Chapter 6
Non-autonomous Single Oscillator

This Chapter analyzes non-autonomous mechanical systems with one degree of freedom whose Lagrange function depends explicitly on time. This involves either some time-dependent parameter or a harmonic excitation. The variational-asymptotic analysis, combined with multi-scaling, belongs again to the arsenal of mostly used analytical methods of solution of variational problems containing small parameters.

6.1 Parametrically-Excited Oscillator

Differential Equation of Motion. If some parameter of an oscillator changes in such a way that the energy supply is synchronized with the period of vibration, the parametric resonance may occur. We consider some examples.

EXAMPLE 6.1. Pendulum with periodically moving support. The support of a pendulum moves in accordance with the equation $x = a(t)$ (see Fig. 6.1). Derive the equation of motion for this pendulum.

In a fixed (x, y)-coordinate system the coordinates of the point-mass are

$$x = a + l\cos\varphi, \quad y = l\sin\varphi.$$

Differentiating these equations with respect to t we obtain the velocity

$$\dot{x} = \dot{a} - l\sin\varphi\,\dot{\varphi}, \quad \dot{y} = l\cos\varphi\,\dot{\varphi}.$$

Therefore the kinetic energy is equal to

$$K = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m(\dot{a}^2 - 2l\sin\varphi\dot{a}\dot{\varphi} + l^2\dot{\varphi}^2).$$

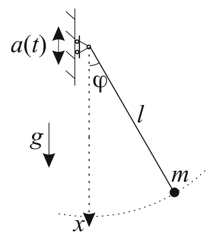

Fig. 6.1 Pendulum with moving support

The potential energy of the point mass is

$$U = -mg(a + l\cos\varphi).$$

Note that the zero level of potential energy corresponds to $x = 0$. Thus, the Lagrange function $L(\varphi, \dot{\varphi}, t) = K - U$ depends explicitly on time through $a(t)$. Mechanical systems with the Lagrange function depending explicitly on time are classified as non-autonomous. Substitution of this Lagrange function into Lagrange's equation gives

$$ml^2\ddot{\varphi} + mgl\sin\varphi - ml\ddot{a}\sin\varphi = 0,$$

or

$$\ddot{\varphi} + \frac{1}{l}(g - \ddot{a})\sin\varphi = 0. \tag{6.1}$$

If $a(t)$ is a periodic function of t, say $a(t) = a_0\cos\omega t$, then (6.1) is a nonlinear equation with periodic coefficients. In order to investigate the stability of one of the equilibrium positions $\varphi = 0$ or $\varphi = \pi$, we would linearize (6.1) about the desired equilibrium. In case $\varphi = 0$ the linearization yields

$$\ddot{\varphi} + \frac{1}{l}(g + a_0\omega^2\cos\omega t)\varphi = 0. \tag{6.2}$$

This is called Mathieu's equation [35]. We will show later that the parametric resonance may occur for some values of ω and a_0.

EXAMPLE 6.2. Stability of a limit cycle.

Assume that $x = x_s(t)$ is a periodic solution of the equation of motion

$$\ddot{x} = f(x, \dot{x}). \tag{6.3}$$

We would like to study the dynamic stability of this periodic solution. For this purpose we investigate the neighboring solutions

$$x = x_s(t) + \xi(t), \quad \dot{x} = \dot{x}_s(t) + \dot{\xi}(t), \quad \ddot{x} = \ddot{x}_s(t) + \ddot{\xi}(t),$$

where $\xi(t)$ and its first derivative are assumed to be small. Substituting these formulas into the equation of motion, we obtain

$$\ddot{x}_s(t) + \ddot{\xi}(t) = f(x_s(t) + \xi(t), \dot{x}_s(t) + \dot{\xi}(t)).$$

Since $\xi(t)$ and $\dot{\xi}(t)$ are small, we expand the right-hand side into the Taylor series and neglect all nonlinear terms

$$\begin{aligned} f(x_s(t) + \xi(t), \dot{x}_s(t) + \dot{\xi}(t)) = f(x_s, \dot{x}_s) &+ \frac{\partial f}{\partial x}(x_s(t), \dot{x}_s(t))\xi(t) \\ &+ \frac{\partial f}{\partial \dot{x}}(x_s(t), \dot{x}_s(t))\dot{\xi}(t) + \dots. \end{aligned}$$

Taking into account that $x_s(t)$ is the solution of (6.3), we obtain for $\xi(t)$

$$\ddot{\xi} = \frac{\partial f}{\partial x}(x_s(t), \dot{x}_s(t))\xi(t) + \frac{\partial f}{\partial \dot{x}}(x_s(t), \dot{x}_s(t))\dot{\xi}(t).$$

Thus, the stability analysis of a limit cycle puts a question, whether or not the linear differential equation with periodic coefficients has bounded solutions. This is quite similar to the problem of parametric resonance.

EXAMPLE 6.3. Pendulum with periodically changeable length $l(t)$ (see Fig. 6.2).

The swing known from our childhood is described by this mechanical model. The kinetic and potential energies of the point-mass are

$$K = \frac{1}{2}ml^2(t)\dot{\varphi}^2, \quad U = mgl(t)(1-\cos\varphi).$$

Lagrange's equation has the form

$$ml^2\ddot{\varphi} + 2ml\dot{l}\dot{\varphi} + mgl\sin\varphi = 0.$$

Dividing this equation by ml^2 we obtain

$$\ddot{\varphi} + 2\frac{\dot{l}}{l}\dot{\varphi} + \frac{g}{l}\sin\varphi = 0.$$

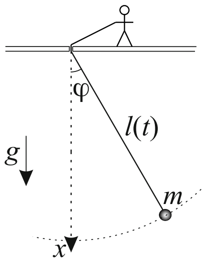

Fig. 6.2 Pendulum with changeable length

In reality, the change of length of the swing is realized by the motion of the swinger. This changes the center of gravity causing the change of the effective length of the physical pendulum. To pump the swing the swinger must raise his or her body as the swing passes through the lowest point and lower themselves near the extremes of the motion.

Solution in a Simplified Model of Swing. To get the "feeling" of how the parametric resonance may occur, we analyze a simplified model of swing, in which the effective length of the pendulum changes abruptly from l_1 to l_2 at $\varphi = 0$, and from l_2 to l_1 as the maximum (or minimum) of φ is achieved at the turning angle. The trajectory of the center of mass is shown in Fig. 6.3 by a loop with arrows. Since $l = l_1 = \text{const}$ in the first quarter of vibration, energy must be conserved if the air resistance is neglected

$$\frac{1}{2}ml_1^2\dot{\varphi}^2 + mgl_1(1-\cos\varphi) = mgl_1(1-\cos\varphi_0),$$

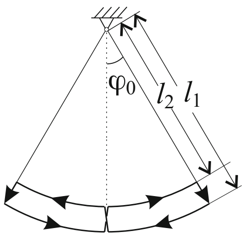

Fig. 6.3 A simplified model of swing

where φ_0 is the starting angle when the swing is released. Using this equation we can compute the angular velocity $\dot{\varphi}$ just before the change of length at $\varphi = 0$

$$\dot{\varphi}_{1-}^2 = \frac{2g}{l_1}(1-\cos\varphi_0). \tag{6.4}$$

Similarly, in the next quarter of vibration in which the swing's length is $l = l_2 =$ const, we have

$$\frac{1}{2}ml_2^2\dot{\varphi}^2 + mgl_2(1-\cos\varphi) = mgl_2(1-\cos\varphi_2),$$

where φ_2 is the turning angle, so the angular velocity immediately after the change of length at $\varphi = 0$ is equal to

$$\dot{\varphi}_{1+}^2 = \frac{2g}{l_2}(1-\cos\varphi_2). \tag{6.5}$$

During the short time when the length of the swing changes abruptly the force in the radial direction is applied. Since the moment of this radial force about the support is zero, the angular momentum must be conserved

$$ml_1^2\dot{\varphi}_{1-} = ml_2^2\dot{\varphi}_{1+}. \tag{6.6}$$

This relation can be used to determine φ_2 through φ_0. Indeed, squaring (6.6) and using (6.4) and (6.5) we get

$$l_1^3(1-\cos\varphi_0) = l_2^3(1-\cos\varphi_2). \tag{6.7}$$

Similar arguments lead to the generalization of this equation for all subsequent halves of vibration

$$l_1^3(1-\cos\varphi_{2(n-1)}) = l_2^3(1-\cos\varphi_{2n}).$$

Thus, the sequence of turning angles can be constructed geometrically as shown in Fig. 6.4. Starting from the point $\mathrm{A}_0 = (\varphi_0, f_1(\varphi_0))$ on the curve $f_1(\varphi) = l_1^3(1-\cos\varphi)$ we find the next turning angle φ_2 at the intersection between the horizontal line going though A_0 and the curve $f_2(\varphi) = l_2^3(1-\cos\varphi)$. Then, starting from $\mathrm{A}_2 = (\varphi_2, f_1(\varphi_2))$ we find the next turning angle φ_4, and the whole process can be continued. We see that after a finite number of halves of vibration the angle may become larger than π.

It is interesting to find out the energy gain after each swing act. Obviously, the energy does not change during the time when $l =$ const. As the length of the swing changes abruptly from l_1 to l_2, the energy gain is

$$E_g = mgh + \frac{1}{2}m(v_{1+}^2 - v_{1-}^2),$$

where $h = l_1 - l_2$. The first term is the gain of potential energy, the second term corresponds to the increase of kinetic energy. Taking into account (6.4)-(6.6) and $v = l\dot{\varphi}$, we express E_g in terms of φ_0

$$E_g = mg\{h + l_1(1-\cos\varphi_0)[(l_1/l_2)^2 - 1]\}.$$

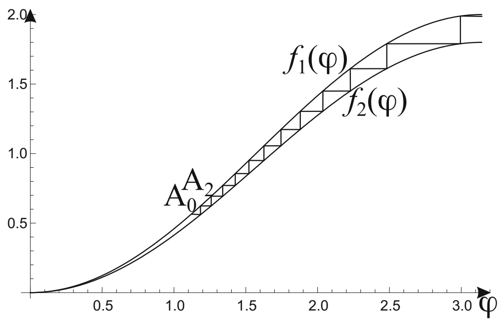

Fig. 6.4 Sequence of turning angles

After the change of the length from l_2 to l_1 at the turning angle φ_2 we have the loss of potential energy

$$E_l = mgh\cos\varphi_2.$$

Thus, the total energy gain in a half of vibration is equal to

$$\Delta E = E_g - E_l = mg\{h(1-\cos\varphi_2) + l_1(1-\cos\varphi_0)[(l_1/l_2)^2 - 1]\}.$$

Recalling (6.7), this can be transformed to

$$\Delta E = \frac{h(l_1^2 + l_1 l_2 + l_2^2)}{l_2^3} mgl_1(1-\cos\varphi_0) = kE_0,$$

where $k = h(l_1^2 + l_1 l_2 + l_2^2)/l_2^3$ and E_0 is the initial energy. Thus, the energy after the first half of vibration is

$$E_2 = E_0 + \Delta E = E_0(1+k).$$

Similar formulas can be derived for the subsequent halves of vibration. Thus, the energy after n halves of vibration becomes

$$E_{2n} = E_0(1+k)^n.$$

We see that the energy grows in a geometrical progression, like an accumulation of a capital invested with the interest rate k. In reality, this energy accumulation is reduced by the energy loss due to the drag force of the air so that a stationary regime may be established under certain conditions.

Numerical Solutions. We turn now to Mathieu's equation (6.2) as the prototype equation describing parametrically excited oscillators. We present it in the form[1]

$$\ddot{x} + (\mu + \varepsilon\cos t)x = 0. \tag{6.8}$$

[1] It is easy to show that equation (6.2) assumes this form with $\mu = (\omega_0/\omega)^2$, $\varepsilon = a_0/l$, and $\omega_0 = \sqrt{g/l}$, if time is replaced by the dimensionless time ωt.

The main concern of this equation is whether or not all solutions are bounded for given values of the parameters μ and ε. If all solutions are bounded, then the corresponding point in the (μ,ε)-plane is said to be stable. In the opposite case we have the parametric resonance and the point is classified as unstable. The problem is to find the stability chart of Mathieu's equation.

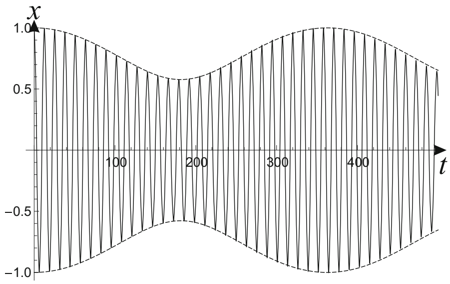

Fig. 6.5 Solution of Mathieu's equation for $\mu = 0.24$ and $\varepsilon = 0.01$

Although equation (6.8) can be solved analytically in terms of Mathieu's functions [3], it is even simpler first to find a solution for some μ and ε by numerical integration. Similar commands in *Mathematica* like those presented in Section 5.3 work quite well. Fig. 6.5 shows the solution $x(t)$ satisfying the initial conditions $x(0) = 1$, $\dot{x}(0) = 0$, for $\mu = 0.24$ and $\varepsilon = 0.01$. We can observe that there are two characteristic time scales: i) one describing the period of fast oscillation of $x(t)$, ii) the other associated with the slow oscillation of amplitude of vibration marked by the dashed envelopes. The solution remains bounded in this case.

If we change parameters μ and ε a little bit, the character of solutions may change radically. For example, if we take $\mu = 0.25$ while keeping $\varepsilon = 0.01$ as before, then the solution satisfying the same initial conditions shown in Fig. 6.6 exhibits the exponential growth of the amplitude. So, it is reasonable to guess that the point $(0.25, 0.01)$ of the (μ,ε)-plane causes the parametric resonance. Also in this case we can observe two characteristic time scales: i) one describing the period of fast

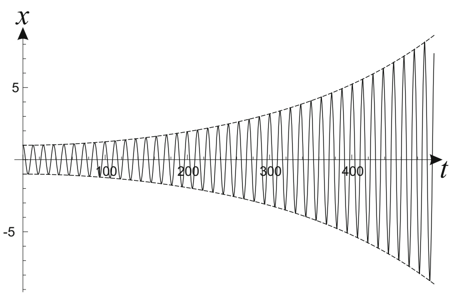

Fig. 6.6 Solution of Mathieu's equation for $\mu = 0.25$ and $\varepsilon = 0.01$

oscillation of $x(t)$, ii) the other associated with the exponential growth of amplitude of vibration marked by the dashed envelopes.

