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Manindra Agrawal
Vikraman Arvind
Editors

Perspectives in Computational Complexity

The Somenath Biswas Anniversary Volume

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Preface

In the summer of 2012, we organized a three day "Computational Complexity" workshop at the Indian Institute of Technology, Kanpur, India, in honor of Professor Somenath Biswas to celebrate his 60th birthday. It was a fitting event for the occasion, well-attended by several well-known experts in the field from different parts of the world.

Professor Biswas is one of the first complexity theorists from India. In his teaching and research career spanning over 30 years, apart from doing quality research, he has contributed immensely to the development of the field in India. We felt that to bring out a *festschrift* volume of articles in complexity theory, based partly on the talks at the workshop, would be a lasting tribute. We are deeply grateful to the eminent researchers who enthusiastically agreed to contribute articles for this project and also helped, in a cross-refereeing process, with refereeing each other's contributed articles. These articles span different aspects of recent complexity theory research, including the isomorphism conjecture, arithmetic circuit complexity, space-bounded complexity classes, proof complexity, applications of entropy, and the complexity of graph isomorphism. As former students of Somenath Biswas, we feel privileged to edit this volume. It is an expression, as it were, of our affection and regard for him.

We are grateful to Eric Allender for his valuable advice and support from the early stages of this book project, and for suggesting the "Progress in Computer Science and Applied Logic" Springer-Birkhäuser series. We would also like to thank Erich Grädel, the chief editor of the series for enthusiastically supporting the project.

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Manindra Agrawal
Vikraman Arvind

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Chapter 1
Complexity Theory Basics: NP and NL

Vikraman Arvind

Abstract We introduce basic concepts and results in computational complexity as background for some of the articles in this volume. Our focus is on the complexity classes nondeterministic polynomial time (NP) and nondeterministic logarithmic space (NL). The presentation is aimed at computer science students at a senior undergraduate level, and assumes some familiarity with algorithm design and theory of computation. The material is covered at a fairly brisk pace. Several results and proof details are incorporated in exercises which the reader is urged to solve or look up in textbooks such as [BDG88, Pap94, AB09].

Keywords Nondeterministic polynomial time · NP-completeness · Nondeterministic logarithmic space

Mathematics Subject Classification (2010) Primary 68Q15.

1.1 NP-completeness

The story of modern computational complexity begins with the advent of stored program computers in the 1940s and the need for efficiently solving optimization problems by programming the computer. It was soon discovered that a brute-force enumerative search for the optimal solution yields only algorithms that take exponential time, since the number of candidate solutions for optimization problems is typically exponential in the input size. Such solutions were not practical even for inputs of moderate size. Cobham [Cob64] and Edmonds [Edm65], around 1965, independently suggested polynomial-time boundedness as an appropriate theoretical criterion for efficient computation. This was an important conceptual contribution.

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Although algorithms with linear or quadratic time bounds are desirable in practice, polynomial time computation is theoretically satisfactory for several reasons. First, it rules out exhaustive search solutions which are typically exponential time. Also, since polynomials are closed under composition, it makes polynomial-time bounded computation closed under procedure calls. Thus, a polynomial-time bounded program making calls to a library of polynomial-time subroutines is still polynomial time. Furthermore, as reasonable models of computation can simulate each other with at most a polynomial-time slowdown,[1] it makes the notion of polynomial-time solvability independent of the computation model.

The next big step was the pioneering research of Cook [Coo71], Levin [Lev73], and Karp [Kar72]. The class NP, consisting of decision problems that have nondeterministic polynomial-time decision procedures, was identified as the class that captures most natural optimization problems of interest. The notion of polynomial-time reductions was used to compare the relative difficulty of problems. Propositional formula satisfiability was shown to be NP-complete under polynomial-time reductions [Coo71, Lev73]. Then several decision problems, arising from optimization problems, were shown NP-complete [Kar72]. NP-complete problems are the hardest problems in the class NP as opposed to decision problems that are polynomial-time solvable.

Let Σ denote a fixed finite alphabet $|\Sigma| \geq 2$. Input instances of decision problems are encoded as finite strings over Σ. Thus, Σ^* comprises of all input instances for a decision problem and the "yes" instances $A \subseteq \Sigma^*$ form a language. Therefore, we can identify decision problems with languages and we use the terms interchangeably. As is customary in computation theory, we will use the standard Turing machine model as the model of computation (see e.g. [HU79]).

Definition 1.1 Let $A, B \subseteq \Sigma^*$ be languages.

1. Then A is said to be polynomial-time *many-one reducible* to B, denoted $A \leq_m^p B$, if there is a polynomial-time computable function f such that for all $x \in \Sigma^*$

$$x \in A \text{ if and only if } f(x) \in B.$$

2. More generally, A is said to be polynomial-time *Turing reducible* to B if there is a polynomial-time bounded oracle Turing machine M such that M^B accepts the language A.

A language $L \subseteq \Sigma^*$ is in the complexity class P if there is a polynomial-time bounded deterministic Turing machine (equivalently, a polynomial-time algorithm) for checking membership in L.

A language $L \subseteq \Sigma^*$ is in the complexity class NP if there is a language $A \in$ P and a polynomial p such that for all $x \in \Sigma^*$

$$x \in L \text{ if and only if } \exists y \in \Sigma^{\leq p(|x|)} \; : \; \langle x, y \rangle \in A,$$

[1] This is a polynomial-time version of the Church-Turing thesis known as the feasibility thesis [vEB90].

where $\Sigma^{\leq p(|x|)}$ denotes strings of length at most $p(|x|)$ over Σ. In other words, the complexity class NP consists of languages L such $x \in L$ has a polynomial-size certificate y of membership in L, and the certificate is polynomial-time verifiable by checking if $\langle x, y \rangle \in A$.

Exercise Show that the following decision problems are in the class NP:

1. Given an undirected graph G and a number k as input, decide if G has a clique of size at least k (known as the CLIQUE problem). The graph G is given by either its adjacency list or adjacency matrix.
2. Given a positive integer m (encoded in binary) decide if it is composite.
3. Given a system of linear equations $Ax = b$ over rationals, where the rational entries of the matrix A and column vector b are encoded in binary, decide if it has a solution.

We now formally define NP-completeness. A language $L \subseteq \Sigma^*$ is said to be NP-complete if L is in NP and each $L' \in$ NP is polynomial-time many-one reducible to L.

Since polynomial-time reducibility between languages is a transitive relation, once a language L' is shown NP-complete, it suffices to show that L' is polynomial-time many-one reducible to L in order to prove that the language L in NP is also NP-complete. A problem that can be directly shown NP-complete is the following:

$$K = \{\langle M, x, 1^t \rangle \mid M \text{ accepts } x \text{ in at most } t \text{ steps}\},$$

where M in the above definition denotes the Turing machine code (as a list of quintuples) of a *nondeterministic* Turing machine.

Exercise Show that K is NP-complete. (Hint: In order to show K is in NP you will need to use a suitable universal Turing machine).

But the first problem shown NP-complete by Cook and Levin was a natural problem, known as the satisfiability problem for propositional formulas, which opened the floodgate to NP-complete problems and the subject of computational complexity.

Theorem 1.2 (Cook-Levin theorem) [Coo71, Lev73] *The satisfiability problem for propositional formulas is NP-complete.*

A propositional formula F is in *conjunctive normal form* (CNF in short) if $F = C_1 \wedge C_2 \wedge \cdots \wedge C_m$ where each C_i is an OR of variables or their negations. The proof of the Cook-Levin theorem actually shows the stronger result that the satisfiability problem for propositional CNF formulas is NP-complete.

Clearly, NP-complete problems are the hardest problems in NP and all NP-complete problems are polynomial-time equivalent. That is to say, if a polynomial-time algorithm is discovered for any NP-complete problem then we have a polynomial-time algorithm for any problem in NP. Whether P equals NP or not is the central open problem in computational complexity.

Exercise

1. Show that the CLIQUE problem (defined in Exercise 2) isNP-complete by giving a reduction to it from propositional CNF formula satisfiability.
2. A *vertex cover* for an undirected graph G is a subset S of vertices such that for each edge (u, v) of G we have $\{u, v\} \cap S \neq \emptyset$. Given an undirected graph G and a number k as input the VC problem is to decide if G has a vertex cover of size at most k. Show that VC is NP-complete. The graph G is given by either its adjacency list or adjacency matrix.

1.2 Inside NP

Thus, within the class NP we have polynomial-time solvable problems on the one hand, which is the subclass P. At the other extreme, we have NP-complete problems, of which there are abundantly many [GJ79], because most natural optimization problems that arise in practice and we wish to solve efficiently turn out to be NP-complete. A natural question that arises is whether NP contains other problems. The answer to this question is given by Ladner's theorem which states that if P $\neq$ NP then there are problems in NP that are neither in P nor NP-complete. We discuss a proof attributed to Russell Impagliazzo [DF03].

Theorem 2.1 (Ladner's theorem)[Lad75] *If* P $\neq$ NP *there is a problem* $A \in$ NP *that is neither in* P *nor* NP*-complete.*

Proof By assumption the NP-complete problem SAT is not in P. Following standard notation [BDG88, Pap94], let DTIME$[g(n)]$ denote the class of languages accepted by deterministic Turing machines that halt in time bounded by $g(n)$ on inputs of length n. Now, if we knew that SAT $\notin$ DTIME$[g(n)]$ for some fixed superpolynomial function $g(n)$, for example $g(n) = n^{\log n}$, then we can easily find a language $A \in$ NP that is neither in P nor NP-complete. Indeed, let

$$A := \{x01^{|x|^{\log\log|x|}} \mid x \in \text{SAT}\}.$$

Clearly, $A \in$ NP because given a string of the form $x01^k$ we can guess and verify a satisfying assignment for the SAT instance x and check in polynomial time that the pad 1^k is of length $|x|^{\log\log|x|}$. Suppose A is NP-complete. Then SAT $\leq_m^p A$ via some polynomial-time computable reduction f. Notice that

$$x \in \text{SAT} \iff f(x) = x'01^{|x'|^{\log\log|x'|}} \in A \iff x' \in \text{SAT}.$$

Since f is polynomial-time computable, $|f(x)| \leq |x|^c$ for some constant c and hence $|x'| < |x|$ for all but finitely many instances $x \in$ SAT. Thus, it suffices to check if the smaller instance $x' \in$ SAT. Repeatedly applying this argument gives a polynomial-time procedure for SAT contradicting P $\neq$ NP. Hence A cannot be NP-complete.

On the other hand, we claim that $A \notin$ P. For, if A were in P that would give a $|x|^{O(\log\log|x|)}$ time algorithm to decide if $x \in$ SAT, contradicting the assumption that SAT is not in DTIME$[n^{\log n}]$.

Unfortunately, we can only assume that SAT is not in P and cannot make the stronger assumption that SAT is not in DTIME$[n^{\log n}]$. So we need to define the padded language A more carefully to make the above argument work. Let

$$A := \{x01^k \mid x \in \text{SAT}, \quad k = f(|x|)\},$$

where the padding function $f(n)$ will be computable in time polynomial in n and constructed by diagonalization. Let $\{M_i\}_{i>0}$ be a recursive enumeration of polynomial-time clocked, deterministic Turing machines; more precisely, let M_i be clocked to run for $n^i + i$ steps on length n inputs. The function f is defined as follows:

```
1. $i := 1$.
2. For $n := 1$ to $\infty$ do
3. Let $f(n) = n^i$.
4. If there is an input $x$ of length at most $\log n$ such that $M_i(x)$ accepts and $x \notin A$ or
   $M_i(x)$ rejects and $x \in A$ then $i := i + 1$.
5. endfor
```

Notice that at the nth iteration of the for-loop, in which $f(n)$ gets defined, the function $f(m)$ is already defined for $m < n$ and hence checking $x \in A$ for $\log n$ length inputs is well defined. Moreover, checking membership of $x \in A$ can be done in polynomial in n time since $|x| \leq \log n$. Hence, f is defined by the above procedure for all n and is computable in time polynomial in n. This defines the set A.

Suppose $A \in$ P. Then $A = L(M_i)$ for some machine M_i. By construction, there are constants n_0 and $k > i$ such that $f(n) = n^k$ for all $n > n_0$. That means, for all but finitely many input lengths, SAT is polynomial-time reducible to A by the map $x \mapsto x01^{|x|^k}$ which contradicts the assumption P $\neq$ NP. It follows that for each constant i we have $f(n) > n^i$ for all but finitely many n.

Suppose A is NP-complete and g is a polynomial-time reduction from SAT to A. We will give a polynomial-time algorithm for SAT contradicting the assumption. Since g is polynomial-time computable, we have $|g(x)| \leq |x|^c$ for some constant $c > 0$ and all but finitely many inputs x. If $g(x)$ is not of the form $y01^{f(|y|)}$ we can reject x. Also, if g(SAT) is finite then SAT is trivially in P by table look-up. Suppose g(SAT) is infinite. Then for all but finitely many $x \in$ SAT we have $g(x) = y01^{f(|y|)}$ where $f(|y|) > |y|^c$. The finitely many exceptions we can keep in a table. Thus, given a SAT instance x we first compute $g(x) = y01^{f(|y|)}$. If x is not in the look-up table, since $f(|y|) < |g(x)| \leq |x|^c$, it follows that $|y| < |x|$ and $x \in$ SAT if and only if $y \in$ SAT. We can now recurse on the instance y. Overall this gives a polynomial-time SAT algorithm contradicting the assumption. This concludes the proof. □

Exercise Suitably adapt the above proof to show for any language $A \notin$ P that there is a language $B \notin$ P such that $B \leq_m^p A$ but $A \not\leq_m^p B$.

1.2.1 The Class NP ∩ co*NP*

The class coNP consists of languages L such that $\Sigma^* \setminus L$ is in NP.

Indeed, for any class of languages $\mathcal{C}$ we can define co$\mathcal{C}$:

$$\text{co}\mathcal{C} = \{L \subseteq \Sigma^* \mid \Sigma^* \setminus L \in \mathcal{C}\}.$$

Remark 2.2 The class coNP consists of all languages whose complements are in NP. For example $\overline{\text{SAT}}$ is in coNP and, by virtue of SAT being NP-complete, $\overline{\text{SAT}}$ is coNP-complete under polynomial-time many-one reductions. The set TAUT consisting of all propositional tautologies is also coNP-complete (exercise: verify this). Whether NP equals coNP is a major open problem. We can view the NP versus coNP question from the logical perspective of propositional proof systems. For any language $L \in$ NP, by definition for each $x \in L$ there is a polynomial-length proof of membership that can be checked in polynomial time. This can be thought of as a "sound and complete proof system" for L. Thus, the question whether NP = coNP amounts to asking if there is a proof system for propositional tautologies in which all tautologies have polynomial length proofs. This leads to a study of propositional proof systems of different strengths with the aim of proving lower bounds for proof lengths in the proof systems. The article by Jacobo Torán in this volume presents aspects of this fascinating topic with pointers to current research and open problems.

We will now discuss decision problems that are in NP ∩ coNP. For any $L \in$ NP ∩ coNP there are languages A and B in P and a polynomial p such that for each $x \in \Sigma^*$

$$\begin{aligned} x \in L \quad &\leftrightarrow \quad \exists y \in \Sigma^{p(|x|)} \langle x, y\rangle \in A \\ &\leftrightarrow \quad \forall z \in \Sigma^{p(|x|)} \langle x, y\rangle \in B \end{aligned}$$

These are the so-called well-characterized problems. They are well characterized in the sense that membership of x in L can be characterized using an existentially quantified predicate, and can also be characterized using a universally quantified predicate. It is a remarkable phenomenon in complexity theory that the discovery of such characterization often precedes (even anticipates) the discovery of a polynomial-time algorithm for the problem. We discuss a few well-known examples.

The language PM $= \{G \mid G$ has a perfect matching$\}$ has a polynomial-time algorithm as shown in the already mentioned famous paper of Edmonds [Edm65]. However, in the 1940s Tutte, in his well-known theorem stated below, had anticipated this by "well-characterizing" the PM problem.

Theorem 2.3 (Tutte's 1-factor theorem) [Tut47] *An undirected graph G has a perfect matching if and only if for every subset S of the vertex set the number of odd-size components in the graph $G \setminus S$ is bounded by $|S|$.*

Similarly, Farkas' lemma [Sch98, Sect. 7.3] stated below "well-characterizes" linear programming which was shown to be in polynomial time many decades later by Kachiyan [Sch98, Chap. 13].

Theorem 2.4 (Farkas' Lemma) *The system of linear inequalities $Ax \leq b$ has a solution if and only if for all $y^T \geq 0$ if $y^T A = 0$ then $y^T b \geq 0$.*

Exercise Use the characterizations in Theorems 2.3 and 2.4 to show that the perfect matching problem and feasibility of linear inequalities problem are in NP ∩ coNP.

Exercise Show that NP = coNP if and only if some problem in NP ∩ coNP is NP-complete.

The above discussion might lead one to believe that perhaps NP ∩ coNP equals P. However, there are problems in NP ∩ coNP that have defied all attempted polynomial-time solutions. The decision version of the Integer factoring problem is a noteworthy example. Each positive integer n can be uniquely factorized as $n = p_1^{e_1} p_2^{e_2}, \ldots, p_k^{e_k}$ where $p_1 < p_2 < \cdots < p_k$ are distinct primes. Let $enc(n)$ denote an encoding in binary of this factorization. Consider the language

$$\text{FACT} = \{\langle n, i, b\rangle \mid \text{ the } i^{th} \text{ bit of } enc(n) \text{ is } b\}.$$

Exercise

1. Assuming primality testing is in P show that FACT is in NP ∩ coNP.
2. Show that the integer factoring problem can be solved in polynomial time with calls to a decision procedure for FACT.

1.3 The Berman-Hartmanis Conjecture

Polynomial-time reductions define a natural order $\leq_m^p$ on NP languages that raises some fundamental questions about the structure of NP languages.

Exercise Show that the order $\leq_m^p$ on NP languages is a binary relation that is reflexive and transitive but not symmetric.

In order to obtain a partial order from $\leq_m^p$ we define the equivalence relation $A \equiv_m^p B$ if and only if $A \leq_m^p B$ and $B \leq_m^p A$.

Exercise

1. Show that $\equiv_m^p$ is an equivalence relation on NP.
2. For $A \in$ NP let $[A]$ denote the equivalence class containing A for the equivalence relation $\equiv_m^p$. Show that $\leq_m^p$, suitably defined on the equivalences classes $[A]$ for $A \in$ NP, yields a partial order.

This partial order has its top element as [SAT], the equivalence class of all NP-complete languages. Its bottom element is P. Notice that class P contains, among all polynomial-time solvable decision problems, all finite languages. In contrast, assuming P $\neq$ NP, all sets in [SAT], being NP-complete, are infinite.

Definition 3.1 [BH77] Let $A, B \subseteq \Sigma^*$. A polynomial-time many-one reduction f from A to B is a *polynomial-time isomorphism* if $f : \Sigma^* \to \Sigma^*$ is a bijection and f^{-1} is also polynomial-time computable.

Berman and Hartmanis [BH77], in 1977 conjectured that all NP-complete sets are polynomial-time isomorphic to each other. Since they could show [BH77] many natural NP-complete problems to be isomorphic, empirically this appears plausible. Although the conjecture is not currently believed to be true, it gave impetus and direction to a lot of interesting complexity theory research. The article by Eric Allender in this volume surveys the interesting complexity theory research related to the Berman-Hartmanis conjecture over the last two decades. Our aim here is to provide some useful background.

A polynomial-time computable function $f : \Sigma^* \to \Sigma^*$ is *1-invertible* if f is injective and f^{-1} is also polynomial-time computable. More precisely, there is a polynomial-time algorithm that on input $y \in \Sigma^*$ computes $f^{-1}(y)$ if y is in the range of f and outputs $\perp$ otherwise.

Now, suppose A and B are NP-complete languages such that A is reducible to B via a 1-invertible function f and B is reducible to A via a 1-invertible function g, can we then conclude that A and B are polynomial-time isomorphic. The motivation for this approach is its analogy to the setting of the Schröder-Bernstein theorem in set theory which we recall with a quick proof in order to generalize it to the isomorphism setting.

Theorem 3.2 (Schröder-Bernstein theorem) *Let A and B be sets and $f : A \to B$ and $g : B \to A$ be injective functions. Then there is a bijection between A and B.*

Proof The proof idea involves examining the alternating *preimage sequence* of points $x, g^{-1}(x), f^{-}(g^{-1}(x)), \ldots$ for each $x \in A$. Since both f and g are injective, notice that for any $x \in A$ and $y \in B$ the preimages $g^{-1}(x)$ and $f^{-1}(y)$, if they exist, are unique. We can partition A into three parts A_1, A_2, and A_3. The part A_1 consists of $x \in A$ such that the preimage sequence is finite and ends in A. The part A_2 consists of $x \in A$ such that it has a finite preimage sequence ending in B, and A_3 consist of the elements $x \in A$ whose preimage sequence is infinite. Likewise, B is partitioned into three parts B_1, B_2, and B_3 of elements $y \in B$ whose pre-image sequence either ends in A or in B or is infinite, respectively. Define a function $h : A \to B$ as follows:

$$\begin{aligned} \forall\, x \in A_1 \quad & h(x) = f(x), \\ \forall\, x \in A_2 \quad & h(x) = g^{-1}(x), \\ \forall\, x \in A_1 \quad & h(x) = f(x). \end{aligned}$$

It is easy to see that h is a bijection. □

Exercise Verify that h defined in the proof is indeed a bijection from A to B.

A function $f : \Sigma^* \to \Sigma^*$ is called *length increasing* if $|f(x)| > |x|$ for all $x \in \Sigma^*$. In order to adapt the Schröder-Bernstein proof strategy for showing polynomial-time isomorphisms between NP-complete sets, it turns out that length-increasing 1-invertible reductions is a suitable notion.

Theorem 3.3 [BH77] *Let $A, B \subseteq \Sigma^*$ be two languages such that there are length-increasing 1-invertible reductions from A to B and from B to A. Then A and B are polynomial-time isomorphic.*

The proof of this theorem is in the following exercise.

Exercise

1. Suppose f and g are length-increasing 1-invertible reductions from A to B and from B to A respectively. Show that the partition corresponding to infinite preimage sequences is empty for both f and g.
2. Verify that the bijection $h : \Sigma^* \to \Sigma^*$ defined in the proof of Theorem 3.2 is a polynomial-time isomorphism between A and B.

The question is how do we get hold of length increasing 1-invertible reductions? Berman and Hartmanis [BH77] discovered another natural property that several NP-complete languages are endowed with. A language $A \subseteq \Sigma^*$ is said to be *paddable* if there is a 1-invertible reduction *pad* from $A \times \Sigma^*$ to A. It turns out that many natural NP-complete problems are paddable.

Exercise Show that SAT, 3-SAT, CLIQUE, VC are all paddable languages.

Now, it turns out that paddable NP-complete languages are all isomorphic [BH77]. Since many NP-complete problems are paddable, it lead Berman and Hartmanis to make their conjecture.

Exercise

1. If $A \leq_m^p B$ and B is paddable then show that A is polynomial-time reducible to B via a 1-invertible length increasing function.
2. Conclude that if A and B are paddable NP-complete languages then they are polynomial-time isomorphic.

1.4 Are There Sparse NP-Complete Sets?

Let $A \subseteq \Sigma^*$ be any language. The *density* of A at length n is defined to be the number $|A^{=n}|$ of length n strings in A. The language A is exponentially dense if $|A^{=n}|$ is at least 2^{n^ϵ} for some constant $\epsilon > 0$. Natural NP-complete problems have exponential density.

Exercise Show that SAT, CLIQUE, VC have exponential density.

A consequence of the Berman-Hartmanis conjecture is that all NP-complete languages have exponential density. Can we show that languages with subexponential density cannot be NP-hard?

A language $A \subseteq \Sigma^*$ is said to be *sparse* if there is a polynomial $p(n)$ such that $|A^{=n}| \leq p(n)$ for all n.

Theorem 4.1 (Mahaney) [Mah82] *No sparse language is* NP*-hard unless* P = NP.

Proof We present a more recent proof [Agr11]. Suppose SAT $\leq_m^p A$ for some arbitrary sparse language A via a polynomial-time reduction f. For some polynomial $p(n)$ we have $|f(x)| \leq p(|x|)$.

We give a polynomial-time algorithm for SAT. Given a formula F of size s in boolean variables $x_1, x_2, \ldots, x_n$ as input, the algorithm proceeds in stages $0 \leq i \leq n$. At stage i, the algorithm maintains a list of formulas $\{F_a \mid a \in I\}$ such that each $F_a, a \in I$ is obtained from F by the truth assignment a to the first i variables $x_1, x_2, \ldots, x_i$ with the property that

$$F \in \text{SAT if and only if } F_a \in \text{SAT for some } a \in I. \tag{1.1}$$

If $|I| > q(s) + 1$, for a suitable polynomial $q(s)$ which will be defined in the course of the proof, then the algorithm applies a pruning operation to discard some a from the list I such that Property (1.1) holds for $I \setminus \{a\}$ as well. We now describe the pruning operation:

Consider all pairs of disjunctions $F_{ab} = F_a \vee F_b$ for $a, b \in I$. If s is the size of formula F then clearly $2s$ bounds the size of F_{ab} for all $a, b \in I$. Fix an $a \in I$ and consider $f(F_{ab})$ for all $b \in I \setminus \{a\}$. If $F_a \in$ SAT then clearly $F_{ab} \in$ SAT for all $b \in I \setminus \{a\}$. Hence, $\mathbb{F}_a \in$ SAT implies $f(F_{ab}) \in A^{\leq p(2s)}$. Since A is sparse we know that $|A^{\leq p(2s)}| \leq q(s)$ for some polynomial q. The algorithm computes $f(F_{ab})$ for all $b \in I \setminus \{a\}$ and does the following:

1. If $f(F_{ab})$ are all distinct for $b \in A \setminus \{a\}$ then F_a cannot be in SAT and a can be discarded from I.
2. Otherwise, for some $b \neq c \in I \setminus \{a\}$ we have $f(F_{ab}) = f(F_{ac})$. The algorithm can discard either b or c from I.

After pruning the list to get $\{F_a \mid a \in I\}$ such that $|I| \leq q(s) + 1$ the algorithm proceeds to the next stage where it first doubles the list of formulas by setting x_{i+1} to 0 and 1 to get $\{F_b \mid b \in J\}$ where J consists of all extensions of assignments in I obtained by setting x_{i+1} to 0 and 1.

Continuing thus, at the nth stage the algorithm accepts SAT if the set I at that stage includes a satisfying assignment. □

Let $A \in$ NP. By definition there are a polynomial-time computable language $B \subseteq \Sigma^* \times \Sigma^*$ and a polynomial bound $p(n)$ such that $x \in A$ if and only if

$$\exists y \in \Sigma^{p(|x|)} \langle x, y \rangle \in B.$$

Define the language $\text{pre}(A) = \{\langle x, w\rangle \mid \exists u \in \Sigma^{p(|x|)-|w|} : \langle x, wu\rangle \in B\}$.

Exercise Show that $\text{pre}(A)$ is in NP and $A \le_m^p \text{pre}(A)$.

A *tally* language is a subset of 0^*. The next exercise is about tally languages and is analogous to Theorem 4.1.

Exercise For a language $A \in \text{NP}$ if $\text{pre}(A)$ is polynomial-time reducible to a tally language then show that $\text{pre}(A)$, and hence A, is in P.

1.4.1 Subexponentially Dense Languages

A subset $S \subseteq \Sigma^*$ is of *subexponential density* if for every $\epsilon > 0$ there is an $n_0 \in \mathbb{N}$ such that for all $n > n_0$ we have $|S^{=n}| \le 2^{n^\epsilon}$. A natural question is whether Mahaney's theorem can be generalized to sets of subexponential density.

The class of languages $\text{SUBEXP} = \cap_{\epsilon>0}\text{DTIME}[2^{n^\epsilon}]$ consists of languages that have subexponential time decision procedures.

In the proof of Theorem 4.1 we gave a polynomial-time algorithm for SAT assuming that it is reducible to some sparse language. Modify the proof to show the following.

Exercise If SAT is polynomial-time many-one reducible to a language of subexponential density then show that $\text{NP} \subseteq \text{SUBEXP}$.

The inclusion $\text{NP} \subseteq \text{SUBEXP}$ is believed unlikely. For instance, it would imply a $2^{o(n)}$ time algorithm for the satisfiability of 3CNF formulas, where n is the number of boolean variables in the input formula. This would, in turn, imply similar subexponential algorithms for certain other NP-complete problems [IPZ01]. For a complexity theory tailored to the problem of designing faster exponential-time algorithms for NP-complete problems see [IPZ01] and related papers.

For the rest of this section we briefly discuss another unlikely complexity-theoretic consequence, assuming SAT is reducible to a set of subexponential density. In the process we will introduce some more basic complexity theory.

We start with the definition of the *polynomial-time hierarchy*. A language L is in the class Σ_k^p if there are a polynomial $p(n)$ and language $A \in \text{P}$ such that

$$\begin{aligned} L = \{x \mid \exists y_1 \forall y_2 \cdots \text{Q} y_k \ : \ & |y_i| \le p(|x|) \text{ for all } i \\ & \text{and } \langle x, y_1, y_2, \ldots, y_k\rangle \in A\}, \end{aligned}$$

where the quantifier "Q" is "$\exists$" if k is odd and "$\forall$" if k is even. The class Π_k^p consists of all languages L such that $\Sigma^* \setminus L$ is in Σ_k^p. The following observations are immediate from the definition:

$$\Sigma_0^p = \Pi_0^p = \mathrm{P},$$
$$\Sigma_1^p = \mathrm{NP},\ \Pi_1^p = \mathrm{coNP},$$
$$\Sigma_k^p \subseteq \Pi_{k+1}^p \text{ and } \Pi_k^p \subseteq \Sigma_{k+1}^p.$$

The entire polynomial-time hierarchy is defined as the union $\mathrm{PH} = \cup_{k\geq 0}\Sigma_k^p$.

Exercise Show that $\Sigma_k^p = \Pi_k^p$ implies $\mathrm{PH} = \Sigma_k^p$.

We now introduce the *nonuniform* analog of P and NP. A polynomially bounded *advice* function is any function $a : \mathbb{N} \to \Sigma^*$, mapping positive integers to strings, such that $|a(n)| \leq p(n)$ for some polynomial p and all $n \in \mathbb{N}$

Let $\mathcal{C}$ be any class of languages. A language L is in the class $\mathcal{C}/\mathrm{poly}$ if there are a polynomially bounded advice function $a(n)$ and a language $A \in \mathcal{C}$ such that for all $n \in \mathbb{N}$ and $\mathbf{x} \in \Sigma^*$:

$$x \in L \text{ if and only if } \langle x, a(|x|)\rangle \in A.$$

Notice that the function $a(n)$ is arbitrary and could even be noncomputable. On the other hand, the advice string $a(|x|)$ is polynomially bounded and is the same for all length n inputs.

The classes P/poly and NP/poly are the usual nonuniform analogues of P and NP respectively. These complexity classes have alternative definitions in terms of boolean circuits which we now introduce in a small digression.

An n-variate *boolean function* is a function $f : \{0,1\}^n \to \{0,1\}$. Boolean circuits are a natural computational model for computing boolean functions. A *boolean circuit* C is a directed acyclic graph whose vertices of indegree 0 are labeled either by a boolean constant (either 0 or 1) or by one of the n input boolean variables $x_1, x_2, \ldots, x_n$. Every other vertex of the graph has indegree either one or two. The vertices of indegree one are labeled as NOT gates and each vertex of indegree two is labeled either as an AND gate or an OR gate. A special vertex of the circuit is designated as the *output* gate. The *size* of the circuit C, denoted $\mathrm{size}(C)$ is the number of gates in C.

The circuit is evaluated in any topologically sorted order of the gates, following the usual semantics of NOT, AND, and OR at each gate. Each gate computes a boolean function and the n-variate boolean function computed at the output gate is defined as the function computed by the circuit.

Let $\Sigma = \{0,1\}$ and $L \subseteq \Sigma^*$ be a language. Consider a family of circuits $\{C_n\}_{n>0}$, where C_n computes an n-variate boolean function for each n. We say that the circuit family $\{C_n\}_{n>0}$ computes the language L if for each n we have

$$L^{=n} = \{x \in \Sigma^n \mid C_n(x) = 1\}.$$

We call $\{C_n\}_{n>0}$ a *polynomial-size* circuit family if $\mathrm{size}(C_n) \leq p(n)$ for some polynomial $p(n)$ and all n.

Exercise Show that a language L is in P/poly if and only if there is a polynomial-size circuit family that computes L. Similarly, characterize NP/poly using circuit families.

As there is no bound on the computational complexity of advice functions, the class P/poly contains even noncomputable languages (exercise: prove this). Are there NP-complete languages in P/poly? This could be useful for efficiently solving NP-complete languages for fixed input lengths. The advice function, computed as a preprocessing step, serves as a small table that can be looked up to efficiently solve instances of that fixed length. However, this seems to be unlikely because it implies that the polynomial hierarchy will collapse to the second level.

Theorem 4.2

1. *If an* NP-*complete language is in* P/poly *then* $\mathrm{PH} = \Sigma_2^p$ [KL80].
2. *If there is a* coNP-*complete language in* NP/poly *then* $\mathrm{PH} = \Sigma_3^p$ [Yap83].

Coming back to sets of subexponential density, it is shown in [BH08], using a nice counting argument from [FS08], that if SAT is polynomial-time many-one reducible to a set S of subexponential density then $\mathrm{coNP} \subset \mathrm{NP/poly}$ which in turn would imply $\mathrm{PH} = \Sigma_3^p$ by the above theorem.

1.5 Nondeterministic Logspace

We now turn to logarithmic space bounded complexity classes. A deterministic logarithmic space-bounded Turing machine is a deterministic Turing machine that has a read-only input tape and one or more worktapes, where n denotes the input size and the workspace on each worktape is bounded by $O(\log n)$. By the tape-compression theorem the constant factor in the space bound is not important. It can be reduced by suitably increasing the tape alphabet size. Furthermore, multiple worktapes can be replaced by a single worktape in space-bounded computation. We denote by DSPACE[$\log n$] the class of decision problems that can be solved in deterministic logspace bounded Turing machines. Likewise, NSPACE[$\log n$] denotes the class of languages accepted by *nondeterministic* logspace bounded Turing machines, where acceptance is defined as usual: the machine accepts an input if there is some accepting computation path for the machine on that input. The definitions and results go back to the seminal work of Hartmanis and Stearns [HS65] and are well treated in the classic textbook [HU79, Chaps. 12 and 13].

The class DSPACE[$\log n$] is denoted as L. Likewise, the nondeterministic class NSPACE[$\log n$] is denoted as NL.

Proposition 5.1 $\mathrm{L} \subseteq \mathrm{NL} \subseteq \mathrm{P}$.

Proof The first containment is obvious. Consider the nondeterministic computation of an NL machine M on an input x of length n. Notice that the configurations of M

have $O(\log n)$ size descriptions. More precisely, a configuration is described by the current state, the head position on the input tape (which requires $\log n$ bits), the head position on the work tape (which requires $\log\log n$ bits), along with the contents of the work tape which is $O(\log n)$ bits. Furthermore, given two configurations I and I' we can easily check from the description of the machine M if M can go from I to I' in one step. Thus, in polynomial in n time we can define a directed graph whose nodes are the configurations and edges are (I, I') if M can move from I to I' in one step. Clearly, M accepts x if and only if there is a directed path from the initial configuration to an accepting configuration. This can be solved in polynomial time by a DFS-based algorithm. Thus, there is a polynomial time algorithm to determine if M accepts x. □

It is an open problem whether L equals NL. The L versus NL problem makes an interesting study in contrast with the P versus NP problem as the results explained in this section will show.

In order to talk about NL-complete problems, we can define logspace many-one reductions: $f : \Sigma^* \to \Sigma^*$ is logspace computable if there is a deterministic logspace-bounded Turing machine that on input $\langle x, i\rangle$ computes the ith bit of $f(x)$ (if $i > |f(x)|$ the machine indicates that the ith bit is undefined). It is easy to check that the composition of logspace computable functions is logspace computable. A language $A \subseteq \Sigma^*$ is NL-complete if $A \in$ NL and for every language $B \in$ NL, B is logspace many-one reducible to A.

Let REACH $= \{\langle G, s, t\rangle \mid G$ is a directed graph with a directed path from s to $t\}$.

Exercise Show that *REACH* is NL-complete (Hint: use the configuration graph from the proof of Proposition 5.1 for the reduction).

Unlike NP and coNP which are believed to be different, the classes NL and coNL are equal. This result was shown independently by Immerman and Szelepcenyi [Imm88, Sze88] in 1987. The Immerman-Szelepcenyi theorem was an important breakthrough in our understanding of nondeterministic complexity classes.

Theorem 5.2 (Immerman-Szelepcenyi) [Imm88, Sze88] *For any space-constructible function* $s(n) \geq \log n$ *the classes* NSPACE$[s(n)]$ *and co*NSPACE$[s(n)]$ *are equal. In particular,* NL $=$ coNL.

A formal detailed proof can be found in a textbook [Pap94, Theorem 7.6]. The following exercise outlines the proof with hints for the ambitious reader. Let $\overline{\text{REACH}} = \{\langle G, s, t\rangle \mid G$ is a digraph with no s-to-t directed path$\}$.

Exercise

1. Show that in order to prove NL $=$ coNL it suffices to give an NL algorithm for the problem $\overline{\text{REACH}}$.
2. Let (G, s, t) be an input instance of $\overline{\text{REACH}}$. Suppose the number N of vertices in G that are reachable from s is given to the algorithm as additional input. Then design an NL algorithm that accepts (G, s, t) precisely when there is no directed s-to-t path in G.

3. Let (G, s, t) be an input instance of $\overline{\text{REACH}}$. Given as additional input the number N_i of vertices in G that are reachable from s by directed paths of length at most i. Design an NL algorithm that on each accepting computation path halts with the number N_{i+1} as the contents of the worktape.
4. Put these parts together to obtain an NL algorithm for $\overline{\text{REACH}}$.

In the 1970s Savitch's theorem [Pap94, Theorem 7.5] had already shown that nondeterministic space is quite different from nondeterministic time.

Theorem 5.3 (Savitch) *For any space-constructible function* $s(n) \geq \log n$ *we have* $\text{NSPACE}[s(n)] \subseteq \text{DSPACE}[s^2(n)]$.

Exercise Show that REACH is in $\text{DSPACE}[\log^2 n]$ using the following divide and conquer strategy: there is an s-to-t directed path in G of length k if and only if for some vertex r of G there is a directed s-to-r path of length $\lceil k/2 \rceil$ and a directed r-to-t path of length $\lfloor k/2 \rfloor$ in G.

The above results definitely encourage the optimist to believe that it is possible to design a deterministic logspace algorithm for REACH and hence show NL $=$ L. Vinodchandran's article surveys recent progress on space-bounded algorithms for REACH. A big breakthrough was obtained by Reingold [Rei08] in 2004 by showing that *undirected* graph reachability is in L. We discuss some aspects of Reingold's result in the Sect. 1.6.

1.6 Undirected Graph Reachability

In this section we develop the background and outline some of the ideas that are involved in the proof of Reingold's theorem [Rei08] which states that undirected graph reachability is in L.

The corresponding language we denote as UREACH: UREACH $= \{\langle G, s, t\rangle \mid G$ is an undirected graph containing a path from s to $t\}$. Clearly, UREACH is a special case of REACH and hence is in NL.

First, we note that without loss of generality we can assume the undirected graph G is 3-regular. That is to say, every vertex in the graph G has degree exactly 3. More precisely, given an instance $\langle G, s, t\rangle$ of UREACH, there is a deterministic logspace computation that we can apply to transform it into another instance $\langle G', s', t'\rangle$ in which G' is 3-regular such that $\langle G', s', t'\rangle \in$ UREACH if and only if $\langle G, s, t\rangle \in$ UREACH. The main idea involved in this transformation is that a vertex v in G of degree $d > 3$ can be replaced by a cycle of length d consisting of vertices $v_1, v_2, \ldots, v_d$, and the ith neighbor of v (say, in lexicographic order) made adjacent to v_i. This transformation can be carried out in logspace and in the transformed graph there is a path between s and t if and only if there is one in G. Repeated application of this step takes care of all vertices in G of degree more than 3. Vertices of degree 1 or 2 can be contracted (taking appropriate care if s or t is encountered in the contraction process).

Exercise Verify the details of the logspace transformation sketched above.

1.6.1 Reachability in Graphs of Logarithmic Diameter

Let G be a connected undirected graph. For every pair of vertices u, v of G let $d_G(u, v)$ denote the length of the shortest u to v path in G. The *diameter* of G is the maximum distance $\max_{u,v} d_G(u, v)$ between vertices in G. Suppose $c, d > 0$ are constants and $\langle G, s, t\rangle$ is an input instance of UREACH such that the maximum degree of G is d and the diameter of G is bounded by $c \log n$. There is a simple deterministic logspace algorithm to check if there is an s to t path in G. Since each vertex u of G has at most d neighbors, its neighborhood $N(u)$ can be indexed by $\{1, 2, \ldots, d\}$, say in increasing order of vertex names. Furthermore, we are looking for a path of length at most $c \log n$ from s to t. We can encode all paths of length $c \log n$ starting from s by sequences $d_1, d_2, \ldots, d_{c \log n}$, where $1 \leq d_i \leq d$ for each i. This sequence can be written down on the worktape using at most $O(c \log d \log n)$ space and the algorithm can cycle through all such sequences one by one in lexicographic order. For a sequence $d_1, d_2, \ldots, d_{c \log n}$ the path in G defined by it is $s = u_0, u_1, \ldots, u_{c \log n}$, where u_i is the d_ith neighbor of u_{i-1} for each i. The algorithm can generate the vertices of this path one by one. In order to generate u_i it only needs to store i and u_{i-1} which is $O(\log n)$ space. The algorithm accepts if $u_i = t$ for some i in some sequence.

The diameter of n-vertex graphs can be $\Omega(n)$ in general. However, it turns out, as shown by Reingold [Rei08] that there is a deterministic logspace computation that transforms an instance $\langle G, s, t\rangle$ of UREACH into an instance $\langle G', s', t'\rangle$ such that

(i) s' and t' are in the same connected component of G' if and only if s and t are connected in G.
(ii) G' is a d-regular graph for some constant d and has diameter bounded by $c \log n$ for some constant c.

Reingold's construction of G' [Rei08] is based on properties of expander graphs and the explicit construction of expanders. We will not describe the details of his algorithm. Instead, we will present a randomized logspace algorithm for UREACH and in the process explain some basic properties of expander graphs.

1.6.2 A Randomized Logspace Algorithm for UREACH

A randomized logspace bounded Turing machine is a Turing machine M that has, in addition to a read-only input tape and a logspace bounded worktape, one-way access to a random tape on which is written the outcome of a sequence of unbiased and independent random bits and in each time instant the random tape head moves one step to the right reading the next random bit.

Definition 6.1 A language $L \subseteq \Sigma^*$ is said to be in the class RL if there is a randomized logspace bounded Turing machine that runs in polynomial time such that

$$x \in L \longrightarrow \text{Prob}[M(x) \text{ accepts}] \geq 1/2$$
$$x \notin L \longrightarrow \text{Prob}[M(x) \text{ accepts}] = 0$$

The class RL stands for randomized logspace with 1-sided error. We can analogously define the more general class BPL which allows two-sided error.

Exercise L $\subseteq$ RL $\subseteq$ NL.

An interesting point in Definition 6.1 is that we need to insist on a polynomial time bound for the randomized machine in order to get a subclass of NL. This is in contrast to the definitions of L and NL where the polynomial time bounds is enforced by detecting repetition of configuration. In the case of randomized logspace computation, allowing long computation paths can influence acceptance probabilities in nontrivial ways. The following exercise explains this aspect.

Exercise Show that randomized logspace machines with one-sided error (and error probability bounded by $1/2$ as in Definition 6.1) that are allowed to run for exponential time accept precisely the class NL.

Our aim is to show that UREACH is in RL. As explained earlier it suffices to consider undirected graphs that are d-regular for some constant d.

Let $G = (V, E)$ be a d-regular graph on n vertices and let A denote its *normalized adjacency matrix*: $A_{ij} = 1/d$ if $ij \in E$ and $A_{ij} = 0$ otherwise.

Since G is undirected, note that A is a real symmetric matrix. An eigenvalue of A is a complex number λ such that $Ax = \lambda x$ for some nonzero vector x. Since A is a symmetric matrix, by standard linear algebra [HK71] it follows that A has all real eigenvalues. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ denote its n eigenvalues. It is easy to see that $-1 \leq \lambda_i \leq 1$ for all i.

The largest eigenvalue of A is 1 and an eigenvector corresponding to eigenvalue 1 is the all 1's vector. We can also think of the uniform distribution $(1/n, 1/n, \ldots, 1/n)^T$ on the vertex set as the eigenvector corresponding to 1.

Exercise

1. Prove that G has k connected components if and only if $1 = \lambda_1 = \lambda_2 = \cdots = \lambda_k > \lambda_{k+1} \geq \cdots \geq \lambda_n$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ denote the n eigenvalues of the normalized adjacency matrix A.
2. Prove that G is bipartite if and only if -1 is an eigenvalue of A.

As a consequence of the statements in the above exercise, if G is connected then $\lambda_2 < 1 = \lambda_1$. Indeed, when G is connected it is possible to give a better upper bound than 1 for λ_2. We will explain this bound and also use it to design a randomized logspace algorithm for UREACH.

Suppose λ and λ' are distinct eigenvalues of A, and x and x' be corresponding eigenvectors. Then we claim x and x' are mutually orthogonal. That is to say, $x^T x' = 0$. The claim holds because $x^T A x' = \lambda x^T x'$ as well as $x^T A x' = \lambda' x^T x'$ and $\lambda \neq \lambda'$.

By the Courant-Fisher-Weyl minmax theorem [CH53, Chap. 1, Sect. 4], we can write $\lambda_2 = \max_{x\perp\mathbf{1},||x||=1} x^T Ax$, where $\mathbf{1}$ stands for the all 1's vector. The condition $x \perp \mathbf{1}$ is equivalent to $\sum_i x_i = 0$. Since $x \neq 0$ there are some negative and some positive entries in x. Let $x_s < 0$ be the smallest entry of x and $x_t > 0$ be the largest. Since G is connected, there is a path P of length at most $n-1$ from s to t in G. Simplifying $\lambda_2 = x^T Ax$ we obtain

$$\begin{aligned}\lambda_2 &= \frac{1}{d}\sum_{ij\in E} 2x_i x_j\\ &= 1 - \frac{1}{d}\sum_{ij\in E}(x_i - x_j)^2\\ &\leq 1 - \frac{1}{d}\sum_{ij\in P}(x_i - x_j)^2.\end{aligned}$$

Now, by the triangle inequality we have $|x_t - x_s| \leq \sum_{ij\in P}|x_i - x_j|$, and by the Cauchy-Schwartz inequality we have $\sum_{ij\in P}|x_i - x_j| \leq \sqrt{n}\sqrt{\sum_{ij\in P}|x_i - x_j|^2}$. Squaring both sides, we obtain $\sum_{ij\in P}|x_i - x_j|^2 \geq (x_t - x_s)^2/n$, and combined with the above inequality for λ_2 this yields

$$\lambda_2 \leq 1 - \frac{1}{dn}(x_t - x_s)^2.$$

Now, since $\sum_i x_i^2 = 1$, it follows that $x_t - x_s \geq 2/\sqrt{n}$. Putting it together we have shown the following:

Lemma 6.2 *For a connected graph G, $\lambda_2(G) \leq 1 - \frac{4}{dn^2}$.*

We now turn to the randomized logspace algorithm for UREACH [AKL+79]. Let $\langle G, s, t\rangle$ be an instance of UREACH where G is a d-regular graph. The algorithm does a simple random walk on G starting at the vertex s. At the ith step of the random walk if the current vertex is u, then in the $(i+1)^{st}$ step the walk picks one of the d neighbors of u uniformly at random and moves to it. Because of the bound on $\lambda_2(G)$, we can show that if t lies in the same connected component as s then with high probability the random walk will visit t within polynomially many steps. Furthermore, this algorithm can be implemented by a randomized logspace Turing machine in the following manner. The current vertex u is stored in the worktape. The next $\log d$ random bits that pick a neighbor of u are read from the random tape by the tapehead moving right and reading a bit for $\log d$ steps.

Theorem 6.3 [AKL+79] UREACH *is in* RL.

Proof Let $\langle G, s, t\rangle$ be an instance of UREACH. We can assume that G is an n-vertex, d-regular graph for some $d \geq 3$. If there is no path from s to t then the random walk starting at s will never visit t. When t is reachable from s, we need to analyze the random walk and lower bound the probability of visiting t in a given number of steps.

At the ith step of the random walk, the state can be described by a probability distribution vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$, where $x_k \geq 0$ for each k and $\sum_k x_k = 1$. Each x_k in this vector denotes the probability of the random walk being at vertex k at the ith step. We note that any probability vector x can be written as $\mathbf{x} = \mathbf{u} + \mathbf{u}^\perp$, where $\mathbf{u} = (1/n, 1/n, \ldots, 1/n)^T$ is the uniform distribution and their difference $\mathbf{u}^\perp = \mathbf{x} - \mathbf{u}$ is clearly orthogonal to $\mathbf{u}$ because the inner product $(\mathbf{u}, \mathbf{x} - \mathbf{u}) = 0$. The normalized adjacency matrix A is the stochastic matrix that governs this random walk. If the probability distribution vector at the ith step is $\mathbf{x}$ then at the $(i+1)^{st}$ step the distribution vector is $A\mathbf{x}$.

Now, the initial distribution vector is $\mathbf{x} = (e_1, e_2, \ldots, e_n)^T$, where $e_s = 1$ and $e_j = 0$ for all $j \neq s$. Thus, after i steps of the random walk the probability distribution is $A^i\mathbf{x}$. Therefore, writing $\mathbf{x} = \mathbf{u} + \mathbf{u}^\perp$ we get

$$A^i\mathbf{x} = \mathbf{u} + A^i\mathbf{u}^\perp.$$

Since $\lambda_2 = \max_{x\perp 1, ||x||=1} x^T Ax$, it follows that

$$||A^i\mathbf{u}^\perp||^2 = |(u^\perp)^T A^{2i} u^\perp)| \leq |\lambda_2(G)|^{2i} ||u^\perp||^2.$$

We can bound $||u^\perp||^2 \leq 2$, since $||u^\perp|| \leq ||\mathbf{x}|| + ||\mathbf{u}||$, $||\mathbf{x}|| = 1$ and $||\mathbf{u}|| = 1/\sqrt{n}$. Letting $i = 2dn^3$ in Lemma 6.2 gives the bound $|\lambda_2(G)|^{4dn^3} \leq e^{-n}$. It follows that

$$||A^i\mathbf{x} - \mathbf{u}||^2 = ||A^i u^\perp)||^2 \leq e^{-n}.$$

Since each entry of $\mathbf{u}$ is $1/n$, the above bound implies that each entry of $A^i\mathbf{x}$ is at least $1/2n$. We can conclude that if t reachable from s then with probability at least $1/2n$ the random walk starting at s visits t within $i = 2dn^3$ steps. Consequently, the probability that the random walk visits t within $4dn^4$ steps is at least $1-(1-1/2n)^{2n} > 1/2$ for sufficiently large n. This bound on the success probability proves that the random walk algorithm is indeed an RL algorithm. □

1.7 Logspace Counting Classes

We now briefly introduce logspace counting classes. These complexity classes provide a tight classification of many natural computational problems. Formal definitions of these classes and their basic properties can be found in [BDHM92, AO96]. The central complexity class here is denoted GapL and captures the complexity of computing the integer determinant.

Definition 7.1 GapL is the class of functions $f : \Sigma^* \to \mathbb{Z}$, for which there is a logspace bounded nondeterministic Turing machine M such that on input $x \in \Sigma^*$, we have $f(x) = \text{acc}_M(x) - \text{rej}_M(x)$, where $\text{acc}_M(x)$ and $\text{rej}_M(x)$ denote the number of accepting paths of M and the number of rejecting paths of M, respectively, on input x.

GapL is a function class and we have an appropriate notion of logspace computable reductions for this class. Given two functions $f, g : \Sigma^* \to \mathbb{Z}$ we say that f is logspace many-one reducible to g (as usual, denoted $f \leq_m^L g$) if there are two logspace computable functions e and v such that for every $x \in \Sigma^*$

$$f(x) = v(g(e(x))).$$

Notice that if g is computable in logspace then f is also logspace computable. A function $g \in$ GapL is GapL-*complete* under logspace reductions if $f \leq_m^L g$ for every $f \in$ GapL. A fundamental result due Toda [Tod91] is that the integer determinant is GapL-complete. The result has an elegant proof by Mahajan and Vinay [MV97]. A problem analogous to REACH that is easily shown to be GapL-complete is in the following exercise.

Exercise Given a directed acyclic graph G with a source vertex s and two sink vertices t_1 and t_2, let $g(G, s, t_1, t_2)$ denote the difference of the number of directed paths from s to t_1 from the number of directed paths from s to t_2. Show that this function is GapL-complete.

The class GapL is analogous to the complexity class #P [Val79] introduced by Valiant to study the complexity of computing the integer permanent. A function $f : \Sigma^* \to \mathbb{N}$ is said to be in #P if there is an NP machine M such that $f(x) = \mathrm{acc}_M(x)$ for all $x \in \Sigma^*$. A celebrated result due to Valiant [Val79] is the #P-completeness of the integer permanent (for an appropriate notion of reductions). The reader can find details of the proof in the textbook [AB09].

Remark 7.2 As an aside, comparing the computational complexity of the determinant and the permanent is a fascinating topic that is central to the area of arithmetic circuits. This book includes several chapters on the topic of arithmetic circuits. Meena Mahajan's article presents a detailed survey on Valiant's algebraic complexity classes.

The complexity of computing the determinant of matrices over the field $\mathbb{F}_2$ is captured by the language class $\oplus$L defined below.

Definition 7.3 A language $L \subseteq \Sigma^*$ is in the complexity class $\oplus$L (called "parity-L") if there is a logspace bounded nondeterministic Turing machine M such that $x \in L \iff \mathrm{acc}_M(x)$ is odd.

Exercise Show that the following language is $\oplus$L complete under logspace many-one reductions:

$$\text{ODD}-\text{REACH} = \{\langle G, s, t\rangle \mid G \text{ is a DAG such that there is an odd number of directed paths from } s \text{ to } t\}.$$

Exercise Assuming that the integer determinant is GapL-complete show that

$$\begin{aligned}\text{NONSINGULAR} = \{M \mid M \ \text{ is a square integer matrix such that}\\ \det(M) \equiv 1 (\text{mod } 2)\}\end{aligned}$$

is complete for $\oplus$L under logspace many-one reductions.

The last complexity class we will introduce in this chapter is *unambiguous logspace* denoted UL. A logspace bounded nondeterministic Turing machine M is said to be *unambiguous* if on every input $x \in \Sigma^*$ the machine M has *at most* one accepting path. A language L is in the complexity class UL if there is an unambiguous logspace-bounded nondeterministic Turing machine M that accepts the language L.

A surprising inclusion of complexity classes shown in [RA00] is that NL $\subseteq$ UL/poly.[2] Whether NL equals UL or not is an intriguing open problem and is discussed in Vinodchandran's article in this volume.

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Chapter 2
Investigations Concerning the Structure of Complete Sets

Eric Allender

Abstract This chapter will discuss developments bearing on three related research directions where Somenath Biswas has made pioneering contributions:

- Isomorphism of Complete Sets
- Creative Sets
- Universal Relations

Some open questions in each of these directions will be highlighted.

Keywords Berman-Hartmanis conjecture · Isomorphism · NP-completeness · Creativity · Universality

Mathematics Subject Classification (2010) Primary 68Q15

2.1 Introduction

How many NP-complete sets are there?

Although there is a trivial and uninteresting answer to this question (namely: there is a countably infinite number of NP-complete sets), there is a large body of work investigating the proposition that in actuality there is *precisely one* NP-complete set (modulo minor encoding details).

Let us clarify what is meant by "minor encoding details": When we consider the set SAT of satisfiable Boolean formulae, it is irrelevant if we encode formulae using round parentheses () or square ones [], or if we write variables in italic font or in bold face. Any of these choices would lead to a reasonable encoding of SAT; they all yield encodings of SAT that are equivalent in some sense.

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One way to attempt to formalize this notion of "equivalence" is to say that two sets A and B, $A, B \subseteq \{0, 1\}^*$, are p-isomorphic if there is a bijection f defined on $\{0, 1\}^*$ computable and invertible in polynomial time, such that $f(A) = B$. This approach leads to the famous Berman-Hartmanis conjecture [BH77], which asserts that all of the sets that are NP-complete under $\leq_m^{\mathrm{p}}$ reductions are p-isomorphic.

The isomorphism conjecture(s) will be discussed in more detail in Sect. 2.2. However, a bit of background about isomorphism of complete sets is necessary here, in order to provide a coherent overview of the current paper. The Berman-Hartmanis conjecture arose, at least in part, because of a cultural inheritance from the study of computability theory. If we accept the rough idea that NP is analogous to the class of computably-enumerable sets, and polynomial time is analogous to the class of computable functions, then the Berman-Hartmanis conjecture is analogous to the Myhill Isomorphism Theorem in computability theory, which states: All of the sets that are complete for the class of computably-enumerable sets under $\leq_m$-reductions are computably-isomorphic to the Halting Problem. (For expositions of this work, see [Rog67] or [Soa87].)

Central to the proof of the Myhill Isomorphism Theorem is the notion of a *creative* set. We postpone until Sect. 2.3 the precise definition of "creativity," but this is an appropriate time to mention that the name was coined by Emil Post [Pos44], who was profoundly influenced by certain consequences of Gödel's incompleteness theorems. Post believed that there was a link between the notion of "mathematical creativity" and the fact that there is a computable function that, given a set of consistent axioms for arithmetic, will produce a true statement that cannot be proved from those axioms. Post's definition of creativity abstracts out this property of the set of theorems provable from a list of axioms.

In the setting of recursion theory, the creative sets turn out to exactly coincide with the sets that are complete for the class of computably enumerable sets under $\leq_m$-reducibility, and this is useful in proving Myhill's Isomorphism Theorem. Thus it was natural for researchers to try to define a resource-bounded analog of creativity. But it is not entirely clear what is the best way to define such an analog. Different definitions were presented by various authors [JY85, Wan91], but the definition of *NP-creative* sets by Agrawal and Biswas [AB96] provides several advantages over other definitions. (For instance, all NP-creative sets are NP-complete; this is not known for some other notions.) Just as all of the NP-complete sets in Garey and Johnson [GJ79] are p-isomorphic to SAT, so also are they all NP-creative. Although Agrawal and Biswas refrain from conjecturing that all NP-complete sets are NP-creative, we may as well consider a "creativity" version of the Berman-Hartmanis conjecture:

The **Creativity Hypothesis**: The class of NP-creative sets coincides with the class of sets that are NP-complete under $\leq_m^{\mathrm{p}}$ reductions.

One might at first guess that, since SAT is NP-creative, then *everything* that is p-isomorphic to SAT would also be NP-creative—but Agrawal and Biswas showed that, if this is true, then the Creativity Hypothesis is true (and hence P $\neq$ NP).

Thus, NP-creativity and p-isomorphism yield two possibly different subclasses of the NP-complete sets, each of which captures a notion of "naturalness" (in the sense that all of the currently-known "natural" examples of NP-complete sets are both creative and p-isomorphic to SAT). Neither the Creativity Hypothesis nor the Berman-Hartmanis Conjecture is known to imply the other. Section 2.3 discusses creativity in more detail.

The final section of this chapter highlights one additional direction in which Somenath Biswas has pushed, in order to give additional insight into the structure underlying completeness. Although the Berman-Hartmanis conjecture focuses on the NP-complete *sets*, let us not forget that much of the practical interest in NP-completeness derives from the desire to find *witnesses* for membership in an NP-complete set. That is, at a fundamental level, it is not a *set*, such as HAMILTONIAN-CIRCUIT, that is of primary interest, but rather the corresponding *relation* consisting of pairs (G, C) such that C is a Hamiltonian cycle in the graph G.

Is it possible that the (string, witness) relations for every NP-complete set are all "the same" in some sense? Note that it is not at all obvious how to formulate this sense of "sameness." For instance, if there is a polynomial-time relation $W(x, y)$ consisting of witnesses y for string x, then there is a relation $W'(x, z)$ such that $W'(x, z)$ is true if and only if $W(x, y)$ holds, where y is the string that results by deleting every second symbol of z. These two relations both serve as witness relations for the same set in NP, but they do give different numbers of witnesses for the same string, and thus they fail to be "the same" on a fairly basic level. And yet, they do contain exactly the same information, in some intuitive sense. Agrawal and Biswas succeeded [AB92] in giving a useful definition of "universal relations", in order to capture the sense in which the defining relations for all known NP-complete sets seem to be "the same." More recently, Chaudhary, Sinha, and Biswas have adapted this notion for nondeterministic logspace [CSB07]. This topic is explored in Sect. 2.4.

2.2 The Isomorphism Conjecture(s)

An outstanding survey of recent developments related to isomorphisms of complete sets is now available [Agr11], and the reader is urged to consult that source for a more complete introduction to the topic and an in-depth discussion of the current state of the field. The discussion here will focus on describing aspects of the topic that are (1) related to the work of Somenath Biswas, or (2) related to some open questions or developments that are not mentioned in [Agr11].

