Chapter 5
Autonomous Single Oscillator

This chapter studies finite amplitude vibrations of the autonomous mechanical systems having one degree of freedom. The character of solutions depends strongly on the type of the system. The solution methods may range from phase portrait and Lindstedt-Poincaré method for conservative systems up to Bogoliubov-Mitropolsky method for systems with weak dissipation.

5.1 Conservative Oscillator

Differential Equation of Motion. As before, Hamilton's variational principle with the Lagrange function $L(q,\dot{q})$, q and $\dot{q}$ being the generalized coordinate and velocity, is our tool for deriving the equation of motion of conservative systems. However, in contrast to the linear theory, we will see that the kinetic energy may now depend on q as well, and the potential energy is no longer quadratic with respect to q. We consider three simple examples.

EXAMPLE 5.1. Mass-spring oscillator. A point-mass m moves horizontally under the action of a non-linear spring (see Fig. 5.1). Derive the equation of motion for this oscillator.

Like the oscillator considered in example 1.1 the kinetic energy is given by $K = \frac{1}{2}m\dot{x}^2$. Concerning the potential energy of the non-linear spring we first consider the most general case, for which $U(x)$ is an arbitrary smooth function. Then Lagrange's equation reads

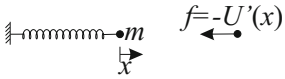

Fig. 5.1 Mass-spring oscillator

$$m\ddot{x} - f(x) = 0, \quad f(x) = -\frac{dU}{dx}.$$

The spring force $f(x)$ is called a restoring force. However, it is quite reasonable to assume that the potential energy of the spring deviates only slightly from that of the linear spring, i.e.,

$$U(x) = \frac{1}{2}kx^2 + \frac{1}{4}\alpha\frac{k}{l_0^2}x^4,$$

where l_0 is the original length of the spring and α a small parameter. If $\alpha > 0$, the spring is called hardening; on the contrary if $\alpha < 0$ it is called softening. Lagrange's equation becomes

$$m\ddot{x} + kx + \alpha\frac{k}{l_0^2}x^3 = 0.$$

Dividing this equation by kl_0 and rewriting it in terms of the dimensionless function $\bar{x} = x/l_0$ and the dimensionless time $\bar{t} = \sqrt{k/m}t$, we obtain[1]

$$\ddot{x} + x + \alpha x^3 = 0. \tag{5.1}$$

Equation (5.1) is known as Duffing's equation.

EXAMPLE 5.2. Derive the equation of motion of the mathematical pendulum considered in example 1.2.

As has been shown already in that example, the Lagrange function is

$$L(\varphi, \dot{\varphi}) = \frac{1}{2}ml^2\dot{\varphi}^2 - mgl(1 - \cos\varphi),$$

but now φ is no longer small. Thus, the finite amplitude vibrations of this pendulum are described by the equation

$$\ddot{\varphi} + \omega_0^2 \sin\varphi = 0, \quad \omega_0 = \sqrt{\frac{g}{l}}.$$

By expanding $\sin\varphi$ in the Taylor series about $\varphi = 0$ and keeping the terms up to φ^3 we obtain the approximate equation

$$\ddot{\varphi} + \omega_0^2(\varphi - \frac{\varphi^3}{6}) = 0,$$

which can be transformed to (5.1) with $\alpha = -1/6$.

EXAMPLE 5.3. A point-mass m is constrained to move along a frictionless path represented by a smooth curve $y = y(x)$ in the (x, y)-plane under the action of gravity (see Fig. 5.2). Derive the equation of motion.

[1] The bar is dropped for short.

This is the typical example of systems with holonomic constraints. Any holonomic constraint like that of the curve $y = y(x)$ can be realized by a strong potential energy $U(x,y)$ which forces the point-mass to move along the path. In the limit when the potential energy goes to infinity in the neighborhood of the path, one gets the Lagrange function evaluated under this constraint [5]. Using in our example x (or, equivalently, the arc-length s along the curve) as the coordinate of the point-mass, we find its constrained velocity along the path

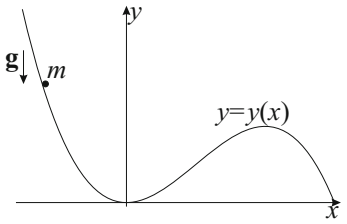

Fig. 5.2 Motion of point-mass along a path

$$v = \dot{s} = \sqrt{1+y'^2}\dot{x}.$$

Thus, the kinetic energy of the point-mass equals

$$K(x,\dot{x}) = \frac{1}{2}mv^2 = \frac{1}{2}m(\sqrt{1+y'^2}\dot{x})^2.$$

Observe that the kinetic energy depends not only on $\dot{x}$, but also on x through the function $y(x)$. Choosing the zero level at $y = 0$, the potential energy is given by

$$U(x) = mgh = mgy(x).$$

Therefore, Lagrange's equation yields

$$\frac{d}{dt}(m\sqrt{1+y'^2}\dot{x}) + mg\frac{y'}{\sqrt{1+y'^2}} = 0.$$

Phase Portrait. As we know from Section 2.4, for conservative oscillators the total energy remains constant during the motion

$$K(x,\dot{x}) + U(x) = E_0.$$

This first integral describes the level curves (phase curves) in the phase plane (x,y), where $y = \dot{x}$. Consider for instance example 5.1 for which

$$\frac{1}{2}m\dot{x}^2 + U(x) = E_0.$$

Solving this equation with respect to $\dot{x}$, we find explicitly

$$\dot{x} = \pm\sqrt{\frac{2}{m}}\sqrt{E_0 - U(x)}. \tag{5.2}$$

The plus or minus sign depends on whether we are in the upper half or lower half of the phase plane.

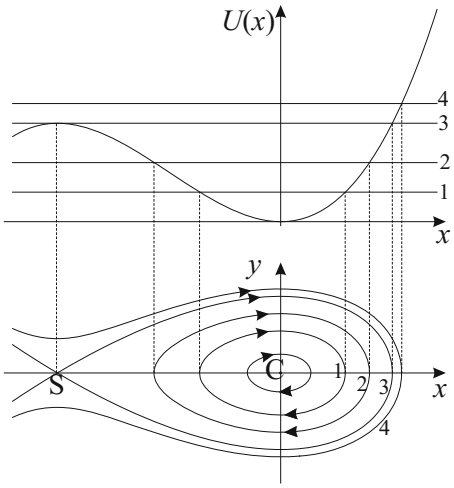

Fig. 5.3 Potential energy and phase portrait of conservative oscillator

Fig. 5.3 shows in its upper part a prototype potential energy as function of x, while in the lower part, with exactly the same x-scale, the corresponding phase portrait. From (5.2) we see that the phase portrait is symmetric with respect to the x-axis and that the phase curves must run from left to right in the upper half-plane and from right to left in the lower half-plane as time increases. The horizontal lines 1,2,3, and 4 in the upper graph label different energy levels E_0 of the oscillator for different types of motions. Since the kinetic energy is non-negative, the potential energy of a particular motion must lie below the corresponding energy level. The intersection points of any E_0-line with the potential energy correspond to the intersection points of the phase curve with the x-axis. For levels 1,2 the phase curves are closed orbits which look like ellipses intersecting the x-axis in two turning points at right angles. These closed orbits describe periodic vibrations of the point-mass about the equilibrium position C. The latter corresponds to the local minimum of the potential energy, so C is the stable center. For level 3 the phase curve is quite special. This curve passes through a saddle point S (corresponding to the local maximum of the potential energy), and consists of four branches, called separatrices[2] which do not intersect the x-axis at right angles. In our case the separatrices separate closed orbits from open phase curves like that of level 4, which describe aperiodic motions of the point-mass. The motion along any separatrix requires infinite amount of time to reach the unstable equilibrium position S. Such motions are called limit motions.

Using equation (5.2), we can now compute the time required to go from the initial point x_0 to point x along a fixed phase curve

[2] The given name originates from the fact that these branches separate regions filled with phase curves of different types.

$$t = t_0 \pm \int_{x_0}^{x} \frac{d\xi}{\sqrt{\frac{2}{m}[E_0 - U(\xi)]}}.$$

Again, the plus or minus sign depends on whether we are in the upper half or lower half of the phase plane. Taking into account the symmetry with respect to the x-axis, we obtain the period of vibration along any closed orbit

$$T = 2\int_{x_m}^{x_M} \frac{d\xi}{\sqrt{\frac{2}{m}[E_0 - U(\xi)]}}, \tag{5.3}$$

where x_m and x_M are the minimum and maximum of x corresponding to the turning points. We see that the period of vibration (and therefore the related frequency) in the nonlinear theory depends on the initial energy, or, in other words, on the amplitude of vibration, in contrast to the linear theory.

Variational-Asymptotic Method. If the action functional contains some small parameter in the nonlinear term, then it is possible to find the correction to the solution and to the frequency without computing complicated integral (5.3). Let us consider for instance Duffing's equation (5.1) which can be obtained as Lagrange's equation of the functional

$$I[x(t)] = \int_0^T (\frac{1}{2}\dot{x}^2 - \frac{1}{2}x^2 - \frac{1}{4}\varepsilon x^4)\, dt,$$

with T being the period of vibration. We assume simply $\alpha = \varepsilon$ as a small parameter. We know that the extremal of this functional depends on ε. On the other hand, the results of the previous paragraph show that the period (and the related frequency $\omega = 2\pi/T$) of vibration depends on the amplitude, and thus, on ε too. We want to make ω enter the action functional explicitly by stretching the time $\tau = \omega t$ so that the functional now takes the form

$$I[x(\tau)] = \frac{1}{\omega}\int_0^{2\pi} (\frac{1}{2}\omega^2 x'^2 - \frac{1}{2}x^2 - \frac{1}{4}\varepsilon x^4)\, d\tau,$$

with prime denoting the derivative with respect to τ. Since the constant factor $1/\omega$ does not influence the extremal, instead of the obtained functional we consider the following one

$$I[x(\tau)] = \int_0^{2\pi} (\frac{1}{2}\omega^2 x'^2 - \frac{1}{2}x^2 - \frac{1}{4}\varepsilon x^4)\, d\tau. \tag{5.4}$$

We try to find the *periodic* extremal of this functional. As $x(\tau)$ is periodic with respect to τ with the period 2π, we call τ phase (or *angular* time). Since the functional contains a small parameter ε, we shall use the variational-asymptotic method to study this variational problem (see [8, 31]). At the first step we put simply $\varepsilon = 0$ to get from (5.4)

$$I_0[x(\tau)] = \int_0^{2\pi} (\frac{1}{2}\omega^2 x'^2 - \frac{1}{2}x^2)\, d\tau.$$

As we know from the linear theory, the 2π-periodic extremal of this functional is

$$x_0(\tau) = a\cos\tau. \tag{5.5}$$

Here a is the amplitude of vibration, the frequency ω is equal to 1 as expected, and we have chosen the initial phase $\phi = 0$ which is possible because functional (5.4) does not depend explicitly on time.

At the second step we seek the periodic extremal and the corresponding frequency in the form

$$x(\tau) = x_0(\tau) + x_1(\tau), \quad \omega = 1 + \omega_1, \tag{5.6}$$

where $x_1(\tau)$ is smaller than $x_0(\tau)$ in the asymptotic sense and $\omega_1 \ll 1$. We may assume that $x_1(\tau)$ and ω_1 are of the order ε of smallness although this is even not necessary. The order of smallness of $x_1(\tau)$ and ω_1 will automatically be determined in this step. Substituting (5.6) into (5.4) and keeping the asymptotically principal terms containing x_1 and the principal cross terms between x_0 and x_1, we obtain[3]

$$I_1[x_1(\tau)] = \int_0^{2\pi} (\frac{1}{2}x_1'^2 + \underline{x_0'x_1'} + 2\omega_1 x_0'x_1' - \frac{1}{2}x_1^2 - \underline{x_0x_1} - \varepsilon x_0^3 x_1)\, d\tau.$$

Integrating the second and the third terms by parts taking into account the periodicity of $x_1(\tau)$, we see that the underlined terms are canceled out. Besides, the cubic of $x_0 = a\cos\tau$ can be transformed into the sum of harmonic cosine functions like that

$$x_0^3 = a^3\cos^3\tau = a^3(\frac{3}{4}\cos\tau + \frac{1}{4}\cos 3\tau).$$

Finally we have

$$I_1[x_1(\tau)] = \int_0^{2\pi} (\frac{1}{2}x_1'^2 - \frac{1}{2}x_1^2 + \underline{(2\omega_1 a - \varepsilon\frac{3}{4}a^3)\cos\tau x_1} - \frac{1}{4}\varepsilon a^3\cos 3\tau x_1)\, d\tau.$$

This functional is reminiscent of that of forced linear oscillator, where the two last terms play the role of the work done by the external forces. The underlined term would lead then to resonance causing non-periodic x_1 with the amplitude tending to infinity as $\tau \to \infty$. However, it is obvious that such resonance cannot appear! Thus, for the consistency of our asymptotic expansion we require the coefficient of $\cos\tau$ in the functional I_1 to vanish.[4] This consistency condition implies

$$2\omega_1 a - \varepsilon\frac{3}{4}a^3 = 0, \quad \text{that is,} \quad \omega_1 = \varepsilon\frac{3}{8}a^2. \tag{5.7}$$

Substituting the result into $(5.6)_2$, we get the correction for the frequency-amplitude relation

$$\omega = 1 + \varepsilon\frac{3}{8}a^2 + O(\varepsilon^2). \tag{5.8}$$

[3] The terms containing only x_0 are dropped because x_0 is not subject to variation at this step.

[4] Allowing some strong expression, we would say that the resonant or secular terms must be "killed".

The period $T = 2\pi/\omega$ may then be written as

$$T = \frac{2\pi}{1+\varepsilon\frac{3}{8}a^2+O(\varepsilon^2)} = 2\pi\left[1-\varepsilon\frac{3}{8}a^2+O(\varepsilon^2)\right]. \tag{5.9}$$

With the underlined resonant term being "killed" we find the extremal of functional I_1

$$x_1(\tau) = \varepsilon\frac{a^3}{32}(\cos 3\tau - \cos\tau). \tag{5.10}$$

Here we have chosen the initial condition such that $x(0) = a$ which is consistent with our choice $\phi = 0$.