It should be noted, however, that the numerical integration, which is quite useful when studying the behavior of particular solutions, is not appropriate for the determination of the stability chart of Mathieu's equation. This is due to two reasons. First, these numerical simulations cannot be provided for an infinite time interval, so the boundedness of solutions cannot strictly be proved. Second, one cannot do infinite number of numerical simulations for all possible values of μ and ε as well as for all possible initial data. Thus, other more "intelligent" methods should be developed for this purpose.

Variational-Asymptotic Method. Let us find the approximate solutions to Mathieu's equation for small ε by using the variational-asymptotic method. These solutions are also the extremals of the functional

$$I[x(t)] = \int_{t_0}^{t_1} [\frac{1}{2}\dot{x}^2 - \frac{1}{2}(\mu + \varepsilon \cos t)x^2]\, dt, \tag{6.9}$$

with t_0 and t_1 being arbitrary time instants. For short we set $t_0 = 0$, $t_1 = T$. At the first step we put simply $\varepsilon = 0$ to get from (6.9)

$$I_0[x(t)] = \int_0^T (\frac{1}{2}\dot{x}^2 - \frac{1}{2}\mu x^2)\, dt.$$

The extremal of $I_0[x(t)]$ satisfies the equation

$$\ddot{x} + \mu x = 0$$

yielding the periodic solution with the period $T = 2\pi/\sqrt{\mu}$

$$x_0(t) = A\cos\sqrt{\mu}t + B\sin\sqrt{\mu}t. \tag{6.10}$$

Taking into account that the coefficients A and B are becoming slightly dependent on time for $\varepsilon \neq 0$, we introduce the slow time $\eta = \varepsilon t$ and seek the corrections to the extremal at the second step in the two-timing fashion

$$x(t) = A(\eta)\cos\sqrt{\mu}t + B(\eta)\sin\sqrt{\mu}t + x_1(t,\eta), \tag{6.11}$$

where $x_1(t,\eta)$ is a periodic function of the period T with respect to t and is much smaller than $x_0(t,\eta)$ in the asymptotic sense. The time derivative of $x(t)$ becomes

$$\dot{x} = -A\sqrt{\mu}\sin\sqrt{\mu}t + \varepsilon A_{,\eta}\cos\sqrt{\mu}t + B\sqrt{\mu}\cos\sqrt{\mu}t + \varepsilon B_{,\eta}\sin\sqrt{\mu}t + x_{1,t} + \varepsilon x_{1,\eta}.$$

Substituting (6.11) into (6.9) and keeping the asymptotically principal terms containing x_1 and the principal cross terms between x_0 and x_1, we obtain

$$
\begin{aligned}
I_1[x_1(t)] = \int_0^T [&\frac{1}{2}x_{1,t}^2 - \underline{A\sqrt{\mu}\sin\sqrt{\mu}t\,x_{1,t}} + \underline{B\sqrt{\mu}\cos\sqrt{\mu}t\,x_{1,t}} \\
&+ \underline{\varepsilon A_{,\eta}\cos\sqrt{\mu}t\,x_{1,t}} + \underline{\varepsilon B_{,\eta}\sin\sqrt{\mu}t\,x_{1,t}} - \frac{1}{2}\mu x_1^2 - \underline{\mu A\cos\sqrt{\mu}t\,x_1} \\
&- \underline{\mu B\sin\sqrt{\mu}t\,x_1} - \varepsilon\cos t(A\cos\sqrt{\mu}t + B\sin\sqrt{\mu}t)x_1]\,dt.
\end{aligned}
$$

Integrating the second up to fifth terms by parts taking into account the periodicity of $x_1(t)$ we see that the underlined terms give $2\varepsilon\sqrt{\mu}(A_{,\eta}\sin\sqrt{\mu}t - B_{,\eta}\cos\sqrt{\mu}t)x_1$. Besides, the products $\cos t\cos\sqrt{\mu}t$ and $\cos t\sin\sqrt{\mu}t$ can be transformed into the sum of harmonic functions like that

$$
\begin{aligned}
\cos t\cos\sqrt{\mu}t &= \frac{1}{2}[\cos(1+\sqrt{\mu})t + \cos(1-\sqrt{\mu})t], \\
\cos t\sin\sqrt{\mu}t &= \frac{1}{2}[\sin(1+\sqrt{\mu})t - \sin(1-\sqrt{\mu})t],
\end{aligned}
$$

so finally we obtain

$$
\begin{aligned}
I_1[x_1(t)] = \int_0^T [&\frac{1}{2}x_{1,t}^2 - \frac{1}{2}\mu x_1^2 + 2\varepsilon A_{,\eta}\sqrt{\mu}\sin\sqrt{\mu}t\,x_1 - 2\varepsilon B_{,\eta}\sqrt{\mu}\cos\sqrt{\mu}t\,x_1 \\
&- \frac{1}{2}\varepsilon A(\cos(1+\sqrt{\mu})t + \cos(1-\sqrt{\mu})t)x_1 \\
&- \frac{1}{2}\varepsilon B(\sin(1+\sqrt{\mu})t - \sin(1-\sqrt{\mu})t)x_1]\,dt. \qquad (6.12)
\end{aligned}
$$

For a general value of μ removal of resonant terms yields the trivial equations

$$A_{,\eta} = 0, \quad B_{,\eta} = 0.$$

Thus, for general μ the $\cos t$ term has no effect. However, if $\sqrt{\mu} = 1 - \sqrt{\mu}$, i.e. $\mu = 1/4$, then there are additional contributions to the resonant terms. In this case removal of resonant terms gives the slow flow

$$
\begin{aligned}
A_{,\eta} &= -\frac{1}{2}B, \\
B_{,\eta} &= -\frac{1}{2}A.
\end{aligned}
$$

These equations lead to $A_{,\eta\eta} = A/4$. Thus, A and B involve exponential growth, and the parameter value $\mu = 1/4$ causes instability. This corresponds to a 2:1 subharmonic resonance in which the driving frequency is twice the natural frequency as in the example of swing.

Let us seek the correction for μ in the neighborhood of $1/4$ in the form

$$\mu = \frac{1}{4} + \mu_1,$$

where μ_1 is much smaller than 1. This brings additional resonant terms of the form $-\mu_1(A\cos\frac{t}{2}+B\sin\frac{t}{2})x_1$ into functional (6.12). Thus, the equations for A and B change to

$$A_{,\eta}=(\frac{\mu_1}{\varepsilon}-\frac{1}{2})B,\quad B_{,\eta}=-(\frac{\mu_1}{\varepsilon}+\frac{1}{2})A. \tag{6.13}$$

This means, $A_{,\eta\eta}+(\mu_1^2/\varepsilon^2-1/4)A=0$, and A and B will be sine and cosine functions of η if $\mu_1^2>\varepsilon^2/4$. That is, if either $\mu_1>\varepsilon/2$ or $\mu_1<-\varepsilon/2$, then A and B remain bounded. Thus, the following two curves in the (μ,ε)-plane represent stability changes, and are called transition curves:

$$\mu=\frac{1}{4}\pm\frac{\varepsilon}{2}+O(\varepsilon^2). \tag{6.14}$$

These two curves emanate from the point $\mu=1/4$ on the μ-axis and define a region of instability called a *tongue*. Inside the tongue, for small ε, x grows exponentially in time. Outside the tongue x is the sum of terms, each of which is the product of two harmonic functions with generally incommensurate frequencies, so x is a bounded quasiperiodic function of t. This confirms our numerical simulations done in the previous paragraph. One can also show that the approximate solution given by equations (6.10) and (6.13) converges to the exact solution of (6.8) in any finite time interval as $\varepsilon\to 0$. The indirect check of this result can be done also by solving equations (6.13) and comparing it with the numerical solutions (see the dashed envelopes in Figs. 6.5 and 6.6 computed by the equations (6.13)).

6.2 Mathieu's Differential Equation

This Section presents the exact treatment of Mathieu's equation based on Floquet's theory of linear differential equations with periodic coefficients and the finding of stability chart.

Floquet's Theory. We first study the general theory of linear differential equations with periodic coefficients (Floquet's theory). Let $\mathbf{x}$ be an $n\times 1$ column vector, and $\mathbf{A}$ an $n\times n$ matrix whose elements are periodic functions with a period T. We consider the following vectorial differential equation

$$\dot{\mathbf{x}}=\mathbf{A}(t)\mathbf{x}. \tag{6.15}$$

Notice that, since $\mathbf{A}(t+T)=\mathbf{A}(t)$, this equation is invariant with respect to the shift of time by a constant period T. Thus, if $\mathbf{x}(t)$ is a solution of (6.15), then $\mathbf{x}(t+T)$ must also be a solution of (6.15).

Now let us consider the fundamental solution matrix of (6.15), $\mathbf{X}(t)$, which is defined as follows. $\mathbf{X}(t)$ is an $n\times n$ matrix, whose columns are solutions of (6.15) such that $\mathbf{X}(0)=\mathbf{I}$, $\mathbf{I}$ being the identity matrix. As the columns of $\mathbf{X}(t)$ are linearly independent, they form a basis for the n-dimensional solution space of (6.15). Since $\mathbf{X}(t+T)$ is also the solution matrix of equation (6.15), each of its columns may be written as a linear combination of the columns of $\mathbf{X}(t)$, so

$$\mathbf{X}(t+T) = \mathbf{X}(t)\mathbf{C}, \tag{6.16}$$

where $\mathbf{C}$ is a $n \times n$ constant matrix. At $t = 0$ we have $\mathbf{X}(T) = \mathbf{X}(0)\mathbf{C} = \mathbf{C}$, so $\mathbf{C}$ is in fact equal to the fundamental matrix evaluated at time T. Thus, $\mathbf{C}$ could be obtained by numerically integrating (6.15) from $t = 0$ to $t = T$, n times, once for each of the n initial conditions satisfied by the i-th column of $\mathbf{X}(0)$. Taking the time instants $2T$, $3T$, and so on and applying similar arguments, we can show that $\mathbf{X}(nT) = \mathbf{C}^n$.

Let us transform (6.16) to normal coordinates. We seek another fundamental solution matrix $\mathbf{Y}(t)$ such that

$$\mathbf{Y}(t) = \mathbf{X}(t)\mathbf{R},$$

where $\mathbf{R}$ is as yet unknown $n \times n$ matrix. Combining this equation with (6.16), we obtain

$$\mathbf{Y}(t+T) = \mathbf{Y}(t)\mathbf{R}^{-1}\mathbf{C}\mathbf{R}. \tag{6.17}$$

Suppose that $\mathbf{C}$ has n linearly independent eingenvectors. If we choose the columns of $\mathbf{R}$ to be these eigenvectors, then the product $\mathbf{R}^{-1}\mathbf{C}\mathbf{R}$ will be a diagonal matrix with the eigenvalues λ_i of $\mathbf{C}$ on its diagonal. With $\mathbf{R}^{-1}\mathbf{C}\mathbf{R}$ diagonal, the matrix $\mathbf{Y}(t)$ satisfying (6.17) will also be diagonal. Indeed, let us construct this diagonal matrix explicitly. Its elements satisfy the equations

$$y_i(t+T) = \lambda_i y_i(t). \tag{6.18}$$

We look for a solution to this functional equation in the form

$$y_i(t) = \lambda_i^{kt} p_i(t),$$

where k is an unknown constant and $p_i(t)$ is an unknown function. Substitution into (6.18) gives

$$y_i(t+T) = \lambda_i^{k(t+T)} p_i(t+T) = \lambda_i(\lambda_i^{kt} p_i(t)).$$

This equation is satisfied if we take $k = 1/T$ and $p_i(t)$ a periodic function of period T. Thus, the constructed matrix with the diagonal elements

$$y_i(t) = \lambda_i^{t/T} p_i(t) \tag{6.19}$$

satisfies (6.17). This implies that the original system (6.15) will be stable if every eigenvalue λ_i of $\mathbf{C}$ has modulus less than 1. In the opposite case the solution will grow exponentially as $t \to \infty$ leading to instability and parametric resonance.

Hill's Equation. Let us first apply Floquet's theory to Hill's equation

$$\ddot{x} + f(t)x = 0, \quad f(t+T) = f(t),$$

which contains Mathieu's equation as a special case. This equation can be written as a system of differential equation

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -f(t) & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

We construct a fundamental solution matrix from two solution vectors satisfying the initial conditions

$$\begin{pmatrix} x_1(0) \\ y_1(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} x_2(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then the matrix $\mathbf{C}$ is the fundamental solution matrix evaluated at time T

$$\mathbf{C} = \begin{pmatrix} x_1(T) & x_2(T) \\ y_1(T) & y_2(T) \end{pmatrix}.$$

From the previous paragraph we know that stability is determined by the eigenvalues of $\mathbf{C}$

$$\lambda^2 - (\mathrm{tr}\mathbf{C})\lambda + \det\mathbf{C} = 0, \tag{6.20}$$

where $\mathrm{tr}\mathbf{C}$ and $\det\mathbf{C}$ are the trace and determinant of $\mathbf{C}$. It turns out that $\det\mathbf{C} = 1$ for Hill's equation. Indeed, let us compute the time derivative of the Wronskian

$$\frac{d}{dt}W(t) = \frac{d}{dt}(x_1y_2 - y_1x_2) = y_1y_2 - f(t)x_1x_2 + f(t)x_1x_2 - y_1y_2 = 0.$$

Thus, $W(T) = \det\mathbf{C} = W(0) = 1$ and equation (6.20) can be written as

$$\lambda^2 - (\mathrm{tr}\mathbf{C})\lambda + 1 = 0,$$

yielding two roots

$$\lambda_{1,2} = \frac{1}{2}(\mathrm{tr}\mathbf{C} \pm \sqrt{(\mathrm{tr}\mathbf{C})^2 - 4}).$$

According to Floquet's theory instability occurs if either eigenvalue has modulus larger than 1. So, if $|\mathrm{tr}\mathbf{C}| > 2$, then we have two real roots, and since their product is 1, one of them has modulus greater than 1. In this case we have instability associated with the exponential growth of solutions. If $|\mathrm{tr}\mathbf{C}| < 2$, then the roots are complex conjugate, and since their product is 1, they lie on the unit circle, with the consequence that the solutions are bounded. The transition from stable to unstable behavior corresponds to those parameter values giving $|\mathrm{tr}\mathbf{C}| = 2$. If $\mathrm{tr}\mathbf{C} = 2$, then we have the double root $\lambda = 1$, and formula (6.19) implies that the solutions must be periodic functions with period T. In case $\mathrm{tr}\mathbf{C} = -2$ we have the double root $\lambda = -1$ corresponding to the periodic solutions with period $2T$. Thus, on the transition curves in parameter space, the motions are periodic with period T or $2T$. In accordance with this theory, the stability of a given pair (μ, ε) can be determined by finding the fundamental solution matrix at $t = T$ through numerical integration and investigating its eigenvalues. However, for the whole (μ, ε)-plane the method is still ineffective.

