Chapter 10
An Entropy-Based Proof for the Moore Bound for Irregular Graphs

S. Ajesh Babu and Jaikumar Radhakrishnan

Abstract We provide proofs of the following theorems by considering the entropy of random walks.

Theorem 1 (Alon, Hoory and Linial) *Let G be an undirected simple graph with n vertices, girth g, minimum degree at least 2 and average degree $\bar{d}$.*

Odd girth *If $g = 2r + 1$, then $n \geq 1 + \bar{d} \sum_{i=0}^{r-1} (\bar{d} - 1)^i$.*

Even girth *If $g = 2r$, then $n \geq 2 \sum_{i=0}^{r-1} (\bar{d} - 1)^i$.*

Theorem 2 (Hoory) *Let $G = (V_L, V_R, E)$ be a bipartite graph of girth $g = 2r$, with $n_L = |V_L|$ and $n_R = |V_R|$, minimum degree at least 2 and the left and right average degrees d_L and d_R. Then,*

$$n_L \geq \sum_{i=0}^{r-1} (d_R - 1)^{\lceil \frac{i}{2} \rceil} (d_L - 1)^{\lfloor \frac{i}{2} \rfloor},$$

$$n_R \geq \sum_{i=0}^{r-1} (d_L - 1)^{\lceil \frac{i}{2} \rceil} (d_R - 1)^{\lfloor \frac{i}{2} \rfloor}.$$

This work was done while S. Ajesh Babu was at School of Technology and Computer Science, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400005, India.

J. Radhakrishnan (✉)
School of Technology and Computer Science, Tata Institute of Fundamental Research, Mumbai 400005, India
e-mail: jaikumar@tifr.res.in

S. A. Babu
Microsoft Research India, Bangalore 560001, India
e-mail: ajesh.babu@gmail.com

M. Agrawal and V. Arvind (eds.), *Perspectives in Computational Complexity*, Progress in Computer Science and Applied Logic 26, DOI: 10.1007/978-3-319-05446-9_10,

Keywords Girth · Moore bound · Graphs

Mathematics Subject Classification (2010) Primary 05C35 · Secondary 05C81

10.1 Introduction

The Moore bound (see Theorem 3.1) gives a lower bound on the order of any simple undirected graph, based on its minimum degree and girth. Alon et al. [AHL02] showed that the same bound holds with the minimum degree replaced by the average degree. Later, Hoory [Hoo02] obtained a better bound for simple bipartite graphs. We reprove the results of Alon et al. [AHL02] and Hoory [Hoo02] using information theoretic arguments based on nonreturning random walks on the graph.

The chapter has three sections: In Sect. 10.2 we introduce the relevant notation and terminology. In Sect. 10.3, we present the information theoretic proof of the result of Alon et al. [AHL02]; in Sect. 10.4, we present a similar proof of the result of Hoory [Hoo02] for bipartite graphs.

10.2 Preliminaries

For an undirected simple graph $G = (V, E)$, let $\vec{G} = (V, \vec{E})$, be the directed version of G, where for each undirected edge of the form $\{v, v\}$ in E, we place two directed edges in $\vec{E}$, one of the form (v, v) and another of the form (v, v). Similarly, for an undirected bipartite graph $G = (V_L, V_R, E)$, let $\vec{G} = (V_L, V_R, \vec{E}_{LR} \cup \vec{E}_{RL})$ be the directed version of G, where for each undirected edge of the form $\{v, v\}$ in E, with $v \in V_L$ and $v \in V_R$, we place one directed edge of the form (v, v) in $\vec{E}_{LR}$, and another of the form (v, v) in $\vec{E}_{RL}$.

We will consider *nonreturning* walks on $\vec{G}$, that is, walks where the edges corresponding to the same undirected edge of G do not appear in succession. For a vertex v, let $n_i(v)$ denote the number of nonreturning walks in $\vec{G}$ starting at v and consisting of i edges. For an edge $\vec{e}$, let $n_i(\vec{e})$ denote the number of nonreturning walks in $\vec{G}$ starting with $\vec{e}$ and consisting of exactly $i + 1$ edges (including $\vec{e}$).

Our proofs will make use of information theoretic ideas. Similar ideas have been employed in various combinatorial proofs to succinctly present arguments that involve averaging and convexity. More examples can be found in the references [CT91, Kah02, LL13, Rad99, Rad01].

