Chapter 4
Linear Waves

This chapter studies linear waves propagating in continuous media. For homogeneous media the method of solution is Fourier's transform which is based entirely on the linear superposition principle. For weakly inhomogeneous media the variational-asymptotic method has to be used instead.

4.1 Hyperbolic Waves

Differential Equation of Wave Propagation. In contrast to vibrations of continuous systems, waves transport disturbances and energy from one part of the medium to another with a recognizable velocity of propagation. Thus, we are dealing locally with transient processes. The equations governing wave propagation remain exactly the same as the equations of motion for continuous oscillators. In addition, the initial and boundary conditions have to be specified. If the influence of the boundary can be neglected, then it is convenient to consider waves propagating in infinite media. In this case the radiation conditions are required to select the physically meaningful solution.

1-D Problem. We begin first with the most simple situation, namely, with the propagation of hyperbolic waves in one dimension governed by the equation

$$u_{,tt} = c^2 u_{,xx}.$$

As one remembers from Section 3.2, this equation describes flexural vibrations of a pre-stretched string, or longitudinal vibrations of an elastic bar. Now instead of vibrations (or standing waves) we want to analyze wave propagation. If the boundaries of the medium are far away from the point of interest so that waves do not still interact with them, we may consider the idealized situation of waves propagating in an equivalent infinite medium. Introducing the characteristic coordinates $\alpha = x - ct$, $\beta = x + ct$, we transform the above equation to

$$\frac{\partial^2 u}{\partial\alpha\partial\beta} = 0,$$

which yields the general solution obtained first by d'Alembert

$$u(x,t) = f(\alpha) + g(\beta) = f(x-ct) + g(x+ct).$$

This formula represents two waves traveling through the medium with the constant velocity c; f to the right, and g to the left. Note that the observer moving to the right (or left) with the velocity c does not see any change of wave shape associated with f (or g). Such waves are called dispersionless.

For the initial value problem

$$u(x,0) = u_0(x), \quad u_{,t}(x,0) = v_0(x),$$

we determine f and g from the initial conditions

$$u(x,0) = f(x) + g(x) = u_0(x), \quad u_{,t}(x,0) = -cf'(x) + cg'(x) = v_0(x),$$

giving

$$u(x,t) = \frac{1}{2}[u_0(x-ct) + u_0(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} v_0(\xi)d\xi.$$

We can also solve the signaling problem for the half-axis $x \geq 0$ of outgoing waves with

$$u_{,x}(0,t) = p(t).$$

In this case the solution reads

$$u(x,t) = -cq(t-x/c),$$

where $q(t)$ is the integral of $p(t)$.

3-D Problem. According to Hadamard's idea, waves propagating in three dimensions will be easier to study than those in two dimensions, so we start with the 3-D case. We first look for particular solutions of the wave equation

$$u_{,tt} = c^2 \Delta u \tag{4.1}$$

in the 3-D space. This equation describes sound waves in fluids and gases, as well as dilatational or shear waves propagating in infinite elastic solids (see Section 3.6 and exercise 4.2). Since equation (4.1) is linear, its particular solutions always exist in form of harmonic (also called monochromatic) waves[1]

$$u(\mathbf{x},t) = e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)},$$

[1] We work directly with the complex form of the solution keeping in mind that the real or imaginary part should be taken when necessary.

where **k** is the wave vector and ω the frequency. Indeed, substituting this Ansatz into (4.1), we obtain the equation

$$(-\omega^2 + c^2|\mathbf{k}|^2)e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} = 0,$$

with $|\mathbf{k}| = \sqrt{k_x^2 + k_y^2 + k_z^2}$ being the magnitude of **k**. As the exponential function is not identically zero, ω must be related to **k** by

$$\omega = \pm c|\mathbf{k}|.$$

Thus, for each non-zero wave vector **k** there are two harmonic waves corresponding to $\omega = c|\mathbf{k}|$ or $\omega = -c|\mathbf{k}|$. We refer to them as *branches*.

For the moment let us concentrate just on one branch since the general solution is simply the linear superposition of them. Taking the real part, we present the monochromatic wave as

$$u(\mathbf{x},t) = \cos(\mathbf{k}\cdot\mathbf{x} - c|\mathbf{k}|t).$$

We call $\theta(\mathbf{x},t) = \mathbf{k}\cdot\mathbf{x} - c|\mathbf{k}|t$ phase; it determines the position on the cycle between a crest, where u has a maximum, and a trough, where u achieves a minimum. This particular solution is called a plane wave because the phase surfaces $\theta = \text{const}$ are parallel planes as shown in Fig. 4.1 in 2-D case. The gradient of θ in the space is the wave vector **k**, whose direction is normal to the phase planes and whose magnitude $\kappa = |\mathbf{k}|$ is the average number of crests per 2π units of distance in that direction. In Fig. 4.1 the wave vector is $\mathbf{k} = (1,1)$ in the (x,y)-plane. Similarly, $-\theta_{,t}$ is the frequency $\omega = c\kappa$, the average number of crests per 2π units of time. The wavelength is $\lambda = 2\pi/\kappa$ and the period is $T = 2\pi/\omega$. The wave motion is recognized from the phase. Any particular phase surface moves in the space with the normal velocity $\omega/\kappa = c$ in the direction of **k**. Thus, for the wave equation $u_{,tt} = c^2\Delta u$ the phase velocity agrees with the usual propagation speed.