Stability Chart. In case of Mathieu's equation the period of $f(t)$ is 2π, so we may seek the periodic solutions on the transition curves in form of a Fourier series

$$x(t) = \sum_{j=0}^{\infty} \left(a_j \cos \frac{jt}{2} + b_j \sin \frac{jt}{2} \right).$$

The factor $1/2$ in the arguments of sine and cosine guarantees that periodic functions of period 4π are included. Substituting this Fourier series into Mathieu's equation (6.8), transforming the products of trigonometric functions into harmonic functions and collecting similar terms gives four sets of homogeneous linear equations on the coefficients a_j and b_j. Each set contains only coefficients a_j or b_j with even or odd indices j. For a nontrivial solution of one set to exist the corresponding determinant must vanish. This gives four infinite determinants known as Hill's determinants. For a_j with even j we have

$$\begin{vmatrix} \mu & \varepsilon/2 & 0 & 0 & \\ \varepsilon & \mu-1 & \varepsilon/2 & 0 & \dots \\ 0 & \varepsilon/2 & \mu-4 & \varepsilon/2 & \\ & & \dots & & \end{vmatrix} = 0.$$

For b_j with even j,

$$\begin{vmatrix} \mu-1 & \varepsilon/2 & 0 & 0 & \\ \varepsilon/2 & \mu-4 & \varepsilon/2 & 0 & \dots \\ 0 & \varepsilon/2 & \mu-9 & \varepsilon/2 & \\ & & \dots & & \end{vmatrix} = 0.$$

For a_j with odd j,

$$\begin{vmatrix} \mu-1/4+\varepsilon/2 & \varepsilon/2 & 0 & 0 & \\ \varepsilon/2 & \mu-9/4 & \varepsilon/2 & 0 & \dots \\ 0 & \varepsilon/2 & \mu-25/4 & \varepsilon/2 & \\ & & \dots & & \end{vmatrix} = 0.$$

Finally, for b_j with odd j,

$$\begin{vmatrix} \mu-1/4-\varepsilon/2 & \varepsilon/2 & 0 & 0 & \\ \varepsilon/2 & \mu-9/4 & \varepsilon/2 & 0 & \dots \\ 0 & \varepsilon/2 & \mu-25/4 & \varepsilon/2 & \\ & & \dots & & \end{vmatrix} = 0.$$

In all four determinants the typical row is

$$\dots \quad 0 \quad \varepsilon/2 \quad \mu - j^2/4 \quad \varepsilon/2 \quad 0 \quad \dots$$

except for the first one or two rows.

Each of these equations represents a relation between μ and ε, which plots as a set of transition curves in the (μ,ε)-plane (see Fig. 6.7). Since the transition curves are symmetric about the μ-axis, only the upper half of chart is shown. The equations obtained at $\varepsilon = 0$ give the intersections of these curves with the μ-axis. For a_j or b_j with even j the transition curves intersect the μ-axis at $\mu = j^2$, $j = 0, 1, 2, \dots$, while those curves obtained from the determinants with odd j intersect the μ-axis at $\mu = (2j+1)^2/4$, $j = 0,1,2,\ldots$. For $\varepsilon > 0$, each of these points give rise to two transition curves, one obtained from the a-determinant, and the other from the b-determinant. Thus, there is a tongue of instability emanating from each of the following points on the μ-axis: $\mu = j^2/4$, $j = 0,1,2,\ldots$. Exception is the case $j = 0$ for which only one transition curve exists.

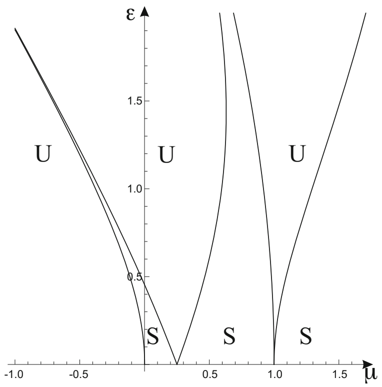

Fig. 6.7 Stability chart of Mathieu's equation (U: unstable, S: stable)

To find the asymptote of a transition curve $\mu = f(\varepsilon)$ emanating from $j^2/4$ on the μ-axis we expand the function $f(\varepsilon)$ in the power series

$$\mu = \frac{j^2}{4} + \mu_1\varepsilon + \mu_2\varepsilon^2 + \ldots \tag{6.21}$$

Substituting this into one of the determinants and equating terms of equal order of ε to zero, we can determine the coefficients μ_i. For example, for $j = 1$ we may take the truncated 3×3 a-determinant with odd j to obtain

$$-\frac{\varepsilon^3}{8} - \frac{\mu\varepsilon^2}{2} + \frac{13\varepsilon^2}{8} + \frac{\mu^2\varepsilon}{2} - \frac{17\mu\varepsilon}{4} + \frac{225\varepsilon}{32} + \mu^3 - \frac{35\mu^2}{4} + \frac{259\mu}{16} - \frac{225}{64} = 0.$$

Substituting (6.21) (with $j = 1$) into the above equation and equating terms with ε and ε^2 to zero, we obtain $\mu_1 = -1/2$ and $\mu_2 = -1/8$. This procedure can be continued to any order of truncation. Here are the asymptotes of first five transition curves computed in this way

$$\mu = \begin{cases} -\frac{\varepsilon^2}{2} & \text{for } j = 0 \\ \frac{1}{4} - \frac{\varepsilon}{2} - \frac{\varepsilon^2}{8} & \text{for } j = 1 \\ \frac{1}{4} + \frac{\varepsilon}{2} - \frac{\varepsilon^2}{8} & \text{for } j = 1 \\ 1 - \frac{\varepsilon^2}{12} & \text{for } j = 2 \\ 1 + \frac{5\varepsilon^2}{12} & \text{for } j = 2 \end{cases}$$

Note that the transition curves (6.14) obtained in Section 6.1 by the variational-asymptotic method corresponds to $j = 1$. Why other tongues of instability were missed? If we continue the next steps of the variational-asymptotic method, the other tongues can be found also (see exercise 6.3).

6.3 Duffing's Forced Oscillator

Differential Equation of Motion. As we know from Section 5.2 the free vibrations of a nonlinear damped oscillator must decay as time increases because of the positive dissipation rate. We want now to analyze the situation, when some external harmonic force acts on such the oscillator. As prototype we consider a damped Duffing's oscillator subjected to a small harmonic excitation. For this forced oscillator the displacement $x(t)$ satisfies the variational equation

$$\delta \int_{t_0}^{t_1} (\frac{1}{2}\dot{x}^2 - \frac{1}{2}x^2 - \frac{1}{4}\varepsilon\alpha x^4 + \varepsilon\hat{f}\cos\omega t\, x)\, dt - \int_{t_0}^{t_1} \varepsilon c\dot{x}\delta x\, dt = 0, \tag{6.22}$$

where ε is a small parameter. This implies the following equation of motion

$$\ddot{x} + x + \varepsilon c\dot{x} + \varepsilon\alpha x^3 = \varepsilon\hat{f}\cos\omega t. \tag{6.23}$$

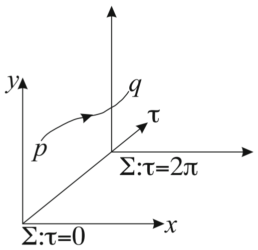

Fig. 6.8 Poincaré map

In contrast to the Duffing's equation (5.1) describing free undamped vibrations, equation (6.23) is nonautonomous, that is, time t appears explicitly in the force term. Therefore the phase plane is no longer appropriate for this equation since the vector field changes in time, allowing a trajectory to return to the previous point and intersect itself. However, the system may be made autonomous by introducing the angular time τ and rewriting (6.23) as follows

$$\begin{aligned}
\dot{x} &= y, \\
\dot{y} &= -x - \varepsilon c y - \varepsilon\alpha x^3 + \varepsilon\hat{f}\cos\tau \\
\dot{\tau} &= \omega.
\end{aligned}$$

This system of three differential equations of first order is defined on a phase space with topology $R^2 \times S$, where the circle S comes from the 2π-periodicity in τ of the vector field. A convenient way to view this 3-D flow in two dimensions is to use the Poincaré map. This map is obtained by the intersection of the trajectory with a plane of section Σ which may be taken as $\tau = 0 \pmod{2\pi}$ as shown schematically in Fig. 6.8. Thus, when $\hat{f} = 0$, the equilibria that would normally lie in the (x, y)-plane, now become periodic orbits of period 2π in this 3-D phase space. For small $\hat{f} > 0$, we may expect by a continuity argument that these periodic orbits persist giving rise to 2π-periodic motions. Such periodic motions correspond to fixed points of the Poincaré map.

Numerical Solutions. Since no analytical solution to the forced Duffing equation (6.23) is available, we first try to find some particular solutions by numerical integration to study their behavior. Take for example $\varepsilon = 0.1$, $c = 0$, $\alpha = 1$. For the harmonic force we choose $\hat{f} = 1$ and $\omega = 1$, together with the initial conditions $x(0) = 1$, $\dot{x}(0) = 0$. The numerical integration with standard commands like those in Section 5.3 gives the graph of $x(t)$ shown in Fig. 6.9. Looking at this solution, we observe that there are two time scales characterizing the vibration: i) one describing the period of fast oscillation of $x(t)$, ii) the other associated with the slow and periodic change of amplitude of vibration.

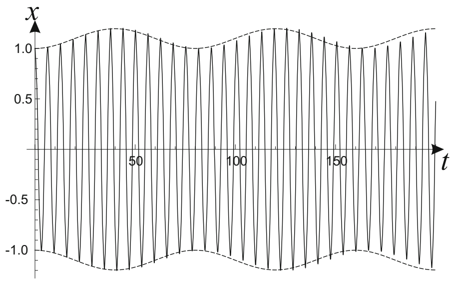

Fig. 6.9 Numerical solution of forced Duffing equation for $\omega = 1$

It turns out that for certain values of frequency and amplitude of force as well as certain initial conditions, purely periodic solutions can be obtained. This corresponds to a fixed point of the Poincaré's map introduced in the previous paragraph. For example, if we take $\omega = 0.9875$ while keeping all other parameters unchanged, the solution is purely periodic as seen in Fig. 6.10.

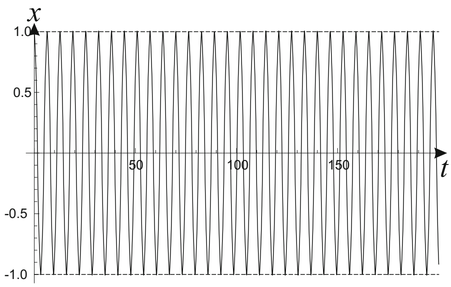

Fig. 6.10 Numerical solution of forced Duffing equation for $\omega = 0.9875$

As soon as the damping becomes nonzero, the character of solutions changes. Take for example $\varepsilon = 0.1$, $c = \alpha = \hat{f} = 1$, $\omega = 1$, together with the initial conditions $x(0) = 1$, $\dot{x}(0) = 0$. Now the amplitude shows first a transient character before approaching a certain steady-state amplitude that depends only on the forcing

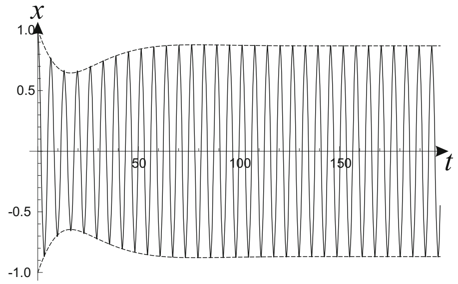

Fig. 6.11 Numerical solution of forced and damped Duffing oscillator for $c = 1$ and $\omega = 1$

frequency and the initial conditions (see Fig. 6.11). The steady-state response frequency coincides with the forcing frequency, what is similar to the linear theory.

If we increase the forcing frequency while keeping all other parameters as well as the initial conditions, the steady-state amplitude may become even smaller as shown in Fig. 6.12 for the case $\omega = 1.2$. Since we cannot do infinite number of numerical simulations to establish the behavior of slow change of amplitude due to the change of frequency and other factors, more "intelligent" methods have to be invented for this purpose.

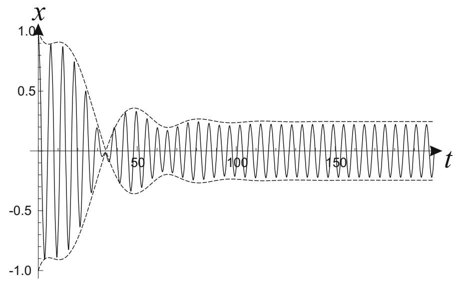

Fig. 6.12 Numerical solution of forced and damped Duffing oscillator for $c = 1$ and $\omega = 1.2$

Variational-Asymptotic Method. For the convenience of our analysis let us introduce the stretched angular time $\tau = \omega t$ explicitly in the variational equation (6.22). Since $\dot{x} = \omega x'$, where prime denotes the derivative with respect to τ, equation (6.22) becomes

$$\delta \int_{\tau_0}^{\tau_0+2\pi} \left(\frac{1}{2}\omega^2 x'^2 - \frac{1}{2}x^2 - \frac{1}{4}\varepsilon\alpha x^4 + \varepsilon\hat{f}\cos\tau x\right) d\tau - \int_{\tau_0}^{\tau_0+2\pi} \varepsilon c\omega x'\delta x\, d\tau = 0, \tag{6.24}$$

where $\tau_1 = \tau_0 + 2\pi$ and τ_0 is an *arbitrary* time instant. For short we set $\tau_0 = 0$.