Then at the next step we seek the corrections to the extremal and the frequency in the form

$$x(\tau) = x_0(\tau) + x_1(\tau) + x_2(\tau), \quad \omega = 1 + \omega_1 + \omega_2,$$

where $x_2(\tau)$ and ω_2 are smaller than $x_1(\tau)$ and ω_1 in the asymptotic sense, and repeat the same procedure as before (see exercise 5.2).

Notice that the similar procedure applied to the differential equations containing a small parameter has first been proposed by Lindstedt and Poincaré (see [32, 40]).

Comparison with the Exact Solution. It turns out that Duffing's equation can be solved exactly in terms of Jacobian elliptic functions [3]. In this paragraph we want to get the frequency from this exact solution and compare it with the result obtained by the variational-asymptotic method.

First of all, let us collect some well known facts about Jacobian elliptic functions. There are three such functions: sn, cn, and dn. They depend on two variables, u and m, where u is called an argument and $m = k^2$ a modulus. In working with Jacobian elliptic functions the modulus m is often dropped, so we write $\mathrm{sn}(u,m) = \mathrm{sn}(u)$. Two of them, sn and cn, are quite similar to trigonometric sine and cosine. For example there are several identities resembling the well-known trigonometric formulas like

$$\mathrm{sn}^2(u) + \mathrm{cn}^2(u) = 1,$$
$$\mathrm{sn}'(u) = \mathrm{cn}(u)\mathrm{dn}(u), \quad \mathrm{cn}'(u) = -\mathrm{sn}(u)\mathrm{dn}(u),$$

where prime denotes the derivative with respect to u. The elliptic function dn satisfies the equations

$$\mathrm{dn}'(u) = -m\,\mathrm{sn}(u)\mathrm{cn}(u), \quad \text{and} \quad m\,\mathrm{sn}^2(u) + \mathrm{dn}^2(u) = 1.$$

The period of sn and cn in their argument u is $4K$ which is the complete elliptic integral of the first kind. The period of dn is $2K$. The asymptotic expansion of $K(m)$ is given by

$$K(m) = \frac{\pi}{2}\left[1 + \left(\frac{1}{2}\right)^2 m + \left(\frac{1.3}{2.4}\right)^2 m^2 + \ldots\right]. \tag{5.11}$$

We look for the solution of Duffing's equation (5.1) in the form

$$x(t) = a\,\mathrm{cn}(u), \quad \text{where } u = bt + c, \tag{5.12}$$

and a, b, c, and m are still unknown constants. Only two of them may be determined from the initial conditions. Let us fix the initial phase $c = 0$. Thus, there must be two relations for the remaining constants. To find these relations we compute the time derivative of $x(t)$

$$\dot{x} = ab\,\mathrm{cn}'(u) = -ab\,\mathrm{sn}(u)\mathrm{dn}(u).$$

Differentiating once again to get

$$\ddot{x} = -ab^2[\mathrm{cn}(u)\mathrm{dn}^2(u) - m\,\mathrm{sn}^2(u)\mathrm{cn}(u)].$$

Using the above identities, this becomes

$$\ddot{x} = -ab^2\mathrm{cn}(u)[1 - 2m + 2m\,\mathrm{cn}^2(u)].$$

Substituting the last equation into Duffing's equation (5.1) (where $\alpha = \varepsilon$) and equating to zero the coefficients of cn and cn^3 gives two equations relating a, b, and m

$$\begin{aligned} a(2b^2m - b^2 + 1) &= 0, \\ -a(2b^2m - \varepsilon a^2) &= 0. \end{aligned}$$

Solving for b and m in terms of a, we obtain finally

$$b^2 = 1 + \varepsilon a^2, \quad m = \frac{\varepsilon a^2}{2(1 + \varepsilon a^2)}. \tag{5.13}$$

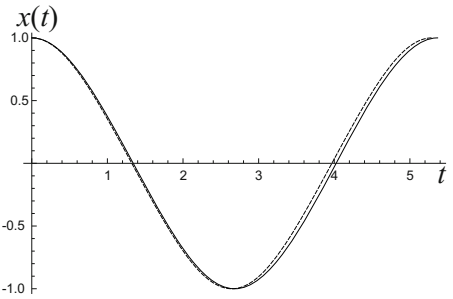

Fig. 5.4 Solution of Duffing's equation for $\varepsilon = 0.5$: i) Bold line: exact solution, ii) Dashed line: approximate solution

Formulas (5.12) and (5.13) give the exact solution of Duffing's equation. Its amplitude a corresponds to the amplitude of the approximate solution (5.5) and (5.10). The period T of the exact solution is $4K/b$ which may be written, using asymptotic formula (5.11),

$$T = \frac{4K(m)}{b} = \frac{2\pi}{b}\left[1 + \frac{1}{4}m + \frac{9}{64}m^2 + O(m^3)\right].$$

Substituting (5.13) into this equation and expanding for small ε, we obtain

$$T = 2\pi\left[1 - \varepsilon\frac{3}{8}a^2 + O(\varepsilon^2)\right]$$

which agrees with formula (5.9). Fig. 5.4 shows the comparison between the exact solution and the approximate one found in the previous paragraph for $\varepsilon = 0.5$. One can find for example in [6,20] the rigorous mathematical proof of convergence of the approximate solution to the exact one as $\varepsilon \to 0$ in any *finite* time interval. However, it is intuitively clear that for any small but finite ε the errors in period and in solution accumulate with the time and become of the order 1 for the time greater than T/ε.

5.2 Dissipative Oscillator

Differential Equation of Motion. For dissipative oscillators there are three types of nonlinearity: i) Non-quadratic energy and quadratic dissipation, ii) Quadratic energy and non-quadratic dissipation, iii) Both energy and dissipation are non-quadratic. The common feature of all dissipative oscillators is the positive definiteness of the dissipation causing the decrease of the energy. Therefore periodic motions in autonomous dissipative systems are clearly impossible. We consider three examples.

EXAMPLE 5.4. Mathematical pendulum with viscous damping. Derive the equation of motion of the mathematical pendulum considered in example 1.2 taking into account the air resistance through viscous damping.

As before the Lagrange function is given by

$$L(\varphi, \dot{\varphi}) = \frac{1}{2}ml^2\dot{\varphi}^2 - mgl(1 - \cos\varphi).$$

For the viscous damping we may assume that the dissipation function is quadratic with respect to the velocity $v = l\dot{\varphi}$

$$D = \frac{1}{2}c(l\dot{\varphi})^2.$$

Thus, generalized Lagrange's equation (2.31) yields

$$\ddot{\varphi} + \omega_0^2 \sin\varphi + \frac{c}{m}\dot{\varphi} = 0. \tag{5.14}$$

This pendulum belongs to the first type of dissipative oscillator with the nonlinear restoring force and the linear damping force.

EXAMPLE 5.5. Mass-spring oscillator with Coulomb's friction. A mass m moves on the rough solid foundation under the action of a linear spring (see Fig. 5.5). Derive the equation of motion for this oscillator.

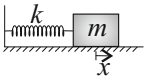

Fig. 5.5 Dry friction

Up to now we have analyzed dissipative oscillators with quadratic dissipation leading to the velocity proportional damping force. However, we are often confronted in reality with another type of damping, namely with the friction between solids with rough and unlubricated surfaces, called Coulomb's (or "dry") friction. The most important features of Coulomb's friction are the existence of a threshold value f_0 for the zero velocity and the constant friction force for nonzero velocities. The force-velocity diagram for Coulomb's friction is shown schematically in Fig. 5.6. We see that this "constant" friction force is constant in magnitude but not in direction since its direction is always opposite to the direction of velocity.

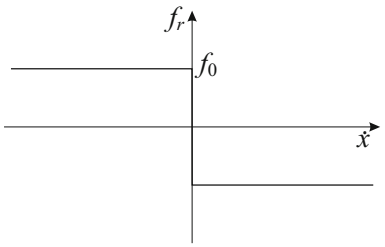

Fig. 5.6 Coulomb's friction force

Looking at the force-velocity diagram we find that Coulomb's friction force can be described by the equation

$$f_r(\dot{x}) = \begin{cases} f_0 & \text{for } \dot{x} < 0, \\ -f_0 & \text{for } \dot{x} > 0. \end{cases}$$

For $\dot{x} = 0$ the friction force may take an arbitrary value in between. Since $f_r = -dD/d\dot{x}$, we have

$$D(\dot{x}) = f_0|\dot{x}|. \tag{5.15}$$

Thus, the dissipation function $D(\dot{x})$ of Coulomb's friction is a positive definite homogeneous function of the first order. Its graph is shown schematically in Fig. 5.7. Mention that $D(\dot{x})$ is non-smooth at $\dot{x} = 0$, but we can still use the constitutive equation $f_r = -dD/d\dot{x}$ for $\dot{x} = 0$ if $dD/d\dot{x}$ is understood in the sense of sub-differential. In this case f_r can take any value between $-f_0$ and f_0.

Now, the equation of motion of this oscillator reads

$$m\ddot{x} = -kx + f_r(\dot{x}). \tag{5.16}$$

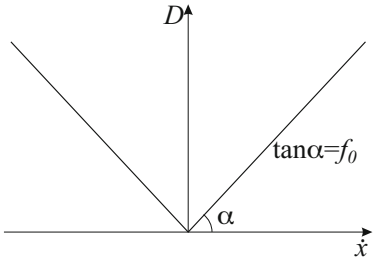

Fig. 5.7 Dissipation function of Coulomb's friction

As the consequence, we see that as long as the magnitude of spring force $|kx|$ is less than f_0, the mass, being released with the zero velocity, cannot move: it is "sticked" to the surface. So we have the "sticky" zone $-f_0/k \le x \le f_0/k$ in which all positions of the oscillator are the equilibrium positions. Released motions are possible only outside of this "sticky" zone. This is the example of oscillators of the second type.

EXAMPLE 5.6. Nonlinear oscillator with a quadratic damping.

If a small mass connected with a non-linear spring moves very fast in a gas or a fluid with a small viscosity, vorticities may occur around it. The resistance from these vorticities on the moving body may sometimes be approximated as proportional to the square of velocity of the point-mass. Such kind of damping is called a "turbulent" damping. Since the damping force acts in the opposite direction to the direction of motion, it must be equal to $f_r = -c|\dot{x}|\dot{x}$. The corresponding dissipation function is

$$D(\dot{x}) = \frac{1}{3}c|\dot{x}|^3.$$

Now the equation of motion reads

$$m\ddot{x} + c|\dot{x}|\dot{x} - f(x) = 0, \quad f(x) = -\frac{dU}{dx}. \tag{5.17}$$

As the spring force $f(x)$ is also non-linear, this oscillator belongs to the third type.

Phase Portrait. Since the energy decreases with time, it is for sure that the amplitude of vibration decays also. There are different methods to determine the evolution to equilibrium for dissipative systems with one degree of freedom. The most general and at the same time most descriptive method remains still that of phase portrait [4]. For all types of dissipative oscillators we may combine the restoring and damping forces in one and present the equation of motion in the form

$$\ddot{x} = f(x, \dot{x}),$$

where $f(x,\dot{x})$ is the resultant force (divided by m) acting on the point-mass. With $y = \dot{x}$ we may reduce this differential equation of second order to the system of equations of first order

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= f(x,y). \end{aligned} \tag{5.18}$$

Thus, at each point (x,y) of the phase plane there is one vector $(y, f(x,y))$ tangent to the phase curve. One can plot this vector field and construct the phase curves by integrating numerically equations (5.18) using for example Euler's or Runge-Kutta's algorithm.

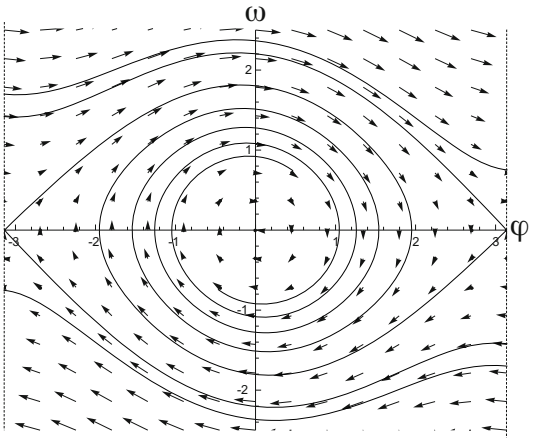

Fig. 5.8 Phase portrait of damped pendulum

Consider for instance the pendulum with viscous damping in example 5.4. The equation of motion (5.14) can be written in the dimensionless form as follows

$$\varphi'' + 2\delta\varphi' + \sin\varphi = 0,$$

where prime denotes the derivative with respect to $\tau = \omega_0 t$, and $\delta = \frac{c}{2m\omega_0}$ is Lehr's damping ratio. Reducing this equation to

$$\begin{aligned} \varphi' &= \omega, \\ \omega' &= -\sin\varphi - 2\delta\omega, \end{aligned}$$

we show the plot of the vector field $(\omega, -\sin\varphi - 2\delta\omega)$ and the phase curves in the phase plane in Fig. 5.8. By wrapping the phase plane onto the cylinder along the lines $\varphi = \pm\pi$ we obtain the phase portrait of the damped pendulum on the cylinder. We see that there are no periodic motions and that almost all phase curves tend to the stable equilibrium position $\varphi = 0$.

Oscillator with Coulomb Friction. For this type of oscillator the solution can directly be found from the energy balance equation. Let us first mention that the energy balance equation (2.32) derived in Section 2.4 should be modified for

Coulomb's friction. Since the dissipation function (5.15) is homogeneous function of the first order, we have merely

$$\frac{dD}{d\dot{x}}\dot{x} = D(\dot{x}).$$

Thus, instead of (2.32) the energy balance equation for the Coulomb's friction reads

$$K + U - E_0 = -\int_{t_0}^{t} D(\dot{x}(s))ds,$$

so that the factor 2 disappears here. With $D(\dot{x})$ from (5.15) we obtain

$$\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = E_0 - f_0|x - x_0|, \tag{5.19}$$

as long as x_0 is found outside of the "sticky" zone, where E_0 is the initial energy at point x_0. It is interesting to note that the dissipation is rate-independent: it depends only on the initial and end coordinates of the point-mass.