The winds of public opinion have blown back and forth, regarding the Berman-Hartmanis Conjecture. It appears to have initially been viewed as fairly plausible. One of the first published accounts questioning whether the isomorphism conjecture is true appears in the work of Joseph and Young [JY85]. They defined a class of NP-complete sets they named the *k-creative sets* (which will also be discussed in Sect. 2.3), and they explicitly conjectured that some k-creative sets are not p-isomorphic to SAT. In particular, for any one-one length-increasing function f computable in polynomial time, they defined a set K_f, and they pointed out that, if f is suitably hard to invert, then it is hard to see how K_f can be p-isomorphic to SAT. Kurtz, Mahaney, and Royer subsequently elaborated on this intuition, and formulated the *Encrypted Complete Set Conjecture*, which states that there is a one-one, length-increasing, one-way function f such that SAT and f(SAT) are not p-isomorphic.

Several papers were then written, all of which tended to buttress support for the Encrypted Complete Set Conjecture (all of which are discussed in the survey [Agr11]). But then attention shifted to some interesting classes of *restricted* $\leq_m^{\mathrm{p}}$-reductions; we will discuss some of these developments in more detail below—but the general trend of these investigations has been to weaken our confidence in the Encrypted Complete Set Conjecture. More recently, there has been a productive series of investigations of *more powerful* classes of reductions, notably including m-reductions computed in P/poly [AW09, Agr02] and in NP $\cap$ coNP [HHP07] (this latter class of reductions is known as SNP-reductions). As a consequence, we now know that some fairly plausible hypotheses imply that all sets complete for NP under P/poly-reductions and SNP-reductions are P/poly-isomorphic and SNP-isomorphic, respectively. Combined with the results about restricted $\leq_m^{\mathrm{p}}$ reductions that will be discussed below, the picture that emerges is that NP-complete sets are either provably isomorphic or at least are reasonably likely to be isomorphic, both when one considers reductions strictly *less* powerful and *more* powerful than $\leq_m^{\mathrm{p}}$ reductions. It remains to be seen whether any of these lessons ultimately shed much light on the case of $\leq_m^{\mathrm{p}}$ reductions themselves.

2.2.1 Restricted Reductions

It is debatable whether $\leq_m^{\mathrm{p}}$ reductions really constitute the most important class of reductions. There is a rich structure of complexity classes within P, and $\leq_m^{\mathrm{p}}$-reducibility is essentially useless in elucidating this structure. This was the motivation for Jones et al. to introduce log-space reductions [Jon75]—but even in that pioneering work, it was realized that a higher precision tool was necessary, in order to investigate the structure of logspace, which is why Jones introduced what he called *log-bounded rudimentary reductions* [Jon75]. This was several years before the modern study of circuit complexity got under way, and it took a while before it was noticed that log-bounded rudimentary reductions actually correspond to many-one reductions computed by uniform AC^0 circuits (that is, constant-depth polynomial-size families of circuits of AND and OR gates) [AG91]. Ultimately, AC^0 reductions have proved to be the most useful notion of reducibility for investigating subclasses of P, surpassing both NC^1 reducibility [CM87] and 1-L reducibility (discussed below). However, AC^0 reducibility posed greater challenges initially, and thus progress was made first with 1-L reducibility.

2.2.2 1-L Reductions

1-L reductions are functions computed by logspace-bounded Turing machines that make a single pass over their input tape (from left to right). They were introduced by Hartmanis, Immerman, and Mahaney [HIM78, HM81] for many of the same reasons that had led Jones to introduce log-bounded rudimentary reductions. 1-L reductions offered the advantages of being significantly more convenient and intuitive (since the original formulation of log-bounded rudimentary reductions lacked the intuitive appeal of the AC^0 formulation). In this brief overview, we avoid giving more detailed definitions of 1-L reductions, but it is appropriate to note that there are some differences in the formulations as presented in [HIM78] and [HM81], and that some of these formulations result in a class of reductions that is not closed under composition (see [All88]).

1-L reductions are easy to invert, and this fact, combined with some diagonalization techniques, enabled a proof that all sets complete for PSPACE under 1-L reductions are p-isomorphic [All88]. They are not isomorphic under 1-L isomorphisms [BH92], but they are complete under isomorphisms computable in nondeterministic logspace [HH93].

Agrawal and Biswas [AB96] succeeded in showing that, even for classes as small as deterministic logspace (and indeed, for any class that is closed under logspace reductions that produce output at most linearly longer than the input) the sets complete under 1-L reductions are complete under one-one, length-increasing, polynomial-time invertible reductions. (Thus by [BH77] all such sets are p-isomorphic.) Finally, Agrawal proved that all such sets are isomorphic via reductions computed by one-way nondeterministic logspace (1-NL) machines [Agr96] (and the same paper proves an analog of the Berman-Hartmanis conjecture for 1-NL reductions).

At this point, study of the structure of sets complete under 1-L reductions effectively stopped.[1] The major open questions had been solved. But this was merely a prelude to a much more exciting and significant development in the history of work on the isomorphism problem, focusing on reductions that are computable by constant-depth circuits. Indeed, although there are problems (such as the PARITY problem) that are computable by 1-L machines but are not computed by AC^0 circuits [FSS84], Agrawal had shown that, for essentially all complexity classes of interest, all sets complete under 1-L reductions *are* complete under reductions computable in AC^0 [Agr96]. And whereas all functions computable by 1-L machines are easy to invert, this is not the case for AC^0. Thus, by considering AC^0 reductions, the research community was moving on to a richer class of complete sets, and was confronting some of the essential issues raised by the Encrypted Complete Set Conjecture.

[1] This is not to suggest that work on 1-L computation stopped. Indeed, much of the very large body of work on *streaming algorithms* consists of the study of 1-L computation.

2.2.3 Constant-Depth Circuits

Attention was first focused on AC^0 isomorphisms by considering a very restricted class of AC^0 reductions: projections (which are reductions computed by circuits with no gates, other than negation gates). Sets complete for NP (and other classes) under uniform projections are isomorphic under uniform AC^0 isomorphisms [ABI97].

The logical next step was to work on extending this result from projections to NC^0 functions (that is, functions computed by constant-depth circuits with fan-in $O(1)$, so that each output bit depends on only $O(1)$ input bits—as contrasted with projections, where each output bit depends on either zero or one input bit). As part of this investigation, it was also discovered that, at least for the class NC^1, the sets complete under AC^0 reductions are also complete under NC^0 reductions, thereby obtaining the first theorem showing that the sets complete under AC^0 reductions are all AC^0 isomorphic [AA96]. Subsequently, the authors were joined by Rudich, in showing that this holds not only for NC^1, but also for NP and for most other complexity classes of interest [AAR98].

These initial AC^0 isomorphism theorems were proved only in the nonuniform setting. After a series of intermediate steps improving the uniformity condition [AAIPR01, Agr01], Agrawal succeeded in overcoming some daunting technical difficulties, in presenting a Dlogtime-Uniform version of the isomorphism theorem [Agr11], which stands as one of the crowning achievements of the study of the structure of complete sets. This work not only shows that a natural re-phrasing of the Berman-Hartmanis Conjecture (in terms of AC^0 reductions and isomorphisms) is true, but also gives a convincing setting where the Encrypted Complete Set Conjecture fails (since even when f is an AC^0 function that provably cannot be inverted in AC^0, $f(\mathrm{SAT})$ is still AC^0-isomorphic to SAT).

2.2.4 Open Questions

Again, please refer to [Agr11] for several interesting open questions. Here are a few additional questions relating to isomorphisms, that are not discussed there.

Two important problems that are not believed to be NP-complete are Factoring and the Minimum Circuit Size Problem:

$$\mathrm{FACT} = \{(x, i, b) : \text{the } i\text{th bit of the prime factorization of } x \text{ is } b\}.$$

$$\mathrm{MCSP} = \{(\chi_f, s) : \chi_f \text{ denotes a string of length } 2^m \text{ (for some } m\text{) that is the truth-table of a Boolean function } f \text{ on } m \text{ variables and } s \text{ denotes an integer such that } f \text{ can be computed by a Boolean circuit of size at most } s.\}$$

(In the definition of FACT, the prime factorization is presented as $p_1^{e_1}, \ldots, p_k^{e_k}$, where each exponent $e_i > 0$, and $p_i < p_{i+1}$, so that each number has a unique prime factorization.)

I suspect that FACT is probably not complete for *any* reasonable complexity class under AC^0 reductions. In an earlier survey [All01] I outlined a possible approach toward proving that this is the case. Namely, I noted that it would suffice to show that there is no one-one length-increasing NC^0 reduction from FACT$\times\{0, 1\}^*$ to FACT (or no isomorphism between these sets, computable and invertible in depth-three AC^0). All of my attempts to construct such a reduction have involved multiplication in some form, and this is not computable in AC^0. Perhaps, I suggested, one could show that multiplication is inherent, in computing such a reduction. Now, however, after some illuminating discussions with Michal Koucký, I no longer think that this is a promising approach. One way to build a padding function would be to map the pair $((x, i, b), y)$ to the triple (z, i, b), where $z = xy'$, where y' has binary representation $10^\ell y_1 0^\ell y_2 0^\ell \ldots y_{n-1} 0^\ell y_n 0^\ell z'$ where ℓ is suitably large, and where z' has $\log^{O(1)} n$ bits. If y' is prime, then it will be the largest prime factor of z, and thus the initial part of the prime factorizations of x and of z will be the same. The product xy' can be computed in AC^0, because of the padding by 0^ℓ and because z' is small. It is reasonably likely that a value of z' exists so that y' will be prime, although number theorists have not yet established that this holds. It would be *very* hard to show that *no* such z' can be found in uniform AC^0. Thus it is reasonably likely that a padding function for (a suitable encoding of) FACT does exist in AC^0. This does not guarantee that such a padding function can be found in NC^0, but it does illustrate some of the difficulties of pursuing this approach.

Kabanets and Cai have presented some arguments, suggesting that MCSP is not complete for NP under $\leq_m^p$ reductions [KC00]. Can one obtain even stronger evidence, suggesting that MCSP is not p-isomorphic to SAT? It is certainly not clear that MCSP should have a padding function (i.e., a polynomial-time computable and invertible function f mapping MCSP$\times\{0, 1\}^*$ onto MCSP). It is even harder to see how to construct a padding function if one fixes the circuit size s to be something exponentially large, but still much smaller than 2^m, such as this set:

MCSP2 = $\{\chi_f : \chi_f$ denotes a string of length 2^m that is the truth-table of a Boolean function f on m variables such that f can be computed by a Boolean circuit of size at most $2^{m/2}$.$\}$

As Kabanets and Cai observe [KC00], if MCSP2 has a padding function computed in polynomial time, then $BPP = P$. The connection between the paddability of MCSP2 and the BPP versus P problem arises through the easy observation that any set C isomorphic to SAT has a P-printable sets contained both in C and in $\overline{C}$, combined with the following equivalence (where the first condition listed is the well-studied Impagliazzo-Wigderson derandomization hypothesis [IW97]):

- There is a set $A \in$ E that requires circuits of size greater than $2^{n/2}$ for all large n iff
- There is a P-printable set B contained in the complement of MCSP2 of the form $B = \{\chi_f : \chi_f$ denotes a string of length 2^m that is the truth-table of $A^{=m}\}$.

The observations above do *not* provide much evidence against MCSP2 being isomorphic to SAT; rather, they merely indicate that it will not be easy to prove that it *is* isomorphic to SAT. What unlikely consequences would follow if MCSP2 (or MCSP) turned out to be isomorphic to SAT?

2.3 Creative Sets

A set is defined to be *creative* if its complement is *productive* (and this holds for all of the variants of "creativity" and "productivity" that have been considered). Thus, in order to discuss creative sets, we must first define productive sets.

A set A is *productive over a class of languages* $\mathcal{C}$ if there is a function (a so-called "*productive function*") witnessing that $A \notin \mathcal{C}$, in some sense. In order to make this definition precise, we must be explicit what notion of Turing machine indices I we are using to represent elements of $\mathcal{C}$, and what class of functions F we will allow as productive functions. Thus we can say that A is (F, I)-productive if there is a function $f \in F$ such that, for every $i \in I$, $f(i) \in A$ if and only if $f(i)$ is not in the language accepted by machine i. That is, given i, f finds an input on which A differs from the i-th element of $\mathcal{C}$.

Agrawal and Biswas define a set A to be NP-creative if its complement is (polynomial-time,I)-productive, where I is an indexing of nondeterministic polynomial-time Turing machines, where machine M_i has the property that, on all inputs, it runs in time $|i|$ [AB96]. It is far from obvious that this is an appropriate definition, since these machines all run in $O(1)$ time! Thus, in particular, $\{L(M_i) : i \in I\}$ is *not* equal to NP, and does not even contain all of the sets in, say, AC^0! Nevertheless, Agrawal and Biswas are able to demonstrate that this definition yields at least as many sets as an earlier notion of creativity (the "k-creative" sets of [JY85], which were re-dubbed "k-completely-creative" sets by Wang [Wan91] to distinguish them from another creativity notion he introduced), and they also show that all NP-creative sets are NP-complete (in contrast to the situation for the "k-creative" sets of [Wan91], which are neither known to include sets such as SAT, nor to be contained in the class of NP-complete sets). Figure 2.1 indicates the inclusion relations among these various classes of "creative" sets for NP.

However, when these creativeness definitions are adapted to larger complexity classes (such as EXP), they all coincide exactly with the class of sets complete under $\leq_m^{\mathrm{p}}$ reductions. (The issue boils down to a question of whether the complexity class $\mathcal{C}$ can diagonalize over the class F of productive functions. This is true when $\mathcal{C}$ = EXP, but is not known to be true for $\mathcal{C}$ = NP.)

Even though the definition of NP-creative sets is less intuitive than the definition of the class of sets that are p-isomorphic to SAT, Agrawal and Biswas make a convincing argument that all "natural" NP-complete sets (including all of the NP-complete sets listed in [GJ79]) are NP-creative. Thus there is some merit in investigating the "Creativity Hypothesis" mentioned in the introduction—the hypothesis that all NP-complete sets are NP-creative—as an alternative to the Berman-Hartmanis

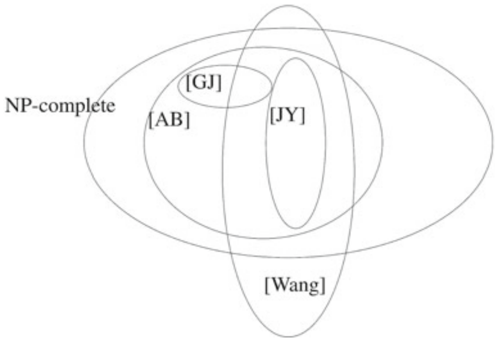

Fig. 2.1 Diagram, showing (likely) inclusions among classes of "creative" sets for *NP*. The region labeled "*[GJ]*" indicates the list of "natural" NP-complete problems catalogued in [GJ79]. It is not known to contain any of the k-creative sets defined by Joseph and Young [JY85], indicated by the region labeled *[JY]*. This same class was called "k-completely-creative" by Wang [Wan91], who also introduced another class of k-creative sets, indicated by the region labeled [Wang]; it is not known whether all of those sets are NP-complete. The region labeled *[AB]* indicates the NP-creative sets of Agrawal and Biswas [AB96]

conjecture. Proving that either of these conditions hold would entail proving P $\neq$ NP. (In the case of the Creativity Hypothesis, Agrawal and Biswas show that any NP-creative set is complete for NP under reductions that are "exponentially honest," in the sense that, for some constant c, $2^{c|f(x)|} > |x|$ for all x [AB96]. Thus, in particular, no finite set can be NP-creative.) It is particularly interesting that Agrawal and Biswas show that, if all of the sets that are p-isomorphic to SAT are NP-creative, then the Creativity Hypothesis holds.

The Creativity Hypothesis has not received much attention. Here are some questions that might yield some interesting insights:

- Are all of the sets that are complete for NP under AC^0 reductions NP-creative? How about the sets that are complete under first-order projections? Or the sets that are complete for NP under 1-L reductions?
- If one assumes that FACT or MCSP are NP-creative, can one derive stronger conclusions than if one merely assumes that these sets are NP-complete?
- Agrawal and Biswas have shown that the complement of any NP-creative set contains an infinite subset in NP. Consider a set such as $\{x :$ the time-n^2-bounded Kolmogorov complexity of x is greater than $|x|/2\}$. Would we expect this set to have an infinite NP-subset? (Actually, the answer is probably *Yes*! It is observed in Sect. 2.2.4 that, under the Impagliazzo-Wigderson derandomization hypothesis, this set even has a P-printable subset.) Can one derive strong and unlikely conclusions from the assumption that this set is the complement of an NP-creative set?

2.4 Universal Relations

All of the NP-complete sets that are p-isomorphic to SAT have a padding function, and even the NP-complete sets that are not known to be p-isomorphic to SAT (such as certain k-creative sets, and sets of the form $f(\mathrm{SAT})$ where f is one-way) have "padding" functions if we drop the requirement of invertibility (i.e., a reduction from $A \times \Sigma^*$ to A that is one–one and length-increasing, but is not necessarily invertible). Similarly, all known NP-complete sets are disjunctive-self-reducible. (A set A is called "disjunctive-self-reducible" if there is a polynomial-time-computable function that takes a string x as input, and produces a list $y_1, \ldots, y_{|x|^{O(1)}}$ as output, such that $x \in A$ iff $\exists i \; y_i \in A$.)

Agrawal and Biswas defined two operators on relations (which they name the *join* and *equivalence* operators) that are related to paddability (without invertibility) and disjunctive-self-reducibility, respectively (in the sense that if the witness relation for a set A has the given operator computable in polynomial time, then A is paddable or disjunctive-self-reducible, respectively). Remarkably, they were able to show that the relations with these two operators are *precisely* the relations from which any other NP-witness relation can be "recovered" in a fairly natural sense. (The details of these definitions will not be repeated here; see [AB92]).

One thing that I particularly like about [AB92] is their presentation of a new class of NP-complete sets. Let f be *any* one–one and size-increasing polynomial function. They define a relation R_f as follows: $(z, w) \in R_f$ if $|w| = 4|z|^3$ and one of the following three conditions hold:

1. $|z| = 1$ and $w \in \{0100, 0101, 0110, 0111, 1001, 1010, 1011\}$.
2. For some $r > 0$, $w = \#^r x_1\#w_1\#\#x_2\#w_2\#\# \ldots \#\#x_n\#w_n$, where $f(1\#x_1\#x_2\# \ldots \#x_n) = z$ and for all $i \leq n$, $(x_i, w_i) \in R_f$.
3. For some $r > 0$, $w = \#^r x\#i_1\#i_2\# \ldots \#i_n\#j_1\# \ldots \#j_n\#\#w'$, where $f(1\#x\#i_1\# \ldots \#i_n\#j_1\# \ldots \#j_n) = z$ and for each $k \leq n$, bits number i_k and j_k of w' are the same.

Agrawal and Biswas show that $\{x : \exists y(x, y) \in R_f\}$ is NP-complete. I know of no *direct* way to see that this set is NP-complete; the proof of completeness presented by Agrawal and Biswas follows because the relation R_f is universal (because it has the required join and equivalence operators). It would be interesting to know if there is any example of a natural NP-complete problem, for which it is easier to prove NP-completeness by presenting the join and equivalence operators, than to present a traditional $\leq_m^{\mathrm{p}}$ reduction.

The theory of Probabilistically-Checkable Proofs tells us that problems in NP have witness relations with very special encoding structure. It would be interesting to know if this body of knowledge can be merged with the theory of universal relations, to obtain any new insights.

Figures 2.2, 2.3 and 2.4 shows inclusion relations among the different notions considered in this survey, all of which present ways to give a mathematically precise definition that can serve as a proxy for the vague concept of what it means to be a "natural" NP-complete set:

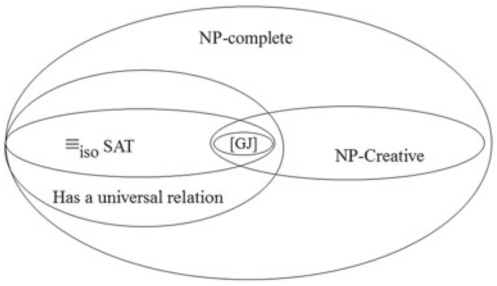

Fig. 2.2 Diagram, showing inclusions among notions of "natural" NP-complete sets discussed in this survey

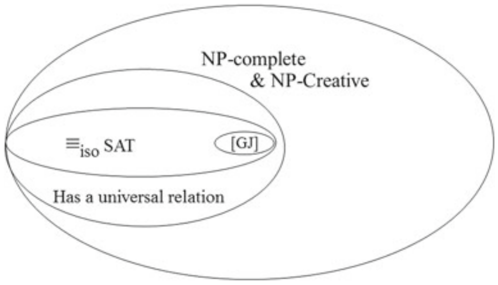

Fig. 2.3 Diagram, showing inclusions among notions of "natural" NP-complete sets, if p-isomorphism preserves NP-Creativity

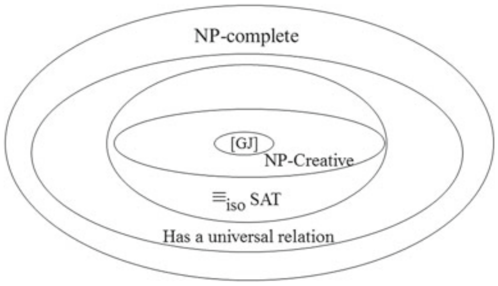

Fig. 2.4 Diagram, showing inclusions among notions of "natural" NP-complete sets, if all NP-Creative sets are p-isomorphic

- being p-isomorphic to SAT.
- being NP-creative.
- having a universal relation.

Do all NP-creative sets have universal relations? (Note in this regard that Agrawal and Biswas show that sets with universal relations are all complete for NP under polynomially honest reductions (i.e., reductions f where there is a polynomial p such that $p(|f(x)|) \geq |x|$ for all x [AB92]), whereas the NP-creative sets are only known to be complete under exponentially honest reductions [AB96]. Thus it might be better to ask first whether all NP-creative sets that are complete under polynomially honest reductions have universal relations.)

Are the standard witness relations for MCSP and FACT universal? (Possibly this question is easy to answer …) Note in this regard that Agrawal and Biswas show that the standard witness relation for Graph Isomorphism is not universal—as well as a (somewhat nonstandard) witness relation for Simple Max Cut. They introduce a more general notion of universal relations, which they call "generalized universal relations," and show that the Simple Max Cut witness relation is generalized universal. If one is able to show that the standard witness relations for MCSP and FACT are not universal, then perhaps one can show that they are also not generalized universal. This would provide some additional evidence that these problems are not NP-complete.

Are there any additional implications that one can prove, regarding the Berman-Hartmanis conjecture, the Creativity Hypothesis, and the question of whether all NP-complete sets have universal witness relations?

(There has been some additional work by other authors, regarding universal relations. The reader is referred to [CSB07] for a discussion of this work.)

2.5 Conclusions

The notions of p-isomorphism, NP-creativity, and universality provide three ways to identify properties that are shared by all of the "natural" NP-complete sets. Although the work of Somenath Biswas and others has given us a body of interesting results regarding these notions, a number of intriguing open questions remain.

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Chapter 3
Space Complexity of the Directed Reachability Problem over Surface-Embedded Graphs

N. Variyam Vinodchandran

Abstract The graph reachability problem, the computational task of deciding whether there is a path between two given nodes in a graph, is the canonical problem for studying space-bounded computations. Three central open questions regarding the space complexity of the reachability problem over directed graphs are: (1) improving Savitch's $O(\log^2 n)$ space bound, (2) designing a polynomial-time algorithm with $O(n^{1-\epsilon})$ space bound, and (3) designing an *unambiguous* non-deterministic log-space (UL) algorithm. These are well-known open questions in complexity theory, and solving any one of them will be a major breakthrough. We discuss some of the recent progress reported on these questions for certain subclasses of surface-embedded directed graphs.

Keywords Graph reachability · Space-bounded computation · Computational complexity

Mathematics Subject Classification (2010) Primary 68Q15

3.1 Introduction

The graph reachability problem, the problem of deciding whether there is a path from a given vertex s to a vertex t in a given graph, is central to space-bounded computations. Various versions of this problem characterize several important space complexity classes. Over directed graphs, it is the canonical complete problem for

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non-deterministic log-space (NL). The breakthrough result of Reingold implies that the undirected reachability problem characterizes the complexity of deterministic log-space (L) [Rei08]. It is also known that a certain restricted promise version of the reachability problem over directed graphs characterizes randomized log-space computations (RL) [RTV06]. Clearly, progress in space complexity studies is directly related to progress in understanding graph reachability problems. We refer the readers to an excellent (albeit two decades old) survey by Avi Wigderson [Wig92] to further understand the significance of reachability problems in complexity theory.

This article is far from an exhaustive survey on the space complexity of the graph reachability problem. In particular, some of the major progress (such as Reingold's algorithm for undirected graph reachability and Saks and Zhou's deterministic simulation of randomized log-space) are not discussed here. Instead, we limit our discussion to some recent progress that the author and his collaborators reported on these questions for certain subclasses of surface-embedded directed graphs. It is mostly an elaboration of the talk that the author gave on Prof. Somenath Biswas's 60th birthday celebrations at IIT Kanpur in August of 2012.

3.1.1 Three Central Questions

We first discuss three central questions concerning the space complexity of the directed graph reachability problem. These are well-known and difficult open questions in the area, and progress on any of these is a step towards the much bigger NL versus L question (the first two problems are discussed in Wigderson's 1992 survey [Wig92]). However, the author feels that the known barriers for attacking these problems are much less severe than those known for many difficult open problems in time-bounded complexity classes and circuit lower bounds, and believes that breakthrough progress on these problems can be made in the near future.

Problem 1: Improving Savitch's bound. About four decades ago Savitch proved that the reachability problem over directed graphs with n vertices can be solved in space $O(\log^2 n)$ deterministically [Sav70]. This implies that problems that can be solved nondeterministically in space $s(n)$ have *deterministic* algorithms with $O(s^2(n))$ space bound. Thus, for polynomial space bounds, nondeterminism does not add any additional power to determinism. For the important case when the space bound is $O(\log n)$, Savitch's theorem implies that all problems in NL can be solved deterministically in $O(\log^2 n)$ space. *Is this quadratic blow-up necessary?* This is one of the most important open problems in this topic.

Problem 2: Improving the BBRS bound. The time complexity of Savitch's algorithm is $\Theta(n^{\log n})$—a super-polynomial bound. The standard breadth first search algorithm (BFS) is another fundamental algorithm for solving graph reachability. BFS can be implemented in linear time but requires linear space. BFS is efficient in time but not in space, and Savitch's algorithm is efficient in space but takes super-polynomial time. Hence a natural and significant question that researchers have considered is whether we can design an algorithm for the directed graph reachability problem that is efficient in both time and space. In particular, can we design a polynomial-time algorithm for the directed graph reachability problem that uses only $O(n^{1-\epsilon})$ space for some constant ϵ? The best known result in this direction is the two decades old bound due to Barnes et al. [BBRS98] (which we call the BBRS bound). By cleverly combining BFS and Savitch's algorithm, Barnes et al. designed a polynomial-time algorithm for reachability that uses space $O(n/2^{\sqrt{\log n}})$—a slightly sub-linear function. Improving the BBRS bound remains another significant open question regarding the space complexity of the directed reachability problem.

Problem 3: NL versus UL Problem. UL denotes an unambiguous subclass of NL. A decision problem L is in UL if and only if there exists a nondeterministic log-space machine M deciding L such that, for every instance x, M has at most one accepting computation on input x [BJLR91, ÀJ93]. Thus UL is a complexity class that is sandwiched between NL and L. Is $\mathsf{NL} = \mathsf{UL}$? While typically such collapse results are unlikely in complexity theory (and even if they are likely, they are nearly impossible to prove), there is an increasing body of evidence that in this specific case the answer is yes, and the author believes that we might be able to prove this equality using known techniques. Reinhardt and Allender showed using the isolation lemma that in the nonuniform setting NL coincides with UL; that is $\mathsf{NL/poly} = \mathsf{UL/poly}$ [RA00]. Further, in a subsequent paper, Allender, Reinhardt, and Zhou showed that, under a certain hardness assumption the construction given in [RA00] can be *derandomized* to show that $\mathsf{NL} = \mathsf{UL}$ [ARZ99]. Thus it is very likely (at least to the author) that $\mathsf{NL} = \mathsf{UL}$, though we do not yet have a proof. Can we show $\mathsf{NL} = \mathsf{UL}$ unconditionally?

3.1.2 Outline

In the next two sections we discuss some progress that we have made towards these three open questions—Sect. 3.2 on Problems 1 and 2, and Sect. 3.3 on the NL versus UL problem. All the results discussed in these sections are for directed graphs embedded on topological surfaces. As an aside, in Sect. 3.4 we reproduce the proof of the BBRS bound from [BBRS98], partly to bring more attention to this nice algorithm.

3.2 Space Efficient Reachability Algorithms for Graphs with Topological Structure

An important sub-case of Problem 1 (and Problem 2) is to design reachability algorithms that beat Savitch's bound (respectively, the BBRS bound) for directed graphs with some topological structure (graphs that are embedded on topological surfaces). We discuss some recent progress reported along this direction. The main results are

(1) algorithms that beat both Savitch's bound and the BBRS bound for a subclass of directed *acyclic* graphs parameterized by the number of sources and the genus of the embedding [SBV10, SV12] (2) an algorithm for directed planar reachability that improves on the BBRS bound [INP+13]. The main approach in both these results is that of space-efficient "kernelization."

Kernelization is a known preprocessing technique in designing algorithms (for example in the area of fixed parameter tractability). Kernelization algorithms are reductions from a problem to itself so that the easy part of the instance is abstracted out and the core part is retained in the reduced instance. The hope is that the core part will be of smaller size and hence known algorithms can be applied to this compressed instance yielding algorithms with better complexity. We first illustrate this in a simple scenario.

Consider a reachability instance $\langle G, s, t\rangle$ where $G = (V, E)$ is a n-vertex graph with the guarantee that it has at most k directed edges (the remaining edges are undirected). Let G_{un} be the undirected graph we get by removing all the directed edges from G. For a directed edge $e = (u, v)$ let $tail(e) = u$ and $head(e) = v$. We show a simple log-space reduction that takes $\langle G, s, t\rangle$ and produces a reachability instance $\langle G', s', t'\rangle$ where G' is a directed graph with $O(k)$ vertices.

In the reduced graph $G' = (V', E')$, $V' = \{s', t'\} \cup \{v_e \mid e$ is a directed edge in $G\}$. The pair $(v_{e_1}, v_{e_2}) \in E'$ if $tail(e_2)$ is in the same connected component as $head(e_1)$ in G_{un}. For a $v_e \in V'$, $(s', v_e) \in E'$ if $tail(e)$ is in the same connected component of s in G_{un} and (v_e, t') if $head(e)$ is in the same connected component of t in G_{un}. Notice that this reduction is log-space since for checking whether two vertices u, v are in the same connected component of G_{un}, we can use Reingold's log-space algorithm for undirected reachability. It is clear that there is a s-t path in G if and only if there is a s'-t' path in G'. Using this reduction together with Savitch's algorithm we get that reachability in graphs with $n^{o(1)}$ directed edges can be solved in $o(\log^2 n)$. Also, by applying BFS to the reduced graph, we get that for any $\epsilon > 0$, reachability in graphs with $O(n^{1-\epsilon})$ directed edges can be solved in polynomial time and $O(n^{1-\epsilon})$ space.

We now describe the main kernelization result of [SBV10, SV12] and its application. Let $\mathcal{G}(m, g)$ denote the class of DAGs with at most $m = m(n)$ source vertices embedded on a surface (orientable or non-orientable) of genus at most $g = g(n)$, where n is the number of vertices. Building on [SBV10], in [SV12] we show the following reduction for graphs in $\mathcal{G}(m, g)$.

Theorem 1 [SV12] *There is a log-space reduction that, given an instance $\langle G, s, t\rangle$ (presented as a combinatorial embedding) where $G \in \mathcal{G}(m, g)$ and s, t are vertices of G, outputs an instance $\langle G', s', t'\rangle$ where G' is a directed graph and s', t' vertices of G', so that (a) there is a directed path from s to t in G if and only if there is a directed path from s' to t' in G', (b) G' has $O(m + g)$ vertices.*

By combining the above reduction with Savitch's theorem (with $m = g = 2^{O(\sqrt{\log n})}$) we get that reachability over graphs with $2^{O(\sqrt{\log n})}$ sources embedded on a surface of genus $2^{O(\sqrt{\log n})}$ can be decided in deterministic log-space. For $m = g = n^{o(1)}$ we get $o(\log^2 n)$ space algorithm for reachability that beats Savitch's bound. For $m = g = O(n^{1-\epsilon})$, we get $O(n^{1-\epsilon})$ space algorithm with polynomial running time for reachability, for any small constant ϵ, improving the BBRS bound.

One of the motivations for investigating the reachability problem for this class of surface-embedded graphs comes from earlier work due to Allender, Barrington, Chakraborthy, Datta, and Roy [ABC+09]. In [ABC+09], the authors considered reachability in planar DAGs with a *single* source vertex. They called this class of graphs Single source Multiple sink Planar Dags (SMPD). SMPD generalizes Single source Single sink Planar Dags (SSPD). SSPDs are interesting since they generalize *series parallel graphs* which is a well-studied restriction of directed acyclic graphs. Allender et al. presented a log-space algorithm for reachability in SMPD and left open whether reachability can be solved using logarithmic space over planar DAGs with multiple source nodes. In [SBV10], building on the SMPD algorithm, we present a log-space algorithm for planar dags with logarithmic number of sources. In the subsequent paper [SV12], via a careful use of techniques developed in [SBV10], we proved the log-space kernelization theorem that in particular implied a log-space algorithm for reachability in graphs with $2^{O(\sqrt{\log n})}$ sources, embedded on a surface of genus $2^{O(\sqrt{\log n})}$. The proof of this theorem is technically involved and we do not discuss it here. It remains a significant open question whether reachability for planar Dags (without any restriction on the number of sources) can be solved deterministically in $o(\log^2 n)$ space.

While improving Savitch's bound even for planar graphs remains open, the question of improving the BBRS bound for planar graphs was settled recently. Using a kernelization approach, in [INP+13], we show that the directed planar reachability problem can be solved in polynomial time using roughly $O(n^{1/2})$ space. This result extends a similar bound for the reachability problem over *grid graphs* due to Asano and Doerr [AD11].

Theorem 2 [INP+13] *For any constant $0 < \epsilon < 1/2$, there is an algorithm that, given a directed planar graph G and two vertices s and t, decides whether there is a path from s to t. This algorithm runs in time $n^{O(1/\epsilon)}$ and uses $O(n^{1/2+\epsilon})$ space, where n is the number of vertices of G.*

For showing this result, we first design a polynomial-time and $\tilde{O}(\sqrt{n})$-space algorithm for computing a "separator" of $O(\sqrt{n})$ size for an undirected planar graph. (For any undirected graph G and for any constant ρ, $0 < \rho < 1$, a *ρ-separator* of G is a a subset of vertices S whose removal disconnects G into two subgraphs A and B, such that $|A|$ and $|B|$ is at most ρn). This algorithm is based on a parallel algorithm for constructing a planar separator due to Gazit and Miller [GM87].

Theorem 3 [INP+13] *There is an algorithm that takes an undirected planar graph G with n vertices as input and outputs a (8/9)-separator of G of size $O(\sqrt{n})$. This algorithm runs in polynomial time and uses $\tilde{O}(\sqrt{n})$ space. (Here $\tilde{O}(s(n))$ denotes $O(s(n)(\log n)^{O(1)})$).*

Proof Sketch While for obtaining the $O(n^{1/2+\epsilon})$ space bound we need a recursive approach, it is easy to illustrate the idea for the case when the space bound is $O(n^{2/3})$. Let $G = (V, E)$ be the input directed planar graph. Let G_u be the underlying undirected graph. The first step is to apply the planar separator algorithm repeatedly k times on the connected components of G_u that are bigger than $n^{2/3}$ to further partition the graph until every component is of size $\leq n^{2/3}$. It is easy to see that after $k = \lceil \frac{2}{3} \times \frac{\log n}{\log(9/8)} \rceil$ applications we get a collection $\mathcal{S}$ of separators with total size $O(n^{2/3})$ so that removing $\mathcal{S}$ partitions the graph into disconnected components where each component is of size $\leq n^{2/3}$. (This is a standard trick used in many separator-based algorithms). Let $C_1, C_2, \ldots, C_l$ be the set of vertices in these components.

Now consider the kernel graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V} = \mathcal{S} \cup \{s, t\}$. For any two nodes x and y in $\mathcal{V}$, (x, y) is a directed edge if and only if there is a directed path from x to y in the subgraph of G that is induced by $\mathcal{V} \cup C_i$ (call this G_i), for some connected component C_i in the partition. Clearly, the number of nodes in $\mathcal{G}$ is $O(n^{2/3})$. Now consider the algorithm that decides whether there is a directed path from s to t in G by performing a BFS on $\mathcal{G}$ starting at s. Whenever BFS queries $(x, y) \in \mathcal{E}$?, the algorithm performs BFSs for each of the graphs G_i starting at x looking for a path from x to y, and returns YES if for some G_i this inner BFS returns true. Notice that since $|\mathcal{V} \cup C_i|$ is at most $O(n^{2/3})$, each of this BFSs can be implemented in $O(n^{2/3})$ space and polynomial time. Hence, overall the algorithm takes $O(n^{2/3})$ space and polynomial time.

For extending this proof to the $O(n^{1/2+\epsilon})$ space bound, we need $|\mathcal{S}| = O(n^{1/2+\epsilon})$. But that will result in large components and a simple application of the inner BFSs will not give the required space bound. Instead, we can apply the algorithm recursively. By limiting the number of recursive applications to a constant, we can make sure that the running time remains polynomial. We omit the details. □

Before we move on to the next section we mention that there is a certain computational model known as NNJAG model in which it is possible to prove lower bounds those match both Savitch's bound and the BBRS bound [Poo93, CR80, EPA99]. The NNJAG model is a branching program model tailored towards the reachability problem with restricted access to the input graph. While all the known algorithms for general reachability (such as BFS, DFS, Savitch's algorithm, BBRS algorithm) can be implemented in the NNJAG without substantial increase in time and space (in comparison to implementations on a random access machine), it is not clear to the author how a general approach such as kernelization can be handled in these models. It is conceivable that algorithms based on kernelization can overcome NNJAG lower bounds and help solve these open problems.

3.3 NL Versus UL Problem

The main progress on this problem has also been on graphs with some topological structure. We first discuss a technique developed by Reinhardt and Allender [RA00] since all the known proofs on this problem use their technique.

A positively and polynomially weighted graph is said to be *min-unique* if, between any two nodes the minimum weight path (if it exists) is unique. Here the weight of a path is the sum of the weights of its edges. Reinhardt and Allender [RA00] showed, using an adaptation of the inductive counting technique of Immerman [Imm88] and Szelepcsényi [Sze88], that the reachability question in min-unique graphs can be decided in UL. They combine this construction with an observation due to Wigderson [Wig94] that the isolation lemma of Mulmuley et al. [MVV87] can be used to nonuniformly assign weights to make a given graph min-unique. These two steps imply the collapse result that NL is in nonuniform UL.

Thus a space-efficient derandomization of the isolation lemma will show that NL = UL. However, derandomizing isolation lemma in its generality will have much deeper consequences and is a well-known and difficult open problem [AM08]. Instead, a viable and concrete approach for showing NL = UL is to first consider a class of graphs over which the reachability problem is NL-complete, and prescribe a *deterministic* log-space computable weight function which will make graphs in this class min-unique.

In [ABC+09], the authors solve this min-unique weight assignment problem for the class of *layered grid graphs*. Layered grid graphs are graphs with vertices on a $n \times n$ grid and the edges that go west-to-east and south-to-north. Subsequently in [BTV09], we show how to extend this weight function to general grid graphs (without restriction on the direction of edges). This implied that directed planar reachability is in UL since the directed planar reachability problem is known to be reducible to the grid graph reachability problem [ADR05]. In fact this even implied that the reachability problem for graphs embedded on constant genus surfaces and graphs that are $K_{3,3}$-free and K_5-free are in UL since the reachability problem for these classes of graphs reduces to the directed planar reachability problem [KV10, TW09] in log-space.

While, in [BTV09] we showed that directed planar reachability is in UL, it was not clear then how to solve the min-unique weight assignment problem for planar graphs. In a subsequent chapter, we solve this problem using *Green's Theorem*, a celebrated result from multivariate calculus [TV12]. Since it is a slightly nonstandard approach to use an analytical result to solve discrete problems, we believe this approach has the potential to solve the general NL versus UL problem. We next present the proof of the min-unique weight assignment problem for directed planar graphs based on Green's theorem.

Green's theorem, stated below, relates a certain curve integral over a closed curve on the plane to a related double integral over the region enclosed by this curve.

Theorem 4 Green's Theorem *Let C be a closed, piecewise smooth, simple curve on the plane which is oriented counterclockwise. Let R_C be the region bounded by C. Let P and Q be functions of (x, y) defined on a region containing R_C that have continuous partial derivatives in the region. Then*

$$\oint_C (P\,dx + Q\,dy) = \iint_{R_C} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

We use the following corollary that we get if we substitute $Q(x, y) = x$ and $P(x, y) = 0$ in Green's theorem.

Corollary 5 (Area by line integrals) *Let C be a closed, piecewise smooth, simple curve on the plane that is oriented counterclockwise. Let R_C be the region bounded by C. Then,*

$$\text{Area}(R_C) = \oint_C x\,dy.$$

Theorem 6 [TV12] *There is a log-space algorithm that, given any planar graph G, assigns weights to the edges so that the resulting weighted graph is min-unique.*

Let us assume that the planar graph $G = (V, E)$ is presented as a straight-line drawing. That is, each vertex v is represented as a point (x_v, y_v) in the plane so that an edge (u, v) is a line between points (x_u, y_u) and (x_v, y_v). In addition, no such lines intersect other than at the vertices. Moreover, we assume that the coordinates are integer points with values bounded by $\text{poly}(n)$ (n is the number of vertices). Typically, planar graphs are presented as a combinatorial embedding and it is not clear how such line drawings can be computed in log-space from a combinatorial embedding. However, this is not critical and in [TV12] we show how to handle this presentation issue.

Let $e = (u, v)$ be a directed edge directed from u to v where u is identified with the point (x_u, y_u) and v is identified with (x_v, y_v). For such a directed edge, define a weight function w as follows:

$$w_{gt}(e) = 2 \times \oint_e x\,dy = (y_v - y_u)(x_v + x_u)$$

The required isolation property of the weight function is proved using the following crucial lemma.

Lemma 7 *Let G be a directed planar graph and let C be any directed simple cycle in G. Let R_C be the region enclosed by C. Then the weight of the cycle C, $|w_{gt}(C)| = 2 \times \text{Area}(\text{R}_\text{c})$. In particular, $w_{gt}(C)$ is nonzero.*

Proof Let $C = (e_1, e_2, \ldots, e_l)$ be a directed cycle-oriented counterclockwise. Then we have

$$w_{gt}(C) = \sum_i w_{gt}(e_i) = 2 \times \sum_i \oint_{e_i} x\,dy = 2 \times \oint_C x\,dy = 2 \times \text{Area}(R_C)$$

The third equality follows from the linearity of integrals and the last equality follows from Corollary 7. If C is oriented clockwise, we get that $w_{gt}(C) = -2 \times \text{Area}(R_C)$. Hence the lemma. □

The following lemma establishes Theorem 6.

Lemma 8 *Let G be a directed planar graph. Then with respect to the weight function w_{gt}, for every pair of nodes u and v, if there is a directed path from u to v, then there is a* unique *path from u to v of minimum weight.*

Proof Suppose there are u, v so that there are two u to v paths P_1 and P_2 of minimum weight. We assume that the paths do not intersect on vertices other than the end points (otherwise we can find two vertices u' and v' along these paths that satisfies this property using a standard cut-and-paste argument and use these vertices instead). We have $w_{gt}(P_1) = w_{gt}(P_2)$. Now consider the graph G' that is same as G except that the path P_2 is reversed so that the set of edges $(P_1, -P_2)$ becomes a simple cycle in G' ($-P_2$ denotes the reversed path). Let C denote this cycle. Then $w_{gt}(C) = w_{gt}(P_1) + w_{gt}(-P_2) = w_{gt}(P_1) - w_{gt}(P_2) = 0$. The second equality because of the skew-symmetry of the weight function. This contradicts Lemma 7. $\square$

It is clear that we can use Green's Theorem to design a class of min-unique weight functions. In fact any "nice" solution to the differential equation $\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = 1$ will yield such a weight function. For example, setting $P(x, y) = \frac{-y}{2}$ and $Q(x, y) = \frac{x}{2}$ to the left-hand side of Green's theorem yields the weight function $w(e) = (x_u y_v - x_v y_u)$ which is also min-unique.

Can we use such geometric techniques to design min-unique weight functions for larger classes of graphs? In [BTV09] it is observed that reachability in layered grid graphs over three dimensions is complete for NL. It might be possible to use generalizations of Green's theorem (such as Stokes' theorem) to design a min-unique weight function for three-dimensional layered grid graphs.

3.4 The BBRS Bound

We present the algorithm due to Barnes, Buss, Ruzzo, and Schieber [BBRS98] that solves the directed graph reachability problem in sub-linear space and polynomial time.

Theorem 9 [BBRS98] *For any k, there is a polynomial-time algorithm that given a directed graph G and two nodes s and t, decides whether there is a path from s to t in space $O(\frac{n}{2^{k\sqrt{\log n}}})$, where n is the number of vertices of G.*

Proof The algorithm uses a combination of BFS and Savitch's algorithm. For a parameter λ (this will be set to $2^{k\sqrt{\log n}}$ to get the desired bound), it constructs the levels of BFS tree that are at λ distance apart. Divide the vertex set into levels according to distance from s. That is, the level i vertex set is defined as:

$$V_i = \{v \mid d(s, v) = i\}, \text{ where } d \text{ is the distance function.}$$

Partition the set of vertices into λ equivalence classes $C_0, C_1, \ldots, C_{\lambda-1}$ where $C_j = \bigcup_{i=0}^{\lfloor n/\lambda \rfloor} V_{j+i\lambda}$. Since the C_is partition the vertex set, we have the following fact.

Fact 10 $\exists j^*$ so that $|C_{j^*}| \leq \lceil \frac{n}{\lambda} \rceil$

The PARTIAL-BFS algorithm (described below) constructs C_{j^*} level by level. Since we do not explicitly know which C_j has $\leq \frac{n}{\lambda}$ nodes, the algorithm will keep a counter to count the number of vertices and try from $j = 0$. At any point of the construction, if $|C_j| > \frac{n}{\lambda}$, it will abandon that j and try the next value for j. The algorithm will succeed for the first such j. This will only increase the space by an additive $O(\log n)$ factor and the time by a multiplicative factor of λ. Hence we assume that the algorithm knows j^*. Following is the description of the PARTIAL-BFS algorithm.

```
PARTIAL-BFS$(G, s)$ /* Outputs $C_{j^*}$*/
    $V_0 = \{s\}$
    $V_{j^*} =$ CONSTRUCT$(G, V_0, j^*)$
    For $i = 1$ to $\lfloor \frac{n}{\lambda} \rfloor$
        $V_{i\lambda+j^*} =$ CONSTRUCT$(G, V_{(i-1)\lambda+j^*}, \lambda)$
        Add $V_{i\lambda+j^*}$ to $C_{j^*}$
    End-For
    Output $C_{j^*}$
```

In general, the procedure CONSTRUCT takes G and a set of nodes S and a parameter λ and returns the set of nodes that are at distance λ from some node in S. CONSTRUCT will use the bounded version of the reachability problem (Barnes et al. calls it *short path problem*) as subroutine.

$$\text{SPATH}(u, v, \lambda) = \text{true} \Leftrightarrow \text{there is a path of length} \leq \lambda \text{ from } u \text{ to } v \text{ in } G.$$

We can use an algorithm for SPATH as subroutine to solve CONSTRUCT as follows. Given (G, S, λ), to check whether $v \in V$ is at distance λ from some vertex in S, first check whether SPATH(u, v, λ) is true for some $u \in S$ and check for all $u \in S$, SPATH$(u, v, \lambda - 1)$ is false.

For a given algorithm for SPATH, let $T(n, \lambda)$ be its time complexity and $S(n, \lambda)$ be its space complexity. Then the time complexity of CONSTRUCT is $O(n^3)T(n, \lambda)$ and its space complexity is $O(\frac{n}{\lambda}) + S(n, \lambda)$. Moreover, once C_{j^*} is constructed, reachability can be solved by making $\lceil \frac{n}{\lambda} \rceil$ calls to SPATH(u, t, λ) (for all $u \in C_{j^*}$). Thus the total running time for the reachability algorithm will be $O(n^4)T(n, \lambda)$ and the space bound will be $O(\frac{n}{\lambda}) + S(n, \lambda)$.

We will now focus on SPATH. We will use a divide and conquer approach as in Savitch's algorithm to design an algorithm for SPATH. The problem with a direct application of Savitch's algorithm is its running time: at each level of recursion it cycles through all n nodes as a candidate for the middle node. This results in $O(n^{\log n})$ time. Since we are interested in keeping the time polynomial, we can not afford to cycle through all n nodes. Instead, we will divide the set of nodes into μ equivalence classes and use a Savitch-like divide and conquer on these equivalence classes (instead of the vertices). For $\mu = 2^{O(\sqrt{\log n})}$ the depth of recursion will be $O(\sqrt{\log n})$ and this approach will result in polynomial time.

For a parameter μ, partition the vertex set into μ equivalence classes [1], [2], ... [μ] where vertex $x \in [a] \Leftrightarrow x \equiv a \pmod{\mu}$. Each equivalence class has $\lceil \frac{n}{\mu} \rceil$ elements (except for the last one whose cardinality may be smaller). We will use $[a]$, $[b]$, $[c]$ etc to denote these equivalence classes of vertices. Although this is not a very standard notation, the i^{th} vertex of the equivalence class $[a]$ (according to some fixed ordering) will be denoted by $[a](i)$.

Consider the procedure MODIFIED-SAVITCH(G, $[a]$, $[b]$, X, l) where $[a]$ and $[b]$ are equivalence classes of vertices, X is an $\lceil \frac{n}{\mu} \rceil$ binary array, and l is a length parameter. This procedure returns a binary vector Y of size $\lceil \frac{n}{\mu} \rceil$, where

$$Y[j] = 1 \Leftrightarrow \exists i \text{ so that } X[i] = 1 \text{and there is a path}$$
$$\text{of length} \leq 2^l \text{from} [a](i)\ to\ [b](j)$$

SPATH(u, v, λ) can be solved by one call to MODIFIED-SAVITCH with parameter ($[a]$, $[b]$, X_u, $\lceil \log_2 \lambda \rceil$) where $[a]$ = the equivalence class containing u, $[b]$ = the equivalence class containing v, and X_u is the vector with 1 in the index corresponding to u and 0 otherwise. There is a path from u to v if and only if there is a 1 in the index corresponding to v in the output vector Y. Below is a recursive version of the algorithm MODIFIED-SAVITCH.

```
MODIFIED-SAVITCH($G$, $[a]$, $[b]$, $X$, $l$)
  If $l = 0$ then
    If $[a] = [b]$ then $Y \leftarrow X$
    Else $Y[j] = 1$ iff $\exists i$ such that there is an edge from $[a](i)$ to $[b](j)$
  Else
    $Y \leftarrow \vec{0}$
    For $c = 1$ to $\mu$
        $Z \leftarrow$ MODIFIED-SAVITCH($G$, $[a]$, $[c]$, $X$, $l - 1$)
        $Y_c \leftarrow$ MODIFIED-SAVITCH($G$, $[c]$, $[b]$, $Z$, $l - 1$)
        $Y \leftarrow Y \vee Y_c$
  Return $Y$
```

Correctness of MODIFIED-SAVITCH is easy to prove. Its time and space bounds can be estimated using the following recurrences:

$$\begin{aligned} S(l) &= O(\frac{n}{\mu}) + S(l-1) \\ &= O(\frac{n}{\mu}) \times l \\ T(l) &= \mu \times 2 \times T(l-1) + O(n) \\ &= (2\mu)^{l+1} \times O(n) \end{aligned}$$

Setting $\mu = 2^{(k+1)\sqrt{\log n}}$ and $l = \lceil \log_2 \lambda \rceil$, we get an algorithm for SPATH with time complexity $T(n, \lambda) = O(2^{\log \lambda} \times 2^{(k+1)\sqrt{\log n}(\log \lambda + 1)} \times n)$ and space complexity $S(n, \lambda) = O(\frac{n}{2^{(k+1)\sqrt{\log n}}} \times \log \lambda)$.

For $\lambda = 2^{k\sqrt{\log n}}$, this results in polynomial time and space $O(\frac{n}{2^{k\sqrt{\log n}}})$ giving an algorithm for the reachability problem with polynomial running time and $O(\frac{n}{2^{k\sqrt{\log n}}})$ space bound. □

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Chapter 4
Algebraic Complexity Classes

Meena Mahajan

Abstract This survey describes, at an introductory level, the algebraic complexity framework originally proposed by Leslie Valiant in 1979, and some of the insights that have been obtained more recently.

Keywords Algebraic complexity · Circuits · Formulas · Branching programs · Determinant · Permanent

Mathematics Subject Classification (2010) Primary 68Q15 · Secondary 68Q05

4.1 Introduction

In this survey, I am going to try and describe the algebraic complexity framework originally proposed by Leslie and Valiant [Val79, Val82], and the insights that have been obtained more recently. This entire article has an "as it appeals to me" flavour, but I hope this flavour will also be interesting to many readers. The article is not particularly in-depth, but it is an invitation to read [BCS97, Bür00a] and many recent papers on the topic, and to start attacking the open problems in the area.

Valiant started out with the mission of understanding the core essence of reductions and completeness, as witnessed in both recursive function theory and in computational complexity theory. He provided an algebraic framework in which to interpret the clustering of natural problems into completeness classes, even for problems of an algebraic rather than combinatorial nature. He had a remarkable hypothesis:

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Linear algebra offers essentially the only fast technique for computing multivariate polynomials of moderate degree.

Clearly, then, we are going to talk about polynomials, not languages or functions.

4.2 Valiant's Original Framework

Let $\mathbb{F}$ be any field, and let $\mathbb{F}[x_1, \ldots, x_n]$ be the ring of polynomials over indeterminates $x_1, \ldots, x_n$ with coefficients from $\mathbb{F}$. Consider a family (f) of polynomials $(f_n)_{n\geq 1}$, where each f_n is in $\mathbb{F}[x_1, \ldots, x_{s(n)}]$ for some function $s : \mathbb{N} \longrightarrow \mathbb{N}$. When should we say that (f) is tractable? Clearly, if there are too many variables to keep track of, there cannot be tractability. So we will henceforth demand that s is a polynomially bounded function ($\exists c, \forall n, s(n) \leq c + n^c$); then the nth polynomial f_n has $O(n^c)$ variables. But that is of course not enough.

There are many ways in which we can set the bar for tractability. Here's a first attempt. Can (f) be computed by a *formula* of reasonable size? To elaborate further, a formula is an expression defined recursively:

1. for each $c \in \mathbb{F}$, "c" is a formula of size 0 computing the polynomial c,
2. for each indeterminate x_i, "x_i" is a formula of size 0 computing the polynomial x_i, and
3. if F_1, F_2 are formulas computing polynomials f_1 and f_2, then "$(F_1 + F_2)$" and "$(F_1 \times F_2)$" are formulas of size size(F_1) + size(F_2) + 1 each, computing the polynomials $f_1 + f_2$ and $f_1 \times f_2$, respectively.

Notice that size(F) is just the number of ring/field operations used to construct F.

Instead of such a recursive definition, we could have a more intuitive picture: a formula is a rooted binary tree where internal nodes are labelled $+$ or $\times$ and leaf nodes are labelled from the set $\mathbb{F} \cup X$, where X is the set of indeterminates. The size is just the number of non-leaf nodes.

Now, for tractability, we could require that there is a polynomially bounded function $t : \mathbb{N} \longrightarrow \mathbb{N}$ and a family of formulas $(F_n)_{n\geq 1}$ such that each F_n computes f_n and has size at most $t(n)$. Let us use the notation VF to denote families of polynomials tractable in this sense. (VF: Valiant's Formulas—of course, Valiant didn't use this name! This class is also referred to as VP$_e$: Valiant's Polynomial-sized Expressions. Personally, I prefer VF. Also note, in formal logic, the formulas/expressions referred to above are called terms, hence VF means families with polynomial "termic complexity".)

Here's a second attempt: Can (f) be computed by a *straight-line program* of reasonable size? As before, we will declare polynomial size to be reasonable. Straight-line programs are programs where instructions involve adding or multiplying previously computed polynomials, no divisions and no conditionals (no if-then-else). In the more intuitive picture, they correspond to directed *acyclic* graphs where each node is a source node (indegree 0) labelled from the set $\mathbb{F} \cup X$, or has indegree 2 and is labelled $+$ or $\times$. A designated sink node (outdegree 0) is the output node. Each node computes a polynomial in the obvious way, and the graph computes the polynomial at the output node. (The acyclicity constraint ensures that there is a linear ordering of the nodes such that each node, or instruction, only uses previously computed polynomials. This gives the straight-line program.) The size is the number of non-source nodes; again, this corresponds to the number of ring/field operations required. Such graphs are in fact exactly *algebraic circuits*, and we now look for polynomial size circuit families.

Clearly, this model generalises formulas. The catch is that it generalises it too much! To see why, consider the following circuit family: C_n has $n+1$ nodes $v_0, v_1, \ldots, v_n$, and the labeling is $v_0 = x_1$, $v_{i+1} = v_i \times v_i$ for $i \in [n]$. The family of polynomials (f_n) computed by (C_n) is $f_n = x_1^{2^n}$. Even for small integer values of x_1, writing down the value of $f_n(x_1)$ is going to require exponentially many bits. How can we say that such a family (f_n) is tractable?

So we need to impose some additional restrictions. The obvious parameter to restrict is the degree of the polynomial. Say that the family (f_n) has moderate degree if for some polynomially bounded function $d : \mathbb{N} \longrightarrow \mathbb{N}$, the degree of each polynomial f_n is at most $d(n)$. If degree$(f_n) = D$ is polynomially bounded, then on integer arguments with b-bit representations, the value of f_n requires no more than poly(n, b) bits. (In general, it needs no more than poly(n, D, b) bits.) Henceforth, to qualify for the label tractable, a family (f_n) must have polynomially bounded degree.

(Why didn't we face this problem when considering *VF*? Simply because a formula of size t cannot compute a polynomial of degree more than $t+1$. Don't just believe me; check this by induction on formula size.)

Now we have our second possible definition of tractability: (f_n) is tractable if the sequence degree(f_n) is polynomially bounded, and there is a polynomially bounded function $t : \mathbb{N} \longrightarrow \mathbb{N}$ and a family of straight-line programs, or algebraic circuits, $(C_n)_{n\geq 1}$, such that each C_n computes f_n and has size at most $t(n)$. Let us use the notation VP to denote families of polynomials tractable in this sense. (VP: Valiant's analogue of the Boolean complexity class P. Valiant called these families p-computable [Val82]).

The well-studied polynomial family from linear algebra, the determinant of a matrix of indeterminates, is known to be tractable in this second sense. (To define the family (Det$_n$), imagine an $n \times n$ matrix A_n with a new indeterminate x_{ij} at each position (i, j), and let Det$_n$ be the polynomial that represents the determinant of A_n. Thus Det$_1 = x_{11}$, Det$_2 = x_{11}x_{22} - x_{12}x_{21}$, and so on. Clearly, this family satisfies the mandatory conditions: Det$_n$ has n^2 variables and is of degree n.) This is not surprising; we know that the determinant can be computed efficiently (in polynomial time) over instantiated matrices using, say, Gaussian elimination. But to compute the symbolic determinant via a straight-line program, Gaussian elimination is apparently not directly of use because we can't search for nonzero pivots and eliminate them (remember, no divisions and no conditionals). However, Strassen [Str73] gave a generic method of converting any straight-line program with divisions to a division-free straight-line program; the resulting program's size is polynomially bounded in the original size, the number of variables and the degree. Thus, we can conclude that there are polynomial-sized straight-line programs for the symbolic determinant. There are more direct algorithms as well; Samuelson, Berkowitz, Csanky, ... See [MV97] for an explicit description of circuits of size $O(n^4)$ (my favourite one—no surprise!).

Whether the determinant can be computed efficiently by formulas (is Det_n in VF?) is still famously open. We know that it needs formula size at least $\Omega(n^3)$, see [Kal85]. But we do know that it can be computed by formulas of subexponential size $2^{O(\log^2 n)}$. This can be shown in many different ways, one of which we will look at a bit later, but the earliest demonstration of this follows from Csanky's algorithm [Csa76], which uses binary arithmetic operations and $O(\log^2 n)$ parallel time. Thus, if we use quasi-polynomial ($2^{\log^c n}$ for some constant c) formula-size as the defining property for tractability (giving a class that we can call VQF), then again the family (Det_n) has long been known to be tractable. We could also use quasi-polynomial circuit size as the defining property for tractability, giving a class that we can call VQP. But VQP obviously contains VP and VQF, so (Det_n) is in VQP; nothing new there. (Note: in defining VQF and VQP, the quasi-polynomial limit on formula or circuit size is over and above the requirement that the degree and number of variables are polynomially bounded.)

Does VP include essentially all interesting and natural polynomial families? We do not know. In fact, there is a large list of such polynomial families not known to be in VP. The most natural one is the permanent family (Perm_n) where Perm_n is the polynomial representing the permanent of an $n \times n$ matrix A_n of indeterminates. It is tantalisingly similar to the determinant; just the sign term is missing.

$$\mathrm{Det}_n = \sum_{\sigma \in S_n} \mathrm{sign}(\sigma) \prod_{i=1}^{n} x_{i\sigma(i)} \qquad \mathrm{Perm}_n = \sum_{\sigma \in S_n} \prod_{i=1}^{n} x_{i\sigma(i)}$$

Yet, it does not seem to be tractable. How "intractable" is it?

