Chapter 8
Nonlinear Waves

This Chapter studies several nonlinear equations of wave propagation which admit the exact solutions by the inverse scattering transform. It analyzes also the amplitude and slope modulations obtained by the variational-asymptotic method which may be applied to non-integrable systems as well.

8.1 Solitary and Periodic Waves

Korteweg-de Vries Equation. Let us begin our study of nonlinear waves with the Korteweg-de Vries (KdV) equation [26]

$$u_{,t} + 6uu_{,x} + u_{,xxx} = 0. \tag{8.1}$$

This equation arose originally in the theory of shallow water waves, but it is now widely used to describe dispersive waves in various nonlinear media.[1] The constant factor 6 in front of the nonlinear term is conventional but of no great significance. The last term accounts for the dispersion. Due to the balanced effects of nonlinearity and dispersion, waves may propagate without changing their shape. To demonstrate this let us seek a particular solution of (8.1) in form of wave traveling with constant velocity c

$$u = \varphi(\xi), \quad \xi = x - ct,$$

which is similar to d'Alembert's solution for linear hyperbolic waves. Substitution of this Ansatz into (8.1) gives

$$-c\varphi' + 6\varphi\varphi' + \varphi''' = 0,$$

with prime denoting the derivative with respect to ξ. The integration yields

[1] Particularly, Zabusky and Kruskal [54] have shown that the KdV equation is the continuum limit of the equations governing the Fermi-Pasta-Ulam chain. Note that the original KdV equation [53] differs from (8.1) but can be brought to this form by a simple transformation.

$$\varphi'' = -3\varphi^2 + c\varphi - g,$$

where g is an integration constant. This resembles the equation of motion of mass-spring oscillator with a unit mass and a nonlinear restoring force derivable from the cubic potential energy $U(\varphi) = \varphi^3 - \frac{1}{2}c\varphi^2 + g\varphi$.

The first integral of the above equation is

$$\frac{1}{2}\varphi'^2 = -\varphi^3 + \frac{1}{2}c\varphi^2 - g\varphi + h.$$

In the special case when φ and its first derivative tend to zero as $\xi \to \pm\infty$, we may set $g = h = 0$. Then the first integral becomes

$$\varphi'^2 = \varphi^2(c - 2\varphi).$$

The corresponding phase curve in the (φ, φ')-plane is the separatrix shown in Fig. 8.1 for $c = 1$. It is seen that φ increases from zero at $\xi = -\infty$, rises to a maximum $\varphi_m = c/2$ and then decreases to zero as $\xi \to \infty$. The solution of the last equation can be found explicitly by quadrature and is given by

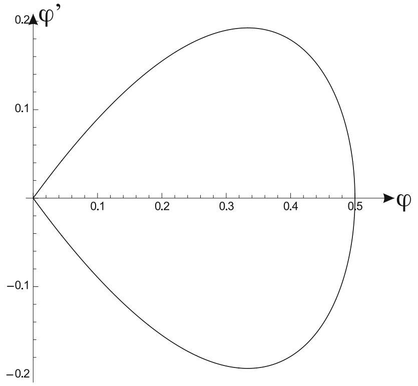

Fig. 8.1 Separatrix

$$\varphi(\xi) = \frac{c}{2}\text{sech}^2(\frac{\xi\sqrt{c}}{2}).$$

This particular solution is called a soliton. Mention that the solution remains still valid if $\xi = x - ct - d$, where d is any constant. Looking at this solution we can observe that: i) the wave speed of the soliton is twice its amplitude, ii) the width of the soliton is inversely proportional to the square root of the wave speed and therefore taller solitons are narrower in width and move faster than shorter ones. The shape of the solitary wave for $c = 1$ is shown in Fig. 8.2.

In general g and h differ from zero and

$$\varphi'^2 = p(\varphi),$$

where $p(\varphi)$ is a cubic polynomial having three simple zeros. For bounded solutions all zeros must be real, and the periodic solution must oscillate between two of them. Let the zeros be b_1, b_2, b_3, and we order them such that $b_1 > b_2 > b_3$. Then

$$p(\varphi) = -2(\varphi - b_1)(\varphi - b_2)(\varphi - b_3).$$

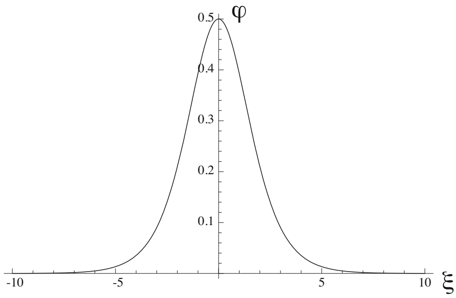

Fig. 8.2 Solitary wave of KdV equation

Since $p(\varphi) > 0$ for $\varphi \in (b_2, b_1)$, the solution oscillates between b_2 and b_1. So, let us define $a = b_1 - b_2$ as the amplitude of the wave. Comparing $p(\varphi)$ with that in the first integral, we find

$$c = 2(b_1 + b_2 + b_3), \quad g = b_1 b_2 + b_1 b_3 + b_2 b_3, \quad h = b_1 b_2 b_3.$$

It can easily be checked that the solution of the first integral is expressed in terms of Jacobian elliptic function cn as follows (see exercise 8.1)

$$\varphi(\xi) = b_2 + (b_1 - b_2)\,\mathrm{cn}^2(\sqrt{(b_1 - b_3)/2}\,\xi, m), \quad m = \frac{b_1 - b_2}{b_1 - b_3}.$$

Such periodic solutions are called cnoidal waves. As the period of $\mathrm{cn}^2(u,m)$ in its argument u is $2K(m)$, with $K(m)$ being the complete elliptic integral of the first kind, the wave length is

$$\lambda = \frac{2K(m)}{\sqrt{(b_1 - b_3)/2}}. \tag{8.2}$$

The phase velocity of this periodic wave packet is $c = 2(b_1 + b_2 + b_3)$. The solution can also be presented in the form

$$\varphi(\xi) = \psi(\theta) = \psi(kx - \omega t),$$

where $\psi(\theta)$ is the periodic function of period 2π. Since $k = 2\pi/\lambda$, we have for the frequency

$$\omega = ck = 2(b_1 + b_2 + b_3)k.$$

From (8.2), $b_1 - b_3$ is a function of λ and $a = b_1 - b_2$. In the special case $b_2 = 0$ the root b_3 can be expressed through a and the dispersion relation for these periodic waves takes the form

$$\omega = \Omega(k, a).$$

We see that the dispersion relation for nonlinear waves involves the amplitude, what is quite similar to nonlinear vibrations where the frequency depends also on the amplitude. If the amplitude of the wave is small, $a \ll 1$ and $m \to 0$ then $2K(m) \simeq \pi$, so $\omega \simeq 2b_3 k \simeq -4\frac{\pi^2}{\lambda^2}k = -k^3$, and we recover the dispersion relation of the linearized KdV equation

$$u_{,t} + u_{,xxx} = 0.$$

In contrary, if $b_3 \to 0$, $m \to 1$, and $a = b_1 \to c/2$, then the wavelength λ tends to infinity, and the solution approaches that of soliton.

Nonlinear Klein-Gordon Equation. We turn next to the nonlinear equation which is derivable from the following Lagrangian

$$L = \frac{1}{2}u_{,t}^2 - \frac{1}{2}u_{,x}^2 - U(u).$$

Euler-Lagrange's equation reads

$$u_{,tt} - u_{,xx} + U'(u) = 0. \tag{8.3}$$

This is the so-called non-linear Klein-Gordon equation which arises in various physical situations. This is especially true of the case $U(u) = 1 - \cos u$ known as the Sine-Gordon equation for which $U'(u) = \sin u$. It describes for instance free torsional vibrations of an elastic rod along which rigid pendulums are attached at close intervals. The pendulums cause additional restoring forces proportional to $\sin u$. Another mechanical problem leading to this equation deals with the motion of dislocations in crystals, where the $\sin u$ term occurs due to the periodic structure of the crystal lattice. Besides, it is used in modeling Josephson junctions, laser pulses and many other phenomena. The alternative choice $U(u) = u^2/2 + \alpha u^4/4$ arises in the problem of free vibrations of a pre-stretched string along which nonlinear springs with the cubic nonlinearity are attached at close intervals. Mention also that the small amplitude expansion of the Sine-Gordon equation leads to this model with $\alpha = -1/6$.

We look first for the soliton traveling with a constant velocity $c < 1$ in the form: $u = \varphi(\xi)$, $\xi = x - ct$. Substitution of this Ansatz into (8.3) gives

$$(1 - c^2)\varphi'' - U'(\varphi) = 0.$$

This resembles the equation of motion of mass-spring oscillator with a mass $m = 1 - c^2$ and a nonlinear restoring force derivable from the potential energy $-U(\varphi)$. The first integral is

$$\frac{1}{2}m\varphi'^2 - U(\varphi) = h.$$

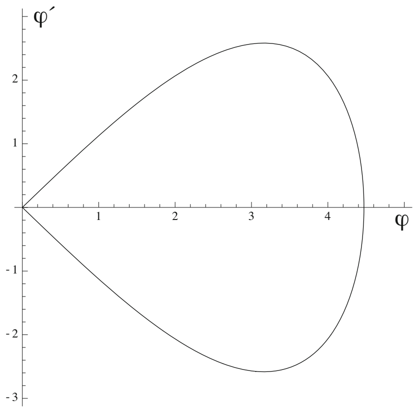

Fig. 8.3 Separatrix

If φ and its first derivative tend to zero as $\xi \to \pm\infty$, then $h = 0$. For definiteness we consider $U(\varphi) = \varphi^2/2 + \alpha\varphi^4/4$ with a negative α. The first integral with $h = 0$

$$\varphi'^2 = \frac{1}{m}\varphi^2(1 + \alpha\varphi^2/2)$$

plots as the separatrix in the (φ, φ')-plane shown in Fig. 8.3 for $c = 1/2$, $\alpha = -0.1$. Thus φ increases from zero at $\xi = -\infty$, rises to a maximum $\varphi_m = \sqrt{2/|\alpha|}$ and then decreases to zero as $\xi \to \infty$. The solution of the last equation can be found explicitly by quadrature and is given by

$$\varphi(\xi) = \sqrt{\frac{2}{|\alpha|}} \frac{2e^{-|\xi|/\sqrt{1-c^2}}}{1 + e^{-2|\xi|/\sqrt{1-c^2}}}.$$

This solitary wave is shown in Fig. 8.4. Mention that the solution remains still valid if $\xi = x - ct - d$, where d is any constant. We can observe that: i) the amplitude of the soliton is constant and independent of the wave speed, ii) the width of the soliton is proportional to $\sqrt{1-c^2}$, so the narrower soliton moves faster than the wider one.

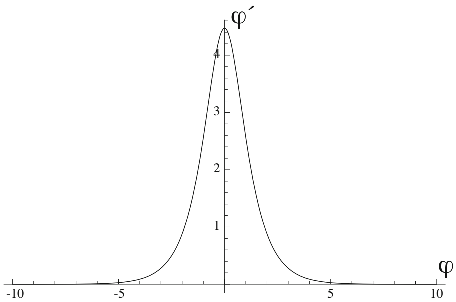

Fig. 8.4 Solitary wave of Klein-Gordon equation

Let us find now the periodic solutions of Klein-Gordon equation. They are obtained by taking $u = \psi(\theta)$, with $\theta = kx - \omega t$, where we assume that $\psi(\theta)$ is 2π-periodic function. Substituting $u = \psi(\theta)$ into (8.3), we get

$$(\omega^2 - k^2)\psi'' + U'(\psi) = 0.$$

The finding of $\psi(\theta)$ is equivalent to searching for the 2π-periodic extremal of the following functional

$$I[\psi] = \int_{\theta_0}^{\theta_0+2\pi} [\frac{1}{2}(\omega^2 - k^2)\psi'^2 - U(\psi)]d\theta, \tag{8.4}$$

where θ_0 may be set equal to zero without limiting the generality. The first integral of Lagrange's equation reads

$$\frac{1}{2}(\omega^2 - k^2)\psi'^2 + U(\psi) = h.$$

Its solution can be found by the separation of variables. The result is

$$\theta = \frac{\sqrt{\omega^2 - k^2}}{\sqrt{2}} \int \frac{d\psi}{\sqrt{h - U(\psi)}}.$$

If $U(\psi)$ is either a cubic, a quartic, or a trigonometric function, then $\psi(\theta)$ can be expressed in terms of standard elliptic functions. Periodic solutions are obtained when ψ oscillates between two simple zeros of $h - U(\psi)$. At the zeros $\psi'(\theta) = 0$, and the solution has a maximum (crest) or a minimum (trough); these points occur at finite values of θ since the above integral converges when the zeros are simple. We denote the zeros by ψ_1 and ψ_2 and consider the case

$$\psi_1 \le \psi \le \psi_2, \quad h - U(\psi) \ge 0, \quad \omega^2 - k^2 > 0.$$

As the period of $\psi(\theta)$ is assumed to be 2π,

$$2\pi = \frac{\sqrt{\omega^2 - k^2}}{\sqrt{2}} \oint \frac{d\psi}{\sqrt{h - U(\psi)}}. \tag{8.5}$$

The contour integral in this formula denotes the integral over a complete oscillation of ψ from ψ_1 up to ψ_2 and back, so it is equal to twice the integral from ψ_1 to ψ_2 because the sign of the square root has to be changed appropriately in the two parts of the contour. This integral may also be interpreted as the contour integral around a cut from ψ_1 to ψ_2 in the complex ψ-plane.

In the linear case $U(\psi) = \frac{1}{2}\psi^2$, and, as we know, the 2π-periodic solution is

$$\psi(\theta) = a\cos\theta, \quad h = \frac{a^2}{2},$$

so the amplitude a cancels out in the integral on the right-hand side of (8.5). Then (8.5) becomes the linear dispersion relation

$$\omega^2 - k^2 = 1,$$

obtained previously for the linear Klein-Gordon equation. This dispersion relation is also the solvability condition of the variational problem (8.4). In the nonlinear case the parameter h does not drop out of (8.5) and we have the typical dependence of the dispersion relation on the amplitude. Consider for example the case $U(\varphi) = \varphi^2/2 + \alpha\varphi^4/4$ with small α. Then (8.4) is exactly the variational problem (5.4) studied by the variational-asymptotic (or Lindstedt-Poincaré) method in Section 5.1, with ω^2 replaced by $\omega^2 - k^2$ and ε by α. Therefore the following asymptotic formulas

$$\sqrt{\omega^2 - k^2} = 1 + \frac{3}{8}\alpha a^2 \quad \Rightarrow \quad \omega^2 - k^2 = 1 + \frac{3}{4}\alpha a^2,$$

and

$$\psi(\theta) = a\cos\theta + \alpha\frac{a^3}{32}(\cos 3\theta - \cos\theta)$$

follow at once.

Behavior of Solitons. Through extensive numerical simulations of the KdV equation[2] the following remarkable behavior of solitons was discovered. If we consider two solitons traveling from left to right with the taller one behind as shown in Fig. 8.5, then since the taller soliton moves faster than the shorter soliton, they will collide. After a short collision time of nonlinear interaction and overlapping the solitons separate again, with the taller one now ahead, and the amplitudes and velocities regain their initial values. The only effect of nonlinear interaction are phase shifts, that is the centers of solitons are slightly shifted from the places where they should have been as if there had been no interaction (see Fig. 8.6). This resembles the collision of particles; so similar to particles the name soliton was given to these special waves.

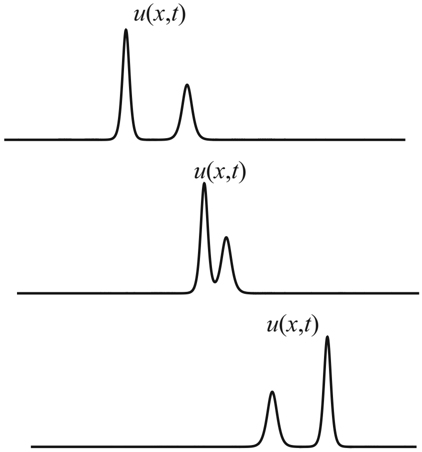

Fig. 8.5 2 traveling solitons

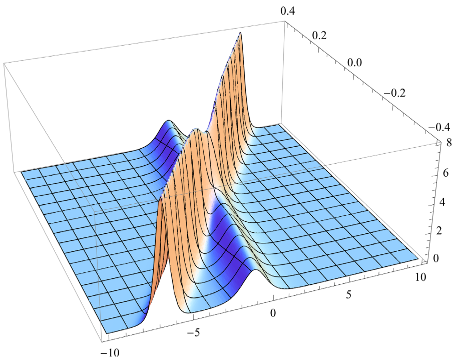

Fig. 8.6 Two-soliton solution of the KdV equation

This remarkable numerical discovery led to a series of first integrals of the KdV equation. All these first integrals are of the form

[2] First initiated by Zabusky and Kruskal in 1965 [54].

$$I_j = \int_{-\infty}^{\infty} P_j(u, u_{,x}, \dots, \frac{\partial^j u}{\partial x^j})\, dx = \text{const},$$

where P_j are polynomials. For example, the first three integrals are (see exercise 8.3)

$$I_{-1} = \int_{-\infty}^{\infty} u\, dx, \quad I_0 = \int_{-\infty}^{\infty} u^2\, dx, \quad I_1 = \int_{-\infty}^{\infty} (u^3 - \frac{1}{2}u_{,x}^2)\, dx.$$

In searching for further first integrals of the KdV equation Miura discovered the following transformation: if v is a solution of the modified KdV equation

$$v_{,t} - 6v^2 v_{,x} + v_{,xxx} = 0,$$

then

$$u = -(v^2 + v_{,x})$$

satisfies KdV equation. This is readily seen from the relation

$$u_{,t} + 6uu_{,x} + u_{,xxx} = -(2v + \partial_x)(v_{,t} - 6v^2 v_{,x} + v_{,xxx}).$$

The equation $u = -(v^2 + v_{,x})$ may be viewed as Riccati's equation for v in terms of u. It can be transformed to a linear equation by substituting $v = \psi_{,x}/\psi$. This yields

$$\psi_{,xx} + u\psi = 0.$$

Since the KdV equation is Galilean invariant, that is invariant under the transformation

$$(x, t, u(x,t)) \to (x - ct, t, u(x,t) + \frac{1}{6}c),$$

it is natural to replace u by $u - \lambda$ and consider the equation

$$\psi_{,xx} + u\psi = \lambda \psi.$$

This is nothing else but the stationary Schrödinger equation which has been studied extensively in context of the scattering problem, where function $-u(x,t)$ plays the role of the scattering potential. The association of the Schrödinger equation with the KdV equation led Gardner, Green, Kruskal, and Miura later [16] to the fruitful development of a beautiful mathematical method called inverse scattering transform which can be used to fully integrate a wide class of nonlinear partial differential equations [1]. We consider this method in the next Section.

8.2 Inverse Scattering Transform

This Section presents the analytical solution of KdV equation based on the inverse scattering transform.[3]

[3] See the detailed derivations in [1].

Lax Pair. Let us consider the KdV equation (8.1) subject to the initial condition

$$u(x,0) = u_0(x),$$

where $u_0(x)$ decays sufficiently rapidly as $|x| \to \infty$. Since the KdV equation is nonlinear, the Fourier transform cannot directly be applied to solve this initial-value problem. However, as motivated in the previous Section, we can relate this equation to the stationary Schrödinger equation

$$L\psi = \lambda \psi, \tag{8.6}$$

where L is the linear operator defined by

$$L\psi = \psi_{,xx} + u(x,t)\psi.$$

The idea is based on the following construction proposed by Lax [30]. Assume that ψ evolves in time in accordance with

$$\psi_{,t} = A\psi. \tag{8.7}$$

Thus, A is the linear operator governing the time evolution of ψ. Now we calculate the time derivative of equation (8.6)

$$L_{,t}\psi + L\psi_{,t} = \lambda_{,t}\psi + \lambda\psi_{,t}.$$

Taking into account (8.7) we transform the above equation to

$$(L_{,t} + LA - AL)\psi = \lambda_{,t}\psi.$$

Thus, if $\lambda_{,t} = 0$, then the so-called Lax equation

$$L_{,t} + [L,A] = 0, \quad [L,A] = LA - AL,$$

holds true. The problem reduces then to finding A so that Lax's equation is compatible with the KdV equation. It is easy to show by the direct inspection (see exercise 8.4) that Lax's equation is compatible with the KdV equation if we choose A as follows

$$A\psi = (\gamma + u_{,x})\psi - (4\lambda + 2u)\psi_{,x}, \tag{8.8}$$

where γ is an arbitrary constant. The byproduct of Lax's construction is that the KdV equation possesses an infinite number of first integrals since all eigenvalues of $L\psi = \lambda\psi$ are such first integrals. The linear operators L and A, called Lax's pair, have been found later on for a wide class of nonlinear partial differential equations, including the Sine-Gordon equation, the nonlinear Schrödinger equation, the Kadomtsev-Petviashvili equation and many other equations of mathematical physics.[4]

[4] The list of fully integrable nonlinear equations can be found in [1].