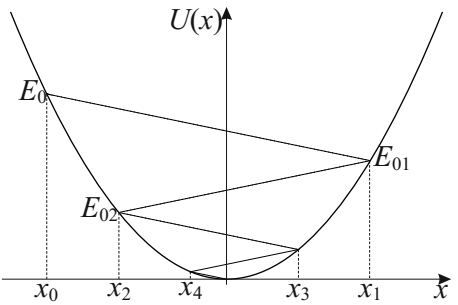

Fig. 5.9 Total energy and turning points of oscillator with Coulomb's friction

Energy balance equation (5.19) gives a clear geometric method for determining the amplitude decay and the turning points of this oscillator. Fig. 5.9 shows the potential energy of the oscillator as well as the total energy during the process of motion. Assume that the point-mass is released from x_0 with the zero initial velocity and then moves to the right. According to (5.19) the total energy at x is $E_0 - f_0(x - x_0)$ since $x > x_0$. This is the straight line with the negative slope $-f_0$ describing the decay rate of the energy. The kinetic energy is the height between the total energy and the potential energy. It becomes zero at the turning point x_1 which is the intersection point between the parabola $U(x)$ and the straight line. Using this point x_1 and the corresponding energy E_{01} as the initial data, we find that the total energy of the motion thereafter must be $E_{01} + f_0(x - x_1)$, since the point mass moves now to the left with $x < x_1$. This is the straight line with the positive slope f_0 which intersects the parabola at the turning point x_2. We can then repeat this geometric construction until $|x_n| < f_0/k$ where the point mass will be sticked there.

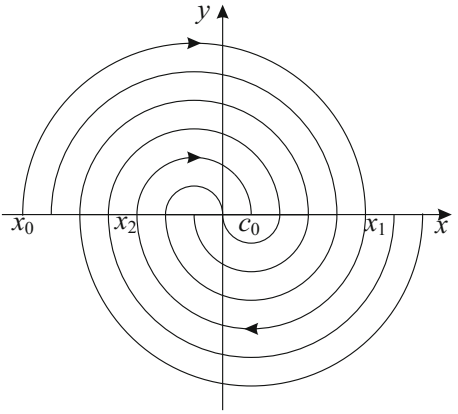

Fig. 5.10 Phase portrait of oscillator with Coulomb's friction

We can also use the energy balance equation (5.19) to plot the phase curves. Indeed, for $\dot{x} > 0$ we have

$$\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = E_0 - f_0(x - x_0).$$

Bringing term $-f_0 x$ to the left-hand side and forming there the square of $x + c_0$ (with $c_0 = f_0/k$), we obtain

$$\frac{1}{2}m\dot{x}^2 + \frac{1}{2}k(x + c_0)^2 = E_0 + f_0 x_0 + \frac{f_0^2}{2k},$$

or with $y = \dot{x}/\omega_0$ (where $\omega_0^2 = k/m$)

$$y^2 + (x + c_0)^2 = r^2, \quad r^2 = \frac{2}{k}(E_0 + f_0 x_0 + \frac{f_0^2}{2k}).$$

We see that the phase curves in the upper half of the phase plane are half-circles with the center at point $-c_0$ on the x-axis. In the lower half-plane they are also half-circles but with the center at point c_0 on the x-axis. The sticky zone lies between these centers. As long as the phase curve does not hit the sticky zone, its continuation in the other half-plane is possible. The sticky zone is the "dead" zone for the phase curves (see Fig. 5.10).

Oscillator with "Turbulent" Damping. The equation of motion for the oscillator considered in example 5.6 can be integrated separately for $\dot{x} > 0$ and $\dot{x} < 0$. Indeed, consider first the case $\dot{x} > 0$ and denote $\dot{x} = v$. Since

$$\ddot{x} = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx} = \frac{1}{2}\frac{dv^2}{dx},$$

we can rewrite equation (5.17) in the form

$$\frac{dv^2}{dx} + \alpha v^2 - \frac{2}{m} f(x) = 0,$$

where $\alpha = 2c/m$. This inhomogeneous differential equation of first order can be integrated by the standard method of variation of coefficients [11] yielding

$$v^2(x) = e^{-\alpha x}\left(C_1 + \frac{2}{m}\int_0^x f(\xi) e^{\alpha\xi}\, d\xi\right).$$

Similarly, for $\dot{x} < 0$ we have

$$v^2(x) = e^{\alpha x}\left(C_2 + \frac{2}{m}\int_0^x f(\xi) e^{-\alpha\xi}\, d\xi\right).$$

The constants C_1 and C_2 are determined from the initial conditions. Let

$$U_+(x) = -\frac{2}{m}\int_0^x f(\xi) e^{\alpha\xi}\, d\xi = \frac{2}{m}\int_0^x U'(\xi) e^{\alpha\xi}\, d\xi,$$
$$U_-(x) = -\frac{2}{m}\int_0^x f(\xi) e^{-\alpha\xi}\, d\xi = \frac{2}{m}\int_0^x U'(\xi) e^{-\alpha\xi}\, d\xi.$$

Assume that the point-mass is released from x_0 with the zero velocity $v_0 = 0$ and that it moves afterward in the positive direction. Then $C_1 = U_+(x_0)$ and

$$v^2(x) = e^{-\alpha x}[U_+(x_0) - U_+(x)], \quad \text{for } \dot{x} > 0.$$

The first turning point x_1 can then be found as the root of the equation $U_+(x_1) = U_+(x_0)$. Choosing now x_1 as the initial coordinate from which the point-mass is released and moves in the negative direction, we find that $C_2 = U_-(x_1)$ and that

$$v^2(x) = e^{\alpha x}[U_-(x_1) - U_-(x)], \quad \text{for } \dot{x} < 0.$$

Therefore, the second turning point, x_2, must be the root of the equation $U_-(x_2) = U_-(x_1)$. Then we can choose x_2 as the initial coordinate from which the point-mass is released and repeat the procedure. So, if functions $U_+(x)$ and $U_-(x)$ are known, then the solution and the turning points can successively be determined.

For illustration let us consider the case of the quadratic potential energy (linear spring) with $U(x) = \frac{1}{2}kx^2$. In this case functions $U_+(x)$ and $U_-(x)$ can easily be computed

$$U_+(x) = \frac{2}{m}\int_0^x k\xi e^{\alpha\xi}\, d\xi = -\frac{2k}{m\alpha^2}[e^{\alpha x}(1-\alpha x) - 1],$$
$$U_-(x) = \frac{2}{m}\int_0^x k\xi e^{-\alpha\xi}\, d\xi = -\frac{2k}{m\alpha^2}[e^{-\alpha x}(1+\alpha x) - 1].$$

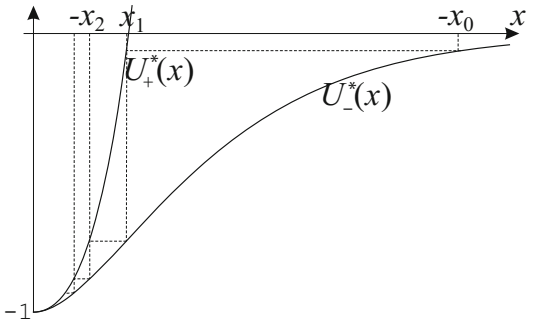

Fig. 5.11 Functions $U_+^*(x)$ and $U_-^*(x)$ and sequence of turning points

The constant factor $\frac{2k}{m\alpha^2}$ and the subtrahend -1 in the square brackets do not obviously influence the determination of the turning points. So, instead of $U_+(x)$ and $U_-(x)$ we can take the following functions

$$U_+^*(x) = -e^{\alpha x}(1-\alpha x) \quad \text{and} \quad U_-^*(x) = -e^{-\alpha x}(1+\alpha x)$$

for this purpose. Besides, $U_+^*(x) = U_-^*(-x)$, so it is enough to plot them for $x > 0$. Fig. 5.11 shows the plot of these functions and the geometric method of determining the sequence of turning points. As function $U_+^*(x)$ cuts the x-axis at point $1/\alpha$, whatever we take for the initial coordinate x_0, the amplitude x_1 is always less than $1/\alpha$.

5.3 Self-excited Oscillator

This Section analyzes self-excited oscillators with one degree of freedom having sustained vibrations. The key features of such oscillators are the presence of an energy source and of a switcher, which switches the energy supply regime to the energy dissipation regime when the amplitude (or velocity) of vibrations becomes large.

Differential Equations of Motion. It was shown in the previous Section that free vibrations of any dissipative system about an equilibrium position decay with time, and in the limit $t \to \infty$ the system approaches equilibrium. Since in reality there are always some sources of small energy dissipation (viscosity, friction, drag force etc.), one might think that permanent vibrations of autonomous mechanical systems are not possible at all. However, the opposite is the case: one can observe everywhere in nature and technique permanent vibrations of living organisms, machines, and constructions. Examples may range from the beating of our hearts to pendulum clocks or flutter of bridges and airplane wings. Let us consider here some simple cases.

EXAMPLE 5.7. Stick-slip oscillator. A mass m connected with a linear spring of stiffness k moves on the rough band of a treadmill that rolls with a constant velocity v_0 (see Fig. 5.12). Derive the equation of motion for this oscillator taking into account Coulomb's friction between the mass and the rolling band. Plot the power of the friction force against the velocity of the mass.

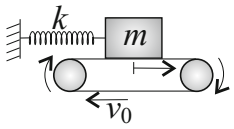

Fig. 5.12 Stick-slip oscillator

This example represents a primitive model of vibrations of a violin string. In terms of the displacement x and velocity $\dot{x}$ the Lagrange function reads

$$L(x,\dot{x}) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2.$$

The dissipation function due to Coulomb's friction between the rolling band and the mass must depend on their *relative* velocity $\dot{x} - v_0$, so

$$D(\dot{x}) = f_0|\dot{x} - v_0|.$$

Thus, generalized Lagrange's equation is

$$m\ddot{x} + kx = f_r(\dot{x} - v_0),$$

where

$$f_r(\dot{x} - v_0) = \begin{cases} f_0 & \text{for } \dot{x} < v_0, \\ -f_0 & \text{for } \dot{x} > v_0. \end{cases}$$

For $\dot{x} = v_0$ the friction force f_r must be equal to the spring force taken with minus sign.

The most interesting property of this oscillator is that the power of the friction force may have both plus and minus sign. Indeed, doing the same calculations as in Section 2.4 for dissipative systems we obtain the balance of energy in the form

$$\frac{d}{dt}(K+U) = -\frac{\partial D(\dot{x} - v_0)}{\partial \dot{x}}\dot{x}.$$

The expression on the right-hand side is the power of the friction force f_r. For the oscillator vibrating about the equilibrium position with $v_0 = 0$ we have shown previously that this is equal to $-D(\dot{x})$, and thus, the energy dissipation rate is positive. In our case, the power may have both signs as one can see in Fig. 5.13.

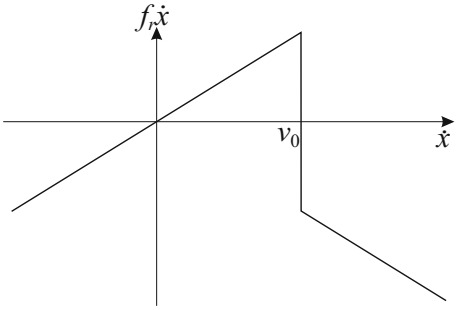

Fig. 5.13 Power of Coulomb's friction force

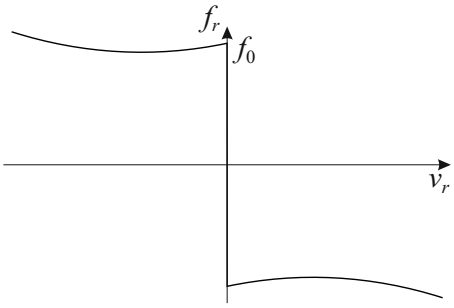

Fig. 5.14 Friction force f_r versus relative velocity $v_r = \dot{x} - v_0$

The positive power of the friction force means that energy is supplied to the oscillator amplifying the vibrations. In contrary, the negative power of the friction force (or, equivalently, the positive dissipation rate) means the energy loss which slows down the vibrations. We see that for our oscillator there is the possibility of amplifying the vibration in the region $0 < \dot{x} < v_0$. This does not still guarantee the self-excitation of small vibrations since we have also the energy loss for $\dot{x} < 0$ plus the air resistance through viscous damping which is always present. It should be mentioned however that the more accurate experiments show a slight dependence of the friction force on the relative velocity as sketched in Fig. 5.14. This, as well as the air resistance through viscous damping in the system may have some influence on the stability of the equilibrium state. We will show later that, under some favorable conditions, the oscillator may develop self-sustained vibrations. From Fig. 5.13 we see also that the two different regimes of energy supply and energy dissipation is switched at the velocities $\dot{x} = 0$ and $\dot{x} = v_0$. Thus, in this case the switcher is velocity sensitive.

EXAMPLE 5.8. Froude's pendulum. A compound pendulum that is rigidly fasten to a sleeve in form of a ring mounted on a shaft rotating with a constant angular velocity v_0 (see Fig. 5.15). Derive the equation of motion of this pendulum taking into account the air resistance as well as Coulomb's friction between the rotating shaft and the sleeve. Plot the power of the friction moment against the angular velocity.

This example is quite similar to the previous one, except for non-quadratic potential energy. The Lagrange function of the pendulum is

$$L(\varphi, \dot{\varphi}) = \frac{1}{2} J \dot{\varphi}^2 - mgl(1 - \cos\varphi),$$

where J is the moment of inertia of the pendulum about the center O of the rotating shaft, and l the distance from the center of mass S to O. The dissipation function includes the dissipation due to the air resistance and the dissipation due to Coulomb's friction between the sleeve of the pendulum and the rotating shaft. The latter must be a function of the *relative* angular velocity. Thus,

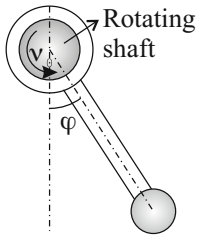

Fig. 5.15 Froude's pendulum

$$D(\dot{\varphi}) = \frac{1}{2} c l^2 \dot{\varphi}^2 + D_c(\dot{\varphi} - \nu_0),$$

and the generalized Lagrange's equation reads

$$J\ddot{\varphi} + mgl\sin\varphi + cl^2\dot{\varphi} - M_r(\dot{\varphi} - \nu_0) = 0.$$

Here $M_r(\dot{\varphi} - \nu_0)$ is the friction moment acting on the pendulum

$$M_r(\dot{\varphi} - \nu_0) = -\frac{\partial D_c(\dot{\varphi} - \nu_0)}{\partial \dot{\varphi}}.$$

If we take $D_c(\dot{\varphi} - \nu_0) = f_0 r|\dot{\varphi} - \nu_0|$ as in the previous case, with r being the radius of the shaft (which is equal to the inner radius of the sleeve), then the plot of the power of the friction moment, $M_r\dot{\varphi}$, is exactly the same as that of the stick-slip oscillator. There exists the energy supply regime for the angular velocity $\dot{\varphi} \in (0, \nu_0)$. This does not still guarantee the self-excitation of small vibrations since we have also the energy loss for $\dot{\varphi} < 0$ plus that due to the air resistance. However, for the more realistic response curve of friction moment versus relative angular velocity similar to that shown in Fig. 5.14, the Froude's pendulum may also develop self-sustained vibrations (see exercise 5.7).