Mirroring the definitions of the Boolean complexity classes P and NP, Valiant proposed a notion of p-definability in [Val79]. A polynomial family (f_n) is p-definable if it can be written as an exponential sum, over partial Boolean instantiations, of another tractable family. Formally, a family (f_n) over $s(n)$ variables and of degree $d(n)$ is p-definable if $s(n)$ and $d(n)$ are polynomially bounded, as always, and further, there exist a polynomially bounded function m, and a family of polynomials (g_n) in VF, such that g_n has $s(n) + m(n)$ variables denoted $\{x_1, \ldots, x_{s(n)}, y_1, \ldots, y_{m(n)}\}$, and

$$f_n(\tilde{x}) = \sum_{y_1=0}^{1} \sum_{y_2=0}^{1} \cdots \sum_{y_{m(n)}=0}^{1} g_n(\tilde{x}, \tilde{y}).$$

This looks like an algebraic analogue $\sum \cdot \mathrm{VF}$ of the boolean class $\exists \cdot F$, where F is the class of languages decided by polynomial-size formulas. But it is well-known that $\exists \cdot F = \mathrm{NP}$, so this should be algebraic NP. Later, Valiant redefined p-definability (no, that is not a circular definition!) as exponential sums of families in VP, rather than VF; that is, VNP $= \sum \cdot$VP. For clarity, let us agree to temporarily refer to these two definitions as VNF (or VNP$_e$) and VNP. However, Valiant [Val82] showed that these two classes are in fact the same, so just VNP will do. The proof involves showing that VP is contained in $\sum \cdot$VF. And it is of course easier to show upper bounds with the definition of VNP rather than VNF.

Now Valiant observed that not only (Det$_n$), even (Perm$_n$) is p-definable. This should be similar to showing that the 0–1 permanent is in #P, right? Almost. We are dealing with symbolic polynomials, so we do not have the liberty of looking at an input value and deciding what to do next. Still, the basic idea is the same. For a statement S, let $[S]$ denote the 0–1 valued Boolean predicate that takes value 1 exactly when S is true. Then

$$\begin{aligned}
\mathrm{Perm}_n = \sum_{\sigma \in S_n} \prod_{i=1}^{n} x_{i\sigma(i)} &= \sum_{Y \in \{0,1\}^{n\times n}} \begin{bmatrix} Y \text{ is a } 0-1 \\ \text{permutation} \\ \text{matrix} \end{bmatrix} \cdot \prod_{i=1}^{n} \left(\sum_{j=1}^{n} Y_{ij} x_{ij} \right) \\
\begin{bmatrix} Y \text{ is a } 0-1 \\ \text{permutation} \\ \text{matrix} \end{bmatrix} &= [Y \text{ has at least one 1 in each row}] \\
&\quad \times \quad [Y \text{ has at most one 1 in each line} \\
&\quad (\text{line = row or column})] \\
&= \left(\prod_{i=1}^{n} \sum_{j=1}^{n} Y_{ij} \right) \left(\prod_{\substack{(i,j) \neq (k.m); \\ i=k \text{ or } j=m}} (1 - Y_{ij} Y_{km}) \right)
\end{aligned}$$

Clearly, the polynomial family

$$g_n = \left(\prod_{i=1}^{n} \sum_{j=1}^{n} Y_{ij} \right) \left(\prod_{\substack{(i,j) \neq (k.m); \\ i=k \text{ or } j=m}} (1 - Y_{ij} Y_{km}) \right) \prod_{i=1}^{n} \left(\sum_{j=1}^{n} Y_{ij} x_{ij} \right)$$

has formulas of size $O(n^3)$, and $\mathrm{Perm}_n(\tilde{x}) = \sum_{Y \in \{0,1\}^{n\times n}} g_n(\tilde{x}, Y)$, so (Perm$_n$) is in VNP.

So we have some families in VP (even VF), and some in VNP but maybe not in VP. How do we compare families? For comparing languages, we have many-one reductions and Turing reductions—what is the algebraic analogue? Valiant proposed projections, a most restrictive kind of reduction when dealing with Boolean classes, but completely natural in the algebraic context. We say that $g \in \mathbb{F}[y_1, \ldots, y_m]$ is a projection of $f \in \mathbb{F}[x_1, \ldots, x_n]$ if g can be obtained from f by substituting a value in $\mathbb{F} \cup Y$ for each variable in X. (For instance, if $f = x_1x_2 + x_3x_4$, then the following are all projections of f: $y_1 + y_2$, $y_1y_2 + 5$, $y_1y_2 + y_2y_3$, $2y^2$. But $y_1^2y_2$, $y_1 + y_2 + y_3$ are not, because a projection cannot increase the degree or number of monomials.) Further, we say that a family (g_n) is a p-projection of a family (f_n) if each g_n is a projection of some f_m for an m not too far from n. That is, there is a polynomially bounded function t, and each g_n is a projection of $f_{t(n)}$. If we allow t to be quasi-polynomially bounded, we obtain qp-projections.

Using these notions of reductions, we have the usual notions of hardness and completeness for algebraic classes. Here's what Valiant showed:

1. (Det_n) is hard for VF under p-projections (and is known to be in VP).
2. (Det_n) is complete for the class of quasi-polynomial size formulas VQF under qp-projections.
3. Over fields with characteristic other than 2, (Perm_n) is complete for VNP under p-projections. Over fields of characteristic 2, Perm_n equals Det_n and hence is in VP and VQF.
4. Polynomial families associated with a number of NP-complete languages are complete for VNP under p-projections.

The first two follow from a proof that a polynomial computed by a size s formula is a projection of Det_{s+2}. (It uses the combinatorial definition of determinant. as the signed weighted sum of cycle covers in an associated graph.) The hardness of (Perm_n) for VNP mirrors the hardness of the Boolean permanent for the counting class $\#P$. As in the case of the upper bound, additional care is needed to take into account non-access to an input instance and fully symbolic computations; in particular, the proof requires a multiplicative inverse of two and hence fails over fields of characteristic 2. See [Val79, BCS97, Bür00a] for various versions of these proofs. See [Blä13] for a simplified gadget construction.

4.3 The Current Status

We now know much more about the classes VF, VP, VQP, VNP defined above, and about other similarly defined classes. Let's review these results one by one.

Say that a family of polynomials (f_n) is a p-family if the number of variables in f_n and the degree of F are polynomially bounded functions of n. We only consider p-families.

Recall that VP consists of p-families with polynomial-sized circuits. Also note that algorithmically, circuit size roughly corresponds to number of processors needed in a parallel algorithm (associate one processor per gate), while circuit depth—the length of a longest path from the output node to an input node—corresponds to parallel time.

A clever construction due to Hyafil [Hya79] shows that any polynomial of degree D in M variables, computable by a circuit of size t, can be computed in parallel time $O(\log D \times \log(D^2t + M))$. This is a depth-reduction of the circuit, and generalises

Csanky's result which was specifically tailored for the determinant. Further, this algorithm has parallel multiplicative depth only $O(\log D)$; that is, any root-to-leaf path goes through at most $O(\log D)$ multiplication nodes. This is worth noting since multiplication seems a more costly operation than addition or subtraction. Unfortunately, the resulting circuit, while shallow and depth-reduced, is rather large, roughly $t^{\log D}$. Applying this construction would take us from polynomial size circuits to shallow quasi-polynomial size circuits. Soon after this, an improved construction was presented by Valiant et al. [VSBR83]; they achieved the same depth-reduction (and also $O(\log D)$ multiplicative depth) with size polynomial in tD. In particular, applying this construction to a circuit family (C_n) witnessing that a polynomial family (f_n) is in VP, we see that (f_n) is in $\mathrm{VSAC}^1 \subseteq \mathrm{VNC}^2$.

Wait, what exactly are these new classes? Again, we can think of them as analogues of Boolean classes. The Boolean circuit class NC^i has circuits of polynomial size and $O(\log^i n)$ depth. The class SAC^i is similarly defined, polynomial size, $O(\log^i n)$ $\wedge$-depth, and negations only at inputs. That is, if $\vee$ nodes are allowed to have unbounded in-degree, but $\wedge$ nodes must have in-degree 2, then these circuits have depth $O(\log^i n)$. (Hence the name SAC, for semi-unbounded alternation.) Clearly, $\mathrm{NC}^i \subseteq \mathrm{SAC}^i \subseteq \mathrm{NC}^{i+1}$. Now define the classes VNC^i and VSAC^i as algebraic analogues of these, with $\times$ and $+$ playing the roles of $\wedge$ and $\vee$, respectively. In the Boolean world, we know that $\mathrm{NC}^1 \subseteq \mathrm{SAC}^1 \subseteq \mathrm{NC}^2 \subseteq \cdots \subseteq \mathrm{NC} \subseteq \mathrm{P}$. In the algebraic world, however, $\mathrm{VNC}^1 \subseteq \mathrm{VSAC}^1 = \mathrm{VNC}^2 = \cdots = \mathrm{VNC} = \mathrm{VP}$.

An important consequence of the depth reduction result of [VSBR83] is that the $(\mathrm{Det}_n) \in \mathrm{VQF}$ result generalises to all of VP; $\mathrm{VP} \subseteq \mathrm{VQF}$. Another important consequence is that at quasi-polynomial size, formulas are as powerful as circuits; VQF equals VQP. Such an equivalence is not known for p-families at polynomial size. (It holds at exponential size, because polynomials in any p-family have only exponentially many monomials. An explicit sum-of-monomials expression gives an exponential-sized formula.)

Even before the results of [Hya79, VSBR83], Brent [Bre74] had shown that depth-reduction is possible for VF. Any formula F can be rebalanced by identifying in it a suitably chosen node N and rewriting F as a linear form in N, say $AN + B$. If N is properly chosen, then the polynomials A and B are computed by small subformulas (size at most half of F) of F, and can be recursively rebalanced. The appropriate N is identified by using the tree separator lemma. This process yields a $O(\log \mathrm{size}(F))$ depth formula. Thus, we conclude $\mathrm{VF} = \mathrm{VNC}^1$.

The depth reduction for VP from [VSBR83] proceeds similarly, but works on "proof-trees" or *parse trees*. Unfolding a circuit into a formula by systematically duplicating reused nodes may yield an exponential-sized formula (recall the example X^{2^n}.) Let us nonetheless do so. Now, a minimal subformula that includes the output node, both children of an included $\times$ node, and exactly one child of an included $+$ node, computes a potential monomial whose degree is the number of leaf nodes in the subformula. Call such a subformula a proof tree. For a circuit computing a p-family of polynomials, we can ignore proof trees of super-polynomial size. For each polynomial-sized proof tree, the balancing technique described above should work. The catch is, there can be too many proof trees (there can be exponentially many monomials), and each proof tree could require cutting at a different node. The clever twist is the following: in the formula depth-reduction, A can be computed recursively because it is the partial derivative of F with respect to N. If F is now a circuit rather than a formula, then F may not be linear in N, so computing the partial derivative will not help. But if N is chosen to have degree more than half the degree of F, then this is indeed the case. So, the algorithm of [VSBR83] computes, for each pair of nodes N, N', a new polynomial $F(N, N')$; these polynomial are recursively constructed, and whenever $2\text{degree}(N) > \text{degree}(N')$, $F(N, N')$ equals the partial derivative of N' with respect to N. Putting this together carefully gives the depth-required circuit. For details, see [VSBR83] itself. Also see [AJMV98a, Vol99] for *uniform* versions, where the task of describing the depth-reduced circuit given the original circuit is achieved using limited computational resources.

A couple of things slipped by almost unnoticed. We know what is meant by the degree of a polynomial, but what do we mean by degree(N)? This should be the degree of the polynomial computed at the node N, and indeed [VSBR83] use degree in this sense. But the uniform versions cannot do so, because computing the degree of a specified node in a given circuit is a completely non-trivial task! See the discussion about DegreeSLP in [ABKPM09, Kay10]. Fortunately, we can equally easily work with an upper bound on the degree of each node. And an upper bound $u(N)$ on the degree at each node N is easy to obtain: $u(N) = 1$ if N is a leaf, $u(N_1 + N_2) = \max\{u(N_1), u(N_2)\}$, $u(N_1 \times N_2) = u(N_1) + u(N_2)$. This upper bound is referred to as the complete formal degree of the circuit (as opposed to the degree of the polynomial it computes). However, just because the output node of C computes a polynomial of degree d, this does not imply that each node computes a polynomial of degree at most d. Higher degree monomials may get computed along the way, and get cancelled finally. Is it necessary, in terms of efficiency, to compute them? No! If C is of size s and computes a polynomial f of degree d, then we can construct a circuit C' of size $O(sd^2)$ computing the same polynomial and with each node computing a polynomial of degree at most d: just compute the homogeneous parts of f separately in the obvious way. Now C' will have complete formal degree $O(d^3 s)$. (See [MP08] for details.) Thus, we could have defined VP in terms of circuits of polynomial size and polynomially bounded complete formal degree as well.

There is a much simpler proof of the fact that VP is contained in VNC. This proof yields a weaker upper bound of VSAC^2 rather than VSAC^1, but is still beautiful, and is still enough to conclude that VP $\subseteq$ VQF. I first saw this proof in a survey talk by Koiran at Dagstuhl [Koi10], and I wish I had come up with it myself! Let (f_n) be in VP, as witnessed by a circuit family (C_n) with complete formal degree bounded by (d_n). To depth-reduce C_n, partition the nodes into $1 + \lceil \log d_n \rceil$ parts; part k has nodes with formal degree in $[2^{k-1}, 2^k)$. Treating the polynomials from parts $i < k$ as variables, the nodes in part k form a *skew* circuit, where each $\times$ node has at most one child that is not an input node. (Multiplying two nodes both in part k would create high degree, giving rise to a node in part $k + 1$.) Now, skew circuits can be depth-reduced to VSAC^1 rather easily, using a divide-and-conquer argument dating back to Savitch [Sav70]. Doing this separately for each part gives a VSAC^2 circuit.

We just introduced a new kind of circuit there: skew circuits. Are they as powerful as general circuits? We do not know! Let's define $\mathrm{VP}_{\mathrm{skew}}$; p-families of polynomials computed by polynomial-sized skew circuits. It turns out this is a great class to study, because it *exactly* characterises the complexity of the determinant. Recall what we have already seen; (Det_n) is hard for $\mathrm{VF} = \mathrm{VNC}^1$ and is in VP. The upper bound proof from [MV97] actually gives a skew circuit of size $O(n^4)$, but skew circuit constructions were known much earlier: in [Ven92], Venkateswaran first defined Boolean skew circuits to capture nondeterministic circuits, and subsequently many authors independently extended that study to arithmetic rings, [Dam91, Tod92, Vin91, Val92]. And the lower bound proof from [Val79] shows that polynomials computed by skew circuits are p-projections of the determinant, though it is not stated this way. Valiant showed that a formula can be converted to a certain kind of graph that we nowadays call an algebraic branching program or ABP (more about this below), and that polynomials computed by ABPs are p-projections of (Det_n). And we now know that ABPs are essentially skew circuits.

Time to define ABPs. These are directed acyclic graphs, with a designated source node s and a designated target sink node t (sometimes there may be multiple target nodes), and with edges labelled from $\mathbb{F} \cup X$ (similar to input nodes in a circuit). For any directed path ρ, the weight of ρ is the product of the labels of the edges on ρ. The polynomial p_v computed at a node v is the sum of the weights of all directed sv paths. The polynomial computed by the ABP is just p_t. Families computed by polynomial-size ABPs form the class VBP. (In some parts of the literature, edge labels are allowed to be linear forms in X. This does not significantly change the properties of ABPs as we discuss here. We'll stick to the convention that labels are in $\mathbb{F} \cup X$.)

So why are ABPs and skew circuits essentially the same? ABPs to skew circuits: clearly, $p_s = 1$, and for any other source node (in-degree 0) s', $p_{s'} = 0$. Look at an edge $u \rightarrow v$ of the ABP with label ℓ. Then p_v has a contribution from $p_u \times \ell$. Summing this over all incoming edges at v gives a small circuit computing p_v from previously computed values, and this circuit is skew. For the reverse simulation, reverse this construction: (1) introduce a source node s, (2) for each input node u labelled ℓ, add an edge $s \rightarrow u$ labelled ℓ, (3) for each node $v = u + u'$, create edges $u \rightarrow v$ and $u \rightarrow v$ labelled 1, and (4) for each node $v = u \times \ell$, create an edge $u \rightarrow v$ labelled ℓ.

So now we can add to the list of results at the end of Sect. 4.2: (Det_n) is complete for $\mathrm{VBP} = \mathrm{VP}_{\mathrm{skew}}$ under p-projections.

In fact, we can add more. What makes the simulation from skew circuits to ABPs possible is the fact that at each $\times$ gate, one argument is *easy*. Toda [Tod92] took this argument further—it is enough if one argument is independent of the rest of the circuit. That is, for each $\times$ node $\alpha = \beta \times \gamma$, the entire sub-circuit rooted at either β or γ has no connection to the rest of the circuit except via this edge to α. (Equivalently, one of the edges into α is a bridge in the circuit.) Call such circuits *weakly skew* circuits. Toda showed that weakly skew circuits can be converted to skew circuits with linear size blow-up. See also [MP08], where Malod and Portier made the size bounds in the conversions even more precise. So now we can say $\mathrm{VBP} = \mathrm{VP}_{\mathrm{skew}} = \mathrm{VP}_{ws}$, where the subscript ws stands for weakly skew.

(Note: Neither [Tod92] not [MP08, Mal03] actually claimed linear size blow-up. However, their constructions from weakly skew circuits to ABPs, with the standard conversion from ABPs to skew circuits, does give linear blow-up. As far as I can see, linear blow-up for weakly skew to skew circuits was explicitly observed in [KK08, Jan08, Gre12a].)

Taking this idea further, Malod and Portier provide a brilliant characterization of the class VP. Say that a circuit is *disjoint* if at every node $\alpha = \beta \circ \gamma$, where $\circ$ could be $+$ or $\times$, the sub-circuits rooted at β and γ are disjoint. This is just a fancy (convoluted?) way of saying that the circuit is a formula. But now relax this constraint a bit. Say that a circuit is *multiplicatively disjoint* or MD if at every $\times$ node $\alpha = \beta \times \gamma$, the sub-circuits rooted at β and γ are disjoint. No restrictions apply to $+$ nodes. Like formulas, MD circuits of size s have complete formal degree bounded by s. But the MD restriction seems to allow more computation than formulas; for instance, weakly skew circuits are MD, and so MD circuits can compute (Det_n) in polynomial size. Malod and Portier showed that in fact polynomial size MD circuits can compute everything in VP, but nothing more. That is, $\mathrm{VP} = \mathrm{VP}_{\mathrm{MD}}$. While this fact can also be deduced once we have depth reduction to VSAC^1, Malod and Portier give a completely self-contained combinatorial proof which is very neat. Basically, imagine that each node in the VP circuit is labelled with its formal degree. Now make multiple copies of each node, inversely proportional to the formal degree. By carefully deciding which copies of its children to use to construct a copy of a node, multiplicative disjointness can be achieved with only polynomial blow-up in size.

A nice consequence of this characterisation of VP is a simpler proof of the fact that VP is contained in $\sum \cdot \mathrm{VF}$. The key observation used is that a circuit is multiplicatively disjoint exactly when every proof tree is already a subgraph of the circuit (even without any unfolding into a formula). See [MP08] for details.

Before we move on, we note another surprising relation between ABPs and formulas: VF equals the class of p-families computed by polynomial-size ABPs of constant width. What is this resource "width"? Recall that an ABP is a DAG with edges going "in the direction from s to t". Suppose we impose a layering constraint. The nodes of the DAG must be laid out at the vertices of a rectangular $w \times \ell$ grid, the node s must be at position $(S, 1)$ for some $S \in [w]$, the node t must be at position (T, ℓ) for some $T \in [w]$, and edges can only go across one layer, from (i, k) to $(j, k + 1)$ for some $i, j \in [w], k \in [\ell - 1]$. Of course, any ABP can be converted to one of this form: just sub-divide edges when necessary and label the sub-division path so that its weight is the original edge's label (use lots of 1s). Now we say that w is the width of the layered ABP and ℓ is the length. A bounded-width branching program family (B_n) is one where for some absolute constant c, each B_n has width at most c. Seems quite a squeeze – if we view moving from s towards t as an incremental computation, then at each stage we can carry forward just c intermediate polynomials. We shouldn't be able to do much this way, right? Wrong! Ben-Or and Cleve [BOC92] showed, in a proof cleverly extending Barrington's famous characterisation [Bar89] of NC^1 by Boolean bounded-with branching programs, that every formula of depth D has an equivalent bounded-width branching program (that's quite a mouthful; let's agree to call it BWBP) of length 4^D and width just 3! Since we already know that formulas can be depth-reduced and VF equals VNC^1, we see that VF is contained in a class that we can name VBWBP: polynomial-sized constant-width ABPs. The converse inclusion is easily seen to hold, again using a Savitch-style divide-and-conquer. Thus, we have another characterisation of VF.

As a matter of curiosity, one may want to know: is the width-3 upper bound tight? Allender and Wang [AW11] recently settled this question affirmatively: they show that a very simple polynomial cannot be computed by any width-2 ABP, no matter what the length. On the other hand, width-3 ABPs are universal, since every polynomial family has some formula family computing it. The question is one of efficiency: which families have polynomial-size width 3 ABPs?

OK, so we've had a plethora of class definitions, but just a handful of distinct classes: $\text{VF} = \text{VP}_e = \text{VNC}^1 = \text{VBWBP}$, $\text{VBP} = \text{VP}_{\text{skew}} = \text{VP}_{\text{ws}}$, $\text{VP} = \text{VP}_{\text{MD}}$, $\text{VQF} = \text{VQP}$, $\text{VNF} = \text{VNP}$.

As stated in [Bür00a], *Valiant's hypothesis* says that $\text{VNP} \not\subseteq \text{VP}$, and *Valiant's extended hypothesis* says that $\text{VNP} \not\subseteq \text{VQP}$. Over fields of characteristic not equal to 2, these imply: Perm_n is not a p-projection of Det_n, and Perm_n is not a qp-projection of Det_n, respectively.

Some miscellaneous results, in no specific order:

1. Let SymDet_n be the polynomial that represents the determinant of a symmetric $n \times n$ matrix of indeterminates B_n. (For instance, $\text{SymDet}_2 = x_{11}x_{22} - x_{12}^2$.) Clearly, (SymDet_n) is a p-projection of (Det_n). The converse is also almost true. As shown by Grenet et al. in [GKKP11], over any field of characteristic other than 2, Det_n is a projection of SymDet_{n^3}. Characteristic 2 is a problem: symmetric matrices correspond to undirected graphs, so each undirected cycle gives rise to two directed cycles, and so to get a projection we need division by 2. In characteristic 2, Det_n itself is provably not a projection of SymDet_m for any m; see [GMT13]. The best that we can currently say in characteristic 2 is that the squared determinant $(\text{Det}_n)^2$ is a projection of SymDet_{2n^3+2}; this is also shown in [GKKP11].
2. VQP is also characterized by quasi-polynomial-size weakly skew circuits of polynomial degree. (From [VSBR83] it follows that $\text{VQP} = \text{VQF}$; hence the above characterization. A direct proof is presented in [MP08].) Several natural polynomials are complete for this class under qp-reductions: the (Det_n) family, of course, but also, the trace of iterated matrix product and the trace of a matrix power. These families are all complete for VBP under p-reductions.
3. While we do not know the exact relationship betweenVQP and VNP, (they both contain VP), we do know that VQP does not equal either VP or VNP. Bürgisser ([Bür00a], Sect. 8.2) has shown that there is an explicit family of polynomials (f_n) in VQP that is provably not in VNP, let alone in VP. This family is defined as follows: Consider numbers in base n. Let μ range over all such numbers with $m(n) = \lceil \log n \rceil$ digits. More precisely, let μ range over length-$m(n)$ sequences over the alphabet $\{0, 1, \ldots, n-1\}$, and let $k_n(\mu)$ denote the value of this sequence,

$k_n(\mu) = \sum_{j=1}^{m(n)} \mu_j n^{j-1}$. Define f_n as:

$$f_n(x_1, \ldots, x_{m(n)}) = \sum_{\mu \in \{0,\ldots,n-1\}^{m(n)}} 2^{2^{k_n(\mu)}} \prod_{j=1}^{m(n)} x_j^{\mu_j}$$

Exploiting the fact that the distinct double exponentials appear as coefficients in f_n, Bürgisser shows that f_n cannot be in VNP.

Furthermore, using $m(n) = \lceil \log^i n \rceil$ gives a family of polynomials f^i in VQP with size $O(n^{\log^i n})$ but provably not in size $O(n^{\log^{i-1} n})$, so within VQP there is a strict hierarchy.

4. From the qp-completeness of (Det$_n$) for VQP, and the p-completeness of (Perm$_n$) for VNP, it follows that VNP $\subseteq$ VQP if and only if (Perm$_n$) is a qp-projection of (Det$_n$). This is a very long-standing open question. Originally, the question of whether (Det$_n$) and (Perm$_n$) are p-equivalent was posed by Pólya [Pól13], who also showed that there is no way of expressing the permanent as the determinant by only changing the signs of selected entries (except for $n = 2$; flip the sign of a_{12} to get matrix B with Det(B) = Perm(A)). (I haven't myself seen Pólya's note, but have seen it referred to in various places.) Marcus and Minc [MM61] showed that there is no size-preserving transformation (Perm$_n$ to Det$_n$), even if we relax the notion of projections to allow linear form substitions for each variable. For many years, a linear lower bound was the best known ($\Omega(\sqrt{2}n)$ due to [vzG87, Cai90, Mes89]), until Mignon and Ressayre [MR04] showed that over the fields of characteristic 0 (eg real or complex numbers), even if linear form substitutions are allowed in projections, to express Perm$_n$ as a projection of Det$_m$, we need $m \geq n^2/2$. The same lower bound was obtained for fields of characteristic other than 2 by Cai et al. [CCL10]. From Ryser's work [Rys63] it follows that Perm$_n$ is a projection of Det$_m$ for some $m < n^2 2^n$. More recently, Grenet showed [Gre12b] via a very simple and neat construction that Perm$_n$ is a projection of Det$_m$ for $m = 2^n - 1$. This is the best known so far. Thus there is a huge gap between the lower and upper bounds on what is called the determinantal complexity of the permanent.
5. It is natural to believe that the complexity of a p-family (f_n) in this framework is closely related to the computational complexity of *evaluating* f_n for a given instantiation of its variables. In [Bür00b], Bürgisser gave this belief a firm footing. Consider a p-family (f_n) where f_n depends on n variables. Define its Boolean part BoolPart(f) as a string function mapping $x \in \{0, 1\}^n$ to the binary encoding of $f_n(x)$. Note that we have considered only Boolean values. Even so, evaluation may seem difficult, because the circuits for (f_n) can involve arbitrary constants from the field. Bürgisser showed that assuming the generalised Reimann hypothesis GRH, over fields of characteristic zero, BoolPart(VP) has non-uniform multi-output NC^3 circuits. Furthermore, assuming GRH, if Valiant's hypothesis is false over such a field, then the entire polynomial hierarchy has (non-uniform) NC circuits.

6. An *extreme* depth reduction result is given by the highly influential paper of Agrawal and Vinay [AV08]. To first see the context, note that any polynomial in n variables with degree d has an unbounded fan-in depth-2 circuit of size $2^{O(d+d\log\frac{n}{d})}$. (If $d \in \Omega(n)$, then $2^{O(d)}$ suffices, otherwise the second term in the exponent makes up.) This is because we can just explicitly compute all monomials of degree at most d, and add up the required ones with suitable weights. Now, can we find circuits substantially better than this, say even $2^{o(d+d\log\frac{n}{d})}$, if we allow depth to be increased a bit? Agrawal and Vinay showed that indeed this is possible, even with depth 4, provided there is some circuit (not necessarily depth-reduced) of that size to begin with. The idea is extremely simple. Peform the depth reduction from [VSBR83] or [AJMV98b], and ensure with some additional care that degree provably drops at $\times$ gates. (The price for this is small: a $\times$ gate may have fanin upto 6, instead of 2.) Now, choose a horizontal cut in the depth-reduced circuit so that for the sub-circuit above it, and for the sub-circuits below it rooted at gates on the cut, the "brute-force" construction described above is small. Obviously there is a trade-off: if the cut is too high up, the lower sub-circuits can have large explicit forms, but if it is too low down, the upper sub-circuit can have large explicit forms. Cut in the right place, and everything works out!

   Subsequently the extreme depth-reductions have been pushed further; see [Koi12, Tav13, GKKS13b]. The lower bound results of [GKKS13a, FLMS13] show that the depth reduction upper bound from [Tav13] is tight and cannot be pushed any further.

   This has significant implications for the quest for derandomizing algorithms for the well-studied problem ACIT (arithmetic circuit identity testing)—checking if a given circuit computes the identically zero polynomial. But that is not directly connected with this survey. One question it raises here is: what kind of extreme depth reduction can we achieve for VQP? Can we stay within quasi-polynomial size?

4.4 The Syntactic Multilinear World

Much of the study concerning VP and VNP involves the families (Det_n) amd (Perm_n). The polynomials in both families are *multilinear*. In principle, to compute a multilinear polynomial via a circuit, we need never compute intermediate polynomials that are not multilinear. Let us call such circuits, where the polynomial computed at each node is multilinear, *multilinear circuits*. However, often it is the case that allowing non-multilinear terms at intermediate stages, and eventually cancelling them out, allows more efficient computation (smaller circuits). This leads to the following quest: what kind of multilinear p-families have efficient multilinear formulas, or even multilinear circuits, where each intermediate polynomial is required to be multilinear? Even for the (Det_n) family, which we know is multilinear and in VP, we do not know of polynomial size multilinear circuits. That being the case, can we prove lower bounds?

This question is trickier than it seems at first glance, because given a circuit, even checking whether it is multilinear is non-trivial. Fournier, Malod and Mengel [FMM12] recently observed that checking multilinearity of a given circuit is computationally equivalent to the well-studied problem arithmetic circuit identity testing (ACIT)—checking if a given circuit computes the identically zero polynomial.

So we may want a notion of certifiably multilinear circuits. One such notion is that of syntactic multilinearity, SM. A circuit is said to be syntactically multilinear if at every $\times$ node $\alpha = \beta \times \gamma$, the sub-circuits rooted at nodes β and γ operate on disjoint sets of variables. Note that this is much more restrictive than multiplicative disjointness. But it certifies multilinearity, since no variable can ever get multiplied by itself. And syntactic multilinearity is easy to check computationally: it is violated if there is some node $\alpha = \beta \times \gamma$, some variable x, two input nodes I, I' labelled x, and paths from I to β and I' to γ.

If a family has efficient (polynomial-sized) SM circuits, then it has efficient multilinear circuits. The converse may not be true. But it is true if we look at formulas. Given a multilinear formula, identify an SM violation α, β, γ, x as above. Then we know by multilinearity of the polynomial $p(\alpha)$ that x does not appear in either $p(\beta)$ or $p(\gamma)$. In the appropriate subformula, set all instances of x to 0; the polynomials computed at and above α remain unchanged. Doing this systematically gives an SM formula of size no more than the original multilinear formula.

In the first major breakthrough, Raz [Raz09] showed that for computation by SM formulas, and hence by multilinear formulas, both (Det_n) and (Perm_n) need size $n^{\Omega(\log n)}$. Clearly, this also means that they are not in SM-VNC^1.

Since (Det_n) is in VP and even in VBP, SM-VF is strictly weaker than VBP. But this is hardly a fair comparison: we have restricted VF to be SM, but not VBP and VP. Can we say that SM-VF is strictly weaker than SM-VBP or SM-VP? We do not know whether (Det_n) is in multilinear VP, let alone SM-VP, so a different family is needed as a separating example. Such an example was provided soon thereafter, again by Raz [Raz06]. He constructed an explicit polynomial family that is in SM-VP and even in SM-VSAC^1, and showed that it needs SM-formula size $n^{\Omega(\log n)}$ and hence is not in SM-VNC^1. Improved lower bounds for constant-depth circuits and subclasses of formulas were subsequently obtained by Raz, Shpilka and Yehudayoff (see for instance [RY09, RSY08]).

Let's step back a bit. Why did we say "in SM-VP, and even in SM-VSAC^1"? Aren't VP and VSAC^1 the same? Well, we know that VP and VF can be depth-reduced. But can we assume that these depth reduction tehniques preserve syntactic multilinearity? Fortunately, they do; Raz and Yehudayoff [RY08] showed that the depth reduction of [VSBR83] preserves SM, so indeed SM-VP= SM-VSAC^1. Similarly, in [JMR12] it is observed that the formula depth reduction of [Bre74] also does preserves SM, so SM-VF= SM-VNC^1.

What about other relationhips between the algebraic classes? We had considered ABPs—what certifies multilinearity there? It is easy to see that a read-once restriction, where on each path in the ABP each variable appears as a label at most once, does so. Let us therefore use read-once as the definition of syntactic multilinearity in ABPs. Then, as observed in [JMR12], the Savitch-style divide-and conquer argument preserves SM. So does the conversion from formulas to ABPs, [Val79]. But the conversion from formulas to width-3 ABPs, [BOC92], does not. In fact, Rao [Rao10] showed that even a significant generalisation of Ben-Or and Cleve's technique, using polynomially many registers instead of just 3, cannot preserve syntactic multilinearity. Of course, there may be other ways of going from SM-VF to SM-VBWBP, but it could equally well be that the classes are distinct.

To get back perspective, in the SM world what we have seen so far is:

$$\text{SM-VBWBP} \subseteq \text{SM-VF} \subseteq \text{SM-VBP} \subseteq \text{SM-VP}$$

As mentioned earlier, Raz [Raz06] showed that the inclusion from SM-VF to SM-VP is proper. Very recently, this was improved by Dvir et al. [DMPY12]. They showed that in fact the inclusion SM-VF $\subseteq$ SM-VBP is strict. Whether the first and the last inclusion are strict is still open.

The proof of [DMPY12] is a clever adaptation of the original technique from [Raz06]. Let us briefly examine this.

The central ingredient in Raz's proof is randomly partitioning the variables and analysing the rank of the resulting partial derivatives matrix. Consider a polynomial f on $2n$ variables $X = \{x_1, \dots, x_{2n}\}$, and consider a partition of X into equi-sized sets Y, Z. Consider a $2^n \times 2^n$ matrix $M_f^{Y,Z}$ where rows and columns are indexed by subsets of Y and Z (equivalently, multilinear monomials over Y and Z, respectively). The entry (m_y, m_z) is the coefficient of the monomial $m_y \cdot m_z$ in f. Intuitively, if $M_f^{Y,Z}$ has high rank, then f should be hard. But high rank with respect to what partition? Raz showed that if multilinear f has small SM-formula size, then for at least one partition (Y, Z) of X, $M_f^{Y,Z}$ will have low rank. (The existence of the partition witnessing low rank is proved using the probabilistic method; choose a partition at random, and analyse the probability that the resulting matrix has rank exceeding some threshold.) He also constructed an explicit family g in SM-VSAC^1 and showed that for every partition (Y, Z) of X, $M_g^{Y,Z}$ has high rank; hence g is not in SM-VF.

The non-trivial adaptation done in [DMPY12] is to consider not all partitions, but a fairly small set of what they call arc-partitions. They showed that if f is in SM-VF, then for at least one arc-partition (Y, Z) of X, $M_f^{Y,Z}$ will have low rank. They consider an explicit family g in SM-VBP and show that for every arc-partition (Y, Z) of X, $M_g^{Y,Z}$ has high rank. Hence g is not in SM-VF. The low-rank proof is again probabilistic, but it has a very appealing combinatorial flavour. So does the very definition of an arc-partition.

4.5 More on Completeness

Assume that completeness is defined with respect to p-projections. If a family (f_n) is complete for a class, then understanding (f_n) better allows us to understand the class better. If a natural family is complete for a class, then this is evidence that the class itself is natural.

Valiant started off with a proof that Perm is VNP-complete. He also showed that polynomial families associated with a number of NP-complete languages are complete for VNP under p-projections. So let us agree that VNP is a natural class.

What about VP? The family that naturally contrasts with Perm is Det, but Det is not yet known to be complete for VP (unless we allow qp-projections; that is not quite satisfactory). If this turns out to be the case, it will solve a major open problem, showing that polynomial-degree polynomial size circuits are no more powerful than polynomial-size branching programs VBP. VBP seems a natural enough class, and Det and many other families are complete for it.

So what problems are complete for VP? One can construct a canonical family complete for VP. By canonical, I mean something similar to saying that

$$\{\langle M, x, 1^t\rangle \mid M \text{ is an NDTM that accepts } x \text{ in } t \text{ or fewer steps}\}$$

is NP-complete. Undoubtedly true, but it doesn't give any new intution about what NP is about. In the case of VP, the canonical family is not so trivial to construct (but not very difficult either).

The first description, with a very general completenes result, appears in [Bür00a] (see Sect. 5.6, Cor 5.32(b)). Bürgisser shows that for every p-family h, the *relativized* classes VP^h and VNP^h have complete families with respect to p-projections. Since $\text{VP}^h = \text{VP}$ and $\text{VNP}^h = \text{VNP}$ whenever h itself is in VP, this gives families complete for VP and VNP as well. (In fact, it shows the existence of VNP-complete families, independent of Valiant's original proof.) These complete families compute homogeneous components separately, to keep the degree small, and then add up the required parts. They are constructed by first defining *generic polynomials*, and then defining the appropriate projection/substitution. The generic polynomials capture the canonical notion referred to above.

Later, a more direct construction tailored for VP (as opposed to VP^h and VNP^h for all h) was described by Raz [Raz10], and also appears in [SY10]. Here, the proof of hardness exploits the fact that we can perform depth reduction on VP circuits. (This was not needed in Bürgisser's proof.) Roughly, here's how it goes: For each natural number N, consider a circuit C_N with nodes arranged in $2\log N + 1$ layers numberd $0, 1, \ldots, 2\log N$. All even layers have exactly N nodes, and compute polynomials $g_{i,j}$ where i is the layer number, $j \in [N]$. Odd layers are used to build these polynomials. At layer 0, the polynomials are just distinct variables, $g_{0,j} = x_j$. At higher layers, we have an inductive definition: $g_{i+1,j} = \sum_{k,\ell\in[N]} g_{i,k} \cdot g_{i,\ell} \cdot y_{i,j,k,\ell}$, where the $y_{i,j,k,\ell}$ are new variables. Thus the nodes at the odd layers are the fanin-3 $\times$ nodes, and nodes at even layers (other than the 0 layer) are $+$ nodes with large fanin. (We can reduce the fanins to 2 later; it won't change the polynomial computed.) The polynomial computed by this circuit at $g_{2\log N,1}$ is p_N. The total number of variables is $O(N^3 \log N)$, and the circuit is also of size $O(N^3 \log N)$. The degree of p_N is $2N - 1$. So (p_N) is in VP. Why is it VP-hard? Take any family (f_n) in VP. By the depth reduction of [VSBR83], it can be computed in VSAC^1. The

VSAC[1] circuit D_n can be normalised to have alternating $+$ and $\times$ nodes, with all $\times$ nodes having fanin 2, and all leaves at the same depth. Choose N at least as large as $\min\{\text{size}(D_n), 2^{\text{depth}(D_n)}\}$, and also at least as large as the number of variables in C_n. Now, the computation of D_n can be embedded into C_N: Choose the right number of $+$ nodes at each even layer, and by carefully assigning 0,1 values to the y variables, ensure that they compute the required combinations of polynomials from the previous even layer.

The circuits described above are called *universal circuits* in [SY10], because every circuit is a projection of the universal circuit of appropriate size. And if we start with VP circuits, the projections are p-projections.

So now we know that VP has complete families under p-projections as well. But generic polynomials, universal circuits, and the polynomials they compute, are rather artifical. Are there other families that are defined independent of circuits and are VP-complete? Actually, we know very few. Recently, Stefan Mengel [Men11] made further progress here, considering polynomial families associated with constraint satisfaction problems CSPs. (This builds on earlier work by Briquel, Koiran, Meer [BK09, BKM11], though they did not explicitly look for VP-completeness.) Let's first review what CSPs are. Think of them as generalising CNF–SAT. In CNF–SAT, each clause forbids one assignment to the variables in it. (e.g the clause $x_1 \vee \overline{x_3}$ forbids $x_1 = 0, x_3 = 1$.) In a CSP, variables can take values from a larger domain, not necessarily 0,1. Each constraint is like a clause; it has a set of variables, and it forbids certain combinations of assignments to these variables. (e.g on domain $\{a, b, c\}$ a constraint on x_1, x_2 could say that $x_1 \neq x_2$. That is, assignments aa, bb, cc are forbidden, the other 6 assignments satisfy this constraint.) As in SAT, we look for assignments satisfying all constraints. If the domain has size 2, the CSP is Boolean. If each constraint involves 2 (or less) variables, the CSP is binary. As usual, consider not just a CSP but a family of CSPs (Φ_n), where Φ_n has domain D_n. For tractability, we will require that the CSP is p-bounded; that is, the CSP has bounded arity (for some fixed constant c, each constraint in every Φ_n looks at no more than c variables), and it has polynomial-sized domains (in Φ_n, the variables take values from a set D_n, where the size of D_n is p-bounded). Now associate with each such CSP (Φ_n) a polynomial family $(Q_n = Q(\Phi_n))$, where Q_n is on the variable set $\{X_d \mid d \in D_n\}$ and is defined as follows:

$$Q(\Phi_n) = \sum_{a:\text{var}(\Phi_n)\to D_n} [a \text{ satisfies } \Phi_n] \prod_{x\in\text{var}(\Phi_n)} X_{a(x)}$$
$$= \sum_{a:\text{var}(\Phi_n)\to D_n} [a \text{ satisfies } \Phi_n] \prod_{d\in D_n} X_d^{|a^{-1}(d)|}$$

(Recall, $[S]$ is Boolean, 1 if and only if statement S is true.) Mengel has this wonderful characterisation of the complexity of the family (Q_n). The characterisation involves associating with the CSP a graph G; this graph has a vertex for each variable and an edge between two variables if they occur simultaneously in some constraint. Now the treewidth and pathwidth of the graph (these parameters describe roughly how tree-like or path-like the graph is, if we can consider blobs of vertices. The smaller the blobs, the better the similarity. See [Bod98] for definitions and an overview.) relate to the complexity. It also involves an assignment bound: a CSP is c-assignment-bounded if for each constraint φ and each variable x in the constraint, the number of distinct values possible for x in assignments satisfying φ is bounded by c, even though the domain may be much larger. This seems like a strong condition, but recall that Boolean CSPs are by definition 2-assignment-bounded.

Enough of definitions! Here's what Mengel shows:

1. For each p-bounded CSP (Φ_n), $(Q(\Phi_n))$ is in VNP. Every family (f_n) in VNP is a p-projection of $(Q(\Phi_n))$ for some p-bounded (Φ_n).
2. For each p-bounded CSP (Φ_n) where G_n has bounded treewidth, $(Q(\Phi_n))$ is in VP. Every family (f_n) in VP is a p-projection of $(Q(\Phi_n))$ for some p-bounded binary (Φ_n) where G is a tree (treewidth 1).
3. For each p-bounded CSP (Φ_n) where G_n has bounded pathwidth, $(Q(\Phi_n))$ is in VBP. Every family (f_n) in VBP is a p-projection of $(Q(\Phi_n))$ for some p-bounded binary (Φ_n) where G is a path (pathwidth 1).
4. For each p-bounded c-assignment-bounded CSP (Φ_n) where G_n has bounded treewidth, $(Q(\Phi_n))$ is in VF. Every family (f_n) in VF is a p-projection of $(Q(\Phi_n))$ for some p-bounded 2-assignment-bounded binary (Φ_n) where G has pathwidth at most 26.

The hardness proofs involve looking at the structure of parse trees for VP, witnessing paths for VBP.

Note that as stated, this falls slightly short of providing a single complete family for VP. However, applying the hardness reduction from universal circuits will yield a single CSP family that is VP-complete. To the best of my knowledge, this is the first instance of a VP-hardness result for a family defined (almost) independent of circuits.

All the above results require that the CSP has bounded arity. Unbounded arity seems to immediately give rise to intractability. If arity is unconstrained, can other types of restrictions still result in families in VP? For further progress in this direction, see [DM11, CDM13].

4.6 Computing Integers

The questions concerning algebraic complexity classes are closely connected to another very intriguing question. Let $N > 1$ be any natural number. Suppose we want to build up N from 1, using only $+$, $-$ and $\times$. The most naive way of doing this would be $N = 1 + 1 + \cdots + 1$. But depending on N there can be many other ways. Which is the *most efficient* way? That is, which way uses the least number of $+$ or $\times$ operations? To do anything non-trivial, we must use $+$ at least once, and the first time we use it we will generate 2. So let us not even count this mandatory $+$. How many more operations are needed?

We can state this as a question about circuits. Each way of building up N is an arithmetic circuit, or a straight-line program (SLP), that uses no constants other than 1 and 2. Let us denote by $\tau(N)$ the size of the smallest such circuit computing N. (This is the τ complexity of N). By definition, $\tau(1) = \tau(2) = 0$, and for all $N > 2$, $\tau(N) > 0$. Algorithms for computing N give upper bounds on $\tau(N)$. For instance, to compute $N = 2^k$, here's an SLP: $g_0 = 2$, $g_{i+1} = 2 \times g_i$ for $0 \le i \le k-2$. Clearly, g_i computes 2^{i+1}, so $\tau(2^k) \le k - 1$. But I'm sure you can already see better ways of doing this. From the circuit viewpoint, an explanation of why this is not the best is that the circuit corresponding to this SLP is skew. Surely, we should be able to use non-skew gates and compute large numbers faster. Here's another SLP that computes big numbers fast: $f_0 = 2$, $f_{i+1} = f_i \times f_i$ for $0 \le i \le \ell - 1$. Clearly, f_i computes 2^{2^i}, so $\tau(2^{2^\ell}) \le \ell$, a much better bound than the earlier $2^\ell - 1$ at least for numbers of this form. Note that the way we used non-skewness, we produced a circuit with exponential formal degree (the degree at f_i is 2^i), but we're not worried about that for now. Now, using these compact circuits for 2^{2^ℓ}, we can build a better circuit for 2^k by just using the binary expansion of k: $k = \sum_{i=0}^{t} b_i 2^i$, where $t = \lfloor \log k \rfloor$ and $b_t = 1$. So $2^k = 2^{\sum_{i=0}^{t} b_i 2^i} = \prod_{i=0}^{t} 2^{b_i \times 2^i} = \prod_{i:b_i=1} 2^{2^i}$. Compute all the double powers using t operations, and then multiply the required ones using at most t operations. Overall, $\tau(2^k) \le 2t = 2\lfloor \log k \rfloor$.

We can use the same binary expansion idea to compute any N, not just a power of 2. Compute all powers of 2 upto $\log N$, and add the required ones. This shows that for all N, $\tau(N) \le 2\lfloor \log N \rfloor - 1$.

So far we have not used any subtractions. But they can be very useful too. For instance, $\tau(2^{2^\ell} - 1) \le \ell + 1$; compute 2^{2^ℓ} and subtract 1.

What about a lower bound? We can actually formalise the intuition that the exponential degree circuits we saw above for 2^{2^ℓ} produce the largest possible number in that size. Hence, for any N, $\tau(N) \ge \log \log N$.

In particular, $\tau(2^{2^\ell}) = \ell$. That sounds impressive – we know the exact value of τ for 2^{2^ℓ}. But essentially just for that; for all other numbers, we still seem to have a pretty large gap. If $N = 2^k$, then $\log \log N \le \tau(N) \le 2\lfloor \log k \rfloor = 2\lfloor \log \log N \rfloor$, so we know $\tau(N)$ within a factor of 2. But for general N, all we know is $\log \log N \le \tau(N) \le 2\lfloor \log N \rfloor - 1$. How can we reduce this gap? An obvious search for an efficient way where the last operation is $+$ or $-$ is to express N as $M \pm k$, compute M, compute $k = \pm(N - M)$, and combine, and to choose M that minimizes $\tau(M) + \tau(k) + 1$. (A similar approach can be used for factors of N and a $\times$ as the last operation.) But in computing M and $\pm(N - M)$ (or N/M), the complexity may be *subadditive* since we can reuse intermediate numbers from the program for M while computing $\pm(N - M)$ or N/M. (We are looking for circuits, not formulas.) It is identifying the extent of this reuse that is a challenge.

Similar to Shannon's bound for functions and circuits (most functions require exponential-sized circuits), de Melo and Svaiter [dMS96] showed that most numbers N have $\tau(N)$ closer to the upper bound. They showed that for every $\epsilon > 0$, most N

satisfy $\tau(N) \geq \frac{\log N}{(\log\log N)^{1+\epsilon}}$. Moreira [Mor97] improved this by showing that this holds even for $\epsilon = 0$. (He also showed that for all $\epsilon > 0$, there is an N_ϵ such that for all $N \geq N_\epsilon$, $\tau(N) \leq \frac{(1+\epsilon)\log N}{(\log\log N)}$). And yet, showing such lower bounds for specific numbers seems quite hard – the classic "searching for hay in a haystack" paradox.

Let's move over from individual numbers to sequences of numbers. Let $(a_n)_{n\geq 1}$ be some sequence of natural numbers. When can we say that the sequence is easy to compute? Each number in the sequence should be "easy" relative to its position in the sequence. That is, the sequence (b_n), where $b_n = \tau(a_n)$, should not grow very fast. One possible definition is that b_n should be polynomially bounded in n. For instance, for $a_n = 2^{2^n}$, we know that $b_n = n$. Is that not moderate growth? Not really. Consider a function that maps a position n to not just the number $\tau(a_n) = b_n$ but to an SLP of size b_n computing a_n. For the sequence (2^{2^n}), this function takes an input n represented in $\Theta(\log n)$ bits, and outputs a circuit of size n, that is, exponential in the size of the input. That's not moderate growth!

OK, so let's say that a sequence (a_n) is easy to compute if for some polynomial $p(.)$, for each n, $\tau(a_n) \leq p(\log n)$, and otherwise it is hard to compute. We've set up this definition so that (2^{2^n}) is hard to compute, while the sequences (n), (2^n) are easy to compute. Makes sense? Now let's ask, what other sequences are easy? And what sequences are hard?

A sequence with famously open status is $(n!)$. The completely naive SLP that constructs the first n numbers with $n-2$ increments and then multiplies them shows that $\tau(n!) \leq 2n-4$. But can this be improved significantly? Or is this sequence hard? The best we know is that $\tau(n!) \in O(\sqrt{n}\log^2 n)$; see [BCS97]. Here is the interesting connection to algebraic circuit complexity. Building on a sequence of constructions by Cheng [Che04] and Koiran [Koi05], Bürgisser [Bür09] showed that if $(n!)$ is hard to compute, then any algebraic circuit for the (Perm_n) family that uses only the constants $-1, 0, 1$ must be of superpolynomial size. If we can't even compute the numbers $n!$ easily, then we cannot compute the polynomials Perm_n efficiently, unless we allow the use of constants that cannot themselves built up efficiently.

Analogous to the τ complexity of natural numbers, we can define the τ complexity of polynomial families. Let $\tau(f)$ denote the size of the smallest algebraic circuit using only the constants $-1, 0, 1$—call such a circuit *constant-free*—and computing f. We say that the family (f_n) has polynomially bounded τ complexity if for some polynomial $p(n)$, and for each n, $\tau(f_n) \leq p(n)$. Bürgisser's result can now be stated as: if $\tau(\mathrm{Perm}_n)$ is polynomially bounded, then $(n!)$ is easy to compute.

Let's examine this a bit closely. Why do we state the hypothesis as "$\tau(\mathrm{Perm}_n)$ is polynomial"? Is this not equivalent to saying (Perm_n) is in VP, and hence VNP $=$ VP? Actually, it may not be equivalent. It is possible that (Perm_n) has polynomial-sized circuits but no polynomial-sized constant-free circuits. Conceivably, using other constants in intermediate computation and then cancelling them out could help. Recall that the proof of VNP-hardness of (Perm_n) uses constants other than $-1, 0, 1$; $1/2$ is needed. (As another example, recall how in showing that Det_n is a projection of SymDet_n, we needed the constant $1/2$, even though all coefficients in Det_n are

$-1, 0, 1$.) So we can define a subclass of VP: families with constant-free circuits of polynomial size.

What can we say about such a subclass? As described above, Bürgisser has shown that if this subclass contains (Perm_n), then $(n!)$ is easy to compute. Under the same hypothesis, he also shows that the sequences $\lfloor 2^n e \rfloor$, $\lfloor (3/2)^n \rfloor$ and $\lfloor 2^n \sqrt{2} \rfloor$ are easy to compute.

Malod [Mal03] observed that unlike in the case of VP, for constant-free circuits we may not be able to bound complete formal degree. For VP, if the polynomial computed by a circuit of size s had degree d, we could find an equivalent circuit with formal degree d, and another with complete formal degree $O(d^3 s)$, with only polynomial blow-up in size. Not so if constants aren't freely available! Consider the polynomial family $f_n = 2^{2^n}(x_1 + \cdots + x_n)$. With arbitrary constants, we have a circuit of size n. With only $-1, 0, 1$, we have a circuit of size $2n+1$: build 2, build 2^{2^n}, build the linear form, multiply. But this circuit has exponential formal degree, and in fact, using only the constants $-1, 0, 1$, any circuit must have exponential formal degree to build up 2^{2^n}. So this polynomial is in VP, it has constant-free circuits of polynomial size, but it does not have constant-free polynomial size circuits with polynomially bounded complete formal degree.

This leads to a definition of a further subclass VP^0, first defined in [Mal03]: polynomial families computed by constant-free circuits with polynomially bounded complete formal degree. Define VNP^0 analogous to VNP as $\sum \cdot \mathrm{VP}^0$. Check back; our proof that (Perm_n) is in VNP also shows that (Perm_n) is in VNP^0.

The hypothesis $(\mathrm{Perm}_n) \in \mathrm{VP}^0$ is stronger than saying that $\tau(\mathrm{Perm}_n)$ is polynomially bounded. What does it imply? Can it lead to more sequences being easier to compute? First, note that $(\mathrm{Perm}_n) \in \mathrm{VP}^0$ does not immediately imply $\mathrm{VP}^0 = \mathrm{VNP}^0$. All we can say is the following, shown by Koiran [Koi05]: If (Perm_n) is in VP^0, then for every family $(f_n) \in \mathrm{VNP}^0$, there is some polynomially bounded function $p(n)$ such that the family $(2^{p(n)} f_n)$ is in VP^0. That is, a "shifted" version of f_n is in VP^0. The precise shift can be described as follows—we know that f_n is a projection of $\mathrm{Perm}_{q(n)}$ for some polynomially bounded $q(n)$, we assumed that $\mathrm{Perm}_{q(n)}$ can be computed by a circuit C_n of size and formal degree bounded by a polynomial function of n, we take $p(n)$ to be the formal degree of C_n. Now C_n can be massaged to compute $2^{p(n)} f_n$ instead of $\mathrm{Perm}_{q(n)}$.

This motivates another variant of easy-to-compute. Let's say that a sequence (a_n) of natural numbers is ultimately easy to compute if at least some shifted version of it is easy to compute. That is, there is some other integer sequence A_n such that the sequence $a_n A_n$ is easy to compute. Note that if (a_n) is not ultimately easy, then for infinitely many n, all nonzero multiples of a_n have large τ complexity. Using this property, under the hypothesis that $n!$ is not even ultimately easy to compute, we can obtain a non-trivial derandomization of the Arithmetic-Circuit-Identity-Testing problem; see the last section of [ABKPM09]. Earlier, Koiran showed in [Koi05] that if $n!$ is not even ultimately easy to compute, then we have some separation: either $\mathrm{VP}^0 \neq \mathrm{VNP}^0$, or $\mathrm{P} \neq \mathrm{PSPACE}$. This is curious: we have a consequence involving Boolean classes as well. But it should not be so surprising. VP^0 and VNP^0 are computed by (sums of) constant-free poly-formal-degree algebraic circuits, and these are the arithmetic circuits that arise when we consider counting classes like #P that count accepting paths of Turing machines. This does not mean that $\mathrm{VNP}^0 = \#\mathrm{P}$; the former is a collection of polynomial families whereas the latter is a collection of functions from strings to whole numbers. But the complexity of evaluating polynomial families in the former collection, at Boolean arguments, is closely related to what the latter collection refers to. Koiran's proof actually shows the contrapositive: he first shows that if $\mathrm{VP}^0 = \mathrm{VNP}^0$ and P = PSPACE, then the sequence $\tau((2^\ell)!)$ is polynomially bounded in ℓ. So consider instead of each $n!$ the possibly larger factorial $(2^{\ell(n)})!$, where $2^{\ell(n)-1} < n \leq 2^{\ell(n)}$. Then the sequence $(b_n) = ((2^{\ell(n)})!)$ is easy to compute, and each b_n is a multiple of $n!$, so $(n!)$ is ultimately easy to compute.

Since Perm_n is not known to be complete for VNP^0, what is? It turns out that for several other VNP-complete families, the hardness proofs use no constants other than $-1, 0, 1$ and the membership proofs use circuits with small formal degree; hence these families are complete for VNP^0 as well. As a concrete example, consider the Hamilton cycle polynomial family HC_n defined as follows: Let distinct variables $x_{i,j}$ label the edges of the complete directed graph D_n. Let C_n denote the set of all directed Hamiltonian cycles in D_n; elements of C_n can be described by cyclic permutations $\sigma \in S_n$. Then

$$\mathrm{HC}_n(x_{11}, \ldots, x_{nn}) = \sum_{\sigma \in C_n} \prod x_{i,\sigma(i)}$$

This family is complete for VNP^0; [Mal03].

Returning to the question "What does $(\mathrm{Perm}_n) \in \mathrm{VP}^0$ imply?"; Koiran [Koi05] showed that it implies the sequence $\lfloor 2^n \ln 2 \rfloor$ is easy to compute. He also improved the earlier-mentioned result in two ways, from "$[(\mathrm{VP}^0 = \mathrm{VNP}^0) \wedge (\mathrm{P} = \mathrm{PSPACE})] \Rightarrow (n!)$ is ultimately easy to compute" to "$[(\mathrm{Perm}_n \in \mathrm{VP}^0) \wedge (\mathrm{P} = \mathrm{PSPACE})] \Rightarrow (n!)$ is easy to compute".

Under the stronger hypothesis that $\mathrm{VP}^0 = \mathrm{VNP}^0$, we can show more (again due to [Koi05]). If $\mathrm{VP}^0 = \mathrm{VNP}^0$, then the sequences $(\sum_{i=1}^{2^n} 2^{i^2-1})$, $\lfloor 2^{2^n} \ln 2 \rfloor$, $\lfloor 2^{2^n} \ln 3 \rfloor$, $\lfloor 2^{2^n} \pi \rfloor$, all have polynomially bounded complexity, something that is not yet known unconditionally.

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Chapter 5
A Selection of Lower Bounds for Arithmetic Circuits

Neeraj Kayal and Ramprasad Saptharishi

It is convenient to have a measure of the amount of work involved in a computing process, even though it may be a very crude one …We might, for instance, count the number of additions, subtractions, multiplications, divisions, recordings of numbers,…

from *Rounding-off errors in matrix processes,*
Alan M. Turing, 1948

Abstract This article is a survey of techniques used in arithmetic circuit lower bounds.

Keywords Arithmetic circuits · Lower bounds · Determinant · Permanent

Mathematics Subject Classification (2010) Primary 68Q25, 68W30, Secondary 12E05

5.1 Introduction

Polynomials originated in classical mathematical studies concerning geometry and solutions to systems of equations. They feature in many classical results in algebra, number theory, and geometry—e.g. in Galois and Abel's resolution of the solvability

To Somenath Biswas, on his 60th Birthday.

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via radicals of a quintic, Lagrange's theorem on expressing every natural number as a sum of four squares and the impossibility of trisecting an angle (using ruler and compass). In modern times, computer scientists have begun to investigate as to what functions can be (efficiently) computed. Polynomials being a natural class of functions, one is naturally led to the following question:

What is the optimum way to compute a given (family of) polynomial(s)?

Now the most natural way to compute a polynomial $f(x_1, x_2, \ldots, x_n)$ over a field $\mathbb{F}$ is to start with the input variables $x_1, x_2, \ldots, x_n$ and then apply a sequence of basic operations such as additions, subtractions, and multiplications[1] in order to obtain the desired polynomial f. Such a computation is called a straight-line program. We often represent such a straight-line program graphically as an arithmetic circuit—wherein the overall computation corresponds to a directed acylic graph whose source nodes are labelled with the input variables $\{x_1, x_2, \ldots, x_n\}$ and the internal nodes are labelled with either $+$ or $\times$ (each internal node corresponds to one computational step in the straight-line program). We typically allow arbitrary constants from the underlying field on the incoming edges of a $+$ gate so that a $+$ gate can in fact compute an arbitrary $\mathbb{F}$-linear combination of its inputs. The complexity of the computation corresponds to the number of operations, also called the size of the corresponding arithmetic circuit. With arithmetic circuits being the relevant model, the informal question posed above can be formalized by defining the optimal way to compute a given polynomial as the smallest arithmetic circuit in terms of the size that computes it. While different aspects of polynomials have been studied extensively in various areas of mathematics, what is unique to computer science is the endeavor to prove upper and lower bounds on the size of arithmetic circuits computing a given (family of) polynomials. Here we give a biased survey of this area, focusing mostly on lower bounds. Note that there are already two excellent surveys of this area—one by Avi Wigderson [Wig02] and the other by Amir Shpilka and Amir Yehudayoff [SY10].[2] Our intention in writing the survey is the underlying hope that revisiting and assimilating the known results pertaining to circuit lower bounds will in turn help us make progress on this beautiful problem. Consequently, we mostly present here those results which we for some reason felt we did not understand comprehensively enough. We conclude with some recent lower bound results for homogeneous bounded depth formulas. Some notable lower bound results that we are unable to present here due to space and time constraints are as follows. A quadratic lower bound for depth three circuits by Shpilka and Wigderson [SW01], for bounded occur bounded depth formulas by Agrawal, Saha, Saptharishi, and Saxena [ASSS12] and the $n^{1+\Omega(1/r)}$ lower bound for circuits of depth r by Raz [Raz10].