Let X be a random variable taking values in a finite set. Let support(X) be the set of values that X takes with positive probability. The entropy of X is

$$H[X] = -\sum_{x \in \text{support}(X)} \Pr[X = x] \log_2 \Pr[X = x].$$

For random variables X and Y, taking values in finite sets according to some joint distribution, and $y \in \text{support}(Y)$, let X_y be the random variable taking values in $\text{support}(X)$ such that $\Pr[X_y = x] = \Pr[X = x \mid Y = y]$. Then, the conditional entropy of X given Y is

$$H[X \mid Y] = \sum_{y \in \text{support}(Y)} \Pr[Y = y] H[X_y].$$

We will use of the following standard facts about entropy [CT91].

$$H[X] \leq \log_2 |\text{support}(X)|;$$
$$H[X_1 X_2 \ldots X_k \mid Y] = \sum_{i=1}^{k} H[X_i \mid X_1 X_2 \ldots X_{i-1} Y].$$

10.3 Moore Bound for Irregular Graphs

In Sect. 10.3.1, we recall the proof of the Moore bound; in Sect. 10.3.2, we review and reprove the theorem of Alon et al. [AHL02] assuming Lemma 3.4. In Sect. 10.3.3, we prove this lemma using an entropy- based argument.

10.3.1 Proof of the Moore Bound

The Moore bound provides a lower bound for the order of a graph in terms of its minimum degree and girth.

Theorem 3.1 (The Moore bound [Big93, p. 180]) *Let G be a simple undirected graph with n vertices, minimum degree δ and girth g.*

Odd girth *If $g = 2r + 1$, then $n \geq 1 + \delta \sum_{i=0}^{r-1} (\delta - 1)^i$.*

Even girth *If $g = 2r$, then $n \geq 2 \sum_{i=0}^{r-1} (\delta - 1)^i$.*

The key observation in the proof of the Moore bound is the following. If the girth is $2r + 1$, then two distinct nonreturning walks of length at most r starting at a vertex v lead to distinct vertices. Similarly, if the girth is $2r$, then nonreturning walks of length at most r starting with (some directed version of) an edge e lead to distinct vertices. We will need this observation again later, so we record it formally.

Observation 3.2 *Let G be an undirected simple graph with n vertices and girth g.*

Odd girth *Let* $g = 2r + 1$. *Then, for all vertices* v,

$$n \geq n_0(v) + n_1(v) + \cdots + n_r(v).$$

Even girth *Let* $g = 2r$. *Let* e *be an edge of* G *and suppose* $\vec{e}_1$ *and* $\vec{e}_2$ *are its directed versions in* $\vec{G}$. *Then,*

$$n \geq \sum_{i=0}^{r-1} [n_i(\vec{e}_1) + n_i(\vec{e}_2)].$$

Proof of Theorem 3.1 The claim follows immediately from Observation 3.2 by noting that for such a graph G, for all vertices $v \in V$ and edges $\vec{e} \in \vec{E}$,

$$n_i(v) \geq \delta(\delta - 1)^{i-1} \quad (\text{for } i \geq 1), \quad n_0(v) = 1; \tag{10.1}$$

$$n_i(\vec{e}) \geq (\delta - 1)^i \quad (\text{for } i \geq 0). \tag{10.2}$$

□

10.3.2 The Alon–Hoory–Linial Bound

Alon, Hoory, and Linial showed that the bound in Theorem 3.1 holds for any undirected graph even when the minimum degree δ is replaced by the average degree $\bar{d}$.

Theorem 3.3 (Alon et al. [AHL02]) *Let* G *be an undirected simple graph with* n *vertices, girth* g, *minimum degree at least* 2 *and average degree* $\bar{d}$.

Odd girth *If* $g = 2r + 1$, *then* $n \geq 1 + \bar{d} \sum_{i=0}^{r-1} (\bar{d} - 1)^i$.

Even girth *If* $g = 2r$, *then* $n \geq 2 \sum_{i=0}^{r-1} (\bar{d} - 1)^i$.

We will first prove this theorem assuming the following lemma, which is the main technical part of Alon et al. [AHL02]. This lemma shows that the bounds (10.1) and (10.2) holds with δ replaced by $\bar{d}$. In Sect. 10.3.3, we will present an information theoretic proof of this lemma.

Lemma 3.4 *Let* G *be an undirected simple graph with* n *vertices, girth* g, *minimum degree at least two and average degree* $\bar{d}$.