EXAMPLE 5.9. van der Pol's and similar oscillators.

If the mass-spring oscillator is connected with some energy source through a switcher, which switches from the energy dissipation to the energy supply regime when a certain combination of amplitude and velocity is less than 1, then the dissipation function may be proposed for example in the form

$$D(x,\dot{x}) = \frac{1}{2}c(\alpha x^2 + \frac{1}{2}\beta\dot{x}^2 - 1)\dot{x}^2,$$

where c, α, and β are positive constants. This formula does not contradict the second law of thermodynamics as the system under consideration is open and is connected with the energy source (or, equivalently, with an external force producing a positive power).[5] Generalized Lagrange's equation takes the form

$$m\ddot{x} + kx + c(\alpha x^2 + \beta\dot{x}^2 - 1)\dot{x} = 0.$$

If $\beta = 0$, the oscillator is called van der Pol's oscillator.[6] In this case the switcher is amplitude sensitive. On the contrary, for the case $\alpha = 0$ corresponding to Rayleigh's equation, the switcher is velocity sensitive like that of the stick-slip oscillator. In the general case (both α and β are non-zero), the switcher is of the mixed type. Mention that Rayleigh's equation can be transformed to van der Pol's equation as well (see exercise 5.9).

In what follows we shall study mainly van der Pol's oscillator as the prototype of self-excited oscillators. It is therefore convenient to bring its governing equation to the dimensionless form. For this purpose let us introduce the dimensionless quantities $\bar{t} = \sqrt{k/m}t$ and $\bar{x} = \sqrt{\alpha}x$. In their terms van der Pol's equation can be written as follow

$$\ddot{x} + x + \mu(x^2 - 1)\dot{x} = 0, \tag{5.20}$$

where $\mu = c/\sqrt{km}$, and the bar is dropped for short. Introducing $y = \dot{x}$, we can rewrite van der Pol's equation as the system of first order differential equations

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x + \mu(1 - x^2)y, \end{aligned} \tag{5.21}$$

which has one fixed point at $(x,y) = (0,0)$.

Energy Household and the Existence of Limit Cycles. To recognize, whether van der Pol's oscillator has a limit cycle in the phase plane or not, all we need is Poincaré-Bendixson theorem proved in the theory of ordinary differential equations (see [11]). Roughly speaking, this theorem states that if there exists a phase curve C of the 2-D continuous dynamical system that is "confined" to stay in some compact ring-shape region R of the phase plane not containing any fixed point, then either C is a limit cycle, or it spirals toward a limit cycle as t goes to infinity (see Fig. 5.16).

[5] Of course, the dissipation function in this model can no longer be interpreted as the pure dissipative potential leading to the energy loss only.

[6] Historically, this equation was deduced by van der Pol in 1920 to describe the self-excited oscillations of an electrical circuit used in the first radios (see [50]). Later on, this type of equation has been widely used in other physical systems as well, i.e., in laser, plasma, or flutter of airplane wings.

The finding of this "trapping" region for van der Pol's oscillator is based on the analysis of energy change as the angle in the phase plane changes on one period 2π. Indeed, let us write down the energy balance equation for van der Pol's oscillator

$$\frac{1}{2}\frac{d}{dt}(\dot{x}^2+x^2)=\mu(1-x^2)\dot{x}^2. \qquad (5.22)$$

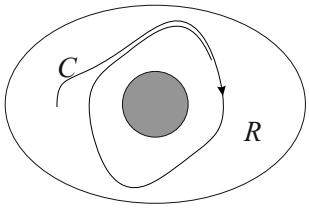

Fig. 5.16 "Trapping" region R

Consider some phase curve starting at point $(x,y)=(a_0,0)$ in the phase plane. As the angle changes on 2π the phase curve cuts the x-axis again at another point $(a_1,0)$. Integrating equation (5.22) over the time interval spent by the phase curve between these two points and taking into account that $y=\dot{x}=0$ at the end-points, we obtain

$$\frac{1}{2}(a_1^2-a_0^2)=\int_{t_0}^{t_1}\mu(1-x^2)\dot{x}^2\,dt=\Delta E. \qquad (5.23)$$

The integral standing on the right-hand side is the energy change ΔE in one angular period 2π. Thus, if the energy is gained in one angular period ($\Delta E>0$), then $|a_1|>|a_0|$. In contrary, if the energy is lost in one angular period ($\Delta E<0$), then $|a_1|<|a_0|$.

Now, let μ be small. Then for the phase curve starting near the origin with small $a_0>0$ we expect that the solution remains small in one angular period. Then $|x(t)|\ll 1$ for $t\in(t_0,t_1)$ and the integrand on the right-hand side of (5.23) is positive. Thus, in one angular period we have the energy gain ($\Delta E>0$), so $a_1>a_0$ and the phase curve must be repelled from the origin. For the phase curve with a large initial amplitude $a_0\gg 1$ the situation is more subtle. Since in one angular period the oscillator may dissipate energy as well as gain it when $x(t)$ comes close to zero, we must compute the energy change precisely. For $\mu=0$ equation (5.20) is the equation of the harmonic oscillator having the solution $x(t)=a_0\cos t$. It is natural to expect that for small μ the solution of (5.20) is close to $a_0\cos t$ in one angular period. Since μ stands also in the integral (5.23), we may use this approximate solution to estimate the energy change in one angular period

$$\Delta E=\mu\int_0^{2\pi}(1-a_0^2\cos^2 t)a_0^2\sin^2 t\,dt=-\mu\frac{\pi}{4}a_0^2(a_0^2-4).$$

Thus, if $a_0>2$ then the energy change ΔE is negative, and the phase curve must be attracted to the origin. So, in the polar coordinates the ring-shape region trapping the phase curve is $r\in(\delta_1,\delta_2)$, with δ_1 a small positive number and $\delta_2>2$. We see also that the amplitude of a limit cycle must be close to 2 for small μ.

Up to now we do not know how many limit cycles van der Pol's oscillator may have. This information can be obtained from Liénard's theorem which is applied to all differential equations of the form

$$\ddot{x} + f(x)\dot{x} + g(x) = 0. \tag{5.24}$$

Equation (5.24) describes the motion of a unit mass subject to a nonlinear damping force $-f(x)\dot{x}$ and a nonlinear restoring force $-g(x)$. The formulation of Liénard's theorem is as follows: If

- $f(x)$ and $g(x)$ are continuously differentiable;
- $g(x)$ is an odd function and $g(x) > 0$ for $x > 0$;
- $f(x)$ is an even function;
- the odd function $F(x) = \int_0^x f(\xi)\,d\xi$ has exactly one positive zero at $x = b$, is negative for $0 < x < b$, is positive and nondecreasing for $x > b$, and $F(x) \to \infty$ as $x \to \infty$;

then equation (5.24) has one stable limit cycle surrounding the origin of the phase plane. Since this limit cycle attracts phase curves to it, it is called an attractor. Now, for van der Pol's oscillator we have

$$g(x) = x, \quad f(x) = \mu(x^2 - 1).$$

Integrating $f(x)$ we obtain $F(x) = \mu x(\frac{1}{3}x^2 - 1)$. Thus, all conditions required in Liénard's theorem are satisfied (for the last condition we have $b = \sqrt{3}$). Consequently, van der Pol's equation has only one stable limit cycle.

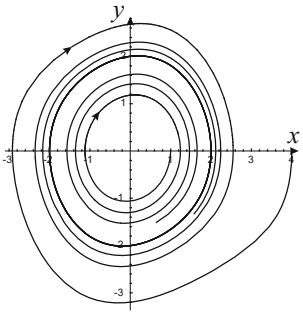

Fig. 5.17 Phase curves and limit cycle of van der Pol's equation with $\mu = 0.1$

Numerical Solutions. Now we know that all phase curves of van der Pol's oscillator are attracted to the limit cycle. But how do they approach this cycle and what does the limit cycle look like? These questions can only be answered by integrating equation (5.20), or equivalently, system (5.21). Unfortunately, analytical solutions are not available. So, let us try to integrate (5.21) numerically by using for instance *Mathematica*. We open a notebook in *Mathematica* and simply write the following commands

```
sol = NDSolve[{x'[t] == y[t], y'[t] == -x[t] + 0.1 (1 - x[t]^2) y[t],
    x[0] == 4, y[0] == 0}, {x, y}, {t, 30}]
ParametricPlot[Evaluate[{x[t], y[t]} /. sol], {t, 0, 30}]
```

In this case we took $\mu = 0.1$ and assumed the initial conditions $x(0) = 4$, $y(0) = 0$. The result of computations is shown in Fig. 5.17, and we see that the phase curve is really attracted to the limit cycle drawn by the thick line which is close in form to the circle of radius 2. If we take a starting point inside the cycle, the phase curve also spirals to the limit cycle from inside.

If we want to know how the solution changes in time, we add a command

```
Plot[Evaluate[x[t] /. sol], {t, 0, 30}]
```

and the computer gives us the curve $x(t)$ shown in Fig. 5.18. The behavior of this solution is quite similar to that of damped oscillator discussed in Section 1.2 except that the amplitude of vibration does not tend to zero but to some value close to 2. As one can observe, there are two characteristic time scales: i) one describing the period of fast oscillation of $x(t)$, ii) the other associated with the monotonic and slow change of amplitude of vibration toward that of the limit cycle shown by the envelopes.

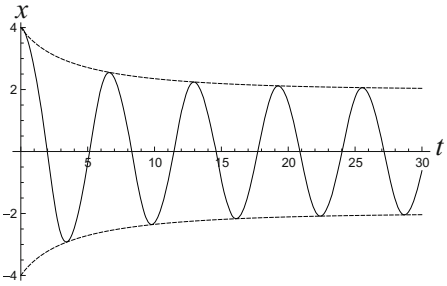

Fig. 5.18 Solution $x(t)$ of van der Pol's equation for $\mu = 0.1$: i) bold line: $x(t)$, ii) dashed lines: envelopes

If we enlarge the parameter μ, the limit cycle deviates more and more from the circle. The motion deviates also from the harmonic motion. For very large μ van der Pol's oscillator exhibits quite interesting type of vibrations called relaxation oscillations. The limit cycle and the corresponding plot of $x(t)$ for $\mu = 10$ are shown in Fig. 5.19. One can see a sequence of slow motions which are quickly switched to other slow motions. This phenomenon will be explained in the next Section.

Limit Cycle of Stick-Slip Oscillator. Poincaré-Bendixson's or Liénard's theorem cannot be applied to oscillators with Coulomb's friction because of the discontinuity of the friction force. So this type of oscillators requires always a special treatment. Let us analyze the stick-slip oscillator considered in example 5.7. Taking also the air resistance into account, we write down the equation of motion

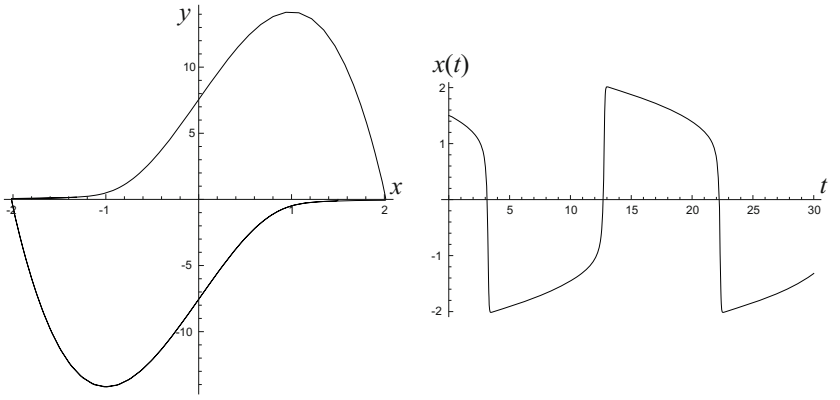

Fig. 5.19 Limit cycle (left) and solution $x(t)$ (right) of van der Pol's equation with $\mu = 10$

$$m\ddot{x} + c\dot{x} + kx = f_r(\dot{x} - v_0).$$

Dividing this equation by k and introducing the notation

$$\frac{k}{m} = \omega_0^2, \quad \frac{c}{2\sqrt{km}} = \delta, \quad \frac{v_0}{\omega_0} = v_0,$$

we rewrite it in the form (compare with (1.15))

$$x'' + 2\delta x' + x = r(x' - v_0),$$

where

$$r(x' - v_0) = \frac{f_r(\omega_0(x' - v_0))}{k},$$

and prime denotes the derivative with respect to $\tau = \omega_0 t$. This second order differential equation is equivalent to the system of equations

$$\begin{aligned} x' &= y, \\ y' &= r(y - v_0) - 2\delta y - x, \end{aligned} \tag{5.25}$$

which has one fixed point S on the x-axis with the coordinate $x_0 = r(-v_0)$. The slope of the phase curve at point (x, y) is equal to

$$\frac{dy}{dx} = \frac{r(y - v_0) - 2\delta y - x}{y}. \tag{5.26}$$

The first interesting thing to know is whether the fixed point S of this dynamical system is a stable equilibrium position or not. To do the stability analysis near the fixed point we seek the neighboring solution of (5.25) in the form

$$x = x_0 + u, \quad y = v,$$

where $u \ll 1$ and $v \ll 1$ and linearize the system (5.25) with respect to u and v. Since

$$r(v - v_0) = r(-v_0) + \left.\frac{dr}{dv}\right|_{-v_0} v + O(v^2),$$

we obtain

$$u' = v, \quad v' = \left(\left.\frac{dr}{dv}\right|_{-v_0} - 2\delta \right) v - u,$$

what is equivalent to the equation

$$u'' - \left(\left.\frac{dr}{dv}\right|_{-v_0} - 2\delta \right) u' + u = 0.$$

Thus, if

$$\left.\frac{dr}{dv}\right|_{-v_0} > 2\delta,$$

then the fixed point is an unstable focus, and the phase curves starting near this fixed point are repelled from it.