[1] One can also allow more arithmetic operations such as division and square roots. It turns out, however, that one can efficiently simulate any circuit with divisions and square roots by another circuit without these operations while incurring only a polynomial factor increase in size.

[2] A more specialized survey by Chen, Kayal, and Wigderson [CKW11] focuses on the applications of partial derivatives in understanding the structure and complexity of polynomials.

Overview. The state of affairs in arithmetic complexity is such that despite a lot of attention we still have only modest lower bounds for general circuits and formulas. In order to make progress, recent work has focused on restricted subclasses. We first present the best-known lower bound for general circuits due to Baur and Strassen [BS83], and a lower bound for formulas due to Kalorkoti [Kal85]. The subsequent lower bounds that we present follow a common roadmap and we articulate this in Sect. 5.4, and present some simple lower bounds to help the reader gain familiarity. We then present (a slight generalization of) an exponential lower bound for monotone circuits due to Jerrum and Snir [JS82]. Moving on to some other restricted (but still nontrivial and interesting) models, we first present an exponential lower bound for depth three circuits over finite fields due to Grigoriev and Karpinski [GK98] and multilinear formulas. We conclude with some recent progress on lower bounds for homogeneous depth four circuits.

Remark Throughout the article, we shall use Det_n and Perm_n to refer to the determinant and permanent, respectively, of a symbolic $n \times n$ matrix $\left(\!\left(x_{ij}\right)\!\right)_{1 \leq i,j \leq n}$.

5.2 Existential Lower Bounds

Before we embark on our quest to prove lower bounds for interesting families of polynomials, it is natural to ask as to what bounds one can hope to achieve. For a multivariate polynomial $f(\mathbf{x}) \in \mathbb{F}[\mathbf{x}]$, denote by $S(f)$ the size of the smallest arithmetic circuit computing f.

Theorem 1 (Folklore) *For "most" polynomials $f(\mathbf{x}) \in \mathbb{F}[\mathbf{x}]$ of degree d on n variables we have*

$$S(f) \geq \Omega\left(\sqrt{\binom{n+d}{d}}\right).$$

Sketch of Proof We prove this here only in the situation where the underlying field $\mathbb{F}$ is a finite field and refer the reader to another survey ([CKW11], Chap. 4) for a proof in the general case. So let $\mathbb{F} = \mathbb{F}_q$ be a finite field. Any line of a straight-line program computing f can be expressed as taking the product of two $\mathbb{F}_q$-linear combinations of previously computed values. Hence the total number of straight-line programs of length s is at most $q^{O(s^2)}$. On the other hand, there are $q^{\binom{n+d}{d}}$ polynomials of degree d on n variables. Hence most n-variate polynomials of degree d require straight-line programs of length at least (equivalently arithmetic circuits of size at least) $s = \Omega\left(\sqrt{\binom{n+d}{d}}\right)$. □

Hrubes and Yehudayoff [HY11] showed that in fact most n-variate polynomials of degree d *with zero-one coefficients* have complexity at least $\Omega\left(\sqrt{\binom{n+d}{d}}\right)$. Now it turns out that this is in fact a lower bound on the number of multiplications in any circuit computing a random polynomial. Lovett [Lov11] complements this nicely by giving a matching upper bound. Specifically, it was shown in [Lov11] that for any polynomial f of degree d on n variables there exists a circuit computing f having at most $\left(\sqrt{\binom{n+d}{d}}\right) \cdot (nd)^{O(1)}$ multiplications.

5.3 Weak Lower Bounds for General Circuits and Formulas

Despite several attempts by various researchers to prove lower bounds for arithmetic circuits or formulas, we only have very mild lower bounds for general circuits or formulas thus far. In this section, we shall look at the two modest lower bounds for general circuits and formulas.

5.3.1 Lower Bounds for General Circuits

The only super-linear lower bound we currently know for general arithmetic circuits is the following result of Baur and Strassen [BS83].

Theorem 2 [BS83] *Any fan-in two circuit that computes the polynomial* $f = x_1^{d+1} + \cdots + x_n^{d+1}$ *has size* $\Omega(n \log d)$.

5.3.1.1 An Exploitable Weakness

Each gate of the circuit Φ computes a local operation on the two children. To formalize this, define a new variable y_g for every gate $g \in \Phi$. Further, for every gate g define a quadratic equation Q_g as

$$Q_g = \begin{cases} y_g - (y_{g_1} + y_{g_2}) & \text{if } g = g_1 + g_2 \\ y_g - (y_{g_1} \cdot y_{g_2}) & \text{if } g = g_1 \cdot g_2. \end{cases}$$

Further if y_o corresponds to the output gate, then the system of equations

$$\left\{Q_g = 0 \;:\; g \in \Phi\right\} \cup \{y_o = 1\}$$

completely characterize the computations of Φ that results in an output of 1.

The same can also be extended for *multi-output* circuits that compute several polynomials simultaneously. In such cases, the set of equations

$$\left\{Q_g = 0 \;:\; g \in \Phi\right\} \cup \left\{y_{o_i} = 1 \;:\; i = 1, \ldots, n\right\}$$

completely characterize computations that result in an output of all ones. The following classical theorem allows us to bound the number of common roots to a system of polynomial equations.

Theorem 3 (Bézout's theorem) *Let* $g_1, \ldots, g_r \in \mathbb{F}[X]$ *such that* $\deg(g_i) = d_i$ *such that the number of common roots of* $g_1 = \cdots = g_r = 0$ *is finite. Then, the number of common roots (counted with multiplicities) is bounded by* $\prod d_i$.

Thus in particular, if we have a circuit Φ of size s that *simultaneously* computes $\{x_1^d, \ldots, x_n^d\}$, then we have d^n inputs that evaluate to all ones (where each x_i must be a d-th root of unity). Hence, Bézout's theorem implies that

$$2^s \geq d^n \implies s = \Omega(d \log n).$$

Observe that $\{x_1^d, \ldots, x_n^d\}$ are all first-order derivatives of $f = x_1^{d+1} + \cdots + x_n^{d+1}$ (with suitable scaling). A natural question here is the following—if f can be computed an arithmetic circuit of size s, what is the size required to compute all first-order partial derivatives of f simultaneously? The naïve approach of computing each derivative separately results in a circuit of size $O(s \cdot n)$. Baur and Strassen [BS83] show that we can save a factor of n.

Lemma 4 [BS83] *Let* Φ *be an arithmetic circuit of size s and fan-in* 2 *that computes a polynomial* $f \in \mathbb{F}[X]$. *Then, there is a multi-output circuit of size* $O(s)$ *computing all first-order derivatives of* f.

Note that this immediately implies that any circuit computing $f = x_1^{d+1} + \cdots + x_n^{d+1}$ requires size $\Omega(d \log n)$ as claimed by Theorem 2.

5.3.1.2 Computing All First-Order Derivatives Simultaneously

Since we are working with fan-in two circuits, the number of edges is at most twice the size. Hence let s denote the number of edges in the circuit Φ, and we shall prove by induction that all first-order derivatives of Φ can be computed by a circuit of size at most $5s$. Pick a non-leaf node v in the circuit Φ closest to the leaves with both its children being variables, and say x_1 and x_2 are the variables feeding into v. In other words, $v = x_1 \odot x_2$ where $\odot$ is either $+$ or $\times$.

Let Φ' be the circuit obtained by deleting the two edges feeding into v, and replacing v by a new variable. Hence, Φ' computes a polynomial $f' \in \mathbb{F}[X \cup \{v\}]$ and has at most $(s-1)$ edges. By induction on the size, we can assume that there is a circuit $\mathbb{D}(\Phi')$ consisting of at most $5(s-1)$ edges that computes all the first-order derivatives of f'.

Observe that since $f' \mid_{(v=x_1 \odot x_2)} = f(\mathbf{x})$, we have that

$$\frac{\partial f}{\partial x_i} = \left(\frac{\partial f'}{\partial x_i}\right)_{v=x_1 \odot x_2} + \left(\frac{\partial f'}{\partial v}\right)_{v=x_1 \odot x_2} \left(\frac{\partial (x_1 \odot x_2)}{\partial x_i}\right).$$

Hence, if $v = x_1 + x_2$ then

$$\begin{aligned}
\frac{\partial f}{\partial x_1} &= \left(\frac{\partial f'}{\partial x_1}\right)_{v=x_1+x_2} + \left(\frac{\partial f'}{\partial v}\right)_{v=x_1+x_2}\\
\frac{\partial f}{\partial x_2} &= \left(\frac{\partial f'}{\partial x_2}\right)_{v=x_1+x_2} + \left(\frac{\partial f'}{\partial v}\right)_{v=x_1+x_2}\\
\frac{\partial f}{\partial x_i} &= \left(\frac{\partial f'}{\partial x_i}\right)_{v=x_1+x_2} \qquad \text{for } i > 2.
\end{aligned}$$

If $v = x_1 \cdot x_2$, then

$$\begin{aligned}
\frac{\partial f}{\partial x_1} &= \left(\frac{\partial f'}{\partial x_1}\right)_{v=x_1\cdot x_2} + \left(\frac{\partial f'}{\partial v}\right)_{v=x_1\cdot x_2} \cdot x_2\\
\frac{\partial f}{\partial x_2} &= \left(\frac{\partial f'}{\partial x_2}\right)_{v=x_1\cdot x_2} + \left(\frac{\partial f'}{\partial v}\right)_{v=x_1\cdot x_2} \cdot x_1\\
\frac{\partial f}{\partial x_i} &= \left(\frac{\partial f'}{\partial x_i}\right)_{v=x_1\cdot x_2} \qquad \text{for } i > 2.
\end{aligned}$$

Hence, by adding at most 5 additional edges to $\mathbb{D}(\Phi')$, we can construct $\mathbb{D}(\Phi)$ and hence size of $\mathbb{D}(\Phi)$ is at most $5s$. □ (Lemma 4)

5.3.2 Lower Bounds for Formulas

This section is devoted to the proof of Kalorkoti's lower bound [Kal85] for formulas computing Det_n, Perm_n.

Theorem 5 [Kal85] *Any arithmetic formula computing* Perm_n *(or* Det_n*) requires* $\Omega(n^3)$ *size.*

The exploitable weakness in this setting is again to use the fact that the polynomials computed at intermediate gates share many polynomial dependencies.

Definition 6 (*Algebraic independence*) A set of polynomials $\{f_1, \ldots, f_m\}$ is said to be *algebraically independent* if there is no nontrivial polynomial $H(z_1, \ldots, z_m)$ such that $H(f_1, \ldots, f_m) = 0$.

The size of the largest algebraically independent subset of $\mathbf{f} = \{f_1, \ldots, f_m\}$ is called the *transcendence degree* (denoted by $\mathrm{trdeg}(f)$).

The proof of Kalorkoti's theorem proceeds by defining a *complexity measure* using the above notion of algebraic independence.

The Measure: For any subset of variables $Y \subseteq X$, we can write a polynomial $f \in \mathbb{F}[X]$ of the form $f = \sum_{i=1}^{s} f_i \cdot m_i$ where m_i's are distinct monomials in the variables in Y, and $f_i \in F[X \setminus Y]$. We shall denote by $\mathrm{td}_Y(f)$ the transcendence degree of $\{f_1, \ldots, f_s\}$

Fix a partition of variables $X = X_1 \sqcup \cdots \sqcup X_r$. For any polynomial $f \in \mathbb{F}[X]$, define the map $\Gamma^{[\mathrm{Kal}]} : \mathbb{F}[X] \to \mathbb{Z}_{\geq 0}$ as

$$\Gamma^{[\mathrm{Kal}]}(f) = \sum_{i=1}^{r} \mathrm{td}_{X_i}(f).$$

The lower bound proceeds in two natural steps:

1. Show that $\Gamma^{[\mathrm{Kal}]}(f)$ is *small* whenever f is computable by a *small* formula.
2. Show that $\Gamma^{[\mathrm{Kal}]}(\mathsf{Det}_n)$ is *large*.

5.3.2.1 Upper Bounding $\Gamma^{[\mathrm{Kal}]}$ for a Formula

Lemma 7 *Let f be computed by a fan-in two formula Φ of size s. Then for any partition of variables $X = X_1 \sqcup \cdots \sqcup X_r$, we have $\Gamma^{[\mathrm{Kal}]}(f) = O(s)$.*

Proof For any node $v \in \Phi$, let $\textsc{Leaf}(v)$ denote the leaves of the subtree rooted at v and let $\textsc{Leaf}_{X_i}(v)$ denote the leaves of the subtree rooted at v that are in the part X_i. Since the underlying graph of Φ is a tree, it follows that the size of Φ is bounded by twice the number of leaves. For each part X_i, we shall show that $\mathrm{td}_{X_i}(f) = O(\left|\textsc{Leaf}_{X_i}(\Phi)\right|)$, which would prove the required bound.

Fix an arbitrary part $Y = X_i$. Define the following three sets of nodes:

$$\begin{aligned}
V_0 &= \left\{v \in \Phi \ : \ \left|\textsc{Leaf}_Y(v)\right| = 0 \quad \text{and} \quad \left|\textsc{Leaf}_Y(\textsc{Parent}(v))\right| \geq 2\right\}\\
V_1 &= \left\{v \in \Phi \ : \ \left|\textsc{Leaf}_Y(v)\right| = 1 \quad \text{and} \quad \left|\textsc{Leaf}_Y(\textsc{Parent}(v))\right| \geq 2\right\}\\
V_2 &= \left\{v \in \Phi \ : \ \left|\textsc{Leaf}_Y(v)\right| \geq 2\right\}.
\end{aligned}$$

Each node $v \in V_0$ computes a polynomial in $f_v \in \mathbb{F}[X \setminus Y]$, and we shall replace the subtree at v by a node computing the polynomial f_v. Similarly, any node $v \in V_1$ computes a polynomial of the form $f_v^{(0)} + f_v^{(1)} y_v$ for some $y_v \in Y$ and $f_v^{(0)}, f_v^{(1)} \in \mathbb{F}[X \setminus Y]$. We shall again replace the subtree rooted at v by a node computing $f_v^{(0)} + f_v^{(1)} y_v$.

Hence, the formula Φ now reduces to a smaller formula Φ_Y with leaves being the nodes in V_0 and V_1 (and nodes in V_2 are unaffected). We would like to show that the size of the reduced formula, which is at most twice the number of its leaves, is $O(\left|\textsc{Leaf}_Y(\Phi)\right|)$.

Observation 8 $|V_1| \leq \left|\textsc{Leaf}_Y(\Phi)\right|$.

Proof Each node in V_1 has a distinct leaf labelled with a variable in Y. Hence, $|V_1|$ is bounded by the number of leaves labelled with a variable in Y. □ (Obs)

This shows that the V_1 leaves are not too many. Unfortunately, we cannot immediately bound the number of V_0 leaves, since we could have a long chain of V_2 nodes each with one sibling being a V_0 leaf. The following observation would show how we can eliminate such long chains.

Observation 9 *Let u be an arbitrary node, and v be another node in the subtree rooted at u with $\textsc{Leaf}_Y(u) = \textsc{Leaf}_Y(v)$. Then the polynomial g_u computed at u and the polynomial g_v computed at v are related as $g_u = f_1 g_v + f_2$ for some $f_1, f_2 \in \mathbb{F}[X \setminus Y]$.*

Proof If $\textsc{Leaf}_Y(u) = \textsc{Leaf}_Y(v)$, then every node on the path from u to v must have a V_0 leaf as the other child. The observation follows as all these nodes are $+$ or $\times$ gates. □ (Obs)

Using the above observation, we shall remove the need for V_0 nodes completely by augmenting each node u (computing a polynomial g_u) in Φ_Y with polynomials $f_u^{(0)}, f_u^{(1)} \in \mathbb{F}[X \setminus Y]$ to enable them to compute $f_u^{(1)} g_u + f_u^{(0)}$. Let this augmented formula be called $\hat{\Phi}_Y$. Using Observation 9, we can now contract any two nodes u and v with $\textsc{Leaf}_Y(u) = \textsc{Leaf}_Y(v)$, and eliminate all V_0 nodes completely. Since all V_2 nodes are internal nodes, the only leaves of the augmented formula $\hat{\Phi}_Y$ are in V_1. Hence, the size of the augmented formula $\hat{\Phi}_Y$ is bounded by $2\,|V_1|$, which is bounded by $2\,\left|\textsc{Leaf}_Y(\Phi)\right|$ by Observation 8.

Suppose Φ computes a polynomial f, which can be written as $f = \sum_{i=1}^t f_i \cdot m_i$ with $f_i \in \mathbb{F}[X \setminus Y]$ and m_i's being distinct monomials in Y. Since $\hat{\Phi}_Y$ also computes f, each f_i is a polynomial combination of the polynomials $S_Y = \left\{f_u^{(0)}, f_u^{(1)} \;:\; u \in \hat{\Phi}_Y\right\}$. Since $\hat{\Phi}_Y$ consists of at most $2\left|\textsc{Leaf}_Y(\Phi)\right|$ augmented nodes, we have that $\mathrm{td}_Y(f) \leq |S_Y| \leq 4\left|\textsc{Leaf}_Y(\Phi)\right|$. Therefore,

$$\mathrm{td}_Y(f) = \mathrm{trdeg}\{f_i \;:\; i \in [t]\} \leq 4\left|\textsc{Leaf}_Y(\Phi)\right|$$

Hence,

$$\Gamma^{[\mathrm{Kal}]}(\Phi) = \sum_{i=1}^{r} \mathrm{td}_{X_i}(f_i) \leq 4\left(\sum_{i=1}^{r} \left|\textsc{Leaf}_{X_i}\right|\right) = O(s).$$

□

5.3.2.2 Lower Bounding $\Gamma^{[\mathrm{Kal}]}(\mathsf{Det}_n)$

Lemma 10 *Let $X = X_1 \sqcup \cdots \sqcup X_n$ be the partition as defined by $X_t = \left\{x_{ij} \;:\; i - j \equiv t \bmod n\right\}$. Then, $\Gamma^{[\mathrm{Kal}]}(\mathsf{Det}_n) = \Omega(n^3)$.*

Proof By symmetry, it is easy to see that $\mathrm{td}_{X_i}(\mathsf{Det}_n)$ is the same for all i. Hence, it suffices to show that $\mathrm{td}_Y(\mathsf{Det}_n) = \Omega(n^2)$ for $Y = X_n = \{x_{11}, \ldots, x_{nn}\}$.

To see this, observe that the determinant consists of the monomials $\left(\frac{x_{11} \cdots x_{nn}}{x_{ii} x_{jj}}\right) \cdot x_{ij} x_{ji}$ for every $i \neq j$. Hence, $\mathrm{td}_Y(\mathsf{Det}_n) \geq \mathrm{trdeg}\left\{x_{ij} x_{ji} \;:\; i \neq j\right\} = \Omega(n^2)$. Therefore, $\Gamma^{[\mathrm{Kal}]}(\mathsf{Det}_n) = \Omega(n^3)$. □

The proof of Theorem 5 follows from Lemma 7 and Lemma 10.

5.4 "Natural" Proof Strategies

The lower bounds presented in Sect. 5.3 proceeded by first identifying a *weakness* of the model, and exploiting it in an explicit manner. More concretely, Sect. 5.3.2 presents a promising strategy that could be adopted to prove lower bounds for various models of arithmetic circuits. The crux of the lower bound was the construction of a good map Γ that assigned a number to every polynomial. The map $\Gamma^{[\mathrm{Kal}]}$ was useful to show a lower bound in the sense that any f computable by a *small* formula had *small* $\Gamma^{[\mathrm{Kal}]}(f)$. In fact, all subsequent lower bounds in arithmetic circuit complexity have more or less followed a similar template of a "natural proof". More concretely, all the subsequent lower bounds we shall see would essentially follow the outlined plan.

Step 1 (normal forms) For every circuit in the circuit class $\mathcal{C}$ of interest, express the polynomial computed as a *small sum of simple building blocks*.

For example, every $\Sigma\Pi\Sigma$ circuit is a *small* sum of *products of linear polynomials* which are the building blocks here. In this case, the circuit model naturally admits such a representation but we shall see other examples with very different representations as sum of building blocks.

Step 2 (complexity measure) Construct a map $\Gamma : \mathbb{F}[x_1, \ldots, x_n] \to \mathbb{Z}_{\geq 0}$ that is *sub-additive* i.e., $\Gamma(f_1 + f_2) \leq \Gamma(f_1) + \Gamma(f_2)$.

In most cases, $\Gamma(f)$ is the rank of a large matrix whose entries are linear functions in the coefficients of f. In such cases, we immediately get that Γ is sub-additive.

The strength of the choice of Γ is determined by the next step.

Step 3 (potential usefulness) Show that if B is a *simple building block*, then $\Gamma(B)$ is *small*. Further, check if $\Gamma(f)$ for a *random polynomial* f is large (potentially).

This would suggest that if any f with large $\Gamma(f)$ is to be written as a sum of $B_1 + \cdots + B_s$, then sub-additivity and the fact that $\Gamma(B_i)$ is small for each i and $\Gamma(f)$ is large immediately imply that s must be large. This implies that the complexity measure Γ does indeed have a potential to prove a lower bound for the class. The next step is just to replace the *random polynomial* by an explicit polynomial.

Step 4 (explicit lower bound) Find an explicit polynomial f for which $\Gamma(f)$ is large.

These are usually the steps taken in almost all the known arithmetic circuit lower bound proofs. The main ingenuity lies in constructing a useful complexity measure, which is really to design Γ so that it is small on the *building blocks*.

Of course, there could potentially be lower bound proofs that do not follow the roadmap outlined. For instance, it could be possible that Γ is not small for a random polynomial, but specifically tailored in a way to make Γ large for the Perm_n. Or perhaps Γ need not even be sub-additive and maybe there is a very different way to argue that all polynomial in the circuit class have small Γ. However, this has been the roadmap for almost all lower bounds so far (barring very few exceptions).

As a warmup, we first present some very simple applications of the above plan to prove lower bounds for some very simple subclasses of arithmetic circuits in the next section. We then move on to more sophisticated proofs of lower bounds for less restricted subclasses of circuits.

5.5 Some Simple Lower Bounds

Let us start with the simplest complete[3] class of arithmetic circuits—depth-2 circuits or $\Sigma\Pi$ circuits.

5.5.1 Lower Bounds for $\Sigma\Pi$ *circuits*

Any $\Sigma\Pi$ circuit of size s computes a polynomial $f = m_1 + \cdots + m_s$ where each m_i is a monomial multiplied by a field constant. Therefore, any polynomial computed by a *small* $\Sigma\Pi$ circuit must have a *small* number of monomials. Hence, it is obvious that any polynomial that has many monomials requires large $\Sigma\Pi$ circuits.

This can be readily rephrased in the language of the outline described in the last section by defining $\Gamma(f)$ to simply be the number of monomials present in f. Hence, $\Gamma(f) \leq s$ for any f computed by a $\Sigma\Pi$ circuit of size s. Of course, even a polynomial like $f = (x_1 + x_2 + \cdots + x_n)^n$ has $\Gamma(f) = n^{\Omega(n)}$ giving the lower bound.

5.5.2 Lower Bounds for $\Sigma\wedge\Sigma$ *Circuits*

A $\Sigma\wedge\Sigma$ circuit of size s computes a polynomial of the form $f = \ell_1^{d_1} + \cdots + \ell_s^{d_s}$ where each ℓ_i is a linear polynomial over the n variables.[4]

Clearly, as even a single ℓ^d could have exponentially many monomials, the Γ defined above cannot work in this setting. Nevertheless, we shall try to design a similar map to ensure that $\Gamma(f)$ is *small* whenever f is computable by a *small* $\Sigma\wedge\Sigma$ circuit.

In this setting, the *building blocks* are terms of the form ℓ^d. The goal would be to construct a *sub-additive* measure Γ such that $\Gamma(\ell^d)$ is *small*. Here is the key observation to guide us towards a good choice of Γ.

Observation 11 *Any k-th order partial derivative of ℓ^d is a constant multiple of ℓ^{d-k}.*

[3] in the sense that any polynomial can be computed in this model albeit of large size.

[4] such circuits are also called *diagonal depth-3 circuits* in the literature.

Hence, if $\partial^{=k}(f)$ denotes the set of k-th order partial derivatives of f, then the space spanned by $\partial^{=k}(\ell^d)$ has dimension 1. This naturally leads us to define Γ exploiting this weakness.

$$\Gamma_k(f) \quad \stackrel{\text{def}}{=} \quad \dim\left(\partial^{=k}(f)\right)$$

It is straightforward to check that Γ_k is indeed sub-additive and hence $\Gamma_k(f) \le s$ whenever f is computable by a $\Sigma \wedge \Sigma$ circuit of size s. For a random polynomial f, we should be expecting $\Gamma_k(f)$ to be $\binom{n+k}{k}$ as there is unlikely to be any linear dependencies among the partial derivatives. Hence, all that needs to be done is to find an explicit polynomial with large Γ_k.

If we consider Det_n or Perm_n, then any partial derivative of order k is just an $(n-k) \times (n-k)$ minor. Also, these minors consist of disjoint sets of monomials and hence are linearly independent. Hence, $\Gamma_k(\mathsf{Det}_n) = \binom{n}{k}^2$. Choosing $k = n/2$, we immediately get that any $\Sigma \wedge \Sigma$ circuit computing Det_n or Perm_n must be of size $2^{\Omega(n)}$.

5.5.3 *Low-Rank* $\Sigma\Pi\Sigma$

A slight generalization of $\Sigma \wedge \Sigma$ circuits is a *rank-r* $\Sigma\Pi\Sigma$ *circuit* that computes a polynomial of the form

$$f = T_1 + \cdots + T_s$$

where each $T_i = \ell_{i1} \dots \ell_{id}$ is a product of linear polynomials such that $\dim \{\ell_{i1}, \dots, \ell_{id}\} \le r$.

Thus, $\Sigma \wedge \Sigma$ is a rank-1 $\Sigma\Pi\Sigma$ circuit, and a similar partial-derivative technique for lower bounds works here as well.

In the setting where r is much smaller than the number of variables n, each T_i is essentially an r-variate polynomial masquerading as an n-variate polynomial using an affine transformation. In particular, the set of n first-order derivatives of T have rank at most r. This yields the following observation.

Observation 12 *Let $T = \ell_1 \dots \ell_d$ with $\dim \{\ell_1, \dots, \ell_d\} \le r$. Then for any k, we have*

$$\Gamma_k(T) \stackrel{def}{=} \dim\left(\partial^{=k}(T)\right) \le \binom{r+k}{k}.$$

Thus once again by sub-additivity, for any polynomial f computable by a rank-r $\Sigma\Pi\Sigma$ circuit of size s, we have $\Gamma_k(f) \le s \cdot \binom{r+k}{r}$. Note that a random polynomial is expected to have $\Gamma_k(f)$ close to $\binom{n+k}{k}$, which could be much larger for $r \ll n$. We already saw that $\Gamma_k(\mathsf{Det}_n) = \binom{n}{k}^2$. This immediately gives the following lower bound, the proof of which we leave as an exercise to the interested reader.

Theorem 13 *Let $r \le n^{2-\delta}$ for some constant $\delta > 0$. For $k = \epsilon n^{\delta}$, where $\epsilon > 0$ is sufficiently small, we have*

$$\frac{\binom{n}{k}^2}{\binom{r+k}{k}} = \exp\left(\Omega\left(n^{\delta}\right)\right).$$

Hence, any rank-r $\Sigma\Pi\Sigma$ *circuit computing* Det_n *or* Perm_n *must have size* $\exp\left(\Omega\left(n^{\delta}\right)\right)$. $\square$

This technique of using the rank of partial derivatives was introduced by Nisan and Wigderson [NW97] to prove lower bounds for *homogeneous depth-*3 *circuits* (which also follows as a corollary of Theorem 13). The survey of Chen, Kayal and Wigderson [CKW11] give a comprehensive exposition of the power of the *partial derivative method.*

With these simple examples, we can move on to other lower bounds for various other more interesting models.

5.6 Lower Bounds for Monotone Circuits

This section presents a slight generalization of a lower bound by Jerrum and Snir [JS82]. To motivate our presentation here, let us first assume that the underlying field is $\mathbb{R}$, the field of real numbers. A monotone circuit over $\mathbb{R}$ is a circuit having $+$, $\times$ gates in which all the field constants are *nonnegative* real numbers. Such a circuit can compute any polynomial f over $\mathbb{R}$ all of whose coefficients are nonnegative real numbers, such as for example the permanent. It is then natural to ask whether there are small monotone circuits over $\mathbb{R}$ computing the permanent. Jerrum and Snir [JS82] obtained an exponential lower bound on the size of monotone circuits over $\mathbb{R}$ computing the permanent. Note that this definition of monotone circuits is valid only over $\mathbb{R}$ (actually more generally over ordered fields but not over say finite fields) and such circuits can only compute polynomials with nonnegative coefficients. Here we will present Jerrum and Snir's argument in a slightly more generalized form such that the circuit model makes sense over any field $\mathbb{F}$ and is complete, i.e., can compute any polynomial over $\mathbb{F}$. Let us first explain the motivation behind the generalized circuit model that we present here. Observe that in any monotone circuit over $\mathbb{R}$, there is no cancellation as there are no negative coefficients. Formally, for a node v in our circuits let us denote by f_v the polynomial computed at that node. For a polynomial f let us denote by $\mathrm{Mon}(f)$ the set of monomials having a nonzero coefficient in the polynomial f.

1. If $w = u + v$ then
$$\mathrm{Mon}(f_w) = \mathrm{Mon}(f_u) \cup \mathrm{Mon}(f_v).$$

2. If $w = u \times v$ then
$$\mathrm{Mon}(f_w) = \mathrm{Mon}(f_u)\cdot\mathrm{Mon}(f_v) \overset{\text{def}}{=} \{m_1 \cdot m_2 \ : \ m_1 \in \mathrm{Mon}(f_u), m_2 \in \mathrm{Mon}(f_v)\}.$$

This means that for any node v in a monote circuit over $\mathbb{R}$ one can determine $\mathrm{Mon}(f_v)$ in a very syntactic manner starting from the leaf nodes. Let us make precise this syntactic computation that we have in mind.

Definition 14 (*Formal Monomials*) Let Φ be an arithmetic circuit. The *formal monomials* at any node $v \in \Phi$, which shall be denoted by $\mathrm{FM}(v)$, shall be inductively defined as follows:

If v is a leaf labelled by a variable x_i, then $\mathrm{FM}(v) = \{x_i\}$. If it is labelled by a constant, then $\mathrm{FM}(v) = \{1\}$.

If $v = v_1 + v_2$, then $\mathrm{FM}(v) = \mathrm{FM}(v_1) \cup \mathrm{FM}(v_2)$.

If $v = v_1 \times v_2$, then

$$\begin{aligned}\mathrm{FM}(v) &= \mathrm{FM}(v_1) \cdot \mathrm{FM}(v_2)\\ &\overset{\text{def}}{=} \{m_1 \cdot m_2 \ : \ m_1 \in \mathrm{FM}(v_1), m_2 \in \mathrm{FM}(v_2)\}\,.\end{aligned}$$

Note that for any node v in any circuit we have $\mathrm{Mon}(f_v) \subseteq \mathrm{FM}(v)$ but in a monotone circuit over $\mathbb{R}$ this containment is in fact an equality at every node. This motivates our definition of a slightly more general notion of a monotone circuit as follows:

Definition 15 (*Monotone circuits*) A circuit C is said to be *syntactically monotone* (simply monotone for short) if $\mathrm{Mon}(f_v) = \mathrm{FM}(v)$ for every node v in C.

The main theorem of this section is the following:

Theorem 16 [JS82] *Over any field $\mathbb{F}$, any syntactically monotone circuit C computing Det_n or Perm_n must have size at least $2^{\Omega(n)}$.*

The proof of this theorem is relatively short assuming the following structural result (which is present in standard depth-reduction proofs [VSBR83, AJMV98]).

Lemma 17 *Let f be a degree d polynomial computed by a monotone circuit of size s. Then, f can be written of the form $f = \sum_{i=1}^{s} f_i \cdot g_i$ where the f_i's and g_i's satisfy the following properties:*

1. *For each $i \in [s]$, we have $\frac{d}{3} < \deg g_i \leq \frac{2d}{3}$.*
2. *For each i, we have $\mathrm{FM}(f_i) \cdot \mathrm{FM}(g_i) \subseteq \mathrm{FM}(f)$.*

We shall defer this lemma to the end of the section and first see how this would imply Theorem 16. The complexity measure $\Gamma(f)$ in this case is just the number of monomials in f, but it is the above *normal form* that is crucial in the lower bound.

Proof of Theorem 16 Suppose Φ is a circuit of size s that computes Det_n. Then by Lemma 17,

$$\mathsf{Det}_n = \sum_{i=1}^{s} f_i \cdot g_i$$

with $\mathrm{FM}(f_i) \cdot \mathrm{FM}(g_i) \subseteq \mathrm{FM}(\mathsf{Det}_n)$. The building blocks are terms of the form $T = f \cdot g$, where $\mathrm{FM}(f) \cdot \mathrm{FM}(g) \subseteq \mathrm{FM}(\mathsf{Det}_n)$.

Since all the monomials in Det_n are products of variables from distinct columns and rows, the rows (and columns) containing the variables f depends on is disjoint from the rows (and columns) containing variables that g depends on. Hence, there exists sets of indices $A, B \subseteq [n]$ such that f depends only on $\left\{x_{jk} \ : \ j \in A, k \in B\right\}$ and g depends only on $\left\{x_{jk} \ : \ j \in \overline{A}, k \in \overline{B}\right\}$.

Further, since Det_n is a homogeneous polynomial of degree n, we also have that both f and g must be homogeneous as well. Also, as all monomials of g using distinct row and column indices from $\overline{A}$ and $\overline{B}$ respectively, we see that $\deg g = |\overline{A}| = |\overline{B}|$ and $\deg f = |A| = |B|$. Using Lemma 17, let $|A| = \alpha n$ for some $\frac{1}{3} \leq \alpha \leq \frac{2}{3}$. This implies that $\Gamma(f) \leq (\alpha n)!$, and $\Gamma(g) \leq ((1-\alpha)n)!$ and hence

$$\Gamma(f \cdot g) \ \leq \ (\alpha n)!((1-\alpha)n)! \leq \frac{n!}{\binom{n}{n/3}}$$

as $\frac{1}{3} \leq \alpha \leq \frac{2}{3}$. Also, Γ is clearly sub-additive and we have

$$\Gamma(f_1 g_1 + \cdots + f_s g_s) \leq s \cdot \frac{n!}{\binom{n}{n/3}}.$$

Since $\Gamma(\mathsf{Det}_n) = n!$, this forces $s \geq \binom{n}{n/3} = 2^{\Omega(n)}$. □

We only need to prove Lemma 17 now.

5.6.1 Proof of Lemma 17

Without loss of generality, assume that the circuit Φ is homogeneous,[5] and consists of alternating layers of $+$ and $\times$ gates. Also, assume that all $\times$ gates have fan-in two, and orient the two children such that the formal degree of the left child is at least as large as the formal degree of the right child. Such circuits are also called *left-heavy* circuits.

Definition 18 (*Proof tree*) A proof tree of an arithmetic circuit Φ is a sub-circuit Φ' such that

- The root of Φ is in Φ'
- If a multiplication gate with $v = v_1 \times v_2 \in \Phi'$, then v_1 and v_2 are in Φ' as well.
- If an addition gate $v = v_1 + \cdots + v_s \in \Phi'$, then exactly one v_i is in Φ'.

[5] It is a forklore result that any circuit can be *homogenized* with just a polynomial blowup in size. Further, this process also preserves monotonicity of the circuit. A proof of this may be seen in [SY10].

Such a sub-circuit Φ', represented as a tree (duplicating nodes if required), shall be called a proof tree of Φ.

Let ProofTrees(Φ) denote the set of all proof trees of Φ. It is easy to see that any proof tree of Φ computes a monomial over the variables. Further, if Φ was a monotone circuit computing a polynomial f, then every proof tree computes a monomial in f. Therefore,

$$f = \sum_{\Phi' \in \text{ProofTrees}(\Phi)} [\Phi']$$

where $[\Phi']$ denotes the monomial computed by Φ'. Of course, the number of proof trees is exponential and the above expression is huge. However, we could use a divide-and-conquer approach to the above equation using the following lemma.

Lemma 19 *Let Φ' be a left-heavy formula of formal degree d. Then there is a node v on the left-most path of Φ' such that $\frac{d}{3} \leq \deg(v) < \frac{2d}{3}$.*

Proof Pick the lowest node on the leftmost path that has degree at least $\frac{2d}{3}$. Then, its left child must be a node of degree less than $\frac{2d}{3}$, and also at least $\frac{d}{3}$ (because the formula is left-heavy). □

For any proof tree Φ' and a node v on its leftmost path, define $[\Phi' : v]$ to be the output polynomial of the proof tree obtained by replacing the node v on the leftmost path by 1. If v does not occur on the leftmost path of Φ', define $[\Phi' : v]$ to be 0. We will denote the polynomial computed at a node v by f_v. Then, the above equation can now be rewritten as:

$$\begin{aligned} f &= \sum_{\Phi' \in \text{ProofTrees}(\Phi)} [\Phi'] \\ &= \sum_{\substack{v \in \Phi \\ \frac{d}{3} \leq \deg v < \frac{2d}{3}}} f_v \cdot \left(\sum_{\Phi' \in \text{ProofTrees}(\Phi)} [\Phi' : v] \right) \\ &= \sum_{\substack{v \in \Phi \\ \frac{d}{3} \leq \deg v < \frac{2d}{3}}} f_v \cdot g_v \quad \text{where } g_v = \sum_{\Phi' \in \text{ProofTrees}(\Phi)} [\Phi' : v]. \end{aligned}$$

Since $\frac{d}{3} \leq \deg v < \frac{2d}{3}$, we also have that $\frac{d}{3} < \deg g_v \leq \frac{2d}{3}$ and the last equation is what was required by Lemma 17. □

5.7 Lower Bounds for Depth-3 Circuits over Finite Fields

This section presents the lower bound of Grigoriev and Karpinski [GK98] for Det_n. A follow-up paper of Grigoriev and Razborov [GR00] extended the result over function fields, also including a weaker lower bound for the permanent, but we shall present a slightly different proof that works for the permanent as well.

Theorem 20 [GK98] *Any depth-3 circuit computing* Det_n *(or* Perm_n*) over a finite field* $\mathbb{F}_q$ *(*$q \neq 2$*) requires size* $2^{\Omega(n)}$.

Main idea: Let $q = |\mathbb{F}|$. Suppose $C = T_1 + \cdots + T_s$, where each T_i is a product of linear polynomials. Define $\operatorname{rank}(T_i)$ as in Sect. 5.5.3 to be the dimension of the set of linear polynomials that T_i is a product of.

In Sect. 5.5.3, we saw that the dimension of partial derivatives would handle *low rank* T_i's. As for the high rank T_i's, since T_i is a product of at least r linearly independent linear polynomials, a random evaluation keeps T_i nonzero with probability at most $\left(1 - \frac{1}{q}\right)^r$. Since q is a constant, we have that a random evaluation of a high rank T_i is almost always zero. Hence, in a sense, C can be "approximated" by just the low-rank components.

Grigoriev and Karpinski [GK98] formalize the above idea as a measure by combining the partial derivative technique seen in Sect. 5.5.3 with evaluations to show that Det_n cannot be approximated by a low-rank $\Sigma\Pi\Sigma$ circuit.

5.7.1 The Complexity Measure

For any polynomial $f \in \mathbb{F}[x_{11}, \ldots, x_{nn}]$, define the matrix $M_k(f)$ as follows—the columns of $M_k(f)$ are indexed by k-th order partial derivatives of f, and rows by elements of $\mathbb{F}^{n^2}$, with the entry being the evaluation of the partial derivative (column index) at the point (row index).

The rank of $M_k(f)$ could be a possible choice of a complexity measure but Grigoriev and Karpinski make a small modification to handle the high rank T_is. Instead, they look at the matrix $M_k(f)$ and remove a few *erroneous* evaluation points and use the rank of the resulting matrix. For any $\mathcal{A} \subseteq \mathbb{F}^{n^2}$, let us define $M_k(f; \mathcal{A})$ to be the matrix obtained from $M_k(f)$ by only taking the rows whose indices are in $\mathcal{A}$. Also, let $\Gamma^{[\mathrm{GK}]}_{k,\mathcal{A}}(f)$ denote $\operatorname{rank}(M_k(f; \mathcal{A}))$.

5.7.2 Upper Bounding $\Gamma^{[\mathrm{GK}]}_{k,\mathcal{A}}$ for a Depth-3 Circuit

Our task here is to give an upper bound on the complexity measure for a $\Sigma\Pi\Sigma$-circuit of size s. We first see that the task reduces to upper bounding the measure for a single term via subadditivity. It follows from the linearity of the entries of the matrix.

Observation 21 (Sub-additivity) $\Gamma^{[\mathrm{GK}]}_{k,\mathcal{A}}(f+g) \le \Gamma^{[\mathrm{GK}]}_{k,\mathcal{A}}(f) + \Gamma^{[\mathrm{GK}]}_{k,\mathcal{A}}(g)$.

Now fix a threshold $r_0 = \beta n$ for some constant $\beta > 0$ (to be chosen shortly), and let $k = \gamma n$ for some $\gamma > 0$ (to be chosen shortly). We shall call a term $T = \ell_1 \dots \ell_d$ to be of *low rank* if $\mathrm{rank}(T) \le r_0$, and *large rank* otherwise. By the above observation, we need to upper bound the measure $\Gamma^{[\mathrm{GK}]}_{k,\mathcal{A}}$ for each term T, for a suitable choice of $\mathcal{A}$.

Low rank terms $(\mathrm{rank}(T) \le r_0)$
Suppose $T = \ell_1 \dots \ell_d$ with $\{\ell_1, \dots, \ell_r\}$ being a maximal independent set of linear polynomials (with $r \le r_0$). Then, T can be expressed as a linear combination of terms from the set $\left\{\ell_1^{e_1} \dots \ell_r^{e_r} \;:\; e_i \le d \quad \forall i \in [r]\right\}$. And since the matrix $M_k(f)$ depends only on evaluations in $\mathbb{F}^{n^2}$, we can use the relation that $x^q = x$ in $\mathbb{F}$ to express the function $T : \mathbb{F}^{n^2} \to \mathbb{F}$ as a linear combination of $\left\{\ell_1^{e_1} \dots \ell_r^{e_r} \;:\; e_i < q \quad \forall i \in [r]\right\}$. Therefore, for any set $\mathcal{A} \subseteq \mathbb{F}^{n^2}$, we have that

$$\Gamma^{[\mathrm{GK}]}_{k;\mathcal{A}}(T) \le \mathrm{rank}(M_k(f)) \le q^r \le q^{\beta n}.$$

High rank terms $(\mathrm{rank}(T) > r_0)$
Suppose $T = \ell_1 \dots \ell_d$ whose rank is greater than $r_0 = \beta n$, and let $\{\ell_1, \dots, \ell_r\}$ be a maximal independent set. We want to use the fact that since T is a product of at least r independent linear polynomials, most evaluations would be zero. We shall be choosing our $\mathcal{A}$ to be the set where all k-th order partial derivatives evaluate to zero.

On applying the product rule of differentiation, any k-th order derivative of T can be written as a sum of terms each of which is a product of at least $r-k$ independent linear polynomials. Let us count the *erroneous points* $\mathcal{E}_T \subseteq \mathbb{F}^{n^2}$ that keep at least $r-k$ of $\{\ell_1, \dots, \ell_r\}$ nonzero, or in other words makes at most k of $\{\ell_1, \dots, \ell_r\}$ zero.

$$\Pr_{\mathbf{a}\in\mathbb{F}^{n^2}}[\text{at most } k \text{ of } \ell_1, \dots, \ell_r \text{ evaluate to zero}] \le \sum_{i=0}^{k} \binom{r}{i} \left(\frac{1}{q}\right)^i \left(1 - \frac{1}{q}\right)^{r-i}$$

Hence, we can upper bound $|\mathcal{E}_T|$ as

$$\begin{aligned} |\mathcal{E}_T| &\le \sum_{i=0}^{k} \binom{r}{i} (q-1)^{r-i} q^{n^2-r} \\ &= O\left(k \cdot \binom{r}{k} \left(1 - \frac{1}{q}\right)^{r-k} q^{n^2}\right) \quad \text{if } r > qk \\ &= q^{n^2} \cdot \alpha^n \quad \text{for some } 0 < \alpha < 1. \end{aligned}$$

By choosing $\mathcal{A} = \mathbb{F}^{n^2} \setminus \mathcal{E}$ where $\mathcal{E} = \bigcup_{T \text{ of large rank}} \mathcal{E}_T$, we have that $M_k(T; \mathcal{A})$ is just the zero matrix and hence $\Gamma^{[\mathrm{GK}]}_{k,\mathcal{A}}(T) = 0$.

Putting it together, if $C = T_1 + \cdots + T_s$, then

$$\Gamma_{k,\mathcal{A}}^{[\mathrm{GK}]}(C) \leq s \cdot q^{\beta n}. \tag{5.1}$$

where $\mathcal{A} = \mathbb{F}^{n^2} \setminus \mathcal{E}$ for some set $\mathcal{E}$ of size at most $s \cdot \alpha^n \cdot q^{n^2}$ for some $0 < \alpha < 1$.

5.7.3 *Lower Bounding* $\Gamma_{k,\mathcal{A}}^{[\mathrm{GK}]}$ *for* Det_n *and* Perm_n

We now wish to show that $M_k(\mathsf{Det}_n; \mathcal{A})$ has large rank. The original proof of Grigoriev and Karpinski is tailored specifically for the determinant, and does not extend directly to the permanent. The following argument is a proof communicated by Srikanth Srinivasan [Sri13] that involves an elegant trick that he attributes to [Kou08]. The following proof is presented for the determinant, but immediately extends to the permanent as well.

Note that if we were to just consider $M_k(\mathsf{Det}_n)$, it would have been easy to show that the rank is full by looking at just those evaluation points that keep exactly one $(n-k) \times (n-k)$ minor nonzero (set the main diagonal of the minor to ones, and every other entry to zero). Hence, $M_k(\mathsf{Det}_n)$ has the identity matrix *embedded inside* and hence must be full rank. However, we are missing a few of the evaluations (since a small set $\mathcal{E}$ of evaluations is removed) and we would still like to show that the matrix continues to have full column-rank.

Lemma 22 *Let $p(X)$ be a nonzero linear combination of $r \times r$ minors of the matrix $X = ((x_{ij}))$. Then,*

$$\Pr_{A \in \mathbb{F}_q^{n^2}} [p(A) \neq 0] \geq \Omega(1).$$

This immediately implies that for every linear combinations of the columns of $M_k(\mathsf{Det}_n)$, a constant fraction of the coordinates have nonzero values. Since we are removing merely a set $\mathcal{E}$ of size $(1-o(1))q^{n^2}$, there must continue to exist coordinates that are nonzero. In other words, no linear combination of columns of $M_k(\mathsf{Det}_n; \mathcal{A})$ results in the zero vector.

The proof of the above lemma would be an induction on the number of minors contributing to the linear combination. As a base case, we shall use a well-known fact about Det_n and Perm_n of random matrices.

Proposition 23 *If A is a random $n \times n$ matrix with entries from a fixed finite field $\mathbb{F}_q$, then for $q \neq 2$ we have*

$$\Pr[\det(A) \neq 0] \geq \frac{q-2}{q-1} = \Omega(1).$$

We shall defer the proof of this proposition for later, and proceed with the proof of Lemma 22.

Proof of Lemma 22 If $p(X)$ is a scalar multiple of a single nonzero minor, then we already have the lemma from Proposition 23. Hence, let us assume that there are at least two distinct minors participating in the linear combination $p(X)$. Without loss of generality, assume that the first row occurs in some of the minors, and does not in others. That is,

$$\begin{aligned} p(X) &= \left(\sum_{i:\mathrm{Row}_1 \in M_i} c_i M_i \right) + \left(\sum_{j:\mathrm{Row}_1 \notin M_j} c_j M_j \right) \\ &= \left(x_{11} M_1' + \cdots + x_{1n} M_n' \right) + M'' \quad \text{(expanding along the first row).} \end{aligned}$$

To understand a random evaluation of $p(X)$, let us first set rows $2, \ldots, n$ to random values, and then setting row 1 to random values.

$$\begin{aligned} \Pr_A[p(A) \neq 0] \geq \Pr[x_{11}M_1' + \cdots + x_{1n}M_n' + M'' \neq 0 \mid \text{some } M_i' \neq 0] \\ \times \Pr[\text{some } M_i' \neq 0] \end{aligned}$$

Note that once we have set rows $2, \ldots, n$ to random values, $p(X)$ reduces to a linear polynomial in $\{x_{11}, \ldots, x_{1n}\}$. Further, a random evaluation of any nonconstant linear polynomial is zero with probability exactly $\left(1 - \frac{1}{q}\right)$. Hence,

$$\begin{aligned} \Pr_A[p(A) \neq 0] &\geq \Pr[x_{11}M_1' + \cdots + x_{1n}M_n' + M'' \neq 0 \mid \text{some } M_i' \neq 0] \\ &\quad \times \Pr[\text{some } M_i' \neq 0] \\ &= \left(1 - \frac{1}{q}\right) \cdot \Pr[\text{some } M_i' \neq 0]. \end{aligned}$$

Now comes Koutis' Trick: the term $\left(1 - \frac{1}{q}\right) \cdot \Pr[\text{ some } M_i' \neq 0]$ is exactly the probability that $x_{11}M_1' + \cdots + x_{1n}M_n'$ is nonzero! Hence,

$$\begin{aligned} \Pr_A[p(A) \neq 0] &= \Pr[x_{11}M_1' + \cdots + x_{1n}M_n' + M'' \neq 0] \\ &\geq \Pr[x_{11}M_1' + \cdots + x_{1n}M_n' \neq 0] \\ &= \Pr\left[\left(\sum_{i:\mathrm{Row}_1 \in M_i} c_i M_i \right) \neq 0 \right]. \end{aligned}$$

which is just the linear combination obtained by only considering those minors that contain the first row. Repeating this process for other rows/columns until only one minor remains, we have

$$\Pr_A[p(A) \neq 0] \geq \Pr_A[\det(A) \neq 0] = \frac{q-2}{q-1} \quad \text{(by Proposition 23).}$$ □

We now give a proof of Proposition 23.

Proof of Proposition 23 We shall fix random values to the first row of A. Then,

$$\begin{aligned}\Pr_A[\mathsf{Det}_n(A) = 0] &\le \Pr[a_{11}M_1 + \cdots + a_{1n}M_n = 0 \mid \text{some } a_{1i} \text{ nonzero}] \\ &\quad + \Pr[a_{11} = \cdots = a_{1n} = 0] \\ &= \Pr[a_{11}M_1 + \cdots + a_{1n}M_n = 0 \mid \text{some } a_{1i} \text{ nonzero}] \\ &\quad + \frac{1}{q^n}.\end{aligned}$$

Whenever there is some a_{1i} that is nonzero, then $a_{11}M_1 + \cdots + a_{1n}M_n$ is a nonzero linear combination of minors. By a similar argument as in the proof of Lemma 22, we have that

$$\Pr[a_{11}M_1 + \cdots + a_{1n}M_n = 0 \mid \text{not all } a_{1i} \text{ are zero}] \;\le\; \Pr[\mathsf{Det}_{n-1}(A) = 0].$$

Unfolding this recursion, we have

$$\begin{aligned}\Pr[\mathsf{Det}_n(A) = 0] &\le \frac{1}{q} + \frac{1}{q^2} + \cdots + \frac{1}{q^n} \;=\; \frac{1}{q-1} \\ \implies \Pr[\mathsf{Det}_n(A) \ne 0] &\ge \left(1 - \frac{1}{q-1}\right) \;=\; \frac{q-2}{q-1}.\end{aligned}$$

□

5.7.4 Putting It All Together

Hence, if Det_n is computed by a depth-3 circuit of top fan-in s over $\mathbb{F}$, then

$$\begin{aligned}s \cdot q^{\beta n} &= \Omega\left(\binom{n}{k}^2\right) \\ &= \Omega\left(2^{2H(\gamma)\cdot n}\right) \\ \implies \log s &= \Omega((2H(\gamma) - \beta \log q)n)\end{aligned}$$

where $H(\gamma)$ is the binary entropy function.[6] By choosing $\gamma < q^{-q/2}$, we can find a β such that $q\gamma < \beta$ (which was required in Sect. 5.7.2) and $2H(\gamma) > \beta \log q$, yielding the lower bound

[6] The binary entropy function is defined as $H(\gamma) \overset{\text{def}}{=} -\gamma \log_2(\gamma) - (1-\gamma)\log_2(1-\gamma)$. It is well known that $\binom{n}{k} \approx 2^{nH(k/n)}$.

$$\begin{aligned} s &= \exp\left(\Omega(q^{-q/2} \cdot q \log q \cdot n)\right) \\ &= 2^{\Omega(n)}. \end{aligned}$$

□ (Theorem 20)

5.8 Lower Bounds for Multilinear Models

Raz [Raz09] showed that multilinear formulas computing the Det_n or Perm_n must be of size $n^{\Omega(\log n)}$. The complexity measure used by Raz also led to exponential lower bounds for constant depth multilinear circuits [RY09] and superlinear lower bounds for syntactic multilinear circuits [RSY08]. We shall first give some intuition behind the complexity measure before actually seeing the lower bounds.

5.8.1 The Partial Derivative Matrix

Intuition A natural first step is to try the simpler task of proving lower bounds for depth-3 multilinear circuits.

$$f \quad = \ell_{11} \ldots \ell_{1d} + \cdots + \ell_{s1} \ldots \ell_{sd}$$

The task is now to construct a measure Γ such that $\Gamma(\ell_1 \ldots \ell_d)$ is small whenever each ℓ_i is a linear polynomial and different ℓ_i's are over disjoint sets of variables. Consider the simplest case of $f = (a_1 + b_1 x)(a_2 + b_2 y)$. An observation is that the coefficients of f are given by the 2×2 matrix obtained as $[a_1 \; b_1]^T [a_2 \; b_2] = \begin{bmatrix} a_1 a_2 & a_1 b_2 \\ a_2 b_1 & b_1 b_2 \end{bmatrix}$. In other words, a polynomial $f = a_0 + a_1 x + a_2 y + a_3 xy$ factorizes into two variable disjoint factors if and only if the matrix $\begin{bmatrix} a_0 & a_1 \\ a_2 & a_3 \end{bmatrix}$ has rank 1. A straightforward generalization of this to multiple variables yields the *partial derivative matrix* (which was first introduced by Nisan [Nis91] in the context of non-commutative ABPs)

Definition 24 For any given partition of variables $X = Y \sqcup Z$, define the partial derivative matrix $M_{Y,Z}(f)$ to be the matrix described as follows—the rows are indexed by monomials in Y, columns indexed by monomials in Z, and the (i, j)-th entry of the matrix is the coefficient of the monomial $m_i(Y) \cdot m_j(Z)$ *in* f. We shall use $\Gamma^{[\text{Raz}]}_{Y,Z}(f)$ to denote $\text{rank}(M_{Y,Z}(f))$. Further, we shall call a polynomial f to be full-rank if $M_{Y,Z}(f)$ is full-rank.

Here are some basic properties of the partial derivative matrix which would be extremely useful in later calculations.

Observation 25 (Sub-additivity) *For any partition $X = Y \sqcup Z$ and any pair of multilinear polynomials f and g in $\mathbb{F}[X]$ we have $\Gamma^{[\mathrm{Raz}]}_{Y,Z}(f+g) \leq \Gamma^{[\mathrm{Raz}]}_{Y,Z}(f) + \Gamma^{[\mathrm{Raz}]}_{Y,Z}(g)$.*

Proof Follows from the linearity of the matrix. □

Observation 26 (Multiplicativity) *If $f_1 \in \mathbb{F}[Y_1, Z_1]$ and $f_2 \in \mathbb{F}[Y_2, Z_2]$ with $Y = Y_1 \sqcup Y_2$ and $Z = Z_1 \sqcup Z_2$, then*

$$\Gamma^{[\mathrm{Raz}]}_{Y,Z}(f_1 \cdot f_2) = \Gamma^{[\mathrm{Raz}]}_{Y_1,Z_1}(f_1) \cdot \Gamma^{[\mathrm{Raz}]}_{Y_2,Z_2}(f_2).$$

Proof It is not hard to see that $M_{Y,Z}(f_1 \cdot f_2)$ is the tensor product $M_{Y_1,Z_1}(f_1) \otimes M_{Y_2,Z_2}(f_2)$, and the rank of a tensor product of two matrices is the product of the ranks. □

Observation 27 $\Gamma^{[\mathrm{Raz}]}_{Y,Z}(f) \leq 2^{\min(|Y|,|Z|)}$.

Proof The number of rows is $2^{|Y|}$ and number of columns is $2^{|Z|}$, and hence the rank is upper bounded by the minimum. □

Let us get back to lower bounds for multilinear models, and attempt to use $\Gamma^{[\mathrm{Raz}]}_{Y,Z}(f)$ defined above. Unfortunately, there are examples of simple polynomials like $f = (y_1 + z_1) \dots (y_n + z_n)$ with $\Gamma^{[\mathrm{Raz}]}_{Y,Z}(f) = 2^n$. Raz's idea here was to look at $\Gamma^{[\mathrm{Raz}]}_{Y,Z}(f)$ for a *random partition*, and show that with high probability the rank of the partial derivative matrix is far from full. As a toy example, we shall see why this has the potential to give lower bounds for depth-3 multilinear circuits.

Lemma 28 *Let $f(X) = \ell_1 \dots \ell_d$ be an n-variate multilinear polynomial. If $X = Y \sqcup Z$ is a random partition with $|Y| = |Z| = |X|/2$, then with high probability we have*

$$\Gamma^{[\mathrm{Raz}]}_{Y,Z}(f) \leq 2^{|X|/2} \cdot 2^{-|X|/16}.$$

It is to be noted that we should expect a random polynomial to be full-rank with respect to any partition, so the measure $\Gamma^{[\mathrm{Raz}]}_{Y,Z}(f)$ is expected to be $2^{|X|/2}$ which should yield a lower bound of $2^{\Omega(|X|)}$.

Sketch of Proof Without loss of generality we can assume that each ℓ_i depends on at least two variables as removing the ℓ_i's that depend on just one variable does not alter $\Gamma^{[\mathrm{Raz}]}_{Y,Z}(f)$ with respect to any partition. Let $|X| = n$.

Using Observation 26, $\Gamma^{[\mathrm{Raz}]}_{Y,Z}(f) \leq 2^d$ and hence if $d < n/3$ then we are done. Hence assume that $d \geq n/3$. By a simple averaging argument, there must hence be at least $d/4$ of the ℓ_i's that depend on at most 3 variables; we shall refer to these as the *small* ℓ_i's.

Since the partition is chosen at random, on expectation a quarter of the small ℓ_i's would have all its variables mapped to either Y or Z, hence not contributing to $\Gamma^{[\mathrm{Raz}]}_{Y,Z}(f)$. Therefore, with high probability,

$$\Gamma^{[\mathrm{Raz}]}_{Y,Z}(f) \leq 2^d \cdot 2^{-d/16} \leq 2^{n/2} \cdot 2^{-n/16}. \qquad \square$$

More generally, if $f = g_1(X_1)\dots g_t(X_t)$ where the X_i's are mutually disjoint, then a random partition is very unlikely to partition all the X_i's into almost equal parts. This is formalized in the next section to prove the lower bound for multilinear formulas.

5.8.2 *Lower Bound for Multilinear Formulas*

We now present the lower bound for multilinear formulas due to [Raz09]. The first step of our roadmap is to find a suitable normal form for multilinear formulas. The normal form that we use is from the survey by Shpilka and Yehudayoff [SY10].

5.8.2.1 Formulas to Log-Product Sums

The following structural lemma shows that any multilinear formula can be converted in to a small sum of *log-product* polynomials. The techniques of the following lemma can also be used in other settings with minor modifications, and we encounter a different version of this lemma later as well.

Definition 29 A multilinear polynomial $f \in \mathbb{F}[X]$ is called a multilinear log-product polynomial if $f = g_1 \dots g_t$ and there exists a partition of variables $X = X_1 \sqcup \cdots \sqcup X_t$ such that

- $g_i \in \mathbb{F}[X_i]$ for all $i \in [t]$.
- $\frac{|X|}{3^i} \leq |X_i| \leq \frac{2|X|}{3^i}$ for all i, and $|X_t| = 1$.