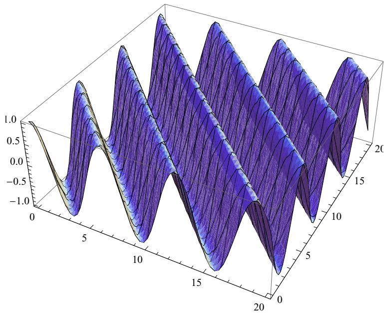

Fig. 4.1 Plot of $\cos(x+y)$

The monochromatic plane waves play a key role in the theory of linear waves propagating in homogeneous media because the general solution can be obtained by the linear superposition of these waves with various wave vectors. This leads to Fourier's integrals, where the contribution of each monochromatic plane wave is Fourier's component of the wave packet. We postpone the derivation of general solution based on this Fourier's analysis to the next Section 4.2. However, in what follows we want to use the monochromatic plane waves to study reflection and refraction of waves.

Reflection and refraction of waves. When a monochromatic plane wave is incident on the boundary between two different media, it undergoes reflection and refraction. The motion in the first medium is a combination of the incident and reflected waves, whereas in the second medium there is only one, the refracted wave. All three waves have the same frequency ω; the relations between their amplitudes and wave vectors are determined by the boundary conditions. Consider for definiteness the reflection and refraction of sound wave at a plane surface separating two media, say air and water, which we take as the (x,y)-plane. Because of the translational invariance in the x- and y-directions, all three waves have the same components k_x, k_y of the wave vector, but not the same component k_z.

For simplicity let us consider wave propagating in the (x,z)-plane. Then $k_y = 0$ in all three waves, so they are coplanar. Let ϑ be the angle between the direction of wave propagation and the z-axis (see Fig. 4.2). From the equality of $k_x = (\omega/c)\sin\vartheta$ for the incident and reflected waves, it follows that $\vartheta_1 = \vartheta_1'$, i.e. the angle of incidence ϑ_1 is equal to that of reflection ϑ_1'. The similar equality of k_x for the incident and refracted waves implies Snell's law

$$\frac{\sin\vartheta_1}{\sin\vartheta_2} = \frac{c_1}{c_2},$$

where c_1 and c_2 are the velocities of sound in these two media.

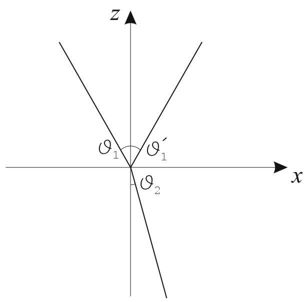

Fig. 4.2 Reflection and refraction of waves

In order to obtain the relation between the intensities of these three waves, we write the velocity potentials as

$$\varphi_1 = A_1 e^{i\omega[(z/c_1)\cos\vartheta_1 + (x/c_1)\sin\vartheta_1 - t]},$$
$$\varphi_1' = A_1' e^{i\omega[(-z/c_1)\cos\vartheta_1 + (x/c_1)\sin\vartheta_1 - t]},$$
$$\varphi_2 = A_2 e^{i\omega[(z/c_2)\cos\vartheta_2 + (x/c_2)\sin\vartheta_2 - t]},$$

where A_1, A_1', and A_2 are the complex amplitudes of waves. At the boundary $z = 0$ the pressure $p = -\rho\varphi_{,t}$ and the normal velocities $v_z = \varphi_{,z}$ in the two media must be equal; these conditions lead to the relations

$$\rho_1(A_1 + A_1') = \rho_2 A_2, \qquad \frac{\cos\vartheta_1}{c_1}(A_1 - A_1') = \frac{\cos\vartheta_2}{c_2} A_2.$$

The reflection coefficient R is defined as the ratio of the average energy flux in the reflected and incident waves. Since the energy flux of sound wave is $c\rho v^2$ (see the general derivation in Section 4.4), we have $R = \overline{v_1'^2}/\overline{v_1^2} = |A_1'|^2/|A_1|^2$, where bar denotes the time average. A simple calculation gives

$$R = \left(\frac{\rho_2 \tan \vartheta_2 - \rho_1 \tan \vartheta_1}{\rho_2 \tan \vartheta_2 + \rho_1 \tan \vartheta_1} \right)^2 .$$

The angles ϑ_1 and ϑ_2 are related by Snell's law; expressing ϑ_2 in terms of ϑ_1, we can put this formula in the form

$$R = \left(\frac{\rho_2 c_2 \cos \vartheta_1 - \rho_1 \sqrt{c_1^2 - c_2^2 \sin^2 \vartheta_1}}{\rho_2 c_2 \cos \vartheta_1 + \rho_1 \sqrt{c_1^2 - c_2^2 \sin^2 \vartheta_1}} \right)^2 .$$

For normal incident ($\vartheta_1 = 0$), this formula gives simply

$$R = \left(\frac{\rho_2 c_2 - \rho_1 c_1}{\rho_2 c_2 + \rho_1 c_1} \right)^2 .$$