To be able to simulate the phase curves numerically let us assume that the dynamic friction force is described by a function $r(v)$ of the form

$$r(v) = \begin{cases} \frac{1}{2} + \frac{1}{2}(v+1)^2 & \text{for } v < 0, \\ -\frac{1}{2} - \frac{1}{2}(v-1)^2 & \text{for } v > 0. \end{cases}$$

The threshold friction force is $r_0 = 1$ in this case, and for $v = 0$ function $r(v)$ can take any value between $-r_0$ and r_0. The constant velocity is $v_0 = 0.5$, while Lehr's damping ratio is chosen to be $\delta = 0.01$. It is easy to check that $r'(-0.5) = 0.5 > 2\delta$.

The vector field and some phase curves of this oscillator are plotted in Fig. 5.20. The phase curves hitting the horizontal line $y = v_0$ must change their slopes abruptly when crossing this line. By this reason the line $y = v_0$ is called a jump line. According to equation (5.26) the jump of the slopes must be equal to $-2r_0/v_0$. Besides, there is a "sticky" zone $-r_0 - 2\delta v_0 < x < r_0 - 2\delta v_0$ on this jump line (the segment AC), where the mass is sticked to the band and move together with it with the constant velocity v_0. When the phase curves hit this sticky zone, they have to move along the horizontal line up to point C with coordinates $(r_0 - 2\delta v_0, v_0)$, where the mass is detached from the band and the slip begins. The phase curve starting from point

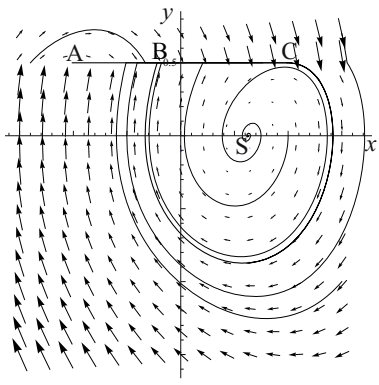

Fig. 5.20 Limit cycle of stick-slip oscillator

C is a spiral, which hit the segment AC at point B if the damping ratio δ is small. This phase curve together with the segment BC correspond to the limit cycle of the stick-slip oscillator. Indeed, the phase curves starting inside this cycle are repelled from S and will hit the jump line at some point between B and C and merge with the limit cycle afterwards. The phase curves starting outside of this cycle will sooner or later hit the segment AC and after a while merge with the limit cycle. The specific feature of the stick-slip oscillator is that the limit cycle is established after a finite time.

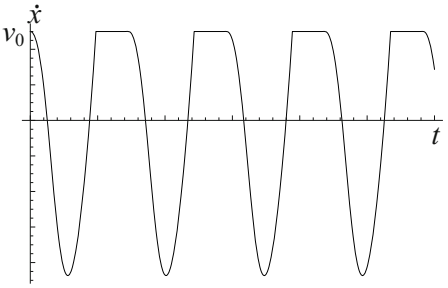

Fig. 5.21 Plot of $\dot{x}(t)$ at the limit cycle

It is interesting to plot the velocity $\dot{x}$ at the limit cycle as function of time. This plot is shown in Fig. 5.21. We can see clearly the sequence of stick and slip regimes, where $\dot{x} = v_0$ in the stick regime.

5.4 Oscillator with Weak or Strong Dissipation

Mathematical Formulation. The results of previous two Sections show that the phase curves of dissipative systems may approach some attractor in the phase plane as time goes to infinity. If the energy dissipation rate is positive definite, then the attractor corresponds just to the equilibrium states. In contrary, if the energy dissipation rate is no longer positive definite, the attractor may become a limit cycle. In both cases the amplitude and phase of vibration change slowly with time. It turns out that if the dissipation function of the system is small in the sense that the governing equation is obtained from the following variational equation

$$\delta \int_{t_0}^{t_1} (\frac{1}{2}\dot{x}^2 - \frac{1}{2}x^2)\, dt - \int_{t_0}^{t_1} \varepsilon \frac{\partial D}{\partial \dot{x}} \delta x\, dt = 0, \tag{5.27}$$

with $\varepsilon D(x,\dot{x})$ being the dissipation function and ε a small parameter, then the evolution of the system to the attractor can be determined analytically in the limit $\varepsilon \to 0$. It is easy to see that the governing equation of this system reads

$$\ddot{x} + x = \varepsilon f(x,\dot{x}),$$

where

$$f(x,\dot{x}) = -\frac{\partial D}{\partial \dot{x}}.$$

We need to find the asymptotic behavior of solution in the limit $\varepsilon \to 0$.

Variational-Asymptotic Method. We have seen from the numerical simulations of van der Pol's oscillator in the previous Section that, if ε is not equal to zero, the amplitude of vibration will change slowly with time due to the energy dissipation. The same can happen also to the period T as well as to the related frequency ω as they are in general amplitude-dependent. Similar to the asymptotic analysis provided for Duffing's equation in Section 5.1 we want to make ω enter the variational equation (5.27) explicitly. For this purpose we multiply (5.27) with ω and rewrite it in terms of the stretched angular time $\tau = \omega t$ for one fixed period 2π

$$\delta \int_{\tau_0}^{\tau_0+2\pi} (\frac{1}{2}\omega^2 x'^2 - \frac{1}{2}x^2)\, d\tau + \int_{\tau_0}^{\tau_0+2\pi} \varepsilon f(x, \omega x')\delta x\, d\tau = 0, \tag{5.28}$$

where prime denotes the derivative with respect to the angular time τ, and τ_0 is an *arbitrary* time instant. For short we set $\tau_0 = 0$.

At the first step of the variational-asymptotic procedure we put simply $\varepsilon = 0$ to get from (5.28)

$$\delta \int_0^{2\pi} (\frac{1}{2}\omega^2 x'^2 - \frac{1}{2}x^2)\, d\tau = 0.$$

This leads to the eigenvalue problem yielding the following 2π-periodic extremal

$$x_0(\tau) = a\cos\tau.$$

Here a is the amplitude of vibration, the frequency ω is equal to 1 as expected, and we have chosen the initial phase $\phi = 0$, which is possible because the governing equation does not depend explicitly on time.

Taking into account that the amplitude a and the frequency ω are becoming slightly dependent on time for $\varepsilon \neq 0$, we introduce the slow time $\eta = \varepsilon\tau$ and seek the corrections to the extremal and to the frequency at the second step in the form[7]

$$x(\tau) = a(\eta)\cos\tau + x_1(\tau,\eta), \quad \omega = 1 + \omega_1(\eta), \tag{5.29}$$

where $x_1(\tau,\eta)$ is a 2π-periodic function with respect to the fast time τ and is much smaller than $x_0(\tau,\eta)$ in the asymptotic sense, and $\omega_1(\eta)$ is much smaller than 1. To make the asymptotic analysis of small terms easier we may assume that x_1 and ω_1 are of the order ε although this is even not necessary. The order of smallness of x_1 and ω_1 will automatically be determined in this step. Since the angular time τ is present also in η, the time derivative of $x(\tau)$ becomes

$$x'(\tau) = -a(\tau)\sin\tau + \varepsilon a_{,\eta}\cos\tau + x_{1,\tau} + \varepsilon x_{1,\eta},$$

[7] This is the crucial idea of two-timing or multi-scaling.

where the comma before an index denotes the partial derivative with respect to the corresponding variable. Let us substitute (5.29) into (5.28) and keep the asymptotically principal terms containing x_1 and the principal cross terms between x_0 and x_1. Because the small parameter ε is already present in the last term of (5.28), it is accurate at this step to replace $f(x,\omega x')$ by $f(a\cos\tau,-a\sin\tau)$. Now the variational equation becomes

$$\delta\int_0^{2\pi}[\frac{1}{2}x_{1,\tau}^2-\underline{a\sin\tau x_{1,\tau}}+\underline{\varepsilon a_{,\eta}\cos\tau x_{1,\tau}}-\underline{2\omega_1 a\sin\tau x_{1,\tau}}$$
$$-\frac{1}{2}x_1^2-a\cos\tau x_1+\varepsilon f(a\cos\tau,-a\sin\tau)x_1]\,d\tau=0.$$

Integrating the underlined terms by parts using the periodicity of x_1 in τ we obtain finally

$$\delta\int_0^{2\pi}[\frac{1}{2}x_{1,\tau}^2-\frac{1}{2}x_1^2+2(\varepsilon a_{,\eta}\sin\tau+\omega_1 a\cos\tau)x_1+\varepsilon f(a\cos\tau,-a\sin\tau)x_1]\,d\tau=0. \tag{5.30}$$

Aside from the negligibly small change of amplitude a in one period we may regard function $f(a\cos\tau,-a\sin\tau)$ as 2π-periodic with respect to τ. Let us expand it in the Fourier series on the interval $(0,2\pi)$

$$f(a\cos\tau,-a\sin\tau)=g_0(a)+\sum_{n=1}^{\infty}[g_n(a)\cos n\tau+h_n(a)\sin n\tau].$$

Substituting this expansion into equation (5.30) we see that there are two resonant terms in this functional, namely

$$\varepsilon[2a_{,\eta}+h_1(a)]\sin\tau x_1 \quad\text{and}\quad [2\omega_1 a+\varepsilon g_1(a)]\cos\tau x_1.$$

From the linear theory we know that such resonant (or secular) terms would lead to nonperiodic x_1 contradicting our asymptotic expansion. To be consistent, we have to remove them. These consistency conditions yield two equations for the amplitude $a(\eta)$ and for the correction of the frequency ω_1

$$\begin{aligned}a_{,\eta}&=-\frac{1}{2}h_1(a)=-\frac{1}{2\pi}\int_0^{2\pi}f(a\cos\tau,-a\sin\tau)\sin\tau\,d\tau,\\ \omega_1&=-\frac{\varepsilon}{2a}g_1(a)=-\frac{\varepsilon}{2\pi a}\int_0^{2\pi}f(a\cos\tau,-a\sin\tau)\cos\tau\,d\tau.\end{aligned} \tag{5.31}$$

Since the change of $a(\eta)$ in one period is of the order ε, we may regard it as "frozen" in the integrals on the right-hand sides.

With the resonant terms being "killed" we can find in principle the extremal $x_1(\tau,\eta)$ in the above variational problem. It has to satisfy the following equation

$$x_{1,\tau\tau}+x_1=\varepsilon\{g_0(a)+\sum_{n=2}^{\infty}[g_n(a)\cos n\tau+h_n(a)\sin n\tau]\}.$$

Then at the next step we seek the corrections to the extremal and to the frequency in the form

$$x(\tau) = x_0(\tau,\eta) + x_1(\tau,\eta) + x_2(\tau,\eta), \quad \omega = 1 + \omega_1(\eta) + \omega_2(\eta),$$

where x_2 and ω_2 are much smaller than x_1 and ω_1 in the asymptotic sense, and repeat the same procedure as before.

Notice that the similar procedure applied to the differential equations containing a small parameter has first been proposed by Bogoliubov and Mitropolsky [9].

Applications. Since the developed variational-asymptotic method does not put any constraint on the dissipation function, we can apply it to both dissipative and self-excited oscillators.

We illustrate how the method works first on the simple example of the linear damped oscillator whose solution is given by formula (1.18). In this case $f(x,\dot{x}) = -\dot{x}$ and $\varepsilon = 2\delta$. Computing the integrals on the right-hand sides of (5.31) we obtain two equations

$$a_{,\eta} = -\frac{a}{2}, \quad \omega_1 = 0.$$

According to the second equation there is no correction to the frequency and the period of vibration within the first approximation. Concerning the amplitude of vibration we obtain the law of its change by integrating the first equation giving $a = a_0 e^{-\eta/2} = a_0 e^{-\delta\tau}$, with a_0 being the initial amplitude. Combining this formula for a with (5.29) we get in the first approximation

$$x(\tau) = a_0 e^{-\delta\tau}\cos\tau.$$

In comparison with the exact solution $x(\tau) = a_0 e^{-\delta\tau}\cos\nu\tau$ (which is obtained from (1.18) when $\phi = 0$), we see only a slight difference in the frequency of vibration: the exact conditional frequency $\nu = \sqrt{1-\delta^2} \approx 1 + O(\varepsilon^2)$. The evolution of the amplitude coincides with that of the exact solution.

Next, let us apply the method to van der Pol's oscillator, for which no analytical solution is available. In this case $f(x,\dot{x}) = (1-x^2)\dot{x}$ and $\varepsilon = \mu$. Similar calculations of integrals in (5.31) give

$$a_{,\eta} = \frac{a}{8}(4-a^2), \quad \omega_1 = 0. \tag{5.32}$$

As in the previous example there is no correction to the frequency and the period of vibration. In contrast to the exact van der Pol equation, equation (5.32) for a can be integrated analytically. Indeed, multiplying this equation with a and noting that $aa_{,\eta} = \frac{1}{2}(a^2)_{,\eta}$ we transform it to the following equation

$$y_{,\eta} = \frac{1}{4}y(4-y),$$

where $y = a^2$. The above equation can be integrated by separating the variables giving

$$\ln \frac{y}{4-y} = \eta + C.$$

The constant of integration can be obtained from the initial condition $y(0) = y_0$: $C = \ln \frac{y_0}{4-y_0}$. Substituting this constant in the last equation and solving it with respect to y we obtain

$$y = \frac{4y_0 e^{\eta}}{4 + y_0(e^{\eta} - 1)},$$

or, in terms of the original amplitude a and time τ

$$a = \frac{2a_0 e^{\varepsilon\tau/2}}{\sqrt{4 + a_0^2(e^{\varepsilon\tau} - 1)}}. \tag{5.33}$$

In order to compare with the numerical solution we plot $a(\tau)$ from (5.33) and show it together with $x(t)$ in Fig. 5.18 (the dashed envelope). The agreement is striking, although $\varepsilon = 0.1$ is not quite small. We can also check that a approaches 2 as $\tau \to \infty$.