Inverse Scattering Transform. Based on the Lax representation we can now solve the KdV equation, corresponding to $u \to 0$ as $|x| \to \infty$, in three steps sketched below. The mathematical justification will be given in the next paragraph.

i) First step. At time $t = 0$ the initial condition $u(x,0) = u_0(x)$ is known. With these given initial data we solve the direct scattering problem: find the eigenvalues and the corresponding eigenfunctions of (8.6). One can show that the spectrum of the Schrödinger equation with $u(x,t) \to 0$ as $|x| \to \infty$ is discrete for $\lambda > 0$ and continuous for $\lambda < 0$. Denote the discrete eigenvalues by $\lambda = \kappa_n^2$, $n = 1,2,\ldots,N$ and the continuous eigenvalues by $\lambda = -k^2$. It turns out that the normalized eigenfunctions corresponding to the discrete eigenvalues behave asymptotically as $x \to \infty$ according to

$$\psi_n(x,t) \sim \sigma_n(t)e^{-\kappa_n x},$$

with the normalization condition

$$\int_{-\infty}^{\infty} \psi_n^2 \, dx = 1.$$

For the continuous spectrum the asymptotic behaviors of the eigenfunctions are described by

$$\begin{aligned} \psi(x,t) &\sim e^{-ikx} + \rho(k,t)e^{ikx} \quad \text{as } x \to \infty, \\ \psi(x,t) &\sim \tau(k,t)e^{-ikx} \quad \text{as } x \to -\infty, \end{aligned}$$

where $\rho(k,t)$ is the reflection coefficient and $\tau(k,t)$ the transmission coefficient. At $t = 0$ the obtained scattering data

$$S(\lambda,0) = \left(\{\kappa_n, \sigma_n(0)\}_{n=1}^N, \rho(k,0), \tau(k,0)\right)$$

serve as the input data for the next step.

ii) Second step. We use now the evolution equation (8.7) with A from (8.8) to determine the time dependence of the scattering data. We know that κ_n are unchanged. It will be shown that, for $n = 1,2,\ldots,N$

$$\sigma_n(t) = \sigma_n(0)e^{4\kappa_n^3 t},$$

and

$$\begin{aligned} \tau(k,t) &= \tau(k,0), \\ \rho(k,t) &= \rho(k,0)e^{8ik^3 t}. \end{aligned}$$

Thus, the scattering data at time t are given by

$$S(\lambda,t) = \left(\{\kappa_n, \sigma_n(t)\}_{n=1}^N, \rho(k,t), \tau(k,t)\right).$$

We use this as the input data for the last step.

iii) Third (last) step. At this final step we solve the inverse scattering problem: reconstruct the potential $u(x,t)$ which is the solution of the KdV equation from the knowledge of the scattering data $S(\lambda,t)$. The results may be summarized as follows. From the scattering data we find the function

$$F(x,t) = \sum_{n=1}^{N} \sigma_n^2(t) e^{-\kappa_n x} + \frac{1}{2\pi} \int_{-\infty}^{\infty} \rho(k,t) e^{ikx} dk.$$

We then solve the linear integral equation

$$K(x,y,t) + F(x+y,t) + \int_x^{\infty} K(x,z,t) F(z+y,t)\, dz = 0, \tag{8.9}$$

called Gelfand-Levitan equation. Finally we compute $u(x,t)$ in accordance with

$$u(x,t) = 2\frac{\partial}{\partial x}[K(x,x,t)].$$

As we see, this method is conceptually quite similar to the Fourier transform used for solving linear equations (cf. Chapter 4), except that the last step of solving the inverse scattering problem is highly nontrivial. Schematically, the described steps may be summarized in the following diagram

$$\begin{array}{ccc} u(x,0) & \xrightarrow{\text{direct scattering}} & S(\lambda,0) \\ & & \Big\downarrow \text{time evolution} \\ u(x,t) & \xleftarrow{\text{inverse scattering}} & S(\lambda,t) \end{array}$$

In this diagram the direct scattering plays the role of the Fourier transform, while the inverse scattering the inverse Fourier transform. The time evolution of the scattering data is similar to the multiplication of the Fourier image with function $e^{i\Omega(k)t}$ which accounts for the dispersion. Note that at each step we have to deal just with linear problems which are "doable".

Mathematical Justification. In this paragraph we present briefly the justification of the above results based on the direct and inverse scattering problems.[5] In the direct scattering problem it is convenient to put $\lambda = -k^2$ and write (8.6) as

$$\psi_{,xx} + [u(x,t) + k^2]\psi = 0.$$

For a given k we let $\phi(x,k)$, $\bar{\phi}(x,k)$ and $\psi(x,k)$, $\bar{\psi}(x,k)$ be the corresponding eigenfunctions which satisfy the following asymptotic behaviors

$$\begin{aligned} \phi(x,k) &\sim e^{ikx}, \quad \bar{\phi}(x,k) \sim e^{-ikx} \quad \text{as } x \to \infty, \\ \psi(x,k) &\sim e^{-ikx}, \quad \bar{\psi}(x,k) \sim e^{ikx} \quad \text{as } x \to -\infty. \end{aligned}$$

[5] See the detailed expositions in [1].

Equation (8.6) is a linear second order differential equation. Therefore, between these eigenfunctions there are linear relationships

$$\psi(x,k) = a(k)\bar{\phi}(x,k) + b(k)\phi(x,k),$$
$$\bar{\psi}(x,k) = -\bar{a}(k)\phi(x,k) + \bar{b}(k)\bar{\phi}(x,k).$$

where $a(k)$ and $b(k)$ satisfy the following symmetry properties

$$\bar{a}(k) = -a(-k) = -a^*(k^*), \quad \bar{b}(k) = b(-k) = b^*(k^*).$$

Besides, the following identity holds true

$$a(k)\bar{a}(k) + b(k)\bar{b}(k) = -1.$$

This can easily be checked by computing the Wronskians giving

$$W(\psi(x,k), \bar{\psi}(x,-k)) = [a(k)\bar{a}(k) + b(k)\bar{b}(k)]W(\phi(x,k), \bar{\phi}(x,k)).$$

We introduce $\tau(k) = 1/a(k)$ and $\rho(k) = b(k)/a(k)$ as the transmission and reflection coefficients, respectively and consider the normalized eigenfunction $\psi(x,k)/a$ as in the previous paragraph. It is easy to see that $|\rho(k)|^2 + |\tau(k)|^2 = 1$.

We turn now to the time dependence of the scattering data. The evolution of $\psi(x,k,t)$ is described by (8.7), with A from (8.8). We introduce the modified eigenfunction $N(x,k,t)$ such that

$$\frac{1}{a}\psi(x,k,t) = N(x,k,t)e^{-ikx}.$$

Then N satisfies the equation

$$N_{,t} = (\gamma - 4ik^3 + u_{,x} + 2iku)N + (4k^2 - 2u)N_{,x}.$$

The asymptotic behavior of $\psi(x,k,t)$ implies that

$$N(x,k,t) \to \tau(k,t) \quad \text{as } x \to -\infty,$$
$$N(x,k,t) \to 1 + \rho(k,t)e^{2ik} \quad \text{as } x \to \infty.$$

By considering the above equation for $N(x,k,t)$ as $x \to -\infty$ and using the fact that u and its first derivative tend to zero in this limit, we obtain

$$\tau_{,t} = (\gamma - 4ik^3)\tau.$$

Thus, the choice $\gamma = 4ik^3$ makes the transmission coefficient $\tau(k)$ independent of t. Then, in the other limit $x \to \infty$ we get

$$\rho_{,t} = 8ik^3\rho \quad \Rightarrow \quad \rho(k,t) = \rho(k,0)e^{8ik^3t}.$$

Concerning the discrete spectrum we know that the eigenvalues $\lambda = \kappa_n^2$ are positive and time independent. Denote by $\chi_n(x,\kappa_n,t)$ the eigenfunctions with the asymptotic behavior $\chi_n \sim e^{-\kappa_n x}$ as $x \to \infty$ and assume that $\psi_n(x,t) = \sigma_n(t)\chi_n(x,\kappa_n,t)$. With (8.6) and (8.7) it is easy to check that

$$\frac{d}{dt}\int_{-\infty}^{\infty} \chi_n^2\,dx = -8\kappa_n^3 \int_{-\infty}^{\infty} \chi_n^2\,dx.$$

Taking into account the normalization condition we have

$$\sigma_n^2(t) = \frac{1}{\int_{-\infty}^{\infty} \chi_n^2\,dx}.$$

Thus,

$$\sigma_n^2(t) = \sigma_n^2(0)e^{8\kappa_n^3 t} \quad \Rightarrow \quad \sigma_n(t) = \sigma_n(0)e^{4\kappa_n^3 t}.$$

The rigorous derivation of the Gelfand-Levitan integral equation requires a deeper insight into the spectral analysis [18] than that provided so far. Let us show nevertheless how to obtain, at least formally, this equation by working with the Schrödinger equation in an equivalent "time domain". We consider equation (8.6) as the Fourier transform of the "wave" equation

$$\varphi_{,xx} - \varphi_{,\theta\theta} + u\varphi = 0, \tag{8.10}$$

where function $\varphi(x,\theta,t)$ is the Fourier image of $\psi(x,k,t)$ with respect to k

$$\varphi(x,\theta,t) = \int_{-\infty}^{\infty} \psi(x,k,t)e^{ik\theta}\,dk.$$

We suppress at present the true time variable t. Consider an incident wave $\varphi = \delta(x+\theta)$ from $x = \infty$ and let the reflected wave be $F(x-\theta)$. Thus,

$$\varphi \sim \varphi_\infty = \delta(x+\theta) + F(x-\theta) \quad \text{as } x \to \infty.$$

We propose that the corresponding solution of (8.10) may be written

$$\varphi(x,\theta) = \varphi_\infty(x,\theta) + \int_x^{\infty} K(x,z)\varphi_\infty(x,\theta)\,dz,$$

what is equivalent to a crucial step in Gelfand-Levitan's work. By direct substitution in (8.10) we verify that there is such a solution provided

$$\begin{aligned} K_{,zz} - K_{,xx} + uK = 0, &\quad z > x, \\ u(x) = 2\frac{d}{dx}K(x,x), & \\ K, K_{,z} \to 0 &\quad \text{as } x \to \infty. \end{aligned}$$

This is a well-posed problem, therefore $K(x,z)$ exists. From the causality property of the wave equation we know that φ must vanish for $x+\theta < 0$. Therefore

$$\varphi_\infty(x,\theta) + \int_x^\infty K(x,z)\varphi_\infty(x,\theta)\,dz = 0 \quad \text{for } x+\theta < 0.$$

Introducing the expression for $\varphi_\infty(x,\theta)$ in this equation we get

$$K(x,-\theta) + F(x-\theta) + \int_x^\infty K(x,z)F(z-\theta)\,dz = 0 \quad \text{for } x+\theta < 0.$$

With $\theta = -y$ this becomes Gelfand-Levitan equation (8.9). At a fixed time t, F is determined from the direct scattering problem in terms of $u(x,t)$ as

$$F(x-\theta) = \sum_{n=1}^{N} \sigma_n^2(t)e^{-\kappa_n(x-\theta)} + \frac{1}{2\pi}\int_{-\infty}^{\infty} \rho(k,t)e^{ik(x-\theta)}dk.$$

With $\theta = -y$ and with the scattering data at time t we obtain the expression for F in the Gelfand-Levitan equation.

Reflectionless Potential. The solution of the Gelfand-Levitan equation simplifies considerably if the reflection coefficient is zero. In this case we obtain the special soliton solutions by the separation of variables. Indeed, if $\rho(k,t) = 0$, then we have for function $F(x,t)$

$$F(x,t) = \sum_{n=1}^{N} \sigma_n^2(t)e^{-\kappa_n x},$$

with $\sigma_n(t) = \sigma_n(0)e^{4\kappa_n^3 t} > 0$ and distinct $\kappa_n > 0$, $n = 1,2,\ldots,N$. So the Gelfand-Levitan equation becomes

$$K(x,y,t) + \sum_{n=1}^{N} \sigma_n^2(t)e^{-\kappa_n(x+y)} + \int_x^\infty K(x,z,t)\sum_{n=1}^{N} \sigma_n^2(t)e^{-\kappa_n(z+y)}\,dz = 0.$$

We seek the solution of this equation in the form

$$K(x,y,t) = \sum_{n=1}^{N} \sigma_n v_n(x)e^{-\kappa_n y}.$$

Substituting this solution Ansatz into the integral equation we get for $m = 1,2,\ldots,N$

$$v_m(x) + \sum_{n=1}^{N} \frac{\sigma_m(t)\sigma_n(t)}{\kappa_m + \kappa_n} e^{-(\kappa_m+\kappa_n)x} v_n(x) = \sigma_m(t)e^{-\kappa_m x}.$$

This is a system of N algebraic equations which can be written in the matrix form as

$$(\mathbf{I} + \mathbf{C})\mathbf{v} = \mathbf{f}, \tag{8.11}$$

where $\mathbf{v} = (v_1, v_2, \ldots, v_N)^T$, $\mathbf{f} = (f_1, f_2, \ldots, f_N)^T$ with $f_m = \sigma_m e^{-\kappa_m x}$, $m = 1,2,\ldots,N$, $\mathbf{I}$ is the identity matrix and $\mathbf{C}$ is a symmetric $N \times N$ matrix with elements

$$C_{mn} = \frac{\sigma_m(t)\sigma_n(t)}{\kappa_m + \kappa_n} e^{-(\kappa_m+\kappa_n)x}, \quad m,n = 1,2,\ldots,N.$$

A sufficient condition for the system (8.11) to have a unique solution is that $\mathbf{C}$ is positive definite. The latter holds true because the quadratic form

$$\boldsymbol{\xi}\cdot\mathbf{C}\boldsymbol{\xi} = \sum_{m=1}^{N}\sum_{n=1}^{N} \frac{\sigma_m(t)\sigma_n(t)\xi_m\xi_n}{\kappa_m+\kappa_n} e^{-(\kappa_m+\kappa_n)x} = \int_x^{\infty} \left(\sum_{n=1}^{N} \sigma_n(t)\xi_n e^{-\kappa_n x} \right)^2 dy$$

is clearly positive for an arbitrary $\boldsymbol{\xi} \neq \mathbf{0}$. The unique solution to the KdV equation in this case is

$$u(x,t) = 2\frac{\partial^2}{\partial x^2}[\ln\det(\mathbf{I}+\mathbf{C})]. \tag{8.12}$$

Soliton Solutions. Consider first the simplest case $N = 1$ for which

$$C = \frac{\sigma_1^2(t)}{2\kappa_1} e^{-2\kappa_1 x} = \frac{\sigma_1^2(0)}{2\kappa_1} e^{-2\kappa_1 x + 8\kappa_1^3 t}.$$

Introducing $\xi = x - ct - d$, where

$$c = 4\kappa_1^2, \quad d = -\frac{1}{\kappa_1}\ln\frac{\sigma_1(0)}{2\kappa_1},$$

we may write $C = e^{-2\kappa_1\xi}$. Then

$$u(x,t) = 2\frac{\partial^2}{\partial x^2}[\ln(1+C)] = 8\kappa_1^2 \frac{C}{(1+C)^2} = 2\kappa_1^2 \operatorname{sech}^2(\kappa_1\xi)$$

coincides with the one soliton solution obtained in Section 8.1.

For $N = 2$ we have

$$\Delta = \det(\mathbf{I}+\mathbf{C}) = 1 + e^{-2\kappa_1\xi_1} + e^{-2\kappa_2\xi_2} + e^{-2\kappa_1\xi_1 - 2\kappa_2\xi_2 + A_{12}},$$

with

$$\xi_n = x - 4\kappa_n^2 t - d_n, \quad A_{12} = 2\ln\left(\frac{\kappa_1-\kappa_2}{\kappa_1+\kappa_2}\right).$$

This formula implies that the only effect of the interaction of two solitary waves is a phase shift. Indeed, consider the trajectory $\xi_1 = \text{const}$, and assume that $\kappa_1 > \kappa_2 > 0$. Then

$$\Delta \sim 1 + e^{-2\kappa_1\xi_1} \quad \text{as } t \to -\infty,$$
$$\Delta \sim e^{-2\kappa_2\xi_2} + e^{-2\kappa_1\xi_1 - 2\kappa_2\xi_2 + A_{12}} \quad \text{as } t \to \infty.$$

Therefore, from (8.12) it follows that for fixed ξ_1

$$u(x,t) = 2\frac{\partial^2}{\partial x^2}(\ln \Delta) \sim 2\kappa_1^2 \text{sech}^2(\kappa_1\xi_1 + \delta_1^{\pm}) \quad \text{as } t \to \pm\infty,$$

with

$$\delta_1^+ = \frac{1}{2}A_{12}, \quad \delta_1^- = 0.$$

Similarly, for fixed ξ_2

$$u(x,t) \sim 2\kappa_2^2 \text{sech}^2(\kappa_2\xi_2 + \delta_2^{\pm}) \quad \text{as } t \to \pm\infty,$$

with

$$\delta_2^+ = 0, \quad \delta_2^- = \frac{1}{2}A_{12}.$$

Thus, for large negative time, the taller soliton is behind the shorter one, and vice-versa for large positive time. The phase shifts of solitons are $A_{12}/2$ and $-A_{12}/2$, respectively.

The calculations for N solitons show the similar behavior. If $\kappa_1 > \kappa_2 > \ldots > \kappa_N > 0$, then for fixed ξ_n

$$u(x,t) \sim 2\kappa_n^2 \text{sech}^2(\kappa_n\xi_n + \delta_n^{\pm}) \quad \text{as } t \to \pm\infty,$$

where

$$\delta_n^+ = \sum_{m=n+1}^{N} \ln\left(\frac{\kappa_n - \kappa_m}{\kappa_n + \kappa_m}\right), \quad \delta_n^- = \sum_{m=1}^{n-1} \ln\left(\frac{\kappa_m - \kappa_n}{\kappa_m + \kappa_n}\right).$$

Therefore, the n-th soliton undergoes a phase shift given by

$$\delta_n = \delta_n^+ - \delta_n^- = \sum_{m=n+1}^{N} \ln\left(\frac{\kappa_n - \kappa_m}{\kappa_n + \kappa_m}\right) - \sum_{m=1}^{n-1} \ln\left(\frac{\kappa_m - \kappa_n}{\kappa_m + \kappa_n}\right).$$

We see that the total phase shift is equal to the sum of phase shifts resulted from pair interaction with every other soliton.