At the first step of the variational-asymptotic procedure we neglect all terms containing ε to get

$$\delta \int_0^{2\pi} (\frac{1}{2}\omega^2 x'^2 - \frac{1}{2}x^2)\, d\tau = 0.$$

The 2π-periodic extremal of this variational problem is

$$x_0(\tau) = A\cos\tau + B\sin\tau.$$

We see that the frequency equals 1 which is not surprising because the external force, the damping and the nonlinear spring force are neglected. The coefficients A and B are still unknown and should be determined from the initial conditions.

From the numerical simulations provided previously we know that, for $\varepsilon \neq 0$, the coefficients A and B are becoming dependent on time. In general the forcing frequency differs from 1 also. Taking all these circumstances into account, we introduce the slow time $\eta = \varepsilon\tau$ and search for the corrections to the extremal and to the frequency at the second step in the form

$$x(\tau) = A(\eta)\cos\tau + B(\eta)\sin\tau + x_1(\tau,\eta), \quad \omega = 1 + \omega_1, \tag{6.25}$$

where $x_1(\tau,\eta)$ is a 2π-periodic function with respect to τ and is much smaller than $x_0(\tau,\eta) = A(\eta)\cos\tau + B(\eta)\sin\tau$ in the asymptotic sense, and ω_1 is assumed to be much smaller than 1. The second equation of (6.25) means that we are restricting to the case where the forcing frequency is nearly equal to 1. The full time derivative of x is equal to

$$x' = x_{0,\tau} + \varepsilon x_{0,\eta} + x_{1,\tau} + \varepsilon x_{1,\eta}. \tag{6.26}$$

We substitute (6.25) and (6.26) into (6.24) and keep the asymptotically principal terms containing x_1 and the principal cross terms between x_0 and x_1. The variational equation becomes

$$\begin{aligned}\delta \int_0^{2\pi} [&\frac{1}{2}x_{1,\tau}^2 + \underline{x_{0,\tau}x_1'} + \underline{\varepsilon x_{0,\eta}x_1'} + 2\omega_1 x_{0,\tau}x_1' \\ &- \frac{1}{2}x_1^2 - \underline{x_0 x_1} - \varepsilon\alpha x_0^3 x_1 + \varepsilon\hat{f}\cos\tau\, x_1 - \varepsilon c x_{0,\tau}x_1]\, d\tau = 0.\end{aligned}$$

Next, we integrate the second, third, and fourth terms by parts using the 2π-periodicity of x_1 with respect to τ. It is easy to see that, since $(x_{0,\tau})' = -x_0 + \varepsilon x_{0,\tau\eta}$, the underlined terms give $-2\varepsilon x_{0,\eta\tau}x_1$. Then we expand x_0^3 of the term $-\varepsilon\alpha x_0^3 x_1$ and transform the products of sine and cosine into the sum of harmonic functions according to

$$\cos^3\tau = \frac{3}{4}\cos\tau + \frac{1}{4}\cos 3\tau, \quad \cos^2\tau\sin\tau = \frac{1}{4}(\sin\tau + \sin 3\tau),$$

$$\sin^3\tau = \frac{3}{4}\sin\tau - \frac{1}{4}\sin 3\tau, \quad \sin^2\tau\cos\tau = \frac{1}{4}(\cos\tau - \cos 3\tau).$$

The variational equation takes the form

$$\delta \int_0^{2\pi} [\frac{1}{2}x_{1,\tau}^2 - \frac{1}{2}x_1^2 + (\dots)\sin\tau x_1 + (\dots)\cos\tau x_1 + \text{nonresonant terms}]\, d\tau = 0.$$

The consistency condition requires the expressions in parentheses to vanish, giving the following equations for A and B

$$\begin{aligned} 2A_{,\eta} + cA + 2\frac{\omega_1}{\varepsilon}B - \frac{3}{4}\alpha B(A^2+B^2) &= 0, \\ 2B_{,\eta} + cB - 2\frac{\omega_1}{\varepsilon}A + \frac{3}{4}\alpha A(A^2+B^2) &= \hat{f}. \end{aligned} \tag{6.27}$$

The fixed points of these equations correspond to periodic motions of the forced Duffing equation (6.23). To find them we set $A_{,\eta}$ and $B_{,\eta}$ equal to zero. Multiplying the first equation of (6.27) by A and adding it to the second equation multiplied by B gives

$$ca^2 = \hat{f}B, \quad \text{where } a^2 = A^2 + B^2.$$

Similarly, multiplying the first equation of (6.27) by B and subtracting it from the second one multiplied by A yields

$$-2\frac{\omega_1}{\varepsilon}a^2 + \frac{3}{4}\alpha a^4 = \hat{f}A.$$

Adding the squares of two last equations together and dividing by a^2 we obtain

$$a^2[c^2 + (-2\frac{\omega_1}{\varepsilon} + \frac{3}{4}\alpha a^2)^2] = \hat{f}^2.$$

Solving the last equation with respect to ω_1 leads to

$$\omega_1 = \frac{3}{8}\varepsilon\alpha a^2 \pm \frac{1}{2}\varepsilon\sqrt{\frac{\hat{f}^2}{a^2} - c^2}.$$

Thus, the correction to the frequency of the external force is of the order ε. Together with (6.25) we have the following nonlinear relation between the frequency ω of the external force and the response amplitude a of the corresponding forced periodic motion

$$\omega = 1 + \frac{3}{8}\varepsilon\alpha a^2 \pm \frac{1}{2}\varepsilon\sqrt{\frac{\hat{f}^2}{a^2} - c^2}. \tag{6.28}$$

Note that if both the force $\hat{f}$ and the damping c are zero, then (6.28) reduces to the well-known formula (5.8) obtained previously for the free undamped Duffing's oscillator. If $\hat{f} > 0$, then there exists ω_c such that for $\omega > \omega_c$ the amplitude a is a multi-valued function of the frequency. However, if $c > 0$, then a is a multi-valued function of ω only in the range $\omega \in (\omega_c, 1 + \frac{3}{8}\varepsilon\alpha(\hat{f}/c)^2)$. Fig. 6.13 shows the amplitude versus frequency curves in these three different cases for $\alpha > 0$ (hardening

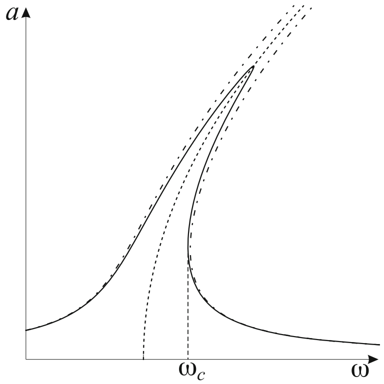

Fig. 6.13 Amplitude versus frequency curves of forced Duffing's oscillator: i) dashed line: $\hat{f} = c = 0$, ii) dotted and dashed line: $c = 0$, $\hat{f} > 0$, iii) bold line: both $\hat{f}$ and c are positive

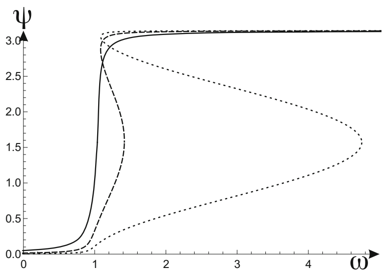

Fig. 6.14 Phase versus frequency curves of forced Duffing's oscillator

spring). Thus, for the fixed frequency ω and magnitude $\hat{f} > 0$ of the external force we may find in general either one or three steady-state amplitudes of forced vibrations. The phase of forced vibrations deviates also from that of the linear theory. Introducing the phase of forced vibrations as

$$\tan\psi = \frac{B}{A} = \frac{c}{\mp\sqrt{\frac{\hat{f}^2}{a^2} - c^2}},$$

we show the plot of ψ versus ω in Fig. 6.14 for $\varepsilon = 0.1$, $\alpha = \hat{f} = 1$, and $c = 1$ (bold line), $c = 0.3$ (dashed line), $c = 0.1$ (dotted line).

Since there are several fixed points of system (6.27) we have to study their stability to select realizable solutions. As these fixed points correspond to the periodic solutions of (6.23), their stability means also the stability of the periodic solutions. For simplicity of our analysis, we consider the case $c = 0$. Denoting $\omega_1/\varepsilon = k_1$, we write (6.27) in the form

$$\begin{aligned} A_{,\eta} &= -k_1 B + \frac{3}{8}\alpha B(A^2+B^2), \\ B_{,\eta} &= k_1 A - \frac{3}{8}\alpha A(A^2+B^2) + \frac{\hat{f}}{2}. \end{aligned} \tag{6.29}$$

For $\omega > \omega_c$ there are three roots of (6.28), a_1, a_2, a_3, such that $a_1 > a_2 > a_3 > 0$. The corresponding fixed points S_1, S_2, S_3 have the coordinates given by

$$(A_1, B_1) = (a_1, 0), \quad (A_2, B_2) = (-a_2, 0), \quad (A_3, B_3) = (-a_3, 0).$$

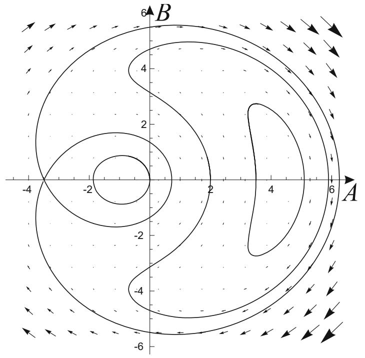

Fig. 6.15 Phase portrait of system (6.29)

To determine the stability of these fixed points we set $A = A_i + u$, $B = v$ and linearize (6.29) in u and v, giving

$$u_{,\eta} = (\frac{3}{8}\alpha A_i^2 - k_1)v, \quad v_{,\eta} = (-\frac{9}{8}\alpha A_i^2 + k_1)u.$$

Thus, if

$$D = (\frac{3}{8}\alpha A_i^2 - k_1)(\frac{9}{8}\alpha A_i^2 - k_1) > 0, \tag{6.30}$$

then the fixed point is a center, and if this same quantity is negative, the fixed point is a saddle point. For S_1 we have

$$k_1 = \frac{3}{8}\alpha a_1^2 - \frac{\hat{f}}{2a_1},$$

and condition (6.30) is satisfied for all ω so that the fixed point is a center. For S_2

$$k_1 = \frac{3}{8}\alpha a_2^2 + \frac{\hat{f}}{2a_2},$$

so in this case

$$D = -\frac{\hat{f}}{2a_2}(\frac{3}{4}\alpha a_2^2 - \frac{\hat{f}}{2a_2}) < 0,$$

and the fixed point is a saddle point. Finally, for S_3

$$D = -\frac{\hat{f}}{2a_3}(\frac{3}{4}\alpha a_3^2 - \frac{\hat{f}}{2a_3}) > 0,$$

so the fixed point is a center. It is interesting to note that, for S_2 and S_3 the sign of D is opposite to the sign of the derivative $\frac{d\omega}{da}$. The vector field and the phase portrait of (6.29) for $\omega > \omega_c$ are shown in Fig. 6.15. If some small damping is included ($c > 0$), then, by the continuity reasoning we expect that the centers would become stable foci attracting phase curves, while the saddle point remains unstable.

Imagine now that we can change the forcing frequency ω so slowly that the steady-state response amplitude a can follow it after a short transient period. Thus, if the forcing frequency is increased starting from zero, then the response amplitude follows first the stable upper branch OA up to point A. After point A no solution of this branch is possible, so the amplitude has to jump to the lower branch (this jump is marked by the vertical line AB) and then follows this stable branch down to point C. If the forcing frequency were now to reverse its course (again quasistatically), then the amplitude would go back along the lower branch CD, after which it jumps to the upper branch (the jump is marked by the vertical line DE), and finally follows this upper curve down to the end point O. This closed loop OABCDEO is called a hysteresis loop.

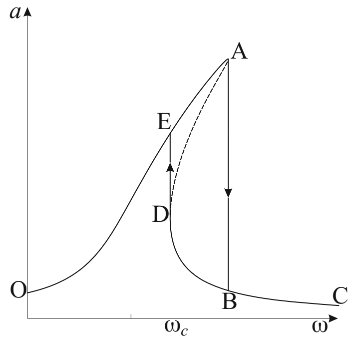

Fig. 6.16 Jump phenomenon and hysteresis

Note that the convergence of the approximate solution given by equations (6.25) and (6.27) to the exact solution of (6.23) as $\varepsilon \to 0$ can be established for any finite time interval. We can also indirectly verify this result by comparing the solutions of (6.27) (presented by the dashed envelopes in Figs. 6.9-6.12) with the numerical solutions of (6.23). The agreement is excellent, although $\varepsilon = 0.1$ is not quite small.

6.4 Forced Vibration of Self-excited Oscillator

Differential Equation of Motion. As we know from Section 5.3 a self-excited oscillator may have limit cycles as attractors of the phase curves. We want now to analyze the situation, when some external harmonic force acts on such the oscillator. As prototype we consider van der Pol's oscillator subjected to a small harmonic excitation. Since we are interested in the primary 1:1 resonance, we order the amplitude of the excitation to be the same as the damping and non-linear term [36]. For this forced oscillator the displacement $x(t)$ satisfies the variational equation

$$\delta \int_{t_0}^{t_1} (\frac{1}{2}\dot{x}^2 - \frac{1}{2}x^2 + \varepsilon \hat{f} \cos \omega t\, x)\, dt + \int_{t_0}^{t_1} \varepsilon (1 - x^2)\dot{x}\delta x\, dt = 0, \tag{6.31}$$

where ε is a small parameter. This implies the following equation of motion

$$\ddot{x} + x - \varepsilon(1 - x^2)\dot{x} = \varepsilon \hat{f} \cos \omega t. \tag{6.32}$$

Equation (6.32) is called forced van der Pol's equation. In the previous Section we have seen that, when a damped Duffing-type oscillator is driven by a harmonic force, the steady-state response will have the same frequency as the forcing frequency. If a self-excited oscillator is driven by some harmonic force, the steady state of vibration may not exist at all and the forced response may include both the unforced limit cycle oscillation as well as a response at the forcing frequency. However, if the amplitude of the force is strong enough, and the frequency difference between the limit cycle oscillation and the harmonic force is small enough, then it may happen that the steady-state response exists and occurs only at the forcing frequency. In this case the forcing function is said to have entrained the limit cycle oscillator, and the system is said to be frequency-locked [36].