Limit Cycle of Relaxation Oscillations. We have seen that for small μ the limit cycle of van der Pol's oscillator is nearly a circle of radius 2, and its frequency is nearly equal to 1. Consider now the opposite case of van der Pol's oscillator with a large parameter μ. As our numerical simulations have shown, the solution corresponding to the limit cycle spends most of time in a slow motion, and then quickly jumps to another slow motion. We analyze this motion by applying the variational-asymptotic method to the variational equation

$$\delta \int_{t_0}^{t_1} (\frac{1}{2}\dot{x}^2 - \frac{1}{2}x^2)\, dt + \int_{t_0}^{t_1} \mu(1 - x^2)\dot{x}\delta x\, dt = 0. \tag{5.34}$$

Let us first concentrate on the slow motion. Introducing the slow time $\eta = t/\mu$, we have $\dot{x} = x_{,\eta}/\mu$, with $x_{,\eta}$ being the derivative with respect to η. Thus, equation (5.34), multiplied by μ, becomes

$$\delta \int_{\eta_0}^{\eta_1} (\frac{1}{2\mu^2}x_{,\eta}^2 - \frac{1}{2}x^2)\, d\eta + \int_{\eta_0}^{\eta_1} (1 - x^2)x_{,\eta}\delta x\, d\eta = 0.$$

Neglecting the first term in this equation[8] as small in accordance with the variational-asymptotic method, we arrive at the equation

$$-x + (1 - x^2)x_{,\eta} = 0, \quad \text{or} \quad x_{,\eta} = \frac{x}{1 - x^2}.$$

[8] This means neglecting the kinetic energy as small compared with the potential energy and dissipation. Thus, the slow motion can be regarded as the motion without inertia.

This differential equation can be solved by separation of variables yielding

$$\ln|x| - \frac{x^2}{2} = \eta + \text{const.}$$

The slow motion proceeds according to this equation until it reaches $x = \pm 1$ where the speed $x_{,\eta}$ is infinite. At this point the assumption of slow motion is violated, so we need to change to another time scale. Introducing now the fast time $\tau = \mu t$, $\dot{x} = \mu x_{,\tau}$, we rewrite (5.34), divided by μ, in the form

$$\delta \int_{\tau_0}^{\tau_1} (\frac{1}{2}\mu^2 x_{,\tau}^2 - \frac{1}{2}x^2)\, d\tau + \int_{\tau_0}^{\tau_1} \mu^2 (1-x^2) x_{,\tau} \delta x\, d\tau = 0.$$

Neglecting the second term as small compared with other terms, we arrive at the differential equation

$$x_{,\tau\tau} - (1-x^2)x_{,\tau} = 0.$$

This equation possesses the first integral

$$x_{,\tau} - x + x^3/3 = \text{const},$$

representing the fast motion (jump). We choose the constant of integration so that this fast motion starting at $x = 1$ as an equilibrium point (with $x_{,\tau} = 0$) will end at another equilibrium point with $x_{,\tau} = 0$. It is easy to see that the constant is equal to $-2/3$ giving the second equilibrium point at $x = -2$. Similarly, the jump starting at $x = -1$ ends up at $x = 2$.

Knowing the solution, we can now easily compute the period of this relaxation oscillation. Since the time spent for jumps is negligible compared to the time spent at slow motions, we compute just the half-period of slow motion as

$$T/2 = \mu\, (\ln|x| - x^2/2)\big|_{x=2}^{x=1} = \mu(\frac{3}{2} - \ln 2).$$

Thus, the period $T = \mu(3 - 2\ln 2)$ tends to infinity as $\mu \to \infty$.

5.5 Exercises

EXERCISE 5.1. A point-mass m moves under the action of gravity along a frictionless circular wire of radius r that is rotating with a constant angular velocity Ω about its vertical diameter (see Fig. 5.22).[9] Derive the equation of motion and plot the potential energy as well as the phase portrait.

Solution. Since the point-mass moves along the wire that rotates about the vertical axis, its absolute velocity equals

$$\mathbf{v} = \mathbf{v}_l + \mathbf{v}_r.$$

[9] A pendulum oscillating on a rotating platform can serve as a similar example.

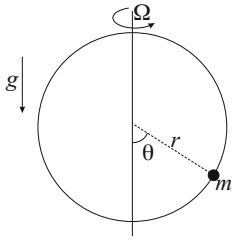

Fig. 5.22 Point-mass on rotating circular wire

In this formula $\mathbf{v}_l$ is the instantaneous velocity of the point-mass moving together with the rotating frame as rigid body, while $\mathbf{v}_r$ is its velocity relative to the rotating frame. As these two vectors are orthogonal to each other and their magnitudes are $\Omega r \sin\theta$ and $r\dot{\theta}$, respectively, the kinetic energy of the point-mass reads

$$K = \frac{1}{2}mv^2 = \frac{1}{2}mr^2(\dot{\theta}^2 + \Omega^2 \sin^2\theta).$$

Choosing the zero level of potential energy at the horizontal plane $\theta = \pi/2$, the potential energy of the point-mass in the gravitation field is given by

$$U = -mgr\cos\theta.$$

Thus, the Lagrange function equals

$$L = K - U = \frac{1}{2}mr^2\dot{\theta}^2 + \frac{1}{2}mr^2\Omega^2 \sin^2\theta + mgr\cos\theta.$$

We may interpret the first term of L as the kinetic energy of the motion relative to the rotating frame, while the term $-\frac{1}{2}mr^2\Omega^2 \sin^2\theta$ as the potential energy of the fictitious "centrifugal force", called centrifugal energy. From Lagrange's equation we derive the equation of motion

$$mr^2\ddot{\theta} - mr^2\Omega^2 \sin\theta\cos\theta + mgr\sin\theta = 0.$$

The above equation of motion can be transformed to the dimensionless form

$$\theta'' - \omega^2 \sin\theta\cos\theta + \sin\theta = 0,$$

where $\omega = \Omega/\omega_0 = \Omega/\sqrt{g/r}$ and where prime denotes the derivative with respect to the dimensionless time $\tau = \omega_0 t$. This equation yields the conservation law

$$\frac{1}{2}\theta'^2 - \cos\theta + \frac{1}{4}\omega^2 \cos 2\theta = E_0.$$

The plots of the modified potential energy $\tilde{U}(\theta) = -\cos\theta + \frac{1}{4}\omega^2 \cos 2\theta$ for different ω are shown in Fig. 5.23, where the points $\theta = -\pi$ and $\theta = \pi$, due to the

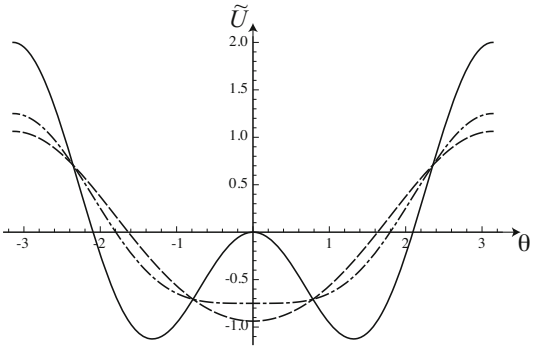

Fig. 5.23 Modified potential energy $\tilde{U}(\theta)$: i) $\omega = 0.5$ (dashed line), ii) $\omega = 1$ (dashed and dotted line), iii) $\omega = 2$ (bold line)

periodicity in θ, have to be identified. For $\omega < 1$ the modified potential energy has one local minimum at $\theta = 0$ corresponding to the stable equilibrium state and one local maximum at $\theta = \pm\pi$ corresponding to the unstable equilibrium state (saddle point). For $\omega > 1$ there are two local minima at $\theta = \pm\theta_0 = \pm\arccos(1/\omega)$, and two local maxima at $\theta = 0$ and at $\theta = \pm\pi$. The saddle-node bifurcation occurs at $\omega = 1$. The phase portrait of this rotating pendulum is plotted in Fig. 5.24 for $\omega = 2$. We see that there are two stable centers corresponding to two minima and two saddle points corresponding to the maxima of the potential energy.

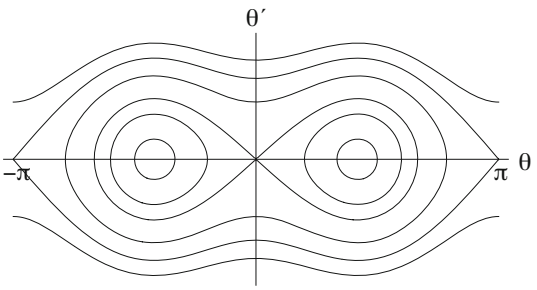

Fig. 5.24 Phase portrait of the rotating pendulum ($\omega = 2$)

EXERCISE 5.2. Do the next step of the variational-asymptotic procedure for Duffing's equation and show that

$$T = 2\pi\left[1 - \varepsilon\frac{3}{8}a^2 + \varepsilon^2\frac{57}{256}a^4 + O(\varepsilon^3)\right].$$

Solution. At the third step we substitute

$$x(\tau) = x_0(\tau) + x_1(\tau) + x_2(\tau), \quad \omega = 1 + \omega_1 + \omega_2$$

into the action functional (5.4). Keeping the asymptotically principal terms containing x_2 and the principal cross terms between x_0, x_1, and x_2, we obtain

$$I_2[x_2(\tau)] = \int_0^{2\pi} [\frac{1}{2}x_2'^2 + \underline{x_0'x_2'} + \underline{2\omega_1 x_0'x_2'} + \underline{(2\omega_2+\omega_1^2)x_0'x_2'} + \underline{x_1'x_2'} + \underline{2\omega_1 x_1'x_2'} - \frac{1}{2}x_2^2 - x_0x_2 - x_1x_2 - \varepsilon x_0^3x_2 - 3\varepsilon x_0^2x_1x_2]\,d\tau.$$

Integrating the underlined terms by parts taking into account the periodicity of $x_0(\tau)$ and $x_1(\tau)$, we reduce this formula to

$$I_2[x_2(\tau)] = \int_0^{2\pi} [\frac{1}{2}x_2'^2 + \underline{2\omega_1 x_0x_2} + (2\omega_2+\omega_1^2)x_0x_2 - \underline{x_1''x_2} - 2\omega_1 x_1''x_2 - \frac{1}{2}x_2^2 - \underline{x_1x_2} - \underline{\varepsilon x_0^3x_2} - 3\varepsilon x_0^2x_1x_2]\,d\tau.$$

Using the equations for x_1 and ω_1, it is easy to check that the underlined terms are canceled out. Besides, the last term can be transformed into the sum of harmonic cosine functions as follows

$$-3\varepsilon x_0^2x_1x_2 = -\frac{3}{128}\varepsilon^2a^5(-2\cos\tau+\cos 3\tau+\cos 5\tau)x_2.$$

Finally, we have

$$I_2[x_2(\tau)] = \int_0^{2\pi} [\frac{1}{2}x_2'^2 - \frac{1}{2}x_2^2 + \underline{(2\omega_2 a+\omega_1^2 a+\frac{3}{64}\varepsilon^2a^5-\frac{1}{16}\omega_1\varepsilon a^3)\cos\tau\, x_2} + \ldots]\,d\tau.$$

Removing the underlined (resonant) term, we get

$$2\omega_2 = -\omega_1^2 - \frac{3}{64}\varepsilon^2a^4 + \frac{3}{128}\varepsilon^2a^4, \quad \text{that is,} \quad \omega_2 = -\frac{21}{256}\varepsilon^2a^4.$$

Thus, the correction for the frequency-amplitude relation reads

$$\omega = 1 + \frac{3}{8}\varepsilon a^2 - \frac{21}{256}\varepsilon^2a^4.$$

The period $T = 2\pi/\omega$ may then be written as

$$T = \frac{2\pi}{1+\frac{3}{8}\varepsilon a^2 - \frac{21}{256}\varepsilon^2a^4} = 2\pi\left[1 - \frac{3}{8}\varepsilon a^2 + \frac{57}{256}\varepsilon^2a^4\right].$$

EXERCISE 5.3. Consider a mass-spring oscillator with an asymmetric spring obeying the equation

$$\ddot{x} + x + \varepsilon x^2 = 0.$$

Find the period of vibration for $\varepsilon = 0.1$ and $x(0) = 1, \dot{x}(0) = 0$ using the numerical integration based on (5.3). Compare it with the result obtained by the variational-asymptotic method.

Solution. The potential energy of the spring is

$$U(x) = \frac{1}{2}x^2 + \frac{\varepsilon}{3}x^3.$$

We compute the period of vibration according to

$$T = 2\int_{x_m}^{x_M} \frac{d\xi}{\sqrt{2[E_0 - U(\xi)]}},$$

where E_0 is the total energy (which is conserved) and x_m and x_M are the turning points. From the initial conditions we find

$$E_0 = \frac{1}{2}v_0^2 + \frac{1}{2}x_0^2 + \frac{\varepsilon x_0^3}{3} = \frac{1}{2} + \frac{\varepsilon}{3} = 0.533.$$

Since the point-mass is released from $x = 1$, this value corresponds to the turning point $x_M = 1$. The other turning point is found as the root of the equation

$$U(x) = U(1) = E_0 \quad \Rightarrow \quad x_m = -1.0718.$$

Using the numerical integration, we find that

$$T = 2\int_{x_m}^{x_M} \frac{d\xi}{\sqrt{2[E_0 - U(\xi)]}} \approx 6.312.$$

To establish the asymptotic formula for the period of vibration we analyze the action functional

$$I[x(\tau)] = \int_0^{2\pi} (\frac{1}{2}\omega^2 x'^2 - \frac{1}{2}x^2 - \frac{1}{3}\varepsilon x^3)\, d\tau.$$

Then the first step of the variational-asymptotic procedure yields

$$x(\tau) = x_0(\tau) = a\cos\tau, \quad \omega = 1.$$

We have chosen the initial phase $\phi = 0$ which is consistent with the above initial conditions. At the second step we look for the solution in the form

$$x(\tau) = x_0(\tau) + x_1(\tau), \quad \omega = 1 + \omega_1.$$

Repeating similar calculations as in Section 5.1, we arrive at the following functional

$$I_1[x_1(\tau)] = \int_0^{2\pi} [\frac{1}{2}x_1'^2 - \frac{1}{2}x_1^2 + \underline{2\omega_1 a\cos\tau\, x_1} - \frac{1}{2}\varepsilon a^2(1+\cos 2\tau)x_1]\, d\tau.$$

Since the underlined term is the only resonant term and since $x_1(0) = x_1'(0) = 0$, we find that

$$\omega_1 = 0, \quad x_1(\tau) = \frac{1}{6}\varepsilon a^2(-3 + 2\cos\tau + \cos 2\tau).$$

At the third step we substitute

$$x(\tau) = x_0(\tau) + x_1(\tau) + x_2(\tau), \quad \omega = 1 + \omega_2$$

into the action functional. Repeating similar calculations as in the previous exercise, we obtain

$$I_2[x_2(\tau)] = \int_0^{2\pi} (\frac{1}{2}x_2'^2 - \frac{1}{2}x_2^2 + 2\omega_2 x_0 x_2 - 2\varepsilon x_0 x_1 x_2)\, d\tau.$$

The last term can be transformed into the sum of harmonic cosine functions as follows

$$-2\varepsilon x_0 x_1 x_2 = -\varepsilon^2 a^3 \frac{1}{6}(2 - 5\cos\tau + 2\cos 2\tau + \cos 3\tau).$$

Thus, the resonant term is

$$(2\omega_2 a + \frac{5}{6}\varepsilon^2 a^3)\cos\tau\, x_2.$$

Removing this resonant term, we get

$$\omega_2 = -\frac{5}{12}\varepsilon^2 a^2.$$

Thus, the refined frequency-amplitude relation reads

$$\omega = 1 - \frac{5}{12}\varepsilon^2 a^2.$$

The period $T = 2\pi/\omega$ may then be written as

$$T = 2\pi\left(1 + \frac{5}{12}\varepsilon^2 a^2\right) \approx 6.309.$$

EXERCISE 5.4. Find and classify the fixed points of equation (5.14) of a damped pendulum for all $c > 0$, and plot the phase portraits for the qualitatively different cases.

