{v}\, \mathrm{d}S, \tag{17}$$

$$\begin{aligned} \ell_h^b(\boldsymbol{v})(t) = & \sum_{K \in \mathcal{T}_h^b} \int_K \boldsymbol{f}(t) \cdot \boldsymbol{v}\, \mathrm{d}x + \sum_{\Gamma \in \mathcal{F}_h^{bN}} \int_\Gamma \boldsymbol{g}_N(t) \cdot \boldsymbol{v}\, \mathrm{d}S \\ & - \Theta \sum_{\Gamma \in \mathcal{F}_h^{bD}} \int_\Gamma (\boldsymbol{\sigma}(\boldsymbol{v}) \cdot \boldsymbol{n}) \cdot \boldsymbol{u}_D(t)\, \mathrm{d}S + \sum_{\Gamma \in \mathcal{F}_h^{bD}} \int_\Gamma \frac{C_W^b}{h_\Gamma} \boldsymbol{u}_D(t) \cdot \boldsymbol{v}\, \mathrm{d}S, \end{aligned} \tag{18}$$

$$(\boldsymbol{u}, \boldsymbol{v})_{\Omega_h^b} = \int_{\Omega_h^b} \boldsymbol{u} \cdot \boldsymbol{v}\, \mathrm{d}x = \sum_{K \in \mathcal{T}_h^b} \int_K \boldsymbol{u} \cdot \boldsymbol{v}\, \mathrm{d}x, \tag{19}$$

where $C_W^b > 0$ is a sufficiently large constant, $\Theta = 1$, $\Theta = 0$ or $\Theta = -1$ and $\boldsymbol{S}_{h\tau}^{b,sq} = \left\{v \in L^2(\Omega_h^b \times (0,T));\, v|_{I_m} = \sum_{i=0}^q t^i \varphi_i \text{ with } \varphi_i \in S_{hs}^b,\ m = 1, \ldots, M\right\}^2$, the ST-DG approximate solution can be defined as a couple $\boldsymbol{u}_{h\tau}, \boldsymbol{z}_{h\tau} \in \boldsymbol{S}_{h\tau}^{b,sq}$ such that

$$
\begin{aligned}
&\text{(a)} \int_{I_m} \left(\rho^b \left(\frac{\partial z_{h\tau}}{\partial t}, \boldsymbol{v}_{h\tau}\right)_{\Omega_h^b} + C\left(\rho^b z_{h\tau}, \boldsymbol{v}_{h\tau}\right)_{\Omega_h^b} + a_h^b(\boldsymbol{u}_{h\tau}, \boldsymbol{v}_{h\tau}) \right. \qquad (20)\\
&\left. + J_h^b(\boldsymbol{u}_{h\tau}, \boldsymbol{v}_{h\tau})\right) \mathrm{d}t + (\{\boldsymbol{u}_{h\tau}\}_{m-1}, \boldsymbol{v}_{h\tau}(t_{m-1}+))_{\Omega_h^b} \\
&= \int_{I_m} \ell(\boldsymbol{v}_{h\tau})\, \mathrm{d}t \ \forall \boldsymbol{v}_{h\tau} \in \boldsymbol{S}_{h\tau}^{b,sq}, \\
&\text{(b)} \int_{I_m} \left(\left(\frac{\partial \boldsymbol{u}_{h\tau}}{\partial t}, \boldsymbol{w}_{h\tau} \right)_{\Omega_h^b} - (z_{h\tau}, \boldsymbol{w}_{h\tau})_{\Omega_h^b} \right) \mathrm{d}t \\
&+ (\{\boldsymbol{u}_{h\tau}\}_{m-1}, \boldsymbol{w}_{h\tau}(t_{m-1}+))_{\Omega_h^b} = 0 \quad \forall \boldsymbol{w}_{h\tau} \in \boldsymbol{S}_{h\tau}^{b,sq}, \\
&\qquad m = 1, \ldots, M.
\end{aligned}
$$

The initial states $\boldsymbol{u}_h(0-), z_h(0-) \in \boldsymbol{S}_{hs}^b$ are defined by $(\boldsymbol{u}_h(0-), \boldsymbol{v}_h)_{\Omega_h^b} = (\boldsymbol{u}^0, \boldsymbol{v}_h)_{\Omega_h^b}$), $(z_h(0-), \boldsymbol{v}_h)_{\Omega_h^b} = (z^0, \boldsymbol{v}_h)_{\Omega_h^b}$ for all $\boldsymbol{v}_h \in \boldsymbol{S}_{hs}^b$.

In the FSI problem the coupling of the discrete flow problem (11) and structural problem (20) are realized via the discrete version of transmission conditions (5). The coupled problem is solved with the aid of the following coupling procedure.

1. Assume that the approximate solution of the flow problem on the time level t_k is known as well as the deformation of the structure $\boldsymbol{u}_{h,k}$.
2. Set $\boldsymbol{u}_{h,k+1}^0 := \boldsymbol{u}_{h,k},\ l := 1$ and apply the iterative process:
   a. Compute the stress tensor τ_{ij}^f and the aerodynamical force acting on the structure and transform it to the interface Γ_{Nh}^b.
   b. Solve the elasticity problem, compute the deformation $\boldsymbol{u}_{h,k+1}^l$ at time t_{k+1} and approximate the domain $\Omega_{ht_{k+1}}^l$.
   c. Determine the ALE mapping $\mathcal{A}_{t_{k+1}h}^l$ and approximate the domain velocity $z_{h,k+1}^l$.
   d. Solve the flow problem on the approximation of $\Omega_{ht_{k+1}}^l$.
   e. If the variation of the displacement $\boldsymbol{u}_{h,k+1}^l$ and $\boldsymbol{u}_{h,k+1}^{l-1}$ is larger than the prescribed tolerance, go to (a) and $l := l + 1$. Else $k := k + 1$ and goto (2).