Numerical Solutions. Similar to the unforced van der Pol's equation, (6.32) does not permit exact analytical treatment. Therefore, to study the behavior of forced vibrations and illustrate the possibility of entrainment (or, equivalently, frequency locking) let us first do some numerical simulations.

We take $\varepsilon = 0.1$, $\hat{f} = 1.06$, and $\omega = 1.02$ and find the solution to (6.32) satisfying the initial conditions $x(0) = 1$, $\dot{x}(0) = 0$ by the numerical integration with *Mathematica*. The result shown in Fig. 6.17 exhibits the entrainment: a steady-state vibration with the forcing frequency is settled after a short transient period.

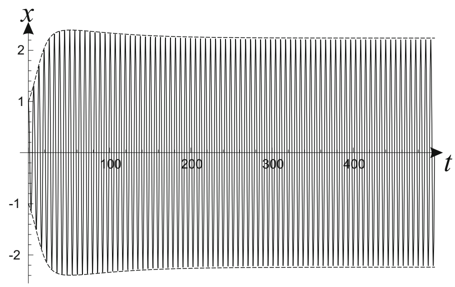

Fig. 6.17 Numerical solution of forced van der Pol's oscillator for $\varepsilon = 0.1$, $\hat{f} = 1.06$, and $\omega = 1.02$

If we increase a little bit the forcing frequency while keeping all other parameters and initial data, the response may change drastically. For example, the solution for $\omega = 1.05$ shown in Fig. 6.18 exhibits a beating behavior typical for the oscillation with two nearly equal frequencies. Thus, in this case entrainment does not occur, and the system is unlocked.

In the next paragraph we will use the variational-asymptotic method to establish the law of slow change of response amplitude as function of the forcing parameters and to predict the entrainment effect for small ε. The outcome of this asymptotic analysis is shown in Figs. 6.17 and 6.18: the dashed envelopes are computed according to the obtained equations of slow change. The agreement is good although $\varepsilon = 0.1$ is not quite small.

Variational-Asymptotic Method. Let us introduce the stretched angular time $\tau = \omega t$ explicitly in the variational equation (6.31). Since $\dot{x} = \omega x'$, where prime denotes the derivative with respect to τ, equation (6.31) becomes

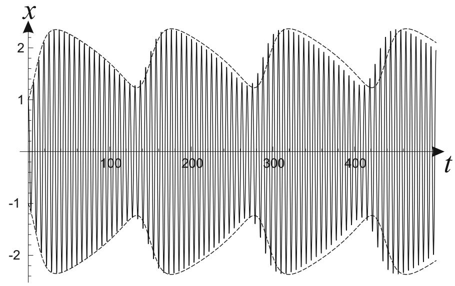

Fig. 6.18 Numerical solution of forced van der Pol's oscillator for $\varepsilon = 0.1$, $\hat{f} = 1.06$, and $\omega = 1.05$

$$\delta \int_{\tau_0}^{\tau_0+2\pi} (\frac{1}{2}\omega^2 x'^2 - \frac{1}{2}x^2 + \varepsilon \hat{f} \cos\tau x)\, d\tau + \int_{\tau_0}^{\tau_0+2\pi} \varepsilon(1-x^2)\omega x' \delta x\, d\tau = 0, \quad (6.33)$$

where $\tau_1 = \tau_0 + 2\pi$ and τ_0 is an *arbitrary* time instant. For short we set $\tau_0 = 0$.

At the first step of the variational-asymptotic procedure we neglect all terms containing ε to get

$$\delta \int_0^{2\pi} (\frac{1}{2}\omega^2 x'^2 - \frac{1}{2}x^2)\, d\tau = 0.$$

The 2π-periodic extremal of this variational problem is

$$x_0(\tau) = A\cos\tau + B\sin\tau.$$

The coefficients A and B in this solution are still unknown.

For $\varepsilon \neq 0$ the coefficients A and B are becoming dependent on time and the forcing frequency deviates from 1. Therefore we introduce the slow time $\eta = \varepsilon\tau$ and search for the corrections to the extremal and to the frequency at the second step in the form

$$x(\tau) = A(\eta)\cos\tau + B(\eta)\sin\tau + x_1(\tau,\eta), \quad \omega = 1 + \omega_1, \quad (6.34)$$

where $x_1(\tau,\eta)$ is a 2π-periodic function with respect to τ and is much smaller than $x_0(\tau,\eta) = A(\eta)\cos\tau + B(\eta)\sin\tau$ in the asymptotic sense, and ω_1 is assumed to be much smaller than 1. The second equation of (6.34) means that we are restricting to the case where the forcing frequency is nearly equal to the unforced limit cycle frequency, which is called a 1:1 resonance. Note that the time derivative of $x(\tau)$ is equal to

$$x' = x_{0,\tau} + \varepsilon x_{0,\eta} + x_{1,\tau} + \varepsilon x_{1,\eta}.$$

Here the comma in index means the partial derivative. We substitute (6.34) into (6.33) and keep the asymptotically principal terms containing x_1 and the principal cross terms between x_0 and x_1. The variational equation becomes

$$\delta \int_0^{2\pi} [\frac{1}{2}x_{1,\tau}^2 + \underline{x_{0,\tau}x_1'} + \underline{\varepsilon x_{0,\eta}x_1'} + 2\omega_1 x_{0,\tau}x_1'$$
$$-\frac{1}{2}x_1^2 - \underline{x_0x_1} + \varepsilon \hat{f}\cos\tau x_1 + \varepsilon(1-x_0^2)x_{0,\tau}x_1]\,d\tau = 0.$$

Next, integrating the second, third, and fourth terms by parts using the periodicity of x_1, we get from the underlined terms $-2\varepsilon x_{0,\eta\tau}x_1$. Finally, reducing the products of sine and cosine in the last term to the sum of harmonic functions,[2] we transform the variational equation to

$$\delta \int_0^{2\pi} [\frac{1}{2}x_{1,\tau}^2 - \frac{1}{2}x_1^2 + (\dots)\sin\tau x_1 + (\dots)\cos\tau x_1 + \text{nonresonant terms}]\,d\tau = 0.$$

The consistency condition requires removal of the resonant terms that leads to

$$\begin{aligned} 2A_{,\eta} &= -2\frac{\omega_1}{\varepsilon}B + A - \frac{A}{4}(A^2+B^2), \\ 2B_{,\eta} &= 2\frac{\omega_1}{\varepsilon}A + B - \frac{B}{4}(A^2+B^2) + \hat{f}. \end{aligned}$$

We see that the correction to the frequency must be of the order ε. Denoting ω_1 by $\omega_1 = k_1\varepsilon$, we rewrite this system of equations in the form

$$\begin{aligned} 2A_{,\eta} &= -2k_1B + A - \frac{A}{4}(A^2+B^2), \\ 2B_{,\eta} &= 2k_1A + B - \frac{B}{4}(A^2+B^2) + \hat{f}. \end{aligned} \tag{6.35}$$

System (6.35) can be simplified by using polar coordinates a and ψ in the phase plane

$$A = a\cos\psi, \quad B = a\sin\psi. \tag{6.36}$$

In terms of a and ψ we can present x_0 as

$$x_0(\tau,\eta) = a(\eta)\cos(\tau - \psi(\eta)).$$

Thus, $a(\eta)$ has the meaning of amplitude of vibration, while $\psi(\eta)$ can be interpreted as the phase; both are slowly changing functions of time. Substituting (6.36) into (6.35) gives

$$\begin{aligned} a_{,\eta} &= \frac{a}{8}(4-a^2) + \frac{\hat{f}}{2}\sin\psi, \\ \psi_{,\eta} &= k_1 + \frac{\hat{f}}{2a}\cos\psi. \end{aligned} \tag{6.37}$$

[2] This can be done quite nicely in *Mathematica* with the help of TrigReduce command. Another way is to use the complex representations of sine and cosine, then multiply everything out and finally collect terms.

When $\hat{f} = 0$, equations (6.37) reduce to the well-known equations (5.32) derived for the self-excited vibrations of val der Pol's oscillator. We seek fixed points of the slow flow (6.37) which correspond to locked periodic motions of (6.32). Setting $a_{,\eta} = \psi_{,\eta} = 0$, solving for $\sin\psi$ and $\cos\psi$ and using the identity $\sin^2 + \cos^2 = 1$, we obtain

$$a^2\left(1-\frac{a^2}{4}\right)^2 + 4k_1^2 a^2 = \hat{f}^2.$$

Expanding this equation and denoting $p = a^2$, we have

$$\frac{p^3}{16} - \frac{p^2}{2} + (4k_1^2+1)p - \hat{f}^2 = 0. \tag{6.38}$$

This cubic equation in p, in view of its 3 sign changes, has either 3 positive roots, or one positive and two complex conjugate roots. The transition between these two cases occurs when there is a double root, which is equivalent to the condition that the derivative of the left-hand side expression vanishes

$$\frac{3p^2}{16} - p + 1 + 4k_1^2 = 0. \tag{6.39}$$

Eliminating p in the last two equations, we obtain

$$\frac{\hat{f}^4}{16} - \frac{\hat{f}^2}{27}(1+36k_1^2) + \frac{16}{27}k_1^2(1+4k_1^2)^2 = 0.$$

This equation gives two curves meeting at a cusp in the $(k_1, \hat{f})$-plane. As one of these curves is traversed quasistatically, a saddle-node bifurcation occurs. At the cusp we have a triple root leading to a further degeneracy. The condition for this is

$$\frac{3p}{8} - 1 = 0 \quad \Rightarrow p = \frac{8}{3}.$$

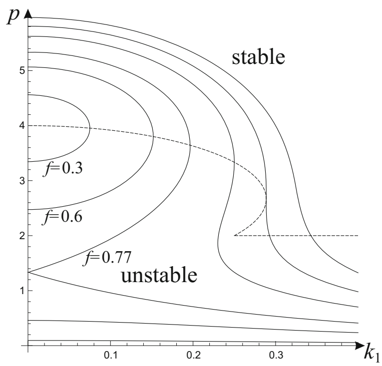

Fig. 6.19 Response-frequency curves

With this value $p = 8/3$ we can easily find the location of the cusp at

$$k_1 = \frac{1}{\sqrt{12}} \approx 0.288, \quad \hat{f} = \sqrt{\frac{32}{27}} \approx 1.088.$$

The square of amplitude versus frequency curves in terms of $p = a^2$ and k_1 are shown in Fig. 6.19 for different forcing amplitudes $\hat{f}$. For $\hat{f} = 0$, the curves degenerate into the k_1-axis and the point $(0,4)$ corresponding to the limit cycle unforced vibration. As $\hat{f}$ increases, the curves first consist of two branches- a branch running near the k_1-axis and a closed curve surrounding the point $(0,4)$. When $\hat{f} = \frac{4}{3\sqrt{3}}$, the two branches coalesce, and the resultant curve has a double point at $(0,4/3)$. As $\hat{f}$ increases beyond this critical value, the response curves are open curves. However, p is still not single-valued function of k_1 until $\hat{f}$ exceeds the second critical value $\hat{f}=\sqrt{32}/\sqrt{27}$. Beyond this critical value the response curves are single-valued for all k_1.

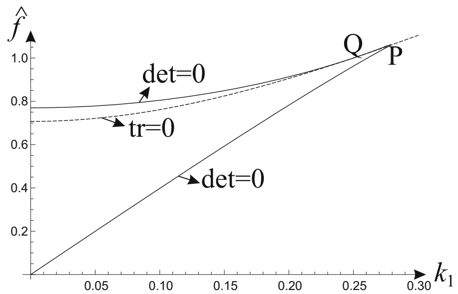

Fig. 6.20 Curves given by: i) $\det \mathbf{M}=0$ (bold lines), ii) $\mathrm{tr}\mathbf{M}=0$ (dashed line)

Since several fixed points of (6.37) are present, we must investigate their stability. Let (a_0,ψ_0) be a fixed point of (6.37). We search for the solutions of (6.37) in the form

$$a=a_0+u, \quad \psi=\psi_0+v,$$

where u and v are small perturbations. Substituting this into (6.37) and linearizing in u and v gives

$$\begin{aligned} u_{,\eta} &= \frac{u}{2}-\frac{3}{8}a_0^2u+\frac{\hat{f}}{2}\cos\psi_0\, v, \\ v_{,\eta} &= -\frac{\hat{f}}{2a_0^2}\cos\psi_0\, u-\frac{\hat{f}}{2a_0}\sin\psi_0\, v. \end{aligned}$$

This system may be simplified by using the following expressions valid at the fixed point

$$\frac{\hat{f}}{2}\sin\psi_0=-\frac{a_0}{2}+\frac{a_0^3}{8}, \quad \frac{\hat{f}}{2}\cos\psi_0=-k_1a_0.$$

Thus, the stability is determined by the eigenvalues of the following matrix $\mathbf{M}$

$$\mathbf{M}=\begin{pmatrix} \frac{1}{2}-\frac{3}{8}a_0^2 & -k_1a_0 \\ \frac{k_1}{a_0} & \frac{1}{2}-\frac{1}{8}a_0^2 \end{pmatrix}.$$

Its eigenvalues λ are the roots of the characteristic equation

$$\lambda^2-\mathrm{tr}\mathbf{M}\,\lambda+\det\mathbf{M}=0,$$

where

$$\mathrm{tr}\mathbf{M} = 1 - \frac{a_0^2}{2} = 1 - \frac{p}{2},$$
$$\det\mathbf{M} = \left(-\frac{1}{2} + \frac{3}{8}a_0^2\right)\left(-\frac{1}{2} + \frac{1}{8}a_0^2\right) + k_1^2 = \frac{1}{4}\left(\frac{3p^2}{16} - p + 1 + 4k_1^2\right).$$

For stability, the eigenvalues of $\mathbf{M}$ must have negative real parts. This puts on the trace and determinant of $\mathbf{M}$ the following conditions

$$\mathrm{tr}\mathbf{M} = 1 - \frac{p}{2} < 0, \quad \det\mathbf{M} = \frac{1}{4}\left(\frac{3p^2}{16} - p + 1 + 4k_1^2\right) > 0.$$

The curves corresponding to $\det\mathbf{M} = 0$ (bold lines) and $\mathrm{tr}\mathbf{M} = 0$ (dashed line) in the $(k_1, \hat{f})$-plane are shown in Fig. 6.20.