To illustrate the relationship between the initial condition and the number of solitons, let us take the initial condition in the form

$$u(x,n) = N(N+1)\text{sech}^2 x.$$

In this case the scattering problem, with $\lambda = \kappa^2$, reads

$$\psi_{,xx} + [N(N+1)\text{sech}^2 x - k^2]\psi = 0.$$

If we make the transformation $\mu = \tanh x$, then this equation becomes

$$(1-\mu^2)\frac{d^2\psi}{d\mu^2} - 2\mu\frac{d\psi}{d\mu} + [N(N+1) - \frac{\kappa^2}{1-\mu^2}]\psi = 0, \tag{8.13}$$

which is the associate Legendre equation (see [3]). Equation (8.13) has N distinct eigenvalues $\kappa_n = 1, 2, \dots, N$ and bounded eigenfunctions in terms of Legendre polynomials

$$\psi_n(x) = \gamma_n P_N^n(\tanh x) \sim c_n e^{-nx} \quad \text{as } x \to \infty,$$

where c_n is determined from the normalization condition. The N-soliton solution of the KdV equation is given by (8.12), where

$$C_{mn} = \frac{c_m + c_n}{m+n} e^{-(m+n)x}.$$

In particular, the two-soliton solution of the KdV equation satisfying the above initial condition for $N = 2$ reads

$$u(x,t) = 12 \frac{3 + 4\cosh(2x - 8t) + \cosh(4x - 64t)}{[3\cosh(x - 28t) + \cosh(3x - 36t)]^2}.$$

If we introduce $\xi_1 = x - 16t$ and $\xi_2 = x - 4t$, then the two-soliton solution can be expressed as

$$u(x,t) = 12 \frac{3 + 4\cosh(2\xi_1 + 24t) + \cosh(4\xi_1)}{[3\cosh(\xi_1 - 12t) + \cosh(3\xi_1 + 12t)]^2},$$

and, alternatively,

$$u(x,t) = 12 \frac{3 + 4\cosh(2\xi_2) + \cosh(4\xi_2 - 48t)}{[3\cosh(\xi_2 - 24t) + \cosh(3\xi_2 - 24t)]^2}.$$

Expanding these formulas, keeping ξ_1 (alternatively ξ_2) fixed, it is easy to see that as $t \to \pm\infty$

$$u(x,t) \sim 2\operatorname{sech}^2(\xi_2 \pm \frac{1}{2}\ln 3) + 8\operatorname{sech}^2(2\xi_1 \mp \frac{1}{2}\ln 3).$$

Thus, the phase shifts are $\pm \ln 3/2$ in this case.

8.3 Energy Method

In this Section we are going to apply the variational-asymptotic method to general variational problems of wave propagation.

Variational-Asymptotic Method. Consider the variational problem in form of Hamilton's variational principle: find the extremal of the action functional

$$I[u_i(\mathbf{x},t)] = \iint_R L(u_i, u_{i,\alpha}, u_{i,t})\, dx\, dt, \tag{8.14}$$

where $R = V \times (t_0, t_1)$ is any finite and fixed region in $(d+1)$-dimensional space-time. We assume that u_i are prescribed at the boundary ∂R. We look for the extremal of this variational problem in form of a slowly varying wave packet[6]

$$u_i = \psi_i(\theta, \mathbf{x}, t), \tag{8.15}$$

where θ is a function of $\mathbf{x}$ and t, ψ_i are 2π-periodic functions with respect to θ. Function θ plays the role of the phase, while $\theta_{,\alpha}$ and $-\theta_{,t}$ correspond to the wave vector k_α and the frequency ω, respectively. As in the linear case we assume that functions $\theta_{,\alpha}$, $\theta_{,t}$ and $\psi_i(\theta, \mathbf{x}, t)|_{\theta=\text{const}}$ change slowly in one wavelength λ and one period τ. The latter are defined as the best constants in the inequalities

$$|\theta_{,\alpha}| \le \frac{2\pi}{\lambda}, \quad |\theta_{,t}| \le \frac{2\pi}{\tau}. \tag{8.16}$$

The characteristic length- and time-scales Λ and T of changes of the functions $\theta_{,\alpha}$, $\theta_{,t}$ and $\psi_i(\theta, \mathbf{x}, t)|_{\theta=\text{const}}$ are defined as the best constants in the inequalities

$$\begin{gathered} |\theta_{,\alpha\beta}| \le \frac{2\pi}{\lambda\Lambda}, \quad |\theta_{,\alpha t}| \le \frac{2\pi}{\lambda T}, \quad |\theta_{,\alpha t}| \le \frac{2\pi}{\tau\Lambda}, \quad |\theta_{,tt}| \le \frac{2\pi}{\tau T}, \\ |\partial_\alpha \psi_i| \le \frac{\bar{\psi}_i}{\Lambda}, \quad |\partial_t \psi_i| \le \frac{\bar{\psi}_i}{T}, \quad |\psi_{i,\theta}| \le \bar{\psi}_i, \end{gathered} \tag{8.17}$$

where $\partial_\alpha \psi_i = \partial \psi_i / \partial x_\alpha$ with $\theta = \text{const}$, and $\partial_t \psi_i = \partial \psi_i / \partial t$ with $\theta = \text{const}$. Therefore it makes sense to call θ "fast" variable as opposed to the "slow" variables x_α and t. Thus, in this variational problem we have two small parameters λ/Λ and τ/T.

We now calculate the derivatives $u_{i,\alpha}$ and $u_{i,t}$. According to (8.15)

$$u_{i,\alpha} = \partial_\alpha \psi_i + \psi_{i,\theta}\theta_{,\alpha}, \quad u_{i,t} = \partial_t \psi_i + \psi_{i,\theta}\theta_{,t}.$$

Because of (8.16) and (8.17) they can be approximately replaced by

$$u_{i,\alpha} = \psi_{i,\theta}\theta_{,\alpha}, \quad u_{i,t} = \psi_{i,\theta}\theta_{,t}.$$

Keeping in the action functional (8.14) the asymptotically principal terms, we obtain in the first approximation

$$I_0[\psi_i] = \iint_R L(\psi_i, \psi_{i,\theta}\theta_{,\alpha}, \psi_{i,\theta}\theta_{,t})\, dx\, dt.$$

Similar to the linear case we decompose the domain R into the $(d+1)$-dimensional strips bounded by the d-dimensional phase surfaces $\theta = 2\pi n$, $n = 0, \pm 1, \pm 2, \ldots$. The integral over R can then be replaced by the sum of the integrals over the strips

[6] The amplitudes a_i appear later.

$$\iint_R L dx dt = \sum \iint L(\psi_i, \psi_{i,\theta}\theta_{,\alpha}, \psi_{i,\theta}\theta_{,t})\kappa \, d\theta \, d\zeta, \tag{8.18}$$

where ζ_α are the coordinates along the phase surface $\theta =$ const, and κ is the Jacobian of transformation from x_α, t to θ, ζ_α. In the first approximation we may regard κ, $\theta_{,\alpha}$ and $\theta_{,t}$ in each strip as independent from θ. Therefore we obtain the same problem in each strip at the first step of the variational-asymptotic procedure [8]: find the extremal of the functional

$$\bar{I}_0[\psi_i] = \int_0^{2\pi} L(\psi_i, \psi_{i,\theta}\theta_{,\alpha}, \psi_{i,\theta}\theta_{,t}) \, d\theta \tag{8.19}$$

among 2π-periodic functions $\psi_i(\theta)$. Since the quantities $k_\alpha = \theta_{,\alpha}$ and $-\omega = \theta_{,t}$ change little within one strip, they are regarded as constants in the functional (8.19). The Euler-Lagrange equation of this functional is a system of n nonlinear second-order ordinary differential equations. Its solutions contain $2n$ arbitrary constants: n of them is determined from the conditions that $\psi_i(\theta)$ are 2π-periodic functions, the other n conditions can be chosen by fixing the amplitudes a_i as follows: $\max \psi_i = |a_i|$, where a_i are arbitrary real constants.[7] We call this variational problem strip problem.

Let us denote by $2\pi\bar{L}$ the value of the functional (8.19) at its extremal. The quantity $\bar{L}$ is a function of $a_i, \theta_{,\alpha}$ and $\theta_{,t}$. The sum (8.18), as $\lambda/\Lambda \to 0$ and $\tau/T \to 0$, can again be replaced by the integral

$$\iint_R \bar{L}(a_i, \theta_{,x}, \theta_{,t}) \, dx dt. \tag{8.20}$$

Euler-Lagrange's equations of the average functional (8.20) read

$$\frac{\partial \bar{L}}{\partial a_i} = 0, \quad \frac{\partial}{\partial t}\frac{\partial \bar{L}}{\partial \theta_{,t}} + \frac{\partial}{\partial x_\alpha}\frac{\partial \bar{L}}{\partial \theta_{,\alpha}} = 0. \tag{8.21}$$

We will see that equations $(8.21)_1$ express the solvability condition for the strip problem leading to the nonlinear dispersion relation, while $(8.21)_2$ is equivalent to the equation of energy propagation.

Strip Problems. As an example let us consider the strip problem for the nonlinear Klein-Gordon equation, whose Lagrangian is given by

$$L = \frac{1}{2}u_{,t}^2 - \frac{1}{2}u_{,x}^2 - U(u).$$

In this case the average Lagrangian must be calculated according to

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[7] This choice is dictated by the phase portrait of the strip problem. We will see later that, in some cases, the constants must be chosen by fixing the slopes rather than the amplitudes.

$$\bar{L} = \frac{1}{2\pi} \min_{\max \psi = a} \int_0^{2\pi} [\frac{1}{2}(\omega^2 - k^2)\psi'^2 - U(\psi)] d\theta,$$

where $\omega = -\theta_{,t}$ and $k = \theta_{,x}$ are regarded as constants. We use the first integral

$$\frac{1}{2}(\omega^2 - k^2)\psi'^2 + U(\psi) = U(a) = h$$

to express $\bar{L}$ in the form

$$\bar{L} = \frac{1}{2\pi} \int_0^{2\pi} (\omega^2 - k^2)\psi'^2 \, d\theta - h.$$

Changing the variable $\theta \to \psi$, we obtain finally

$$\bar{L} = \frac{1}{2\pi}(\omega^2 - k^2) \int_0^{2\pi} \psi' \, d\psi - h = \frac{1}{2\pi}\sqrt{2(\omega^2 - k^2)} \oint \sqrt{h - U(\psi)} d\psi - h. \quad (8.22)$$

The contour integral in (8.22) denotes the integral over a complete oscillation of ψ from b, with $U(b) = U(a)$, up to a and back, so it is equal to twice the integral from b to a because the sign of the square root has to be changed appropriately in the two parts of the contour. This integral may also be interpreted as the contour integral around a cut from b to a in the complex ψ-plane, where ψ plays the role of the variable of integration.

Now let us consider the average variational problem (8.20) in which $\bar{L}$ is given by (8.22) with $h = U(a)$, $\omega = -\theta_{,t}$, and $k = \theta_{,x}$. Euler-Lagrange's equations of this problem read

$$\frac{\partial \bar{L}}{\partial h}\frac{dh}{da} = 0, \qquad -\frac{\partial}{\partial t}\frac{\partial \bar{L}}{\partial \omega} + \frac{\partial}{\partial x_\alpha}\frac{\partial \bar{L}}{\partial k} = 0. \quad (8.23)$$

It is easy to see that differentiation of $\bar{L}$ with respect to h gives

$$\frac{\partial \bar{L}}{\partial h} = \frac{1}{2\pi}\frac{\sqrt{\omega^2 - k^2}}{\sqrt{2}} \oint \frac{d\psi}{\sqrt{h - U(\psi)}} - 1.$$

Thus, the first equation of (8.23) is nothing else but the nonlinear dispersion relation (8.5) for the nonlinear Klein-Gordon equation. Together with the kinematic relation

$$k_{,t} + \omega_{,x} = 0, \quad (8.24)$$

they form a system of nonlinear coupled equations describing the amplitude modulations.

The strip problems for two or more unknown functions reduce to the problem of finding the nonlinear normal modes already solved in Chapter 7. Consider for example the wave equations which are Euler-Lagrange's equations of the following Lagrangian

$$L = \frac{1}{2}(u_{1,t}^2 + u_{2,t}^2) - \frac{1}{2}(u_{1,x}^2 + u_{2,x}^2) - U(u_1, u_2).$$

This Lagrangian arises in the problem of coupled vibrations of two pre-stretched strings along which nonlinear springs with the cubic nonlinearity are attached at close intervals, where function $U(u_1,u_2)$ describes the potential energy density of the springs. The strip problem becomes: find the 2π-periodic functions ψ_1 and ψ_2 which minimize the following functional

$$I_0[\psi_1,\psi_2]=\int_0^{2\pi}[\frac{1}{2}(\omega^2-k^2)(\psi_{1,\theta}^2+\psi_{2,\theta}^2)-U(\psi_1,\psi_2)]\,d\theta.$$

Denoting $\omega^2-k^2=m$, we write the corresponding Lagrange's equations in the form

$$m\psi_{1,\theta\theta}=-\frac{\partial U}{\partial\psi_1},\quad m\psi_{2,\theta\theta}=-\frac{\partial U}{\partial\psi_2}.$$

This is nothing else but equations (7.7) studied in connection with the nonlinear normal modes in Section 7.2. If we seek the nonlinear normal modes as 2π-periodic solutions by assuming ψ_2 as a function of ψ_1, then the problem reduces to solving the modal equation

$$2(h-U)\psi_2''+(1+\psi_2'^2)(\frac{\partial U}{\partial\psi_2}-\psi_2'\frac{\partial U}{\partial\psi_1})=0,$$

which is the ordinary differential equation of second order, where the prime denotes the derivative with respect to ψ_1 and h is a constant in the first integral

$$\frac{1}{2}m\psi_{1,\theta}^2(1+\psi_2'^2)+U(\psi_1,\psi_2)=h.$$

Particularly, if $U(\psi_1,\psi_2)$ equals

$$U(\psi_1,\psi_2)=\frac{1}{2}[\psi_1^2+\frac{\alpha}{2}\psi_1^4+\psi_2^2+\frac{\alpha}{2}\psi_2^4+\frac{\beta}{2}(\psi_2-\psi_1)^4],$$

then the normal modes become similar modes $\psi_2=c\psi_1$, with

$$c=1,-1,1-\frac{1}{2\kappa}\pm\frac{1}{\kappa}\sqrt{1/4-\kappa},$$

where $\kappa=\beta/\alpha$ is the coupling factor. The strip problem reduces then to the problem with one unknown function admitting the analytical solution (see exercise 8.7). Thus, for $\kappa<1/4$, there are two additional normal modes bifurcated out of the antisymmetric mode $\psi_2=-\psi_1$ (vibrations in counter-phases) at $\kappa=1/4$. This indicates the bifurcation of amplitude modulations in our original problem of wave propagation.

Hamilton's Equations for the Strip Problem. It is quite straightforward to transform Lagrange's equations of the strip problem to the equivalent Hamilton's form. We take the differential of the Lagrange function $\Lambda(\psi_i,\psi_i')=L(\psi_i,k_\alpha\psi_i',-\omega\psi_i')$ as function of ψ_i and $\psi_i'=\psi_{i,\theta}$

$$d\Lambda = \sum_{i=1}^{n} (\frac{\partial \Lambda}{\partial \psi_i} d\psi_i + \frac{\partial \Lambda}{\partial \psi_i'} d\psi_i').$$

We introduce new variables $p_i = \partial \Lambda / \partial \psi_i'$ and the Hamilton function $H(\psi_i, p_i)$ as Legendre's transform of $\Lambda(\psi_i, \psi_i')$ with respect to ψ_i'

$$H(\psi_i, p_i) = \sum_{i=1}^{n} p_i \psi_i' - \Lambda.$$

The Lagrange's equations of the strip problem are equivalent to

$$\psi_i' = \frac{\partial H}{\partial p_i}, \quad p_i' = -\frac{\partial H}{\partial \psi_i},$$

for all $i = 1, 2, \dots, n$. These replace n differential equations of second order by the system of $2n$ differential equations of first order. The functional (8.19) may now be written as

$$\bar{I}_0[\psi_i, p_i] = \int_0^{2\pi} [\sum_{i=1}^{n} p_i \psi_i' - H(\psi_i, p_i)]\, d\theta.$$

It is easy to check that the extremal of this functional among 2π-periodic functions $\psi_i(\theta)$ and $p_i(\theta)$ corresponds to the extremal of the functional (8.19). If the Hamilton function does not depend explicitly on θ, then the first integral follows

$$H(\psi_i, p_i) = h.$$

Adiabatic Invariants. If we consider wave propagation in weakly inhomogeneous media or wave propagation under some external forces which change slowly in space and time, then the Lagrangian depends explicitly on $\mathbf{x}$ and t. This is quite similar to the vibrations of a non-autonomous mechanical system where one parameter of the system changes slowly in time.[8] It turns out that some quantities, called adiabatic invariants, remain constant in this situation. The finding of these adiabatic invariants can be done by the variational-asymptotic method. For simplicity let us consider a nonlinear oscillator with one degree of freedom $q(t)$ and one slowly varying parameter $\lambda(t)$. Hamilton's variational principle states that

$$\delta \int_{t_0}^{t_1} L(q, \dot{q}, \lambda)\, dt = 0.$$

We first calculate the average Lagrange function for the periodic motion with λ held fixed. Since the period is $T = 2\pi/\omega$, we have

$$\bar{L} = \frac{\omega}{2\pi} \int_0^T L(q, \dot{q}, \lambda)\, dt.$$

[8] For example, the vibration of a pendulum with slowly changing length.

With $\lambda = \text{const}$ the conservation of energy

$$\dot{q}\frac{\partial L}{\partial \dot{q}} - L = h$$

holds true. This equation may be solved with respect to $\dot{q}$ so that the generalized momentum $p = \partial L/\partial \dot{q}$ can be expressed as

$$p = p(q, h, \lambda).$$

Using the same conservation of energy we may calculate the average Lagrange function as follows

$$\bar{L} = \frac{\omega}{2\pi}\int_0^T p\dot{q}\,dt - h = \frac{\omega}{2\pi}\oint p(q,h,\lambda)\,dq - h, \tag{8.25}$$

where $\oint p dq$ means the integral over one complete period of vibration which corresponds to the close orbit in the phase plane. We now allow a slow change of λ in time, and consider the average variational problem obtained as the particular case of (8.20)

$$\delta \int_{t_0}^{t_1} \bar{L}(a, \theta_{,t}, \lambda)\,dt = 0.$$

Here $\theta_{,t} = -\omega$, with ω being the frequency of vibration. Lagrange's equations of this variational problem read

$$\frac{\partial \bar{L}}{\partial a} = \frac{\partial \bar{L}}{\partial h}\frac{dh}{da} = 0, \quad \frac{\partial}{\partial t}\frac{\partial \bar{L}}{\partial \theta_{,t}} = -\frac{\partial}{\partial t}\frac{\partial \bar{L}}{\partial \omega} = 0. \tag{8.26}$$

The first equation is nothing else but the frequency-amplitude equation of this nonlinear oscillator (see exercise 8.8). The second equation leads to the conservation of the action variable

$$I(\omega, h) = \frac{\partial \bar{L}}{\partial \omega} = \frac{1}{2\pi}\oint p(q,h,\lambda)\,dq = \text{const},$$

which is just the classical result of the adiabatic invariant [5]. From (8.25) and (8.26) the period is given by

$$T = \frac{2\pi}{\omega} = \frac{\partial I}{\partial h},$$

which is also classical.

From this analysis we see that for waves the quantities $\partial \bar{L}/\partial \omega$ and $\partial \bar{L}/\partial k_\alpha$ are akin to the adiabatic invariants with respect to time and space. If the wave packet is uniform in space but responding to changes of the medium in time, then we must have

$$\frac{\partial \bar{L}}{\partial \omega} = \text{const}.$$

Alternatively, for a wave packet of fixed frequency moving in a weakly inhomogeneous medium dependent only on one coordinate x, we have

$$\frac{\partial \bar{L}}{\partial k} = \text{const.}$$

In general, modulations in space and time balance according to the equation

$$\frac{\partial}{\partial t}\frac{\partial \bar{L}}{\partial \omega} - \frac{\partial}{\partial x_\alpha}\frac{\partial \bar{L}}{\partial k_\alpha} = 0,$$

which describes the propagation of the amplitude modulations.