Solution as a Superposition of Spherical Waves. There is a simple way to obtain the solution of the wave equation in 3-D case as a superposition of spherical waves. We start by assuming first the spherical symmetry of a particular solution about the origin: $u = u(r,t)$, where r is the distance from the origin. The wave equation reduces to

$$\frac{1}{c^2} u_{,tt} = u_{,rr} + \frac{2}{r} u_{,r}.$$

This equation can be rewritten as

$$\frac{1}{c^2} (ru)_{,tt} = (ru)_{,rr}$$

which is exactly the 1-D wave equation for ru. Thus, the particular solution reads

$$u(r,t) = \frac{f(r-ct)}{r}.$$

Here we select only the outgoing wave. This selection is equivalent to posing the radiation condition which requires that waves can only propagate from sources to infinity. If the source generating waves is found at point $\boldsymbol{\xi}$, then the particular solution takes the form

$$u(\mathbf{x},t) = \frac{f(|\mathbf{x} - \boldsymbol{\xi}| - ct)}{|\mathbf{x} - \boldsymbol{\xi}|}.$$

Now the particular solution of (4.1) can be constructed as a linear superposition of spherical waves

$$\phi(\mathbf{x},t) = \int \psi(\boldsymbol{\xi}) \frac{\delta(|\mathbf{x} - \boldsymbol{\xi}| - ct)}{|\mathbf{x} - \boldsymbol{\xi}|} d\xi, \qquad (4.2)$$

where $d\xi = d\xi_1 d\xi_2 d\xi_3$. In the integrand we take Dirac's delta function representing the unit source, while function $\psi(\boldsymbol{\xi})$ accounts for the fact that waves coming from different points will have in general different intensities. The form (4.2) suggests the introduction of spherical coordinates $(\rho, \vartheta, \varphi)$ with the origin at $\mathbf{x}$ yielding

$$\begin{aligned}\phi(\mathbf{x},t) &= \int_0^\infty \int_0^\pi \int_0^{2\pi} \psi(\mathbf{x}+\rho\mathbf{l})\delta(\rho-ct)\rho\sin\vartheta\,d\varphi\,d\vartheta\,d\rho \\ &= ct\int_0^\pi \int_0^{2\pi} \psi(\mathbf{x}+ct\mathbf{l})\sin\vartheta\,d\varphi\,d\vartheta,\end{aligned} \tag{4.3}$$

where $\mathbf{l}$ is the unit vector from $\mathbf{x}$ to $\boldsymbol{\xi}$ having the cartesian components

$$\mathbf{l} = (\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta).$$

As $t \to 0$ the right-hand side of (4.3) tends to zero. For a continuously differentiable function $\psi(\mathbf{x})$ we may differentiate this expression with respect to t and find the limit as $t \to 0$

$$\phi_{,t}(\mathbf{x},0) = 4\pi c\psi(\mathbf{x}).$$

Thus, the integral

$$\phi(\mathbf{x},t) = \frac{t}{4\pi}\int_0^\pi \int_0^{2\pi} v_0(\mathbf{x}+ct\mathbf{l})\sin\vartheta\,d\varphi\,d\vartheta$$

solves equation (4.1) with the initial conditions

$$u(\mathbf{x},0) = 0, \quad u_{,t}(\mathbf{x},0) = v_0(\mathbf{x}).$$

Note that this solution can also be represented as a surface integral

$$\phi(\mathbf{x},t) = \frac{1}{4\pi c^2 t}\int_{S(t)} v_0(\mathbf{x}+ct\mathbf{l})\,da,$$

where $S(t)$ is the spherical surface with center at $\mathbf{x}$ and radius ct.

To satisfy the remaining initial condition $u(\mathbf{x},0) = u_0(\mathbf{x})$ we use the following property: if ϕ is a solution of (4.1), then its time derivative $\phi_{,t}$ is also the solution. Consider the solution of the form

$$\chi(\mathbf{x},t) = \phi_{,t},$$

where ϕ is given by (4.3). In this case it is easy to check that, as $t \to 0$,

$$\chi(\mathbf{x},0) = 4\pi c\psi(\mathbf{x}), \quad \chi_{,t}(\mathbf{x},0) = \phi_{,tt} = c^2\Delta\phi = 0.$$

Therefore we choose now $\psi(\mathbf{x}) = u_0(\mathbf{x})/4\pi c$ and get for χ

$$\chi(\mathbf{x},t) = \frac{\partial}{\partial t}\left[\frac{1}{4\pi c^2 t}\int_{S(t)} u_0(\mathbf{x}+ct\mathbf{l})\,da\right].$$

The complete solution reads

$$u(\mathbf{x},t) = \frac{\partial}{\partial t}\left[\frac{1}{4\pi c^2 t}\int_{S(t)} u_0(\mathbf{x}+ct\mathbf{l})\,da\right] + \frac{1}{4\pi c^2 t}\int_{S(t)} v_0(\mathbf{x}+ct\mathbf{l})\,da. \tag{4.4}$$

Equation (4.4), called Poisson's formula, represents the total contribution of the instantaneous sources which send spherical waves to point $\mathbf{x}$ at time t; they are all exactly a distance ct away and their contributions traveling with speed c arrive at $\mathbf{x}$ just at time t. Notice that sources inside $S(t)$ do not contribute to the solution at $\mathbf{x}$. Thus, there is no "tail" for spherical waves. This is no longer so in 2-D case as will be seen in the next paragraph.