Lemma 30 *Let Φ be a multilinear formula of size s computing a polynomial p. Then f can be written as a sum of $(s+1)$ log-product multivariate polynomials.*

Proof Similar to Lemma 19, let v be a node in Φ such that set of variables X_v that it depends on satisfies $\frac{|X|}{3} \leq |X_v| \leq \frac{2|X|}{3}$. If Φ_v is the polynomial computed at this node, then f can be written as

$$f = \Phi_v \cdot g_1 + \Phi_{v=0} \quad \text{for some } g_1 \in \mathbb{F}[X \setminus X_v].$$

where $\Phi_{v=0}$ is the formula obtained by replacing the node v by zero. Note that the subtree at the node v is completely disjoint from $\Phi_{v=0}$. Hence the sum of the sizes of Φ_v and $\Phi_{v=0}$ is at most s. Hence, $g_1 \in \mathbb{F}[X \setminus X_v]$ and $\frac{|X|}{3} \leq |X \setminus X_v| \leq \frac{2|X|}{3}$. Inducting on the formulas Φ_v and $\Phi_{v=0}$ gives the lemma. $\square$

5.8.2.2 Log-Products Are Far from Full-Rank on a Random Partition

The main technical part of the proof is to show that log-product multivariate polynomials are far from full-rank under a random partition of variables. This would let us show that a sum of log-product multivariate polynomials cannot be full rank unless it is a very large sum.

{**Main idea** Suppose $f = g_1 \dots g_t$ where each $g_i \in \mathbb{F}[X_i]$. Let $X = Y \sqcup Z$ be a random partition with $|Y| = |Z| = |X|/2$, and $Y_i = Y \cap X_i$ and $Z_i = Z \cap X_i$. Let $d_i = \left|\frac{|Y_i|-|Z_i|}{2}\right|$ measure the imbalance between the sizes of Y_i and Z_i, and we shall say X_i is k-imbalanced if $d_i \geq k$. Let $b_i = \frac{|Y_i|+|Z_i|}{2} = \frac{|X_i|}{2}$.

By Observation 26, we know that

$$\begin{aligned}\Gamma^{[\mathrm{Raz}]}_{Y,Z}(f) &= \Gamma^{[\mathrm{Raz}]}_{Y_i,Z_i}(g_1)\dots\Gamma^{[\mathrm{Raz}]}_{Y_i,Z_i}(g_t)\\ &\leq 2^{\min(|Y_1|,|Z_1|)}\cdot\dots\cdot 2^{\min(|Y_t|,|Z_t|)}\\ &= 2^{b_1-d_1}\dots 2^{b_t-d_t} = \frac{2^{|X|/2}}{2^{d_1+\dots+d_t}}.\end{aligned}$$

Hence, even if one of the X_i's is a little imbalanced, the product is far from full-rank.

Lemma 30 shows that the size of X_i decreases slowly with i, and it is not hard to show that $|X_i| \geq \sqrt{|X|}$ for $i \leq t' \overset{\text{def}}{=} \frac{\log|X|}{100}$. We wish to show that the probability that none of g_i (for $i \leq t'$) is k-unbalanced for $k = |X|^{1/20}$ is very small. Let $\mathcal{E}_i$ be the event that X_i is not k-unbalanced. The goal is to upper bound the probability that all the events $\mathcal{E}_i$ hold. These probability calculations would follow from this lemma about the *hypergeometric distribution.*

Hypergeometric Distribution Fix parameters $n, g, r \geq 0$, and let $G \subseteq [n]$ with $|G| = g$. Informally, the hypergeometric distribution is the distribution obtained on the intersection sizes of a random set of size r with a fixed set of size g from a universe of size n. Formally, the random variable $\mathcal{H}(n, g, r)$ is defined as:

$$\Pr[\mathcal{H}(n,g,r) = k] = \Pr_{R\subseteq[n],|R|=r}[|R \cap G| = k] = \frac{\binom{g}{k}\binom{n-g}{r-k}}{\binom{n}{r}}.$$

The following lemma shows that for a fairly large range of parameters, the hypergeometric distribution does not put too much mass on any value.

Lemma 31 *Let n, g, r be parameters such that $\frac{n}{4} \leq r \leq \frac{3n}{4}$ and $0 \leq g \leq \frac{2n}{3}$. Then for any $t \leq g$,*

$$\Pr[\mathcal{H}(n,g,r) = t] \leq O\left(\frac{1}{\sqrt{g}}\right).$$

The proof of this lemma follows from standard binomial coefficient estimates on the probability.

Let us go back to estimating the probability that all the events $\mathcal{E}_i$ hold.

$$\Pr\left[\mathcal{E}_1 \wedge \cdots \wedge \mathcal{E}_{t'}\right] = \Pr[\mathcal{E}_1] \cdot \Pr[\mathcal{E}_2 \mid \mathcal{E}_1] \cdots \Pr[\mathcal{E}_{t'} \mid \mathcal{E}_1 \wedge \cdots \wedge \mathcal{E}_{t'-1}].$$

The event $\mathcal{E}_1$ is just the probability that a random set Y of size $|X|/2$ intersects X_1 in t places where $t \in \left[\frac{|X_i|}{2} - k, \frac{|X_i|}{2} - k\right]$. This is just a particular setting of the hypergeometric distribution and Lemma 31 asserts that

$$\Pr[\mathcal{E}_1] \le O\left(\frac{2k}{\sqrt{|X|}}\right).$$

To apply a similar bound for the other terms, consider the event $\mathcal{E}_i$ given that $\mathcal{E}_1, \ldots, \mathcal{E}_{i-1}$ hold. Let $X' = X \setminus (X_1 \cup \ldots \cup X_{i-1})$ and $Y' = Y \cap X'$. The fact that $\mathcal{E}_1, \ldots, \mathcal{E}_{i-1}$ hold means that the partition has been fairly balanced in the first $(i-1)$ parts and hence $|Y'| \le |X'| + ik$. Hence, we would still be in the range of parameters in Lemma 31 to also get that

$$\begin{aligned}
\forall i \le t' \quad \Pr[\mathcal{E}_i \mid \mathcal{E}_1 \wedge \cdots \wedge \mathcal{E}_{i-1}] &\le O\left(\frac{2k}{\sqrt{|X|}}\right)\\
\implies \Pr\left[\mathcal{E}_1 \wedge \cdots \wedge \mathcal{E}_{t'}\right] &\le |X|^{-\epsilon \log |X|} \quad \text{for some } \epsilon > 0\\
\implies \Pr\left[\Gamma^{[\mathrm{Raz}]}_{Y,Z}(g_1 \ldots g_t) \le 2^{(|X|/2) - |X|^{1/20}}\right] &\le |X|^{-\epsilon \log |X|}.
\end{aligned}$$

Hence, if $g_1 \ldots g_t$ is a log-product multilinear polynomial, then with probability at least $\left(1 - |X|^{-\epsilon \log |X|}\right)$ we have that $\Gamma^{[\mathrm{Raz}]}_{Y,Z}(g_1 \ldots g_t) \le 2^{(|X|/2) - |X|^{1/20}}$. Further, if f is computable by a multilinear formula of size s then, by Lemma 30, f can be written as a sum of $(s+1)$ log-product multilinear polynomials. Hence, with probability at least $\left(1 - (s+1)|X|^{-\epsilon \log |X|}\right)$ we have that

$$\Gamma^{[\mathrm{Raz}]}_{Y,Z}(f) \le (s+1) \cdot 2^{(|X|/2) - |X|^{1/20}}.$$

Hence, if $(s+1) < |X|^{(\epsilon/2) \log |X|}$, then with high probability a random partition would ensure $\Gamma^{[\mathrm{Raz}]}_{Y,Z}(f) \ll 2^{|X|/2}$. Let us record this as a lemma.

Lemma 32 *Let $f \in \mathbb{F}[X]$ be computed by a multilinear formula of size $s < |X|^{(\epsilon/2) \log |X|}$ for a small enough constant $\epsilon > 0$. Then with probability at least $(1 - |X|^{-(\epsilon/2) \log |X|})$ we have*

$$\Gamma^{[\mathrm{Raz}]}_{Y,Z}(f) \le (s+1) \cdot 2^{|X|/2} \cdot 2^{-|X|^{1/20}}$$

for a random partition $X = Y \sqcup Z$ with $|Y| = |Z| = |X|/2$.

5.8.2.3 Det_n and Perm_n have Large Rank

The last step of the proof would be to find an explicit polynomial whose partial derivative matrix under a random partition has large rank. As earlier, our candidate polynomial would be Det_n or Perm_n. Unfortunately, both these polynomials are over n^2 variables and degree n. It is not hard to verify that the rank of the partial derivative matrix of Det_n or Perm_n can never be greater than 2^{2n}. Hence directly using Lemma 32, we would have $2^{O(n)}$ competing with $2^{n^2/2 - n^{O(1)}}$ which is simply futile. A simple fix is to first randomly restrict ourselves to fewer variables and then apply Lemma 32.

Let $m = n^{1/3}$. Let σ be a random restriction that assigns random values to all but $2m$ randomly chosen variables. We shall call this set of $2m$ variables as X, and randomly partition this into two sets Y and Z of size m each. Hence, $\sigma(\mathsf{Det}_n)$ reduces to a multilinear polynomial over $2m$ variables. It is also worth noting that a multilinear formula remains a multilinear formula under this restriction. The following claim is easy to verify.

Claim 33 *With probability at least* $1/2$*, the variables in* X *belong to distinct rows and columns.*

We shall restrict ourselves to only these random restrictions, and without loss of generality let the sets be $Y = \{x_{1,1}, x_{3,3}, \ldots, x_{2m-1,2m-1}\}$ and $Z = \{x_{2,2}, x_{4,4}, \ldots, x_{2m,2m}\}$. For ease of notation, we shall refer to $x_{2i-1,2i-1}$ as y_i and $x_{2i,2i}$ as z_i for $i = 1, \ldots, m$.

Consider the following restriction:

$$f = \mathsf{Det}\begin{bmatrix} y_1 & 1 & & & & & & \\ 1 & z_1 & & & & & & \\ & & \ddots & & & & & \\ & & & y_m & 1 & & & \\ & & & 1 & z_m & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{bmatrix}$$
$$= (y_1 z_1 - 1) \ldots (y_m z_m - 1).$$

It is easy to check that $\Gamma^{[\mathrm{Raz}]}_{Y,Z}(f) = 2^m$. Although this is a single restriction with large rank, the Schwartz-Zippel-DeMillo-Lipton lemma immediately gives that random restriction would also have rank 2^m with high probability.[7] We shall record this as a lemma.

[7] provided the underlying field is large, but this isn't really a concern as we can work with a large enough extension if necessary.

Lemma 34 *With probability at least* $1/100$*, we have that* $\Gamma^{[\mathrm{Raz}]}_{Y,Z}(\sigma(\mathsf{Det}_n)) = 2^m$ *where* σ *is a random restriction to* $2m$ *variables for* $m = n^{1/3}$.

Combining Lemma 34 with Lemma 32, we have the following theorem.

Theorem 35 [Raz09] *Any multilinear formula computing* Det_n *or* Perm_n *must be of size* $n^{\Omega(\log n)}$. □

5.8.3 Stronger Lower Bounds for Constant Depth Multilinear Formulas

Looking back at Lemma 32, we see that whenever $f(X)$ is computable by a size s multilinear formula $\Gamma^{[\mathrm{Raz}]}_{Y,Z}(f)$ is exponentially smaller than $2^{|X|/2}$ with probability $\left(1 - s \cdot |X|^{-\epsilon \log |X|}\right)$. Hence we had to settle for a $n^{\Omega(\log n)}$ lower bound not because of the rank deficit but rather because of the bounds in the probability estimate. Unfortunately, this lower bound technique cannot yield a better lower bound for multilinear formulas as there are explicit examples of polynomials computable by poly-sized multilinear circuits with $\Gamma^{[\mathrm{Raz}]}_{Y,Z}(f) = 2^{|X|/2}$ under *every* partition [Raz06]. However, the probability bound can be improved in the case of constant depth multilinear circuits to give stronger lower bounds.

Note that Lemma 32 was proved by considering *multilinear log-products* (Definition 29) as the building blocks. To show that a multilinear log product $g_1(X_1)\ldots g_\ell(X_\ell)$ has small rank under a random partition, we argued that the probability that all the X_i's are partitioned in a roughly balanced fashion is quite small. This was essentially done by thinking of this as $\ell = O(\log n)$ close-to-independent events, each with probability $1/\mathrm{poly}(n)$.

If ℓ was much larger than $\log n$ (with other parameters being roughly the same), it should be intuitively natural to expect a much lower probability of all the X_i's being partitioned in a roughly balanced manner. This indeed is the case for constant depth multilinear circuits, and we briefly sketch the key points where they differ from the earlier proof. The first is an analog of Definition 29 in this setting.

Definition 36 A multilinear polynomial f is said to be a multilinear t-product if f can be written as $f = g_1 \ldots g_t$ with the following properties:

- The variable sets of the g_i are mutually disjoint
- Each g_i non-trivially depends on at least t variables

Lemma 37 *Let* f *be a multilinear polynomial of degree* d *over* n *variables that is computed by a depth-*Δ *multilinear formula* Φ *of size* s*. Then,* f *can be written as a sum of at most* s *multilinear* t*-products for* $t = (n/100)^{1/2\Delta}$*, and a multilinear polynomial of degree at most* $n/100$.

Proof If $d < n/100$, then the lemma is vacuously true. Since Φ is a formula of depth Δ and computes a polynomial of degree $d > n/100$, there must be at least one product gate v of fan-in at least $\left(\frac{n}{100}\right)^{1/\Delta} = t^2$. Then similar to Lemma 30,

$$f = \Phi_v \cdot f' + \Phi_{v=0}$$

As Φ_v is a product of t^2 polynomials, by grouping the factors together we have that $\Phi_v \cdot f'$ is a multilinear t-product. Further, $\Phi_{v=0}$ is a multilinear polynomial that is computable by a depth-Δ formula of smaller size and we can induct on $\Phi_{v=0}$. $\square$

Lemma 38 *Let $f(X)$ be an n-variate polynomial computed by a depth-Δ multilinear formula of size s. If $X = Y \sqcup Z$ is a randomly chosen partition with $|Y| = |Z| = n/2$, then with probability at least $(1 - s \cdot \exp(-n^{\Omega(1/\Delta)}))$ we have*

$$\Gamma^{[\mathrm{Raz}]}_{Y,Z}(f) \leq (s+1) \cdot 2^{n/2} \cdot \exp(-n^{\Omega(1/\Delta)}).$$

Sketch of Proof By Lemma 37, we have that f can be written as $g_0 + g_1 + \cdots + g_s$ where $\deg(g_0) \leq n/100$ and $g_1, \ldots, g_s$ are multilinear t-products. Note that since g_0 is a multilinear polynomial of degree at most $(n/100)$, the number of monomials in g_0 is at most $\binom{n}{n/100} \leq 2^{n/10}$. Hence, $\Gamma^{[\mathrm{Raz}]}_{Y,Z}(g_0) \leq 2^{n/10}$.

For the other g_i's, we can bound the probability that $\Gamma^{[\mathrm{Raz}]}_{Y,Z}(g_i)$ is large in a very similar fashion as in Lemma 32, as the probability that all the factors of g_i are partitioned in a balanced manner is roughly the intersection of t independent events. By very similar estimates, this probability can be bounded by $(1/\mathrm{poly}(n))^t$. Hence, with high probability

$$\Gamma^{[\mathrm{Raz}]}_{Y,Z}(f) \leq \Gamma^{[\mathrm{Raz}]}_{Y,Z}(g_0) + \cdots + \Gamma^{[\mathrm{Raz}]}_{Y,Z}(g_s) \leq (s+1) \cdot 2^{n/2} \cdot \exp(-n^{\Omega(1/\Delta)}).$$

$\square$

Combining Lemma 38 with Lemma 34, we have the following theorem of Raz and Yehudayoff.

Theorem 39 [RY09] *Any multilinear formula of depth Δ computing Det_n or Perm_n must be of size* $\exp(n^{\Omega(1/\Delta)})$. $\square$

5.9 Lower Bounds for Depth-4 Circuits

This section addresses a recent technique for proving lower bounds for some depth-4 circuits.

Definition 40 A depth-4 circuit, also referred to as a $\Sigma\Pi\Sigma\Pi$ circuit, computes a polynomial of the form

$$f = Q_{11} \ldots Q_{1d} + \cdots + Q_{s1} \ldots Q_{sd}.$$

The number of summands s is called the top fan-in of the circuit.

Further, a $\Sigma\Pi^{[a]}\Sigma\Pi^{[b]}$ circuit is a depth-4 circuit computing a polynomial of the form

$$f = Q_{11}\dots Q_{1a} + \cdots + Q_{s1}\dots Q_{sa} \quad \text{where } \deg Q_{ij} \le b \text{ for all } i, j.$$

5.9.1 Significance of the Model

In a surprising series of results on depth reduction, Agrawal and Vinay [AV08] and subsequent strengthenings of Koiran [Koi12] and Tavenas [Tav13] showed that depth-4 circuits more or less capture the complexity of general circuits.

Theorem 41 [AV08, Koi12, Tav13] *If f is an n variate degree-d polynomial computed by a size s arithmetic circuit, then f can also be computed by a $\Sigma\Pi^{[O(\sqrt{d})]}\Sigma\Pi^{[\sqrt{d}]}$ circuit of size* $\exp\left(O(\sqrt{d}\log s)\right)$.

Conversely, if an n-variate degree-d polynomial requires $\Sigma\Pi^{[O(\sqrt{d})]}\Sigma\Pi^{[\sqrt{d}]}$ circuits of size $\exp\left(\Omega(\sqrt{d}\log s)\right)$*, then it requires arbitrary depth arithmetic circuits of size $n^{\Omega(\log s/\log n)}$ to compute it.*

Thus proving strong enough lower bounds for this special case of depth-4 circuits imply lower bounds for general circuits. The main results of the section is some recent lower bound [GKKS13, KSS13, FLMS13] that comes very close to the required threshold.

5.9.2 Building the Complexity Measure

As a simpler task, let us first attempt to prove lower bounds for expressions of the form

$$f = Q_1^d + \cdots + Q_s^d$$

where each of the Q_i's are quadratics. This is exactly the problem studied by Kayal [Kay12], which led to the complexity measure for proving depth-4 lower bounds.

The goal is to construct a measure Γ such that $\Gamma(f)$ is small whenever f is a power of a quadratic. As a first attempt, let us look at the space of k-th order partial derivatives of Q^d (for a suitable choice of k). Unlike the case of $\Sigma\wedge\Sigma$-circuits where the space of k-th order partial derivatives of ℓ^d had dimension 1, the space of partial derivatives of Q^d could be as large as it can be expected. Nevertheless, the following simple observation would provide the key intuition.

Observation 42 *Any k-th order partial derivative of Q^d is of the form $Q^{d-k}p$ where p is a polynomial of degree at most k. Hence, if $k \ll d$, then all k-th order partial derivatives of Q^d share large common factors.*

This suggests that instead of looking at linear combinations of the partial derivatives of Q^d, we should instead be analyzing *low-degree polynomial combinations* of them.

Definition 43 Let $\partial^{=k}(f)$ refer to the set of all k-*th* order partial derivatives of f, and $\mathbf{x}^{\leq \ell}$ refer to the set of all monomials of degree at most ℓ. The shifted partials of f, denoted by $\left\langle \partial^{=k}(f)\right\rangle_{\leq \ell}$, is the vector space spanned by $\left\{\mathbf{x}^{\leq \ell} \cdot \partial^{=k}(f)\right\}$. The dimension of this space shall be denoted by $\Gamma^{[\mathrm{Kay}]}_{k,\ell}(f)$.

The above observation shows that any element of $\left\langle \partial^{=k}\left(Q^d\right)\right\rangle_{\leq \ell}$ is divisible by Q^{d-k} and we thereby have the following lemma.

Lemma 44 *If $f = Q^d$ where Q is a quadratic, then $\Gamma^{[\mathrm{Kay}]}_{k,\ell}(f) \leq \binom{n+k+\ell}{n}$, the number of monomials of degree $(k+\ell)$.*

Note that if f was instead a random polynomial, we would expect the measure $\dim\left(\left\langle \partial^{=k}(f)\right\rangle_{\leq \ell}\right)$ to be about $\binom{n+k}{n} \cdot \binom{n+\ell}{n}$, which is *much* larger than $\binom{n+k+\ell}{n}$ for suitable choice of k, ℓ. Hence this measure $\Gamma^{[\mathrm{Kay}]}_{k,\ell}$ is certainly potentially useful for this model. Very similar to the above lemma, one can also show the following upper bound for the *building blocks* of $\Sigma\Pi^{[a]}\Sigma\Pi^{[b]}$ circuits.

Lemma 45 *Let $f = Q_1 \ldots Q_a$ with $\deg Q_i \leq b$ for all i. Then,*

$$\Gamma^{[\mathrm{Kay}]}_{k,\ell}(f) = \dim\left(\left\langle \partial^{=k}(f)\right\rangle_{\leq \ell}\right) \leq \binom{a}{k}\binom{n+(b-1)k+\ell}{n}.$$

It is easy to check that $\Gamma^{[\mathrm{Kay}]}_{k,\ell}$ is a sub-additive measure, and we immediately have this corollary.

Corollary 46 *Let f be an n-variate polynomial computed by a $\Sigma\Pi^{[a]}\Sigma\Pi^{[b]}$ circuit of top fan-in s. Then,*

$$\Gamma^{[\mathrm{Kay}]}_{k,\ell}(f) \leq s \cdot \binom{a}{k}\binom{n+(b-1)k+\ell}{n}.$$

Or in other words for any choice of k, ℓ, we have that any $\Sigma\Pi^{[a]}\Sigma\Pi^{[b]}$ circuit computing a polynomial f must have top fan-in s at least

$$\frac{\Gamma^{[\mathrm{Kay}]}_{k,\ell}(f)}{\binom{a}{k}\binom{n+(b-1)k+\ell}{n}}.$$

Intuition from Algebraic Geometry Another perspective for the shifted partial derivatives comes from algebraic geometry. Any zero $a \in \mathbb{F}^n$ of Q is a zero of *multiplicity* d of Q^d. This implies that the set of common zeroes of all k-th order partial derivatives of Q^d (for $k \approx \sqrt{d}$) is *large*. On the other hand if f is a random polynomial, then with high probability there are no roots of large multiplicity.

In algebraic geometry terminology, the common zeroes of a set of polynomials is called the *variety* of the ideal generated by them. Further, there is also a well-defined notion of a *dimension of a variety* which measures how large a variety is. Let $\mathbb{F}[\mathbf{x}]_{\leq r}$ refer to the set of polynomials of degree at most r, and let $\gamma_I(r) = \dim\left(I \cap \mathbb{F}[\mathbf{x}]_{\leq r}\right)$. Intuitively, if $\gamma_I(r)$ is large, then there are *many constraints* and hence the variety is *small*. In other words the growth of $\gamma_I(r)$ is inversely related to the dimension of the variety of I, and this is precisely captured by what is known as the *Affine Hilbert function of I*. More about the precise definitions of the Affine Hilbert function, etc., can be found in any standard text in algebraic geometry such as [CLO07].

In our setting, the ideal we are interested in is $I = \left\langle \partial^{=k} f \right\rangle$. If f is a homogeneous polynomial, then $I \cap \mathbb{F}[\mathbf{x}]_{\leq r} = \left\langle \partial^{=k}(f) \right\rangle_{\leq \ell}$ where $\ell = r - (\deg(f) - k)$. Hence studying the dimension of shifted partial derivatives is exactly studying $\gamma_I(r)$ which holds all information about the dimension of the variety.

5.9.3 *Lower Bounding Shifted Partials of Explicit Polynomials*

For a random polynomial $R(\mathbf{x})$, we would expect that

$$\Gamma_{k,\ell}^{[\mathrm{Kay}]}(R) \approx \min\left\{\binom{n+\ell+d-k}{n}, \binom{n+k}{n}\binom{n+\ell}{n}\right\}.$$

The terms on the RHS correspond to trivial upper bounds, where the first term is the total number of monomials of degree $(\ell + d - k)$ and the second term is the total number shifted partials.

Claim 47 *For $k = \epsilon\sqrt{d}$ for a small enough $\epsilon > 0$, and $\ell = \frac{cn\sqrt{d}}{\log n}$ for a large enough constant c, we have*

$$\frac{\min\left\{\binom{n+\ell+d-k}{n}, \binom{n+k}{n}\binom{n+\ell}{n}\right\}}{\binom{O(\sqrt{d})}{k}\binom{n+(\sqrt{d}-1)k+\ell}{n}} = 2^{\Omega(\sqrt{d}\log n)}.$$

The proof of this claim is easily obtained by using standard asymptotic estimates of binomial coefficients. Note that using Corollary 46, the above claim shows that if we can find an explicit polynomial whose dimension of shifted partials are as large as above, then we would have an $\exp(\Omega(\sqrt{d}\log n))$ lower bound for the top fan-in of $\Sigma\Pi^{[\sqrt{d}]}\Sigma\Pi^{[\sqrt{d}]}$ circuits computing this polynomial.

If we have a set of polynomials with distinct leading monomials, then they are clearly linearly independent. Hence, one way of lower bounding the dimension of a space of polynomials is to find a sufficiently large set of polynomials with distinct monomials in the space. The vector space of polynomials we are interested is $\left\langle \partial^{=k} (f) \right\rangle_{\leq \ell}$, and if we choose a structured polynomial f we can hope to be able to estimate the number of distinct leading monomials in this vector space.

5.9.3.1 Shifted Partials of the Determinant and Permanent

The first lower bound for $\Sigma\Pi^{[\sqrt{d}]}\Sigma\Pi^{[\sqrt{d}]}$ circuits was by Gupta, Kamath, Kayal, and Saptharishi [GKKS13] for the determinant and the permanent polynomials. We shall describe the lower bound for Det_n, although it would carry over immediately to Perm_n as well. As mentioned earlier, we wish to estimate the number of distinct leading monomials in $\left\langle \partial^{=k} (\mathsf{Det}_n) \right\rangle_{\leq \ell} = \text{span}\left\{\mathbf{x}^{\leq \ell}\partial^{=k}\mathsf{Det}_n\right\}$. [GKKS13] made a relaxation to merely count the number of distinct leading monomials among the generators $\left\{\mathbf{x}^{\leq \ell}\partial^{=k}\mathsf{Det}_n\right\}$ instead of their span.

The first observation is that any k-th order partial derivative of Det_n is just an $(n-k)\times(n-k)$ minor. Let us fix a monomial ordering induced by the lexicographic ordering on the variables:

$$x_{11} \succ x_{12} \cdots \succ x_{1n} \succ x_{21} \succ \cdots \succ x_{nn}.$$

Under this ordering, the leading monomial of any minor is just the product of variables on the main diagonal of the submatrix corresponding to the minor, and hence is a term of the form $x_{i_1 j_1} \ldots x_{i_{(n-k)}, j_{(n-k)}}$ where $i_1 < \cdots < i_{n-k}$ and $j_1 < \cdots < j_{n-k}$; let us call such a sequence of indices as an $(n-k)$-increasing sequence in $[n] \times [n]$. Further, for any $(n-k)$-increasing sequence, there is a unique minor M whose leading monomial is precisely the product of the variables indexed by the increasing sequence. Therefore, the task of lower bounding distinct leading monomials in $\left\{\mathbf{x}^{\leq \ell}\partial^{=k}\mathsf{Det}_n\right\}$ reduces to the following combinatorial problem:

Claim 48 *For any $k, \ell > 0$, we have*

$$\Gamma_{k,\ell}^{[\text{Kay}]}(\mathsf{Det}_n) \geq \# \left\{ \begin{array}{c} \textit{monomials of degree } (\ell + n - k) \textit{ that} \\ \textit{contain an } (n-k)\textit{-increasing sequence} \end{array} \right\}.$$

We could start with an $(n-k)$-increasing sequence, and multiply by a monomial of degree ℓ to obtain a monomial containing an increasing sequence. Of course, the issue is that this process is not invertible and hence we might overcount. To fix this issue, [GKKS13] assign a *canonical increasing sequence* to every monomial that contains an increasing sequence and multiply by monomials of degree ℓ that do not change the canonical increasing sequence.

Definition 49 Let $D_2 = \left\{x_{1,1}, \ldots, x_{n,n}, x_{1,2}, x_{2,3}, \ldots, x_{n-1,n}\right\}$, the main diagonal and the diagonal just above it. For any monomial m define the canonical increasing sequence of m, denoted by $\chi(m)$, as $(n-k)$-increasing sequence of m that is entirely contained in D_2 and is ordered highest according to the ordering '$\succ$'. If m contains no $(n-k)$-increasing sequence entirely in D_2, then we shall say the canonical increasing sequence is empty.

The reason we restrict ourselves to D_2 is because it is easier to understand which monomials change the canonical increasing sequence and which monomials do not.

Lemma 50 *Let S be an $(n-k)$-increasing sequence completely contained in D_2, and let m_S be the monomial obtained by multiplying the variables indexed by S. There are at least $(2(n-k)-1)$ variables in D_2 such that if m is any monomial over these variables, then $\chi(m_S) = \chi(m \cdot m_S)$.*

Proof Note that for any $x_{i,j} \in D_2$ other than $x_{n,n}$, exactly one of $x_{i+1,j}$ or $x_{i,j+1}$ is in D_2 as well; let us refer to this element in D_2 as the *companion* of $x_{i,j}$. It is straightforward to check that for any $(n-k)$-increasing sequence S, the elements of S and their companions do not alter the canonical increasing sequence. $\square$

It is a simple exercise to check that the number of $(n-k)$-increasing sequences contained in D_2 is $\binom{n+k}{2k}$. Further, as we are free to use the n^2-2n+1 variables outside D_2, and the $2(n-k)-1$ variables that don't alter the canonical increasing sequence, we have the following lemma.

Lemma 51 *For any $k, \ell \geq 0$,*

$$\dim\left(\left\langle \partial^{=k}\,(\mathsf{Det}_n)\right\rangle_{\leq \ell}\right) \geq \binom{n+k}{2k}\binom{(n^2-2n+1)+2(n-k)-1+\ell}{\ell}.$$

Although this lower bound is not as large as expected for a random polynomial, this is still sufficient to give strong lower bounds for depth-4 circuits. By choosing $k = \epsilon\sqrt{n}$ for a small enough $\epsilon > 0$, and $\ell = n^2\sqrt{n}$, Lemma 51 with Corollary 46 yields the lower bound of Gupta, Kamath, Kayal and Saptharishi [GKKS13]

Theorem 52 *Any $\Sigma\Pi^{[O(\sqrt{n})]}\Sigma\Pi^{[\sqrt{n}]}$ circuit computing Det_n or Perm_n has top fanin $2^{\Omega(\sqrt{n})}$.* $\square$

It is worth noting that although Claim 47 suggests that we should be able to obtain a lower bound of $\exp(\Omega(\sqrt{n}\log n))$ for Det_n, [GKKS13] also showed that the above estimate for the dimension of shifted partial derivatives for the determinant is fairly tight. Hence the dimension of shifted partials cannot give a stronger lower bound for the determinant polynomial. However, it is possible that the estimate is *not* tight for the permanent and the dimension of shifted partial derivatives of the permanent is provably strictly larger than that of the determinant! It is conceivable that one should be able to prove an $\exp(\Omega(\sqrt{n}\log n))$ lower bound for the permanent using this measure.

Indeed, subsequently an $\exp(\Omega(\sqrt{d}\log n))$ was proved [KSS13, FLMS13] for other explicit polynomials which we now outline.

5.9.3.2 Shifted Partials of the Nisan-Wigderson Polynomial

Very shortly after [GKKS13]'s $2^{\Omega(\sqrt{n})}$ lower bound, Kayal, Saha, and Saptharishi [KSS13] gave a stronger lower bound for a different polynomial. Their approach was to engineer an explicit polynomial F for which the dimension of shifted partial derivatives is easier to estimate. The main idea was that, if any k-th order partial derivative of the engineered polynomial is a monomial, then once again estimating $\dim\left(\left\langle\partial^{=k}(F)\right\rangle_{\leq \ell}\right)$ reduces to a monomial counting problem. If we could ensure that no two monomials of F have a gcd of degree k or more, then we would immediately get that all k-th order partial derivatives of F are just monomials (albeit possibly zero). If we were to interpret the set of nonzero monomials of F as just subsets over the variables, then the above constraint can be rephrased as a set system with *small pairwise intersection*. Such systems are well studied and are known as Nisan-Wigderson designs [NW94]. With this in mind, [KSS13] studied the following polynomial family inspired by an explicit construction of a Nisan-Wigderson design.

Definition 53 (*Nisan-Wigderson Polynomial*). Let n be a power of 2 and let $\mathbb{F}_n$ be the finite field with n elements that are identified with the set $\{1, \ldots, n\}$. For any $0 \leq k \leq n$, the polynomial NW_k is a n^2-variate polynomial of degree n defined as follows:

$$\mathrm{NW}_k(x_{1,1}, \ldots, x_{n,n}) = \sum_{\substack{p(t) \in \mathbb{F}_n[t] \\ \deg(p) < k}} x_{1,p(1)} \cdots x_{n,p(n)}.$$

It is easy to show that the above family of polynomials is in VNP. Further, since any two distinct univariate polynomials of degree less than k intersects in less than k places, we have the following observation.

Observation 54 *Any two monomials of* NW_k *intersect in less than k variables. Hence, any k-th order partial derivative of* $\mathrm{NW}_k(\mathbf{x})$ *is a monomial (which could possibly be zero).* □

Hence, the problem of lower bounding the shifted partials of NW_k reduces to the problem of counting distinct monomials of degree $\ell + d - k$ that are divisible by one of these k-th order derivatives. [KSS13] additionally used the observation that two random k-th order partial derivatives of NW_k are monomials that are *far* from each other. Using this, they estimate the number of distinct shifts of these monomials and showed that the dimension of shifted partial derivatives of NW_k is very close to the trivial upper bound as in Claim 47. We sketch the argument by Chillara and Mukhopadhyay [CM14]. Formally, for any two multilinear monomials m_1 and m_2, let the $\Delta(m_1, m_2)$ denote $\min\{|m_1| - |m_1 \cap m_2|, m_2 - |m_1 \cap m_2|\}$ (abusing notation by identifying the multilinear monomials with the set of variables that divide it).

Lemma 55 [CM14] *Let $m_1, \ldots, m_s$ be monomials over N variables such that $\Delta(m_i, m_j) \geq d$ for all $i \neq j$. Then the number of distinct monomials that may* *be obtained by multiplying some m_i by arbitrary monomials of degree ℓ is at least* $s\binom{N+\ell}{N} - \binom{s}{2}\binom{N+\ell-d}{N}$.

Proof For $i = 1, \ldots, s$, let A_i be the set of monomials that can be obtained by multiplying m_i with a degree ℓ monomial. By inclusion-exclusion,

$$\left|\bigcup_{i=1}^{s} A_i\right| \geq \sum_{i=1}^{s} |A_i| - \sum_{i<j} \left|A_i \cap A_j\right|.$$

Note that each A_i is of size exactly $\binom{N+\ell}{N}$. Further, since $\Delta(m_i, m_j) \geq d$, any monomial that is divisible by m_i and m_j must necessarily be divisible by m_i and the variables in m_j not in m_i. Hence, $\left|A_i \cap A_j\right| \leq \binom{N+\ell-d}{N}$. The lemma follows by substituting these above. □

Note that any two distinct monomials of NW_k intersect in at most k places. For each monomial m_i of NW_k, let m_i' be any nonzero k-th order partial derivative of m_i. Therefore, $\Delta(m_i', m_j') \geq n - 2k \geq \frac{n}{2}$ for $k = \epsilon\sqrt{n}$. Since we have n^k monomials of pairwise distance at least $n/2$, the above lemma immediately yields a lower bound for the shifted partials of NW_k.

Theorem 56 [KSS13] *Let* $k = \epsilon\sqrt{d}$ *for some constant* $\epsilon > 0$. *Then for any* $\ell = \Theta\left(\frac{n^2\sqrt{n}}{\log n}\right)$,

$$\dim\left(\left\langle \partial^{=k}\left(\mathrm{NW}_k\right)\right\rangle_{\leq \ell}\right) \geq \frac{n^k}{2} \cdot \binom{n^2+\ell}{n^2}$$

Sketch of Proof As mentioned earlier, we have n^k monomials $\{m_i'\}$ with pairwise distance at least $\frac{n}{2}$. Using Lemma 55, it suffices to show that

$$n^k \cdot \binom{n^2+\ell}{n^2} \geq 2 \cdot \binom{n^k}{2} \cdot \binom{n^2+\ell-\frac{n}{2}}{n^2}$$

and this follows easily from standard binomial coefficient estimates. □

Combining with Corollary 46, we have the lower bound of [KSS13] using standard estimates.

Theorem 57 [KSS13] *Any* $\Sigma\Pi^{[O(\sqrt{n})]}\Sigma\Pi^{[\sqrt{n}]}$ *computing the* NW_k *polynomial, where* $k = \epsilon\sqrt{n}$ *for a sufficiently small* $\epsilon > 0$, *must have top fan-in* $\exp(\Omega(\sqrt{n}\log n))$. □

[KSS13] used the above lower bound to give an $n^{\Omega(\log n)}$ lower bound for a subclass of formulas called *regular formulas*. The interested reader can refer to [KSS13] for more details.

5.9.3.3 Shifted Partials of the Iterated-Matrix-Multiplication Polynomial

Fourier, Limaye, Malod and Srinivasan [FLMS13] showed the same lower bound as [KSS13] but for the *iterated matrix multiplication* polynomial which is known to have polynomial sized circuits computing it.

Definition 58 (*Iterated matrix multiplication polynomial*) *Let* $M_1, \ldots, M_d$ *be* $n \times n$ *matrices with distinct variables as entries, i.e.,* $M_k = \left(\!\left(x_{ij}^{(k)}\right)\!\right)_{i,j \le n}$ *for* $k = 1, \ldots, d$. *The polynomial* $\mathrm{IMM}_{n,d}$ *is a* $(n^2 d)$*-variate degree-*d *polynomial defined as the* $(1, 1)$*-th entry of the matrix product* $M_1 \ldots M_d$*:*

$$\mathrm{IMM}_{n,d}(\mathbf{x}) = (M_1 \ldots M_d)_{1,1} .$$

A more useful perspective is to interpret this as a *canonical algebraic branching program*.

Definition 59 (*Algebraic branching program*) An algebraic branching program (ABP) comprises a layered directed graph G with $(d+1)$ layers of vertices, where the first and last layer consists of a single node (called source and sink respectively), all other layers consist of n vertices, and edges are only between successive layers and have linear polynomials as edge-weights. The ABP is set to compute the polynomial f defined as

$$f(\mathbf{x}) = \sum_{\textit{source-sink path } \rho} \mathrm{weight}(\rho)$$

where the weight of any path is just the product of the edge weights on the path.

The canonical ABP comprises a graph where the i-th vertex of layer $(\ell - 1)$ is connected to the j-th vertex of layer ℓ with edge-weight $x_{ij}^{(\ell)}$ for every choice of i, j and ℓ. It is easy to see that the polynomial computed by the canonical ABP is in fact $\mathrm{IMM}_{n,d}$.

To lower bound the dimension of shifted partial derivatives of $\mathrm{IMM}_{n,d}$, first note that a derivative with respect to any variable (or edge) simply results in the sum of all source-sink paths that *pass* through this edge. [FLMS13] uses the following simple but crucial observation to assist in bounding the dimension of shifted partials.

Observation 60 *Assume that* d *is even. Let* $e_1, e_3, \ldots, e_{d-1}$ *be an arbitrary set of edges such that* e_i *is between layer* i *and* $i + 1$. *Then, there is a unique path from source to sink that passes through all these edges.*

Proof Since these are edges in alternate layers, their starting and ending points uniquely determine the edges that are picked up from the even-numbered layers to complete the source-sink path. □

Since we are interested in k-th order derivatives for $k \approx \epsilon\sqrt{d}$, [FLMS13] consider the following restriction by removing some edges from the underlying graph:

- Select $(2k-1)$ layers $\ell_1, \ldots, \ell_{2k-1}$ that are roughly equally spaced between the first and the last layer. These layers, and the first and the last layers, shall be untouched and shall be called *pristine layers*.
- In all the other layers, retain only those edges connecting vertex i of this layer to vertex i of the next.

This restriction effectively makes the graph similar to an ABP with $2k+1$ layers. Let the polynomial computed by the restricted ABP be $\mathrm{IMM}'_{n,d}(\mathbf{x})$. Since $\mathrm{IMM}'_{n,d}$ was obtained by just setting some variables of $\mathrm{IMM}_{n,d}$ to zero, the dimension of shifted partial derivatives of $\mathrm{IMM}'_{n,d}$ can only be smaller than that of $\mathrm{IMM}_{n,d}$. Similar to Observation 60, we have the following observation.

Observation 61 *For every choice of k edges from odd-numbered pristine layers, there is a unique source-sink path that passes through them.*

In other words, for any choice of k variables chosen by picking one from each odd-numbered pristine layer, then the k-th order partial derivative of $\mathrm{IMM}'_{n,d}$ *with respect to these k variables is a nonzero monomial.*

Once again, we can lower bound the dimension of shifted partial derivatives of $\mathrm{IMM}'_{n,d}$ by a monomial counting problem. Similar to the earlier case, [FLMS13] show that the monomials thus obtained are *far* from one another. We state their main lemma below without proof.

Lemma 62 [FLMS13] *There are at least $n^{k/2}$ monomials of* $\mathrm{IMM}'_{n,d}$ *of pairwise distance at least $\frac{n}{4}$.*

Again, using Lemma 55 and standard binomial coefficient estimates, this implies that the shifted partial derivatives of $\mathrm{IMM}'_{n,d}$ is almost as large as the trivial upper bound.

Theorem 63 [FLMS13] *Let $k = \epsilon\sqrt{d}$ for a sufficiently small $\epsilon > 0$ and ℓ be an integer such that $n^{1/16} \leq \frac{N+\ell}{\ell} \leq n^{1/4}$ where N is the number of variables* $\mathrm{IMM}'_{n,d}$ *depends on. Then,*

$$\dim\left(\left\langle \partial^{=k}\left(\mathrm{IMM}_{n,d}\right)\right\rangle_{\leq \ell}\right) \geq \dim\left(\left\langle \partial^{=k}\left(\mathrm{IMM}'_{n,d}\right)\right\rangle_{\leq \ell}\right) = \Omega\left(n^{k/2}\cdot \binom{N+\ell}{\ell}\right).$$

□

Combining with Corollary 46, we get the lower bound of [FLMS13].

Theorem 64 [FLMS13] *Any $\Sigma\Pi^{[O(\sqrt{d})]}\Sigma\Pi^{[\sqrt{d}]}$ circuit computing* $\mathrm{IMM}_{n,d}$, *with $d \leq n^{\delta}$ for a sufficiently small $\delta > 0$, has top fan-in* $\exp(\Omega(\sqrt{d}\log n))$. □

Similar to [KSS13], the above result also implies $n^{\Omega(\log n)}$ lower bounds for regular formulas computing $\mathrm{IMM}_{n,d}$.

5.10 Conclusion

The field of arithmetic complexity, like Boolean complexity, abounds with open problems and proving lower bounds for almost any natural subclass of arithmetic circuits is interesting especially if the currently known techniques/ complexity measures do not apply to that subclass.[8] The surveys [Wig02, SY10, CKW11] mark out the frontiers of this area in the form of many open problems and we invite the reader to try some of them.

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Chapter 6
Explicit Tensors

Markus Bläser

Abstract This is an expository article the aim of which is to introduce interested students and researchers to the topic of tensor rank, in particular to the construction of explicit tensors of high rank. We try to keep the mathematical concepts and language used as simple as possible to address a broad audience. This article is thought to be an appetizer and does not provide by any means a complete coverage of this topic.

Keywords Algebraic complexity theory · Tensor rank · Lower bounds

Mathematics Subject Classification (2010) 68Q17 · 15A69

6.1 Tensors and Rank

Let U and V be vector spaces over some field k. It is a well-known fact that every linear map $\alpha : U \to V$ is represented by a matrix $A = (a_{i,j}) \in k^{\ell \times m}$, where $\ell = \dim U$ and $m = \dim V$. The rank of the matrix A is the maximum number of rows that are linearly independent. There are a lot of equivalent definitions of the rank of a matrix, for instance,

- the maximum number of columns that are linearly independent,
- $\ell - \dim \ker \alpha$,
- $\dim \operatorname{im} \alpha$,

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- the minimum number of matrices of the form $x^T \cdot y$ with $x \in k^\ell$ and $y \in k^m$, so-called rank-one-matrices, such that A can be written as the sum of these matrices,

just to mention a few. The rank of linear maps and matrices is well understood. There are efficient algorithms to compute the rank, the most famous method is Gaussian elimination.

Let W be another vector space over k, $\dim W = n$. Let $\beta : U \times V \to W$ be a bilinear map. By choosing bases $u_1, \dots, u_\ell$ of U, $v_1, \dots, v_m$ of V and $w_1, \dots, w_n$ of W, we can associate structural constants $b_{h,i,j}$ with β:

$$\beta(u_h, v_i) = \sum_{j=1}^{n} b_{h,i,j} w_j, \qquad 1 \le h \le \ell, \quad 1 \le i \le m. \tag{6.1}$$

We can view $B = (b_{h,i,j}) \in k^{\ell \times m \times n}$ as a three-dimensional matrix, a so-called *tensor*. As we have seen, there are many ways to define the rank of a matrix, which is nothing but a tensor in $k^{\ell \times m \times 1}$. From the many equivalent definitions of rank of a matrix given above, it turns out that the appropriate one for tensors is the last one. We call a tensor $S = (s_{h,i,j})$ a rank-one tensor or triad, if there are vectors $a = (a_1, \dots, a_\ell)^T \in k^\ell$, $b = (b_1, \dots, b_m)^T \in k^m$, and $c = (c_1, \dots, c_n)^T \in k^n$ such that

$$s_{h,i,j} = a_h b_i c_j, \qquad 1 \le h \le \ell, \quad 1 \le i \le m, \quad 1 \le j \le n.$$

We will write $S = a \otimes b \otimes c$. Then the *rank of a tensor* T is the minimum number r such that there are rank-one tensors $S_1, \dots, S_r$ with

$$T = S_1 + \dots + S_r.$$

We denote the rank of T by $R(T)$. The question of the rank of tensors of order three or equivalently, of the rank of the corresponding bilinear mapping is a central question in algebraic complexity theory. The flagship problem is of course matrix multiplication, which is a bilinear mapping $k^{n \times n} \times k^{n \times n} \to k^{n \times n}$. The current best upper bounds are bounds are $O(n^{2.38})$, see [CW90, Sto10, Wil12], whereas the best lower bound is the recent $3n^2 - o(n^2)$ by Landsberg [Lan12].

6.1.1 What is So Special About Matrices?

The rank of a matrix is a well-understood quantity. The maximum rank of a matrix in $k^{\ell \times m}$ is $\min\{\ell, m\}$ and it is easy to come up with a matrix that achieves this maximum. For instance, the identity matrix padded with rows or columns of zeroes does the job. Or Vandermonde matrices. The fact that we have a lot of equivalent characterizations of the rank of the matrix seems to be crucial for the fact that it is so easy to come up with explicit matrices of high rank. We can compute the rank of a matrix in polynomial time.

For tensors the situation is more complicated. Per se, it is even not clear what the maximum rank of tensors in $k^{\ell \times m \times n}$ is.[1] We will discuss this briefly in the next section. How to construct explicit tensors of high rank is a widely open problem. And finally, computing the rank of a tensor is an NP-hard problem.

Theorem 1 (Håstad [Hås90]) *Let k be a field that can be represented over* $\{0, 1\}^*$. *Let* `Tensor-Rank` *(over k) be the following problem: Given a tensor $T \in k^{\ell \times m \times n}$ and a bound b, decide whether $R(T) \le b$.*

1. *Over finite fields,* `Tensor-Rank` *is* NP*-complete.*
2. *Over* $\mathbb{Q}$, `Tensor-Rank` *is* NP*-hard.*

What does it mean that "a field can be represented over $\{0, 1\}^*$"? Traditional complexity classes like P or NP are defined over some fixed alphabet, so we need to be able to encode the field elements by $\{0, 1\}$-strings. For instance, elements from finite fields can be represented in binary and rational numbers by tuples of integers represented in binary. The actual encoding does not matter as long as it is "reasonably nice", that is, all operations like addition, multiplication, etc., can be performed in polynomial time.

The hardness proof is the same over finite fields and over $\mathbb{Q}$. Over finite fields, the problems is also NP-easy; we just have to guess b rank-one tensors and check whether their sum is T. Over $\mathbb{Q}$, it is not clear whether this is possible, since we do not know an upper bound on the number of bits of the representation of the entries of the rank-one tensors. It could be the case that the rank of T is b, but all sums of b rank-one tensors involve rational numbers with a huge number of bits. ("Huge" means superpolynomial in the size of the representation of the input tensor.) To the best of my knowledge, it is even not known whether `Tensor-Rank` over $\mathbb{Q}$ is decidable.

Over $\mathbb{R}$, the situation is somewhat better: $\mathbb{R}$ itself is not representable over $\{0, 1\}$, but since $\mathbb{Q} \subseteq \mathbb{R}$, we can look at tensors T over $\mathbb{Q}$ and ask what is the minimum number of rank-one tensor with entries from $\mathbb{R}$ such that T is the sum of these rank-one tensors. This problem is decidable and even in PSPACE, since it can be reduced to the existential theory over the reals [Can88, Ren92].

Open Problem 1 What can you say about the approximability of `Tensor-Rank`? Is there a constant factor approximation algorithm? A PTAS? As far as I know, nothing in this direction is known.

Open Problem 2 What is the complexity of `Tensor-Rank` over $\mathbb{Q}$?

Another important property of matrix rank is that it is semicontinuous. If $(M_i) \in k^{n \times n}$ is a sequence of matrices that converges to a matrix M (componentwisely), then

$$R(M_i) \le r \quad \text{for all } i \qquad \Rightarrow \qquad R(M) \le r.$$

[1] Of course, ℓmn is an upper bound. However, it is not clear—and not true—that this is necessary.

Why does this hold? The fact that the rank of $R(M_i) \leq r$ is equivalent to the fact that all $(r+1) \times (r+1)$ minors vanish. These minors are polynomials and hence continuous functions. Therefore all $(r+1) \times (r+1)$ minors of M vanish, too.

For tensors, this is not the necessarily true. Consider the following tensor t given by the following two slices:

$$(t_{1,i,j}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad (t_{2,i,j}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{6.2}$$

Define

$$t_\epsilon = \frac{1}{\epsilon} \cdot (\epsilon, 1) \otimes (1, \epsilon) \otimes (1, \epsilon) - \frac{1}{\epsilon} \cdot (0, 1) \otimes (1, 0) \otimes (1, 0)$$

The two slices of t_ϵ are

$$\begin{pmatrix} 1 & \epsilon \\ \epsilon & \epsilon^2 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 & 1 \\ 1 & \epsilon \end{pmatrix}.$$

So $t_\epsilon \to t$ when $\epsilon \to 0$. And $R(t_\epsilon) \leq 2$ for all $\epsilon > 0$. On the other hand, $R(t) = 3$. We can prove this using the so-called *substitution method*. This method was first introduced by Pan [Pan66] to prove the optimality of the Horner scheme. See [BCS97] for more applications of this method and more references.

Let

$$t = \sum_{i=1}^{r} u_i \otimes v_i \otimes w_i \tag{6.3}$$

with

$$u_i = (u_{i,1}, u_{i,2}), \quad v_i = (v_{i,1}, v_{i,2}), \quad w_i = (w_{i,1}, w_{i,2}), \quad 1 \leq i \leq r,$$

be an optimal decomposition of t into rank-one tensors. Since $t_{1,1,1} = 1$, there is an i_0 such that $u_{i_0,1} \neq 0$, w.l.o.g. $i_0 = r$. Think of t as consisting of two slices as in (6.2). From the decomposition (6.3), we will construct a new tensor in $k^{1\times 2\times 2}$, which is a linear combination of the two slices, in such a way that the rank drops by one. Specifically,

$$\sum_{i=1}^{r} (\alpha u_{i,1} + u_{i,2}) \otimes v_i \otimes w_i = \begin{pmatrix} \alpha & 1 \\ 1 & 0 \end{pmatrix} := t'.$$

If we set $\alpha = -u_{r,2}/u_{r,1}$, this kills the rth rank-one tensor. Therefore,

$$R(t) \geq R(t') + 1.$$

But t' is a matrix whose rank is obviously two. Therefore, $R(t) \geq 3$. Since t has only three entries that are nonzero, there is a trivial decomposition of t of length 3.

6.2 Explicit Tensors of High Rank Imply Circuit Lower Bounds

6.2.1 Higher Order Tensors

We can generalize the concept of tensors of order three to higher orders. Let $V_1, \dots, V_n$ be vector spaces, $\dim V_i = d_i$, $1 \leq i \leq n$. The tensor product $V_1 \otimes \cdots \otimes V_n$ can be built as follows: Choose bases $v_{i,j}$, $1 \leq j \leq d_i$, for each V_i. Then we formally built the elements $v_{1,j_1} \otimes \cdots \otimes v_{n,j_n}$, $1 \leq j_1 \leq d_1$, ..., $1 \leq j_n \leq d_n$. They form a basis of the vector space $V_1 \otimes \cdots \otimes V_n$. If we have arbitrary vectors $x_i = \alpha_{i,1} v_{i,1} + \cdots \alpha_{i,d_i} v_{i,d_i} \in V_i$, $1 \leq i \leq d_i$, then

$$x_1 \otimes \cdots \otimes x_n = \sum_{j_1=1}^{d_1} \cdots \sum_{j_n=1}^{d_n} \alpha_{1,j_1} \cdots \alpha_{n,j_n} v_{1,j_1} \otimes \cdots \otimes v_{n,j_n}.$$

An element $v_1 \otimes \cdots \otimes v_n$ with $v_i \in V_i$ is a rank-one tensor. As before, the rank of a tensor $t \in V_1 \otimes \cdots \otimes V_n$ is minimum number of rank-one tensors $s_1, \dots, s_r$ such that

$$t = s_1 + \cdots + s_r.$$

The definition above is a coordinate-free definition of tensor rank. You can also think in coordinates, if you prefer that: V_i is isomorphic to k^{d_i}, simply choose bases. These isomorphims naturally extend to an isomorphism between $V_1 \otimes \cdots \otimes V_n$ and $k^{d_1} \otimes \cdots \otimes k^{d_n}$. There is also a way of defining tensor products without choosing bases at all by the universal property of turning multilinear mappings into linear ones.

6.2.2 Basic Properties

Let $\pi \in S_n$ be a permutation of $\{1, \dots, n\}$. If $v_i \in V_i$, then $v_{\pi(1)} \otimes \cdots \otimes v_{\pi(n)} \in V_{\pi(1)} \otimes \cdots \otimes V_{\pi(n)}$. So π identifies the rank-one tensors of $V_1 \otimes \cdots \otimes V_n$ with the rank-one tensors of $V_{\pi(1)} \otimes \cdots \otimes V_{\pi(n)}$. We can extend this mapping to a linear mapping $V_1 \otimes \cdots \otimes V_n \to V_{\pi(1)} \otimes \cdots \otimes V_{\pi(n)}$. This mapping clearly is surjective and by comparing dimensions, we see that is is in fact an isomorphism. The image of any tensor t under this mapping is denoted by t^{π}.

If you think in coordinates, then the entries $t'_{i_1,\dots,i_n}$ of t^{π} are defined by $t'_{i_1,\dots,i_n} = t_{i_{\pi^{-1}(1)},\dots,i_{\pi^{-1}(n)}}$, where $t = (t_{j_1,\dots,j_n})$.

Fact 1 $R(t) = R(t^{\pi})$.

Let $U_1, \dots, U_n$ be vector spaces. Let $h_i : V_i \to U_i$ be homomorphism of vector spaces, $1 \leq i \leq n$. We get a mapping that maps the rank-one tensors of $V_1 \otimes \cdots \otimes V_n$ to the rank-one tensors of $U_1 \otimes \cdots \otimes U_n$ by

$$v_1 \otimes \cdots \otimes v_n \mapsto h_1(v_1) \otimes \cdots \otimes h_n(v_n).$$

Again, we can extend this to a linear mapping $V_1 \otimes \cdots \otimes V_n \to U_1 \otimes \cdots \otimes U_n$. We denote this mapping by $h_1 \otimes \cdots \otimes h_n$.

Fact 2 *$R(t) \geq R(h_1 \otimes \cdots \otimes h_n(t))$ for any tensor t. If all h_i are isomorphisms, then equality holds.*

Let $t \in V_1 \otimes \cdots \otimes V_n$ and $s \in U_1 \otimes \cdots \otimes U_n$. We can embed both tensors into the larger space $(V_1 \oplus U_1) \otimes \cdots \otimes (V_n \oplus U_n)$ as follows: Since each V_i is a subspace of $V_i \oplus U_i$, each rank-one tensor of $V_1 \otimes \cdots \otimes V_n$ is also a rank-one tensor of $(V_1 \oplus U_1) \otimes \cdots \otimes (V_n \oplus U_n)$. Every tensor t in $V_1 \otimes \cdots \otimes V_n$ is a sum of rank-one tensors, so t embeds into $(V_1 \oplus U_1) \otimes \cdots \otimes (V_n \oplus U_n)$ as well. The same works for s. $t \oplus s$ denotes the tensor that we get by viewing t and s as tensors in $(V_1 \oplus U_1) \otimes \cdots \otimes (V_n \oplus U_n)$ and forming their sum. The following fact follows immediately.

Fact 3 *$R(t \oplus s) \leq R(t) + R(s)$.*

Open Problem 3 Does $R(t \oplus s) = R(t) + R(s)$ hold for all tensors t and s? This is known as *Strassen's additivity conjecture*.

Finally, we define the tensor product of t and s, which is an element of $(V_1 \otimes U_1) \otimes \cdots \otimes (V_n \otimes U_n)$. For two rank-one tensors $x = v_1 \otimes \cdots \otimes v_n$ and $y = u_1 \otimes \cdots \otimes u_n$, their tensor product is defined as

$$x \otimes y = (v_1 \otimes u_1) \otimes \cdots \otimes (v_n \otimes u_n).$$

If we write $t = x_1 + \cdots + x_r$ as a sum of rank-one tensors and $s = y_1 + \cdots + y_p$, then we define their tensor product as

$$t \otimes s = \sum_{i=1}^{r} \sum_{j=1}^{p} x_i \otimes y_j. \tag{6.4}$$

It is easy to verify that this is well defined.

If you think in coordinates, then the tensor product of $t = (t_{i_1,\ldots,i_n}) \in k^{d_1 \times \cdots \times d_n}$ and $s = (s_{j_1,\ldots,j_n}) \in k^{e_1 \times \cdots \times e_n}$ is given by

$$t \otimes s = (t_{i_1,\ldots,i_n} s_{j_1,\ldots,j_n}) \in k^{d_1 e_1 \times \cdots \times d_n e_n}.$$

The pair (i_1, j_1) is interpreted as a number from $\{1, \ldots, d_1 e_1\}$ and is used to index the first coordinate, (i_2, j_2) for the second, and so on.

From (6.4), the next fact follows easily.

Fact 4 *$R(t \otimes s) \leq R(t)R(s)$.*

Note that in this case, the inequality may be strict. For instance, the rank of 2×2-matrix multiplication is 7, however, the rank of $2^m \times 2^m$-matrix matrix multiplication is strictly less than 7^m for large enough m, since there are algorithms for matrix multiplication that are asymptotically faster than Strassen's algorithm.

6.2.3 From Tensor Rank Bounds to Formula Size Bounds

With a tensor $t = (t_{i_1,\dots,i_d}) \in k^n \otimes \cdots \otimes k^n$, we associate the following polynomial in the nd variables $X_{i,j}$, $1 \le i \le d$, $1 \le j \le n$:

$$f_t = \sum_{i_1=1}^{n} \cdots \sum_{i_d=1}^{n} t_{i_1,\dots,i_d} X_{1,i_1} \cdots X_{n,i_n}.$$

Raz [Raz10] proved the following result:

Theorem 2 *For any family of tensors t_n of order $d(n)$ such that $\omega(1) \le d(n) \le o(\log n / \log\log n)$ and $R(t_n) \ge n^{(1-o(1))d(n)}$, the polynomials f_{t_n} have superpolynomial formula size.*

Note that $R(t_n) \le n^{d(n)}$ by the trivial decomposition. Therefore, the family t_n has "almost" highest rank possible. It is a major open problem to find a family of polynomials with superpolynomial formula size. So finding high rank tensors might be a way of doing so. The best lower bounds we have are due to Kalorkoti [Kal85] and are quadratic.

Several decades earlier, Strassen [St73] proved the following result.

Theorem 3 *For any family of tensors t_n of order* 3, *the circuit complexity of the trilinear forms f_{t_n} are bounded by $\Omega(R(t_n))$.*

This means that a family of tensors of superlinear rank yields a family of polynomials with superlinear circuit complexity, something which we do not know for general circuit models.

But there is a catch, as we will see in the next section. Finding some family of tensors/polynomials with the desired properties is easy, a random choice does the job. So what we really want is an *explicit* tensor. We call a family of tensors $t_n = (t_{n;i_1,\dots,i_d})$ explicit if the mapping $(n; i_1, \dots, i_d) \mapsto t_{n;i_1,\dots,i_d}$ can be computed by an arithmetic circuit of size polynomial in d and $\log n$. One can think also of other notion of explicitness. For the purpose of this appetizer, any notion that prevents random tensors is fine. If the entries of the tensors are rational, we could also require that the mapping is computable in $\mathsf{P/poly}$. Then, by using Valiant's criterion, we can use high rank tensors to separate classes in Valiants model, in particular, we could show that the permanent does not have polynomial size formulas, see [Bür00].