(a) *If* $v \in V(G)$ *is chosen with distribution* π, *where* $\pi(v) = d_v/(2|E(G)|) = d_v/(\bar{d}n)$, *then* $\mathbb{E}[n_i(v)] \geq \bar{d}(\bar{d} - 1)^{i-1}$ $(i \geq 1)$.
(b) *If* $\vec{e}$ *is a uniformly chosen random edge in* $\vec{E}$, *then* $\mathbb{E}[n_i(\vec{e})] \geq (\bar{d} - 1)^i$ $(i \geq 0)$.

Proof of Theorem 3.3 First, consider graphs with odd girth. From Observation 3.2, Lemma 3.4 (a) and linearity of expectation we obtain

$$n \geq \mathbb{E}[n_0(v) + n_1(v) + \cdots + n_r(v)] \geq 1 + \bar{d} \sum_{i=0}^{r-1} (\bar{d} - 1)^i,$$

where $v \in V(G)$ is chosen with distribution π (defined in Lemma 3.4 (a)).

Now, consider graphs with even girth. Let $\vec{e}_1$ be chosen uniformly at random from $\vec{E}$ and let $\vec{e}_2$ be its companion edge (going in the opposite direction). Note that $\vec{e}_2$ is also uniformly distributed in $\vec{E}$. Then, from Observation 3.2, Lemma 3.4 (b) and linearity of expectation we obtain

$$n \geq \mathbb{E}\left[\sum_{i=0}^{r} [n_i(\vec{e}_1) + n_i(\vec{e}_2)]\right] \geq 2 \sum_{i=0}^{r-1} (\bar{d} - 1)^i.$$

10.3.3 The Entropy-Based Proof of Lemma 3.4

The proof of Lemma 3.4 below is essentially the same as the one originally proposed by Alon, Hoory, and Linial but is stated more naturally using the language of entropy.

Proof of Lemma 3.4 (a) Consider the Markov process $v, \vec{e}_1, \vec{e}_2, \ldots, \vec{e}_i$, where v is a random vertex of G chosen with distribution π, $\vec{e}_1$ is a random edge of $\vec{G}$ leaving v (chosen uniformly from the d_v choices), and for $1 \leq j < i$, $\vec{e}_{j+1}$ is a random successor edge for $\vec{e}_j$ chosen uniformly from among the nonreturning possibilities. (If $\vec{e}_j$ has the form (x, y), then there are $d_y - 1$ possibilities for $\vec{e}_{j+1}$). Let $v_0 = v, v_1, v_2, \ldots, v_i$ be the vertices visited by this non-returning walk. We observe that each $\vec{e}_j$ is distributed uniformly in the set $E(\vec{G})$ and each v_j has distribution π. Then,

$$\begin{aligned}
\log_2 \mathbb{E}[n_i(v)] &\geq \mathbb{E}[\log_2 n_i(v)] \\
&\geq H[\vec{e}_1\vec{e}_2 \ldots \vec{e}_i \mid v] \\
&= H[\vec{e}_1|v] + \sum_{j=1}^{i-1} H[\vec{e}_{j+1} \mid \vec{e}_1\vec{e}_2 \ldots \vec{e}_j v] \\
&= \mathbb{E}[\log_2 d_v] + \sum_{j=1}^{i-1} \mathbb{E}[\log_2(d_{v_j} - 1)] \\
&= \mathbb{E}[\log_2 d_v(d_v - 1)^{i-1}] \\
&= \frac{1}{\bar{d}n} \sum_v d_v \log_2 d_v(d_v - 1)^{i-1} \\
&\geq \log_2 \bar{d}(\bar{d} - 1)^{i-1},
\end{aligned}$$

where to justify the first inequality we use Jensen's inequality for the concave function log, to justify the second we use the fact that the entropy of a random variable is at most the log of the size of its support, and to justify the last we use Jensen's inequality for the convex function $x \log_2 x(x-1)^{i-1}$ ($x \geq 2$). The claim follows by exponentiating both sides.