Comparing the last condition with equation (6.39), we see that $\det\mathbf{M}$ vanishes on the curve (6.39) along which there are saddle-node bifurcations. This is a typical feature of nonlinear vibrations, namely that a change in stability is accompanied by a bifurcation. The first condition on the trace of $\mathbf{M}$ requires that $p > 2$ for the stability. Substitute $p = 2$ in (6.38), we obtain

$$\hat{f}^2 = \frac{1}{2} + 8k_1^2. \tag{6.40}$$

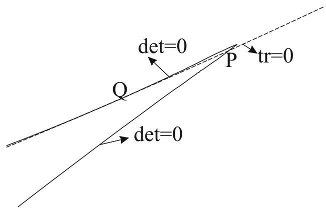

Fig. 6.21 Bifurcation curves

Hopf bifurcations occur along the curve represented by (6.40), provided $\det\mathbf{M} > 0$. This curve intersects the lower curve of saddle-node bifurcations obtained from (6.39) at point P, and touches the upper curve of saddle-node bifurcations at point Q in the $(k_1, \hat{f})$-plane with the coordinates (see Fig. 6.21)

$$\mathrm{P}: k_1 = \frac{\sqrt{5}}{8}, \quad \hat{f} = \frac{3}{\sqrt{8}}, \quad \mathrm{Q}: k_1 = \frac{1}{4}, \quad \hat{f} = \frac{5}{\sqrt{27}}.$$

Thus, the stability analysis predicts that the forced van der Pol oscillator exhibits stable entrainment solutions everywhere in the first quadrant of the $(k_1, \hat{f})$-plane except in that region bounded by i) the lower curve of saddle node bifurcations corresponding to $\det\mathbf{M} = 0$ from the origin to point P, ii) the curve of Hopf bifurcation corresponding to $\mathrm{tr}\mathbf{M} = 0$ from point P to infinity, and iii) the k_1-axis. In terms of p and k_1 the boundary between stable and unstable solutions is marked by the dashed lines shown in Fig. 6.19. This means that for a given detuning k_1 there is a minimum value of forcing $\hat{f}$ required in order for entrainment to occur. Note that, since k_1 always appears in the form k_1^2 in the equations of the bifurcation and stability curves, the above conclusions are independent of whether we are above or below the 1:1 resonance. The discussions about other resonances can be found in [36].

The entrainment is widely used in engineering to synchronize nonlinear oscillators (for instance clocks). Another positive and pleasant spillover effect of entrainment often takes place when an orchestra is playing music. Various string- and wind-instruments are self-excited oscillators, which may experience extra excitations through the sound waves generated by other players of the orchestra. When one of these musical instruments produces a tone which is not quite clean, it may be entrained by the remaining instruments so that only one tone will be heard, provided the forcing amplitude of the sound waves is strong enough.

6.5 Exercises

EXERCISE 6.1. A point-mass m is constrained to move in the (x,y)-plane and is restrained by two linear springs of equal stiffness k and equal unstretched length l. The anchor points of the springs are located on the x-axis at $x=-b$ and $x=b$ (see Fig. 6.22). Study the stability of the motion along the x-axis, $x = a\cos\omega_0 t$, $y=0$ under the assumption that $a \ll b$.

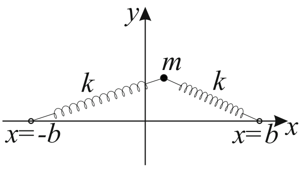

Fig. 6.22 Point-mass in (x,y)-plane

Solution. Let $q=(x,y)$. To derive the equations of motion we write down the Lagrange function

$$L(q,\dot{q}) = K(\dot{q}) - U(q),$$

where the kinetic energy is

$$K(\dot{q}) = \frac{1}{2}m(\dot{x}^2+\dot{y}^2),$$

and the potential energy of the springs is

$$U(q) = \frac{1}{2}k(\sqrt{(x+b)^2+y^2}-l)^2 + \frac{1}{2}k(\sqrt{(b-x)^2+y^2}-l)^2.$$

From Lagrange's equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0, \quad j=1,2,$$

follow

$$m\ddot{x} = f_x, \quad m\ddot{y} = f_y,$$

where

$$f_x = -\frac{\partial U}{\partial x} = -\frac{k(\sqrt{(x+b)^2+y^2}-l)(x+b)}{\sqrt{(x+b)^2+y^2}} + \frac{k(\sqrt{(b-x)^2+y^2}-l)(b-x)}{\sqrt{(b-x)^2+y^2}},$$

$$f_y = -\frac{\partial U}{\partial y} = -\frac{k(\sqrt{(x+b)^2+y^2}-l)y}{\sqrt{(x+b)^2+y^2}} - \frac{k(\sqrt{(b-x)^2+y^2}-l)y}{\sqrt{(b-x)^2+y^2}}.$$

One possible particular solution of these equations is obtained when $y = 0$. In this case the second equation is identically satisfied, while the first equation reduces to

$$m\ddot{x} + 2kx = 0,$$

yielding

$$x_0 = a\cos(\omega_0 t - \phi), \quad \omega_0 = \sqrt{2k/m},$$

where the initial phase ϕ can be set equal to zero. To study the stability of this motion along the x-axis, $q_0(t) = (a\cos\omega_0 t, 0)$, we consider the neighboring solutions in the form

$$x = a\cos\omega_0 t + u, \quad y = v,$$

where u and v are assumed to be small. Substituting these formulas into the equations of motion, expanding the right-hand sides in the Taylor series in terms of u and v, and taking into account the equations for $q_0(t)$, we obtain

$$m\ddot{u} = \left.\frac{\partial f_x}{\partial x}\right|_{q_0} u + \left.\frac{\partial f_x}{\partial y}\right|_{q_0} v,$$
$$m\ddot{v} = \left.\frac{\partial f_y}{\partial x}\right|_{q_0} u + \left.\frac{\partial f_y}{\partial y}\right|_{q_0} v.$$

Computing the partial derivatives of f_x and f_y and evaluating them at $q_0 = (x_0, 0)$, it is easy to check that

$$\left.\frac{\partial f_x}{\partial x}\right|_{q_0} = -2k, \quad \left.\frac{\partial f_x}{\partial y}\right|_{q_0} = 0, \quad \left.\frac{\partial f_y}{\partial x}\right|_{q_0} = 0,$$
$$\left.\frac{\partial f_y}{\partial y}\right|_{q_0} = -2k\frac{1-\lambda-\alpha^2\cos^2\omega_0 t}{1-\alpha^2\cos^2\omega_0 t},$$

where

$$\lambda = \frac{l}{b}, \quad \alpha = \frac{a}{b}.$$

The equation for u turns out to be the equation for a harmonic oscillator, $m\ddot{u} + 2ku = 0$, and cannot produce instability. The equation for v is

$$m\ddot{v} + 2k\frac{1-\lambda-\alpha^2\cos^2\omega_0 t}{1-\alpha^2\cos^2\omega_0 t}v = 0.$$

Expanding the second term for small α and setting $\tau = 2\omega_0 t$, we obtain

$$\frac{d^2 v}{d\tau^2} + (\frac{2 - 2\lambda - \lambda\alpha^2}{8} - \frac{\lambda\alpha^2}{8}\cos\tau)v = 0,$$

which is the Mathieu's equation. Thus, the stability chart of Mathieu's equation can be used to investigate the stability of this motion.

EXERCISE 6.2. The support of a pendulum considered in example 6.1 moves in accordance with the equation $x = a_0 \cos\omega t$, where $a_0 = 0.1l$. How large must the frequency ω be to stabilize the vertical position $\varphi = \pi$.

Solution. The equation of motion of this pendulum is

$$\ddot{\varphi} + \frac{1}{l}(g + a_0\omega^2 \cos\omega t)\sin\varphi = 0.$$

To analyze the stability of the vertical position, it is enough to linearize this equation about $\varphi = \pi$. Let

$$\varphi = \pi + x,$$

where $x \ll 1$ is a small perturbation. The linearization with respect to x leads to

$$\ddot{x} - \frac{1}{l}(g + a_0\omega^2 \cos\omega t)x = 0.$$

This equation can be transformed to Mathieu's equation

$$\ddot{x} + (\mu + \varepsilon\cos t)x = 0,$$

with $\mu = -(\omega_0/\omega)^2$, $\varepsilon = -a_0/l$, and $\omega_0 = \sqrt{g/l}$, if the time is replaced by the dimensionless time ωt. Thus, we can use the stability chart of Mathieu's equation to study the stability of the vertical position. Since μ is negative, we use the first transition curve lying in the left half-plane of the (μ, ε)-plane described by

$$\mu = -\varepsilon^2/2$$

to find the condition for stability. As $\varepsilon = -0.1$, the vertical position is stable if

$$(\omega_0/\omega)^2 < 0.1^2/2 \quad \Rightarrow \quad \omega > \frac{\sqrt{2}}{0.1}\omega_0.$$

Thus, the vertical position will be stabilized if the frequency of the vibration of the support is at least 14.14 times larger than the eigenfrequency of the pendulum.

EXERCISE 6.3. Apply the variational-asymptotic method to find the asymptotes of the transition curves of Mathieu's equation emanating from the point $\mu = 1$.

Solution. The first step of the variational-asymptotic method yields

$$x(t) = x_0(t) = A\cos t + B\sin t.$$

At the second step the obtained functional for x_1 near $\mu = 1$, as seen from (6.12), does not contains additional resonant terms except the third and the fourth. Thus, $A_{,\eta} = B_{,\eta} = 0$ at this step, and the extremal of (6.12) is easily found to be

$$x_1(t) = \varepsilon(-\frac{1}{2}A + \frac{1}{6}A\cos 2t + \frac{1}{6}B\sin 2t).$$

At the third step we look for $x(t)$ and μ in the form

$$x(t) = x_0(t,\eta) + x_1(t,\eta) + x_2(t,\eta), \quad \mu = 1 + \mu_2,$$

where $\eta = \varepsilon t$ is the slow time and $x_0(t,\eta)$ and $x_1(t,\eta)$ are given by the above formulas with A and B being now functions of the slow time. Function $x_2(t,\eta)$ and μ_2 are assumed to be much smaller than $x_1(\eta,t)$ and 1, respectively. Besides, $x_2(t,\eta)$ is 2π-periodic with respect to t. The time derivative of $x(t)$ becomes

$$\dot{x} = x_{0,t} + \varepsilon x_{0,\eta} + x_{1,t} + \varepsilon x_{1,\eta} + x_{2,t} + \varepsilon x_{2,\eta}.$$

We substitute x and $\dot{x}$ into (6.9) and keep the asymptotically principal terms containing x_2 and the principal cross terms. The functional becomes

$$\begin{aligned} I_2[x_2(t)] = \int_0^{2\pi} [\frac{1}{2}x_{2,t}^2 + \underline{x_{0,t}x_{2,t}} + \varepsilon x_{0,\eta}x_{2,t} + \underline{\underline{x_{1,t}x_{2,t}}} + \varepsilon x_{1,\eta}x_{2,t} \\ - \frac{1}{2}x_2^2 - \underline{x_0x_2} - \mu_2 x_0 x_2 - \underline{\underline{x_1x_2}} - \underline{\underline{\varepsilon\cos t\, x_0x_2}} - \varepsilon\cos t\, x_1x_2]\, dt. \end{aligned}$$

Integrating the cross terms containing $x_{2,t}$ by parts and taking into account the periodicity of x_2 in t and the equations for x_0 and x_1, we see that the underlined and doubly underlined terms are canceled out. Among the remaining terms only $\varepsilon x_{0,\eta}x_{2,t}$, $-\mu_2 x_0 x_2$ and the last term contributes to the resonant terms. The products $\cos t\cos 2t$ and $\cos t\sin 2t$ in the last term can be transformed into the sum of harmonic functions as follows

$$\cos t\cos 2t = \frac{1}{2}(\cos t + \cos 3t), \quad \cos t\sin 2t = \frac{1}{2}(\sin t + \sin 3t).$$

Requiring that the resonant terms must vanish, we obtain for $A(\eta)$ and $B(\eta)$ the equations

$$-\varepsilon B_{,\eta} + \frac{5}{12}\varepsilon^2 A - \mu_2 A = 0, \quad \varepsilon A_{,\eta} - \frac{1}{12}\varepsilon^2 B - \mu_2 B = 0.$$

From these equations we obtain the resulting equation for A (and the similar for B)

$$\varepsilon^2 A_{,\eta\eta} + (\mu_2 - \frac{5}{12}\varepsilon^2)(\mu_2 + \frac{1}{12}\varepsilon^2)A = 0,$$

which shows that the stability (or instability) is determined by the sign of $(\mu_2 - \frac{5}{12}\varepsilon^2)(\mu_2 + \frac{1}{12}\varepsilon^2)$. Thus, the transition occurs at $\mu_2 = -\frac{1}{12}\varepsilon^2$ and at $\mu_2 = \frac{5}{12}\varepsilon^2$, or, in terms of μ, at

$$\mu = 1 - \frac{1}{12}\varepsilon^2 \quad \text{and at} \quad \mu = 1 + \frac{5}{12}\varepsilon^2,$$

which are in full agreement with the asymptotic formulas obtained from the vanishing Hill's determinants.