This represents the so-called strong coupling. If in the step (e) we set $k := k + 1$ and go to (2) already in the case when $l = 1$, then we get the weak (loose) coupling.

3 Numerical Results

We consider a 2D model of gas flow past an elastic airfoil. For testing our method we assume that the material of the airfoil is very soft. It is characterized by the Lamè parametres $\lambda^b = 2 \cdot 10^7$ Pa and $\mu^b = 5 \cdot 10^6$ Pa. The structural damping coefficients

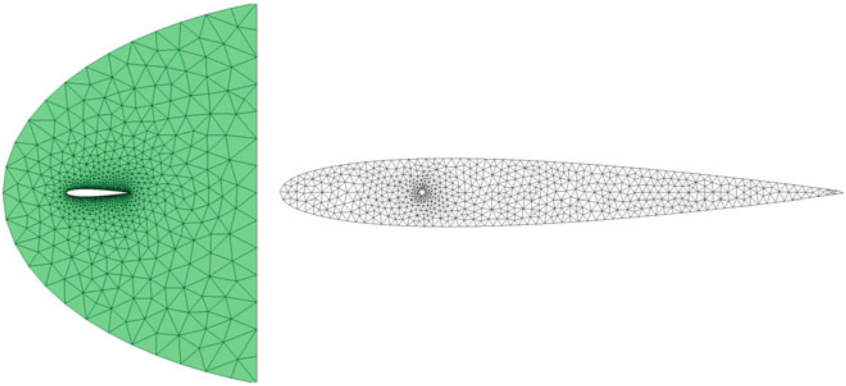

Fig. 1 Triangulation at time $t = 0$ used for the computation of fluid flow and triangulation for the elasticity problem

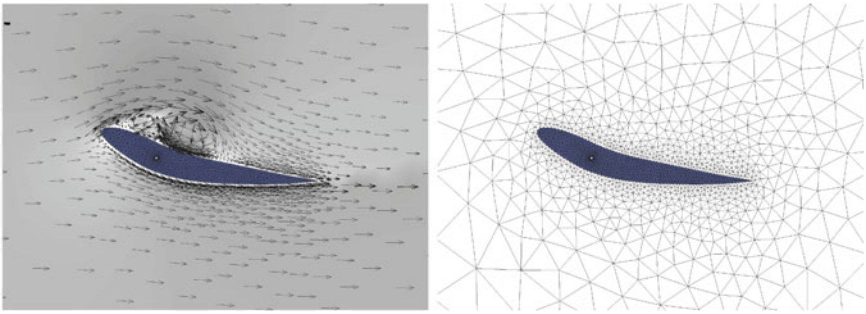

Fig. 2 Visualization of velocity vectors and of the deformed elastic airfoil at time $t = 0.15$ s

are chosen as $c_M = 0.1\,\mathrm{s}^{-1}$ and $c_K = 0.1$ s and the material density is given by $\rho^b = 10^4$ kg m^{-3}.

The fluid flow simulation was carried out using the following data: $\mu = 1.72 \cdot 10^{-5}$ kg m^{-1}.s, far-field pressure $p = 101250$ Pa, far-field density $\rho = 1.225$ kg m^{-3}, Poisson adiabatic constant $\gamma = 1.4$, specific heat $c_v = 721.428\,\mathrm{m}^2\,\mathrm{s}^{-2}$ K^{-1}, heat conduction coefficient $k = 2.428 \cdot 10^{-2}$ kg m . s^{-2} K^{-1}. The far-field velocity was 40 m s^{-1}. Figure 1 shows the triangulation at the initial time $t = 0$.

Fluid flow is solved by the ST-DGM with quadratic polynomials in space and linear polynomials in time. For the elasticity problem we also used the ST-DGM, but with linear polynomials in space and constant polynomials in time. For both problems the non-symmetric version (NIPG) was used. For flow problem we set $C_W = 1000$ on the interior elements and $C_W = 10000$ on the boundary elements in order to keep the prescribed Dirichlet boundary conditions, particularly in the boundary layer. For elasticity we set $C_W^b = 10^{10}$ in order to match the magnitude of the Lamè parametres. We used the time step $\tau = 2.25 \cdot 10^{-6}$ s. The strong coupling was used for the FSI process. The accuracy 10^{-6} was achieved with at most 5 iteration on each time level. Figure 2 shows the visualization of the deformed airfoil and the velocity vectors.

Acknowledgments This work was supported by the grants P101/11/0207 (J. Horáček) and 13-00522S (M. Feistauer) of the Czech Science Foundation. The work of M. Hadrava and A. Kosík was supported by the grant SVV–2014-260106, financed by the Charles University in Prague.

References

1. Česenek, J., Feistauer, M., Kosík, A.: DGFEM for the analysis of airfoil vibrations induced by compressible flow. Z. Angew. Math. Mech. **93**(6–7), 387–402 (2013)
2. Feistauer, M., Felcman, J., Straškraba, I.: Mathematical and Computational Methods for Compressible Flow. Clarendon Press, Oxford (2003)
3. Feistauer, M., Horáček, J., Kučera, V., Prokopová, J.: On numerical solution of compressible flow in time-dependent domains. Mathematica Bohemica **137**, 1–16 (2011)
4. Nomura, T., Hughes, T.J.R.: An arbitrary Lagrangian-Eulerian finite element method for interaction of flow and a rigid body. Comput. Meth. Appl. Mech. Eng. **95**, 115–138 (1992)