Solution. Equation (5.14) can be written in the dimensionless form as

$$\varphi'' + 2\delta\varphi' + \sin\varphi = 0,$$

where prime denotes the derivative with respect to $\tau = \omega_0 t$, and $\delta = \frac{c}{2m\omega_0}$ is Lehr's damping ratio. Transforming this equation to

$$\varphi' = \omega,$$
$$\omega' = -\sin\varphi - 2\delta\omega,$$

we see that the fixed points should be

$$(\varphi_n, \omega_n), \quad \text{where} \quad \varphi_n = n\pi, \quad \omega_n = 0,$$

and n are integers. Due to the periodicity in φ with the period 2π only the fixed points corresponding to $n = 0$ and $n = 1$ need be analyzed. For $n = 0$ linearization near the fixed point $(0,0)$ gives

$$\varphi' = \omega,$$
$$\omega' = -\varphi - 2\delta\omega,$$

which is equivalent to

$$\varphi'' + 2\delta\varphi' + \varphi = 0.$$

Thus, if $0 < \delta < 1$ the fixed point is a stable focus, and if $\delta > 1$ it is a stable node. For $n = 1$ linearization near the fixed point $(\pi, 0)$ with $\varphi = \pi + u$ and $\omega = v$ gives

$$u' = v,$$
$$v' = u - 2\delta v.$$

The corresponding characteristic equation

$$\lambda^2 + 2\delta\lambda - 1 = 0$$

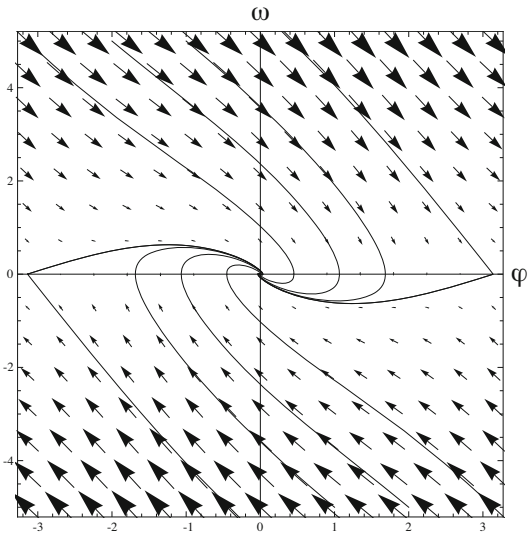

Fig. 5.25 Phase portrait of overdamped pendulum

has one negative real root $\lambda_1 = -\delta - \sqrt{\delta^2+1}$ and one positive real root $\lambda_2 = -\delta + \sqrt{\delta^2+1}$. Thus, the fixed point is a saddle point for all positive δ. The phase portrait for $\delta < 1$ (underdamped pendulum) was shown in Fig. 5.8. For the case $\delta > 1$ corresponding to the overdamped pendulum, the phase portrait is shown in Fig. 5.25. It is seen that the fixed point $(0,0)$ is a stable node, while the fixed point $(\pi,0)$ remains a saddle point.

EXERCISE 5.5. The motion of a mass-spring oscillator with the linear restoring force $-kx$ ($k = 2$N/cm) is damped by a constant braking force $f_r = 1$N; this force acts however only in the region $-1\text{cm} \le x \le 1\text{cm}$. Outside this region the oscillator carries out a free vibration. Find the sequence of turning points and the number of halves of vibrations for the initial conditions $x = -3$cm and $\dot{x} = 0$.

Solution. Outside the region $-1\text{cm} \le x \le 1\text{cm}$ there is no braking force, so the energy must be conserved

$$\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = E_0.$$

In the region $-1\text{cm} \le x \le 1\text{cm}$ the braking force causes the energy dissipation according to

$$\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = E_0 - f_r|x - x_0|.$$

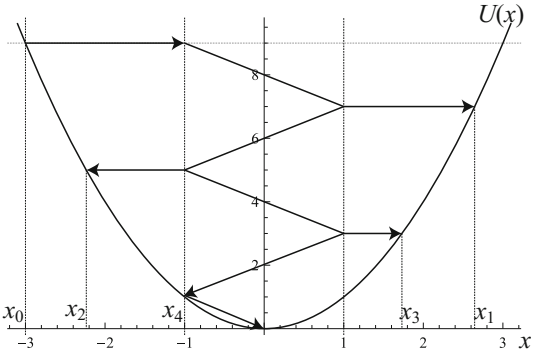

Fig. 5.26 Energy change and the sequence of turning points

With these two equations we can construct the sequence of the turning points as shown in Fig. 5.26. Being released from $x_0 = -3$cm with the zero velocity, the oscillator has the initial energy $E_0 = \frac{1}{2}kx_0^2 = 9$Ncm. From -3cm to -1cm the total energy does not change. After passing the zone with braking force the total energy is reduced by 2Ncm. Then the oscillator moves to the turning point without changing the energy $E_1 = 7$Ncm. Thus, the turning point $x_1 = \sqrt{7}$cm. Similar arguments can again be applied for the oscillator moving now to the left. After passing the zone with braking force the total energy becomes $E_2 = 5$Ncm, so the next turning

point is $x_2 = -\sqrt{5}$cm. Analogously, the next two turning points are $x_3 = \sqrt{3}$cm and $x_4 = -1$cm corresponding to the total energies $E_3 = 3$Ncm and $E_4 = 1$Ncm. The oscillator ends its motion at point $x = 0$cm which is inside the sticky zone $-0.5\text{cm} \le x \le 0.5\text{cm}$.

EXERCISE 5.6. Consider a damped pendulum with "turbulent" damping described by the equation

$$\ddot{\varphi} + c\dot{\varphi}|\dot{\varphi}| + \omega_0^2 \sin\varphi = 0.$$

Find the sequence of turning angles.

Solution. As shown in example considered in Section 5.2, the above equation can be integrated separately for $\dot{\varphi} > 0$ and $\dot{\varphi} < 0$. For $\dot{\varphi} > 0$ this equation can be rewritten in the form

$$\frac{dv^2}{d\varphi} + \alpha v^2 + \beta \sin\varphi = 0,$$

with $v = \dot{\varphi}$, $\alpha = 2c$, $\beta = 2\omega_0^2$. Applying the method of variation of coefficients, we obtain

$$v^2(\varphi) = e^{-\alpha\varphi}\left(C_1 - \beta\int_0^{\varphi} \sin\xi\, e^{\alpha\xi}\, d\xi\right).$$

Similarly, for $\dot{\varphi} < 0$ we have

$$v^2(\varphi) = e^{\alpha\varphi}\left(C_2 - \beta\int_0^{\varphi} \sin\xi\, e^{-\alpha\xi}\, d\xi\right).$$

The constants C_1 and C_2 should be determined from the initial conditions. Let

$$U_+(\varphi) = \beta\int_0^{\varphi} \sin\xi\, e^{\alpha\xi}\, d\xi = \frac{\beta}{1+\alpha^2}[1 + e^{\alpha\varphi}(-\cos\varphi + \alpha\sin\varphi)],$$
$$U_-(\varphi) = \beta\int_0^{\varphi} \sin\xi\, e^{-\alpha\xi}\, d\xi = \frac{\beta}{1+\alpha^2}[1 - e^{-\alpha\varphi}(\cos\varphi + \alpha\sin\varphi)].$$

Assume that the pendulum is released from φ_0 with the zero angular velocity $v_0 = 0$ and that it rotates afterward in the positive direction. Then $C_1 = U_+(\varphi_0)$ and

$$v^2(\varphi) = e^{-\alpha\varphi}[U_+(\varphi_0) - U_+(\varphi)] \quad \text{for } \dot{\varphi} > 0.$$

The first turning point φ_1 can then be found as the root of the equation $U_+(\varphi_1) = U_+(\varphi_0)$. Choosing now φ_1 as the initial angle from which the pendulum is released and rotates in the negative direction, we find that $C_2 = U_-(\varphi_1)$ and that

$$v^2(\varphi) = e^{\alpha\varphi}[U_-(\varphi_1) - U_-(\varphi)] \quad \text{for } \dot{\varphi} < 0.$$

Therefore, the second turning point, φ_2, must be the root of the equation $U_-(\varphi_2) = U_-(\varphi_1)$. Then we can choose φ_2 as the initial angle from which the pendulum is released and repeat the procedure. For the above functions the constant factors and the constant summands do not obviously influence the determination of the turning points. So, instead of $U_+(\varphi)$ and $U_-(\varphi)$ we can take the following functions

$$U_+^*(\varphi) = e^{\alpha\varphi}(-\cos\varphi + \alpha\sin\varphi) \quad \text{and} \quad U_-^*(\varphi) = -e^{-\alpha\varphi}(\cos\varphi + \alpha\sin\varphi)$$

for this purpose. Besides, $U_+^*(\varphi) = U_-^*(-\varphi)$, so it is enough to plot them for $\varphi > 0$. Fig. 5.27 shows the plots of these functions (for $c = 1/2$) and the geometric method of determining the sequence of turning points.

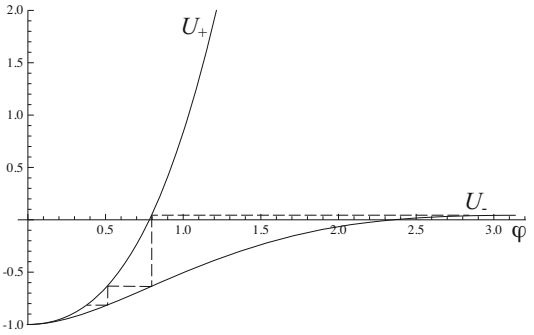

Fig. 5.27 Functions $U_+^*(\varphi)$ and $U_-^*(\varphi)$ and sequence of turning points

EXERCISE 5.7. Consider Froude's pendulum described by the following dimensionless equation

$$\ddot{\varphi} + 2\delta\dot{\varphi} + \omega_0^2\sin\varphi = m_r(\dot{\varphi} - \nu_0),$$

where

$$2\delta = \frac{c}{J}, \quad \omega_0^2 = \frac{mgl}{J}, \quad m_r = \frac{M_r}{J}.$$

Find conditions under which this oscillator develops self-sustained vibrations.

Solution. The above equation of motion is equivalent to the system of equations

$$\begin{aligned} \varphi' &= y, \\ y' &= m_r(y - \nu_0) - 2\delta y - \omega_0^2\sin\varphi, \end{aligned}$$

which has one fixed point S on the φ-axis with the coordinate $\varphi = \varphi_0$, where

$$\varphi_0 = \arcsin\frac{m_r(-\nu_0)}{\omega_0^2}.$$

To know whether this equilibrium position is stable or not, we study the neighboring solutions assumed in the form

$$\varphi = \varphi_0 + u, \quad y = v,$$

and linearize the system with respect to u and v. Since

$$m_r(v-v_0) = m_r(-v_0) + \left.\frac{dm_r}{dv}\right|_{-v_0} v + O(v^2), \quad \sin\varphi = \sin\varphi_0 + u\cos\varphi_0 + O(u^2),$$

and $m_r(-v_0) - \omega^2 \sin\phi_0 = 0$ at the fixed point S, we obtain

$$u' = v, \quad v' = \left(\left.\frac{dm_r}{dv}\right|_{-v_0} - 2\delta \right) v - \omega_0^2 \cos\varphi_0\, u,$$

what is equivalent to the equation

$$u'' - \left(\left.\frac{dm_r}{dv}\right|_{-v_0} - 2\delta \right) u' + \omega_0^2 \cos\varphi_0\, u = 0.$$

Thus, if

$$\left.\frac{dm_r}{dv}\right|_{-v_0} > 2\delta,$$

then the fixed point is an unstable focus, and the phase curves starting near this fixed point are repelled from it.

The vector field and phase curves of this pendulum are quite similar to those of the stick-slip oscillator shown in Fig. 5.20. The phase curves hitting the horizontal line $y = v_0$ must change their slopes abruptly when crossing this line. By this reason the line $y = v_0$ is called a jump line. Besides, there is a "sticky" zone $\arcsin((-r_0 - 2\delta v_0)/\omega_0^2) < \varphi < \arcsin((r_0 - 2\delta v_0)/\omega_0^2)$ on this jump line (the segment AC), where r_0 denotes the critical threshold moment of the friction. When the phase curves hit this sticky zone, they have to move along the horizontal line up to point C with coordinates $(\arcsin((r_0 - 2\delta v_0)/\omega_0^2), v_0)$, where the pendulum is detached from the rotating shaft and the slip begins.[10] The phase curve starting from point C is a spiral, which hit the segment AC at point B if the damping ratio δ is small. This phase curve together with the segment BC correspond to the limit cycle of the Froude's pendulum.

EXERCISE 5.8. Consider the mechanical system governed by the differential equation

$$\ddot{x} - \varepsilon \sin\dot{x} + x = 0.$$

Construct several phase curves for $\varepsilon = 0.1$ using numerical integration. Show that more than one limit cycle exist. Use the variational-asymptotic method to calculate the amplitudes of limit cycles.