Effect of Damping. If the medium in which waves propagate possesses some viscosity, then the energy is not only transported by the waves, but also dissipated during the process of wave propagation. The average equations of amplitude modulations can be obtained by the variational-asymptotic method for the case of small dissipation. Let us illustrate this on the example of the nonlinear Klein-Gordon equation with a small resistance force

$$u_{,tt} - u_{,xx} + U'(u) = f(u, u_{,t}),$$

where $f(u,u_{,t}) = -\partial D/\partial u_{,t}$ is a small term of the order $(\tau/T)u$, with $D(u,u_{,t})$ being the dissipation function assumed as homogeneous of order 2 with respect to $u_{,t}$. It is easy to show that this equation can be obtained from the variational equation

$$\delta \iint [\frac{1}{2}u_{,t}^2 - \frac{1}{2}u_{,x}^2 - U(u)]\, dx\, dt + \iint f(u,u_{,t})\delta u\, dx\, dt = 0. \tag{8.27}$$

In the first step of the variational-asymptotic method we neglect the last term in (8.27) as small compared with other terms and seek for the solution in the form

$$u = u_0(\theta, x, t),$$

where u_0 and θ behave in the same way as in (8.15). So, the analysis provided in the first paragraph of this Section leads to the following strip problem

$$\min_{\max u_0 = a} \langle \frac{1}{2}(\omega^2 - k^2)u_{0,\theta}^2 - U(u_0)\rangle,$$

where $\langle . \rangle = \frac{1}{2\pi}\int_0^{2\pi} .\, d\theta$ denotes the averaging over the strip, and where $\omega = -\theta_{,t}$ and $k = \theta_{,x}$ are treated as constants. Let $\bar{L}(a,\omega,k)$ be the minimum and $u_0 = \psi(a,\theta)$ the corresponding minimizer of this strip problem.

It can be shown that the second step brings correction of the order $u_1 \simeq (\tau/T)u_0$ in u and corrections of the order $(\tau/T)^2 u_0^2$ in the average Lagrangian and dissipation causing no influence on the average equations for a and θ.

To find the average equations let us substitute $u = \psi(a,\theta)$ into the original variational equation (8.27) and keep the asymptotically principal terms up to the order

$(\tau/T)\psi^2$ of smallness. Replacing the sums over the strips by the integrals in the limit $\lambda/\Lambda \to 0$ and $\tau/T \to 0$, we obtain

$$\delta \iint \bar{L}(a, -\theta_{,t}, \theta_x)\, dx dt + \iint \langle f(\psi, -\psi_{,\theta}\omega)\delta\psi(a,\theta)\rangle\, dx dt = 0.$$

Substitution of $\delta\psi = \psi_{,a}\delta a + \psi_{,\theta}\delta\theta$ into this equation yields

$$\delta \iint \bar{L}(a, -\theta_{,t}, \theta_x)\, dx dt + \iint \langle f(\psi, -\psi_{,\theta}\omega)(\psi_{,a}\delta a + \psi_{,\theta}\delta\theta)\rangle\, dx dt = 0.$$

It is easy to see that the term containing δa in the second integral brings just a small correction to the dispersion relation, so it can be neglected. Since the dissipation function $D(u, u_{,t})$ is a homogeneous function of second order with respect to $u_{,t}$,

$$\langle f(\psi, -\psi_{,\theta}\omega)\psi_{,\theta}\delta\theta\rangle = \frac{2}{\omega}\langle D(\psi, -\psi_{\theta}\omega)\rangle\delta\theta = \frac{2}{\omega}\bar{D}\delta\theta,$$

where $\bar{D}$ is the average dissipation function. Thus, the average variational equation reads

$$\delta \iint \bar{L}(a, -\theta_{,t}, \theta_{,x})\, dx dt + \iint \frac{2}{\omega}\bar{D}\delta\theta\, dx dt = 0. \tag{8.28}$$

Varying equation (8.28) with respect to a, we obtain

$$\frac{\partial \bar{L}}{\partial a} = 0,$$

which shows that the dispersion relation remains exactly the same as in the case without dissipation. Varying (8.28) with respect to θ, we derive the following equation

$$\frac{\partial}{\partial t}\frac{\partial \bar{L}}{\partial \omega} - \frac{\partial}{\partial x}\frac{\partial \bar{L}}{\partial k} = -\frac{2}{\omega}\bar{D}.$$

This equation shows the loss in wave action due to dissipation. We see also that the term on the right-hand side must be maintained because it is of the same order of smallness as the terms standing on the left-hand side. The energy balance equation, which can easily be obtained from here, reads

$$(\omega\bar{L}_{,\omega} - \bar{L})_{,t} - (\omega\bar{L}_{,k})_{,x} = -2\bar{D}.$$

We see that some portion of energy is transported by the energy flux $-\omega\bar{L}_{,k}$, and some is simply dissipated against the resistance due to viscosity. To complete the system of average equations of amplitude modulations we have to include also

$$k_{,t} + \omega_{,x} = 0,$$

which is simply the kinematic relation. It is easy to generalize this result to higher dimensions and more unknown functions.

8.4 Amplitude and Slope Modulation

This Section studies the theory of amplitude (or slope) modulation of nonlinear dispersive waves and presents some of its selected applications.[9]

The Near-Linear Case. As we know already from the previous Section, the amplitude modulation in 1-D case is described by the equations

$$\begin{aligned} \bar{L}_{,a} = 0, \quad k_{,t} + \omega_{,x} = 0, \\ (\bar{L}_{,\omega})_{,t} - (\bar{L}_{,k})_{,x} = 0. \end{aligned} \tag{8.29}$$

The first equation corresponds to the nonlinear dispersion relation. The near-linear theory is obtained by expanding $\bar{L}$ in powers of the amplitude. This expansion may be taken as

$$\bar{L} = G(\omega,k)a^2 + G_2(\omega,k)a^4 + \dots.$$

Computing $\bar{L}_{,a}$, we may solve $(8.29)_1$ with respect to ω to have explicitly

$$\omega = \Omega(k,a) = \Omega_0(k) + \Omega_2(k)a^2 + \dots, \tag{8.30}$$

where

$$G(\Omega_0,k) = 0, \quad \Omega_2(k) = -\frac{2G_2(\Omega_0(k),k)}{G_{,\omega}(\Omega_0(k),k)}.$$

We see that the dispersion relation $\omega = \Omega(k,a)$ couples the remaining equations (8.29). With (8.30) equation $(8.29)_2$ becomes

$$k_{,t} + [\Omega_0'(k) + \Omega_2'(k)a^2]k_{,x} + \Omega_2(k)(a^2)_{,x} = 0.$$

The important coupling term is $\Omega_2(k)(a^2)_{,x}$ because it leads to the correction $O(a)$ to the characteristic velocities. The other new term $\Omega_2'(k)a^2k_{,x}$ merely contributes the correction of oder $O(a^2)$. Concerning $(8.29)_3$ the inclusion of terms of order a^4 would provide corrections of order a^2 to the existing terms. Therefore in the first assessment of nonlinear effects we leave in the dispersion relation only one additional term $\Omega_2(k)a^2$ and consider

$$\begin{aligned} k_{,t} + \Omega_0'(k)k_{,x} + \Omega_2(k)(a^2)_{,x} = 0, \\ (a^2)_{,t} + (\Omega_0'(k)a^2)_{,x} = 0. \end{aligned} \tag{8.31}$$

This system of equations admits the characteristic form. To see this let us multiply the first equation by p and the second by q and then add them together. The resulting equation is

$$[pk_{,t} + (p\Omega_0' + q\Omega_0''a^2)k_{,x}] + [q(a^2)_{,t} + (p\Omega_2 + q\Omega_0')(a^2)_{,x}] = 0.$$

[9] Various applications of the theory of amplitude modulations to laser beams and water waves can be found in [53].

We want to choose p and q so that the expressions in the square brackets represent full derivatives of k and a^2 along the *same* characteristic curve. This is possible if

$$\frac{p}{p\Omega_0' + q\Omega_0'' a^2} = \frac{q}{p\Omega_2 + q\Omega_0'} \quad \Rightarrow \quad p = \pm\sqrt{\frac{\Omega_0''(k)}{\Omega_2(k)}}aq.$$

We may choose $p = 1$. Then, the characteristic form of (8.31) read

$$\frac{1}{2}\sqrt{\frac{\Omega_0''(k)}{\Omega_2(k)}}(\mathrm{sign}\Omega_0''(k))dk \pm da = 0$$

on the characteristics

$$\frac{dx}{dt} = \Omega_0'(k) \pm \sqrt{\Omega_2(k)\Omega_0''(k)}a. \tag{8.32}$$

This simple near-linear version of the theory of amplitude modulation already shows some interesting results. In the case $\Omega_2(k)\Omega_0''(k) > 0$, the characteristics are real and the system is hyperbolic. The double characteristic velocity splits under the nonlinear correction and we have the two velocities given by (8.32). In general, an initial disturbance or modulating source will introduce disturbances on both families of characteristics. If the initial disturbance is concentrated in a compact domain, it will eventually split into two.

When $\Omega_2(k)\Omega_0''(k) < 0$, the characteristics are imaginary and the system is elliptic. This leads to ill-posed problems in the wave propagation context. It means that small perturbations will grow in time leading to the instability of the wave packet. This case turns out to be not rare. For example, the Klein-Gordon equation with $U(\varphi) = \varphi^2/2 + \alpha\varphi^4/4$, α being small, gives

$$\Omega_0 = \sqrt{1+k^2}, \quad \Omega_2 = \frac{3}{8}\alpha/\sqrt{1+k^2}.$$

Thus, the sign of $\Omega_2(k)\Omega_0''(k)$ coincides with the sign of α; the modulation equations are hyperbolic for $\alpha > 0$ and elliptic for $\alpha < 0$. For waves of small up to moderate amplitudes, the Sine-Gordon equation has $\alpha = -1/6 > 0$. Thus, the waves of small amplitudes governed by the Sine-Gordon equation are unstable. This consequence of the near-linear theory, already non-trivial, is not easy to obtain by the direct stability analysis of the Sine-Gordon equation.

Characteristic Form of the Equations of Amplitude Modulation. Also the governing equations (8.29) of fully nonlinear theory of amplitude modulation admit the characteristic form. This can be obtained by doing Legendre transform of the average Lagrangian $\bar{L}(a,k,\omega)$ with respect to ω

$$H(a,k,I) = \omega\bar{L}_{,\omega} - \bar{L} = \omega I - \bar{L},$$

where $I = \bar{L}_{,\omega}$. Due to the property of Legendre transform we have

$$\bar{L}_{,k} = -H_{,k} = -J, \quad \omega = H_{,I}. \tag{8.33}$$

Therefore equations $(8.29)_{2,3}$ become

$$\begin{aligned} k_{,t} + (H_{,I})_{,x} &= 0, \\ I_{,t} + (H_{,k})_{,x} &= 0. \end{aligned}$$

Recalling that $\bar{L}_{,a} = -H_{,a} = 0$ due to the first equation of (8.29), we rewrite the above equations in the vector form as

$$\mathbf{v}_{,t} + \mathbf{M}\mathbf{v}_{,x} = 0,$$

where

$$\mathbf{v} = \begin{pmatrix} k \\ I \end{pmatrix}, \quad \mathbf{M} = \begin{pmatrix} H_{,Ik} & H_{,II} \\ H_{,kk} & H_{,kI} \end{pmatrix}.$$

Proceeding similarly as for equations (8.31) we get the characteristic equations

$$\sqrt{H_{,kk}}dk \pm \sqrt{H_{,II}}dI = 0$$

on the characteristics

$$\frac{dx}{dt} = H_{Ik} \pm \sqrt{H_{,kk}H_{,II}}.$$

If the characteristics are real, then the system (8.29) is hyperbolic. In the opposite case the system is elliptic. The type of the equations depends thus on the sign of $H_{,kk}H_{,II}$.

Slope Modulation of Waves Governed by Sine-Gordon Equation. The phase portrait of the strip problem for the Sine-Gordon equation

$$u_{,tt} - u_{,xx} = \sin u \tag{8.34}$$

exhibits in the subsonic regime quite different behavior than that of non-linear Klein-Gordon equation with $\alpha > 0$. This behavior dictates the fixing of slope rather than amplitude for its solution. To see this, let us start from the variational formulation of (8.34): find the extremal of the functional

$$I[u(x,t)] = \iint [\frac{1}{2}u_{,t}^2 - \frac{1}{2}u_{,x}^2 - (1 - \cos u)]\, dxdt.$$

The variational asymptotic procedure using the multi-scale Ansatz $u = \psi(\theta, x, t)$, developed in the previous Section, leads to the following strip problem: find the extremal of the functional

$$I_0[\psi(\theta)] = \frac{1}{2\pi} \int_0^{2\pi} [\frac{1}{2}(\omega^2 - k^2)\psi_{,\theta}^2 - (1 - \cos\psi)]\, d\theta$$

among functions $\psi(\theta)$ satisfying the conditions

$$\psi(2\pi) = \psi(0) + 2\pi, \quad \psi_{,\theta}(2\pi) = \psi_{,\theta}(0). \tag{8.35}$$

In this strip problem, the wave number $k = \theta_{,x}$ and the frequency $\omega = -\theta_{,t}$ are regarded as constants. Since we are interested in the subsonic regime $\omega^2 < k^2$, it is convenient to change the sign of this functional which does not influence Euler-Lagrange's equation. Thus, we need to find the extremal of the functional

$$\frac{1}{2\pi}\int_0^{2\pi}[\frac{1}{2}m\psi_{,\theta}^2 - (\cos\psi - 1)]\,d\theta \tag{8.36}$$

among functions $\psi(\theta)$ satisfying the conditions (8.35), where $m = k^2 - \omega^2$. Variational problem (8.36) possesses an obvious first integral

$$\frac{1}{2}m\psi_{,\theta}^2 + (\cos\psi - 1) = h$$

resembling that of the mathematical pendulum in the upward position. The corresponding phase portrait is plotted in Fig. 8.7. Looking at this phase portrait, we see that the determination of the phase curves as extremals of (8.36) outside the separatrix requires, in addition to (8.35), the fixing of the maximal slope of ψ as follows:

$$\max_\theta |\psi_{,\theta}| = p, \tag{8.37}$$

where p is an arbitrary real and positive number.

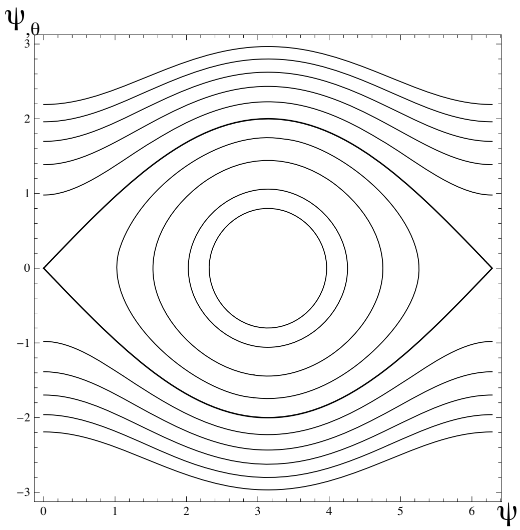

Fig. 8.7 Phase portrait of a pendulum with $m = 1$

Let us denote by $\bar{L}$ the average Lagrangian (extremum of functional (8.36)) which is a function of p, $k = \theta_{,x}$, and $\omega = -\theta_{,t}$. The sum of integrals over the strips, as the wave length goes to zero, can again be replaced by the integral

$$\bar{I}_0[p,\theta] = \iint \bar{L}(p,\theta_{,x},-\theta_{,t})\,dxdt.$$

Euler-Lagrange's equations of this average functional read

$$\frac{\partial \bar{L}}{\partial p} = \frac{\partial \bar{L}}{\partial h}\frac{\partial h}{\partial p} = 0, \qquad \frac{\partial}{\partial t}\frac{\partial \bar{L}}{\partial \omega} - \frac{\partial}{\partial x}\frac{\partial \bar{L}}{\partial k} = 0.$$

The first equation yields the nonlinear dispersion relation, while the second equation is the equation of slope modulation.

Using the above integral, we find the solution in terms of elliptic functions and then the average Lagrangian according to

$$\bar{L} = \frac{1}{2\pi}\int_0^{2\pi} m\psi_{,\theta}^2 d\theta - h = \frac{1}{2\pi}\int_0^{2\pi} m\psi_{,\theta} d\psi - h.$$

Now, to find the explicit dependence of $\bar{L}$ on p and m we use condition (8.37). Since the maximal slope of ψ is achieved at $\theta = \pi$ (see Fig. 8.7), this condition implies that $\frac{1}{2}mp^2 - 2 = h$. We require $h \ge 0$, so $p \ge 2/\sqrt{m}$. Then, from the first integral it follows

$$\bar{L}(p,k,\omega) = \frac{\sqrt{2m}}{2\pi} f(h) - h, \tag{8.38}$$

where $f(h)$ is the function expressed in terms of the complete elliptic integral

$$f(h) = \int_0^{2\pi} \sqrt{h - \cos\psi + 1}\,d\psi = 2[\sqrt{h}E(-2/h) + \sqrt{2+h}E(2/(2+h))].$$

According to (8.38) the dispersion relation reads

$$\frac{\sqrt{2m}}{2\pi} f'(h) - 1 = 0. \tag{8.39}$$

Keeping in mind this dispersion relation, let us find the derivatives

$$\frac{\partial \bar{L}}{\partial k} = \frac{\sqrt{2}}{2\pi}\frac{m_{,k}}{2\sqrt{m}} f(h) + \left(\frac{\sqrt{2m}}{2\pi} f'(h) - 1\right) h_{,k} = \frac{\sqrt{2}}{2\pi}\frac{k}{\sqrt{m}} f(h),$$

and

$$\frac{\partial \bar{L}}{\partial \omega} = \frac{\sqrt{2}}{2\pi}\frac{m_{,\omega}}{2\sqrt{m}} f(h) + \left(\frac{\sqrt{2m}}{2\pi} f'(h) - 1\right) h_{,\omega} = -\frac{\sqrt{2}}{2\pi}\frac{\omega}{\sqrt{m}} f(h),$$

where the last terms in these formulas vanish due to (8.39). Now we substitute these derivatives into the equation of slope modulation and compute the partial derivatives with respect to x and t. After some algebra we get

$$\frac{f(h)}{m\sqrt{m}}\mathscr{I}_1(\omega,k)+\frac{2\pi}{\sqrt{2}m}q(k^2k_{,x}-2k\omega\omega_{,x}-\omega^2\omega_{,t})+\frac{\pi}{\sqrt{2}}(kq_{,x}+\omega q_{,t})=0, \quad (8.40)$$

where $q=p^2$ and

$$\mathscr{I}_1(\omega,k)=k^2\omega_{,t}-\omega^2k_{,x}+2k\omega\omega_{,x}.$$

The equation of slope modulation in terms of θ is obtained if we replace in (8.40) $k=\theta_{,x}$ and $\omega=-\theta_{,t}$. Equivalently, equation (8.40) can be solved together with the consistency condition

$$\omega_{,x}+k_{,t}=0.$$

Asymptotic Solution to the Equation of Slope Modulation. From numerous numerical simulations and n-soliton exact solutions of the Sine-Gordon equation we know that, at large time, the solitons become non-interacting and propagating along the straight lines $x/t=$ const. Since the phase increases by 2π when one soliton is passed, let us look for the phase in the following form

$$\theta(x,t)=g(\xi(x,t)),\quad \xi(x,t)=x/t.$$

According to this Ansatz we have

$$k=\theta_{,x}=g'(\xi)\frac{1}{t},\quad \omega=-\theta_{,t}=g'(\xi)\frac{x}{t^2},\quad k_{,x}=g''(\xi)\frac{1}{t^2}, \quad (8.41)$$
$$\omega_{,x}=g''(\xi)\frac{x}{t^3}+g'(\xi)\frac{1}{t^2},\quad \omega_{,t}=-\left(g''(\xi)\frac{x^2}{t^4}+2g'(\xi)\frac{x}{t^3}\right).$$

It is now straightforward to check that $\mathscr{I}_1(\omega,k)=0$, so the equation of slope modulation takes the form

$$2q\mathscr{I}_2(\omega,k)+m(kq_{,x}+\omega q_{,t})=0, \quad (8.42)$$

where

$$\begin{aligned}\mathscr{I}_2(\omega,k)&=k^2k_{,x}-2k\omega\omega_{,x}-\omega^2\omega_{,t}\\&=g'(\xi)^2g''(\xi)\left(\frac{t^2-x^2}{t^4}\right)^2-2g'(\xi)^3\frac{x}{t^5}\left(\frac{t^2-x^2}{t^2}\right),\end{aligned}$$

and

$$m=k^2-\omega^2=g'(\xi)^2\frac{t^2-x^2}{t^4}. \quad (8.43)$$

Substituting these formulas into (8.42), we obtain

$$2q[g''(\xi)(t^2-x^2)-2g'(\xi)xt]+g'(\xi)t^2(tq_{,x}+xq_{,t})=0. \quad (8.44)$$

Equation (8.44) is the partial differential equation of first order which can be solved by the method of characteristics. The characteristic curves are determined by the equation

$$\frac{dx}{dt} = \frac{t}{x},$$

yielding

$$t^2 - x^2 = \alpha > 0.$$

Along any characteristic curve (8.44) becomes an ordinary differential equation

$$\frac{dQ_\alpha}{dt} + 2A_\alpha(t)Q_\alpha(t) = 0, \tag{8.45}$$

where α remains constant along each curve, $Q_\alpha(t) = q(x_\alpha(t),t)$, and

$$A_\alpha(t) = \frac{\alpha}{x_\alpha(t)t^2}\frac{g''(\xi_\alpha(t))}{g'(\xi_\alpha(t))} - \frac{2}{t}, \quad \xi_\alpha(t) = \frac{x_\alpha(t)}{t}, \quad x_\alpha(t) = \pm\sqrt{t^2-\alpha}.$$

A standard integration of (8.45) leads to

$$Q_\alpha(t) = C(\alpha)^2 \frac{t^4}{g'(\xi_\alpha(t))^2},$$

with $C(\alpha)$ being a function of α. Turning back to the original coordinates x and t, we obtain

$$q(x,t) = C(t^2-x^2)^2 \frac{t^4}{g'(\xi(x,t))^2}, \quad \xi(x,t) = \frac{x}{t},$$

and thus,

$$p(x,t) = \sqrt{q(x,t)} = C(t^2-x^2)\frac{t^2}{g'(\xi(x,t))}. \tag{8.46}$$

As $g(\xi)$ describes the phase, function $g'(\xi)$ can be identified with $2\pi\rho(\xi)$, where $\rho(\xi)$ is the density of solitons (or the number of solitons per unit length).