EXERCISE 6.4. Consider the damped Mathieu's equation

$$\ddot{x} + \varepsilon c\dot{x} + (\mu + \varepsilon\cos t)x = 0,$$

with ε being a small parameter. Apply the variational-asymptotic method to find the asymptotes of the transition curves near the point $\mu = 1/4$.

Solution. The variational equation corresponding to this damped Mathieu's equation reads

$$\delta\int_0^T [\frac{1}{2}\dot{x}^2 - \frac{1}{2}(\mu + \varepsilon\cos t)x^2]\,dt - \int_0^T \varepsilon c\dot{x}\delta x\,dt = 0.$$

At the first step of the variational asymptotic procedure we put simply $\varepsilon = 0$ to get from this equation

$$\delta\int_0^T (\frac{1}{2}\dot{x}^2 - \frac{1}{2}\mu x^2)\,dt = 0.$$

The periodic extremal (with the period $T = 2\pi/\sqrt{\mu}$) reads

$$x_0(t) = A\cos\sqrt{\mu}t + B\sin\sqrt{\mu}t.$$

Let $\mu = 1/4 + \varepsilon\mu_1$. Taking into account that the coefficients A and B are becoming slightly dependent on time for $\varepsilon \neq 0$, we introduce the slow time $\eta = \varepsilon t$ and seek the corrections to the extremal at the second step in the two-timing fashion

$$x(t) = A(\eta)\cos\frac{1}{2}t + B(\eta)\sin\frac{1}{2}t + x_1(t,\eta),$$

where $x_1(t,\eta)$ is a periodic function of the period T with respect to t and is much smaller than $x_0(t,\eta)$ in the asymptotic sense. Substituting this Ansatz into the variational equation and proceeding similarly as in Section 6.1, we obtain for A and B the following equations

$$A_{,\eta} = -\frac{c}{2}A + (\mu_1 - \frac{1}{2})B, \quad B_{,\eta} = -\frac{c}{2}B - (\mu_1 + \frac{1}{2})A.$$

Thus, the last term in the variational equation contributes additional resonant terms. The last equations are linear equations with constant coefficients which may be solved by assuming a solution in the form $A(\eta) = A_0\exp(\lambda\eta)$, $B(\eta) = B_0\exp(\lambda\eta)$. Nontrivial solutions exist if the following determinant vanishes

$$\begin{vmatrix} -\frac{c}{2}-\lambda & -\frac{1}{2}+\mu_1 \\ -\frac{1}{2}-\mu_1 & -\frac{c}{2}-\lambda \end{vmatrix} = 0.$$

Thus,

$$\lambda = -\frac{c}{2} \pm \sqrt{-\mu_1^2 + \frac{1}{4}}.$$

At the transition curve $\lambda = 0$, so

$$\mu_1 = \pm\frac{\sqrt{1-c^2}}{2}.$$

This gives the following expressions for the transition curves near $\mu = 1/4$:

$$\mu = \frac{1}{4} \pm \varepsilon\frac{\sqrt{1-c^2}}{2}.$$

This formula predicts that for a given value of c there is a minimum value of ε which is required for instability to occur. The tongue, which, for $c = 0$, emanates from the μ-axis, becomes detached from the μ-axis for $c > 0$.

EXERCISE 6.5. Non-linear parametric resonance. Consider the following equation

$$\ddot{x} + \omega^2 x + \varepsilon \cos t\, x^3 = 0,$$

with ε being a small parameter. Apply the variational-asymptotic method to study the behavior of solutions near the frequency $\omega_0 = 1/2$.

Solution. The solution of the above equation is the extremal of the following action functional

$$\int_0^T [\frac{1}{2}\dot{x}^2 - \frac{1}{2}\omega^2 x^2 - \frac{1}{4}\varepsilon \cos t x^4]\, dt.$$

It is convenient to change to the new variable $\tau = \omega_0 t$, in terms of which the action functional takes the form

$$I[x(\tau)] = \int_0^{2\pi} [\frac{1}{2}\omega_0^2 x'^2 - \frac{1}{2}\omega^2 x^2 - \frac{1}{4}\varepsilon \cos\frac{\tau}{\omega_0} x^4]\, d\tau.$$

At the first step of the variational-asymptotic method we put $\varepsilon = 0$ in this functional to obtain $\omega = \omega_0 = 1/2$ and

$$x(\tau) = x_0(\tau) = A\cos\tau + B\sin\tau.$$

At the second step we look for $x(\tau)$ and ω in the form

$$x(\tau) = x_0(\tau,\eta) + x_1(\tau,\eta), \quad \omega = \omega_0 + \omega_1,$$

where $\eta = \varepsilon\tau$ and $x_0(\tau,\eta)$ is given by the previous equation with A and B being now the functions of η. We assume also that x_1 is 2π-periodic in τ. Note that the derivative of x equals

$$x' = x_{0,\tau} + \varepsilon x_{0,\eta} + x_{1,\tau} + \varepsilon x_{1,\eta}.$$

We substitute these formulas into the action functional and keep the asymptotically principal terms containing x_1 and the principal cross terms between x_0 and x_1. The functional becomes

$$\begin{aligned} I_1[x_1(\tau)] = \int_0^{2\pi} [&\frac{1}{2}\omega_0^2 x_{1,\tau}^2 + \underline{\omega_0^2 x_{0,\tau} x_{1,\tau}} + \omega_0^2 \varepsilon x_{0,\eta} x_{1,\tau} \\ &- \frac{1}{2}\omega_0^2 x_1^2 - \underline{\omega_0^2 x_0 x_1} - 2\omega_0\omega_1 x_0 x_1 - \varepsilon \cos\frac{\tau}{\omega_0} x_0^3 x_1]\, d\tau. \end{aligned}$$

Integrating the cross terms containing $x_{1,\tau}$ by parts and taking into account the 2π-periodicity of x_1 in τ and the equations for x_0, we see that the underlined terms are canceled out. Besides, with the TrigReduce command in *Mathematica* one can show that the last term contributes two resonant terms, namely, $-\frac{1}{2}\varepsilon A^3 \cos\tau x_1$ and $\frac{1}{2}\varepsilon B^3 \sin\tau x_1$. Requiring that the resonant terms must vanish, we obtain for $A(\eta)$ and $B(\eta)$ the equations

$$\frac{1}{4}\varepsilon A_{,\eta} - \omega_1 B + \frac{1}{2}\varepsilon B^3 = 0, \quad \frac{1}{4}\varepsilon B_{,\eta} + \omega_1 A + \frac{1}{2}\varepsilon A^3 = 0.$$

Dividing these equations by $\varepsilon/4$ and introducing $k_1 = 2\omega_1/\varepsilon$, we rewrite them in the form

$$A_{,\eta} - 2k_1 B + 2B^3 = 0, \quad B_{,\eta} + 2k_1 A + 2A^3 = 0.$$

These equations have one fixed point $(0,0)$ which is a stable center, and two other fixed points

$$(0, \sqrt{k_1}), \quad (0, -\sqrt{k_1}), \tag{6.41}$$

provided $k_1 > 0$, or

$$(\sqrt{-k_1}, 0), \quad (-\sqrt{-k_1}, 0), \tag{6.42}$$

provided $k_1 < 0$. It is easy to check that the fixed points lying on the B- or A-axis are saddle points.

EXERCISE 6.6. Solve the slow flow system (6.26) numerically for $\varepsilon = 0.1$, $c = 0$, $\alpha = \hat{f} = 1$ and for two detuning values $k_1 = 0$ and $k_1 = -0.125$, with the initial conditions $A(0) = 1$ and $B(0) = 0$. Plot the curves $a(\tau) = \sqrt{A^2 + B^2}$ together with the numerical solutions shown in Figs. 6.8 and 6.9.

Solution. Remembering that the slow time $\eta = \varepsilon t$, we rewrite equations (6.26) for $c = 0$ in terms of the real time

$$\begin{aligned} A_{,t} &= -\omega_1 B + \frac{3}{8}\alpha\varepsilon B(A^2 + B^2), \\ B_{,t} &= \omega_1 A - \frac{3}{8}\alpha\varepsilon A(A^2 + B^2) + \varepsilon\frac{\hat{f}}{2}. \end{aligned}$$

This system of nonlinear first-order differential equations can be integrated numerically by using the standard command NDSolve in *Mathematica*. Based on this numerical integration function $a(t) = \sqrt{A^2+B^2}$ can then be plotted. For $\varepsilon = 0.1$, $\alpha = \hat{f} = 1$ and $\omega_1 = 0$ the commands are

```
sol = NDSolve[{a'[t] == 3/8 ϵ b[t] (a[t]^2 + b[t]^2),
   b'[t] == -3/8 ϵ a[t] (a[t]^2 + b[t]^2) + ϵ f / 2,
   a[0] == 1, b[0] == 0}, {a, b}, {t, 200}]
Plot[{Evaluate[√(a[t] /. sol)^2 + (b[t] /. sol)^2],
  Evaluate[-√(a[t] /. sol)^2 + (b[t] /. sol)^2]},
 {t, 0, 200}, PlotRange → All, PlotStyle → Black]
```

Here, in accordance with the recommendation of *Mathematica*, the lower case letters for functions are used everywhere. The plot of $a(t) = \sqrt{A^2+B^2}$ based on this numerical integration is shown together with the solution of the forced Duffing's equation in Fig. 6.9.

The case $\omega = 0.9875$ corresponding to the detuning value $k_1 = -0.125$ can be studied in a similar manner. The plot of $a(t) = \sqrt{A^2+B^2}$ based on the numerical integration is shown together with the solution of the forced Duffing's equation in Fig. 6.10. One can see that the solution is purely periodic which corresponds to the constant amplitude $a = \sqrt{A^2+B^2}$.

EXERCISE 6.7. Find the steady-state amplitude versus frequency curve of the forced Duffing's equation with the softening spring ($\alpha < 0$). Discuss the jump phenomenon and the hysteresis loop.

Solution. The plot of the amplitude-frequency curves according to the formula

$$\omega = 1 + \frac{3}{8}\varepsilon\alpha a^2 \pm \frac{1}{2}\varepsilon\sqrt{\frac{\hat{f}^2}{a^2} - c^2}$$

is shown in Fig. 6.23 for $\varepsilon = 0.1$, $\alpha = -1$, $\hat{f} = 1$, and $c = 0.3$. We see that, for negative α the amplitude-frequency curves are bent to the left. There exists $\omega_c < 1$ such that for $\omega < \omega_c$ the amplitude a is a multi-valued function of the frequency. However, if $c > 0$, then a is a multi-valued function of ω only in the range $\omega \in (1 + \frac{3}{8}\varepsilon\alpha(\hat{f}/c)^2, \omega_c)$. Imagine now that we can change the forcing frequency ω so slowly that the steady-state response amplitude a can follow it after a short transient period. Thus, if the forcing frequency is decreased starting from some value larger than ω_c, then the response amplitude follows first the stable upper branch OA up to point A (it can be shown that the middle branch between points A and D contains unstable solutions). After point A no solution of the upper branch is possible, so the amplitude has to jump to the lower branch (this jump is marked by the vertical line

AB) and then follows this stable branch down to point C. If the forcing frequency were now to reverse its course (again quasistatically), then the amplitude would go back along the lower branch CD, after which it jumps to the upper branch (the jump is marked by the vertical line DE), and finally follows this upper curve down to the end point O. This closed loop OABCDEO is called a hysteresis loop for the Duffing's oscillator with the softening spring.

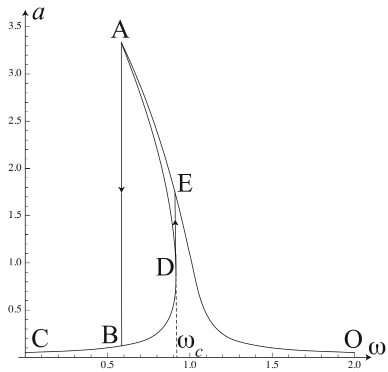

Fig. 6.23 Amplitude-frequency curve and hysteresis

EXERCISE 6.8. Consider the forced oscillator with the quadratic damping described by the equation

$$\ddot{x}+x+\varepsilon c\dot{x}|\dot{x}| = \varepsilon\hat{f}\cos\omega t,$$

where ε is small. Apply the variational-asymptotic method to find the amplitude versus frequency curve near the 1:1 resonant frequency.

Solution. The above equation can be derived from the variational equation

$$\delta\int_0^T (\frac{1}{2}\dot{x}^2-\frac{1}{2}x^2+\varepsilon\hat{f}\cos\omega t\, x)\,dt-\int_0^T \varepsilon c|\dot{x}|\dot{x}\delta x\,dt=0.$$

Introducing $\tau=\omega t$, we rewrite the latter in the form

$$\delta\int_0^{2\pi} (\frac{1}{2}\omega^2x'^2-\frac{1}{2}x^2+\varepsilon\hat{f}\cos\tau x)\,d\tau-\int_0^{2\pi} \varepsilon c\omega^2|x'|x'\delta x\,d\tau=0.$$

At the first step of the variational-asymptotic procedure we put $\varepsilon=0$ which leads to $\omega=1$ and

$$x=A\cos\tau+B\sin\tau.$$

At the second step we introduce the slow time $\eta=\varepsilon\tau$ and look for the solution and correction to the frequency in the form

$$x(\tau)=x_0(\tau,\eta)+x_1(\tau,\eta),\quad \omega=1+\omega_1,$$

where

$$x_0(\tau,\eta) = A(\eta)\cos\tau + B(\eta)\sin\tau,$$

and x_1 and ω_1 are much smaller than x_0 and 1, respectively. Substituting this into the variational equation, one can reduce it to

$$\delta\int_0^{2\pi}[\frac{1}{2}x_{1,\tau}^2 - \frac{1}{2}x_1^2 + (2\varepsilon A_{,\eta} + 2\omega_1 B)\sin\tau x_1 \\ + (-2\varepsilon B_{,\eta} + 2\omega_1 A + \varepsilon\hat{f})\cos\tau x_1 + \text{nonresonant terms}]\,d\tau = 0.$$

Equating the resonant terms to zero, we obtain

$$2A_{,\eta} + 2\frac{\omega_1}{\varepsilon}B = 0,$$
$$2B_{,\eta} - 2\frac{\omega_1}{\varepsilon}A = \hat{f}.$$

The fixed point of this system, $B = 0$ and $A = -\varepsilon\hat{f}/2\omega_1$, corresponds to the steady-state vibration. Thus, the steady-state amplitude is given by

$$a = \sqrt{A^2 + B^2} = \varepsilon\hat{f}/2|\omega_1|.$$

Taking into account that $\omega = 1 + \omega_1$, we get the following amplitude-frequency relation

$$a = \begin{cases} \frac{\varepsilon\hat{f}}{2(1-\omega)} & \text{for } \omega < 1\ , \\ \frac{\varepsilon\hat{f}}{2(\omega-1)} & \text{otherwise.} \end{cases}$$

EXERCISE 6.9. Resonant excitation. Consider the forced Duffing's oscillator described by the equation

$$\ddot{x} + x + \varepsilon c\dot{x} + \varepsilon\alpha x^3 = \hat{f}\cos\omega t,$$

where ε is small, but $\hat{f}$ is finite (sometimes called a "hard excitation"). Apply the variational-asymptotic method to show that, to $O(\varepsilon)$, the only resonant excitation frequencies are 1,3, and 1/3.