Solution. The above equation can be written as the system

$$\dot{x} = y, \quad \dot{y} = -x + \varepsilon \sin y,$$

[10] Note, however, that this detach point C exists only if $r_0 \le \omega_0^2 + 2\delta v_0$. For larger threshold values r_0 the pendulum will rotate with the shaft without being detached from it.

for which the numerical integration with the command NDSolve in *Mathematica* is applicable as demonstrated in Section 5.3. If we set $\varepsilon = 0.1$ and take the initial conditions $x(0) = 3$, $y(0) = 0$, and $x(0) = 5$, $y(0) = 0$, the corresponding phase curves, shown in Fig. 5.28, enable one to guest that there exists one stable limit cycle whose amplitude is near the value 4. Likewise, the phase curves starting at $x(0) = 9$, $y(0) = 0$, and $x(0) = 11$, $y(0) = 0$ show that another stable limit cycle whose amplitude near the value 10 exists.

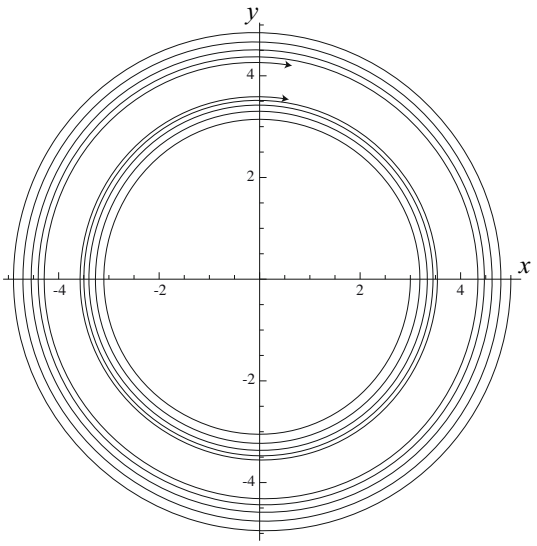

Fig. 5.28 Two phase curves starting at $x(0) = 3$, $y(0) = 0$, and $x(0) = 5$, $y(0) = 0$

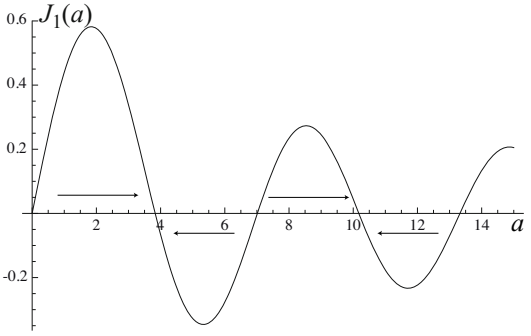

Fig. 5.29 The plot of $J_1(a)$ and the directions of change of a according to $a_{,\eta} = J_1(a)$ as η goes to infinity

According to the variational-asymptotic method, the slow evolution of the amplitudes of vibrations to those of the limit cycles and the corrections to the frequencies for small ε are determined from the equations

$$a_{,\eta} = -\frac{1}{2\pi}\int_0^{2\pi} f(a\cos\tau, -a\sin\tau)\sin\tau\, d\tau,$$
$$\omega_1 = -\frac{\varepsilon}{2\pi a}\int_0^{2\pi} f(a\cos\tau, -a\sin\tau)\cos\tau\, d\tau,$$

where

$$f(x,\dot{x}) = \sin\dot{x}.$$

The integrals on the right-hand sides yield

$$a_{,\eta} = J_1(a), \quad \omega_1 = 0,$$

where $J_1(a)$ is the Bessel function, whose plot is shown in Fig. 5.29. Thus, there is no correction to the frequency. The evolution equation for a cannot be solved in terms of special functions, but as seen from Fig. 5.29, the zeros of $J_1(a)$ at which $J_1'(a)$ is negative give the amplitudes of the stable limit cycles. This yields

$$a_1 = 3.83171, \quad a_2 = 10.1735, \ldots.$$

EXERCISE 5.9. Show that Rayleigh's equation

$$\ddot{x} + x - \varepsilon(1 - \frac{1}{3}\dot{x}^2)\dot{x} = 0$$

can be rewritten as van der Pol's equation

$$\ddot{u} + u - \varepsilon(1 - u^2)\dot{u} = 0,$$

where $u = \dot{x}$. Find the amplitude of its limit cycle for small ε.

Solution. Let us differentiate Rayleigh's equation using the definition $u = \dot{x}$

$$\ddot{u} + u - \varepsilon\dot{u} + \varepsilon u^2\dot{u} = 0.$$

We see that this is exactly van der Pol's differential equation in terms of u. On the other side, Rayleigh's equation in terms of x contains a small parameter ε, so the slow evolution of the amplitude of vibration and the correction to the frequency are determined from the equations

$$a_{,\eta} = -\frac{1}{2\pi}\int_0^{2\pi} f(a\cos\tau, -a\sin\tau)\sin\tau\, d\tau,$$
$$\omega_1 = -\frac{\varepsilon}{2\pi a}\int_0^{2\pi} f(a\cos\tau, -a\sin\tau)\cos\tau\, d\tau,$$

where

$$f(x,\dot{x}) = (1-\frac{1}{3}\dot{x}^2)\dot{x}.$$

We compute the integrals on the right-hand sides of these equations

$$-\frac{1}{2\pi}\int_0^{2\pi} f(a\cos\tau, -a\sin\tau)\sin\tau\, d\tau = \frac{1}{2\pi}\int_0^{2\pi}(1-\frac{1}{3}a^2\sin^2\tau)a\sin^2\tau\, d\tau$$
$$= \frac{a}{8}(4-a^2),$$

and

$$\int_0^{2\pi} f(a\cos\tau, -a\sin\tau)\cos\tau\, d\tau = -\int_0^{2\pi}(1-\frac{1}{3}a^2\sin^2\tau)a\sin\tau\cos\tau\, d\tau = 0.$$

Thus, there is no correction to the frequency. The evolution equation for a is exactly the same as that obtained for the van der Pol oscillator

$$a_{,\eta} = \frac{a}{8}(4-a^2).$$

The solution to this equation reads

$$a = \frac{2a_0 e^{\varepsilon\tau/2}}{\sqrt{4+a_0^2(e^{\varepsilon\tau}-1)}},$$

where a_0 is the initial amplitude of vibration. So, the amplitude of vibration tends to the value 2 as $\tau \to \infty$. This result is also confirmed by the numerical integration of Rayleigh's equation with $\varepsilon = 0.1$ as shown in Fig. 5.30.

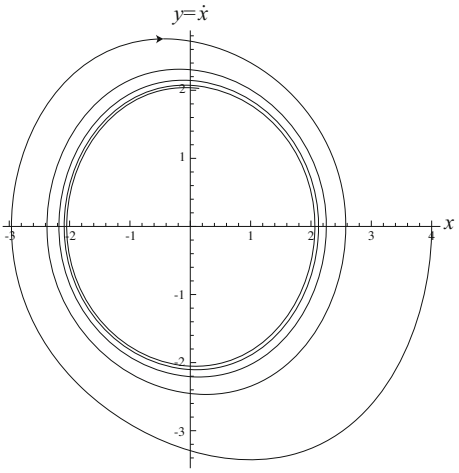

Fig. 5.30 A phase curve of Rayleigh's equation beginning from $(x,y) = (4,0)$

EXERCISE 5.10. Consider the equation

$$\ddot{x}+x+\mu(|x|-1)\dot{x}=0.$$

Find the approximate period and amplitude of the limit cycle for small μ.

Solution. For small μ let the slow time be $\eta=\mu\tau$. Then the slow evolution of the amplitude of vibration and the correction to the frequency are determined from the equations

$$a_{,\eta}=\frac{1}{2\pi}\int_0^{2\pi}(1-a|\cos\tau|)a\sin^2\tau\,d\tau=-\frac{a(4a-3\pi)}{6\pi},$$
$$\omega_1=\frac{\varepsilon}{2\pi a}\int_0^{2\pi}(1-a|\cos\tau|)a\sin\tau\cos\tau\,d\tau=0.$$

Thus, there is no correction to the frequency. The equation for a can be integrated to give

$$\ln\frac{4a-3\pi}{4a}=-\eta/2+C.$$

The constant of integration can be obtained from the initial condition $a(0)=a_0$: $C=\ln\frac{4a_0-3\pi}{4a_0}$. Substituting this constant in the last equation and solving it with respect to a we obtain

$$a=\frac{3\pi}{4(1-\frac{4a_0-3\pi}{4a_0}e^{-\eta/2})}.$$

Thus, a approaches $3\pi/4$ as $\tau\to\infty$.

EXERCISE 5.11. Use the variational-asymptotic method to study the equation

$$\ddot{x}+x-\varepsilon(1-x^4)\dot{x}=0$$

for small ε. Find the approximate amplitude of the limit cycle.

Solution. The slow evolution of the amplitude of vibration and the correction to the frequency are determined from the equations

$$a_{,\eta}=\frac{1}{2\pi}\int_0^{2\pi}(1-a^4\cos^4\tau)a\sin^2\tau\,d\tau=\frac{a}{16}(8-a^4),$$
$$\omega_1=\frac{\varepsilon}{2\pi a}\int_0^{2\pi}(1-a^4\cos^4\tau)a\sin\tau\cos\tau\,d\tau=0.$$

Thus, there is no correction to the frequency. The equation for a can be transformed by multiplying it with $4a^3$

$$z_{,\eta}=\frac{1}{4}z(8-z),$$

where $z=a^4$. This equation can be integrated by separating the variables giving

$$\ln\frac{z}{8-z}=2\eta+C.$$

The constant of integration can be obtained from the initial condition $z(0) = z_0$: $C = \ln \frac{z_0}{8-z_0}$. Substituting this constant into the last equation and solving it with respect to z, we obtain

$$z = \frac{8z_0 e^{2\eta}}{8 - z_0 + z_0 e^{2\eta}},$$

or, in terms of the original amplitude a and time τ

$$a^4 = \frac{8a_0^4 e^{2\varepsilon\tau}}{8 - a_0^4 + a_0^4 e^{2\varepsilon\tau}}.$$

Thus, a approaches $\sqrt[4]{8} \approx 1.682$ as $\tau \to \infty$. This result is also confirmed by the numerical integration of the equation for x (with $\varepsilon = 0.1$) as shown in Fig. 5.31.

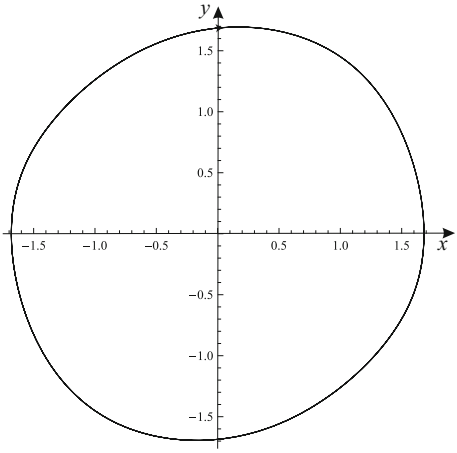

Fig. 5.31 Limit cycle of the equation $\ddot{x} + x - 0.1(1 - x^4)\dot{x} = 0$

EXERCISE 5.12. Use the variational-asymptotic method to study the equation

$$\ddot{x} + x - \mu(1 + x - x^2)\dot{x} = 0,$$

where μ is a large parameter. Find the amplitude and period of the limit cycle. Compare the results with those obtained by numerical integration for $\mu = 10$.

Solution. The above equation is obtained from the following variational equation

$$\delta \int_{t_0}^{t_1} (\frac{1}{2}\dot{x}^2 - \frac{1}{2}x^2)\, dt + \int_{t_0}^{t_1} \mu(1 + x - x^2)\dot{x}\delta x\, dt = 0.$$

Numerical simulations for large μ (see, for instance, Fig. 5.32 in case $\mu = 10$) show that the solution corresponding to the limit cycle spends most of time in a slow

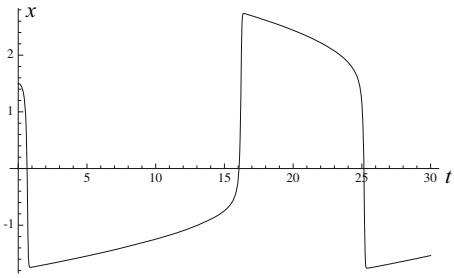

Fig. 5.32 Numerical solution of the equation $\ddot{x}+x-10(1+x-x^2)\dot{x}=0$

motion, and then quickly jumps to another slow motion. To analyze slow motions we introduce the slow time $\eta = t/\mu$ and transform the variational equation to

$$\delta \int_{\eta_0}^{\eta_1} (\frac{1}{2\mu^2} x_{,\eta}^2 - \frac{1}{2}x^2)\, d\eta + \int_{\eta_0}^{\eta_1} (1+x-x^2) x_{,\eta} \delta x\, d\eta = 0.$$

Neglecting the first term in this equation as small in accordance with the variational-asymptotic method, we arrive at the equation

$$-x+(1+x-x^2)x_{,\eta}=0, \quad \text{or} \quad x_{,\eta} = \frac{x}{1+x-x^2}.$$

This differential equation can be solved by separation of variables yielding

$$\ln|x| + x - \frac{x^2}{2} = \eta + \text{const}.$$

The slow motion proceeds according to this equation until it reaches $x_{1,2} = (1 \pm \sqrt{5})/2$ where the speed $x_{,\eta}$ is infinite. At these points the assumption of slow motion is violated, so we need to change to another time scale. Introducing now the fast time $\tau = \mu t$, we rewrite the variational equation as

$$\delta \int_{\tau_0}^{\tau_1} (\frac{1}{2}\mu^2 x_{,\tau}^2 - \frac{1}{2}x^2)\, d\tau + \int_{\tau_0}^{\tau_1} \mu^2 (1+x-x^2) x_{,\tau} \delta x\, d\tau = 0.$$

Neglecting the second term as small compared with other terms, we arrive at the differential equation

$$x_{,\tau\tau} - (1+x-x^2)x_{,\tau} = 0.$$

This equation possesses the first integral

$$x_{,\tau} - x - x^2/2 + x^3/3 = \text{const},$$

representing the fast motion (jump). We choose the constant of integration so that this fast motion starting at $x = x_1$ ($x = x_2$) as an equilibrium point (with $x_{,\tau} = 0$) will end at another equilibrium point with $x_{,\tau} = 0$. It is easy to see that the second

equilibrium point corresponding to x_1 is $x = 2.736$. Similarly, the jump starting at x_2 ends up at $x = -1.736$. Knowing the solution, we can easily compute the period of this relaxation oscillation. The comparison with the solution obtained by the numerical integration for $\mu = 10$ shown in Fig. 5.32 yields a good agreement.