The unknown function $C(t^2-x^2)$ should be determined from the dispersion relation (8.39). Using the solution given by (8.46) and formula (8.43) for m, we obtain

$$h = \frac{1}{2}mp^2 - 2 = \frac{1}{2}(t^2-x^2)C(t^2-x^2)^2 - 2.$$

Since m goes to zero as t goes to infinity, the dispersion relation is fulfilled at large time if and only if h goes to zero. [10] Thus,

$$C(t^2-x^2) = \frac{2}{\sqrt{t^2-x^2}},$$

and the final asymptotic formula for the slope reads

[10] Strictly speaking, the exact fulfillment of the dispersion relation is warranted if h is of the order $m/2$ as $t \to \infty$, but this does not affect the asymptotically leading term for p.

$$p(x,t) = \frac{2t^2}{g'(\xi(x,t))\sqrt{t^2 - x^2}}. \tag{8.47}$$

Comparison with the Exact Solution. Let us compare the asymptotic solution (8.47) with the exact solution of Sine-Gordon equation obtained by the inverse scattering transform. It turns out that the exact solution of Sine-Gordon equation is intimately related to that of KdV equation. Note, however, that while the exact solution of KdV equation is given explicitly, the solution of Sine-Gordon equation is only obtainable through its slope $u_{,x}$. It is convenient to solve the Sine-Gordon equation in cone coordinates

$$X = \frac{1}{2}(x+t), \quad T = \frac{1}{2}(x-t). \tag{8.48}$$

Knowing the solution in X and T, the solution in the physical coordinates x and t can easily be found through a simple change of variables.

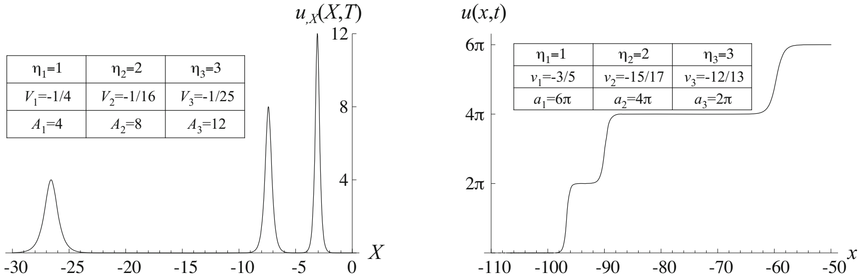

Fig. 8.8 3-soliton solution in physical coordinates (right) and its slope in cone coordinates (left). The eigenvalues, velocities and amplitudes of solitons and their slope are presented in the respective tables.

The solution reads (see [2])

$$\frac{1}{4}\left(\frac{\partial u}{\partial X}\right)^2 = \frac{\partial^2}{\partial X^2} \ln\left[\det\left(\mathbf{I} + \mathbf{A}\mathbf{A}^*\right)\right], \tag{8.49}$$

where

$$A_{lm} = \frac{\sqrt{c_l(T)c_m^*(T)}}{\zeta_l - \zeta_m^*} \exp\left[i(\zeta_l - \zeta_m^*)X\right],$$

and $c_l(T) = c_{l0}\exp(-iT/2\zeta_l)$. In the above formulas the asterisk is used to denote complex conjugate, while $\mathbf{I}$ corresponds to the identity matrix. Constants c_{l0} characterize the initial state of solitons, while $\zeta_l = i\eta_l$ are different purely imaginary eigenvalues of the linear operator associated with the Sine-Gordon equation (see [1,2] for the setting of the eigenvalue problem). Distinct types of solutions of this equation are determined by different choices of pairs of eigenvalues ζ_l and $\zeta_m = \zeta_l^*$. We shall concentrate on the traveling solitons, so the discrete and distinct imaginary eigenvalues are henceforth sufficient for our comparison purpose. Fig. 8.8 shows a 3-soliton solution (with solitons propagating to the left) together with its slope. By adding three other solitons (dislocations) propagating to the right and having the negative slope we may get the shape of the symmetrically propagating crack.

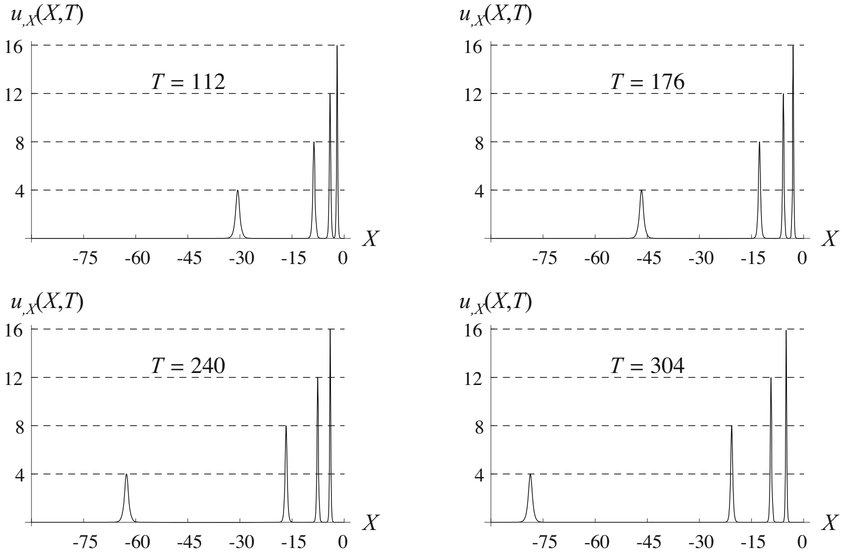

Fig. 8.9 Slope of 4-soliton solution in cone coordinates

As seen from Fig. 8.8 the slope of n-soliton solution to Sine-Gordon equation is itself n solitons having different shape. For this reason it makes sense to denote by V_j the velocities of solitons, which mark the velocities of points where maxima are achieved (centers of solitons), and by A_j the corresponding maxima. They are computed according to the following formulas

$$V_j = -\frac{1}{(2\eta_j)^2}, \quad A_j = 4\eta_j,$$

in which the minus sign indicates that the solitons travel to the left. The velocities of the j-th soliton in real space-time can be obtained through the change of variable (8.48)

$$v_j = -\frac{1+V_j}{1-V_j}.$$

In Fig. 8.9, several snap-shots at different time instants of the slope of 4-soliton solution constructed with the eigenvalues $\eta_j = j$, velocities $V_j = -1/4j^2$ and amplitudes $A_j = 4j$ are shown in cone coordinates.

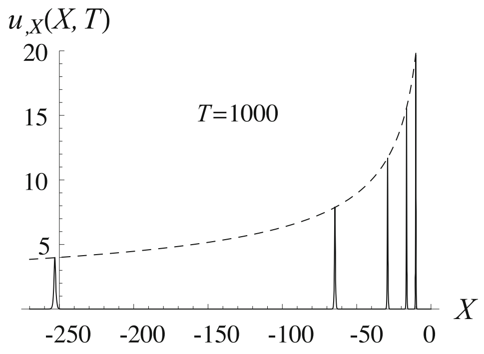

Fig. 8.10 Slope of 5-soliton in cone coordinates versus slope modulation: a) exact solution $u_{,X}$ (bold line), b) asymptotic law $2\sqrt{T/|X|}$ (dashed line)

To compare with the asymptotic solution obtained in the previous paragraph we note that for the slowly varying wave packet and to the first approximation,

$$u_{,X} = u_{,x} + u_{,t} = \psi_{,\theta}(k - \omega).$$

Since the maximum of $\psi_{,\theta}$ in one wavelength is chosen to be p, we expect that $p(k-\omega)$, with p being given by (8.47), should serve as the asymptotic envelope for the exact slope of soliton solution. Using (8.41) and (8.48), this quantity is given in cone coordinates by

$$p(k - \omega) = 2\frac{t - x}{\sqrt{t^2 - x^2}} = 2\sqrt{-\frac{T}{X}} \tag{8.50}$$

Formula (8.50) holds true for solitons traveling to the left. For solitons traveling to the right and having the negative slope, the signs inside and in front of the square root should be changed. Note also that this asymptotic law which can be used to predict, among others, the shape of the propagating crack regarded as the wave packet of moving dislocations in crystals, is universal and does not depend on the distribution of solitons. Fig. 8.10 shows the slope of the exact 5-soliton solution and the graph of $2\sqrt{T/|X|}$ (see exercise 8.10). From this Figure it is seen that, at large time, the curve $2\sqrt{T/|X|}$ can serve as the asymptotic envelope for the slope of solitons.

8.5 Amplitude Modulations for KdV Equation

This last Section studies Whitham's theory of amplitude modulations of waves governed by the KdV equation.

Derivation of Whitham's Equations. In view of the exact analytical solution of KdV equation by the inverse scattering transform, it is tempting to develop the theory of amplitude modulations for the KdV equation and to compare its asymptotic law with the exact solution. Unfortunately, in contrast to the Sine-Gordon equation the KdV equation does not admit a direct variational formulation. However, keeping in mind that the KdV equation is derivable from the Boussinesq's equation which admits a variational formulation, we may associate this equation with a variational principle by substituting $u = \eta_{,x}$ into (8.1) to get the equation

$$\eta_{,xt} + 6\eta_{,x}\eta_{,xx} + \eta_{,xxxx} = 0.$$

The latter can be obtained from the stationarity of the following functional

$$I[\eta(x,t)] = \iint (-\frac{1}{2}\eta_{,t}\eta_{,x} - \eta_{,x}^3 + \frac{1}{2}\eta_{,xx}^2)\, dxdt. \tag{8.51}$$

We shall use this indirect variational formulation through η to derive the equations of amplitude modulations for u. We look for the extremal of this variational problem in form of slowly varying wave packet

$$\eta(x,t) = \varphi(\theta,x,t) + \chi(x,t),$$

with φ a function of fast variable θ and slow variables x and t. We assume that φ is 2π-periodic with respect to θ. The fast variable θ, being itself a function of slow variables x and t, plays the role of the phase, with $\theta_{,x}$ and $-\theta_{,t}$ corresponding to the wave number k and the frequency ω, respectively. Besides, the derivative $\beta = \chi_{,x}$ accounts for the mean value of u over one θ-period. We calculate the partial derivatives of η in accordance with this Ansatz

$$\eta_{,x} = \varphi_{,\theta}\theta_{,x} + \underline{\partial_x\varphi} + \chi_{,x}, \quad \eta_{,t} = \varphi_{,\theta}\theta_{,t} + \underline{\partial_t\varphi} + \chi_{,t},$$
$$\eta_{,xx} = \varphi_{,\theta\theta}\theta_{,x}^2 + \underline{\varphi_{,\theta}\theta_{,xx}} + \underline{2\partial_x\varphi_{,\theta}\theta_{,x}} + \underline{\partial_x^2\varphi} + \underline{\chi_{,xx}}.$$

Based on the assumptions similar to those in (8.17), one recognizes immediately that the underlined terms are negligibly small compared with their first respective terms. Besides, the wave number and the frequency change slowly in one wave length and one period. We assume further that the mean value $\beta = \chi_{,x}$ changes also slowly in one wavelength so that its derivative $\beta_{,x} = \chi_{,xx}$ can be neglected in the first approximation. Taking all these circumstances into account, the derivatives of η can approximately be replaced by

$$\eta_{,x} = \varphi_{,\theta}\theta_{,x} + \chi_{,x}, \quad \eta_{,t} = \varphi_{,\theta}\theta_{,t} + \chi_{,t}, \quad \eta_{,xx} = \varphi_{,\theta\theta}\theta_{,x}^2,$$

where $\gamma = -\chi_{,t}$ is assumed to change slowly in one period. Substituting these formulas into (8.51), we obtain the functional

$$I_0[\varphi,\theta] = \iint [-\frac{1}{2}(\varphi_{,\theta}\theta_{,t} + \chi_{,t})(\varphi_{,\theta}\theta_{,x} + \chi_{,x}) - (\varphi_{,\theta}\theta_{,x} + \beta)^2 + \frac{1}{2}\theta_{,x}^4\varphi_{,\theta\theta}^2]dxdt.$$

In accordance with the method developed in Section 8.3, we formulate the strip problem as follows: find the extremal of the functional

$$\frac{1}{2\pi}\int_0^{2\pi}[\frac{1}{2}(\omega\varphi_{,\theta}+\gamma)(k\varphi_{,\theta}+\beta)-(k\varphi_{,\theta}+\beta)^3+\frac{1}{2}k^4\varphi_{,\theta\theta}^2]\,d\theta \tag{8.52}$$

among functions $\varphi(\theta)$ satisfying 2π-periodicity conditions

$$\varphi(2\pi)=\varphi(0),\quad \varphi_{,\theta}(2\pi)=\varphi_{,\theta}(0),\quad \varphi_{,\theta\theta}(2\pi)=\varphi_{,\theta\theta}(0).$$

In this strip problem k, ω, β, and γ are considered as constants. Let us denote by $c=\omega/k$ the phase velocity.

To reduce the order of the resulting differential equations, let us make a change of unknown function

$$\phi=k\varphi_{,\theta}+\beta.$$

It is natural to use this new unknown function in the strip problem because it represents the approximate solution of KdV equation. According to it we have

$$\varphi_{,\theta}=\frac{\phi-\beta}{k},\quad \phi_{,\theta}=k\varphi_{,\theta\theta}.$$

As function $\varphi(\theta)$ is 2π-periodic, the introduced new function $\phi(\theta)$ should satisfy the constraint

$$\frac{1}{2\pi}\int_0^{2\pi}\phi\,d\theta=\frac{1}{2\pi}\int_0^{2\pi}(k\varphi_{,\theta}+\beta)\,d\theta=\beta. \tag{8.53}$$

Thus, we replace the functional (8.52) by

$$\begin{aligned}&\frac{1}{2\pi}\int_0^{2\pi}[\frac{1}{2}(\frac{\phi-\beta}{k}\omega+\gamma)\phi-\phi^3+\frac{1}{2}k^2\phi_{,\theta}^2]\,d\theta\\&=\frac{1}{2\pi}\int_0^{2\pi}[\frac{1}{2}c\phi^2-\phi^3+\frac{1}{2}k^2\phi_{,\theta}^2]\,d\theta+\frac{1}{2}(\gamma-c\beta)\beta,\end{aligned}$$

which must be minimized among 2π-periodic functions $\phi(\theta)$ satisfying the constraint (8.53). To get rid of constraint (8.53) we introduce the Lagrange multiplier and consider the following equivalent variational problem: find the extremal of the functional

$$\begin{aligned}&\frac{1}{2\pi}\int_0^{2\pi}[\frac{1}{2}k^2\phi_{,\theta}^2+\frac{1}{2}c\phi^2-\phi^3]\,d\theta+\frac{1}{2}(\gamma-c\beta)\beta-\lambda(\frac{1}{2\pi}\int_0^{2\pi}\phi\,d\theta-\beta)\\&=\frac{1}{2\pi}\int_0^{2\pi}[\frac{1}{2}k^2\phi_{,\theta}^2-U(\phi,c,\lambda)]\,d\theta+\frac{1}{2}(\gamma-c\beta)\beta+\lambda\beta\end{aligned}$$

among λ and $\phi(\theta)$ satisfying the periodicity conditions

$$\phi(2\pi)=\phi(0),\quad \phi_{,\theta}(2\pi)=\phi_{,\theta}(0),$$

where the function of three arguments $U(\phi,c,\lambda)$ is given by

$$U(\phi,c,\lambda)=\phi^3-\frac{1}{2}c\phi^2+\lambda\phi.$$

This variational problem leads to Lagrange's equation of second order in terms of $\phi(\theta)$, which possesses an obvious first integral

$$\frac{1}{2}k^2\phi_{,\theta}^2+U(\phi,c,\lambda)=h.$$

Using this formula and introducing an elliptic integral

$$W(c,\lambda,h)=\frac{1}{2\pi}\oint\sqrt{2h-2U(\phi,c,\lambda)}d\phi=\frac{1}{2\pi}\oint\sqrt{2h-2\lambda\phi+c\phi^2-2\phi^3}d\phi,$$

we find the average Lagrangian as the minimum of the above functional in the form

$$\bar{L}(\lambda,\beta,\gamma,h,k,\omega)=kW(\frac{\omega}{k},\lambda,h)+\frac{1}{2}(\gamma-\frac{\omega}{k}\beta)\beta+\lambda\beta-h.$$

Then the variational-asymptotic analysis leads to the following average variational problem

$$\delta\iint\bar{L}(\lambda,\chi_{,x},-\chi_{,t},h,\theta_{,x},-\theta_{,t})\,dxdt=0.$$

The Euler-Lagrange's equations for λ and χ read

$$\beta=-kW_{,\lambda},\quad \frac{1}{2}\beta_{,t}-(\lambda+\frac{1}{2}\gamma-c\beta)_{,x}=0.$$

From the last equation and from the consistency condition $\beta_{,t}+\gamma_{,x}=0$ it follows that γ can be taken as $\gamma=c\beta-\lambda$. Thus, $\beta=-kW_{,\lambda}$, $\gamma=-ckW_{,\lambda}-\lambda$, and the consistency condition becomes

$$(kW_{,\lambda})_{,t}+(ckW_{,\lambda}+\lambda)_{,x}=0. \tag{8.54}$$

For h and θ we have

$$kW_{,h}=1,\quad (\bar{L}_{,\omega})_{,t}-(\bar{L}_{,k})_{,x}=0.$$

Multiplying the last equation by k and using the chain rule of differentiation together with the consistency condition $k_{,t}+\omega_{,x}=0$, we obtain

$$(k\bar{L}_{,\omega})_{,t}-(k\bar{L}_{,k})_{,x}+\bar{L}_{,\omega}\omega_{,x}+\bar{L}_{,k}k_{,x}=0. \tag{8.55}$$

On the other hand, differentiation of the average Lagrangian $\bar{L}$ with respect to x gives

$$\begin{aligned}\bar{L}_{,x}&=\bar{L}_{,k}k_{,x}+\bar{L}_{,\omega}\omega_{,x}+\bar{L}_{,\beta}\beta_{,x}+\bar{L}_{,\gamma}\gamma_{,x}\\&=\bar{L}_{,k}k_{,x}+\bar{L}_{,\omega}\omega_{,x}+(\beta\bar{L}_{,\beta})_{,x}-\beta(\bar{L}_{,\beta})_{,x}-(\beta\bar{L}_{,\gamma})_{,t}+\beta(\bar{L}_{,\gamma})_{,t},\end{aligned}$$

which implies

$$\bar{L}_{,\omega}\omega_{,x}+\bar{L}_{,k}k_{,x}=\bar{L}_{,x}-(\beta\bar{L}_{,\beta})_{,x}+(\beta\bar{L}_{,\gamma})_{,t}+\beta[(\bar{L}_{,\gamma})_{,t}-(\bar{L}_{,\beta})_{,x}].$$

The last term vanishes due to the Lagrange's equation for χ yielding

$$\bar{L}_{,\omega}\omega_{,x}+\bar{L}_{,k}k_{,x}=\bar{L}_{,x}-(\beta\bar{L}_{,\beta})_{,x}+(\beta\bar{L}_{,\gamma})_{,t}. \tag{8.56}$$

Substituting (8.56) into (8.55), we obtain the so-called wave momentum equation

$$(k\bar{L}_{,\omega}+\beta\bar{L}_{,\gamma})_{,t}+(\bar{L}-k\bar{L}_{,k}-\beta\bar{L}_{,\beta})_{,x}=0,$$

which can replace the Lagrange's equation for θ. In our case, this equation becomes

$$(kW_{,c})_{,t}+(ckW_{,c}-h)_{,x}=0. \tag{8.57}$$

Again, the consistency condition

$$k_{,t}+(ck)_{,x}=0 \tag{8.58}$$

has to be included. Equations (8.54), (8.57), and (8.58) may be viewed as three equations for h, λ, and c, with k given by the dispersion relation $k=1/W_{,h}$. A more symmetric equivalent form of this system is

$$\frac{D}{Dt}(W_{,h})=W_{,h}c_{,x},\quad \frac{D}{Dt}(W_{,\lambda})=-W_{,h}\lambda_{,x},\quad \frac{D}{Dt}(W_{,c})=W_{,h}h_{,x}, \tag{8.59}$$

where

$$\frac{D}{Dt}=\frac{\partial}{\partial t}+c\frac{\partial}{\partial x}.$$

In terms of these unknown functions the wave number, the frequency, and the mean value of u, $\bar{u}=\beta$, are given by

$$k=\frac{1}{W_{,h}},\quad \omega=\frac{c}{W_{,h}},\quad \beta=-\frac{W_{,\lambda}}{W_{,h}}.$$

The amplitude is obtained by relating the zeros of the cubic polynomial in W to the coefficients h, λ, and c.