2-D Problem. The solution to the 2-D problem can be obtained from the 3-D solution by assuming $u_0(\mathbf{x})$ and $v_0(\mathbf{x})$ to be independent of x_3. Suppose now that nonzero values of $u_0(x_1,x_2)$, $v_0(x_1,x_2)$ are specified in a finite domain C_0 of the (x_1,x_2)-plane. From the 3-D point of view, the non-zero initial data occupy the cylinder C with generators parallel to the x_3-axis based on the cross section C_0. Thus, the domain of initial disturbances is no longer compact in the space. For a point outside the cylinder C, the construction of wavefront is as before, but the spheres with the center at $\mathbf{x}$ will intersect C at all time after the first time of intersection (see Fig. 4.3). This accounts for the "tail" in the 2-D case and shows clearly the difference between 2-D and 3-D cases.

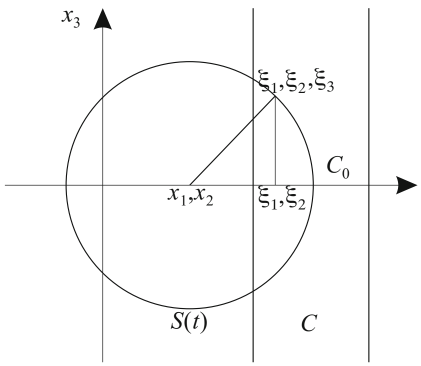

Fig. 4.3 Reduction of wavefront from three to two dimensions

Let us consider now the integrals in (4.4) at some fixed point $(x_1,x_2,0)$. At point (ξ_1,ξ_2,ξ_3) on $S(t)$ (see Fig. 4.3) the value of u_0 is $u_0(\xi_1,\xi_2)$. The outward normal to the sphere has a component n_3 given by

$$n_3 = \frac{\xi_3}{ct} = \pm\frac{\sqrt{c^2t^2-(x_1-\xi_1)^2-(x_2-\xi_2)^2}}{ct}.$$

The surface element da is equal to $d\xi_1 d\xi_2/|n_3|$, where $d\xi_1 d\xi_2$ is its projection in the (x_1,x_2)-plane. Therefore, taking into account the two equal contributions from above and below the (x_1,x_2)-plane, we have

$$\frac{1}{4\pi c^2 t}\int_{S(t)} u_0(\mathbf{x}+ct\mathbf{l})\,da = \frac{1}{2\pi c}\int_{\sigma(t)} \frac{u_0(\xi_1,\xi_2)d\xi_1 d\xi_2}{\sqrt{c^2t^2-(x_1-\xi_1)^2-(x_2-\xi_2)^2}},$$

where $\sigma(t)$ is the interior of the projection of $S(t)$ onto the (x_1,x_2)-plane:

$$\sigma(t)=\{(\xi_1,\xi_2)\,|\,(x_1-\xi_1)^2+(x_2-\xi_2)^2\le c^2t^2\}.$$

Thus, the solution of 2-D problem reads

$$u(x_1,x_2,t)=\frac{\partial}{\partial t}\left[\frac{1}{2\pi c}\int_{\sigma(t)}\frac{u_0(\xi_1,\xi_2)d\xi_1 d\xi_2}{\sqrt{c^2t^2-(x_1-\xi_1)^2-(x_2-\xi_2)^2}}\right]+\frac{1}{2\pi c}\int_{\sigma(t)}\frac{v_0(\xi_1,\xi_2)d\xi_1 d\xi_2}{\sqrt{c^2t^2-(x_1-\xi_1)^2-(x_2-\xi_2)^2}}.$$

Since the integrals are taken over the whole domain inside the circle $(x_1-\xi_1)^2+(x_2-\xi_2)^2=c^2t^2$, not just its boundary, the disturbance continues even after this circle completely surrounds the initial domain C_0.

Geometrical Optics. Although the exact solution to the wave equation has been found, the computation of Poisson's integrals is not always easy, even if we do it numerically. A simplification is possible if the wave packet may be regarded as plane in any small region of the medium. For this to be so it is necessary that the amplitude and the direction of propagation vary only slightly in one wavelength. If this condition holds, we can introduce the idea of *rays* as lines whose tangent at any point coincides with the direction of wave propagation. Then, to find the wavefront we need just to find the rays while ignoring the nature of wave propagation. This task will be done within the so-called geometrical optics which turns out to be valid in the high frequency (short wave) approximation.