6.2.4 Random Tensors

Let V be a vector space of dimension n. A generic rank-one tensor in $V^{\otimes d}$ is described by dn variables,

$$(x_{1,1}, \dots, x_{n,1}) \otimes \cdots \otimes (x_{1,d}, \dots, x_{n,d}).$$

The sum of r generic rank-one tensors is described by rdn varibles. Its entries are multilinear polynomials in these variables. A generic tensor in $V^{\otimes d}$ is described by n^d variables, all of them begin algebraically independent. Therefore, $rnd \geq n^d$ is required to write every tensor as a sum of r rank-one tensor, that is,

$$r \geq \frac{n^{d-1}}{d}.$$

This is a very simple argument, but sufficient for our needs and almost optimal. With more sophisticated ones, we can get tighter bounds, see the work of Lickteig and Strassen for three-dimensional tensors [Lic85, Str83], see Landsberg's book for the general case [Lan11].

From the argument above, it follows that there is a tensor with rank at least n^{d-1}/d. But even random tensors have at least this rank with high probability. The entries of tensors that can be written as the sum of fewer rank-one tensors are algebraically dependent, since they can we written as polynomials in less than n^d variables. Therefore, these entries fulfill some polynomial relation. It is well known that random points do not fufill polynomial relations with high probability. In theoretical computer science, this fact is known as the Schwartz-Zippel lemma.[2]

Lemma 1 (Schwartz–Zippel) *Let F be a field. Let p be a nonzero polynomial in $F[X_1, \dots, X_n]$ of total degree d. Let $S \subseteq F$. Then*

$$\Pr_{r_1,\dots,r_n \in S}[p(r_1, \dots, r_n) = 0] \leq |S|/d.$$

Even if we do not know a bound on the polynomial describing the algebraic dependence, if the underlying field is large enough or even infinite, a random tensor will have rank $\geq n^{d-1}/d$ with high probability.

Although it sounds very simple, it is a major open problem to find a tensor of high rank that is explicit, i.e., whose entries can be constructed by a *deterministic* polynomial time algorithm.

6.3 Explicit Tensors from Bilinear Mappings

This section shows the present (poor) knowledge of how to construct explicit tensors of high rank.

6.3.1 The Rank of Bilinear Mappings and Algebras

Let $\phi : U \times V \to W$ be a bilinear mapping. Every bilinear map corresponds in a unique way to a tensor t_ϕ in $U^* \otimes V^* \otimes W$, see (6.1). Since a vector space and its

[2] The name of the lemma is justified because Schwartz and independently Zippel were the *last* to prove this lemma.

dual are isomorphic, we can also think if t_ϕ living in $U \otimes V \otimes W$. We define the rank of a bilinear map ϕ to be the rank of the corresponding tensor t_ϕ. If A is an finite dimensional associative algebra with unity, that is, A is a ring which is also a finite dimensional vector space over some field k, then the multiplication map in A is a bilinear mapping $A \times A \to A$. The rank $R(A)$ of A is the rank of its multiplication map.

If we think in coordinates, we get the tensor that corresponds to A as follows. Choose a basis $x_1, \dots, x_n$ of A. The product of any two elements of A is again an element of A and can be written as a linear combination of $x_1, \dots, x_n$. In particular

$$x_i \cdot x_j = \sum_{k=1}^{n} \alpha_{i,j,k} x_k.$$

The so-called *structural constants* $\alpha_{i,j,k}$ are the entries of the tensor (with respect to the chosen basis). Since a change of basis is an isomorphim of vector spaces, we get that the rank of this tensor is independent of the chosen basis.

The best lower bounds for the rank of an algebra and for any other tensor of order three are of the form $3 \dim A - o(\dim A)$. Very recently, Landsberg proved this for the algebra $k^{n\times n}$ of $n \times n$-matrices. An earlier example with an easier proof is the algebra $k[X_1, \dots, X_n]/I_d$ where I_d is the ideal generated by all monomials of degree d, see [Blä01]. Because the families of algebras have a "regular" structure, it is clear that the corresponding tensors are explicit. We just have to compute the structural constants. For instance, in the case of the algebra $k^{n\times n}$ with the standard basis, we have

$$e_{i,i'} e_{j,j'} = \begin{cases} e_{i,j'} & \text{if } i' = j, \\ 0 & \text{otherwise.} \end{cases}$$

(Note that we use double indices since $\dim k^{n\times n} = n^2$.) In the second case, if we take all monomials of degree $< d$ as a basis, we get a similar expression.

It is a major open problem to find explicit tensors or explicit families of algebras with a larger rank.

Open Problem 4 1. Is there an explicit family of tensors $t_n \in k^n \otimes k^n \otimes k^n$ with $R(t_n) \geq (3 + \epsilon)n$ for some $\epsilon > 0$.
2. Can we even achieve this for tensors corresponding to the multiplication in an algebra, i.e., is there an explicit family of algebras A_n with $R(A_n) \geq (3 + \epsilon) \dim A_n$ for some $\epsilon > 0$. Of course, $\dim A_n$ should go to infinity.

6.3.2 From Tensors of Order Three to Higher Order Tensors

We can use the lower bounds of the rank of tensors of order three to obtains bounds for the rank of higher order tensors. Up to lower order terms, they match the current best lower bounds (see the next section).

Let $t \in V_1 \otimes \cdots \otimes V_n$. Let $I_1, \dots, I_m$ be a partition of $\{1, \dots, n\}$, that is, the I_j are pairwise disjoint and their union is $\{1, \dots, n\}$. Let $U_j = \bigotimes_{i \in I_j} V_i$ for $1 \le j \le m$. We can view t as an element of $U_1 \otimes \cdots \otimes U_m$. Note that the rank of t as an element of $U_1 \otimes \cdots \otimes U_m$ is a lower bound for the rank of t as an element of $V_1 \otimes \cdots \otimes V_n$. Why? Any rank-one tensor $v_1 \otimes \cdots \otimes v_n \in V_1 \otimes \cdots \otimes V_n$ induces a rank-one tensor $u_1 \otimes \cdots \otimes u_m \in U_1 \otimes \cdots \otimes U_m$ by setting $u_j = \bigotimes_{k \in I_j} v_k$. When it is not clear from context, whether we think of t being a tensor in $U_1 \otimes \cdots \otimes U_m$ or $V_1 \otimes \cdots \otimes V_n$, we add it as a subscript.

Lemma 2 $R_{U_1 \otimes \cdots \otimes U_m}(t) \le R_{V_1 \otimes \cdots \otimes V_n}(t)$. □

The rank can indeed become smaller. Consider $\langle n, n, n \rangle \in k^{n \times n} \otimes k^{n \times n} \otimes k^{n \times n}$, the tensor of matrix multiplication. If we consider it as a tensor in $(k^{n \times n} \otimes k^{n \times n}) \otimes k^{n \times n}$, then it is a matrix of size $n^4 \times n^2$. Its rank is at most n^2. However, we know a lower bound of $3n^2 - o(n^2)$ for the rank of $\langle n, n, n \rangle$ as a tensor in $k^{n \times n} \otimes k^{n \times n} \otimes k^{n \times n}$ [Lan12]. In fact, it is an old open problem, whether the so-called exponent of matrix multiplication is two.

6.3.3 Explicit Tensors of Higher Order

Let d be even and let $N = n^{d/2}$ Take any full rank matrix $M \in k^{N \times N}$, for instance the identity matrix. It has rank $n^{d/2}$. By Lemma 6.2,

$$R_{\bigotimes_{i=1}^{d} k^n}(M) \ge n^{d/2}. \tag{6.5}$$

The tensor M is obviously explicit, an entry $m_{i_1,\dots,i_d} = 1$ if $(i_1, \dots, i_{d/2}) = (i_{d/2+1}, \dots, i_d)$ and 0 otherwise. Note that if we could achieve $n^{(1-o(1))d}$, then this will lead to formula lower bounds.

It is a sad state of affairs that (6.5) is the asymptotically best lower bound for an explicit tensor that we currently know; further improvements just concern the constant factor.

Here is one such improvement which uses a lower bound by Hartmann [Har85]: Let k be a field and K be an extension field of dimension n. Consider the multiplication of the K-left module $K^{1 \times m}$ as a bilinear map over k. (We take $x \in K$ and $(y_1, \dots, y_m) \in K^{1 \times m}$ and map them to $(xy_1, \dots, xy_m) \in K^{1 \times m}$). However, we view this as a k-bilinear map and not as a K-bilinear map. Let $\hat{s}$ be the corresponding tensor. Hartmann showed that

$$R(K^{1 \times m}) \ge (2n - 1)m = 2nm - m. \tag{6.6}$$

If we now set $m = n^{e-1}$ and let $s \in K^{\otimes(2e+1)}$ be the tensor corresponding to $\hat{s}$, we get

$$R(s) \ge 2n^e - n^{e-1}$$

for a tensor of order $d = 2e + 1$. Note that if d is odd, then the approach above using an invertible matrix only gives the lower bound n^e.

If we take $K = k[X]/(X^n)$ instead of an extension field, then we can show the same bound as (6.6) and get another example of an explicit tensor. As a basis of K, we choose the basis $1, X, \dots, X^{n-1}$. This induces a basis of $K^{1,m}$ in the natural way. How does the tensor of the multiplication of $K^{1\times m}$ look like? First consider the case $m = 1$. The tensor looks like

$$\begin{pmatrix} 1 & 2 & 3 & \dots & n \\ 2 & 3 & & \iddots & 0 \\ 3 & & \iddots & 0 & 0 \\ \vdots & \iddots & \iddots & \vdots & \vdots \\ n & 0 & \dots & 0 & 0 \end{pmatrix} \tag{6.7}$$

How to interpret this? It is a $\{0, 1\}$-valued tensor of size $n \times n \times n$. An entry $k > 0$ in position (i, j) means that the entry in position (i, j, k) is 1. All other entries are 0 whether it is explicitly indicated or not. The tensor for arbitrary m looks like follows:

$$T = \begin{pmatrix} T_1 \\ T_2 \\ \vdots \\ T_m \end{pmatrix}.$$

Each T_j is a copy of the tensor in (6.7). However, these tensors T_j live in different slices. T is a tensor in $k^{n\times nm\times nm}$, each T_j lives in the slices $(j-1)n+1, \dots, jn$ in the third component. Now, as in the beginning, we want to apply substitution method. We will only work with the copy T_1, kill $2n - 1$ products, and then we can simply apply induction. In an optimal decomposition of T into rank-one tensors

$$T = \sum_{i=1}^{r} u_i \otimes v_i \otimes w_i,$$

we can assume that $w_1, \dots, w_{n-1}$ restricted to the first $n-1$ coordinates, are linearly independent. Let h be the projection along the linear span of $w_1, \dots, w_{n-1}$ onto $\langle e_n, e_{n+1}, \dots, e_{mn}\rangle$. Here, $e_i \in k^{mn}$ is the ith unit vector and $\langle \dots \rangle$ the linear span. Applying h, we kill $n - 1$ products in the decomposition of T. What happens to T under this homomorphism? Note that only the first n rows of T are affected, which just contain the copy T_1. In (6.7), h maps multiples of the slices $1, \dots, n-1$ onto the nth one. The result is a lower triangular matrix with all 1s on the diagonal. Therefore, the matrix has full rank. Like before, we can now kill another n products and the tensor that is still computed is

$$T' = \begin{pmatrix} T_2 \\ T_3 \\ \vdots \\ T_m \end{pmatrix}.$$

Therefore, we can proceed by induction and get a lower bound of $(2n-1)m$.

6.4 Explicit Tensors by Combinatorial Constructions

The currently best lower bound for an explicit tensor is due to [AF11]. It improves on the lower order term of the construction in the last section.

Let $\ell = \lfloor \log_2 n \rfloor$. For $i \in \{0, \dots, \ell\}$, we recursively define the following matrices:

1. $S_{1,0} = (1)$.
2. For even $n = 2m > 1$,

$$S_{n,i} = \begin{cases} \begin{pmatrix} 0 & 0 \\ S_{m,i} & 0 \end{pmatrix} & \text{if } i < \ell \\ \begin{pmatrix} I_m & 0 \\ 0 & I_m \end{pmatrix} & \text{otherwise} \end{cases}.$$

3. For odd $n = 2m + 1 > 1$,

$$S_{n,i} = \begin{cases} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ S_{m,i} & 0 & 0 \end{pmatrix} & \text{if} i < \ell \\ \begin{pmatrix} 0 & 0 & 0 \\ I_m & 0 & 0 \\ 0 & I_m & 0 \end{pmatrix} & \text{otherwise} \end{cases}.$$

Finally, let T_n be the tensor consisting of slices $S_{n,0}, \dots, S_{n,\ell}$. The format of T_n is $n \times n \times (\ell + 1)$. T_n is certainly explicit, we can determine the entries by following the recursive structure.

Theorem 4 *$R(T_n) \geq 2n - 2h(n) + 1$ where $h(n)$ is the number of 1s in the binary expansion of n.*

Note that $h(n) \leq \log n$. We can use the substitution method to prove the theorem. Certainly $R(T_1) = 1$ holds. If $n = 2m$ is even, then we can "substitute away" the two identity matrices, killing n products. The remaining tensor is $T_{n/2}$. Therefore, we get the recurrence

$$R(T_{2m}) \geq R(T_m) + 2m.$$

In the same way, we get

$$R(T_{2m+1}) \geq R(T_m) + 2m.$$

It is easy to verify that the bound stated in the theorem is the solution of this recurrence. We can now extend this to a tensor of size $n \times (n+1) \times n$ by extending the slices above by one column and then add $n - (\ell + 1)$ linearly independent slices, which just have a one in the extra column. These slices can be substituted away and we get a tensor of size $n \times (n+1) \times n$ with rank bounded by $3n - \Theta(\log n)$. If we set $n = m^e$ and add just m extra slices, we get a tensor of size $m^e \times (m^e + 1) \times m^e$ the rank of which can be lower bounded by $2n + m - \Theta(\log n) = 2m^e + m - \Theta(d \log m)$. As before, we can interpret this tensor as a tensor of order $2e + 1$. (The $+1$ in $m^e + 1$ the second component disturbs this construction a little bit, to remedy this, we can start with a tensor of size $(n-1) \times (n-1) \times (n-1)$ and then extend this to a tensor of size $n \times n \times n$ by adding zeros.)

If you look at the construction closely, we can view this again as a tensor related to an algebra, as pointed by Landsberg [Lan13]. For simplicity, assume that n is a power of 2. Otherwise, the tensor will just have some additional zeros. It is quite easy to see, that up to permutations, the tensor T_n are just the slices $1, 2, \ldots, 2^i, \ldots, 2^\ell$ of the tensor of the algebra $k[X]/(X^n)$. For instance, T_8 has the form

$$T_8 = \begin{pmatrix} 1 & 2 & & 3 & & & & 4 \\ 2 & & 3 & & & & 4 & \\ & 3 & & & & 4 & & \\ 3 & & & & 4 & & & \\ & & & 4 & & & & \\ & & 4 & & & & & \\ & 4 & & & & & & \\ 4 & & & & & & & \end{pmatrix}$$

Looking at this, it is easy to see that we can project away the lower four rows and four columns to the righthand side. This reduces the rank by 8 (in general by n) and affects only the fourth slice. We can remove this slice and get T_4. Using induction, we get a lower bound of $1 + 2 + 4 + 8 = 15$ (in general $2n - 1$, note that n is a power of 2).

6.5 Conclusions

In the examples we have seen, the bounds on the rank are proven via the substitution method. The bounds that are achievable with this method are usually limited by the sum of the dimension of the vector spaces. Up to lower order terms, our constructions reach this limit. To get better bounds, new lower bound techniques are needed. One promising approach for this is the geometric complexity approach by Bürgisser and Ikenmeyer [BI11, BI12].

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Chapter 7
Progress on Polynomial Identity Testing-II

Nitin Saxena

To my grand-advisor Professor Somenath Biswas

Abstract We survey the area of algebraic complexity theory; with the focus being on the problem of polynomial identity testing (PIT). We discuss the key ideas that have gone into the results of the last few years.

Keywords Arithmetic circuit · Identity testing · Hitting-set · Rank · Lower bound · Jacobian · Concentration · Shift · Morphism

Mathematics Subject Classification (2010) Primary 68Q25, 68W30, Secondary 12Y05, 13P25

7.1 Introduction

Algebraic complexity theory is the study of computation via *algebraic* models, hence, algebraic techniques. In this article we work with only one model—*arithmetic circuit* (in short, *circuit*). A circuit $C(x_1, \ldots, x_n)$, over a ring R, computes a polynomial f in $R[x_1, \ldots, x_n]$. Its description is in the form of a rooted tree; with the *leaves* having the variables or constants as input, the internal *nodes* computing addition or multiplication, and the *root* having the f as output. The edges in C, called *wires*, carry the intermediate polynomials and could also be used to multiply by a constant (from R). By the *size*, respectively the *depth*, of C we mean the natural notions (sometimes to avoid "trivialities" we might want to take into account the bit-size needed to represent an element in R).

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A moment's thought would suggest that a circuit is a rather compact way of representing polynomials. Example a circuit of size s could produce a polynomial of degree 2^s (hint: repeated squaring). In fact, a single product gate could multiply s linear polynomials and produce $n^{\Omega(s)}$ many monomials. Thus, a circuit is an 'exponentially' compact representation of some polynomial families (as opposed to simply writing it as a sum of monomials). Conversely, are there 'explicit' polynomial families (say n-variate n-degree) that require exponential (i.e. $2^{\Omega(n)}$) sized circuits? We "expect" almost every polynomial to be this hard, but, the question of finding an *explicit* family is open and is the main goal motivating the development of algebraic complexity.

One can try to directly give a good *lower bound* against circuits by designing an explicit polynomial family $\{f_n\}$ and prove that it requires a 'large'-sized circuit family $\{C_n\}$. The other, indirect, way is to design an efficient *hitting-set* $\mathcal{H}$ for the circuit family, i.e. if $C_n \neq 0$ then $\exists a \in \mathcal{H}, C_n(a) \neq 0$. This 'flip' from lower bounds to algorithms was first remarked by [HS80] and now it has several improved versions [KI04, Agr05, Agr06]. This is a remarkable phenomenon and is one of the primary motivations to study the question of PIT: Given a circuit C test it for zeroness, in time polynomial in size(C). The hitting-set version of PIT is also called *blackbox* PIT (contrasted with *whitebox* PIT).

The last 10 years have seen a decent growth of algebraic tools and techniques to understand the properties of polynomials that a circuit computes. The feeling is that these polynomials are special, different from general polynomials, but a strong enough algebraic 'invariant' or a combinatorial 'concept' is still lacking. There have been several articles surveying the known techniques and the history of PIT [Sax09, AS09, SY10, CKW11, Sap13]. In this survey we will attempt not to repeat what those surveys have already covered. So, we will focus only on the new ideas and assume that the reader has given at least a cursory glance at the older ones. We directly move on to the Leitfaden.

7.1.1 Survey Overview

This article deals mainly with three broad topics—the 'universality' of depth-3 circuits, the design of hitting-sets via 'faithful' morphisms and that via rank 'concentration'. A major emerging area that we skip in this article is that of PIT vis à vis GCT (geometric complexity theory) program [Mul11, Mul12a, Mul12b]; the algebraic-geometry interpretations there are interesting though any concrete PIT algorithm, or application, is yet to emerge.

Shallow circuits A depth-2 circuit (top $+$ gate) of size s, over a field, essentially computes a sum of s monomials. Such polynomials are called *sparse* polynomials; blackbox PIT for them was solved few decades ago. So, our next stop is depth-3: Polynomials of the form

$$C = \sum_{i=1}^{k} \prod_{j=1}^{d} L_{i,j},$$

where $L_{i,j}$ are linear polynomials in $\mathbb{F}[x_1, \ldots, x_n]$. Significant research has been done with this model, but both subexponential PIT and exponential lower bounds are open here. Recently, a remarkable universality result was shown for depth-3 [GKKS13]: If an n-variate poly(n)-degree polynomial can be nontrivially computed by a circuit, then it can be nontrivially computed in depth-3. This 'squashing' of depth means that it suffices to focus on depth-3 for PIT purposes.

If we consider a depth-2 circuit (top $\times$ gate), over a *ring* R, then again we get some remarkable connections. Fix R to be the 2×2 matrix algebra $M_2(\mathbb{F})$, and consider the circuit

$$D = \prod_{i=1}^{d} L_i,$$

where L_i are linear polynomials in $R[x_1, \ldots, x_n]$. Traditionally, D is called a *width-2 algebraic branching program* (ABP). It was shown by [SSS09] that depth-3 PIT efficiently reduces to width-2 ABP PIT.

Faithful morphisms It was observed in the last few years that in all the known hitting-sets, the key idea in the proof is to work with a *homomorphism* φ and an algebraic *property* that the image of φ should preserve. [SS12] used a (Vandermonde-based) map $\varphi : \mathbb{F}[x_1, \ldots, x_n] \to \mathbb{F}[y_1, \ldots, y_k]$ that preserves the 'linear' rank of any k linear polynomials. This gave the first blackbox PIT for bounded top fanin depth-3, over any field.

Beecken et al. [BMS13] and Agrawal et al. [ASSS12] used a (Vandermonde and Kronecker-based) map $\varphi : \mathbb{F}[x_1, \ldots, x_n] \to \mathbb{F}[y_1, \ldots, y_k]$ that preserves the 'algebraic' rank (formally, *transcendence degree*) of certain k polynomials. This gave the first blackbox PIT (and lower bounds) for several well-studied classes of constant-depth circuits. One drawback of the technique is that it requires zero/large characteristic fields.

Rank concentration Inspired from the tensors, a restricted circuit model called multilinear read-once ABP (ROABP) has been intensively studied. Let R be the $w \times w$ matrix algebra $M_w(\mathbb{F})$ and let $\{S_i\}$ be a partition of $[n]$. Consider the circuit $D = \prod_{i=1}^{d} L_i$, where L_i are linear polynomials in $R[x_{S_i}]$ (i.e. the linear factors have disjoint variables). For D [FSS13] gave a hitting-set in time poly$(wn)^{\log w \cdot \log n}$, i.e. quasi-poly-time. The proof is based on the idea, following [ASS13], that after applying a small (Kronecker-based) 'shift', D gets the following property: The rank of its coefficients (viewed as $\mathbb{F}$-vectors) is concentrated in the 'low' support monomials. Thus, checking the zeroness of these low monomials is enough!

We conjecture that rank concentration, after a 'small' shift, should be attainable in any ABP D. But currently the proof techniques are not that strong. Recently, [AGKS13] have achieved rank concentration in multilinear depth-3 circuits where the partitions (corresponding to each product gate) are 'close' to each other in the sense of 'refinement'.

7.2 Shallow Circuits, Deep Interconnections

In this section, we exhibit the key ideas behind the universality of two shallow circuits.

7.2.1 The Depth-3 Chasm

In the study of circuits one feels that low-depth should already hold the key. This feeling was confirmed in a series of work [VSBR83, AV08, Koi12, Tav13]: Any $\mathrm{poly}(n)$-degree n-variate polynomial computed by a $\mathrm{poly}(n)$-sized circuit C can also be computed by a $n^{O(\sqrt{n})}$-sized depth-4 circuit!

The idea for this is, in retrospect, simple—since the degree is only $\mathrm{poly}(n)$, first, squash the depth of C to $O(\log n)$ by only a polynomial blowup in the size. This is done in a way so as to make the product gates quite *balanced*, i.e. their two inputs are roughly of the same degree. Next, identify a subcircuit C_2 by picking those gates whose output polynomial has degree at least $\sqrt{n}$, and call the remaining subcircuit C_1. We view C_2 as our circuit of interest that takes gates of C_1 as input. It can be shown that C_2 computes a polynomial of degree $\approx\sqrt{n}$ of its input variables (which are $\mathrm{poly}(n)$ many). Obviously, each gate of C_1 also computes a polynomial of degree $\approx\sqrt{n}$ of its input variables (which are $x_1, \ldots, x_n$). Thus, C_2 finally computes a sum of $\approx\binom{\mathrm{poly}(n)+\sqrt{n}}{\sqrt{n}}$ products, each product has $\sqrt{n}$ factors, and each factor is itself a sum of $\approx\binom{n+\sqrt{n}}{\sqrt{n}}$ degree-$\sqrt{n}$ monomials. To put it simply (and ignoring the constant factors), C can be expressed as a $\sum\prod^{\sqrt{n}}\sum\prod^{\sqrt{n}}$ circuit of size $n^{O(\sqrt{n})}$. The details of this proof can be seen in [Tav13].

The strength of depth-4 is surprising. Recently, an even more surprising reduction has been shown [GKKS13]—that to depth-3 (again, $n^{O(\sqrt{n})}$ sized). We will now sketch the proof. It ties together the known results in an unexpected way.

Essentially, the idea is to modify a $\sum\prod^{a}\sum\prod^{a}$ circuit C of size $s := n^a$ (where $a := \sqrt{n}$) by using two polynomial identities that are in a way "inverse" to each other, and are to do with powers-of-linear-forms. First, replace the product gates using Fischer's identity:

Lemma 2.1 [Fis94] *Any degree a monomial can be expressed as a linear combination of 2^{a-1} ath powers of linear polynomials, as:*

$$y_1 \cdots y_a \;=\; (2^{a-1}\cdot a!)^{-1} \cdot \sum_{r_2,\ldots,r_a\in\{\pm 1\}} \left(y_1 + \sum_{i=2}^{a} r_i y_i\right)^{a} \cdot (-1)^{\#\{i\,|\,r_i=-1\}}.$$

We denote this type of a circuit by the notation $\sum \bigwedge^a \sum$, where the wedge signifies the powering by a. The above identity transforms the $\sum \prod^a \sum \prod^a$ circuit C to a $\sum \bigwedge^a \sum \bigwedge^a \sum$ circuit, of size $\text{poly}(s)$. We reuse s for this size estimate.

Next, the two power gates are 'opened' up using an identity introduced by the author:

Lemma 2.2 [Sax08] *For any a, m, there exist degree-a univariate polynomials $f_{i,j}$ such that*

$$(y_1 + \cdots + y_m)^a \;=\; \sum_{i=1}^{ma+1} \prod_{j=1}^{m} f_{i,j}(y_j).$$

Let us carefully see the jugglery on C. The $\sum \bigwedge^a \sum \bigwedge^a \sum$ circuit C has the expression $C = \sum_i T_i$, where each T_i has the form $(\sum_{j=1}^{s} \ell_{i,j}^{e_{i,j}})^a$ with linear $\ell_{i,j}$'s. We want to open up the top power gate of C. By Lemma 2.2 we get

$$T_i \;=\; \sum_{u=1}^{sa+1} \prod_{j=1}^{s} f_{u,j}(\ell_{i,j}^{e_{i,j}}).$$

Since $f_{u,j}$ is a univariate, it splits into linear polynomials when the base field $\mathbb{F}$ is *algebraically closed*. As $\ell_{i,j}$ is already a linear polynomial, we deduce that T_i, and hence C, is a $\sum \prod \sum$ circuit of size $\text{poly}(s)$.

Finally, note that for the above arguments to work, we require $\mathbb{F}$ to be algebraically closed and $\text{char}(\mathbb{F}) > a$. Lemma 2.2 has been generalized to all characteristics by [FS13b], so it is likely that this depth-3 reduction can be extended to *all* algebraically closed fields.

The optimality of $n^{\sqrt{n}}$-size, in this reduction, is open. However, [KSS13] showed that any decent reduction in this size bound would imply *VNP* $\neq$ *VP*.

7.2.2 The Width-2 Chasm

Here we look at $\prod \sum$ circuits over a matrix algebra. Though the model $D = \prod_i L_i$, with linear $L_i \in R[x_1, \ldots, x_n]$, seems innocuous at first sight, a closer look proves the opposite! It can be shown fairly easily that a polynomial computed by a constant-depth circuit (over a field) can as well be computed by a D over a 3×3 matrix algebra [BC88]. On the other extreme, by taking $R = M_n(\mathbb{F})$ we can compute the *determinant* of a matrix in $\mathbb{F}^{n \times n}$ [MV97], hence, arithmetic *formulas* (not general circuits!) can be simulated in this model [Val79].

Perhaps surprisingly, [SSS09] showed that: A polynomial C computed by a depth-3 circuit (over a field) can be "almost"[1] computed by a D over a 2×2 matrix algebra.

[1] We are able to compute only a *multiple* of C. However, the extra factor is simply a product of poly-many linear polynomials. So, it suffices for PIT purposes.

This, togetherwith the previous subsection, makes the $\prod\sum$ circuits over $M_2(\mathbb{F})$ quite strong.

Say, we want to express the depth-3 circuit $C = \sum_{i=1}^{k} T_i$ in a 2×2 matrix product. First, we express a product $T_i = \prod_{j=1}^{d} \ell_{i,j}$ as:

$$\begin{bmatrix} \ell_{i,1} & 0 \\ 0 & 1 \end{bmatrix} \cdots \begin{bmatrix} \ell_{i,d-1} & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & \ell_{i,d} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} T_i' & T_i \\ 0 & 1 \end{bmatrix}, \text{ where } T_i' := T_i/\ell_{i,d}.$$

Once we have such k 2×2 matrices, each containing T_i in the $(1, 2)$th place, we would like to sum the T_i's in a 'doubling' fashion (instead of one-by-one).

We describe one step of the iteration. Let $\begin{bmatrix} L_1 & L_2 f \\ 0 & L_3 \end{bmatrix}$ and $\begin{bmatrix} M_1 & M_2 g \\ 0 & M_3 \end{bmatrix}$ be encapsulating two intermediate summands f and g. With the goal of getting (a multiple of) $f + g$ we consider the following, carefully designed, product:

$$\begin{bmatrix} L_1 & L_2 f \\ 0 & L_3 \end{bmatrix} \cdot \begin{bmatrix} L_2 M_3 & 0 \\ 0 & L_1 M_2 \end{bmatrix} \cdot \begin{bmatrix} M_1 & M_2 g \\ 0 & M_3 \end{bmatrix}$$
$$= \begin{bmatrix} L_1 M_1 L_2 M_3 & L_2 M_3 L_1 M_2 (f + g) \\ 0 & L_3 M_3 L_1 M_2 \end{bmatrix}$$

After $\log k$ such iterations, we get a *multiple* of C in the $(1, 2)$-th entry of the final 2×2 matrix product. Note that the middle matrix, introduced in the LHS above, potentially doubles (in the degree of the entry polynomials) in each iteration. Thus, finally, D is a product of poly$(d 2^{\log k})$ linear polynomials over $M_2(\mathbb{F})$. Thus, the size blowup is only polynomial in going from depth-3 to width-2.

7.3 Faithful Morphisms, Hitting-Sets

In algebraic complexity the study of certain maps has been fruitful—homomorphisms $\varphi : \mathcal{R} := \mathbb{F}[x_1, \ldots, x_n] \to \mathbb{F}[y_1, \ldots, y_k] =: \mathcal{R}'$ such that the algebraic 'relationship' of certain polynomials $\{f_1, \ldots, f_k\}$ does not change in the image of φ. When f_i's are linear this boils down to a linear algebra question and we can easily design φ in time poly(n) (hint: employ Vandermonde matrix). This business becomes complicated when f_i's are nonlinear. Then we have to ask how are f_i's represented. If they are given via monomials then we invoke the Jacobian criterion to design φ, but the time complexity becomes exponential in k. Several variants of such faithful maps are discussed in the Ph.D thesis [Mit13]. We sketch the ideas behind two basic maps here.

7.3.1 Bounded Fanin Depth-3 Blackbox PIT

Let $C = \sum_{i\in[k]} T_i$ be a depth-3 circuit. When k is constant, C is naturally called *bounded fanin* depth-3. This case of PIT has, by now, a rich history [DS07, KS07, KS11, SS11, KS09, SS13, SS12]. Several new techniques have sprung up from this model— a locally decodable code structure, a rank-preserving map via extractors, Sylvester-Gallai configurations (higher dimensions and all fields) and rank bounds. We will sketch here the main idea behind the poly-time blackbox PIT of bounded fanin depth-3. The details are quite technical and could be seen in [SS13, SS12].

Vandermonde map We define a homomorphism Ψ_β, for a $\beta \in \mathbb{F}$, as:

$$\forall i \in [n], \quad \Psi_\beta : x_i \mapsto \sum_{j=1}^{k} \beta^{ij} y_j ,$$

and $\Psi_\beta(\alpha) = \alpha$ for all $\alpha \in \mathbb{F}$. This (naturally) defines the action of Ψ_β, on *all* the elements of $\mathcal{R}$, that preserves the ring operations. We have the following nice property, as a consequence of [GR08, Lemma 6.1]:

Lemma 3.1 [Ψ_β preserves k-rank] *Let S be a subset of linear forms in $\mathcal{R}$ with* $\mathrm{rk}(S) \le k$, *and* $|\mathbb{F}| > nk^2$. *Then* $\exists \beta \in \mathbb{F}$, $\mathrm{rk}(\psi_\beta(S)) = \mathrm{rk}(S)$.

Intuitively, Ψ_β is *faithful* to any algebraic object involving the elements in $\mathrm{span}(S)$. The proof of this lemma is by studying the coefficient-matrix of the linear polynomials in S, and its change under Ψ_β. This map has a role to play in bounded fanin depth-3 owing to a certain structural theorem from [SS13]—*certificate for a non-identity*.

To discuss this certificate we need a definition, that of 'paths' of 'nodes' in C (assumed to be nonzero). A *path* $\mathbf{p}$ with respect to an ideal I is a sequence of terms $\{p_1, p_2, \ldots, p_b\}$ (these are products of linear forms) with the following property. Each p_i divides T_i, and each p_i is a 'node' of T_i with respect to the ideal $\langle I, p_1, p_2, \ldots, p_{i-1}\rangle$.[2] So p_1 is a node of T_1 wrt I, p_2 is a node of T_2 wrt $\langle I, p_1\rangle$, etc.

Let us see an example of a path $(\langle 0\rangle, p_1, p_2, p_3)$ in Fig. 7.1. The oval bubbles represent the list of forms in a product gate, and the rectangles enclose forms in a node. The arrows show a path. Starting with the zero ideal, nodes $p_1 := x_1^2$, $p_2 := x_2(x_2 + 2x_1)$, and $p_3 := (x_4 + x_2)(x_4 + 4x_2 - x_1)(x_4 + x_2 + x_1)(x_4 + x_2 - 2x_1)$ form a path. Initially, the path is just the zero ideal, so x_1^2 is a node. Note how p_2 is a power of x_2 modulo $\mathrm{radsp}\langle p_1\rangle$, and p_3 is a power of x_4 modulo $\mathrm{radsp}\langle p_1, p_2\rangle$.

The non-identity certificate theorem [SS13, Theorem 25] states that for any non-identity C, there exists a path $\mathbf{p}$ such that modulo $\langle \mathbf{p}\rangle$, C reduces to a single nonzero multiplication term.

[2] By a *node* p_i we mean that some nonzero constant multiple of p_i is identical to a power-of-a-linear-form modulo $\mathrm{radsp}\langle I, p_1, p_2, \ldots, p_{i-1}\rangle$, where radsp is the ideal generated by the set of all the linear polynomials that divide p_j, $j \in [i-1]$ and the generators of I.

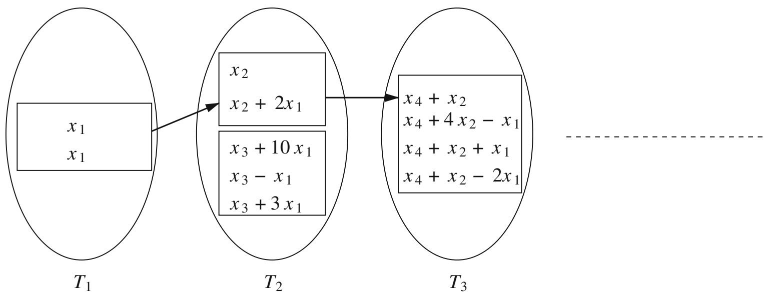

Fig. 7.1 Nodes and paths in $C = T_1 + T_2 + T_3 + \cdots$

Theorem 3.2 (Certificate for a non-identity) *Let I be an ideal generated by some multiplication terms. Let $C = \sum_{i\in[k]} T_i$ be a depth-3 circuit that is nonzero modulo I. Then $\exists i \in \{0,\ldots,k-1\}$ such that $C_{[i]}$[3] mod I has a path $\mathbf{p}$ satisfying: $C \equiv \alpha \cdot T_{i+1} \not\equiv 0 \pmod{I + \langle \mathbf{p}\rangle}$ for some $\alpha \in \mathbb{F}^*$.*

The proof of this theorem involves an extension of Chinese remaindering to ideals that are generated by multiplication terms. Once we have this structural result about depth-3, observe that we would be done if we could somehow ensure $T_{i+1} \notin \langle \mathbf{p}\rangle$ (in our application I is zero). How do we preserve this ideal non-membership under a cheap map?

Notice that the rank of the set S_0 of linear polynomials that divide the nodes in the path $\mathbf{p}$ is $<k$ (since path length is below k). Moreover, T_{i+1} factors into at most d linear polynomials, denote the set by S_1. So if we apply a map that preserves the rank of each of the d sets $S_0 \cup \{\ell\}$, $\ell \in S_1$, then, intuitively, the ideal non-membership should be preserved. As $\text{rk}(S_0 \cup \{\ell\}) \le k$ we can employ the previously discussed map Ψ_β (over a field satisfying $|\mathbb{F}| > dnk^2$). This idea could be easily turned into a proof; details are in [SS12].

Finally, what we have achieved is the construction of a map Ψ_β, in time poly(dnk), that reduces the variables of C from n to k and preserves nonzeroness. Once this is done, the poly(nd^k) blackbox PIT follows from the brute-force hitting-set.

7.3.2 Depth $\ge$ 3 Results

Looking at the success of bounded fanin depth-3 one wonders about the analogous depth-4 model:

$$C = \sum_{i\in[k]} \prod_{j\in[d]} f_{i,j}, \quad \text{where } f_{i,j} \text{ are sparse polynomials.} \tag{7.1}$$

[3] We mean $C_{[i]} := \sum_{j\in[i]} T_j$.

Here we are thinking of a bounded k. But now even $k = 2$ seems non-trivial! In fact, a simpler PIT case than this is an old open question in a related area [vzG83].

This *bounded top fanin* depth-4 PIT is an important open question currently. What is doable are other restricted models of depth-4. Inspired from the last subsection we ask: Is there a notion of 'rank' for general polynomials, are there easy 'faithful' maps, and finally is all this useful in PIT?

There are several notions of rank in commutative algebra. The one we [BMS13] found useful is—*transcendence degree* (trdeg). We say that a set S of polynomials $\{f_1, \dots, f_m\} \subset \mathbb{F}[x_1, \dots, x_n]$ is *algebraically dependent* if there exists a nonzero annihilating polynomial $A(y_1, \dots, y_m)$, over $\mathbb{F}$, such that $A(f_1, \dots, f_m) = 0$. The largest number of algebraically *in*dependent polynomials in S is called $\mathrm{trdeg}(S)$. With this notion we call a homomorphism φ *faithful* if $\mathrm{trdeg}(S) = \mathrm{trdeg}(\varphi(S))$. The usefulness of φ (assuming that one can come up with it efficiently) was first proved in [BMS13]:

Lemma 3.3 (Faithful is useful) *Let φ be a homomorphism faithful to $\mathbf{f} = \{f_1, \dots, f_m\} \subset \mathbb{F}[\mathbf{x}]$. Then for any $C \in \mathbb{F}[\mathbf{y}]$, $C(\mathbf{f}) = 0 \Leftrightarrow C(\varphi(\mathbf{f})) = 0$.*

This implies that we can use a faithful map to 'reduce' the number of variables n without changing the nonzeroness of C. The strategy can be used in cases where $\mathrm{trdeg}(\mathbf{f})$ is small, say, smaller than a constant r.

The only missing piece is the efficiency of φ.[4] To do this we need three fundamental ingredients—an efficient criterion for algebraic independence (Jacobian), its behaviour under φ (chain rule), and standard maps (Vandermonde and Kronecker-based).

Lemma 3.4 (Jacobian criterion) *Let $\mathbf{f} \subset \mathbb{F}[\mathbf{x}]$ be a finite set of polynomials of degree at most d, and* $\mathrm{trdeg}(\mathbf{f}) \le r$. *If* $\mathrm{char}(\mathbb{F}) = 0$ *or* $\mathrm{char}(\mathbb{F}) > d^r$, *then* $\mathrm{trdeg}(\mathbf{f}) = \mathrm{rk}_{\mathbb{F}(\mathbf{x})} \mathcal{J}_{\mathbf{x}}(\mathbf{f})$, *where* $\mathcal{J}_{\mathbf{x}}(\mathbf{f}) := \left(\partial f_i / \partial x_j\right)_{m \times n}$ *is the* Jacobian matrix.

There are several proofs of this, see [Jac41, For91, BMS13, MSS12]. This gives us an efficient way to capture trdeg, when the characteristic is zero/large. Let us now see how the Jacobian matrix changes under φ.

Lemma 3.5 (Chain rule) $\mathcal{J}_{\mathbf{y}}(\varphi(\mathbf{f})) = \varphi\left(\mathcal{J}_{\mathbf{x}}(\mathbf{f})\right) \cdot \mathcal{J}_{\mathbf{y}}(\varphi(\mathbf{x}))$, *where φ applied to a matrix/set refers to the matrix/set obtained by applying φ to every entry.*

This is a simple consequence of the chain rule of 'derivatives'. It suggests that for φ to preserve the trdeg of the polynomials, we need to control—(1) the image of the original Jacobian under φ, and (2) the Jacobian of the image of $\mathbf{x}$. In our applications, the former is achieved by a Kronecker-based map (i.e. sparse PIT tricks, e.g. [BHLV09]) and the latter by Vandermonde map (as seen in the previous subsection).

This general 'recipe' has been successfully implemented to various circuit models. The case of the circuit $C'(\mathbf{x}) := C(\mathbf{f})$, where $\mathrm{trdeg}(\mathbf{f}) \le r$ and f_i's are polynomials

[4] It can be shown, from first principles, that a faithful r-variate map always *exists* [BMS13].

of sparsity at most s, was worked out in [BMS13]. The proof follows exactly the above strategy. The time complexity is polynomial in $\text{size}(C')$ and $(s \cdot \deg(C'))^r$, where the exponential dependence comes from the sparsity estimate of $\mathcal{J}_{\mathbf{x}}(\mathbf{f})$ (and of course the final brute-force hitting-set for the r-variate $\varphi(C')$).

Agrawal et al. [ASSS12] extended the recipe to depth-4 circuits (7.1) where the number of $f_{i,j}$'s where any variable appears is bounded by r.[5] This model is called *occur-r* depth-4; it generalizes the well-studied multilinear read-r depth-4. Interestingly, slightly modified techniques also provided *exponential* lower bounds against these special models. This required proving some combinatorial properties of the derivatives of immanant (e.g. permanent, determinant).

The faithful maps recipe has been able to unify all the assorted *poly-time* hitting-sets known. However, one drawback is that it needs the characteristic to be zero/large. Baby steps in resolving that issue have been taken by [MSS12].

7.4 Rank Concentration, Shift, Hitting-Sets

The hitting-sets that we saw till now were for models where some parameter was kept bounded. But we could also study models with a 'structural' restriction, e.g. multilinearity. This route has also been successful and enlightening. We call a depth-3 circuit $C = \sum_i T_i$ *multilinear* if the linear factors in T_i involve disjoint variables. Hence, each product gate T_i induces a partition $\mathcal{P}_i$ on the variables (or indices) $[n]$. Moreover, we call C *set-multilinear* if these partitions are the same across all T_i's.

There is a large body of work on the set-multilinear model [RS05, AMS10, FS12, FS13b, ASS13, FS13a, FSS13, AGKS13]. The motivation for this model is, on the one hand, the algebraic concept of *tensors*, and, on the other hand, the interest in read-once *boolean* branching programs [Nis92, IMZ12, Vad12]. Interestingly, [FSS13] has shown (extending the ideas of [ASS13]) that the *current* situation in the arithmetic world is exponentially better than that in the boolean one!

Here we will exhibit the key ideas of [ASS13] and [AGKS13] on two *toy* cases that are already quite instructive; this saves us from the gory technical machinery that drives the more general cases.

7.4.1 Multilinear ROABP

Agrawal et al. [ASS13] gave the first quasi-poly-time hitting-set for set-multilinear depth-3 (and extensions to constant-depth, non-multilinear versions). This was generalized by [FSS13] to *any* depth; in fact, they dealt directly with the *multilinear ROABP* $D = \prod_i L_i$ over $M_w(\mathbb{F})$, where L_i's are linear polynomials in disjoint variables. Both the papers proved 'low-support rank concentration' in their models.

[5] Note that this does not mean that $\text{trdeg}(f_{i,j} | i, j)$ is bounded.

For the following discussion we fix a base commutative ring $R = H_w(\mathbb{F})$ called the *Hadamard* algebra (instead of the $w \times w$ matrix algebra). This is basically $(\mathbb{F}^k, +, \star)$, where $+$ is the vector addition and $\star$ is the coordinate-wise vector product (called the Hadamard product).

ℓ-**concentration.** We say that a polynomial $f \in R[x_1, \ldots, x_n]$ is ℓ-*concentrated* if

$$\mathrm{rk}_{\mathbb{F}}\{\mathrm{coef}_f(x_S) \mid S \subseteq [n], |S| < \ell\} = \mathrm{rk}_{\mathbb{F}}\{\mathrm{coef}_f(x_S) \mid S \subseteq [n]\},$$

where coef_f extracts a coefficient in f.

I.e. the coefficient-vectors of 'lower' monomials already span every possible coefficient-vector in f. We are interested in studying whether circuits compute an ℓ-concentrated polynomial for small ℓ (say, $\log n$ instead of n). By itself this is not true, e.g. the trivial circuit $D = x_1 \cdots x_n$ is not even n-concentrated. But, maybe we can transform f a bit and then attain $(\log n)$-concentration? In this case, $D' := D(x_1 + 1, \ldots, x_n + 1)$ is suddenly 1-concentrated!

It was shown by [ASS13] that any D, above R, becomes $(\log k)$-concentrated after applying a 'small' shift; the price of which is $n^{\log k}$ time. Once we have this it directly applies to the set-multilinear depth-3 model. Since, a depth-3 $C = \sum_{i \in [k]} T_i$ can be rewritten as $C = [1, \ldots, 1] \cdot D$, where $D = \begin{bmatrix} T_1 \\ \vdots \\ T_k \end{bmatrix}$ is of the promised sort over $R = H_k(\mathbb{F})$ (since D completely factorizes into disjoint-variate linear polynomials). So, ℓ-concentration in D implies an easy way to check C for zeroness—test the coefficients of the monomials below ℓ-support in C.

Glimpse of a proof We now show how to achieve ℓ-concentration, $\ell = O(\log k)$, in the following toy model:

$$D = \prod_{i \in [n]} (1 + z_i x_i), \quad \text{where } z_i \in H_k(\mathbb{F}). \tag{7.2}$$

Because of the disjointness of the factors it can be seen, as a simple exercise, that: D is ℓ-concentrated iff $D_S := \prod_{i \in S}(1 + z_i x_i)$ is ℓ-concentrated, for all $S \in \binom{[n]}{\ell}$. Thus, from now on we assume, wlog, $n = \ell$.

Shift D by formal variables $\mathbf{t}$, and normalize, to get a new circuit:

$$D' = \prod_{i \in [\ell]} (1 + z_i' x_i), \quad \text{where } z_i' \in H_k(\mathbb{F}(\mathbf{t})).$$

We can express the new coefficients as:

$$z_i' = z_i/(1 + z_i t_i), \ \forall i \in [\ell].$$

Conversely, we write:

$$z_i = z_i'/(1 - z_i' t_i), \ \forall i \in [\ell]. \tag{7.3}$$

We write z_S for $\prod_{i\in S} z_i$. Now the goal is to 'lift' an $\mathbb{F}$-dependence of z_S's to the z'_S; which ultimately shows the condition on the shift that shall yield concentration.

Consider the 2^ℓ vectors $\{z_S \mid S \subseteq [\ell]\}$. If $\ell > \log k$ then there is a nontrivial linear dependence amongst these vectors, say,

$$\sum_{S\subseteq[\ell]} \alpha_S z_S = 0, \quad \text{where } \alpha_S \in \mathbb{F}.$$

Rewriting this in terms of z'_S we get:

$$\sum_{S\subseteq[\ell]} \alpha_S \cdot \prod_{i\in S} z'_i/(1 - z'_i t_i) = 0.$$

$$\text{Or,} \sum_{S\subseteq[\ell]} \alpha_S \cdot z'_S \cdot \prod_{i\in[\ell]\setminus S} (1 - z'_i t_i) = 0. \tag{7.4}$$

Let us collect the 'coefficient' of $z'_{[\ell]}$ in the above expression. It comes out to,

$$\sum_{S\subseteq[\ell]} \alpha_S \cdot (-1)^{|[\ell]\setminus S|} \cdot t_{[\ell]\setminus S}. \tag{7.5}$$

If we can ensure this expression to be nonzero then Eq. (7.4) tells us that $z'_{[\ell]}$ is in the $\mathbb{F}(\mathbf{t})$-span of the 'lower' z'_S. But, ensuring the nonzeroness of Eq. (7.5) is easy—use t_i's such that all the $(\leq\ell)$-support monomials t_S are *distinct*. We can use standard sparse PIT tricks [BHLV09] for this, in time poly(n^ℓ).

What we have shown is that, after applying a Kronecker-based shift, the circuit D becomes ℓ-concentrated; all this in time $n^{O(\log k)}$. This 'recipe' of studying the generic shift, via some combinatorial properties of the 'transfer' equations (7.3), is generalized in [ASS13] to other D; and further improved in [FSS13] to multilinear ROABP. The latter use a 'primal' interpretation of the 'transfer' matrix and show that the linear transformation– corresponding to a Kronecker shift together-with the truncation of the high-support monomials –behaves like a *rank-extractor*.

It is not known how to design such hitting-sets, even for the toy case, in *poly*-time.

7.4.2 Towards Multilinear Depth-3

It is tantalizing to achieve ℓ-concentration in multilinear depth-3 (before embarking on the general depth-3!). A partial result in that direction was obtained in [AGKS13]. We will sketch their ideas in a toy model.

Consider a multilinear depth-3 circuit C with only *two* partitions being induced by the product gates—$\mathcal{P}_1 = \{\{1\}, \cdots, \{n\}\}$ and an arbitrary partition $\mathcal{P}_2$. Say, the number of the corresponding product gates is k_1 respectively k_2 (summing to k).

We can say, naturally, that $\mathcal{P}_1$ is a *refinement* of $\mathcal{P}_2$ (denoted $\mathcal{P}_1 \le \mathcal{P}_2$) because: For every color (or part) $S \in \mathcal{P}_2$ there exist colors in $\mathcal{P}_1$ whose union is *exactly* S. In this refinement situation [AGKS13] showed that, again, a suitable shift in the $\prod\sum$ circuit D (corresponding to C) achieves ℓ-concentration in time poly$(n^{\log k})$.

Glimpse of a proof We can assume $\mathcal{P}_2$ different from $\mathcal{P}_1$, otherwise this case is no different from the last subsection. We assume that the first k_1 product gates in $C = \sum_{i\in[k]} T_i$ respect $\mathcal{P}_1$ and the rest k_2 respect $\mathcal{P}_2$. The corresponding circuit D where we desire to achieve concentration is $D = \begin{bmatrix} T_1 \\ \vdots \\ T_k \end{bmatrix}$ over $R = H_k(\mathbb{F})$. But now the linear factors of D are not necessarily in disjoint variables. Example $\begin{bmatrix} x_1x_2 \\ x_1+x_2 \end{bmatrix} = \left(x_1 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot x_2\right) \cdot \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot x_2\right)$ over $H_2(\mathbb{F})$.

To get some kind of a reduction to the set-multilinear case, we prove rank concentration in parts. First, we consider those monomials (called $\mathcal{P}_1$-*type*) that could only be produced by the 'upper' part of D (i.e. the first k_1 product gates of C). Such a monomial, say indexed by $S \subseteq [n]$, is characterized by the presence of $i, j \in S$ that are in the same color of $\mathcal{P}_2$. For a fixed such i, j we can "access" all such monomials by the derivative $\partial^2 D/\partial x_i \partial x_j =: \partial_{i,j} D$. Notice that this differentiation kills the 'lower' part of D and only the $\mathcal{P}_1$-part remains. So, we can prove $(2 + \log k_1)$-concentration in the monomials containing i, j as in Sect. 7.4.1. This proves $O(\log k_1)$-concentration in the monomials of $\mathcal{P}_1$-type.

Next, we want to understand the remaining monomials (called $\mathcal{P}_2$-type); those that could be produced by the 'lower' part of D (i.e. the last k_2 product gates of C). These, obviously, could also be produced by the upper part of D. Let us fix such a monomial, say $x_1 \cdots x_\ell$. Assume that $S_1, \ldots, S_\ell \in \mathcal{P}_2$ are the colors that contain one of the indices $1, \ldots, \ell$. Consider the subcircuit D_ℓ that in its i-th coordinate, $\forall i \in [k]$, simply drops those factors of T_i that are free of the variables $S_1 \cup \cdots \cup S_\ell$. The problem here is that D_ℓ may be a 'high' degree circuit ($\approx n$ instead of ℓ) and so we cannot use a proof like in Sect. 7.4.1.

But, notice that all the degree-($\ge \ell$) monomials in D_ℓ are $\mathcal{P}_1$-type; where we know how to achieve ℓ-concentration. So, we only have to care about degree-($\le\ell$) $\mathcal{P}_2$-type monomials in D_ℓ. There, again, $(\log k)$-concentration can be shown using Sect. 7.4.1 and the well-behaved transfer equations.

This sketch, handling two refined partitions, can be made to work for significantly generalized models [AGKS13]. But, multilinear depth-3 PIT is still open (nothing better than exponential time known).

Remark 4.1 Using a different technique [AGKS13] also proves *constant*-concentration, hence designs *poly*-time hitting-sets, for certain constant-width ROABP. These models are arithmetic analogs of the *boolean* ones—width-2 read-once branching programs [AGHP92, NN93] and constant-width read-once permutation branching programs [KNP11].

7.5 Open Ends

The search for a strong enough technique to study arithmetic circuits continues. We collect here some easy-to-state questions that interest us.

Top fanin-2 depth-4 Find a faithful map φ that preserves the algebraic independence of two products-of-sparse polynomials $\prod_i f_i$ and $\prod_j g_j$. If we look at the relevant 2×2 Jacobian determinant, say wrt variables $X := \{x_1, x_2\}$, then the question boils down to finding a hitting-set for the special *rational* function $\sum_{i,j} \frac{\det \mathcal{J}_X(f_i, g_j)}{f_i g_j}$. Can this version of *rational sparse* PIT be done in subexponential time?

Independence over $\mathbb{F}_p$ Currently, there is no subexponential time algorithm/heuristic known to test two given circuits for algebraic independence over a 'small' finite field $\mathbb{F}_p$. The reason is that something as efficient as the Jacobian criterion is not readily available, see [MSS12].

Model in Eqn (7.2) Find a *poly*-time hitting-set for this simple model. Note that a poly-time whitebox PIT is already known [RS05].

Multilinear depth-3 Achieve $o(n)$-concentration in multilinear depth-3 circuits, in $n^{o(n)}$ time. Here, the presence of an exponential lower bound against the model [RY09] is quite encouraging.

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Chapter 8
Malod and the Pascaline

Bruno Poizat

Presented to Somenath Biswas on the occasion of his 60th birthday

Abstract We make explicit the central role played by the binomial coefficients in the description of the coefficient-function of a polynomial computed in polynomial time (no bound on the degree), following the works of Guillaume Malod. Our results are obtained with the help of a universal polynomial which is simpler than the one used by Malod. As a corollary, with the help of a result of Peter Bürgisser, we establish in characteristic zero a connection between Leslie Valiant's question $\mathsf{VNP} = ?\mathsf{VP}$ (bounded degree) and its unbounded degree version, generalizing what Malod had previously done in finite characteristic.

Keywords Complexity · Polynomials · Summations

8.1 Polynomials, Functions and Arithmetic Circuits

We compute polynomials, in several variables, with integer coefficients. For that, we consider *arithmetic circuits*, with input gates labeled either by a variable or by the constant -1; the other gates are binary (they receive two arrows) and perform either

Guillaume Malod defended his doctorate in 2003 in Lyon; he visited Kanpur as a participant to a cooperation program between IITK and Claude Bernard University. Blaise Pascal (1623–1662), philosopher, mathematician and physicist; he discovered the recurrence relation between the binomial coefficients ("Triangle de Pascal"), and invented the Pascaline, a mechanical device performing arithmetic operations.

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an addition or a multiplication; there is only one output gate, where is obtained the polynomial computed by the circuit.

The size of the circuit is the number of its operation gates, and the *complexity* of a polynomial is the minimal size of a circuit computing it. We note that, contrarily to a widely admitted convention in Valiant's style calculus, we do not consider that the constants are given for free: under our convention, if a big integer N is needed for the computation of a polynomial f, the number of steps necessary to obtain N starting from the constant -1 should be included in the complexity of f.

A polynomial of complexity c has at most $c + 1$ variables; its degree is bounded by 2^c; it is the sum of less than $2^{c(c+1)}$ monomials, whose coefficients are integers with absolute value bounded by 2^{2^c}. For more details on circuits you may consult [Poi95] or [Bür00], and the survey chapter of Meena Mahajan in this volume.

A special kind of circuits are the *terms* (some say *formulae*), of an arborescent nature, in which each gate is allowed to throw only one arrow. The degree of a polynomial computed by a term of size c is bounded by $c + 1$, and its coefficients are bounded by $2c$. The *termic complexity* of a polynomial is the minimal size of a term which computes it.

Thank to the parallelization lemma of [MP76], terms correspond to computations in logarithmic depth: an (arithmetic) term of size c can be replaced by an equivalent circuit (or term!) of depth $O(\log c)$, computing the same polynomial.

We also consider functions, which by definition are applications from $\{0, 1\}^n$ into $\mathbb{Z}$; a function is *boolean* if it takes its values in $\{0, 1\}$. It is easily seen that any function is the restriction to $\{0, 1\}^n$ of some polynomial with integer coefficients, and we define the complexity of the function as the minimal complexity of such a polynomial, that is, as the minimal size of an arithmetic circuit computing the function when we restrict its entry variables to the $\ll$ boolean $\gg$ values 0 and 1. This gives a fair evaluation of the usual complexity of a boolean function since, on one hand, one can consider the arithmetic circuit modulo 2, and on the other addition and multiplication modulo 2 can be simulated in characteristic zero by $x + y - 2 \cdot x \cdot y$ and $x \cdot y$, respectively. This is also the case for termic complexity, thanks to parallelization.

We use letters like $x, y, z, \dots$ to denote the variables in our polynomials; letters like $u, v, w, \dots$ are used for *boolean variables*, that is, variables that are assumed to take only the values 0 and 1.

8.2 Two Functions, Pascaline and Factorial

To avoid a conflict between french and english notations, we denote by $C(m, n)$ the number of subsets with n elements of a set with m elements. The n'th instance of the Pascaline is the following function, depending of $2n + 2$ boolean variables: $\mathrm{Pasc}_n(u_0, \dots, u_n; v_0, \dots, v_n) = C(U, V)$, where U and V are the two numbers whose developments in binary figures are respectively $\underline{u}$ and $\underline{v}$; in other words, $U = u_0 + 2 \cdot u_1 + \cdots + 2^n \cdot u_n$ and $V = v_0 + 2 \cdot v_1 + \cdots + 2^n \cdot v_n$.

We consider also the Factorial, which is defined by: $\mathrm{Fact}_n(u_0, \dots, u_n) = U!$.

These numbers are admittedly big, but their size (they are twice exponential) is not in itself an obstacle to a polynomial complexity. Nevertheless, it seems very unlikely, or at least highly unwishable, that they be computable in a polynomial number of steps, since this would provoke a cryptographic tsunami.

Let us first observe that, if the Pascaline has a polynomial complexity, the same is true of the Factorial. Indeed, from the identity $(2 \cdot U)! = C(2 \cdot U, U) \cdot (U!)^2$ we deduce the following induction formula: $\text{Fact}_n(u_0, u_1, \ldots, u_n) = (1 + u_0(u_1 \cdot 2 + \cdots + u_n \cdot 2^n)) \cdot C_n(0, u_1, \ldots, u_n; u_1, \ldots, u_n, 0) \cdot (\text{Fact}_{n-1}(u_1, \ldots, u_n)^2)$.

In the other direction, it is not clear that a fast computation for the Factorial would provide the same for the Pascaline, in the absence of division.

If the Factorial were easy to compute, then factorization would be possible using the following well-known algorithm: consider two positive integers $U < V$, given in figures; compute $U!$ modulo V, and $\gcd(U!, V)$: this decides whether or not V has a factor smaller than U; in a small number of steps we localize the smallest factor of V, and finally discompose it.

We also consider two boolean functions, depending on some more boolean variables, which are the Pascaline and the Factorial in figures, defined by:

$\text{Pascfig}_n(u_0, \ldots, u_n; v_0, \ldots, v_n; w_0, \ldots, w_n) =$ the W° digit of the representation of $\text{Pasc}_n(u_0, \ldots, u_n; v_0, \ldots, v_n)$ in base 2, where $W = w_0 + 2 \cdot w_1 + \cdots + 2^n \cdot w_n$

$\text{Factfig}_n(u_0, \ldots, u_n; w_0, \ldots, w_{n+\log(n)}) =$ the W° digit of $\text{Fact}_n(u_0, \ldots, u_n)$

These two (sequences of) functions belong to PSPACE. The best argument for that is that every sequence of boolean functions appearing in an algorithmic context belongs to PSPACE, unless it is manufactured to be a counter-example; for something more formal, you may consult this time [Poi08] or [Bür09].