The Characteristic Equations. It turns out that the system (8.59) is hyperbolic and can be written in the characteristic form. If the zeros b_1, b_2, b_3 of the cubic equation

$$\phi^3-\frac{1}{2}c\phi^2+\lambda\phi-h=0 \tag{8.60}$$

are used as new unknown functions instead of h, λ, c, Whitham's equations may be put in a simple characteristic form as

$$\frac{D}{Dt}r_j + V_j r_{j,x} = 0, \quad j = 1,2,3, \quad \text{(no sum!)}$$

where $r_1 = b_2 + b_3$, $V_1 = W_{,h}/(W_{,h})_{,b_1}$, together with similar equations for r_2 and r_3 in cyclic permutation. In the following we sketch a brief proof for the first of these equations. Let us factorize the cubic polynomial (8.60) as

$$\phi^3 - \frac{1}{2}c\phi^2 + \lambda\phi - h = (\phi - b_1)(\phi - b_2)(\phi - b_3).$$

According to this identity the unknowns c, λ, h are related to the zeros b_1, b_2, b_3 by

$$c = 2(b_1 + b_2 + b_3), \quad \lambda = b_1 b_2 + b_1 b_3 + b_2 b_3, \quad h = b_1 b_2 b_3.$$

Differentiating these relations with respect to x, we rewrite Whitham's equations (8.59) in terms of new unknown functions b_j in the form

$$\begin{aligned}
(W_{,h})_{,b_1}\frac{Db_1}{Dt} + (W_{,h})_{,b_2}\frac{Db_2}{Dt} + (W_{,h})_{,b_3}\frac{Db_3}{Dt} &= 2W_{,h}(b_{1,x} + b_{2,x} + b_{3,x}), \\
(W_{,\lambda})_{,b_1}\frac{Db_1}{Dt} + (W_{,\lambda})_{,b_2}\frac{Db_2}{Dt} + (W_{,\lambda})_{,b_3}\frac{Db_3}{Dt} &= -W_{,h}[(b_2 + b_3)b_{1,x} \\
&\quad + (b_1 + b_3)b_{2,x} + (b_1 + b_2)b_{3,x}], \\
(W_{,c})_{,b_1}\frac{Db_1}{Dt} + (W_{,c})_{,b_2}\frac{Db_2}{Dt} + (W_{,c})_{,b_3}\frac{Db_3}{Dt} &= W_{,h}(b_2 b_3 b_{1,x} + b_1 b_3 b_{2,x} + b_1 b_2 b_{3,x}).
\end{aligned} \tag{8.61}$$

We introduce

$$f(\phi) = 2h - 2\lambda\phi + c\phi^2 - 2\phi^3 = -2(\phi - b_1)(\phi - b_2)(\phi - b_3),$$

and denote the elliptic integral as follows

$$W(c,\lambda,h) = \frac{1}{2\pi}\oint\sqrt{2h - 2\lambda\phi + c\phi^2 - 2\phi^3}\,d\phi = \frac{1}{2\pi}\oint\sqrt{f(\phi)}\,d\phi.$$

With this notation at hand we compute $W_{,h}$, $W_{,\lambda}$, $W_{,c}$

$$W_{,h} = \frac{1}{2\pi}\oint\frac{d\phi}{\sqrt{f(\phi)}}, \quad W_{,\lambda} = -\frac{1}{2\pi}\oint\frac{\phi\,d\phi}{\sqrt{f(\phi)}}, \quad W_{,c} = \frac{1}{2\pi}\oint\frac{\phi^2 d\phi}{2\sqrt{f(\phi)}}.$$

Next, differentiating these formulas with respect to b_1, we obtain

$$\begin{aligned}
(W_{,h})_{,b_1} &= \frac{1}{4\pi}\oint\frac{1}{(\phi - b_1)\sqrt{f(\phi)}}d\phi, \\
(W_{,\lambda})_{,b_1} &= -\frac{1}{4\pi}\oint\frac{\phi}{(\phi - b_1)\sqrt{f(\phi)}}d\phi, \\
(W_{,c})_{,b_1} &= \frac{1}{4\pi}\oint\frac{\phi^2}{2(\phi - b_1)\sqrt{f(\phi)}}d\phi.
\end{aligned}$$

Similar formulas hold true for the derivatives with respect to b_2 and b_3. Now the trick comes to play at this step. We multiply the first equation of (8.61) by p, the second by q, and the third by r, add them, and choose p, q, r in such a way that the coefficient of $b_{1,x}$ on the right-hand side vanishes, while those of $b_{2,x}$ and $b_{3,x}$ are equal. This leads to the two following conditions

$$\begin{aligned}
&2p-q(b_2+b_3)+rb_2b_3=0,\\
&2p-q(b_1+b_3)+rb_1b_3=2p-q(b_1+b_2)+rb_1b_2.
\end{aligned}$$

The solution of the above system reads

$$q=rb_1, \quad p=\frac{r}{2}(b_1b_2+b_1b_3-b_2b_3),$$

in which r can be chosen arbitrarily. Let us choose $r=2$ for convenience and obtain the explicit expressions for q and p

$$r=2, \quad q=2b_1, \quad p=b_1b_2+b_1b_3-b_2b_3.$$

With this choice, the right-hand side of the equation resulted from these operations takes the form

$$\begin{aligned}
\text{RHS}&=[2(b_1b_2+b_1b_3-b_2b_3)-2b_1(b_1+b_3)+2b_1b_3]W_{,h}(b_2+b_3)_{,x}\\
&=-2(b_1-b_2)(b_1-b_3)W_{,h}(b_2+b_3)_{,x}.
\end{aligned} \tag{8.62}$$

Let us turn now to the left-hand side and denote by K_1, K_2, and K_3 the coefficients of Db_1/Dt, Db_2/Dt, and Db_3/Dt, respectively. Then we have

$$\begin{aligned}
K_1&=p(W_{,h})_{,b_1}+q(W_{,\lambda})_{,b_1}+r(W_{,c})_{,b_1}\\
&=\frac{1}{4\pi}\oint\frac{b_1b_2+b_1b_3-b_2b_3-2b_1\phi+\phi^2}{(\phi-b_1)\sqrt{f(\phi)}}d\phi,\\
K_2&=p(W_{,h})_{,b_2}+q(W_{,\lambda})_{,b_2}+r(W_{,c})_{,b_2}\\
&=\frac{1}{4\pi}\oint\frac{b_1b_2+b_1b_3-b_2b_3-2b_1\phi+\phi^2}{(\phi-b_2)\sqrt{f(\phi)}}d\phi,\\
K_3&=p(W_{,h})_{,b_3}+q(W_{,\lambda})_{,b_3}+r(W_{,c})_{,b_3}\\
&=\frac{1}{4\pi}\oint\frac{b_1b_2+b_1b_3-b_2b_3-2b_1\phi+\phi^2}{(\phi-b_3)\sqrt{f(\phi)}}d\phi.
\end{aligned}$$

One can prove the following identities (see exercise 8.11)

$$K_1=0, \quad K_2=K_3. \tag{8.63}$$

Furthermore, we can rewrite the coefficients K_2 and K_3 as

$$K_2 = -\frac{b_1 - b_2}{2\pi} \oint \frac{\phi - b_3}{(\phi - b_2)\sqrt{f(\phi)}} d\phi + \frac{1}{4\pi} \oint \frac{\phi^2 - 2b_2\phi + b_1 b_2 + b_2 b_3 - b_1 b_3}{(\phi - b_2)\sqrt{f(\phi)}} d\phi,$$
$$K_3 = -\frac{b_1 - b_3}{2\pi} \oint \frac{\phi - b_2}{(\phi - b_3)\sqrt{f(\phi)}} d\phi + \frac{1}{4\pi} \oint \frac{\phi^2 - 2b_3\phi + b_1 b_3 + b_2 b_3 - b_1 b_2}{(\phi - b_3)\sqrt{f(\phi)}} d\phi.$$

The last terms in K_2 and K_3 vanish because their integrands are again full differentials. Thus,

$$K_2 = -(b_1 - b_2)W_{,h} - 2(b_1 - b_2)(b_2 - b_3)(W_{,h})_{,b_2},$$
$$K_3 = -(b_1 - b_3)W_{,h} - 2(b_1 - b_3)(b_3 - b_2)(W_{,h})_{,b_3}.$$

Equality $K_2 = K_3$ gives

$$W_{,h} = 2[(b_1 - b_2)(W_{,h})_{,b_2} + (b_1 - b_3)(W_{,h})_{,b_3}],$$

which implies

$$K_2 = K_3 = -2(b_1 - b_2)(b_1 - b_3)[(W_{,h})_{,b_2} + (W_{,h})_{,b_3}].$$

Due to the identity

$$(W_{,h})_{,b_1} + (W_{,h})_{,b_2} + (W_{,h})_{,b_3} = \frac{1}{4\pi} \oint \frac{f'(\phi)}{f^{3/2}(\phi)} d\phi = 0,$$

we can write the last formula in the form

$$K_2 = K_3 = 2(b_1 - b_2)(b_1 - b_3)(W_{,h})_{,b_1}. \tag{8.64}$$

With (8.62) and (8.64) we get one of the Whitham's equations in the characteristic form

$$\frac{D}{Dt}(b_2 + b_3) + \frac{W_{,h}}{(W_{,h})_{,b_1}}(b_2 + b_3)_{,x} = 0,$$

which shows that $b_2 + b_3$ is the Riemann's invariant. The other two equations for r_2 and r_3 in cyclic permutations can be established in the same manner.

Alternative Representation of Whitham's Equations. Whitham's equations involve three unknown functions, namely c, λ, and h. In order to find the amplitude modulation in particular cases such as wave of small up to moderate amplitudes or trains of solitons one have to relate them to the amplitude $a = b_1 - b_2$. Then, using Whitham's equation in the characteristic form, different types of solution can be found. Here and below we consider an alternative but equivalent version of system of equations which directly involves the amplitude. For this purpose let us define the amplitude in a slightly different way

$$a = \max \phi .$$

Note that the amplitude defined in this way is nothing else but b_1. Using this definition, we rewrite the average Lagrangian as follows

$$\bar{L} = \frac{k\sqrt{2}}{\pi} \int_{b_2}^{a} \sqrt{U(a,c,\lambda) - U(\phi,c,\lambda)} d\phi - U(a,c,\lambda) + \lambda\beta + \frac{1}{2}(\gamma - c\beta)\beta, \quad (8.65)$$

where the energy level h has been replaced by $U(a,c,\lambda)$, namely $h = U(a,c,\lambda)$. Observe that the integrand in the above integral vanishes at three zeros a, b_2, b_3 due to

$$U(a,c,\lambda) - U(\phi,c,\lambda) = (a-\phi)(\phi - b_2)(\phi - b_3).$$

This circumstance will be used later when one differentiates the average Lagrangian. The Euler-Lagrange's equations associated with this average Lagrangian read

$$\begin{aligned} &\frac{\partial \bar{L}}{\partial a} = 0, \quad \frac{\partial}{\partial t}\frac{\partial \bar{L}}{\partial \omega} - \frac{\partial}{\partial x}\frac{\partial \bar{L}}{\partial k} = 0, \\ &\frac{\partial \bar{L}}{\partial \lambda} = 0, \quad \frac{\partial}{\partial t}\frac{\partial \bar{L}}{\partial \gamma} - \frac{\partial}{\partial x}\frac{\partial \bar{L}}{\partial \beta} = 0. \end{aligned} \quad (8.66)$$

The first equation is nothing else but the dispersion relation, whereas the third equation is equivalent to the constraint (8.53). To express these equations in terms of a, c, and λ let us compute the derivative of $\bar{L}$ from (8.65) with respect to a and λ

$$\begin{aligned} \frac{\partial \bar{L}}{\partial a} &= \frac{\partial U}{\partial a}(a,c,\lambda)\Big[\frac{k\sqrt{2}}{2\pi}\int_{b_2}^{a} \frac{d\phi}{\sqrt{U(a,c,\lambda) - U(\phi,c,\lambda)}} - 1\Big], \\ \frac{\partial \bar{L}}{\partial \lambda} &= \frac{\partial U}{\partial \lambda}(a,c,\lambda)\Big[\frac{k\sqrt{2}}{2\pi}\int_{b_2}^{a} \frac{d\phi}{\sqrt{U(a,c,\lambda) - U(\phi,c,\lambda)}} - 1\Big] \\ &\quad - \frac{k\sqrt{2}}{2\pi}\int_{b_2}^{a} \frac{\phi d\phi}{\sqrt{U(a,c,\lambda) - U(\phi,c,\lambda)}} + \beta. \end{aligned}$$

Thus, the dispersion relation and the constraint associated with λ follow at once

$$\frac{k\sqrt{2}}{2\pi}\int_{b_2}^{a} \frac{d\phi}{\sqrt{U(a,c,\lambda) - U(\phi,c,\lambda)}} - 1 = 0, \quad (8.67)$$

$$\frac{k\sqrt{2}}{2\pi}\int_{b_2}^{a} \frac{\phi d\phi}{\sqrt{U(a,c,\lambda) - U(\phi,c,\lambda)}} - \beta = 0. \quad (8.68)$$

Let us turn now to the equation of amplitude modulation. First, we compute the derivative of $\bar{L}$ with respect to ω

$$\frac{\partial \bar{L}}{\partial \omega}=\frac{\partial U}{\partial c}(a,c,\lambda)\frac{\partial c}{\partial \omega}[\frac{k\sqrt{2}}{2\pi}\int_{b_2}^{a}\frac{d\phi}{\sqrt{U(a,c,\lambda)-U(\phi,c,\lambda)}}-1]$$
$$+\frac{\sqrt{2}}{4\pi}F(a,c,\lambda)-\frac{\beta^2}{2k}=\frac{\sqrt{2}}{4\pi}F(a,c,\lambda)-\frac{\beta^2}{2k},$$

where

$$F(a,c,\lambda)=\int_{b_2}^{a}\frac{\phi^2 d\phi}{\sqrt{U(a,c,\lambda)-U(\phi,c,\lambda)}}.$$

Differentiation of $\bar{L}$ with respect to k gives

$$\frac{\partial \bar{L}}{\partial k}=\frac{\sqrt{2}}{\pi}\int_{b_2}^{a}\sqrt{U(a,c,\lambda)-U(\phi,c,\lambda)}d\phi$$
$$+\frac{\partial U}{\partial c}(a,c,\lambda)\frac{\partial c}{\partial k}[\frac{k\sqrt{2}}{2\pi}\int_{b_2}^{a}\frac{d\phi}{\sqrt{U(a,c,\lambda)-U(\phi,c,\lambda)}}-1]$$
$$+\frac{\partial c}{\partial k}[\frac{k\sqrt{2}}{4\pi}\int_{b_2}^{a}\frac{\phi^2 d\phi}{\sqrt{U(a,c,\lambda)-U(\phi,c,\lambda)}}-\frac{1}{2}\beta^2]$$
$$=\frac{\sqrt{2}}{\pi}\int_{b_2}^{a}\sqrt{U(a,c,\lambda)-U(\phi,c,\lambda)}d\phi-c\frac{\partial \bar{L}}{\partial \omega}.$$

Plugging these derivatives into $(8.66)_2$, we obtain the Euler-Lagrange's equation for θ

$$\frac{\sqrt{2}}{4\pi}[\frac{\partial F}{\partial a}(a_{,t}+ca_{,x})+\frac{\partial F}{\partial c}(c_{,t}+cc_{,x})+\frac{\partial F}{\partial \lambda}(\lambda_{,t}+c\lambda_{,x})]+\frac{\sqrt{2}}{4\pi}F(a,c,\lambda)c_{,x}$$
$$-\frac{1}{2}[(\frac{\beta^2}{k})_{,t}+(c\frac{\beta^2}{k})_{,x}]-\frac{\sqrt{2}}{\pi}\frac{\partial}{\partial x}\int_{b_2}^{a}\sqrt{U(a,c,\lambda)-U(\phi,c,\lambda)}d\phi=0.$$

Further, using the dispersion relation (8.67) and constraint (8.68), we compute the derivative

$$J=\frac{\partial}{\partial x}\int_{b_2}^{a}\sqrt{U(a,c,\lambda)-U(\phi,c,\lambda)}d\phi$$
$$=\frac{\pi}{k\sqrt{2}}\frac{\partial U}{\partial a}(a,c,\lambda)a_{,x}-\frac{1}{2}\frac{\pi}{k\sqrt{2}}a^2c_{,x}+\frac{\pi}{k\sqrt{2}}a\lambda_{,x}-\frac{\pi}{k\sqrt{2}}\beta\lambda_{,x}+\frac{1}{4}F(a,c,\lambda)c_{,x}.$$

Finally, substituting this expression into the above equation and dividing the latter by $\sqrt{2}/4\pi$, we obtain the equation of amplitude modulation in terms of a, c and λ

$$\frac{\partial F}{\partial a}(a_{,t}+ca_{,x})+\frac{\partial F}{\partial c}(c_{,t}+cc_{,x})+\frac{\partial F}{\partial \lambda}(\lambda_{,t}+c\lambda_{,x})$$
$$+\frac{\pi\sqrt{2}}{k}\{a^2c_{,x}-2\partial_a U(a,c,\lambda)a_{,x}+2(\beta-a)\lambda_{,x}-k[(\frac{\beta^2}{k})_{,t}+(c\frac{\beta^2}{k})_{,x}]\}=0. \quad (8.69)$$

As indicated in the previous paragraph, the equation for χ will be automatically satisfied if the parameters are chosen such that

$$\gamma = c\beta - \lambda. \tag{8.70}$$

The four equations (8.67)-(8.70) constitute a system of differential equations which is equivalent to (8.59) plus the dispersion relation $kW_{,h} = 1$.