We derive the equations of geometrical optics by assuming the periodic solution with a given frequency ω: $u(\mathbf{x},t)=w(\mathbf{x})e^{-i\omega t}$. Then the wave equation reduces to Helmholtz's equation

$$\Delta w+\frac{\omega^2}{c^2}w=0.$$

For large value of ω/c, a standard method of finding the asymptotic solutions[2] is to take

$$w=e^{i\omega\sigma(\mathbf{x})}[w_0(\mathbf{x})+\frac{1}{\omega}w_1(\mathbf{x})+\dots], \tag{4.5}$$

where functions $\sigma(\mathbf{x})$ and $w_j(\mathbf{x})$ are to be determined. Substituting (4.5) into Helmholtz's equation and keeping the asymptotically leading terms only, we obtain

$$e^{i\omega\sigma(\mathbf{x})}[\omega^2(-\sigma_{,\alpha}\sigma_{,\alpha}+\frac{1}{c^2})(w_0+\frac{1}{\omega}w_1)+i\omega(\Delta\sigma w_0+2\sigma_{,\alpha}w_{0,\alpha})+\dots]=0.$$

The exponential function can be dropped in this equation. Then, equating the asymptotically leading terms at ω^2 and ω to zero, we obtain

[2] Which is called WKB-method [6].

$$\begin{aligned}\sigma_{,\alpha}\sigma_{,\alpha} &= \frac{1}{c^2}, \\ \Delta\sigma w_0 + 2\sigma_{,\alpha}w_{0,\alpha} &= 0.\end{aligned} \tag{4.6}$$

The first equation is called eikonal equation which determines $\sigma(\mathbf{x})$. The second equation, called transport equation, can be used to find $w_0(\mathbf{x})$.

The eikonal equation $(4.6)_1$, as a nonlinear partial differential equation of first order, may be solved by the method of characteristic curves [12]. If we introduce $p_\alpha = \sigma_{,\alpha}$ and write this equation as

$$H \equiv \frac{1}{2}cp_\alpha p_\alpha - \frac{1}{2}c^{-1} = 0,$$

the characteristic curves are defined by the equation

$$\frac{dx_\alpha}{ds} = cp_\alpha.$$

Parameter s is the arc-length along the characteristic curve, because $c^2 p_\alpha p_\alpha = 1$. The full set of characteristic equations reads

$$\frac{dx_\alpha}{ds} = cp_\alpha, \quad \frac{dp_\alpha}{ds} = 0, \quad \frac{d\sigma}{ds} = \frac{1}{c}.$$

Looking at the asymptotic solution (4.5), one may recognize that $\theta = \omega(\sigma(\mathbf{x}) - t)$ is the phase of the wave packet. Let us choose the initial phase such that $\theta = 0$ corresponds to the wave front. Thus, the equation of the wave front is $\sigma(\mathbf{x}) = t$. Since the vector $\mathbf{p} = \nabla\sigma$ is normal to the wavefront, the first equation for the characteristics tells us that the rays are also normal to it. The second equation shows that $\mathbf{p}$ is constant on the ray, so the rays must be straight lines. The new wavefront at time $t + t_1$ (with small t_1) can be constructed by drawing the family of straight lines normal to the wavefront at time t, and by the third equation, $\sigma = s/c$, so $t = s/c$, and the new wavefront is a distance ct_1 out along the rays (see Fig. 4.4). This is Huygens' principle which agrees also with Poisson's exact solution (4.4) found previously.

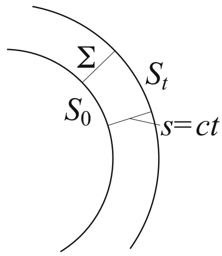

Fig. 4.4 Wavefront and rays

It remains to solve the transport equation which is the linear equation for w_0. Its characteristics are the same rays, so we can write this equation as

$$\frac{1}{w_0}\frac{dw_0}{ds} = -\frac{1}{2}\Delta\sigma.$$

The integration is straightforward once $\sigma(\mathbf{x})$ has been determined. But due to the implicit form of $\sigma(\mathbf{x})$ we proceed a little differently. First we note that $(4.6)_2$ takes the divergence form

$$(\sigma_{,\alpha}w_0^2)_{,\alpha}=0.$$

Let us consider a tube formed by rays going from the initial wavefront S_0 to the current wavefront S_t as shown in Fig. 4.4. We integrate this equation over the volume of the tube. The use of Gauss' theorem gives

$$\int n_\alpha \sigma_{,\alpha} w_0^2 \, da = 0,$$

where $\mathbf{n}$ is the outward normal and the surface integral must be taken over the sides Σ and ends S_0, S_t of the ray tube. As the rays are orthogonal to the wavefronts $\sigma = t$, $n_\alpha \sigma_{,\alpha} = 0$ on Σ. On S_t the normal $\mathbf{n}$ and $\nabla\sigma$ are in the same direction, so $n_\alpha \sigma_{,\alpha} = |\nabla\sigma| = 1/c$. Similarly, $n_\alpha \sigma_{,\alpha} = -|\nabla\sigma| = -1/c$ on S_0. Thus,

$$\int_{S_t} w_0^2 \, da = \int_{S_0} w_0^2 \, da.$$

This equation expresses the conservation of energy flux along the ray tube.

The geometrical optics can also be developed for anisotropic and inhomogeneous media [53]. However, one should be cautious near the point where $c = 0$ (called the turning point) as well as near the caustics, where this type of approximation needs to be modified.