Solution. The above differential equation can be derived from the variational equation

$$\delta\int_0^T(\frac{1}{2}\dot{x}^2 - \frac{1}{2}x^2 - \frac{1}{4}\varepsilon\alpha x^4 + \hat{f}\cos\omega t\, x)\,dt - \int_0^T \varepsilon c\dot{x}\delta x\,dt = 0.$$

At the first step of the variational-asymptotic method we put $\varepsilon = 0$ to obtain

$$\delta\int_0^T(\frac{1}{2}\dot{x}^2 - \frac{1}{2}x^2 + \hat{f}\cos\omega t\, x)\,dt = 0.$$

The extremal of this functional satisfies the equation

$$\ddot{x} + x = \hat{f}\cos\omega t$$

yielding

$$x = \frac{\hat{f}}{1-\omega^2}\cos\omega t + A\cos t + B\sin t.$$

We see that the resonance occurs at $\omega = 1$. Consider now the case $\omega \neq 1$. At the second step we introduce the slow time $\eta = \varepsilon t$ and look for the solution in the form

$$x(t) = x_0(t,\eta) + x_1(t,\eta),$$

where

$$x_0(t,\eta) = \frac{\hat{f}}{1-\omega^2}\cos\omega t + A(\eta)\cos t + B(\eta)\sin t,$$

and x_1 is much smaller than x_0 in the asymptotic sense. We assume that x_1 is 2π-periodic with respect to t. We substitute $x(t)$ into the above variational equation for $T = 2\pi$ and keep the asymptotically principal terms containing x_1 and principal cross terms between x_0 and x_1

$$\delta\int_0^{2\pi}(\frac{1}{2}\dot{x}_1^2 + \underline{\dot{x}_0\dot{x}_1} - \frac{1}{2}x_1^2 - \underline{x_0x_1} - \varepsilon\alpha x_0^3x_1 + \underline{\hat{f}\cos\omega t\, x_1} - \varepsilon c\dot{x}_0x_1)\,dt = 0.$$

Integrating the second term by part using the periodicity of x_1, we see that the underlined terms give $2\varepsilon(A_{,\eta}\sin t - B_{,\eta}\cos t)x_1$. We expand the fifth term containing x_0^3 and transform the products of sine and cosine into the sum of harmonic functions. As a result, we get among others the following terms

$$c_1\cos 3\omega t\, x_1 \quad \text{and} \quad [c_3\cos(2-\omega)t + c_4\sin(2-\omega)t]x_1.$$

They become resonant if $\omega = 1/3$ or $\omega = 3$. Thus, we have, in addition to $\omega = 1$, two other resonant excitation frequencies $\omega = 1/3$ and $\omega = 3$.

EXERCISE 6.10. Study the excitation of 3:1 subharmonic resonance in the previous exercise by setting $\omega = 3 + k\varepsilon$. Obtain a slow flow of the coefficients $A(\eta)$ and $B(\eta)$. Then transform to the polar coordinates $a(\eta)$ and $\psi(\eta)$ and look for fixed points of those equations. Eliminate ψ in order to find a relation between a^2 and other parameters. For $\varepsilon = 0.1$, $\alpha = c = \hat{f} = 1$, $k = 0$ simulate the exact and approximate solutions and compare them.

Solution. We continue the solution of the previous exercise by setting $\omega = 3 + k\varepsilon$ and write $x_0(t,\eta)$ in the form

$$x_0(t,\eta) = \lambda\cos(3t + k\eta) + A(\eta)\cos t + B(\eta)\sin t, \quad \lambda = \frac{\hat{f}}{1-\omega^2}.$$

Substituting the Ansatz $x(t) = x_0(t,\eta) + x_1(t,\eta)$ into the variational equation and keeping the asymptotically principal terms containing $x_1(t,\eta)$, we reduce it to

$$\delta\int_0^{2\pi}[\frac{1}{2}x_{1,t}^2 - \frac{1}{2}x_1^2 + (\dots)\sin t\, x_1 + (\dots)\cos t\, x_1 + \text{nonresonant terms}]\,dt = 0.$$

The consistency condition requiring the vanishing resonant terms leads to

$$2A_{,\eta} = \alpha\left[\frac{3}{4}B(A^2+B^2)+\frac{3}{2}B\lambda^2-\frac{3}{4}\lambda(A^2-B^2)\sin k\eta-\frac{3}{2}\lambda AB\cos k\eta\right]-cA,$$
$$2B_{,\eta} = -\alpha\left[\frac{3}{4}A(A^2+B^2)+\frac{3}{2}A\lambda^2+\frac{3}{4}\lambda(A^2-B^2)\cos k\eta-\frac{3}{2}\lambda AB\sin k\eta\right]-cB.$$

The obtained system of equations can be simplified by using polar coordinates a and ψ according to

$$A = a\cos\psi, \quad B = a\sin\psi.$$

Multiplying the first equation by A, the second one by B, and adding them, we obtain

$$a_{,\eta} = -\frac{3}{8}\alpha\lambda a^2\sin(3\psi+k\eta)-\frac{c}{2}a.$$

Multiplying the first equation by B and subtracting it from the second multiplied by A, we get, after some algebra,

$$\psi_{,\eta} = -\frac{3}{4}\alpha(\lambda^2+\frac{1}{2}a^2)-\frac{3}{8}\alpha\lambda a\cos(3\psi+k\eta).$$

To transform this system of equations into an autonomous system we introduce a new unknown function $\varphi = 3\psi+k\eta$ and write

$$a_{,\eta} = -\frac{3}{8}\alpha\lambda a^2\sin\varphi-\frac{c}{2}a,$$
$$\varphi_{,\eta} = k-\frac{9}{4}\alpha(\lambda^2+\frac{1}{2}a^2)-\frac{9}{8}\alpha\lambda a\cos\varphi.$$

In terms of a and φ the approximate solution of the original equation, to the order $O(\varepsilon)$, reads

$$x(t) = \lambda\cos(3t+k\varepsilon t)+a(\varepsilon t)\cos[t-\frac{1}{3}(\varphi-k\varepsilon t)].$$

The steady-state vibrations due to the second term correspond to the fixed points of the slow flow system for which

$$\frac{c}{2}a = -\frac{3}{8}\alpha\lambda a^2\sin\varphi,$$
$$(k-\frac{9}{4}\alpha\lambda^2)a-\frac{9}{8}\alpha a^3 = \frac{9}{8}\alpha\lambda a^2\cos\varphi.$$

Eliminating φ from this system, we obtain the frequency-amplitude equation

$$\left[\frac{9}{4}c^2+\left(k-\frac{9}{4}\alpha\lambda^2-\frac{9}{8}\alpha a^2\right)^2\right]a^2 = \frac{81}{64}\alpha^2\lambda^2a^4.$$

Thus, either $a = 0$ or

$$\frac{9}{4}c^2 + \left(k - \frac{9}{4}\alpha\lambda^2 - \frac{9}{8}\alpha a^2\right)^2 = \frac{81}{64}\alpha^2\lambda^2 a^2,$$

which is quadratic in a^2. Its roots are

$$a^2 = p \pm \sqrt{p^2 - q},$$

where

$$p = \frac{8}{9}\frac{k}{\alpha} - \frac{3}{2}\lambda^2, \quad q = \frac{64}{81\alpha^2}\left[\frac{9}{4}c^2 + \left(k - \frac{9}{4}\alpha\lambda^2\right)^2\right].$$

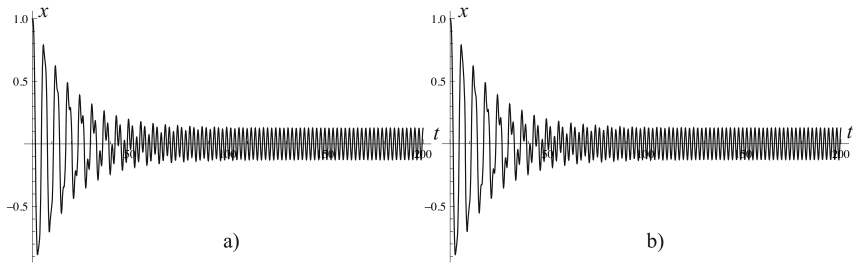

Fig. 6.24 Simulation of the exact and approximate solution: a) exact solution of the forced Duffing's equation with hard excitation, b) approximate solution based on the slow flow system

Fig. 6.24 represents the results of numerical simulation of the exact and approximate solutions for $\varepsilon = 0.1$, $\alpha = c = \hat{f} = 1$, $k = 0$ which shows a good agreement. One can see that the steady-state amplitude a goes to zero as t goes to infinity in this case. Note that q is always positive, and thus, non-trivial steady-state free-oscillation amplitudes occur when $p > 0$ and $p^2 \geq q$.

EXERCISE 6.11. Solve the slow flow system (6.37) numerically for $\varepsilon = 0.1$, $k_1 = 0.2$ and $k_1 = 0.5$, with the initial conditions $a(0) = 1$ and $\psi(0) = 0$. Plot the curves $a(\tau)$ together with the numerical solutions shown in Figs. 6.17 and 6.18.

Solution. The slow flow system

$$a_{,\eta} = \frac{a}{8}(4 - a^2) + \frac{\hat{f}}{2}\sin\psi,$$
$$\psi_{,\eta} = k_1 + \frac{\hat{f}}{2a}\cos\psi,$$

can be solved numerically in *Mathematica* by the following commands

```
sol = NDSolve[{a'[t] == ε a[t]/8 (4 - a[t]^2) + f/2 ε Sin[ψ[t]],
   ψ'[t] == 0.2 ε + f/(2 a[t]) ε Cos[ψ[t]], a[0] == 1, ψ[0] == 0}, {a, ψ}, {t, 500}]
Plot[{Evaluate[a[t] /. sol], Evaluate[-a[t] /. sol]}, {t, 0, 500}]
```

Since we want to plot a as function of the real time t, we use the relation $d/d\eta = (1/\varepsilon)d/dt$ and solve the above system multiplied by ε. The values of parameters have been chosen as $\varepsilon = 0.1$, $f = 1.06$, and $k_1 = 0.2$ ($k_1 = 0.5$). The plotted curves were shown together with the corresponding numerical solutions of equation (6.32) in Figs. 6.17 and 6.18.

EXERCISE 6.12. Resonant excitation. Consider the forced van der Pol's oscillator described by the equation

$$\ddot{x} + x - \varepsilon(1-x^2)\dot{x} = \hat{f}\cos\omega t,$$

where ε is small, but $\hat{f}$ is finite. Apply the variational-asymptotic method to show that to $O(\varepsilon)$, the only resonant excitation frequencies are 1,3, and 1/3.

Solution. The above differential equation can be derived from the variational equation

$$\delta\int_0^T (\frac{1}{2}\dot{x}^2 - \frac{1}{2}x^2 + \hat{f}\cos\omega t\, x)\, dt + \int_0^T \varepsilon(1-x^2)\dot{x}\delta x\, dt = 0.$$

At the first step of the variational-asymptotic method we put $\varepsilon = 0$ to obtain

$$\delta\int_0^T (\frac{1}{2}\dot{x}^2 - \frac{1}{2}x^2 + \hat{f}\cos\omega t\, x)\, dt = 0.$$

The extremal of this functional reads

$$x = \frac{\hat{f}}{1-\omega^2}\cos\omega t + A\cos t + B\sin t.$$

We see that the resonance occurs at $\omega = 1$. Consider now the case $\omega \neq 1$. At the second step we introduce the slow time $\eta = \varepsilon t$ and look for the solution in the form

$$x(t) = x_0(t,\eta) + x_1(t,\eta),$$

where

$$x_0(t,\eta) = \frac{\hat{f}}{1-\omega^2}\cos\omega t + A(\eta)\cos t + B(\eta)\sin t,$$

and x_1 is much smaller than x_0 in the asymptotic sense. We assume that x_1 is 2π-periodic with respect to t. We substitute $x(t)$ into the above variational equation for $T = 2\pi$ and keep the asymptotically principal terms containing x_1 and the principal cross terms between x_0 and x_1

$$\delta \int_0^{2\pi} [\frac{1}{2}\dot{x}_1^2 + \underline{\dot{x}_0 \dot{x}_1} - \frac{1}{2}x_1^2 - \underline{x_0 x_1} + \underline{\hat{f}\cos\omega t\, x_1} + \varepsilon(1 - x_0^2)\dot{x}_0 x_1]\, dt = 0.$$

Integrating the second term by parts using the periodicity of x_1, we see that the underlined terms give $2\varepsilon(A_{,\eta}\sin t - B_{,\eta}\cos t)x_1$. Expanding the last term and transforming products of sine and cosine into the harmonics, we obtain among others the following terms

$$(c_1 \cos 3\omega t + c_2 \sin 3\omega t)x_1 \text{ and } [c_3 \cos(2-\omega)t + c_4 \sin(2-\omega)t]x_1.$$

They become resonant if $\omega = 1/3$ or $\omega = 3$. Thus, we have, in addition to $\omega = 1$, two other resonant excitation frequencies $\omega = 1/3$ and $\omega = 3$.