Trains of Solitons. In the limit $\lambda \to 0$ and $h \to 0$, the wave packet becomes a train of solitary waves. For the wave packet consisting of n solitons we know that the solitons cease to interact at large time in such a way that each of them propagates with a constant velocity along the line $x/t = \text{const}$. Based on this observation we look for the solution $a = a(x,t)$ of (8.69) using the following Ansatz for θ and χ

$$\theta(x,t) = q(\xi(x,t)), \quad \chi(x,t) = p(\xi(x,t)), \quad \xi(x,t) = x/t.$$

Differentiating $\theta(x,t)$ and $\chi(x,t)$ in accordance with these Ansatz, we find k, ω, c, β, and γ in the form

$$k = \frac{1}{t}q'(\xi), \quad \omega = \frac{x}{t^2}q'(\xi), \quad c = \frac{x}{t},$$
$$\beta = \frac{1}{t}p'(\xi), \quad \gamma = \frac{x}{t^2}p'(\xi).$$

It is easy to see that β and γ from the last equations satisfy (8.70), provided $\lambda = 0$. Besides, the following equation

$$(\frac{\beta^2}{k})_{,t} + (c\frac{\beta^2}{k})_{,x} = 0$$

is fulfilled identically. Furthermore, if the amplitude is searched among functions of the form

$$a(x,t) = g(\xi(x,t)),$$

the term $\frac{\partial F}{\partial a}(a_{,t} + ca_{,x})$ vanishes, so (8.69) reduces to

$$g(\xi)^2 - 2\partial_a U(g(\xi),\xi,0)g'(\xi) = 0,$$

where $\partial_a U(a,c,\lambda) = 3a^2 - ca + \lambda$. The last equation can also be rewritten as

$$g(\xi)^2 - (6g(\xi)^2 - 2\xi g(\xi))g'(\xi) = 0,$$

which is equivalent to

$$g(\xi) - (6g(\xi) - 2\xi)g'(\xi) = 0. \tag{8.71}$$

General solution of (8.71) contains one constant of integration that should be determined from the dispersion relation. In the following we shall guess a particular solution of (8.71) and prove its validity by verifying the fulfillment of dispersion relation at large time. Looking at this equation, we see that one of its possible solutions is the linear function

$$g(\xi) = C_1\xi + C_2.$$

Substituting this guess into (8.71) and equating the coefficient of first power of ξ and the free one to zero, we find

$$C_1 = 1/2, \quad C_2 = 0.$$

Thus, a simple particular solution of (8.69) reads

$$a(x,t) = \frac{x}{2t}. \tag{8.72}$$

To see the fulfillment of the dispersion relation at large time we rewrite (8.67) in an equivalent form

$$k = \frac{\pi\sqrt{(a-b_2)/2}}{mK(m)}, \quad m = \sqrt{\frac{a-b_2}{a-b_3}},$$

where $K(m)$ is the complete elliptic integral. In the limit $\lambda \to 0$, $h \to 0$, the roots b_2 and b_3 go to 0, so $m \to 1$. Provided the derivative $q'(x/t)$ is finite, the left- and right-hand sides of the dispersion relation tend to 0 as $t \to \infty$, so the dispersion relation is satisfied asymptotically at large time.

Recalling that a soliton of amplitude a moves with the velocity $2a$, one can easily recognize that (8.72) represents a large-time asymptotic envelope of a sequence of solitons each retaining a constant amplitude and moving on the path $x = 2at$ as shown in Fig. 8.11. Another way of obtaining this amplitude modulation of soliton solution is to derive the system of equations

$$\bar{k}_{,t} + (2a\bar{k})_{,x} = 0, \quad a_{,t} + 2aa_{,x} = 0, \quad \bar{k} = \frac{k}{2\pi} \tag{8.73}$$

directly from the conservation laws of the KdV equation and integrate it. Note that, due to the nonlinearity and hyperbolicity of the system (8.73), the shock wave will develop sooner or later which violates the amplitude modulation. However, in this case equations (8.73) can be used to justify the jump conditions at the shock waves (see exercise 8.12).

Thus, we are now at the end of these lectures. Before closing, let us summarize shortly. Looking back, one sees that we have learned a lot of things. Among them, we would put on the first place Hamilton's variational principle of least action and its generalizations for the derivation of the equations of motion. We have studied also various methods of solving these equations and finding laws of behavior of the solutions. Some of the numerical methods, in particular finite element method, were not touched at all. But fortunately there are other excellent courses where one can learn those methods (see, for instance, [41, 55]). One thing is for sure: with numerical methods *alone* one can hardly establish any *behavioral law* for the solutions.

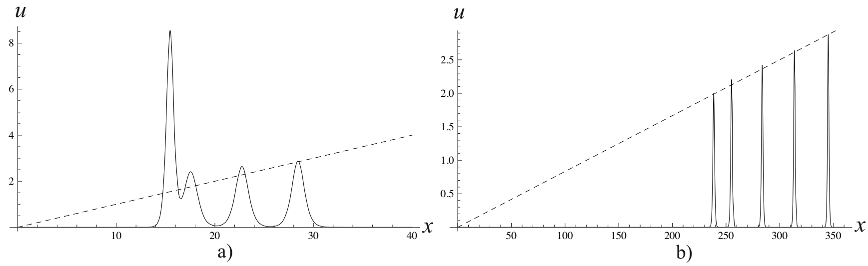

Fig. 8.11 A train of solitons (bold line) and the amplitude modulation (dashed line): a) at small time, b) at large time

To establish such laws, which are often quite useful in engineering applications, analytical skills have to be trained and cultivated. For those problems containing small parameters, the variational-asymptotic method turns out quite effective, and it is hoped that this course has helped students a little bit in mastering it. Last but not least, one should not forget about the exercises. Just remember "Übung macht den Meister" (practice makes perfect), as Germans say.

8.6 Exercises

EXERCISE 8.1. Use the identities for the Jacobian elliptic functions sn, cn, and dn given in Section 5.1 to check that $\varphi(\xi) = a\,\mathrm{cn}^2(\sqrt{b/2}\xi, a/b)$, with $\xi = x - ct$, is the periodic solution of the KdV equation (in this case $b_1 = a$, $b_2 = 0$, $b_3 = a - b$).

Solution. In the special case

$$b_1 = a, \quad b_2 = 0, \quad b_3 = a - b,$$

where a and $b > a$ are two real and positive numbers, the first integral for the periodic solution of the KdV equation reduces to

$$\varphi'^2 = 2(a - \varphi)\varphi(\varphi + b - a).$$

Let us check that

$$\varphi(\xi) = a\,\mathrm{cn}^2(\sqrt{b/2}\xi, a/b)$$

satisfies this equation. Differentiating φ with respect to ξ and using the formulas for the Jacobian elliptic functions sn, cn, and dn given in Section 5.1, we get

$$\varphi' = -2a\sqrt{b/2}\,\mathrm{cn}(\sqrt{b/2}\xi, a/b)\,\mathrm{sn}(\sqrt{b/2}\xi, a/b)\,\mathrm{dn}(\sqrt{b/2}\xi, a/b).$$

Squaring both sides of this formula and using the identities $\mathrm{sn}^2 = 1 - \mathrm{cn}^2$ and $\mathrm{dn}^2 = 1 - m + m\,\mathrm{cn}^2$, with $m = a/b$, one can easily show that $\varphi(\xi) = a\,\mathrm{cn}^2(\sqrt{b/2}\xi, a/b)$ satisfies the above equation.

EXERCISE 8.2. Show that

$$u(x,t) = 4\arctan e^{\gamma(x-ct)},$$

with $\gamma = 1/\sqrt{1-c^2}$ is the soliton solution of the Sine-Gordon equation.

Solution. Consider the Sine-Gordon equation

$$u_{,tt} - u_{,xx} + \sin u = 0.$$

We look for the soliton solution in the form

$$u(x,t) = \varphi(x-ct),$$

with c being a constant. Substitution in the above equation gives

$$(c^2-1)\varphi'' + \sin\varphi = 0,$$

where prime denotes the derivative with respect to $\xi = x - ct$. The last equation can be presented in the form

$$m\varphi'' - U'(\varphi) = 0,$$

with

$$m = 1-c^2, \quad U(\varphi) = 1 - \cos\varphi.$$

This resembles the equation of motion of mass-spring oscillator with a mass $m = 1-c^2$ and a nonlinear restoring force derivable from the potential energy $-U(\varphi)$. The first integral is

$$\frac{1}{2}m\varphi'^2 - U(\varphi) = h.$$

If φ and its first derivative tend to zero as $\xi \to \pm\infty$, then $h = 0$. In this case

$$\varphi' = \frac{2}{\sqrt{m}}\sin(\varphi/2).$$

Integrating this equation by separating the variables ξ and φ, we obtain

$$\sqrt{m}\ln[\tan(\varphi/4)] = \xi, \tag{8.74}$$

and, thus,

$$u(x,t) = \varphi(\xi) = 4\arctan e^{\gamma(x-ct)}.$$

EXERCISE 8.3. Use the conservation law of the KdV equation

$$u_{,t} + (3u^2 + u_{,xx})_{,x} = 0$$

to show that

$$I_{-1} = \int_{-\infty}^{\infty} u\,dx$$

is the first integral. Show that the conservation laws of the KdV equation for I_0 and I_1 are

$$(u^2)_{,t} + (4u^3 + 2uu_{,xx} - u_{,x}^2)_{,x} = 0,$$
$$(u^3 - \frac{1}{2}u_{,x}^2)_{,t} + (\frac{9}{2}u^4 + 3u^2u_{,xx} - 6uu_{,x}^2 - u_{,x}u_{,xxx} + \frac{1}{2}u_{,xx}^2)_{,x} = 0.$$

Solution. It is easy to see that the equation

$$u_{,t} + (3u^2 + u_{,xx})_{,x} = 0$$

follows at once from the KdV equation. Integrating this equation over x from $-\infty$ to ∞ and taking into account the behavior of the solution at infinity, we obtain

$$\frac{d}{dt}\int_{-\infty}^{\infty} u\,dx = 0,$$

so I_{-1} is conserved. Differentiating the second equation, we have

$$2uu_{,t} + 12u^2u_{,x} + 2u_{,x}u_{,xx} + 2uu_{,xxx} - 2u_{,x}u_{,xx} = 0,$$

and it is again the consequence of the KdV equation. Integrating this conservation law over x from $-\infty$ to ∞, we can establish that I_0 is conserved. To show that the third conservation law also follows from the KdV equation, we differentiate the expressions in the brackets to obtain

$$3u^2u_{,t} - \underline{u_{,x}u_{,xt}} + 18u^3u_{,x} + \underline{6uu_{,x}u_{,xx}} + 3u^2u_{,xxx}$$
$$- \underline{6u_{,x}^3} - \underline{12uu_{,x}u_{,xx}} - u_{,xx}u_{,xxx} - \underline{u_{,x}u_{,xxxx}} + u_{,xx}u_{,xxx} = 0.$$

The underlined terms represent the product of $u_{,x}$ with the derivative of the KdV equation with respect to x, taken with minus sign, while the remaining terms give the product of $3u^2$ with the KdV equation. So, the third conservation law is also the consequence of the KdV equation, and hence, I_1 is conserved.

EXERCISE 8.4. With the Lax's pair

$$L\psi = \psi_{,xx} + u(x,t)\psi, \quad A\psi = (\gamma + u_{,x})\psi - (4\lambda + 2u)\psi_{,x},$$

show that the Lax equation $L_{,t} + [L,A] = 0$ (which expresses the compatibility condition between $L\psi = \lambda\psi$ and $\psi_{,t} = A\psi$) is satisfied if and only if the KdV equation is fulfilled.

Solution. As shown in Section 8.2, the Lax equation is fulfilled if and only if $\lambda_{,t} = 0$. But if $\lambda_{,t} = 0$, then the differentiation of the equation $\psi_{,xx} + u\psi = \lambda\psi$ with respect to t yields

$$\psi_{,xxt} + u\psi_{,t} + u_{,t}\psi = \lambda\psi_{,t}.$$

Replacing $\psi_{,t}$ in this equation by $A\psi = (\gamma + u_{,x})\psi - (4\lambda + 2u)\psi_{,x}$, we obtain

$$\psi_{,xxt} = [(\lambda - u)(\gamma + u_{,x}) - u_{,t}]\psi - (\lambda - u)(4\lambda + 2u)\psi_{,x}.$$

On the other side, if we differentiate the evolution equation $\psi_{,t} = A\psi$ with respect to x, then

$$\psi_{,tx} = (\gamma + u_{,x})\psi_{,x} + u_{,xx}\psi - (4\lambda + 2u)\psi_{,xx} - 2u_{,x}\psi_{,x}.$$

Replacing $\psi_{,xx}$ by $(\lambda - u)\psi$, we obtain

$$\psi_{,tx} = (\gamma + u_{,x})\psi_{,x} + u_{,xx}\psi - (4\lambda + 2u)(\lambda - u)\psi - 2u_{,x}\psi_{,x}.$$

Differentiating this again with respect to x with the use of the condition $\psi_{,xx} = (\lambda - u)\psi$ leads to

$$\psi_{,txx} = [(\gamma + u_{,x})(\lambda - u) + u_{,xxx} + 6uu_{,x}]\psi - (\lambda - u)(4\lambda + 2u)\psi_{,x}.$$

Thus, the two equations for $\psi_{,xxt}$ and $\psi_{,txx}$ are compatible ($\psi_{,xxt} = \psi_{,txx}$) if and only if u satisfies KdV equation.

EXERCISE 8.5. Consider two linear equations

$$\mathbf{v}_{,x} = \mathbf{X}\mathbf{v}, \quad \mathbf{v}_{,t} = \mathbf{T}\mathbf{v},$$

where $\mathbf{v}$ is an n-dimensional vector and $\mathbf{X}$ and $\mathbf{T}$ are $n \times n$ matrices. Provided these equations are compatible, that is $\mathbf{v}_{,xt} = \mathbf{v}_{,tx}$, show that $\mathbf{X}$ and $\mathbf{T}$ satisfy

$$\mathbf{X}_{,t} - \mathbf{T}_{,x} + [\mathbf{X}, \mathbf{T}] = 0.$$

The pair $\mathbf{X}$ and $\mathbf{T}$ is similar to Lax's pair L and A, and the last equation may lead to various interesting equations of mathematical physics [1].

Solution. Let us differentiate the first equation with respect to t

$$\mathbf{v}_{,xt} = \mathbf{X}_{,t}\mathbf{v} + \mathbf{X}\mathbf{v}_{,t}.$$

Replacing $\mathbf{v}_{,t}$ by $\mathbf{T}\mathbf{v}$ in accordance with the second equation, we obtain

$$\mathbf{v}_{,xt} = (\mathbf{X}_{,t} + \mathbf{X}\mathbf{T})\mathbf{v}.$$

Analogously, the differentiation of the second equation with respect to x leads to

$$\mathbf{v}_{,tx} = (\mathbf{T}_{,x} + \mathbf{T}\mathbf{X})\mathbf{v}.$$

Thus, the above equations are compatible ($\mathbf{v}_{,xt} = \mathbf{v}_{,tx}$) if

$$\mathbf{X}_{,t} - \mathbf{T}_{,x} + [\mathbf{X}, \mathbf{T}] = 0.$$

EXERCISE 8.6. Consider the two-soliton solution

$$u(x,t) = 12\frac{3 + 4\cosh(2x - 8t) + \cosh(4x - 64t)}{[3\cosh(x - 28t) + \cosh(3x - 36t)]^2}.$$

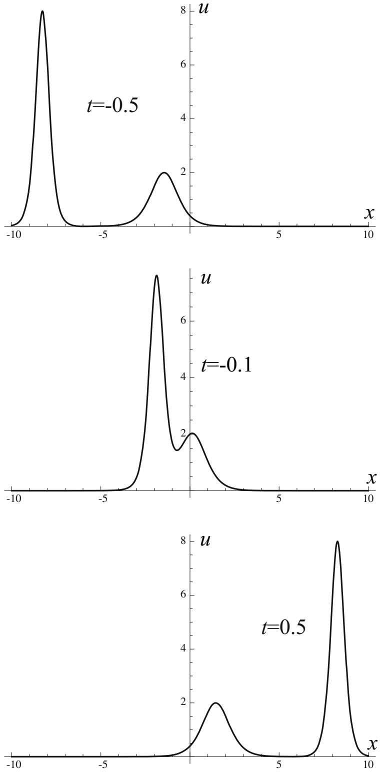

Fig. 8.12 Two solitons before, during, and after collision

Plot this function for the time instants before, during, and after the collision. Observe the behavior of the amplitudes and phases.

Solution. After opening a notebook in *Mathematica* we first define function $u(x,t)$ given above representing the two-soliton solution. Then, by typing the following command

Plot[u[x, -0.5], $\{x,-10,10\}$, PlotRange $\to$ All] ,

we plot this function at time instant $t=-0.5$. Doing the same for the time instants $t=-0.1$ and $t=0.5$, we obtain the sequence of graphs representing two solitons moving to the right before, during, and after collision as shown in Fig. 8.12. One can observe that the solitons maintain their original shapes after the collision. The only change is the phase shift. The graph of this function in the (x,t)-plane was shown in Fig. 8.6.

EXERCISE 8.7. Find the average Lagrangian by solving the minimization problem

$$\bar{L}=\frac{1}{2\pi}\min_{\psi_1,\psi_2}\int_0^{2\pi}[\frac{1}{2}(\omega^2-k^2)(\psi_{1,\theta}^2+\psi_{2,\theta}^2)-U(\psi_1,\psi_2)]\,d\theta,$$

where

$$U(\psi_1,\psi_2)=\frac{1}{2}[\psi_1^2+\frac{\alpha}{2}\psi_1^4+\psi_2^2+\frac{\alpha}{2}\psi_2^4+\frac{\beta}{2}(\psi_2-\psi_1)^4],$$

among 2π-periodic functions for which $\psi_2=c\psi_1$.

Solution. Let $\psi_1=\psi$. Substitute the relation $\psi_2=c\psi$ (which describes the similar normal mode) into the above variational problem for the average Lagrangian, we reduce it to

$$\bar{L}=\frac{1}{2\pi}\min_{\max\psi=a}\int_0^{2\pi}[\frac{1}{2}(\omega^2-k^2)(1+c^2)\psi_{,\theta}^2-U_1(\psi,c)]\,d\theta,$$

where

$$U_1(\psi,c)=\frac{1}{2}[(1+c^2)\psi^2+\frac{\alpha}{2}(1+c^4)\psi^4+\frac{\beta}{2}(1-c)^4\psi^4].$$

Let prime denote the derivative with respect to θ. We use the first integral

$$\frac{1}{2}(\omega^2-k^2)(1+c^2)\psi'^2+U_1(\psi,c)=U_1(a,c)=h$$

to express $\bar{L}$ in the form

$$\bar{L}=\frac{1}{2\pi}\int_0^{2\pi}(\omega^2-k^2)(1+c^2)\psi'^2\,d\theta-h.$$

Changing the variable $\theta\to\psi$, we obtain finally

$$\begin{aligned}\bar{L}&=\frac{1}{2\pi}(\omega^2-k^2)(1+c^2)\oint\psi'\,d\psi-h\\&=\frac{1}{2\pi}\sqrt{2(\omega^2-k^2)(1+c^2)}\oint\sqrt{h-U_1(\psi,c)}d\psi-h.\end{aligned}$$

The contour integral in this formula denotes the integral over a complete oscillation of ψ from b, with $U_1(b,c)=U_1(a,c)$, up to a and back, so it is equal to twice the integral from b to a because the sign of the square root has to be changed appropriately in the two parts of the contour. This integral may also be interpreted as the contour integral around a cut from b to a in the complex ψ-plane, where ψ plays the role of the variable of integration.