4.2 Dispersive Waves

Differential Equation and Dispersion Relation. Typically, the differential equation governing the propagation of dispersive waves in a homogeneous medium can be written as

$$P(\partial_t, \partial_\alpha)u = 0. \tag{4.7}$$

Here $P(r, s_\alpha)$ is a polynomial of the variables r and s_α with constant coefficients, ∂_t and ∂_α are the partial derivatives with respect to t and x_α, respectively. Some examples in 1-D case which will be used as illustration are

$$\begin{aligned} u_{,tt} + \omega_0^2 u - c^2 u_{,xx} = 0, &\quad P(\partial_t, \partial_x) = \partial_t^2 + \omega_0^2 - c^2\partial_x^2, \\ u_{,tt} + \gamma^2 u_{,xxxx} = 0, &\quad P(\partial_t, \partial_x) = \partial_t^2 + \gamma\partial_x^4, \\ u_{,t} + \alpha u_{,x} + \beta u_{,xxx} = 0, &\quad P(\partial_t, \partial_x) = \partial_t + \alpha\partial_x + \beta\partial_x^3. \end{aligned} \tag{4.8}$$

The first equation describes free vibrations of a string with an additional restoring force proportional to u, or thickness vibrations of a rod [31]. It is also the Klein-Gordon equation of quantum mechanics. The second equation of (4.8) corresponds to Bernoulli-Euler's beam theory (3.27) with $\gamma = \sqrt{EI/\mu}$. The last equation is the linearized version of Korteweg-de Vries equation describing small amplitude long water waves and various other dispersive waves.

Since (4.7) is a linear differential equation with constant coefficients, its particular solutions always exist in form of harmonic waves

$$u(\mathbf{x},t)=e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)},$$

where $\mathbf{k}$ is the wave vector and ω the frequency. Indeed, substituting this Ansatz into (4.7) and using the property of exponential function, we see that $\mathbf{k}$ and ω have to be related by the equation

$$P(-i\omega, ik_\alpha)=0.$$

This is the so called dispersion relation which contains all the information about the differential equation. Knowing this dispersion relation we can restore the governing equation by using the correspondence: $\partial_t \leftrightarrow -i\omega$, $\partial_\alpha \leftrightarrow ik_\alpha$. Note that the above derivation can easily be generalized for the situation when $\mathbf{u}$ is a vector. In this case P becomes a matrix, whose elements are polynomials of r and s_α. The harmonic wave form of particular solutions remains, with a small modification that a constant vector $\mathbf{a}$ as a factor has to be included. Nontrivial solutions exist for the vanishing determinant of the matrix, whose elements are polynomials of $-i\omega$ and ik_α, yielding the dispersion relation (see exercise 4.4).

We assume that the dispersion relation may be solved with respect to ω giving real roots

$$\omega=\Omega(\mathbf{k}). \tag{4.9}$$

In general there will be a number of such solutions, with different functions $\Omega(\mathbf{k})$. We refer to them as *branches*. For example, if u satisfies Bernoulli-Euler's beam equation $(4.8)_2$, then the dispersion relation reads

$$-\omega^2+\gamma^2k^4=0.$$

Solving this with respect to ω, we obtain two branches

$$\omega=\gamma k^2, \quad \omega=-\gamma k^2.$$

In contrary, the linearized Korteweg-de Vries equation $(4.8)_3$ yields only one branch given by

$$\omega=\alpha k-\beta k^3.$$

For the present we study just one branch since the general solution is simply the linear superposition of them. The monochromatic plane wave corresponding to this branch is

$$u=\cos(\mathbf{k}\cdot\mathbf{x}-\Omega(\mathbf{k})t).$$

We call as before $\theta=\mathbf{k}\cdot\mathbf{x}-\Omega(\mathbf{k})t$ phase which determines the wave motion. Any particular phase surface moves in the space with the normal velocity $\Omega(\mathbf{k})/\kappa$ in the direction of $\mathbf{k}$, where $\kappa=|\mathbf{k}|$. We define the phase velocity as

$$\mathbf{c}=\frac{\Omega(\mathbf{k})}{\kappa}\mathbf{n},$$

where $\mathbf{n}$ is the unit vector in the $\mathbf{k}$-direction. For the hyperbolic waves governed by the equation $u_{,tt} = c^2 \Delta u$ considered in the previous Section the phase velocity is constant and agrees with the propagation speed c. In general $\mathbf{c}$ depends on κ, so different waves propagate with different velocities causing the change of shape. This explains the adjective "dispersive" for such waves. We classify waves as dispersive if $\Omega(\mathbf{k})$ is real and the determinant of the matrix $\frac{\partial^2 \Omega}{\partial k_\alpha \partial k_\beta}$ is not identically zero (see [53]). This definition excludes hyperbolic waves.