(b) This time we consider the Markov process $\vec{e}_0 = \vec{e}, \vec{e}_1, \ldots, \vec{e}_i$, where $\vec{e}$ is chosen uniformly at random from $\vec{E}$, and for $0 \leq j < i$, $\vec{e}_{j+1}$ is a random successor edge for $\vec{e}_j$ chosen uniformly from among the nonreturning possibilities. Let $v_0, v_1, v_2, \ldots, v_{i+1}$ be the vertices visited by this nonreturning walk. As before observe that each v_j has distribution π. Then,

$$\begin{aligned}
\log_2 \mathbb{E}[n_i(e)] &\geq \mathbb{E}[\log_2 n_i(e)] \\
&\geq H[\vec{e}_1 \vec{e}_2 \ldots \vec{e}_i \mid \vec{e}_0] \\
&= \sum_{j=1}^{i} \mathbb{E}[\log_2 (d_{v_j} - 1)] \\
&= \mathbb{E}[\log_2 (d_{v_0} - 1)^i] \\
&= \frac{1}{\bar{d}n} \sum_{v} d_v \log_2 (d_v - 1)^i \\
&\geq \log_2 (\bar{d} - 1)^i,
\end{aligned}$$

where we justify the first two inequalities as before, and the last using Jensen's inequality applied to the convex function $x \log_2 (x-1)^i$ ($x \geq 2$). The claim follows by exponentiating both sides. □

Remark 3.5 We assumed above that the minimum degree is at least 2. It is possible to eliminate vertices of small degree and show that Theorem 3.3 holds for any graph with *average* degree at least 2. For details, see the proof of Theorem 1 in [AHL02].

10.4 Moore Bound for Bipartite Graphs

Following the proof technique of [AHL02], Hoory [Hoo02] obtained an improved Moore bound for bipartite graphs. In this section, we provide an information theoretic proof of Hoory's result.

10.4.1 The Hoory Bound

Theorem 4.1 (Hoory [Hoo02]) *Let $G = (V_L, V_R, E)$ be a bipartite graph of girth $g = 2r$, with $n_L = |V_L|$ and $n_R = |V_R|$, minimum degree at least 2 and the left and right average degrees d_L and d_R. Then,*

$$n_L \geq \sum_{i=0}^{r-1} (d_R - 1)^{\lceil \frac{i}{2} \rceil} (d_L - 1)^{\lfloor \frac{i}{2} \rfloor},$$

$$n_R \geq \sum_{i=0}^{r-1} (d_L - 1)^{\lceil \frac{i}{2} \rceil} (d_R - 1)^{\lfloor \frac{i}{2} \rfloor}.$$

For bipartite graphs the girth is always even. We then have the following variant of Observation 3.2.

Observation 4.2 *Let $G = (V_L, V_R, E)$ be an undirected bipartite graph with $|V_L| = n_L$ and $|V_R| = n_R$ and girth $g = 2r$. Let e be an edge of G and suppose $\vec{e}_1$ and $\vec{e}_2$ be its directed versions in $\vec{G}$, such that $\vec{e}_1 \in \vec{E}_{LR}$ and $\vec{e}_2 \in \vec{E}_{RL}$. Then,*

$$n_L \geq \sum_{i=0}^{\lfloor \frac{r}{2} \rfloor - 1} n_{2i+1}(\vec{e}_1) + \sum_{i=0}^{\lceil \frac{r}{2} \rceil - 1} n_{2i}(\vec{e}_2).$$

We will prove the Theorem 4.1, assuming the following lemma, which is the main technical part of Hoory [Hoo02]. In Sect. 10.4.2, we will present the proof of this lemma using the language of entropy.

Lemma 4.3 *Let $G = (V_L, V_R, E)$ be an undirected simple bipartite graph with n_L vertices on the left and n_R vertices on the right, girth g, minimum degree at least two and average left and right degrees, respectively d_L and d_R.*

(a) *If $\vec{e}$ is a uniformly chosen random edge in $\vec{E}_{LR}$, then* $\mathbb{E}[n_{2i+1}(\vec{e})] \geq (d_R - 1)^{i+1}(d_L - 1)^i$ $(i \geq 1)$.
(b) *If $\vec{e}$ is a uniformly chosen random edge in $\vec{E}_{RL}$, then* $\mathbb{E}[n_{2i}(\vec{e})] \geq (d_R - 1)^i(d_L - 1)^i$ $(i \geq 1)$.