8.3 Polynomials as Sums of Monomials

A Valiant summation is an expression of the form:

$$f(x_1, \ldots, x_n) = \sum_{\underline{u} \text{ boolean}} \varphi(u_0, \ldots, u_m; x_1, \ldots, x_n)$$

where φ is a polynomial in which we have separated the variables into two blocks. This summation of exponentially many terms should have an explosive effect on the complexity of polynomials, but this is an open question, We define the Σ-*complexity* of f as the minimal complexity of φ. It seems unlikely that the Pascaline be of polynomial Σ-complexity, but this fact, if false, should be hard to refute since it is implied by P = PSPACE (the Pascaline is obtained from the Pascaline in figures by a summation; see below the details on the exponential polynomial).

If it were true, it is not clear that it would imply a low Σ-complexity for the Factorial, because the induction formula contains a square; the Fubini formula $\left(\sum_{\underline{u}} \varphi(\underline{u})\right) \cdot \left(\sum_{\underline{v}} \psi(\underline{v})\right) = \sum_{\underline{u},\underline{v}} \varphi(\underline{u}) \cdot \psi(\underline{v})$ is valid only if the tuples of variables $\underline{u}$ and $\underline{v}$ are disjoint; in case of a squaring, $\left(\sum_{\underline{u}} \varphi(\underline{u})\right)^2 = \sum_{\underline{u},\underline{v}} \varphi(\underline{u})\cdot\varphi(\underline{v})$, so that the computation of φ must be done twice: when we repeat n times, we explode. Similarly, we can define the Π-complexity, replacing summations by productions; it is equally unclear if the Pascaline, as the Factorial, has a polynomial Π-complexity.

A polynomial is the sum of its monomials. This basic truth explains the importance of summations. Let us first define exponential as the following polynomial, where $\underline{y} = (y_0, \dots, y_m)$:

$$x^y = (y_0 \cdot x + 1 - y_0) \cdot (y_1 \cdot x^2 + 1 - y_1) \cdots (y_i \cdot x^{2^i} + 1 - y_i) \cdots (y_m \cdot x^{2^m} + 1 - y_m)$$

Observe that if we give boolean values to $\underline{y}$, then $x^{\underline{u}} = x^U$, where U is the integer $U = u_0 + 2 \cdot u_1 + \cdots + 2^n \cdot u_n$. In these conditions, a polynomial $f(\underline{x})$ depending of n variables of degree at most 2^m can be written as:

$$f(\underline{x}) = \sum_{\underline{u}} \mathrm{Cf}(\underline{u}_1; \dots; \underline{u}_n) \cdot x_1^{\underline{u}_1} \dots x_n^{\underline{u}_n}$$

where $\mathrm{Cf}(\underline{u})$ is a function depending of $m \cdot n$ boolean variables, that we call the coefficient-function of $f(\underline{x})$.

Since polynomials correspond bijectively to their coefficient-functions, it is tempting to establish a correlation between the complexities of these two kinds of objects. It is easy to bound the complexity of $x^{\underline{y}}$ by 6 m, so that a polynomial Σ-complexity for the coefficient-function implies a polynomial Σ-complexity for the polynomial. What about the other direction? We shall see that the Pascaline plays a crucial role in this question.

8.4 Operations on the Polynomials and their Effect on the Coefficient-Functions

Given a polynomial $f(y, \underline{x})$, what is the effect of a projection, i.e., of the substitution of some of the variables by constants, on the coefficient-function? When we replace the variable y by the constant a, how do we obtain the coefficient-function of $g(\underline{x}) = f(a, \underline{x})$ from the coefficient-function of $f(y, \underline{x})$? We must reconstruct partially the polynomial: if we note $\underline{v}$ the boolean variables describing the degree of the variable y, and leave the others in the dark, $\mathrm{Cg} = \sum_{\underline{v}} \mathrm{Cf}(\underline{v}) \cdot a^{\underline{v}}$. The substitution has therefore only a polynomial effect on the Σ-complexity of the coefficient-functions.

This is not the case of the termic Σ-complexity, since the exponential involves a^{2^n} which is obtained by a succession of n squarings. Fortunately enough, there are three integers whose powers remain at a reasonable distance:

- $a = 0$; Cg is the constant term, $\mathrm{Cg} = \mathrm{Cf}(\underline{0})$
- $a = 1$; all the powers are equal to 1, $\mathrm{Cg} = \sum_{\underline{v}} \mathrm{Cf}(\underline{v})$
- $a = -1$; we change the sign according to parity, $\mathrm{Cg} = \sum_{\underline{v}} (1 - 2v_0) \cdot \mathrm{Cf}(\underline{v})$

We have no increase of Σ-complexity, even termic, in the first two cases, and a minuscule one in the third. Similarly, when we reconstruct a polynomial with boolean variables, satisfying $u^n = u$ when $n > 0$, there is no need to rise them to powers, so that a summation will have only a benign effect on the Σ-complexity, and on the termic Σ-complexity, of the coefficient-functions.

We note in passing, although we shall not use this kind of operations, that replacements of variables also have a mild effect: for instance, when we substitute z to x and y in $f(x, y)$, to obtain $g(z) = f(z, z)$, $\mathrm{Cg}(\underline{w}) = \sum_{\underline{u},\underline{v}} S(\underline{u}, \underline{v}, \underline{w}) \cdot \mathrm{Cf}(\underline{u}, \underline{v})$, where S is an easily computable boolean function taking the value 1 if $W = U + V$, and 0 if not.

There is a similar formula for the product of two polynomials, so that the coefficient-function of a polynomial computed by a circuit of size c will be computed by a circuit of comparable size with summation gates deeply buried in it; but in general there is no known way to drag smoothly these summation gates at the output of the circuit (this is possible when the circuit is multiplicatively disjoint; see the definition in Sect. 8.6). Circuits with summation gates define $\mathsf{VPSPACE}$, the analog for polynomials of PSPACE for boolean functions: this is the only class above the class VNPmd^0 (that will be defined in Sect. 8.6) that we are certain to be closed by taking coefficient-functions; for the details, see [Poi08]; see also [KP07] for an approach of the class $\mathsf{VPSPACE}$ via coefficient-functions in figures.

Now the Pascaline again. Consider the polynomial $b(x, \underline{y}) = (1 + x)^{\underline{y}} = (y_0 \cdot (1+x)+1-y_0) \cdot \dots \cdot (y_n \cdot (1+x)^{2^n} + 1 - y_n)$, whose complexity is less than $6(n+1)$. Giving to $\underline{y}$ a boolean value $\underline{w}$, by identification in the binomial formula $(1+x)^{\underline{w}} = \sum_{\underline{v} \leq \underline{w}} \mathrm{Pasc}_n(\underline{w}; \underline{u}) \cdot x^{\underline{u}}$ we obtain $\mathrm{Pasc}_n(\underline{w}, \underline{u}) = \sum_{\underline{v}_0,\dots,\underline{v}_n} \mathrm{Cb}(\underline{u}, v_0, \dots, v_n) \cdot w_0^{v_0} \cdot \dots \cdot w_n^{v_n}$, so that we have an easy polynomial bound of the Σ-complexity of the Pascaline in function of the Σ-complexity of the coefficient-function of $b(x, \underline{y})$. Note in passing that such a simple operation as the substitution of x by $1 + x$ may have a drastic effect on the complexity of the coefficient-function.

In conclusion, if the Σ-complexity of the coefficient-function can be polynomially bounded in function of the Σ-complexity of the polynomial, then the Pascaline has a polynomial Σ-complexity; to establish the reciprocal, we shall construct in the next section a universal polynomial whose coefficient-function has a simple expression in function of the Pascaline.

Remark Since $b(x, \underline{y})$ is a product, its coefficient-function can be expressed as a summation from the coefficient-functions of the factors. This give an expression, with a summation, of the Pascaline in function of $\Gamma_n(\underline{v}) = C(2^n, V)$: the Pascaline has a polynomial Σ-complexity iff this is true also for this function. There is no mystery in this reduction: it is closely related to the formula $C(U_1 + \dots + U_k, V) = \sum_{V_1+\dots+V_k=V} C(U_1, V_1) \cdot \dots \cdot C(U_k, V_k)$, which generalizes Pascal's Triangle. There is no apparent induction on V leading to a computation of the Pascaline with the help of the constants $C(2^i, 2^j)$.

8.5 A Universal Polynomial

We consider the polynomial P_n depending of $n(n^2 - 1)/6$ variables x_{ijk}, where $0 \leq k < j < i \leq n$, which is defined by the following induction:

$$P_0 = P_1 = 1$$
$$P_n(\dots x_{ijk}, \dots) = \sum_{\{(i,j) | 0 \leq j < i < n\}} x_{nij} \cdot P_i \cdot P_j$$

We say that n is the *head* of the variable x_{nij}, and that i and j are its *tails*.

This polynomial, of complexity less than $n^3/2$, is able to simulate any circuit of size less than $n/4$ just by replacing some of its variables by 0, 1 or -1. To see that, we consider circuits as straight line programs, that is, we order their gates in such a way that each operation gate receive its two arrows from anterior gates. If we want that P_i simulate an input gate labeled by the variable x_{i10}, we equate all the other variables of head i to 0. If we want to simulate the product of P_j and P_k, where $1 < k < j < i$, we equate x_{ijk} to 1 and the other variables of head i to 0. If we want to simulate the addition of P_j and P_k, where $1 < j < i$ and $1 < k < i$, we equate x_{ij1} and x_{ik0} to 1, and the other variables of head i to 0. The only constraint in the simulation is that we cannot multiply directly a gate by itself: we must before duplicate it by a neutral operation, such that a multiplication by 1, tripling (yes) the size of the circuit; note that the entry gates count in the length of the straight line program, but not in the size of the circuit: all this explains the bound $n/4$.

There is no need to make substitutions of variables, since we can assume that, in a circuit, the variables are associated injectively to the input gates; but of course one of the variable has to be replaced by -1.

Let us now evaluate the coefficient-function of P_n. When we develop brutally P_n, just by distributing the product on the sum, we express it has the sum of a twice exponential family of products of variables that we call *arborescent monomials*; the actual expression of P_n as a sum of monomials is obtained by grouping together the arborescent monomials corresponding to a same monomial, those which have the same degree in each of the variables.

The arborescent monomials are obtained as follows: we start from a variable x_{nij} whose head is n, then we multiply it by a variable of head i and a variable of head j (if possible, that is if $i > j \geq 2$), and repeat the process till we reach at the extremity of every branch a terminal variable x_{k01}; when the small tail is 0 or 1, but not the big one, there is only one variable that follows.

In the following example, for $n = 6$, we do not write the x's, but only their indexes; the monomial associated to this arborescent monomial is:

$$654.540.(432)^2.321.310.(210)^3.$$

$$\begin{tikzcd}
 & & & 321 \arrow[r] & 210 \\
 & 540 \arrow[r] & 432 \arrow[ur] \arrow[dr] & & \\
 & & & 210 & \\
654 \arrow[ruu] \arrow[rdd] & & & & \\
 & & 310 & & \\
 & 432 \arrow[ur] \arrow[dr] & & & \\
 & & 210 & &
\end{tikzcd}$$

The condition for a monomial, given by the expansion in figures of the degree of each of the variables, to be associated to an arborescent monomial is easily determined: the largest index n must appear only once, and in head position, and all the others i, $n > i \geq 2$, must appear as many times in head position than in tail position. This condition can be expressed by a boolean function of the degrees of low termic complexity, taking the value 1 when satisfied and 0 if not; let us call it the compatibility condition.

To check that this condition is sufficient, we observe that, before placing the variables of head i, we can place the variables of head bigger than i; this done, we can add to the tree the variables of head i, since the number of tail position corresponds, and then the variables of tail $i - 1$, etc.

For instance, in the example above there is only one second arborescent monomial giving the same monomial (therefore the coefficent of this monomial is 2), the only degree of freedom being the choice for the placement of the two variables with head 3. This second arborescent monomial is:

$$\begin{tikzcd}
 & & & 310 \\
 & 540 \arrow[r] & 432 \arrow[ur] \arrow[dr] & \\
 & & & 210 \\
654 \arrow[ruu] \arrow[rdd] & & & \\
 & & 321 \arrow[r] & 310 \\
 & 432 \arrow[ur] \arrow[dr] & & \\
 & & 210 &
\end{tikzcd}$$

To compute the coefficient-function of P_n, we must count how many arborescent monomials are produced by a given monomial. If we note C_i the number of ways to place the variables of head i once the variables of bigger head are placed, number that is independent of the anterior (and posterior) placements, then the coefficient of the monomial will be the product of the C_i: this is only a matter of matching heads and tails.

To compute C_i, let us note k the number of occurencies of i as tail; k is less than 2^n. The variables with head i are splitted in s sorts of respective degrees $k_1, \dots, k_s$, and $k = k_1 + \cdots + k_s$; s is smaller than $i(i-1)/2$. We must choose the k_1 i's in tail position that will be followed by a variable of the first kind, making $C(k, k_1)$ choices; then we must affect the variables of the second sort, making $C(k - k_1, k_2)$ choices, and so on. Taking into account the fact that $C(0, 0) = 1$, which permits the neutralization of certain binomials coefficients, we see that C_i is expressible as the product of $i(i-1)/2$ pascalines of size n, whose arguments are numbers in figures easily computable from the degrees: they have a polynomial term complexity.

In short, the coefficient-function of P_n is expressible as the product of the compatibility condition and of less than $n^3/6 \geq \sum i(i-1)/2$ pascalines of size n.

Since the coefficient-function of a polynomial of complexity n is obtained by the following operations from the coefficient-function of P_{4n}: equating to 0 the variables representing the degrees of the variables that we replace by 0, summing over the degrees of the variables that we replace by 1 or -1 (after a small multiplication concerning the unique variable that we replace by -1), it is now clear that if the Pascaline has a polynomial Σ-complexity, then the Σ-complexity of the coefficient-function of an arbitrary polynomial is polynomially bounded in function of the Σ-complexity of the polynomial.

8.6 A Short Review of Malod's Thesis

We shall further on follow a common (and vicious) tendency, to which we have resisted till now, to express complexity hypothesises with the help of sequences of polynomials, or functions, with asymptotic properties. We adopt the notations of [Mal03, Mal07].[1]

VPnb^0 denotes the class of sequences of polynomials whose complexity is polynomially bounded; V is for Valiant, P for polynomial time, nb indicates that there is no bound on the degree of the polynomials in the sequence, and ^{0} indicates that -1 is the only constant that is used in the computations. Note that the class is nonuniform: a sequence of boolean functions is in VPnb^0 if and only if it is in $\mathsf{P}/poly$.[2]

VNPnb^0 is the class of sequences of polynomials which are obtained by a summation from a sequence in VPnb^0; the letter N, which is not a fortunate choice, has been introduced in this context by Valiant as an analogy with the class NP. We have shown that VNPnb^0 is closed by taking the coefficient-functions if and only if the Pascaline belongs to it.

[1] We have resisted to a crave for changing them!.

[2] It is customary in Valiant's calculus to use nonuniform classes, based on circuit size; were we insist on uniformity for their Boolean counterparts, we could uniformise the Valiant's classes as well; but for what concerns us here, uniformity would play only a decorative role.

The classes VP and VNP in general use by Valiant and his followers differs from them by two aspects: on one hand, there is a polynomial bound on the degree of the polynomials in the sequence; on the other, the entries of the circuits may be labeled by arbitrary constants, living in an unprecised loka (integers are not always sufficient, since the inversion of 2 is essential in the celebrated proof of the universality of the Permanent).

Various devices have been invented preventing a circuit to produce a polynomial of exponential degree, but the truly adequate one has been defined by Malod: a circuit is multiplicatively disjoint if each of its multiplication gates receives its two arrows from disjoint subcircuits (note that a term is both additively and multiplicatively disjoint). A *multiplicatively disjoint* circuit of size c produces a polynomial of degree at most c, and in fact, when we allow arbitrary gratuitous constants, the class VP is equal to the class VPmd of sequences of polynomials with a polynomially bounded md-complexity (the first step of the reduction uses a computation by homogeneous parts, renouncing to any control on the constants; see the details in [MP06, MP08]).

We shall use the class VPmd^0 of sequences computed by multiplicatively disjoint circuits of polynomial size, using only the constant -1, and also its termic analogue VPt^0; it is not known if VPt^0 is a proper subclass of VPmd^0, but terms and md-circuits are equivalent in the presence of summations; in other words $\mathsf{VNPmd}^0 = \mathsf{VNPt}^0$; this last class is closed under taking coefficient-functions.

To study the unbounded case, Malod constructs a universal polynomial whose coefficient-function is computable provided that the Pascaline is so. He then observes that an old theorem of [Luc78] shows that the Pascaline is computable modulo p,[3] where p is a *fixed* prime number, to obtain the following results concerning computations of polynomials with coefficients in $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$:

(i) the class $\mathsf{VNPnb}^0 \pmod{p}$ is closed for taking coefficient-functions
(ii) any sequence in $\mathsf{VNPnb}^0 \pmod{p}$ is obtained from a sequence in $\mathsf{VNPmd}^0 \pmod{p}$ by replacing variables by (simply exponential) powers of variables
(iii) $\mathsf{VNPmd}^0 \pmod{p} = \mathsf{VPmd}^0 \pmod{p} \iff \mathsf{VNPnb}^0 \pmod{p} = \mathsf{VPnb}^0 \pmod{p}$.

The right to left implication of (iii) rests on the fact that the only constants appearing in a computation are the integers modulo p, which form a finite set.

With the help of our simpler universal polynomial, and a result of Peter Bürgisser, we shall say more on what happens in characteristic zero.

8.7 Characteristic Zero

Bürgisser [Bür09] shows that the hypothesis $\mathsf{VNPmd}^0 = \mathsf{VPmd}^0$ collapses a certain hierarchy within $\mathsf{PSPACE}/poly$, to which belongs the Pascaline in figures; in short, $\mathsf{VNPmd}^0 = \mathsf{VPmd}^0 \implies \text{Pascfig} \in \mathsf{P/poly}$.

[3] If we develop the integers m and n in base p , $C(m, n)$ is equal modulo p to the product of the binomial coefficients $C(u_i, v_i)$ of their digits.

Let us remind that, by the Parsimonious Reduction Lemma, to any boolean function $f(\underline{u})$, computed by an arithmetic circuit modulo 2 of complexity c, is associated a boolean function $g(\underline{u}, \underline{v})$, computed by an arithmetic term modulo 2 of size $4c$, such that, if $f(\underline{u}) = 0$, then $g(\underline{u}, \underline{v}) = 0$ for every $\underline{v}$, and if $f(\underline{u}) = 1$, then $g(\underline{u}, \underline{v}) = 0$ for every $\underline{v}$, except for one for which it takes the value 1 (the length of $\underline{v}$ is equal to c, and the exceptional $\underline{v}$ corresponds to the values that are computed at the gates of the circuit). After parallelization, the computation of g can be simulated in characteristic zero by a term of small complexity, and, because of the uniqueness of the exceptional $\underline{v}$, $f(\underline{u})$ is obtained by a summation in front of this last term (we do not claim that there is a way to simulate in characteristic zero summations in characteristic 2; see [Poi]). As a consequence, any sequence of boolean functions which is in VPnb^0, i.e., in $\mathsf{P/poly}$, is in VNPmd^0.

Assume that $\mathsf{VNPmd}^0 = \mathsf{VPmd}^0$. If, in the expression of the Pascaline as a summation from the Pascaline in figure we replace the numbers 2^{2^i} by variables y_i, we obtain therefore a (sequence of) multilinear polynomial(s) in VNPmd^0; using a second time our hypothesis, we see that this polynomial is in VPmd^0, and, after substitution of the y_i by the adequate powers of 2, that the Pascaline is in VPnb^0 (we remind that this implies an easy factorization). This is more than enough for VNPnb^0 to be closed for the coefficient-function.

Moreover, in the expression of a polynomial as the summation of its monomials, we can also replace the power $x_i^{2^j}$ of a variable x_i by a new variable x_{ij}. From our description of the coefficient-functions of the projections of the universal polynomial, we conclude that any sequence of polynomials in VNPnb^0 is obtained by plugging in a sequence in VNPmd^0 simply exponential powers of 2 and of variables. We reach the same conclusion as Malod in characteristic p, this time not as a fact, but as a consequence of an hazardous hypothesis.

The conclusion is that $\mathsf{VNPmd}^0 = \mathsf{VPmd}^0$ implies $\mathsf{VNPnb}^0 = \mathsf{VPnb}^0$.

The reciprocal is problematic: if $\mathsf{VNPnb}^0 = \mathsf{VPnb}^0$, VNPmd^0 is included in VPnb^0; since a polynomial in VNPmd^0 has a small degree, a computation by homogeneous parts truncates its VPnb-computation into a small multiplicatively disjoint circuit, which unfortunately may use at its entry gates some twice exponential integers: we know no way to get rid of these big numbers.

Nevertheless we should keep in a corner of our mind that all these hypothesizes are probably equivalent, and false!

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Chapter 9
A Tutorial on Time and Space Bounds in Tree-Like Resolution

Jacobo Torán

Abstract Tree-like resolution is a well-known method for proving the unsatisfiability of a given formula. Lower bounds for the size and space in tree-like resolution imply lower bounds for many of the algorithms used in practice to solve satisfiability problems. We review a combinatorial game that can be used to prove lower and upper bounds for size and space in tree-like resolution and show some of its applications.

Keywords Resolution · Combinatorial games

9.1 Introduction

Let us first introduce some notation needed to understand the tutorial. The Boolean formulas considered here are in conjunctive normal form (CNF), that is, they are a conjunction of clauses. A clause is a disjunction of literals (variables or negated variables). We will also talk about a set of clauses to refer to a formula. An assignment α for a formula F is (partial) mapping from the set of variables in the formula to $\{0, 1\}$. We denote by $F\alpha$ the result of substituting in F the variables assigned by α by the corresponding value and reducing the formula in the intuitive way. If $F\alpha$ is 1 then we say that α is a satisfying assignment for F. If some satisfying assignment for F exists, then we say that F is satisfiable, otherwise it is unsatisfiable.

Robinson introduced in [Rob65] the concept of *resolution*, a method for deciding if a given formula in conjunctive normal form is unsatisfiable. Due to its simplicity and to its importance in automatic theorem proving and logic programming systems, resolution is one of the best studied refutation systems. The only inference rule in this proof system is the resolution rule:

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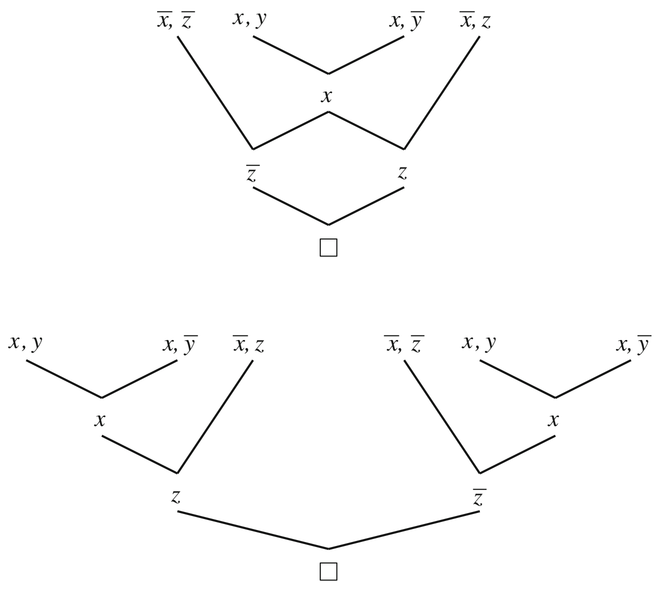

Fig. 9.1 A refutation and a tree-like refutation of the set of clauses $\{x \vee y, x \vee \overline{y}, \overline{x} \vee z, \overline{x} \vee \overline{z}\}$

$$\frac{C \vee x \qquad D \vee \bar{x}}{C \vee D}.$$

Cutting variable x from clauses $C \vee x$ and $D \vee \bar{x}$ we get the *resolvent* clause $C \vee D$. It is easy to see that if a set of clauses S is satisfiable and a clause C is the resolvent of two clauses in S, then $S \cup C$ is also satisfiable. A resolution refutation of a CNF formula F is a sequence of clauses $C_1, \dots, C_s$ where each C_i is either a clause from F or is inferred from earlier clauses by the resolution rule, and C_s is the empty clause (denoted by $\square$). By the above observation, if $\square$ can be derived from a set S of clauses by resolution, then S is unsatisfiable. In other words, the resolution system is *correct*. We will soon observe that the system is also *complete*, that is, if a set of clauses in unsatisfiable then there is always a way to derive the empty clause from it by resolution.

A resolution refutation can be seen as a directed acyclic graph, a *dag*, in which the clauses are the vertices, and if two clauses are resolved then there is a directed edge going from each of the two clauses to the resolvent. If the underlying graph in a refutation happens to be a tree, we call it a *tree-like* resolution. See Fig. 9.1. Tree-like resolution is also a complete refutation system.

Theorem 1.1 *Let F be a set of clauses. If F is unsatisfiable, then there is a tree-like resolution refutation for it.*

Proof We argue by induction on the number n of variables in F. If $n = 1$, then there is only one variable x_1 and since F is unsatisfiable it must contain the clauses x_1

and $\overline{x_1}$. Resolving them we obtain $\Box$. For the induction step, by giving variable x_n the values 0 and 1 we obtain two new sets of clauses $F\{x_n = 0\}$ and $F\{x_n = 1\}$ containing at most $n - 1$ variables. Since F is unsatisfiable, so are the new sets of clauses and applying the induction hypothesis we know that there exist tree-like resolution refutations R_0 and R_1 for $F\{x_n = 0\}$ and $F\{x_n = 1\}$. The clauses of $F\{x_n = 0\}$ are either clauses of F (not containing variable x_n) or are clauses C for which $C \vee x_n$ is in F. By adding x_n to the initial clauses of this type in R_0 we either get a tree-like derivation of x_n from clauses in F (variable x_n is never resolved) or a derivation of $\Box$ (in case none of the clauses in R_0 contain variable x_n). In the later case we are done, otherwise we can do the same process on R_1 obtaining either a tree-like derivation of $\overline{x_n}$ from F or a refutation of F. Again, in the second case we are done, otherwise we put together the tree-like resolution derivations of x_n and $\overline{x_n}$ and the resolvent of this two clauses to obtain a tree-like refutation of F. $\Box$

From the proof of the above Theorem it follows that the number of clauses needed in a tree-like refutation of a formula F is at most exponential in the number of its variables.

It is known that for certain formulas general resolution can produce shorter refutations than tree-like resolution [BEGJ02, BIW04]. The reason for this is that, contrary to general resolution, in tree-like resolution if a clause is needed more than once it must be re-derived from the initial clauses each time. Even if tree-like refutations can be larger than in general resolution, the study of this proof system is well motivated by many methods used in practice for testing satisfiability (SAT solvers). These methods are in fact concrete implementations of tree-like resolution. Most SAT solvers are based on the basic backtracking algorithm:

```
**proc** Backtracking ($F$ : set of clauses, $\alpha$ : assignment) : **bool**
// outputs 1, if F$\alpha$ is satisfiable and 0 otherwise
**if** $\Box \in F\alpha$ **then return** 0
**if** $F\alpha = \emptyset$ **then return** 1
**else** (choose a variable $x \in \mathrm{Var}(F\alpha)$)
  **if** Backtracking ($F, \alpha\{x = 0\}$) **then return** 1
    **else return** Backtracking ($F, \alpha\{x = 1\}$)
```

In order to test if a formula is satisfiable, the algorithms starts with the empty assignment. This backtracking algorithm is in principle more efficient than a straightforward test of all possible total assignments for the variables because when for a partial assignment α, the algorithms obtains $\Box \in F\alpha$, then it follows $F\beta = 0$ for all extensions β from α. The Algorithm does not need to explore the sub-tree with the possible extensions from α and this decreases the search space.

In the algorithm we have not specified the way to choose the next variable to be considered. There are many ways to do this. Modern SAT solvers use refined methods for choosing the next variable to be assigned as well as many other techniques for efficiently pruning the search tree (see e.g., [ST13]).

Exercise 1.2 *Show that in the backtracking algorithm there is a strategy for choosing the next variable to be assigned so that on input a formula F in CNF with n variables and at most k variables in each clause, the algorithm produces at most $(2^k - 1)^{n/k}$ recursive calls.*

A proof of the unsatisfiability of a formula with the backtracking algorithm is in fact a tree-like resolution refutation for the formula as can be seen in the next well-known result:

Theorem 1.3 *Let F be a nonsatisfiable formula in CNF and let r be the minimum number of recursive calls made by the backtracking algorithm on input $(F, \emptyset)$ (by any ordering of the variables). Then there is a tree-like resolution refutation of F of size at most r.*

Proof The backtracking algorithm running on the input $(F, \emptyset)$ defines a tree. The nodes of the tree indicate what variable is being assigned. From an interior node two edges come out, one for each of the two possible values (0 and 1) that the variable can take. We can associate a resolution refutation with this tree. Every node in the tree is reached from the root by a partial assignment and can be identified with an initial clause falsified by this assignment in the following way: Every leaf in the recursion tree is identified with a clause of F falsified by the assignment reaching the leaf. Such a clause must exist because otherwise this particular assignment would satisfy the formula. For an inner node corresponding to a variable x and whose children are identified with clauses K_i and K_j, we associate a clause K. If both clauses K_i and K_j contain variable x, then in one of the clauses the variable is positive and in the other clause it is negated. We define then the clause K as the resolvent of K_i and K_j. Otherwise we associate with the node the clause not containing variable x from one of its children. One can see that in both cases K is falsified by the assignment that goes from the root of the tree to the node. Moreover, every clause in the tree is either the resolvent of its children or one of the clauses from the children. The root in the recursion tree is associated to the empty clause $\square$ since this is the only clause falsified by the empty assignment. The tree with its associated clauses defines a tree-resolution refutation for F with at most r clauses. $\square$

In this result, we have already used the natural complexity measure of *size*. The size of a refutation is the number of clauses it contains. It is known that certain families of propositional formulas need (general) resolution refutations with a number of clauses that is exponential in the formula size [Hak85, Urq87, CS88, BP96].

Another natural complexity measure is the *space*. Intuitively, the resolution space of a CNF formula is the minimal number of clauses that must be kept simultaneously in order to refute a formula. The formal definition [ET01, ABRW02] is the following:

Definition 1.4 Let $k \in \mathbb{N}$, we say that an unsatisfiable CNF formula F has resolution refutation bounded by space k if there is a series of CNF formulas $F_1, \ldots, F_s$, such that $F_1 \subseteq F$, $\square \in F_s$, in any F_i there are at most k clauses, and for each $i < s$, F_{i+1} is obtained from F_i by:

(1) Deleting a clause from F_i.
(2) Adding the resolvent of two clauses from F_i.
(3) Adding a clause from F (initial clause).

The space needed for the resolution refutation of an unsatisfiable formula is the minimum k for which the formula has a refutation bounded by space k. Note that initial clauses do not need much space because they can be added at any moment and at most two of them are needed simultaneously. The only clauses that consume space are the ones derived at intermediate stages. From the proof of Theorem 1.1, it can be observed that the space needed in a refutation of a formula with n variables is at most $n+1$. In [Tor99, ET01, ABRW02] it is shown that the refutations for certain families of formulas need linear space in the number of variables. It was observed in [ET01] that the space required for the refutation of a CNF formula F, corresponds to the minimum number of pebbles needed in the following game played on the graph of a refutation of F.

Definition 1.5 Given a connected directed acyclic graph G with one sink the aim of the pebble game is to put a pebble on the sink of the graph, the only node with no outgoing edges, following this set of rules:

(1) A pebble can be placed in any initial node, that is, a node with no predecessors.
(2) Any pebble can be removed from any node at any time.
(3) A node can be pebbled provided all its predecessors are pebbled.
(4) If all the predecessors of node are pebbled, instead of placing a new pebble on it, one can shift a pebble from a predecessor.

We denote by $\mathrm{Peb}(G)$ the minimum number of pebbles needed in order to put a pebble on the sink of G following the above rules.

Exercise 1.6 *Show that for a complete binary tree of depth d, T_d (with the directed edges pointing to the root), $Peb(T_d) = d+1$. Show that this is also true for any tree T with the property that d is the depth of the biggest complete binary tree embedded*[1] *in T.*

Lemma 1.7 [ET01] *Let F be an unsatisfiable CNF formula. The space needed in a resolution refutation of F coincides with the number of pebbles needed for the pebble game played on the graph of a resolution refutation of F.*

In this tutorial we will concentrate on tree-like refutations. The size of such a refutation is the number of nodes in the resolution tree, and the space is the number of pebbles needed to play the game in such a refutation tree. There is also a way to define the space measure for tree-like resolution without using the pebbling game [ET01].

There is a relation between the space and size in tree-like resolution. As the following result shows, a lower bound on the tree-like space of a formula implies an exponentially larger lower bound on the size of a tree-like resolution refutation.

[1] We say that a graph G is embedded in another graph H if H can be obtained from G by adding new vertices and edges and placing new vertices in the middle of an edge.

Theorem 1.8 *Let F be an unsatisfiable CNF formula with a tree-like refutation of size s, then F has a tree-like resolution refutation of space $\lceil \log s \rceil + 1$.*

Proof As mentioned in Exercise 1.6, the resolution tree in the refutation of F can be pebbled with $d + 1$ pebbles, where d is the depth of the biggest complete binary tree embedded in the refutation tree. As the biggest possible complete binary tree embedded in a tree of size s has depth $\lceil \log s \rceil$, the result follows. □

Exercise 1.9 *Show that if in an unsatisfiable formula F all its clauses have at most 2 literals, then there is a tree-like resolution refutation for F with constant space and polynomial size.*

9.2 A Combinatorial Game for Proving Lower Bounds in Tree-Like Resolution

Impagliazzo and Pudlák introduced in [PI00] a combinatorial game for proving lower bounds on the size of tree-like refutations. This game was also used in [BIW04]. We will show that this game exactly characterizes tree-like resolution space.

The Prover-Delayer game:
The game is played in rounds on an unsatisfiable set of clauses F by two players: Prover and Delayer.[2] Prover wants to falsify some initial clause and Delayer tries to retard this as much as possible. In each round Prover chooses a variable in F and asks Delayer for a value for this variable. Delayer can answer either 0,1 or $*$. In this last case Prover can choose the truth value (0 or 1) for the variable and Delayer scores one point. The variable is set to the selected value and the next round begins. The game ends when a clause in F is falsified (all its literals are set to 0) by the partial assignment constructed in this way. The goal of Delayer is to score as many points as possible and Prover tries to prevent this. The outcome of the game is the number of points scored by Delayer.

Definition 2.1 Let F be an unsatisfiable formula in CNF. We denote by $g(F)$ the maximum number of points that Delayer can score while playing the game on F with an optimal strategy of Prover.

As an example we show how this game applies to the family of formulas for the general Pigeonhole Principle PHP_n^m. These formulas express the fact that it is not possible to fit m pigeons in n pigeonholes (for $m > n$). For the case $m = n + 1$, $\neg\mathrm{PHP}_n^m$ was the first example of a family of formulas with an exponential resolution size lower bound [Hak85]. The contradiction $\neg\mathrm{PHP}_n^m$ can be written as a CNF formula in the following way: The variables of the formula are $x_{i,j}$, $i \in [m]$, $j \in [n]$. $x_{i,j}$ has the intuitive meaning that pigeon i is mapped to hole j. There are mn variables. The clauses of the formula are:

[2] For clarity in the exposition Delayer is a female player.

(1) $x_{i,1} \vee x_{i,2} \vee \cdots \vee x_{i,n}$ for $i \in [m]$, and
(2) $\overline{x}_{i,k} \vee \overline{x}_{j,k}$ for $i, j \in [m], k \in [n],\ i < j$.

Clauses of type (1) express the fact that every pigeon is mapped to some hole, while the clauses of type (2) indicate that at most one pigeon can be mapped to any hole.

The number of clauses in $\neg\text{PHP}^m_n$ is $m + \binom{m}{2}n < m^2 n$.

Lemma 2.2 $g(\neg PHP^m_n) \geq n$

Proof We give a strategy for Delayer scoring at least n points. When asked for the value of a variable $x_{i,j}$ Delayer answers 0 if for some $i' \neq i$, $x_{i',j}$ has been assigned value 1, otherwise she assigns value $*$ to $x_{i,j}$. With this strategy the game can only end when a clause of Type 1 has been falsified. Observe that since Delayer does not assign any 1's, for every 0 she assigns to a variable $x_{i,j}$, she must have assigned before a $*$ to another variable $x_{i',j}$ that is set to 1 by Prover. When a clause $\bigvee^n_{j=1} x_{i,j}$ is falsified because all its variables have value 0, Delayer has assigned a $*$ to at least one variable $x_{i',j}$ corresponding to position j for each position, and therefore she has scored at least n points. □

Exercise 2.3 *Show by designing a suitable strategy for Prover that it also holds* $g(\neg PHP^m_n) \leq n$.

We show now that for an unsatisfiable CNF formula F, the space needed in a tree-like resolution refutation of F is exactly $g(F) + 1$. In our example this implies that the tree-like resolution space for $\neg\text{PHP}^m_n$ is exactly $n + 1$ independently of m.

We show first that $g(F) + 1$ is an upper bound for the tree-like resolution space.

Theorem 2.4 [ET03] *If a CNF formula F requires tree-like resolution space s, then Delayer has a strategy in which at least $s - 1$ points can be scored.*

Proof Let s be the minimum space needed in any tree-like resolution refutation of F. We give a strategy for Delayer for playing the combinatorial game on F that scores at least $s - 1$ points with any strategy of Prover. We prove the result by induction on n, the number of variables in F.

For the base case $n = 1$, F contains just one variable and therefore $s \leq 2$. Delayer just needs to answer $*$ to the only variable asked by Prover.

For $n > 1$, let x be the first variable asked by Prover and let $F\{x = 1\}$ and $F\{x = 0\}$ the CNF formulas obtained after given value 1 and 0 respectively to variable x in F. Any tree-like refutation of F requires s pebbles and therefore either

(i) the tree-like space for refuting each of $F\{x = 1\}$ and $F\{x = 0\}$ is at least $s-1$ or
(ii) for one of the formulas (say $F\{x = 1\}$) the tree-like resolution space is at least s.

Any other possibility would imply that F could be refuted in space less than s. In the first case, Delayer can answer $*$ and she scores one point. By induction hypothesis Delayer can score $s-2$ more points playing the game in any of the formulas $F\{x = 1\}$ or $F\{x = 0\}$. In the second case, Delayer answers the value leading to the formula that requires tree-like resolution space s ($x = 1$ in this case) and the game is played on $F\{x = 1\}$ in the next round. □

On the other hand $g(F)$ is also a lower bound for the tree-like resolution space.

Theorem 2.5 [ET03] *The tree-like space needed for refuting a CNF F is at least $g(F) + 1$.*

Proof Let s be the minimum space needed in a tree-like refutation of F. We describe a Strategy for Prover in which the number p of points scored by Delayer is at most $s - 1$. Let R be a tree-like resolution refutation for F with $\text{Peb}(R) = s$. Prover chooses the variables in the order induced by the refutation in the following way: He starts at the empty clause in R and in general at the end of a round moves to a clause C that is falsified by the partial assignment constructed so far. In the next round, Prover chooses the resolved variable x from the two parent clauses of C. Let α_i be the partial assignment constructed after i rounds of the game and $R\alpha_i$ be the subtree of the refutation that has its root at the node reached from the root of R by the path specified by α_i and let p_i be the number of points scored by Delayer after round i. If Delayer assigns to x a value 0 or 1 (by setting the weights in the way indicated above) then Prover moves to the parent clause that is falsified by the constructed partial assignment and the new round starts. When Delayer assigns value $*$ to a variable x at step i then Prover gives x value 0 if $\text{Peb}(R\alpha_{i-1}\{x = 0\}) \leq \text{Peb}(R\alpha_{i-1}\{x = 1\})$ and assigns x value 1 otherwise. We show by induction on the number i of rounds that with this strategy

$$p_i \leq s - \text{Peb}(R\alpha_i).$$

The result follows from this fact because when the game has reached a contradiction of an initial clause after constructing an assignment α, then $\text{Peb}(R\alpha) = 1$ and the inequality shows $p \leq s - 1$.

In the beginning of the game $R\alpha_0$ is the whole tree and Delayer has scored 0 points.

For the inductive step, if at round $i + 1$ Prover chooses variable x and Delayer assigns values 0 or 1 to it then she does not score any points, and we get

$$p_{i+1} = p_i \leq s - \text{Peb}(R\alpha_i) \leq s - \text{Peb}(R\alpha_{i+1})$$

since $R\alpha_{i+1}$ is a subtree of $R\alpha_i$.

If Delayer assigns $*$ to x, since Prover select the value corresponding to the subtree from $R\alpha_i$ with the smallest pebbling number and $\min\{\text{Peb}(R\alpha_i\{x = 0\}), \text{Peb}(R\alpha_i\{x = 1\})\} < \text{Peb}(R\alpha_i)$ we get

$$p_{i+1} = p_i + 1 \leq s - \text{Peb}(R\alpha_i) - 1 \leq s - \text{Peb}(R\alpha_{i+1})$$

and the result follows. □

As mentioned before, the combinatorial game was defined in [PI00] as a tool for proving lower bound for the size of tree-like resolution refutation. This application can be seen considering the relationship between tree-like space and size from Theorem 1.8:

Corollary 2.6 *For any unsatisfiable CNF formula F, if Delayer has a strategy on F which scores r points then any tree-like resolution refutation of F has size at least $2^{r+1} - 1$.*

Buss and Pitassi [BP97] showed that for for any $m > n$, $\neg\mathrm{PHP}^m_n$ needs tree-like resolution refutation of size at least 2^n. The given estimation for the points obtained by Delayer on the formula $\neg\mathrm{PHP}^m_n$ with the space characterization and the above corollary provide an alternative proof for this result. The bound has been improved to $2^{\Omega(n \log n)}$ in [IM99, DR01].

Computing the number of points that can be scored by Delayer on a formula is in general not an easy task. Hertel and Urquhart [HU07] have shown that in fact this problem is PSPACE complete. We finish this section with some exercises to compute the points scored on certain formulas related to pebbling.

Exercise 2.7 *Consider the following pebbling formulas defined by Ben-Sasson and Wigderson in* [BW01]*. They express the principle that in a directed acyclic graph, pebbling the source nodes and following the rule that if all the predecessors of a node v contain a pebble then v also gets one, implies that a pebble will be placed on the sink.*

Let $G = (V, E)$ be a directed acyclic graph in which every vertex has fan-in 2 or 0 with a unique sink s. We call a graph with these properties a circuit graph. We associate 2 distinct Boolean variables v_1, v_2 with every vertex $v \in V$. Peb_G, the pebbling contradiction of G, is the conjunction of:

- **Source axioms:** $v^1 \vee v^2$ *for each source v.*
- **Pebbling axioms:** $\overline{u}^i \vee \overline{v}^j \vee w^1 \vee w^2$ *for u and v the two predecessors of w and $i, j \in \{1, 2\}$.*
- **Sink axioms:** $\overline{s}^i$ *for the sink s and $i \in \{1, 2\}$.*

Show that for a constant c, if G is a tree then $g(Peb_G) = Peb(G) + c$

Exercise 2.8 [Nor12] *Consider now a variation of the pebbling formulas, defined by Ben-Sasson and Nordström in* [BN08] *expressing the same principle but using the parity function instead of disjunction. For simplicity we just define these formulas for line graphs $G = (V, E)$ with n vertices $v_1 \dots v_n$ with unique source v_1, unique sink v_n and with a directed edge (v_1, v_{i+1}) for $1 \leq i \leq n-1$. We associate 2 distinct Boolean variables v^1_i, v^2_i with every vertex $v_i \in V$. $Peb^{\oplus}_G$ is the conjunction of (the clauses expressing in conjunctive normal form the formulas):*

- **Source axioms:** $v^1_1 \oplus v^2_1$.
- **Pebbling axioms:** $v^1_i \oplus v^2_i \to v^1_{i+1} \oplus v^2_{i+1}$ *for* $1 \leq i \leq n-1$.
- **Sink axioms:** $\neg(v^1_n \oplus v^2_n)$.

Show that for a line graph G with n vertices, $g(Peb^{\oplus}_G) = \Theta(\log n)$.

9.3 The Asymmetric Prover-Delayer Game

We have seen in the previous section that the original Prover-Delayer game gives an exact characterization of tree-like resolution space and provides a good tool for proving size lower bounds in tree-like resolution size. However, this lower bound might not be tight in case the refutation trees are not well balanced. Beyersdorff, Galesi, and Lauria [BGL10, BGL11, BGL13] have developed a refinement of the game that improves in some cases the size lower bounds obtained by using the original game.

The asymmetric game is defined exactly as the original one, only the way of scoring points by Delayer is different. In each round, Prover selects an unassigned variable x as before, and Delayer assigns two weights p_0 and p_1 to the two possible variable values, satisfying:

$$p^0 \geq 0, \quad p^1 \geq 0, \quad \text{and } p^0 + p^1 = 1.$$

Prover selects one of the values in $\{0, 1\}$ for x and the number of points that Delayer gets is $-\log p^b$.

Definition 3.1 Let F be an unsatisfiable formula in CNF. We denote by $G(F)$ the maximum number of points that Delayer can score while playing the game on F with an optimal strategy of Prover.

By setting the weights to either $(0, 1)$ or $(1, 0)$, Delayer forces Prover to choose one of the two values, since by choosing the other one he would loose an infinite amount of points. Also, by setting the weights to be $(\frac{1}{2}, \frac{1}{2})$ Delayer scores one point no matter what the choice of Prover is. These two cases show that this game is an extension of the original one.

The new game can be used to obtain a characterization of tree-like resolution size. The next theorem shows that the size of a resolution tree implies an upper bound on the number of points scored by Delayer.

Theorem 3.2 [BGL11] *Let F be an unsatisfiable formula in CNF. If F has a tree-like refutation of size at most S then $G(F) \leq \log\lceil\frac{S}{2}\rceil$.*

Proof The proof is very similar to that of Theorem 2.5 but considering the new scoring rules. Let R be a tree-like resolution refutation of size S for F and let $L(R)$ be the number of leaves in R. Observe that $L(R) = \lceil\frac{S}{2}\rceil$. We describe a Strategy for Prover in which the number p of points scored by Delayer is at most $\log(L(R))$.

Prover chooses the variables in the order induced by the refutation starting at the empty clause in R and at the end of a round moves to the parent clause C that is falsified by the partial assignment constructed so far. Let α_i be the partial assignment constructed after i rounds of the game and $R\alpha_i$ be the subtree of the refutation that has its root at the node reached from the root of R by the path specified by α_i and let

p_i be the number of points scored by Delayer after round i. When Delayer assigns values (p^0, p^1) to a variable x at step i then Prover gives x value 0 if $L(R\alpha_{i-1}\{x = 0\}) \leq p^0 L(R\alpha_{i-1})$ and assigns x value 1 otherwise. By the property of the weights p^0 and p^1 and by the fact that $L(R\alpha_{i-1}) = L(R\alpha_{i-1}\{x = 0\}) + L(R\alpha_{i-1}\{x = 1\})$ it follows that in this last case $L(R\alpha_{i-1}\{x = 1\}) \leq p^1 L(R\alpha_{i-1})$.

We show by induction on the number i of rounds that with this strategy

$$2^{p_i} \leq \frac{L(R)}{L(R\alpha_i)}.$$

Again the result follows from this because when the game has reached a contradiction of an initial clause after constructing an assignment α then $L(R\alpha) = 1$ and the inequality shows $2^p \leq L(R)$.

In the beginning of the game $R\alpha_0$ is the whole tree and Delayer has scored 0 points. For the inductive step, if at round $i + 1$ Prover chooses variable x and Delayer assigns weights p^0, p^1 to it, and Prover selects value $b \in \{0, 1\}$ for the variable, we get

$$2^{p_{i+1}} = 2^{p_i - \log p^b} = \frac{2^{p_i}}{p^b} \leq \frac{L(R)}{p^b L(R\alpha_i)} \leq \frac{L(R)}{L(R\alpha_{i+1})}$$

and the result follows. □

The extended game completely characterizes the size of a tree-like resolution proof. This is a consequence of Theorem 3.2 and the converse result:

Theorem 3.3 [BGL13] *Let F be an unsatisfiable formula in CNF. If the smallest tree-like refutation for F has size S then $G(F) \geq \log\lceil \frac{S}{2} \rceil$.*

Proof Let $L(F)$ be the number of leaves in the shortest tree-like refutation of F. When a new variable x is selected by Prover and α is the partial assignment computed so far, Delayer assigns weights according to the following rules: for $b \in \{0, 1\}$,

$$p^b = \frac{L(F\alpha\{x = b\})}{L(F\alpha\{x = 0\}) + L(F\alpha\{x = 1\})}.$$

We show by induction on n, the number of variables in F, that Delayer scores at least $\log L(F)$ points. This implies the result, since a tree-like refutation of size S has exactly $\lceil \frac{S}{2} \rceil$. leaves. The base case is trivial; if there is only one variable, the resolution tree has two leaves, and Delayer can always score 1 point. For $n > 1$, let x be the first variable chosen by Prover and let $b \in \{0, 1\}$ be the value assigned by him to it. The score of the game is $-\log p^b + X$ where X is the score achieved in the subsequent steps. By induction hypothesis we have $X \geq \log L(F\{x = b\})$. The total score is then at least

$$
\begin{aligned}
G(F) &\geq -\log p^b + \log L(F\{x = b\}) \\
&= \log\left(\frac{L(F\alpha\{x = 0\}) + L(F\alpha\{x = 1\})}{L(F\alpha\{x = b\})}\right) + \log L(F\{x = b\}) \\
&= \log(L(F\alpha\{x = 0\}) + L(F\alpha\{x = 1\})) \geq \log L(F).
\end{aligned}
$$

□

We can see that the asymmetric game can improve the size lower bounds for tree-like resolution for the pigeon hole principle achieved by using the original game [BGL10]. For this we analyze the asymmetric game on PHP formulas and then use Theorem 3.2.

Theorem 3.4 *For $m > n$, $G(\neg PHP^m_n, c_0, c_1) = \Omega(n \log n)$*

Proof We give a simplified proof from [Bey11] describing a strategy for Delayer for which the number of scored points is $\Omega(n \log n)$.

For $i \in [m]$ we define the function h_i that for a partial assignment α

$$h_i(\alpha) = |\{k \in [n] \mid \alpha(x_{i,k}) = 0 \text{ and } \alpha(x_{j,k}) \neq 1 \text{ for all } j \in [m]\}|.$$

$h_i(\alpha)$ indicates the number of holes that are still free but are excluded for pigeon i under α.

The strategy of Delayer for the variable $x_{i,j}$ when the partial assignment constructed so far is α, is the following:

If there is some other variable $x_{i',j}$ assigned to 1 (hole j is already used) or there is some $x_{i,j'}$ assigned to 1 (pigeon i already has a hole) then set $x_{i,j}$ to 0 (by setting the weights to $(1, 0)$). Otherwise if $h_i(\alpha) \geq \frac{n}{2}$ then set $x_{i,j}$ to 1. Otherwise set the weights of $x_{i,j}$ to (p^0, p^1). The values of p^0 and p^1 are the same for all the variables satisfying this property and will be specified later.

Intuitively, when the number of free holes excluded for pigeon i are at least $\frac{n}{2}$ then Delayer tries to put i in a free hole. As in Lemma 2.2 with this strategy only a clause of type 1 $\bigvee_{j=1}^{n} x_{i,j}$ can be falsified. We show that when this happens at the end of the game, at least $\frac{n}{2}$ variables have been assigned to 1. Consider the last round r after which $h_i(\alpha_r) < \frac{n}{2}$. If at the end of the game $h_i(\alpha_r) < \frac{n}{2}$, then there are less than $\frac{n}{2}$ holes that are free for pigeon i and therefore more than $\frac{n}{2}$ are occupied. The corresponding variables are set to 1. Otherwise let z be the number of variables in $\bigvee_{j=1}^{n} x_{i,j}$ set to 0 after round r. There are exactly $\frac{n}{2}$ of them that correspond to free holes excluded for pigeon i. For the other $z - \frac{n}{2}$ variables the corresponding hole is already occupied, that is, there is a variable for each of these holes set to 1. After round r every time one of the $n - z$ remaining variables $x_{i,j}$ is set to 0 in $\bigvee_{j=1}^{n} x_{i,j}$ this is done by Delayer and because there is some other variable $x_{i',j}$ already set to 1. The number of 1's at the end of the game is then at least $z - \frac{n}{2} + n - z = \frac{n}{2}$. W.l.o.g. let us call the first $\frac{n}{2}$ variables that are set to 1 at the end of the game by x_{i,j_i}, $i \in 1 \ldots \frac{n}{2}$. We analyze now how many points Delayer scores for each of them. If variable x_{i,j_i} has been assigned by Prover, then Delayer scores $-\log(p^1)$ points. If it is Delayer the one who gave the variable value 1, then at that point there were at least $\frac{n}{2}$ free holes that were excluded for pigeon i. All the 0's indicating the exclusion, had been set by Prover because Delayer only sets 0's when a hole is not free or a pigeon has already a hole. Therefore, Delayer scores at least $-\frac{n}{2}\log(p^0)$ points for this. Observe that since Delayer does not allow a pigeon to be in more than one hole, the 0's set by Prover are different for every pigeon i. Summarizing, Delayer receives either $-\log(p^1)$ or $-\frac{n}{2}\log(p^0)$ points for each of the $\frac{n}{2}$ variables x_{i,j_i} set to 1.

Let us now define the values for p^0 and p^1. Intuitively Delayer has to score more points when Prover sets a variable to 1 than we he sets it to 0 because the former brings the game quicker to an unsatisfying assignment. We define:

$$p^1 = \frac{\log n}{n} \text{ and } p^0 = 1 - \frac{\log n}{n} = \Omega\big(e^{-\frac{\log n}{n}}\big) = 2^{\Omega\left(\frac{-\log n}{n}\right)}$$

The number of points scored for each of the $\frac{n}{2}$ variables is then either $-\log(p^1) = \Omega(\log n)$ or $-\frac{n}{2}\log(p^0) = \Omega(\log n)$, and the total number of points $\Omega(n\log n)$. $\square$

With more complicated arguments the authors of the original paper [BGL10] provide a lower bound of $\frac{n}{2}\log(\frac{n}{2}+1)$ for $G(\neg\text{PHP}_n^m)$. The best existing lower bounds for the tree-like resolution size for PHP can be seen as consequences of this result and Theorem 3.2:

Theorem 3.5 [IM99, DR01] *For $m > n$ the size of a tree-like resolution refutation of $\neg PHP_n^m$ is at least $2^{\frac{n}{2}\log(\frac{n}{2}+1)}$.*

9.4 Conclusions

We have shown in this tutorial that variations of a combinatorial game played on formulas, characterize exactly the concepts of space and size in tree-like resolution. It is interesting to observe that the outcome of these games depends only on the structure of the input formulas. This means that the concepts of tree-like resolution space and size are complexity measures intrinsic to the formulas and completely independent of the notion of resolution. It is an important open question whether such game characterizations also exist for the case of general resolution.

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Chapter 10
An Entropy-Based Proof for the Moore Bound for Irregular Graphs

S. Ajesh Babu and Jaikumar Radhakrishnan

Abstract We provide proofs of the following theorems by considering the entropy of random walks.

Theorem 1 (Alon, Hoory and Linial) *Let G be an undirected simple graph with n vertices, girth g, minimum degree at least 2 and average degree $\bar{d}$.*

Odd girth *If $g = 2r + 1$, then $n \geq 1 + \bar{d}\sum_{i=0}^{r-1}(\bar{d} - 1)^i$.*

Even girth *If $g = 2r$, then $n \geq 2\sum_{i=0}^{r-1}(\bar{d} - 1)^i$.*

Theorem 2 (Hoory) *Let $G = (V_L, V_R, E)$ be a bipartite graph of girth $g = 2r$, with $n_L = |V_L|$ and $n_R = |V_R|$, minimum degree at least 2 and the left and right average degrees d_L and d_R. Then,*

$$n_L \geq \sum_{i=0}^{r-1}(d_R - 1)^{\lceil \frac{i}{2} \rceil}(d_L - 1)^{\lfloor \frac{i}{2} \rfloor},$$

$$n_R \geq \sum_{i=0}^{r-1}(d_L - 1)^{\lceil \frac{i}{2} \rceil}(d_R - 1)^{\lfloor \frac{i}{2} \rfloor}.$$

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10.1 Introduction

The Moore bound (see Theorem 3.1) gives a lower bound on the order of any simple undirected graph, based on its minimum degree and girth. Alon et al. [AHL02] showed that the same bound holds with the minimum degree replaced by the average degree. Later, Hoory [Hoo02] obtained a better bound for simple bipartite graphs. We reprove the results of Alon et al. [AHL02] and Hoory [Hoo02] using information theoretic arguments based on nonreturning random walks on the graph.

The chapter has three sections: In Sect. 10.2 we introduce the relevant notation and terminology. In Sect. 10.3, we present the information theoretic proof of the result of Alon et al. [AHL02]; in Sect. 10.4, we present a similar proof of the result of Hoory [Hoo02] for bipartite graphs.

10.2 Preliminaries

For an undirected simple graph $G = (V, E)$, let $\vec{G} = (V, \vec{E})$, be the directed version of G, where for each undirected edge of the form $\{v, v\}$ in E, we place two directed edges in $\vec{E}$, one of the form (v, v) and another of the form (v, v). Similarly, for an undirected bipartite graph $G = (V_L, V_R, E)$, let $\vec{G} = (V_L, V_R, \vec{E}_{LR} \cup \vec{E}_{RL})$ be the directed version of G, where for each undirected edge of the form $\{v, v\}$ in E, with $v \in V_L$ and $v \in V_R$, we place one directed edge of the form (v, v) in $\vec{E}_{LR}$, and another of the form (v, v) in $\vec{E}_{RL}$.

We will consider *nonreturning* walks on $\vec{G}$, that is, walks where the edges corresponding to the same undirected edge of G do not appear in succession. For a vertex v, let $n_i(v)$ denote the number of nonreturning walks in $\vec{G}$ starting at v and consisting of i edges. For an edge $\vec{e}$, let $n_i(\vec{e})$ denote the number of nonreturning walks in $\vec{G}$ starting with $\vec{e}$ and consisting of exactly $i + 1$ edges (including $\vec{e}$).

Our proofs will make use of information theoretic ideas. Similar ideas have been employed in various combinatorial proofs to succinctly present arguments that involve averaging and convexity. More examples can be found in the references [CT91, Kah02, LL13, Rad99, Rad01].

Let X be a random variable taking values in a finite set. Let support(X) be the set of values that X takes with positive probability. The entropy of X is

$$H[X] = -\sum_{x \in \text{support}(X)} \Pr[X = x] \log_2 \Pr[X = x].$$

For random variables X and Y, taking values in finite sets according to some joint distribution, and $y \in \text{support}(Y)$, let X_y be the random variable taking values in support(X) such that $\Pr[X_y = x] = \Pr[X = x \mid Y = y]$. Then, the conditional entropy of X given Y is

$$H[X \mid Y] = \sum_{y \in \text{support}(Y)} \Pr[Y = y] H[X_y].$$

We will use of the following standard facts about entropy [CT91].

$$H[X] \leq \log_2 |\text{support}(X)|;$$
$$H[X_1 X_2 \ldots X_k \mid Y] = \sum_{i=1}^{k} H[X_i \mid X_1 X_2 \ldots X_{i-1} Y].$$

10.3 Moore Bound for Irregular Graphs

In Sect. 10.3.1, we recall the proof of the Moore bound; in Sect. 10.3.2, we review and reprove the theorem of Alon et al. [AHL02] assuming Lemma 3.4. In Sect. 10.3.3, we prove this lemma using an entropy- based argument.

10.3.1 Proof of the Moore Bound

The Moore bound provides a lower bound for the order of a graph in terms of its minimum degree and girth.

Theorem 3.1 (The Moore bound [Big93, p. 180]) *Let G be a simple undirected graph with n vertices, minimum degree δ and girth g.*

Odd girth *If $g = 2r + 1$, then $n \geq 1 + \delta \sum_{i=0}^{r-1} (\delta - 1)^i$.*

Even girth *If $g = 2r$, then $n \geq 2 \sum_{i=0}^{r-1} (\delta - 1)^i$.*

The key observation in the proof of the Moore bound is the following. If the girth is $2r + 1$, then two distinct nonreturning walks of length at most r starting at a vertex v lead to distinct vertices. Similarly, if the girth is $2r$, then nonreturning walks of length at most r starting with (some directed version of) an edge e lead to distinct vertices. We will need this observation again later, so we record it formally.

Observation 3.2 *Let G be an undirected simple graph with n vertices and girth g.*

Odd girth *Let $g = 2r + 1$. Then, for all vertices v,*

$$n \geq n_0(v) + n_1(v) + \cdots + n_r(v).$$

Even girth *Let $g = 2r$. Let e be an edge of G and suppose $\vec{e}_1$ and $\vec{e}_2$ are its directed versions in $\vec{G}$. Then,*

$$n \geq \sum_{i=0}^{r-1} [n_i(\vec{e}_1) + n_i(\vec{e}_2)].$$

Proof of Theorem 3.1 The claim follows immediately from Observation 3.2 by noting that for such a graph G, for all vertices $v \in V$ and edges $\vec{e} \in \vec{E}$,

$$n_i(v) \geq \delta(\delta - 1)^{i-1} \quad (\text{for } i \geq 1), \quad n_0(v) = 1; \tag{10.1}$$

$$n_i(\vec{e}) \geq (\delta - 1)^i \quad (\text{for } i \geq 0). \tag{10.2}$$

□

10.3.2 The Alon–Hoory–Linial Bound

Alon, Hoory, and Linial showed that the bound in Theorem 3.1 holds for any undirected graph even when the minimum degree δ is replaced by the average degree $\bar{d}$.