Solution. The general solution of (4.7) can be obtained by the linear superposition of particular solutions using Fourier's integral

$$u(\mathbf{x},t) = \int \psi(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x} - i\Omega(\mathbf{k})t} dk,$$

where $dk = dk_1 dk_2 dk_3$. Function $\psi(\mathbf{k})$ accounts for the intensity of waves with different $\mathbf{k}$ and may be chosen to satisfy arbitrary initial data, provided these data are described by regular functions admitting the Fourier transform. For illustration let us consider the first two equations in (4.8). Each of them has two branches $\omega = \pm\Omega(k)$, and since we are in 1-D situation,

$$u(x,t) = \int_{-\infty}^{\infty} \psi_1(k) e^{ikx - i\Omega(k)t} dk + \int_{-\infty}^{\infty} \psi_2(k) e^{ikx + i\Omega(k)t} dk. \tag{4.10}$$

As there are two branches, $u(x,t)$ must satisfy two initial conditions

$$u(x,0) = u_0(x), \quad u_{,t}(x,0) = v_0(x).$$

This leads to

$$\int_{-\infty}^{\infty} [\psi_1(k) + \psi_2(k)] e^{ikx} dk = u_0(x),$$
$$\int_{-\infty}^{\infty} -i\Omega(k)[\psi_1(k) - \psi_2(k)] e^{ikx} dk = v_0(x).$$

Applying the Fourier transform to these equations, we obtain

$$\psi_1(k) + \psi_2(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} u_0(x) e^{-ikx} dx = U_0(k),$$
$$-i\Omega(k)[\psi_1(k) - \psi_2(k)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} v_0(x) e^{-ikx} dx = V_0(k).$$

Solving the above equations with respect to $\psi_1(k)$ and $\psi_2(k)$ gives

$$\psi_1(k) = \frac{1}{2}\left[U_0(k) + \frac{iV_0(k)}{\Omega(k)}\right], \quad \psi_2(k) = \frac{1}{2}\left[U_0(k) - \frac{iV_0(k)}{\Omega(k)}\right].$$

Since $u_0(x)$ and $v_0(x)$ are real, their Fourier images $U_0(k)$ and $V_0(k)$ satisfy the properties

$$U_0(-k) = U_0^*(k), \quad V_0(-k) = V_0^*(k),$$

where asterisks denote complex conjugates. Thus, if $\Omega(k)$ is odd function, then

$$\psi_1(-k) = \psi_1^*(k), \quad \psi_2(-k) = \psi_2^*(k).$$

If $\Omega(k)$ is even function, we have

$$\psi_1(-k) = \psi_2^*(k), \quad \psi_2(-k) = \psi_1^*(k).$$

It is easy to check that the solution is real in both cases as expected.

Large Time Asymptotics. Although Fourier's integrals give the exact solution, its behavior is still difficult to analyze. For wave propagation it is important to know the behavior of solution in the limits $t \to \infty$ and $x \to \infty$ while x/t is held fixed. Let us analyze first the typical integral

$$u(x,t) = \int_{-\infty}^{\infty} \psi(k) e^{ikx - i\Omega(k)t} dk$$

in 1-D case. In the limit $t \to \infty$ at fixed x/t we can write this integral as

$$u(x,t) = \int_{-\infty}^{\infty} \psi(k) e^{-i\chi(k)t} dk, \tag{4.11}$$

where $\chi(k)$ is the following function

$$\chi(k) = \Omega(k) - k\frac{x}{t}.$$

Here x/t is regarded as a fixed parameter. The asymptotic behavior of integral (4.11) as $t \to \infty$ can be studied by the method of stationary phase [6], according to which the main contribution to the integral comes from the neighborhood of stationary points of $\chi(k)$ such that

$$\chi'(k) = \Omega'(k) - \frac{x}{t} = 0. \tag{4.12}$$

Otherwise, the integrand oscillates rapidly and makes little net contribution to $u(x,t)$.

Assume first that $\chi(k)$ has one stationary point at $k = k_s$. To find the leading contribution we expand $\psi(k)$ and $\chi(k)$ in Taylor's series near $k = k_s$

$$\psi(k) \simeq \psi(k_s), \quad \chi(k) \simeq \chi(k_s) + \frac{1}{2}\chi''(k_s)(k-k_s)^2,$$

provided $\chi''(k_s) \neq 0$. Substitution of these formulas in (4.11) leads to

$$u(x,t) \simeq \psi(k_s) e^{-i\chi(k_s)t} \int_{-\infty}^{\infty} e^{-\frac{i}{2}(k-k_s)^2 \chi''(k_s)t} dk.$$

The remaining integral can be reduced to the standard integral

$$\int_{-\infty}^{\infty} e^{-\alpha z^2} dz = \sqrt{\frac{\pi}{\alpha}}$$

by rotating the path of integration through $\pm\pi/4$; the sign should be chosen to be the same as that of $\chi''(k_s)$. Thus,

$$u(x,t) \simeq \psi(k_s)\sqrt{\frac{2\pi}{t|\chi''(k_s)|}} e^{ik_s x - i\Omega(k_s)t - \frac{i\pi}{4}\operatorname{sign}\chi''(k_s)}.$$

If there are several stationary points, the contributions from their neighborhoods have to be summed up to get the final result.