EXERCISE 8.8. For the average Lagrange function

$$\bar{L}=\frac{\omega}{2\pi}\int_0^T p\dot{q}\,dt-h=\frac{\omega}{2\pi}\oint p(q,h,\lambda)\,dq-h$$

of an oscillator depending on the slowly changing parameter λ show that $\partial\bar{L}/\partial h=0$ coincides with the amplitude-frequency equation.

Solution. We use the conservation of energy

$$\frac{1}{2m}p^2 + U(q,\lambda) = h$$

to express the impulse p in terms of q

$$p = \sqrt{2m}\sqrt{h - U(q,\lambda)}.$$

Substitute this into the formula for the average Lagrange function to obtain

$$\bar{L} = \frac{\omega}{2\pi}\sqrt{2m}\oint \sqrt{h - U(q,\lambda)}\,dq - h.$$

Let us differentiate this average Lagrange function with respect to h

$$\frac{\partial \bar{L}}{\partial h} = \frac{\omega}{2\pi}\sqrt{\frac{m}{2}}\oint \frac{dq}{\sqrt{h - U(q,\lambda)}} - 1.$$

Thus, the equation $\bar{L}_{,h} = 0$ is equivalent to

$$\oint \frac{dq}{\sqrt{2/m}\sqrt{h - U(q,\lambda)}} = \frac{2\pi}{\omega} = T.$$

The last equation is nothing else but the amplitude-frequency (or amplitude-period) relation; cf. (5.3).

EXERCISE 8.9. Show that the Sine-Gordon equation in cone coordinates takes the form

$$u_{,XT} = \sin u.$$

Develop the theory of slope modulation for this equation.

Solution. Using the cone-coordinates (8.48), we can establish, in our case, the following chain rule of differentiation

$$\begin{aligned}
\frac{\partial}{\partial t} &= \frac{\partial}{\partial X}\frac{\partial X}{\partial t} + \frac{\partial}{\partial T}\frac{\partial T}{\partial t} = \frac{1}{2}\Big(\frac{\partial}{\partial X} - \frac{\partial}{\partial T}\Big),\\
\frac{\partial}{\partial x} &= \frac{\partial}{\partial X}\frac{\partial X}{\partial x} + \frac{\partial}{\partial T}\frac{\partial T}{\partial t} = \frac{1}{2}\Big(\frac{\partial}{\partial X} + \frac{\partial}{\partial T}\Big).
\end{aligned}$$

Then the second derivatives follow

$$\frac{\partial^2}{\partial t^2} = \frac{1}{4}\Big(\frac{\partial^2}{\partial X^2} - 2\frac{\partial^2}{\partial X \partial T} + \frac{\partial^2}{\partial T^2}\Big), \quad \frac{\partial^2}{\partial x^2} = \frac{1}{4}\Big(\frac{\partial^2}{\partial X^2} + 2\frac{\partial^2}{\partial X \partial T} + \frac{\partial^2}{\partial T^2}\Big).$$

Thus, the left-hand side of Sine-Gordon can be replaced by

$$\Big(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\Big)u = -\frac{\partial^2}{\partial X \partial T}u,$$

and consequently, the Sine-Gordon equation in cone coordinates reads

$$-u_{,XT} + \sin u = 0 \quad \Rightarrow \quad u_{,XT} = \sin u.$$

The strip problem associated with this form of Sine-Gordon equation is stated as follows: find the extremal of the functional

$$\int_0^{2\pi} [\frac{1}{2} k\omega \psi_{,\theta}^2 - (1 - \cos\psi)] d\theta \tag{8.75}$$

among functions $\psi(\theta)$ satisfying

$$\psi(2\pi) = \psi(0) + 2\pi, \quad \psi_{,\theta}(2\pi) = \psi_{,\theta}(0). \tag{8.76}$$

The maximal slope of solution is defined as before: $p = \max|\psi_{,\theta}|$, with p being an arbitrary real and positive number. The construction of average Lagrangian as well as the associated functional has been discussed in Section 8.4. However, there are two modifications. Firstly, the first integral should read now

$$\frac{1}{2} m \psi_{,\theta}^2 + (1 - \cos\psi) = h, \quad m = k\omega.$$

The phase portrait is shown in Fig. 8.13, where it can be seen that the maximal slope is achieved at $\psi = 0$. This implies $mp^2/2 = h$. Secondly, the average Lagrangian need be slightly modified as follows

$$\bar{L}(p, k, \omega) = \frac{\sqrt{2m}}{2\pi} f(h) - h,$$

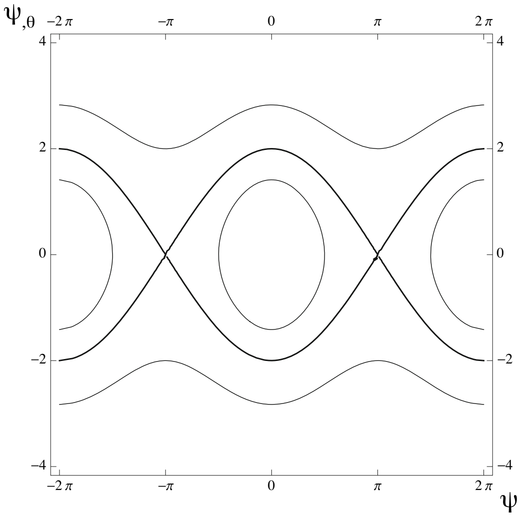

Fig. 8.13 Phase portrait associated with the strip problem with $m = 1$

where $f(h)$ is the function expressed in terms of the complete elliptic integral.

$$f(h)=\int_0^{2\pi}\sqrt{h-(1-\cos\psi)}d\psi.$$

The latter is nothing else but the time required for a pendulum with mass $m=2$ and unit length to complete its full circular motion in the gravitational field. Note that the average Lagrangian does not change its form compared to (8.38), so the dispersion relation remains unchanged in its form. To derive the equation of slope modulation let us compute the derivatives

$$\frac{\partial\bar{L}}{\partial k}=\frac{\sqrt{2}}{2\pi}\frac{m_{,k}}{2\sqrt{m}}f(h)+(\frac{\sqrt{2m}}{2\pi}f'(h)-1)h_{,k}=\frac{\sqrt{2}}{4\pi}\frac{\omega}{\sqrt{k\omega}}f(h)=\frac{\sqrt{2}}{4\pi}\sqrt{c}f(h),$$
$$\frac{\partial\bar{L}}{\partial\omega}=\frac{\sqrt{2}}{4\pi}\frac{m_{,\omega}}{2\sqrt{m}}f(h)+(\frac{\sqrt{2m}}{2\pi}f'(h)-1)h_{,\omega}=\frac{\sqrt{2}}{4\pi}\frac{k}{\sqrt{k\omega}}f(h)=\frac{\sqrt{2}}{4\pi}\frac{1}{\sqrt{c}}f(h),$$

where $c=\omega/k$ and the dispersion relation (8.38) has been used in two steps. Next, we compute their derivatives with respect to X and T

$$\frac{\partial}{\partial X}\frac{\partial\bar{L}}{\partial k}=\frac{\sqrt{2}}{4\pi}[\frac{c_{,X}}{2\sqrt{c}}f(h)+\frac{\sqrt{c}}{2}f'(h)(k_{,X}\omega+\omega_{,X}k)p^2+\frac{\sqrt{c}}{2}f'(h)m(p^2)_{,X}],$$
$$\frac{\partial}{\partial T}\frac{\partial\bar{L}}{\partial\omega}=\frac{\sqrt{2}}{4\pi}[-\frac{c_{,T}}{2c\sqrt{c}}f(h)+\frac{1}{2\sqrt{c}}f'(h)(k_{,T}\omega+\omega_{,T}k)p^2+\frac{1}{2\sqrt{c}}f'(h)m(p^2)_{,T}].$$

Subtracting the second equation from the first one and dividing the result by the common factor $\sqrt{2}/4\pi$, we get, after some algebra,

$$\frac{f(h)}{2c\sqrt{c}}(c_{,T}+cc_{,X})+\frac{f'(h)}{2\sqrt{c}}q(k_{,X}\frac{\omega^2}{k}+2\omega_{,X}\omega-\omega_{,T}k)+\frac{f'(h)}{2\sqrt{c}}(\omega^2q_{,X}-k\omega q_{,T})=0,$$

where the square of maximal slope is denoted by $q=p^2$.

We shall find only a particular solution to this equation using the Ansatz for the phase as before: $\theta(X,T)=g(\xi(X,T))$, $\xi(X,T)=X/T$. With this Ansatz the above equation reduces to

$$4[Tg'(\xi)+Xg''(\xi)]q(X,T)+Tg'(\xi)(Xq_{,X}-Tq_{,T})=0.$$

The last equation is the partial differential equation of first order which can be solved by the method of characteristics and whose solution is given by

$$q(X,T)=W(XT)^2\frac{T^4}{g'(\xi(X,T))^2},\quad \xi(X,T)=\frac{X}{T},$$

and thus,

$$p(X,T)=\sqrt{q(X,T)}=W(XT)\frac{T^2}{g'(\xi(X,T))}. \tag{8.77}$$

The unknown function $W(XT)$ should be determined from the expression for h in the limit $h \to 2$, which corresponds to the separatrix in the phase portrait

$$h = \frac{1}{2}mp^2 = \frac{1}{2}XT \times W(XT)^2 \quad \Rightarrow \quad W(XT) = \frac{2}{\sqrt{XT}}.$$

The final asymptotic formula for the slope reads

$$p(X,T) = \frac{2T\sqrt{T}}{\sqrt{|X|}g'(\xi(X,T))}.$$

EXERCISE 8.10. Use the analytical soliton solution for the Sine-Gordon equation given in cone coordinates by (8.49) to simulate the 5-soliton solution and compare it with the asymptotic formula $2\sqrt{T/|X|}$ at large time.

Solution. We use formula (8.49) representing the exact analytical solution of the Sine-Gordon equation. The *Mathematica* code which enables one to simulate this solution is reproduced below.

```
createDelta[η_, c0_] := Block[{matrixC, γ, num, m, k, n}, num = Length[η];
  γ = Table[Exp[-(η[[k]] + η[[n]]) X - T/4 (1/η[[k]] + 1/η[[n]])], {k, num}, {n, num}];
  matrixC = Table[Null, {k, num}, {n, num}];
  For[k = 1, k ≤ num, k++,
   For[n = 1, n ≤ k, n++,
    matrixC[[k, n]] = Sum_{m=1}^{num} c0[[m]] Sqrt[c0[[k]] c0[[n]]] (γ[[k, m]] γ[[m, n]])/((η[[k]] + η[[m]]) (η[[m]] + η[[n]]));
    matrixC[[n, k]] = matrixC[[k, n]]]];
  Return[Det[IdentityMatrix[num] + matrixC]];
 ];
Δ = createDelta[Table[k, {k, 5}], Table[1, {5}]];
du = Sqrt[Simplify[4 (D[Δ, {X, 2}] Δ - (∂_X Δ)^2)/Δ^2]];
```

To explain this code let us first consider elements of matrix $\mathbf{C} = \mathbf{A}\mathbf{A}^*$. Since the eigenvalues are purely imaginary, we have

$$\zeta_k - \zeta_n^* = i(\eta_k + \eta_n).$$

Therefore, function $c_k(T)$ and, consequently, its conjugate, turn out to be real functions

$$c_k(T) = c_{k0}\exp(-T/2\eta_k), \quad c_k(T)^* = c_k(T).$$

It is now easy to write the elements of matrix $\mathbf{A}$ and $\mathbf{A}^*$

$$A_{kn} = -\frac{i\sqrt{c_{k0}c_{n0}}}{\eta_k + \eta_n}\gamma_{kn}(X,T), \quad A^*_{kn} = \frac{i\sqrt{c_{k0}c_{n0}}}{\eta_k + \eta_n}\gamma_{kn}(X,T),$$
$$\gamma_{kn}(X,T) = \exp\left[-(\eta_k + \eta_n)X - \frac{T}{4}(\frac{1}{\eta_k} + \frac{1}{\eta_n})\right].$$

Thus,

$$C_{kn} = \sum_{m=1}^{N} A_{km}A^*_{mn} = \sum_{m=1}^{N} c_{m0}\sqrt{c_{k0}c_{n0}}\frac{\gamma_{km}(X,T)\gamma_{mn}(X,T)}{(\eta_k + \eta_m)(\eta_m + \eta_n)}.$$

Denoting $\Delta = \det[\mathbf{I} + \mathbf{C}]$, we differentiate the right-hand side of the expression for the slope solution to get

$$\frac{\partial^2}{\partial X^2}\ln\Delta = \frac{\Delta\partial_X^2\Delta - (\partial_X\Delta)^2}{\Delta^2},$$

which implies further

$$\frac{\partial u}{\partial X} = 2\frac{\sqrt{\Delta\partial_X^2\Delta - (\partial_X\Delta)^2}}{\Delta}.$$

The first piece of the above code is used to generate the determinant $\Delta = \det[\mathbf{I} + \mathbf{C}]$, while the next one is aimed at computing the slope $\partial u/\partial X$. The graph of $\partial u/\partial X$ is plotted with the usual Plot Command. Using this *Mathematica* code, one can reproduce Fig. 8.10 shown at the end of Section 8.4.

EXERCISE 8.11. Prove the identities $K_1 = 0$, $K_2 = K_3$.

Solution. Since K_1 is given as an integral over a closed contour, K_1 vanishes if its integrand is a full differential of a function. To show that this is the case let us compute the following derivative

$$\begin{aligned}
D_1 &= \frac{d}{d\phi}\left[2\sqrt{\frac{(\phi - b_2)(\phi - b_3)}{b_1 - \phi}}\right] \\
&= \sqrt{\frac{\phi - b_3}{(\phi - b_2)(b_1 - \phi)}} + \sqrt{\frac{\phi - b_2}{(\phi - b_3)(b_1 - \phi)}} + \frac{1}{b_1 - \phi}\sqrt{\frac{(\phi - b_2)(\phi - b_3)}{b_1 - \phi}} \\
&= \frac{1}{\phi - b_1}\sqrt{\frac{2}{f(\phi)}}[(\phi - b_3)(\phi - b_1) + (\phi - b_2)(\phi - b_1) - (\phi - b_2)(\phi - b_3)] \\
&= \frac{\sqrt{2}(\phi^2 - 2b_1\phi + b_1b_2 + b_1b_3 - b_2b_3)}{(\phi - b_1)\sqrt{f(\phi)}},
\end{aligned}$$

where $f(\phi)$ is equal to

$$f(\phi) = 2(b_1 - \phi)(\phi - b_2)(\phi - b_3).$$

We see that $D_1/\sqrt{2}$ is exactly the integrand standing in the integral of K_1, which implies that K_1 vanishes.

Now, we prove that the difference $K_2 - K_3$ also vanishes using the same argument. Subtracting K_3 from K_2, we obtain

$$K_2 - K_3 = \frac{b_2 - b_3}{4\pi} \oint \frac{\phi^2 - 2b_1\phi + b_1 b_2 + b_1 b_3 - b_2 b_3}{(\phi - b_2)(\phi - b_3)\sqrt{f(\phi)}} d\phi.$$

Then we consider the following derivative

$$\begin{aligned}
D_2 &= \frac{d}{d\phi}\left[\sqrt{\frac{2(b_1 - \phi)}{(\phi - b_2)(\phi - b_3)}}\right] \\
&= -\frac{1}{\sqrt{f(\phi)}} - \frac{b_1 - \phi}{(\phi - b_2)\sqrt{f(\phi)}} - \frac{b_1 - \phi}{(\phi - b_3)\sqrt{f(\phi)}}. \\
&= -\frac{(\phi - b_2)(\phi - b_3) + (b_1 - \phi)(\phi - b_2) + (b_1 - \phi)(\phi - b_3)}{(\phi - b_2)(\phi - b_3)\sqrt{f(\phi)}} \\
&= \frac{\phi^2 - 2b_1\phi + b_1 b_2 + b_1 b_3 - b_2 b_3}{(\phi - b_2)(\phi - b_3)\sqrt{f(\phi)}}.
\end{aligned}$$

Thus, the integrand in the above formula for $K_2 - K_3$ is again the full differential and consequently, the integral vanishes.

EXERCISE 8.12. Derive equations (8.73) directly from the conservation law of KdV equation

$$u_{,t} + (3u^2 + u_{,xx})_{,x} = 0.$$

Find its solution.

Solution. Let us average the above equation over a unit length (having $\bar{k}$ solitons) to obtain

$$\bar{u}_{,t} + 3(\overline{u^2})_{,x} = 0.$$

Since there are $\bar{k}$ solitons in a unit length, the average values should be

$$\bar{u} = \bar{k}\int_{-\infty}^{\infty} u_1 dx, \quad \overline{u^2} = \bar{k}\int_{-\infty}^{\infty} u_1^2 dx,$$

where u_1 is a single soliton solution having the amplitude a, and the integrals are computed approximately by extending the unit interval to the whole real axis. Now, for the single soliton given by

$$u_1 = a\,\mathrm{sech}^2[\sqrt{a/2}(x - 2at)],$$

the integration yields

$$\bar{u} = 2\sqrt{2}\bar{k}\sqrt{a}, \quad \overline{u^2} = \frac{4\sqrt{2}}{3}\bar{k}a^{3/2}.$$

The average equation becomes

$$(\bar{k}\sqrt{a})_{,t} + (2\bar{k}a^{3/2})_{,x} = 0.$$

Keeping in mind that the phase velocity c, in case of solitons, is $c = 2a$, we obtain from the kinematic condition $\bar{k}_{,t} + (c\bar{k})_{,x} = 0$ the following equation

$$\bar{k}_{,t} + (2a\bar{k})_{,x} = 0.$$

We rewrite the above equation as $(\bar{k}\sqrt{a})_{,t} + (c\bar{k}\sqrt{a})_{,x} = 0$, expand the derivatives and factorize appropriately to obtain

$$\sqrt{a}[\bar{k}_{,t} + (c\bar{k})_{,x}] + \frac{\bar{k}}{2\sqrt{a}}(a_{,t} + ca_{,x}) = 0.$$

The second equation of (8.73) follows from the above equation plus the consistency condition.

In this approximation the system is not strictly hyperbolic, but a may be found by integration along the characteristics $dx/dt = 2a$. Along this curve, the amplitude a remains constant, due to

$$\frac{da}{dt} = a_{,t} + \frac{dx}{dt}a_{,x} = 0.$$

Thus,

$$\frac{dx}{dt} = 2a = 2C \quad \Rightarrow \quad \frac{x}{t} = 2C,$$

where C is a constant characterizing such a curve. By varying this constant, one can obtain the solution $a = a(x,t)$ spanned in the whole plane (x,t)

$$a(x,t) = \frac{x}{2t}.$$

With this solution the first equation, after some algebra, is reduced to

$$\partial_t(t\bar{k}) + x\bar{k}_{,x} = 0.$$

Changing the unknown function $q = t\bar{k}$, we obtain

$$q_{,t} + \frac{x}{t}q_{,x} = 0,$$

which admits a simple solution $q(x,t) = f(x/t)$, where f is an arbitrary function. Thus, the average number of solitons is

$$\bar{k} = \frac{1}{t}f(\frac{x}{t}).$$

However, to achieve the full agreement, solution $a(x,t) = x/2t$ has to be cut off at some leading solitary wave in the sequence. This is equivalent to posing jump conditions on the shock waves. If we accept (8.73), the jump conditions have to be

$$-V[\![\bar{k}\sqrt{a}]\!] + [\![2\bar{k}a^{3/2}]\!] = 0,$$
$$-V[\![\bar{k}]\!] + [\![2a\bar{k}]\!] = 0,$$

where V is the velocity of the discontinuity and $[\![\cdot]\!]$ denotes the jump. A jump from $a = 0$ to a nonzero value $a^{(0)}$ would therefore have $V = 2a^{(0)}$. This is the phase velocity and the result indicates that the solution $a(x,t) = x/2t$ may be cut off at a leading solitary wave in the sequence.