Theorem 3.3 (Alon et al. [AHL02]) *Let G be an undirected simple graph with n vertices, girth g, minimum degree at least 2 and average degree $\bar{d}$.*

Odd girth *If $g = 2r + 1$, then $n \geq 1 + \bar{d} \sum_{i=0}^{r-1} (\bar{d} - 1)^i$.*

Even girth *If $g = 2r$, then $n \geq 2 \sum_{i=0}^{r-1} (\bar{d} - 1)^i$.*

We will first prove this theorem assuming the following lemma, which is the main technical part of Alon et al. [AHL02]. This lemma shows that the bounds (10.1) and (10.2) holds with δ replaced by $\bar{d}$. In Sect. 10.3.3, we will present an information theoretic proof of this lemma.

Lemma 3.4 *Let G be an undirected simple graph with n vertices, girth g, minimum degree at least two and average degree $\bar{d}$.*

(a) *If $v \in V(G)$ is chosen with distribution π, where $\pi(v) = d_v/(2|E(G)|) = d_v/(\bar{d}n)$, then $\mathbb{E}[n_i(v)] \geq \bar{d}(\bar{d} - 1)^{i-1}$ ($i \geq 1$).*
(b) *If $\vec{e}$ is a uniformly chosen random edge in $\vec{E}$, then $\mathbb{E}[n_i(\vec{e})] \geq (\bar{d} - 1)^i$ ($i \geq 0$).*

Proof of Theorem 3.3 First, consider graphs with odd girth. From Observation 3.2, Lemma 3.4 (a) and linearity of expectation we obtain

$$n \geq \mathbb{E}[n_0(v) + n_1(v) + \cdots + n_r(v)] \geq 1 + \bar{d} \sum_{i=0}^{r-1} (\bar{d} - 1)^i,$$

where $v \in V(G)$ is chosen with distribution π (defined in Lemma 3.4 (a)).

Now, consider graphs with even girth. Let $\vec{e}_1$ be chosen uniformly at random from $\vec{E}$ and let $\vec{e}_2$ be its companion edge (going in the opposite direction). Note that $\vec{e}_2$ is also uniformly distributed in $\vec{E}$. Then, from Observation 3.2, Lemma 3.4 (b) and linearity of expectation we obtain

$$n \geq \mathbb{E}\left[\sum_{i=0}^{r} [n_i(\vec{e}_1) + n_i(\vec{e}_2)]\right] \geq 2 \sum_{i=0}^{r-1} (\bar{d} - 1)^i.$$

10.3.3 The Entropy-Based Proof of Lemma 3.4

The proof of Lemma 3.4 below is essentially the same as the one originally proposed by Alon, Hoory, and Linial but is stated more naturally using the language of entropy.

Proof of Lemma 3.4 (a) Consider the Markov process $v, \vec{e}_1, \vec{e}_2, \ldots, \vec{e}_i$, where v is a random vertex of G chosen with distribution π, $\vec{e}_1$ is a random edge of $\vec{G}$ leaving v (chosen uniformly from the d_v choices), and for $1 \leq j < i$, $\vec{e}_{j+1}$ is a random successor edge for $\vec{e}_j$ chosen uniformly from among the nonreturning possibilities. (If $\vec{e}_j$ has the form (x, y), then there are $d_y - 1$ possibilities for $\vec{e}_{j+1}$). Let $v_0 = v, v_1, v_2, \ldots, v_i$ be the vertices visited by this non-returning walk. We observe that each $\vec{e}_j$ is distributed uniformly in the set $E(\vec{G})$ and each v_j has distribution π. Then,

$$\begin{aligned}
\log_2 \mathbb{E}[n_i(v)] &\geq \mathbb{E}[\log_2 n_i(v)] \\
&\geq H[\vec{e}_1 \vec{e}_2 \ldots \vec{e}_i \mid v] \\
&= H[\vec{e}_1 | v] + \sum_{j=1}^{i-1} H[\vec{e}_{j+1} \mid \vec{e}_1 \vec{e}_2 \ldots \vec{e}_j v] \\
&= \mathbb{E}[\log_2 d_v] + \sum_{j=1}^{i-1} \mathbb{E}[\log_2 (d_{v_j} - 1)] \\
&= \mathbb{E}[\log_2 d_v (d_v - 1)^{i-1}] \\
&= \frac{1}{\bar{d}n} \sum_v d_v \log_2 d_v (d_v - 1)^{i-1} \\
&\geq \log_2 \bar{d}(\bar{d} - 1)^{i-1},
\end{aligned}$$

where to justify the first inequality we use Jensen's inequality for the concave function log, to justify the second we use the fact that the entropy of a random variable is at most the log of the size of its support, and to justify the last we use Jensen's inequality for the convex function $x \log_2 x(x-1)^{i-1}$ $(x \geq 2)$. The claim follows by exponentiating both sides.

(b) This time we consider the Markov process $\vec{e}_0 = \vec{e}, \vec{e}_1, \ldots, \vec{e}_i$, where $\vec{e}$ is chosen uniformly at random from $\vec{E}$, and for $0 \leq j < i$, $\vec{e}_{j+1}$ is a random successor edge for $\vec{e}_j$ chosen uniformly from among the nonreturning possibilities. Let $v_0, v_1, v_2, \ldots, v_{i+1}$ be the vertices visited by this nonreturning walk. As before observe that each v_j has distribution π. Then,

$$\begin{aligned}\log_2 \mathbb{E}[n_i(e)] &\geq \mathbb{E}[\log_2 n_i(e)] \\ &\geq H[\vec{e}_1\vec{e}_2 \ldots \vec{e}_i \mid \vec{e}_0] \\ &= \sum_{j=1}^{i} \mathbb{E}[\log_2(d_{v_j} - 1)] \\ &= \mathbb{E}[\log_2(d_{v_0} - 1)^i] \\ &= \frac{1}{\bar{d}n} \sum_v d_v \log_2(d_v - 1)^i \\ &\geq \log_2(\bar{d} - 1)^i,\end{aligned}$$

where we justify the first two inequalities as before, and the last using Jensen's inequality applied to the convex function $x \log_2(x-1)^i$ $(x \geq 2)$. The claim follows by exponentiating both sides. □

Remark 3.5 We assumed above that the minimum degree is at least 2. It is possible to eliminate vertices of small degree and show that Theorem 3.3 holds for any graph with *average* degree at least 2. For details, see the proof of Theorem 1 in [AHL02].

10.4 Moore Bound for Bipartite Graphs

Following the proof technique of [AHL02], Hoory [Hoo02] obtained an improved Moore bound for bipartite graphs. In this section, we provide an information theoretic proof of Hoory's result.

10.4.1 The Hoory Bound

Theorem 4.1 (Hoory [Hoo02]) *Let $G = (V_L, V_R, E)$ be a bipartite graph of girth $g = 2r$, with $n_L = |V_L|$ and $n_R = |V_R|$, minimum degree at least 2 and the left and right average degrees d_L and d_R. Then,*

$$n_L \geq \sum_{i=0}^{r-1} (d_R - 1)^{\lceil \frac{i}{2} \rceil} (d_L - 1)^{\lfloor \frac{i}{2} \rfloor},$$

$$n_R \geq \sum_{i=0}^{r-1} (d_L - 1)^{\lceil \frac{i}{2} \rceil} (d_R - 1)^{\lfloor \frac{i}{2} \rfloor}.$$

For bipartite graphs the girth is always even. We then have the following variant of Observation 3.2.

Observation 4.2 *Let $G = (V_L, V_R, E)$ be an undirected bipartite graph with $|V_L| = n_L$ and $|V_R| = n_R$ and girth $g = 2r$. Let e be an edge of G and suppose $\vec{e}_1$ and $\vec{e}_2$ be its directed versions in $\vec{G}$, such that $\vec{e}_1 \in \vec{E}_{LR}$ and $\vec{e}_2 \in \vec{E}_{RL}$. Then,*

$$n_L \geq \sum_{i=0}^{\lfloor \frac{r}{2} \rfloor - 1} n_{2i+1}(\vec{e}_1) + \sum_{i=0}^{\lceil \frac{r}{2} \rceil - 1} n_{2i}(\vec{e}_2).$$

We will prove the Theorem 4.1, assuming the following lemma, which is the main technical part of Hoory [Hoo02]. In Sect. 10.4.2, we will present the proof of this lemma using the language of entropy.

Lemma 4.3 *Let $G = (V_L, V_R, E)$ be an undirected simple bipartite graph with n_L vertices on the left and n_R vertices on the right, girth g, minimum degree at least two and average left and right degrees, respectively d_L and d_R.*

(a) *If $\vec{e}$ is a uniformly chosen random edge in $\vec{E}_{LR}$, then $\mathbb{E}[n_{2i+1}(\vec{e})] \geq (d_R - 1)^{i+1}(d_L - 1)^i$ $(i \geq 1)$.*
(b) *If $\vec{e}$ is a uniformly chosen random edge in $\vec{E}_{RL}$, then $\mathbb{E}[n_{2i}(\vec{e})] \geq (d_R - 1)^i (d_L - 1)^i$ $(i \geq 1)$.*

Proof of Theorem 4.1 We will prove the bound for n_L. The proof for n_R case is similar. Let $\vec{e}_1$ be chosen uniformly at random from $\vec{E}_{LR}$ and let $\vec{e}_2$ be its companion edge (going in the opposite direction). Note that $\vec{e}_2$ is also uniformly distributed in $\vec{E}_{RL}$. Then, from Observation 4.3, Lemma 4.3 and linearity of expectation we obtain

$$n_L \geq \mathbb{E}\left[\sum_{i=0}^{\lfloor \frac{r}{2} \rfloor - 1} n_{2i+1}(\vec{e}_1) + \sum_{i=0}^{\lceil \frac{r}{2} \rceil - 1} n_{2i}(\vec{e}_2) \right] \geq \sum_{i=0}^{r-1} (d_R - 1)^{\lceil \frac{i}{2} \rceil} (d_L - 1)^{\lfloor \frac{i}{2} \rfloor}. \qquad \square$$

10.4.2 The Entropy-Based Proof of Lemma 4.3

The proof of Lemma 4.3 below is essentially the same as the one originally proposed by Hoory, but is stated in the language of entropy.

Proof of Lemma 4.3 (a) Consider a Markov process $\vec{e}_0, \vec{e}_1, \vec{e}_2, \ldots, \vec{e}_{2i+1}$, where $\vec{e}_0$ is a uniformly chosen random edge from $\vec{E}_{LR}$, and for $0 \le j < 2i+1$, $\vec{e}_{j+1}$ is a random successor edge for $\vec{e}_j$ chosen uniformly from among the nonreturning possibilities. Let $v_0, v_1, v_2, \ldots, v_{2i+2}$ be the vertices visited by this nonreturning walk. We observe that for $0 \le j \le i$ each $\vec{e}_{2j}$ and $\vec{e}_{2j+1}$ is respectively distributed uniformly in the set $\vec{E}_{LR}$ and $\vec{E}_{RL}$. Furthermore, for j even, $\Pr[v_j = v] = d_v/|E(G)|$ for all $v \in V_L$, and for j odd, $\Pr[v_j = v] = d_v/|E(G)|$ for all $v \in V_R$. Then,

$$\begin{aligned}
\log_2 \mathbb{E}[n_{2i+1}(e)] &\ge \mathbb{E}[\log_2 n_{2i+1}(e)] \\
&\ge H[\vec{e}_0\vec{e}_1 \ldots \vec{e}_{2i+1} \mid \vec{e}_0] \\
&= \sum_{j=0}^{i} H[\vec{e}_{2j+1}|\vec{e}_{2j}] + \sum_{j=1}^{i} H[\vec{e}_{2j}|\vec{e}_{2j-1}] \\
&= \sum_{j=0}^{i} \mathbb{E}[\log_2(d_{v_{2j+1}} - 1)] + \sum_{j=1}^{i} \mathbb{E}[\log_2(d_{v_{2j}} - 1)] \\
&\ge (i+1)\log_2(d_R - 1) + i\log_2(d_L - 1) \\
&= \log_2(d_R - 1)^{i+1}(d_L - 1)^i.
\end{aligned}$$

where to justify the first inequality we use Jensen's inequality for the concave function log, to justify the second we use the fact that the entropy of a random variable is at most the log of the size of its support, and to justify the last we use Jensen's inequality for the convex function $x\log_2(x-1)$ $(x \ge 2)$. The claim follows by exponentiating both sides.

(b) Similarly,

$$\log_2 \mathbb{E}[n_{2i}(e)] \ge \log_2(d_L - 1)^i(d_R - 1)^i. \qquad \square$$

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Chapter 11
Permutation Groups and the Graph Isomorphism Problem

Sumanta Ghosh and Piyush P Kurur

Abstract In this article we discuss various algorithms for permutation group-theoretic problems and their connections to Graph Isomorphism. In the last part we examine the group representability problem on graphs, its connection to Graph Isomorphism, and discuss some open problems that arise in this context.

Keywords Graph isomorphism · Permutation groups

11.1 Introduction

One of the core ideas in mathematics is the notion of an isomorphism, i.e. structure preserving bijections between mathematical objects like groups, rings and fields. A natural computational question is to decide, given two such objects as input, whether they are isomorphic or not. In the context of undirected finite graphs, this problem is called the graph isomorphism problem and is the subject matter of this article. Informally, we say that two graphs are isomorphic if they are the same up to a renaming of their vertices, i.e. we have a bijection between the vertex sets that preserve the adjacency relation of edges. Many other isomorphism problems for explicitly presented finite mathematical structures like groups, for example, reduce to the graph isomorphism problem. Also, many problems that arise in practice, like studying the structure of chemical compounds, are essentially graph isomorphism in disguise. Hence, understanding this problem computationally is important.

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There are efficient programmes and libraries (see NAUTY for instance) that can solve large instances of graph isomorphism that arise in practice. However, there are no known polynomial-time algorithm for the general case. In complexity theory, ever since the notion of NP-completeness has been formalised, the graph isomorphism problem has had an important status as it is believed to be a natural example of a problem of intermediate complexity [GJ79, Chap. 7], i.e. neither in P nor NP-complete: It is known that the graph isomorphism problem is in the complexity class co-AM [BHZ87], a randomised version of co-NP, and its NP hardness will result in the collapse of the polynomial hierarchy [BHZ87, Sch87]. Furthermore, Köbler et al. [KST92] showed that graph isomorphism is low for the counting class PP by showing its membership in LWPP. This was further improved by Arvind and Kurur [AK02] to SPP. As a result graph isomorphism is in and low for various counting complexity classes like classes like ⊕P etc. Thus, under reasonable complexity theoretic assumptions, graph isomorphism is *not* NP-hard. Ladner [Lad75] proved the existence of an infinite hierarchy of problems of intermediate complexity assuming that P is different from NP. The graph isomorphism problem, for reasons stated above, is believed to be a natural example.

In this article, we study graph isomorphism and related problems. There is now a vast literature on graph isomorphism and we really cannot do justice to the topic in such a short article. For a detailed study of graph isomorphism, mainly from a complexity theoretic view point, we refer the reader to the excellent book by Köbler et al. [KST93]. This paper concentrates on one of the aspects of the graph isomorphism problem, namely its intimate connection to permutation group algorithms. Permutation groups arise in the study of graph isomorphism problem because of its close relation to the graph automorphism problem. For a graph X, the automorphisms, i.e. isomorphisms from X to itself, forms a group under function composition. We can identify this group as a subgroup of the set of all permutations of the vertex set $V(X)$. Automorphisms, thus are symmetries of the graph. The computational problem of computing a generating set of the automorphism group is equivalent to the graph isomorphism problem [Mat79]. Most algorithms for graph isomorphism that make use of permutation group theory makes use of this connection. Understanding the automorphism group of a graph is also in tune with what is now a guiding principle in much of modern mathematics: understanding objects by understanding their symmetries.

11.2 Preliminaries

We briefly review the group theory required for this article mainly to fix notation and convention. For details please refer any standard book on group theory, for example, Marshall Hall [Hal59]. The *trivial group* that contains only the identity is denoted by 1. For a group G, we use the notation $H \leqslant G$ (or $G \geqslant H$) to denote that H is a subgroup of G. The *right coset* of the subgroup H of G associated with an element $g \in G$ is the set $Hg = \{hg | h \in H\}$. The set of all right cosets form a partition of G

and any subset T of G that has a unique coset representative from each right coset is called a *right transversal* of H in G. Analogously, we define left cosets and left transversals. In general, the right coset Hg and the left coset gH is different. We say that H is a *normal subgroup* if $gH = Hg$. We use the notation $H \trianglelefteq G$ (or $G \trianglerighteq H$) to denote that H is a normal subgroup of G.

A *simple group* is a group that has no non-trivial normal subgroups. A *composition series* of a group G is a tower of subgroups $G = G_0 \trianglerighteq G_1 \trianglerighteq \cdots \trianglerighteq G_t = 1$ such that each of the factor groups G_i/G_{i+1}, called the *composition factors*, are simple. The Jordan-Hölder theorem states that for any group G, its composition series is essentially unique, i.e. any two composition series are of equal length and the list of composition factors are equal up to a reordering. *Solvable groups* are those whose composition factors are abelian.

The set of all permutation of n elements forms a group called the *symmetric group* which we denote by S_n. In algorithmic settings, it is often useful to make the domain of n elements explicit: For a finite set Ω, the set $\mathrm{Sym}\,(\Omega)$ denote the group of permutations on Ω. By a *permutation group* on Ω we mean a subgroup of the symmetric group $\mathrm{Sym}\,(\Omega)$. As is customary, we use Wielandt's notation [Wie64]: Let α be any element of Ω and let g be a permutation in $\mathrm{Sym}\,(\Omega)$, the image of α under g is denoted by α^g. The advantage of this notation is that it follows the familiar laws of exponentiation: $(\alpha^g)^h = \alpha^{gh}$. We can extend this notation to (i) subsets of permutations: $\alpha^A = \{\alpha^g | g \in A\}$, or to (ii) subsets of Ω: $\Sigma^g = \{\alpha^g | \alpha \in \Sigma\}$. In particular, for a permutation group on Ω, the set α^G is called the *G-orbit* of α. Given any two elements α and β of Ω the G-orbits α^G and β^G are either disjoint or are the same. Thus, orbits of G partition the underlying set Ω. A subset Σ of G is said to be *G-stable* if $\Sigma^g = \Sigma$. Clearly any G-orbit is G-stable. In general, a G-stable set is a union of G-orbits.

Let G be a permutation group acting on the set Ω and let Σ be a subset of Ω. The *point-wise stabiliser* of Σ is the subgroup of all g in G that is trivial on Σ, i.e. $\alpha^g = \alpha$ for all α in Σ. The *setwise stabiliser* is the subgroup that fixes the set Σ as a whole, i.e it is the subgroup of all g in G such that $\Sigma^g = \Sigma$.

A permutation group G is said to be *transitive* if the entire set Ω is a single orbit. Equivalently, G is transitive if for any two elements α and β in Ω there is a permutation g in G such that $\alpha^g = \beta$. For a transitive permutation G on Ω, a subset Σ is said to be a *G-block* if for any permutation g in G, the set Σ^g is either *identical to* or *disjoint from* the set Σ. Any singleton set is a block and so is the entire set Ω. These blocks are called the *trivial blocks* of G. For a transitive permutation group G on Ω and a permutation g in G, the set Σ^g is a G-block whenever Σ itself is one. Such a block Σ^g is called a *conjugate block* of Σ. The family of conjugate blocks $\{\Sigma^g | g \in G\}$ forms a partition of the set Ω which is called the *block system* associated with the block Σ. A permutation group that has no non-trivial block is called a *primitive permutation group*. An example of a primitive group is the group $\mathrm{Sym}\,(\Omega)$. We have the following lemma about block systems which is more or less direct from the definition.

Lemma 2.1 *Let G be a transitive permutation group on Ω and let Σ be a block. Let N denote the subgroup of G that setwise stabilises all the elements in the Σ-block system $\mathcal{B}(\Sigma) = \{\Sigma^g | g \in G\}$. Then N is a normal subgroup of G and G/N acts as a permutation on Σ-block system $\mathcal{B}(\Sigma)$. In addition, if Σ is a maximal G-block then this action of the group G/N is primitive.*

In algorithms that deal with permutation groups, we need a succinct way to encode them which we now describe. Any permutation of Ω can be presented by an array of $\#\Omega$ elements and hence can be encoded as a string of size $O(n \lg n)$. A permutation group is presented via a list of permutations that generate the group. It is a well-known fact that any group G has a generating set of size less than $\lceil \lg \#G \rceil$ and hence this presentation of permutation group is reasonable. Thus, we assume that the input size, for an algorithm that takes a generating set S of a permutation group G on Ω, is $\#S + \#\Omega$. Similarly, an algorithm that is expected to produce a permutation group as output, should output a generating set of size polynomial in $\#\Omega$. For example, the strong generating set that we describe in the next section, is of size at most $\#\Omega^2$.

By a graph, we mean an undirected graph, i.e. a finite set of vertices and an edge set which is a subset of unordered pairs of vertices. We use $V(X)$ and $E(X)$ to denote the set of vertices and the set of edges of a graph X, respectively. A bijection f from $V(X)$ to $V(Y)$ is an *isomorphism* if for every two vertices u and v of X, the unordered pair $\{u, v\}$ is an edge of X if and only if $\{f(u), f(v)\}$ is an edge of Y. An *automorphism* of a graph X is an isomorphism from the graph to itself. The set of automorphism of a graph X, denoted by Aut (X), form a group under composition. In fact, Aut (X) is a permutation group on $V(X)$.

In the article, we assume that a graph of n vertices is encoded as an n^2-bit strings that represent its $n \times n$ adjacency matrix. We now define the *graph isomorphism problem*.

Problem 2.2 (*Graph isomorphsim problem*) The *graph isomorphism problem* (GI for short) is defined as follows: Given two undirected graphs X and Y via their adjacency matrix, decide whether they are isomorphic.

The counting version of the graph isomorphism problem, denoted by #GI, is the problem of computing the number of isomorphism between the two input graphs (0 when they are not isomorphic).

Graph isomorphism problem is closely related to the *automorphism problem* that we define next.

Problem 2.3 (*Automorphism problem*) The *automorphism problem* (AUT for short) is the problem of computing a strong generating set of the automorphism group Aut (X) of an input graph X.

Mathon [Mat79] proved that the problems GI, #GI and AUT are all polynomial-time Turing reducible to each other. Therefore, in the setting of permutation group algorithms, it is often the automorphism problem that is attacked for solving graph isomorphism.

A graph X is said to be *rigid* if it has no non-trivial automorphism, i.e. if Aut (X) is the trivial group. We now define the graph rigidity problem.

Problem 2.4 (*Graph rigidity problem*) Given an input graph X via its adjacency matrix, check whether the graph is rigid.

Clearly, an oracle for the automorphism problem, or by Mathon's result [Mat79], the graph isomorphism problem, is sufficient to decide the rigidity of a graph. However, the other direction is open.

Open problem 2.5 Is the graph rigidity problem polynomial-time equivalent to the graph isomorphism problem.

An important variant of graph isomorphism is the isomorphism of coloured graphs. For this article, a *c-colouring* of a graph X, where c a positive integer, is a map from the vertex set $V(X)$ to the set of integers $1, \dots, c$. Given a c-colouring ψ, the ith *colour class* is subset $\psi^{-1}(i)$ of $V(X)$. A *coloured graph* is a tuple (X, ψ) of a graph X and colouring ψ. We often suppress the colouring ψ when it is understood from the context and just denote the coloured graph by X. Given two c-coloured graphs (X, ψ) and (Y, φ), an isomorphism f between the underlying graphs X and Y is a *coloured graph isomorphism* if it respects the vertex colours, i.e. for any vertex v of X, $\psi(v) = \varphi(f(v))$. An automorphism of a coloured graph is analogously defined. Clearly coloured graph isomorphism generalises graph isomorphism as we can assume an ordinary graph as 1-coloured graph. In the other direction, coloured graph isomorphism polynomial-time Turing reduces to the graph isomorphism problem. The key idea is the following *gadget construction*. For a coloured graph X, we construct a new graph $\tilde{X}$ by first adding, for each colour class i, a long path L_i (say of length $n + i + 1$). We then connect all the vertices of the colour class i to one of the end points of L_i. Given coloured graphs X and Y, any isomorphism between the modified graphs $\tilde{X}$ and $\tilde{Y}$ forces the vertices in a given colour class of X to be mapped to the vertices of the same colour class in Y due to the graph gadgets L_i. Therefore, the coloured graphs X and Y are isomorphic if and only if the modified graphs $\tilde{X}$ and $\tilde{Y}$ are isomorphic. The rigidity problem and the automorphism problem generalise naturally to coloured graphs as well.

The graph isomorphism problem and the automorphism problem can be defined for directed graphs as well. It turns out that these variants are polynomial-time Turing reducible to the undirected case. Therefore, in this article, we mostly concentrate on undirected graph isomorphism. Nonetheless, from the perspective of the isomorphism problem, there is an important subclass of directed graphs called *tournaments* that we define below.

Definition 2.6 A directed graph X is a *tournament* if for every two distinct vertices u and v, exactly one of the directed edge (u, v) or (v, u) exists in $E(X)$.

The automorphism group of a tournament cannot have a 2-cycle (why?), and hence has to be of odd order. This forces it to be *solvable* by Feit-Thompson theorem [FT63]. This property has been exploited by Babai and Luks [BL83] to give significantly efficient algorithms for tournament isomorphism.

11.3 Basic Polynomial-Time Algorithms

In this section, we mention some well-known polynomial-time algorithms for permutation group problems. The very first polynomial-time algorithm is the algorithm to compute the orbits of a permutation group. Let S be a generating set of the permutation group G then define a relation $\alpha \rightarrow_S \beta$ if there exists a g in S such that $\alpha^g = \beta$. It is easy to see that the symmetric, transitive closure of the relation $\rightarrow_s$ gives us all the G-orbits. We can thus compute the orbits efficiently by computing reachability.

Lemma 3.1 *There is a polynomial-time algorithm, which given a generating set S of a permutation group G on Ω and an $\alpha \in \Omega$, computes the orbit α^G.*

Many permutation group algorithms follows the general scheme of first reducing the problem to the transitive case by finding all the orbits of the group using the above lemma, and then restricting the group to the orbit. This is followed by a divide can conquer that is done on the blocks of the transitive action of the group. Thus, finding the blocks of a transitive permutation group is a crucial step in various algorithms. Let G be a transitive permutation group over Ω. Fix any two elements α and β in Ω and consider the graph $X_{\alpha,\beta}$ whose vertices are Ω and edges are $\{\alpha, \beta\}^G$. Let Σ be the smallest G-block containing both α and β then Sim's observed [Sim67] that vertices in any connected component C of the graph $X_{\alpha,\beta}$ is a G-block in the block system $\{\Sigma^g | g \in G\}$ associated with Σ. By running this algorithm on all pairs one can compute the set of minimal (as well as maximal) blocks of G-blocks.

Lemma 3.2 *There is a polynomial-time algorithm that takes as input the generating set of a transitive permutation group G on Ω and decides whether G is primitive or not. If the input group G is not primitive, then it computes a minimal (or maximal) G-block system.*

We already argued that a generating set of a group is a natural way to present a permutation group. A *strong generating set* is a special generating set of a permutation group that makes many computational tasks easy. Consider a permutation group G on the set Ω. Fix an ordering $\{\alpha_1, \cdots, \alpha_n\}$ on the set Ω and let $G^{(i)}$ denote the subgroup of G that fixes the first i elements of Ω, i.e. the subgroup of all elements g of G such that $\alpha_j^g = \alpha_j$ for all $1 \leq j \leq i$. Consider the *tower* $G = G^{(0)} \geqslant \cdots \geqslant G^{(n-1)} = 1$ of subgroups of G. Let C_i be any set of permutations that has exactly one element from each right coset of $G^{(i)}$ in $G^{(i-1)}$, i.e. C_i is a *right transversal* of $G^{(i)}$ in $G^{(i-1)}$. Given any permutation g in G, there is an unique element, say h_1, in C_1 which is in the same right coset of $G^{(1)}$ as that of g. It is easy to see that $g' = gh_1^{-1}$ is in $G^{(1)}$. Continuing this argument with g' and the group $G^{(1)}$, it is easy to see that any element g can be expressed as a product $h_1 \ldots h_{n-1}$, $h_i \in C_i$. In fact, if the transversals C_i's are fixed, the above product representation is unique. Thus, $\cup_i C_i$ forms a generating set of G which we call the *strong generating* set of G. Many computational tasks become trivial once the strong generating set is calculated. For example, the uniqueness of the product representation of g shows that the order of the group $\#G$ is the product of the sizes $\prod_i \#C_i$.

We now describe the algorithm to compute the strong generating set of a permutation group that was given in its complete form by Furst et al. [FHL80] based on ideas from Sims [Sim78]. It is based on the following lemma due to Schreier and hence it (and similar algorithms) are some times called *Schreier-Sims algorithm*.

Lemma 3.3 (Schreier's Lemma) *Let G be a group and H be a subgroup of G. Let T be any right transversal of H in G that contains 1 as a coset representative. For each g in G, let $\overline{g}$ denote the unique coset representative of Hg in T. Let S be a generating set for G then set*

$$S' = \{ts(\overline{ts})^{-1} | t, s \in S\}$$

generates the group H.

Let S be the generating set of a permutation group G. The main idea of the algorithm is that once we have a right transversal C_1 of $G^{(1)}$ in $G^{(0)} = G$, we can use Schreier's Lemma 11.3.3 to compute the Schreier generating set for $G^{(1)}$. We then recursively compute the strong generating set for $G^{(1)}$. At each stage of the algorithm, we compute the right transversal C_{i+1} and recurse on the Schreier generating set of $G^{(i+1)}$ obtained in that stage.

The right transversal C_1 is computed by starting with the set $T_0 = 1$ and inductively compute T_{i+1} as follows: T_{i+1} is the union of T_i and a subset of $T_i S$ such that T_{i+1} does not contain any redundant representative of same right coset of $G^{(1)}$, i.e. T_{i+1} does not contain two distinct elements g_1 and g_2 such that $\alpha_1^{g_1} = \alpha_1^{g_2}$. If at some point $T_{i+1} = T_i$, we stop the procedure. Since the set C_1 can at most have n elements this procedure has to terminate in polynomially many steps. The actual algorithm [FHL80] can be significantly more efficient by computing all the transversals C_i's simultaneously through a sifting procedure. We summarise all the polynomial-time solvable tasks that uses the Schreier-Sims procedure in the following lemma.

Lemma 3.4 (Furst, Hopegraft and Luks) *There are polynomial-time algorithms for the computational tasks:*

1. *computing a strong generating set,*
2. *computing the order of a permutation group,*
3. *checking the membership of a permutation $g \in Sym(\Omega)$ in a given permutation group G.*

The Schreier-Sims algorithm can be generalised to find the generating set of a subgroup H of a permutation group G, given indirectly by a membership oracle, provided the index $\frac{\#G}{\#H}$ is small. We state this in the next lemma.

Lemma 3.5 *There is algorithm that takes as input a permutation group G on Ω via a generating set S and computes the generating set of a subgroup H of G given via a membership oracle, i.e. a procedure to test whether a given element g of G is actually an element of H. The algorithm takes time polynomial in $\#S$, $\#\Omega$ and the index $\frac{\#G}{\#H}$.*

Consider a permutation group G on Ω and let Δ be any subset of Ω. The *point-wise stabiliser* of the set Δ, which we denote by $G(\Delta)$ is the subgroup of all elements of G that fix every element of G, i.e. $G(\Delta) = \{g | \delta^g = \delta, \ \forall \delta \in \Delta\}$. It is easy to see that finding the point-wise stabiliser of any subset of Ω can be done in polynomial-time by adapting the Schreier-Sims algorithm.

11.4 Divide and Conquer Algorithms for Permutation Groups

We now illustrate a general technique that is used in many permutation group algorithms by giving an algorithm to find the setwise stabiliser for special groups. Although superficially similar to point-wise stabiliser, computing the setwise stabiliser is a different ball game. It is at least as hard as graph isomorphism: For a graph X, consider the group $G = \mathrm{Sym}\,(V(X))$ acting on the set $\Omega = \binom{V(X)}{2}$. The automorphism group of the graph X is the set-wise stabiliser of the subset $E(X)$ of Ω. The setwise stabiliser problem is a variant of a more general problem which we define below.

Problem 4.1 (*Colour preserving subgroup*) Let G be a permutation group on Ω which is partitioned into k-colour classes $\mathcal{C} = \{C_i\}_{i=1}^{k}$. Compute the subgroup of G that stabilises each of the colour class C_i, i.e. compute the subgroup $\{g \in G | C_i^g = C_i\}$.

The setwise stabiliser problem is the special case when the number of colours is 2. While we cannot expect a polynomial-time algorithm for this problem in general without solving the graph isomorphism problem, for special groups, we can solve the above in polynomial-time. For example, if we know that the input group G is solvable then we have a polynomial-time algorithm. In fact, the polynomial-time algorithm of Luks [Luk82] for trivalent graphs uses such a subroutine as the group that occurs there is a 2-group and hence is solvable.

We now given a sketch of the algorithm, detail of which can be found in the paper by Luks [Luk82]. To avoid notation clutter we fix an input group G and the colouring $\mathcal{C}$ of Ω. We say that a permutation g preserves colours of all elements in the subset Σ if for all $\alpha \in \Sigma$, α and α^g are in the same colour class. Let H be a subgroup of G and Σ an H-stable subset of Ω. We denote $\mathrm{CP}\,(H, \Sigma)$ to be the subset of H that preserves the colours of elements of Σ. Our task is to compute $\mathrm{CP}\,(G, \Omega)$. For the divide and conquer algorithm to work, we need to generalise the problem to cosets of permutation groups: We need to compute $\mathrm{CP}\,(Hg, \Sigma)$ for the coset Hg of the subgroup H of G where the set Σ is H-stable. Note that Σ is not necessarily stabilised by elements of Hg.

The set $\mathrm{CP}\,(Hg, \Sigma)$ has the following crucial properties which follows more or less directly from the definitions.

Lemma 4.2 1. *The set* $\mathrm{CP}\,(H, \Sigma)$ *is a subgroup of* H.
2. *The set* $\mathrm{CP}\,(Hg, \Sigma)$ *is either empty or is a coset of the group* $\mathrm{CP}\,(H, \Sigma)$.

3. *Suppose Σ is the disjoint union $\Sigma_1 \uplus \Sigma_2$ both of which are H-stable then* $\mathrm{CP}(Hg, \Sigma) = \mathrm{CP}(\mathrm{CP}(Hg, \Sigma_1), \Sigma_2)$.

It is crucial that $\mathrm{CP}(Hg, \Sigma)$ is a coset (item 2 in the above lemma) because we can then succinctly represent the set by giving a generating set of $\mathrm{CP}(H, \Sigma)$ and the coset representative.

We are now ready to give the outline of the divide and conquer algorithm for computing $\mathrm{CP}(Hg, \Sigma)$.

Reduction to transitive case Let $\Sigma' \subset \Sigma$ be any H-orbit. We first compute the group $\mathrm{CP}\left(Hg, \Sigma'\right)$ which is the transitive case of the above problem. Let $\Sigma = \Sigma' \uplus \Sigma''$. We use the fact that $\mathrm{CP}(Hg, \Sigma) = \mathrm{CP}\left(\mathrm{CP}\left(Hg, \Sigma'\right), \Sigma''\right)$.
Transitive case For this case H acts transitively on Σ. Let Δ be a *maximal* H-block of H and let $\mathcal{B}(\Delta) = \{\Delta_1, \ldots, \Delta_k\}$ be the associated block system. Let N be the normal subgroup of H that fixes all the blocks $\mathcal{B}(\Delta)$ setwise. Then we have $H = \uplus_x Nx$ as a disjoint union of cosets of N. We can then compute the set $\mathrm{CP}(Hg, \Sigma)$ by taking the union of all the cosets $\mathrm{CP}(Nxg, \Sigma)$ which are not empty.

If the number of cosets Nx are polynomially bounded then we can compute a generating set for N using Lemma 11.3.5. It then amounts to recursively computing the polynomially many cosets $\mathrm{CP}(Nxg, \Sigma)$ and combining the nonempty ones. The number of cosets Nx that is considered in the transitive case is the same as the order of the quotient group H/N. Since the block Δ that we choose is the maximal block, the quotient group H/N, as a permutation group on the set $\mathcal{B}(\Delta)$, is a primitive group (See Sect. 2.1). Thus, we need a bound on the size of a primitive permutation group. While the order of a primitive permutation group on n elements can be exponential in n, consider the case of the primitive group S_n for example, for solvable primitive permutation groups, a result by Pálfy [Pál82, Theorem 1] gives us the polynomial bound we are looking for.

Theorem 4.3 (pálfy) *There are absolute constants C and c such that any solvable primitive permutation group on Ω is of size less than $C \cdot \#\Omega^c$.*

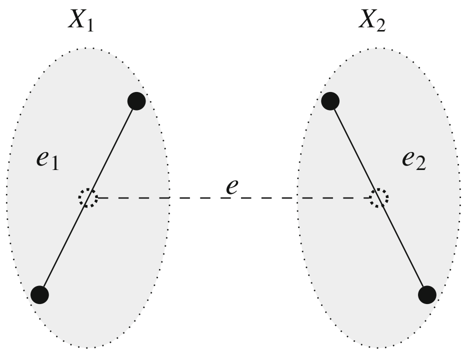
The above bound has a generalisation to groups with bounded non-abelian composition factors: Let Γ_d denote the class of groups such that each composition factor is either abelian or is isomorphic to a subgroup of S_d. Babai et al. [BCP82] generalised the Pálfy's bound to the class Γ_d.

Theorem 4.4 (Babai, Cameron and pálfy) *There are absolute constants C and c such that for any positive integer d, any primitive permutation group on Ω in the class Γ_d is of size less than $C\#\Omega^{cd}$.*

As a result, the colour preserving subgroup problem is solvable in polynomial-time for groups that are in the class Γ_d.

Lemma 4.5 *Colour preserving subgroup problem is solvable in polynomial-time for the class of solvable groups and the class of groups in the family Γ_d for constant d.*

Fig. 11.1 Luks gadget for bounded valence graphs



We now discuss a natural context where the colour stabiliser problem for groups in the class Γ_d occurs. Consider the graphs of valence d, i.e. all vertices are of degree less than or equal to d. Luks [Luk82] gives a polynomial-time algorithm for this class of graphs by reducing it to the colour preserving subgroup problem where the input group G is in the class Γ_{d-1}. We quickly give a sketch.

Fix the two input graphs X_1 and X_2. We assume that the graphs are connected, otherwise we run the algorithm for each pair of connected components. Furthermore, we restrict our attention to checking whether X_1 and X_2 are isomorphic via an isomorphism that maps a particular edge e_1 of X_1 to an edge e_2 of X_2: We just need to repeat the procedure for all such pairs of edges to know whether X_1 and X_2 are isomorphic.

Consider the new graph which we denote by Z which is essentially the disjoint union of the graph X_1 and X_2 with an additional edge e that connects the mid points of e_1 and e_2 (see Fig. 11.1). If $d > 2$ is the maximum degree of any vertex in the input graph then Z also has degree bounded by d. Luks algorithm for bounded valence computes the group subgroup $\text{Aut}\,(Z)_e$ of $\text{Aut}\,(Z)$ that maps the auxiliary edge e to itself. It is clear that the input graphs X_1 and X_2 are isomorphic if and only if there is at least one element in $\text{Aut}\,(Z)_e$ (and therefore in any generator set) that flips the edge e. The algorithm then proceeds to compute the group $\text{Aut}\,(Z)_e$. First the graph Z is layered as follows: Let the ith layer of Z, denoted by Z_i, be the subgraph which contains all the edges (as well as the end points) at a distance i from the auxiliary edge e. In particular, the graph Z_0 consists of just the edge e and its end points. All automorphisms of Z that stabilises the edge e has to preserve this layered structure. The group $\text{Aut}\,(Z)_e$ is then computed by inductively computing the groups $G_i = \text{Aut}\,(Z_i)_e$.

Inductively, Luks proves that G_i's are in the class Γ_{d-1} as follows: Let H_{i+1} denote the subgroup of G_i that is obtained by restricting elements of G_{i+1} on Z_i and let K_{i+1} be the associated kernel, i.e. the subgroup of G_{i+1} that is trivial when restricted to Z_i. For a fixed vertex u in layer i, i.e. in the graph $Z_i \setminus Z_{i-1}$, is connected to at most $d - 1$ vertices in the layer $i + 1$ and these vertices have to be mapped within themselves by elements of K_{i+1}. Therefore, K_{i+1} is a subgroup of a product of m many copies of the symmetric groups S_{d-1} for some positive integer m. The quotient group G_{i+1}/K_{i+1} is the group H_{i+1}, which itself is a subgroup G_i, a group in the class Γ_{d-1}. This is possible only if G_{i+1} is in Γ_{d-1}: Consider any composition series of G_{i+1} which passes through the normal subgroup K_{i+1}. The composition factors are either composition factors of H_i or that K_{i+1}.

Having computed G_i the algorithm computes G_{i+1} by computing (1) a generating set for the kernel K_{i+1} and (2) a set of elements of G_{i+1} whose restriction to G_i generates H_i. It is this inductive step that requires the solution of colour preserving subgroup problem and luckily the input group (G_i in our case) turns out to be in the class Γ_d and hence solvable by the algorithm in Lemma 11.4.5. Thus, we have the following theorem.

Theorem 4.6 (Luks) *Consider the family $\mathcal{G}_d$ of graphs whose vertices are of degree bounded by d. There is a $n^{O(d)}$ algorithm to decide isomorphism of graphs in $\mathcal{G}_d$.*

The current fastest algorithm for graph isomorphism [ZKT85] is based on a valence reduction step together with the application of the above theorem of Luks for bounded valence graphs. Therefore, any improvement on the bounded valence case will improve the state of the art for the general graph isomorphism problem.

11.5 Lexicographically Least Permutations

We now mention some results that make use of the ordering of permutations induced by the ordering on the domain Ω. First, note that a total ordering on the set Ω gives a total ordering on Sym (Ω): for distinct permutations g and h, $g < h$ if at the first (in the ordering on Ω) element α in Ω where they differ, we have $\alpha^g < \alpha^h$. We call this ordering the *lexicographic ordering* on the permutations. Under this ordering the lexicographically least permutation is the identity permutation. The first problem that we study is the problem of computing the lexicographically least element in a coset.

Problem 5.1 (*lexicographically least in a coset*) Given a permutation group G on Ω as a set of generators and an arbitrary permutation x on Ω, compute the lexicographically least element in the coset Gx.

Clearly if x is in G then the coset is the group G itself and the lexicographically least element of the coset is the identity element. We now sketch a polynomial-time algorithm for this problem.

Let α be the least element of Ω. The set of images of α under permutations in the coset Gx is given by the set $\alpha^{Gx} = (\alpha^G)^x$ which can be computed easily once the orbit α^G of α is computed. Clearly, the lexicographically least element of Gx should map α to the least element β of α^{Gx}. We can also compute, using the transitive closure algorithm for orbits, an element x_1 in the coset Gx such that $\alpha^{x_1} = \beta$. Therefore, the lexicographically least element of Gx is also the lexicographically least element in the coset $G_\alpha x_1$ as this coset is precisely the set of elements of Gx that maps α to β. A similar algorithm can be given for left cosets xG as well. We thus have the following lemma:

Lemma 5.2 *Computing the lexicographically least element in a coset can be done in polynomial-time.*

The above lemma is a key step in proving that the graph isomorphism problem is in the complexity class SPP.

Definition 5.3 (*SPP*) For a non-deterministic polynomial-time Turing machine let $\text{gap}(M, x)$ denote the *difference* in the number of accepting paths and rejecting paths of M on the input x. A language L is in the complexity class SPP if there is a polynomial time non-deterministic Turing machine M such that for all strings x in the language L, the $\text{gap}(M, x)$ is 1 and for all strings not in the language L, $\text{gap}(M, x)$ is 0.

Languages in SPP are believed to be of low complexity and are unlikely to be NP-hard. In particular, any *gap definable complexity* [FFK91] class not only contain SPP but also derive no extra computational power with a language in SPP as oracle, i.e. SPP is *low* for all these complexity classes. Gap definable complexity classes [FFK91] are counting classes defined using GapP functions, i.e. functions that are differences of accepting and rejecting paths of an NP machine, and contain many common counting complexity class like PP and ⊕P, etc.

The main idea involved in the proof is to design a polynomial-time algorithm A that makes queries to an NP language L with some restriction on the queries that A makes to L. We design a non-deterministic polynomial-time machine M for L such that for all queries the algorithm A makes, the machine M has at most one accepting path. Such an oracle machine can be converted to an SPP algorithm, i.e. an NP machine whose gap function is the characteristic function of GI, using standard techniques [KST92].

The base algorithm A is an inductive algorithm that builds the strong generating set of the automorphism group by computing the group G_i of all automorphisms that fix the first i vertices of the graph. To compute G_{i-1} from G_i, the algorithm has to query the NP-language L which essentially checks, given a $j > i$, whether there is an automorphism that maps i to j. The base polynomial-time machine can then find one such by doing a prefix search. However, we need to design an NP machine M for L such that for all queries asked by A, there is at most one accepting path. This we achieve as follows: The algorithm A also provides to L the generator set of G_i, i.e. queries to L are (encoding of) pairs $\langle S, j \rangle$ where S is a generating set of G_i at the i-th stage. We know that if there is an automorphism, say g in G_{i-1}, that maps i to j then the set of all such automorphisms form the coset $G_i g$. The machine M essentially guess the automorphism g that maps i to j if it exists and accepts only if g is the lexicographically least permutation in $G_i g$. Since there is only one such guess g, we know that for all queries that the algorithm A makes to L the machine M has at most one accepting path. The SPP result then follows as mentioned above.

Theorem 5.4 (Arvind and Kurur) *The graph isomorphism problem is in SPP.*

While computing the lexicographically least element in a coset has an efficient algorithm, consider the following generalisation to a double coset.

Problem 5.5 (*Lex-least in a double coset*) Given the generating sets of permutation groups G and H on a totally ordered set Ω and an arbitrary permutation x on Ω, compute the lexicographically least element in GxH.

The problem of computing the lex-least element in a double coset is intimately connected to the problem of graph canonisation which we define below.

Definition 5.6 (*Canonical forms for graphs*) Consider the class $\mathcal{G}(\Omega)$ of all graphs on the vertex set Ω. A function **CF** on $\mathcal{G}(\Omega)$ is a *canonical form* if it satisfies the following properties:

1. For every graph X in $\mathcal{G}(\Omega)$, $\mathbf{CF}(X)$ is isomorphic to X.
2. If X and Y are two isomorphic graphs then $\mathbf{CF}(X)$ is the same as $\mathbf{CF}(Y)$.

In other words, a canonical form **CF** picks a unique representative from each isomorphism class of graphs on Ω. Clearly graph isomorphism is solvable given a canonisation procedure. Therefore, one way of attacking the graph isomorphism problem is to give fast canonisation procedure. For many classes of graphs, Babai and Luks [BL83] gave an efficient canonisation procedure which is also currently the best general purpose algorithms. In particular, they were able to give an $O\left(n^{c \log n}\right)$ for tournaments. This canonisation procedure makes use of the fact that tournaments have a solvable automorphism group. They also show how Luks' polynomial-time algorithm for bounded valance [Luk82] can be modified and extended to a canonisation algorithm with essentially the same running time.

As opposed to computing the lexicographically least element in a coset, computing it in a double coset is known to be NP-hard [Luk93, Theorem 5.1] even when one of the group is solvable. However, in many contexts, particularly in relation with graph isomorphism and canonisation, we have some freedom to choose the underlying ordering of the set Ω. Can we reorder the set Ω so as to make it possible to apply the divide and conquer technique similar to that of the colour preserving subgroup problem that we saw in Sect. 11.4. Indeed this is the case provided the reordering is "compatible" with the divide and conquer structure of the group G. First, we need to generalise the lexicographically least element as follows: Consider an ordering $<$ on Ω. Let Δ be a G-stable set. We consider the restriction of the order $<$ on the set Δ. This gives a partial order on elements of permutations, we say that $g < h$ if for the least δ in Δ on which g and h differ, we have $\delta^g < \delta^h$. Under this restricted ordering there will be multiple lexicographically minimal elements. We now describe how to build a new ordering $\prec$ on Ω under which it is feasible to compute the lexicographically least element of the double coset GxH.

Ordering the orbit Fix an ordering between the G-orbits by picking say the least element in each of them. If Ω_1 and Ω_2 are two orbits such that $\Omega_1 < \Omega_2$ in the above chosen order, then we set every element of Ω_1 to be less than that of Ω_2 under the new ordering $\prec$. The motivation of this reordering is the following: Let $\Omega = \Omega_1 \uplus \Omega_2$ then computing the lexicographically least element with respect to the new ordering $\prec$ can be done by first computing the lexicographically minimal elements with respect to the restriction of $\prec$ on Ω_1 and then from them picking the lexicographically least element with respect to the restriction of $\prec$ on Ω_2.

Ordering within orbits When G is transitive, we do the reordering with respect to the blocks. We pick a maximal G-block Δ in a canonical way. The Δ block system partition the set Ω so reorder it pretty much the same way as in the previous case using the Δ block system instead. If N denotes the normal subgroup of G that fixes all the blocks in the Δ block system, we can recursively find the lex-least elements in double cosets $NgxH$ and find the minimal ones out of it. If G is in the class Γ_d, the number of subproblems are polynomially bounded by Theorem 11.4.4.

The above reordering can be formalised in terms of the *structure forest* of the group G. The structure forest is the collection of *structure trees* one for each G-orbit. For an orbit Σ, the structure tree is a tree where the leaves are elements of Σ. Each internal node v is associated with a G-block Δ_v with the following properties.

1. For any child u of v, the G-block Δ_u is a maximal block contained in Δ_v.
2. If $\{u_1, \ldots, u_k\}$ are the set of children of v then the blocks Δ_{u_i}'s are all conjugates of each other and partition the parent block Δ_v.

This structure forest captures the divide and conquer on G. The elements of Ω can be reordered once we compute the structure forest of G: The structure trees are ordered in the order of the least element in them. For each internal node v and children u_1 and u_2, $u_1 \prec u_2$ if the least element of the associated block in u_1 is smaller than that of u_2 according to the original ordering $<$. This will finally give an ordering $\prec$ on the entire set Ω. Thus, we have the following theorem (See the survey article by Luks [Luk93] for details).

Theorem 5.7 *Given a totally ordered set Ω, permutation groups G and H on Ω and a permutation x on Ω. Suppose G is in the class Γ_d then in time polynomial in n we can compute a new ordering $\leq_G$ such that computing the lex-least element of the double coset GxH can be done in $n^{O(d)}$.*

Notice that we do not have any restriction whatsoever on the group H.

11.6 Structure of Primitive Groups

In the last two sections, we saw how bounds on the order of primitive permutation group can be crucial in the runtime analysis of various divide and conquer algorithms for permutation groups. We now mention how knowing the actual structure of primitive groups are computationally useful. This section is mainly motivated by the study of *bounded colour class graph isomorphism problem.*

Problem 6.1 (*Bounded colour class graph isomorphism*) Fix a constant b. Given two coloured graph X and Y such that the number of vertices in any given colour class is bounded by b decide whether the graphs are isomorphic.

We abbreviate this problem as BCGI_b. This restricted graph isomorphism problem does have polynomial-time algorithm but what about fast parallel algorithms? Luks [Luk86] answered this question affirmatively by giving a reduction to a restricted point-wise stabiliser problem and solving it in NC. Further careful analysis by Arvind et al. [AKV05] showed that the problem lies in the $\mathrm{Mod}_k\mathrm{L}$ hierarchy. Together with the hardness for this class [Tor04], we have a fairly tight classification of this variant of graph isomorphism.

We now explain the overall structure of the algorithm for BCGI_b. Both Luks [Luk86] and Arvind et al. [AKV05] reduce the bounded colour graph isomorphism problem to a restricted version of point-wise stabiliser problem which we now define.

Problem 6.2 (*bounded orbit point-wise stabiliser problem*) Given as input a set Ω, a subset Δ of Ω and a permutation group G on Ω such that the G-orbits are all of cardinality bounded by a constant c. Compute generating set of the point-wise stabiliser subgroup $G(\Delta)$.

We abbreviate this problem as PWS_c. As mentioned before, the above problem is solvable in polynomial-time. However, in this context, we are interested in providing a parallel algorithm. What needs to be exploited is that the G-orbits are bounded and thus G is actually a subgroup of a product of small symmetric groups, the symmetric groups on each of the G-orbits.

To see the connection of the PWS_c and BCGI_b isomorphism we consider the equivalent automorphism problem which we denote by AUT_b.

Lemma 6.3 (Luks) *The* AUT_b *problem logspace reduces to* $\mathrm{PWS}_{2^{b^2}}$ *problem.*

Here is the sketch of this reduction. Let X be the coloured graph and let $C_1, \ldots, C_m$ denote the colour classes into which the vertex set $V(X)$ is partitioned. The automorphism group is a subgroup of the product group $G = \prod_i \mathrm{Sym}\,(C_i)$. We expressed the automorphism group $\mathrm{Aut}\,(X)$ as a point-wise stabiliser of G on its action on a different set Ω that we construct as follows: Define the set $C_{i,j}$ to be the set of unordered pairs $\{u, v\}$ where $u \in C_i$ and $v \in C_j$ and let $E_{i,j}$ be the subset of edges of X between the colour classes C_i and C_j. Define the set $\Omega_{i,j}$ to be the power set of $C_{i,j}$ then the edge sets $E_{i,j}$ are actually *points* or element of $\Omega_{i,j}$. Consider the natural action of G on the union $\Omega = \cup_{i,j}\Omega_{i,j}$ and let Δ be the subset of all the *points* $E_{i,j}$ of Ω. It is easy to see that the point-wise stabiliser of G with respect to the subset Δ is actually the automorphism group $\mathrm{Aut}\,(X)$. Notice that each of the set $\Omega_{i,j}$ are G-stable and hence the orbits of G are at most of size $2^{\binom{b}{2}}$.

Both Luks [Luk86] and Arvind et al. [AKV05] then solve the PWS_c problem. While the actual algorithms of Luks [Luk86] and Arvind et al. [AKV05] are fairly technical, we attempt to explain the essence of the algorithm and the permutation group theory involved in those results.

The group G in question can be seen as a product of groups $G = \prod_i G_i$ where G_i is the action of G on the ith orbit. For each of the groups G_i, compute a special *normal series* $G_i = N_{i,0} \rhd \cdots \rhd N_{i,t}$ and let N_k denote the produce $\prod_i N_{i,k}$. The algorithm does a divide and conquer to compute $N_k(\Delta)$ going one level at a time. The base case of this divide and conquer is when the group $N_{i,s}$ hits a socle. The *socle* of a group G is the group generated by the set of all minimal normal subgroups of G. The main group theoretic result that is used is the O'Nan-Scott theorem (See the book by Dixon and Mortimer [DM91] for a proof of this result) on the structure of the socle of a primitive permutation group and its point-wise stabiliser.

For the details of the algorithm, we refer the reader to the conference paper of Arvind et al. [AKV05]. A detailed version is available in the Ph.D thesis of Kurur [Kur06, Chap. 5]

11.7 Representation of Groups on Graphs

In this section, we look at the group representability problem. This problem was defined and studied by Dutta and Kurur [DK09] to explore the connection between graph isomorphism and permutation group algorithms from a representation theoretic point of view. Representation theory is the study of homomorphisms from a group to the group $\mathrm{GL}(V)$, the automorphisms of a vector space V. In the context of graph isomorphism, we would like to understand homomorphisms between groups and automorphisms of graphs.

Definition 7.1 A *representation* of a group G *on a* graph X is a homomorphism from the group G to the automorphism group $\mathrm{Aut}\,(X)$ of the graph X.

There is always a trivial representation that sends all the elements of the group to the identity automorphism. What we are interested in is a non-trivial homomorphisms. The main problem of interest in this section is the following [DK09].

Problem 7.2 (*Group representability problem*) Given a group G and a graph X, decide whether G has a non-trivial representation on X.

The hardness of the problem depends on how the group G is presented. We assume, unless otherwise mentioned, that G is provided to the algorithm via a multiplication table. Therefore, one can assume that the input size is $\#G + \#V(X)$. In studying its connection to graph isomorphism, we can restrict the problem in two ways: (1) restrict the groups to come from a natural class like, for example, solvable or abelian or (2) restrict the class of graphs to say planar graphs or trees.

The very first result that we have in this context is the following [DK09].

Lemma 7.3 (Dutta and Kurur) *The graph isomorphism problem is log-space many-one reducible to the abelian group representability problem.*

The main idea behind the proof is the following: Consider an instances of graph isomorphism where we want to check whether the graphs X and Y are isomorphic. We assume they are connected and have n vertices. For a prime p, consider the graph Z which is the disjoint union of $p - 1$ copies of X and 1 copy of Y. Suppose that the group Aut (Z) has a p-cycle say g. Consider any vertex u of Z such that $u^g \neq u$. It is easy to see that the orbit of u under the cyclic group generated by g has to have p-elements. Furthermore, if any two of the elements in this orbit is in the same connected component of Z then the entire orbit is. If the prime p is chosen to be greater than n then such a p-cycle necessarily has to permutes the components as each of the connected components of Z are of cardinality at most $n < p$. This is only possible if some copy of X in Z is mapped to the copy of Y and hence X and Y have to be isomorphic. Conversely, for any prime p, if X and Y are isomorphic, then the group Aut (Z) has a p-cycle. Thus, to decide whether X and Y are isomorphic, we need to check the group representability of the additive group of $\mathbb{Z}/p\mathbb{Z}$, on the graph Z for some prime p greater than the number of vertices in X. By Bertrand's postulate (it is actually a theorem but the name seems to be stuck) there is always a prime p between n and $2n$ which we chose for this purpose.

Notice that for the previous lemma, all we needed is to pick a prime p such that Aut (X) does not have p-cycle. Recall that tournaments have odd order automorphism group and hence we have the following result.

Theorem 7.4 *Tournament isomorphism is reducible to $\mathbb{Z}/2\mathbb{Z}$ representability.*

In this context, we have the following open problem.

Open problem 7.5 Is graph isomorphism reducible to $\mathbb{Z}/2\mathbb{Z}$-representability (or for that matter any fixed group representability).

What about the other direction, i.e. reduction from representability to isomorphism? Dutta and Kurur [DK09] prove the following result for solvable group representability.

Lemma 7.6 (Dutta and Kurur) *The representability problem for solvable groups is polynomial-time Turing reducible to graph isomorphism.*

For a group G, let G' be the commutator subgroup. The main idea is the following group theoretic fact.

Lemma 7.7 *A solvable group G is representable on X if and only if there is a prime p that divides both the orders G/G' and* #Aut (X).

From the above lemma it follows that to check representability for solvable groups, all we need is a way to compute the orders $\#G/G'$ and that of #Aut (X). Clearly the former can be computed easily as the group is presented as a multiplication table and the latter using an oracle to the automorphism problem (or equivalently the graph isomorphism problem). We can even assume that the group is presented as a permutation group because there are efficient algorithms to compute the commutator subgroup of a permutation group [FHL80, Theorem 4].

Thus as far as group representability is concerned, as long as we restrict the problem to solvable groups, we are within the realm of graph isomorphism. However, even for the simplest of the non-solvable case we do not have a satisfactory answer:

Open problem 7.8 (A_5 representability problem) Given a graph X decide whether there is a subgroup of Aut (X) which is isomorphic to the alternating group of A_5.

The importance of A_5 here is that it is the smallest example of a non-solvable group. Since A_5 is a simple group, non-trivial homomorphisms from it to Aut (X) can only be injections.

Torán [Tor04] showed that graph isomorphism is hard for a lot of parallel complexity classes like ⊕L etc. An important open problem in the context of graph isomorphism is whether it is hard for the complexity class P (under suitable reductions). If this is the case, it would give evidence that it is unlikely to have efficient parallel algorithms for graph isomorphism. We would like to pose the same question for the group representability problem

Open problem 7.9 Is the group representability problem hard for the complexity class P.

Are there reasons to believe that the group representability problem is harder than graph isomorphism? We really do not know. However, for the restricted case of representability on trees, we already have a difficulty. Graph isomorphism on trees can be done in polynomial time. In fact, even for planar graphs isomorphism testing can be done in linear time [HW74] or, if one is interested in the space-bounded classes, in logarithmic space [DLNTW09]. In contrast, [DK09] proved the following result for representability on trees.

Theorem 7.10 (Dutta and Kurur) *The problem of group representability on trees is Turing equivalent to the problem of testing, given an integer n in unary and group G via multiplication table, whether there is a non-trivial homomorphism to the symmetric group S_n or not.*

We call the problem of checking whether a group G has a homomorphism to S_n as *permutation representability problem* and is motivated by Cayley's theorem that states that every finite group is a subgroup of a symmetric group. However, finding the smallest n for which G is a subgroup seems to be hard although we admit that there are no known hardness result for the above problem.

11.8 Conclusion

In this article, we discussed the complexity of some permutation group algorithms and its close connection to graph isomorphism. Most of these algorithms perform a divide and conquer and it is here the structure of permutation groups plays a crucial role. Of particular interest are permutation group theoretic structures like orbits and blocks whose computation allows us to often reduce the general case to the primitive case. Also in most of these cases the primitive case is solvable if the group is known to be in some special class like solvable or Γ_d. This makes use of bounds on the sizes of primitive groups or, in some cases, their explicit structure. We also saw how these classes naturally arose in study of restricted versions of graph isomorphism. We therefore believe that a better understanding of permutation groups and is relation to graph isomorphism is crucial in pinning down the computational complexity of this elusive problem.

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