Proof of Theorem 4.1 We will prove the bound for n_L. The proof for n_R case is similar. Let $\vec{e}_1$ be chosen uniformly at random from $\vec{E}_{LR}$ and let $\vec{e}_2$ be its companion edge (going in the opposite direction). Note that $\vec{e}_2$ is also uniformly distributed in $\vec{E}_{RL}$. Then, from Observation 4.3, Lemma 4.3 and linearity of expectation we obtain

$$n_L \geq \mathbb{E}\left[\sum_{i=0}^{\lfloor \frac{r}{2} \rfloor - 1} n_{2i+1}(\vec{e}_1) + \sum_{i=0}^{\lceil \frac{r}{2} \rceil - 1} n_{2i}(\vec{e}_2) \right] \geq \sum_{i=0}^{r-1} (d_R - 1)^{\lceil \frac{i}{2} \rceil} (d_L - 1)^{\lfloor \frac{i}{2} \rfloor}. \qquad \square$$

10.4.2 The Entropy-Based Proof of Lemma 4.3

The proof of Lemma 4.3 below is essentially the same as the one originally proposed by Hoory, but is stated in the language of entropy.

Proof of Lemma 4.3 (a) Consider a Markov process $\vec{e}_0, \vec{e}_1, \vec{e}_2, \ldots, \vec{e}_{2i+1}$, where $\vec{e}_0$ is a uniformly chosen random edge from $\vec{E}_{LR}$, and for $0 \le j < 2i+1$, $\vec{e}_{j+1}$ is a random successor edge for $\vec{e}_j$ chosen uniformly from among the nonreturning possibilities. Let $v_0, v_1, v_2, \ldots, v_{2i+2}$ be the vertices visited by this nonreturning walk. We observe that for $0 \le j \le i$ each $\vec{e}_{2j}$ and $\vec{e}_{2j+1}$ is respectively distributed uniformly in the set $\vec{E}_{LR}$ and $\vec{E}_{RL}$. Furthermore, for j even, $\Pr[v_j = v] = d_v/|E(G)|$ for all $v \in V_L$, and for j odd, $\Pr[v_j = v] = d_v/|E(G)|$ for all $v \in V_R$. Then,

$$\begin{aligned}
\log_2 \mathbb{E}[n_{2i+1}(e)] &\ge \mathbb{E}[\log_2 n_{2i+1}(e)] \\
&\ge H[\vec{e}_0\vec{e}_1 \ldots \vec{e}_{2i+1} \mid \vec{e}_0] \\
&= \sum_{j=0}^{i} H[\vec{e}_{2j+1}|\vec{e}_{2j}] + \sum_{j=1}^{i} H[\vec{e}_{2j}|\vec{e}_{2j-1}] \\
&= \sum_{j=0}^{i} \mathbb{E}[\log_2(d_{v_{2j+1}} - 1)] + \sum_{j=1}^{i} \mathbb{E}[\log_2(d_{v_{2j}} - 1)] \\
&\ge (i+1)\log_2(d_R - 1) + i\log_2(d_L - 1) \\
&= \log_2(d_R - 1)^{i+1}(d_L - 1)^i.
\end{aligned}$$

where to justify the first inequality we use Jensen's inequality for the concave function log, to justify the second we use the fact that the entropy of a random variable is at most the log of the size of its support, and to justify the last we use Jensen's inequality for the convex function $x\log_2(x-1)$ $(x \ge 2)$. The claim follows by exponentiating both sides.

(b) Similarly,

$$\log_2 \mathbb{E}[n_{2i}(e)] \ge \log_2(d_L - 1)^i(d_R - 1)^i. \qquad \square$$

References

[AHL02] N. Alon, S. Hoory, N. Linial, The Moore bound for irregular graphs. Graphs Comb. **18**(1), 53–57 (2002)

[Big93] N. Biggs, *Algebraic Graph Theory*, 2nd edn. (Cambridge University Press, Cambridge, 1993)

[CT91] T.M. Cover, J.A. Thomas, *Elements of Information Theory* (Wiley-Interscience, New York, 1991)

[Hoo02] S. Hoory, The size of bipartite graphs with a given girth. J. Comb. Theory, Ser. B, **86**(2):215–220 (2002)
[Kah02] J. Kahn, Entropy, independent sets and antichains: a new approach to Dedekind's problem. Proc. Amer. Math. Soc. **130**, 371–378 (2002)
[LL13] N. Linial, Z. Luria, Upper bounds on the number of Steiner triple systems and 1-factorizations. Random Struct. Algorithms **43**, 399–406 (2013)
[Rad99] J. Radhakrishnan, An entropy proof of Bregman's theorem. J. Comb. Theor. A **77**(1), 161–164 (1999)
[Rad01] J. Radhakrishnan, Entropy and counting, in *IIT Kharagpur Golden Jubilee Volume on Computational Mathematics, Modelling and Algorithms*, ed. by J.C. Mishra (Narosa Publishers, New Delhi, 2001)