For the case of two branches with $\omega = \pm\Omega(k)$, the solution is given by (4.10). Assuming further that $\Omega'(k)$ is monotonic and positive for $k > 0$, we analyze the asymptotic behavior of (4.10) for $x > 0$. If $\Omega(k)$ is even, then $\Omega'(k)$ is odd and there is only one positive stationary point for the first branch denoted by $k_s(x,t)$: $\Omega'(k) = x/t$ for $x/t > 0$. The second branch has also one stationary point equal to $-k_s(x,t)$. Combining two contributions of the branches, we get

$$u(x,t) \simeq 2\mathrm{Re}\left[\psi_1(k_s)\sqrt{\frac{2\pi}{t|\chi''(k_s)|}} e^{ik_s x - i\Omega(k_s)t - \frac{i\pi}{4}\operatorname{sign}\chi''(k_s)}\right] \quad \text{for } x/t > 0. \quad (4.13)$$

It is easy to see that the case of odd function $\Omega(k)$ leads to the same result.

Group Velocity. At any point (x,t) formula (4.13) determines a local wave number $k_s(x,t)$ and the corresponding local frequency $\omega_s(x,t) = \Omega(k_s(x,t))$. By introducing a phase

$$\theta(x,t) = k_s(x,t)x - \omega_s(x,t)t,$$

we may present (4.13) in the form

$$u(x,t) \simeq \mathrm{Re}[A(x,t)e^{i\theta(x,t)}], \quad (4.14)$$

where the complex amplitude is

$$A(x,t) = 2\psi_1(k_s)\sqrt{\frac{2\pi}{t|\chi''(k_s)|}} e^{-\frac{i\pi}{4}\operatorname{sign}\chi''(k_s)}.$$

The difference between (4.14) and the monochromatic waves is that A, k, and ω are no longer constants. However, this asymptotic formula still represents a nonuniform wave packet, with a phase θ describing the oscillations between crests and troughs. It is natural to define the local wave number and frequency as $\theta_{,x}$ and $-\theta_{,t}$, respectively. In our nonuniform case we have

$$\theta_{,x} = k_{s,x}x + k_s - \Omega'(k_s)k_{s,x}t = k_s(x,t),$$
$$\theta_{,t} = k_{s,t}x - \Omega'(k_s)k_{s,t}t - \Omega(k_s) = -\omega_s(x,t),$$

so the local wave number and frequency introduced above agree with these definitions. Moreover, the local wave number and frequency satisfy the dispersion relation even in the nonuniform wave packet. Mention that the relative changes of the local wave number k_s in one period and in one wavelength are small. Indeed, from (4.12) we see that the quantities

$$\frac{k_{s,x}}{k_s} = \frac{\Omega'}{k_s\Omega''}\frac{1}{x}, \quad \frac{k_{s,t}}{k_s} = -\frac{1}{k_s\Omega''}\frac{1}{t}$$

are small for large x and t. Thus, $k_s(x,t)$ is a slowly changing function in one period and one wavelength. The same is true of the frequency ω_s and amplitude A.

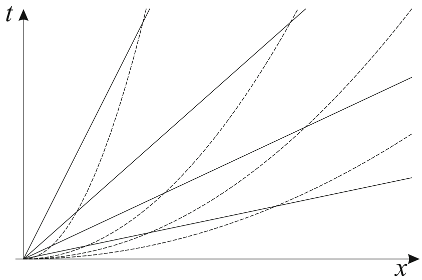

Fig. 4.5 Group (solid) and phase (dashed) lines for waves in beam

Let us have a closer look at the equation (4.12) determining $k_s(x,t)$. According to that equation an observer moving with the velocity $\Omega'(k_s)$ will see the wave number k_s and the frequency ω_s. Therefore we call the velocity

$$\Omega'(k) = \frac{d\omega}{dk}$$

group velocity, or the velocity for a group of waves. To illustrate the distinction between the phase and the group velocities we consider equation $(4.8)_2$ for Bernoulli-Euler's beam. The dispersion relation for the branch of waves propagating to the right is

$$\Omega(k) = \gamma k^2.$$

Therefore the equation determining k becomes $x/t = \Omega'(k) = 2\gamma k$. Thus,

$$k = \frac{x}{2\gamma t}, \quad \omega = \frac{x^2}{4\gamma t^2}, \quad \theta = kx - \omega t = \frac{x^2}{4\gamma t}.$$

The group lines of constant k and ω are the straight lines $\frac{x}{2\gamma t} = \text{const}$. The lines of constant phase $\theta = \text{const}$ are the parabola $\frac{x^2}{4\gamma t} = \text{const}$. These two families of lines are shown in Fig. 4.5. We see that the group velocity $\Omega'(k) = 2\gamma k$ is twice the phase velocity $\omega/k = \gamma k$ for waves propagating in Bernoulli-Euler's beam.

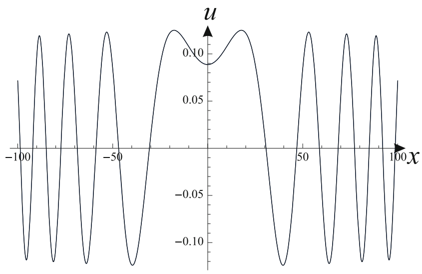

Fig. 4.6 Comparison of exact and approximate solutions

To compare the solution obtained by the numerical integration of Fourier's integrals with the asymptotic solution (4.14) let us set $\gamma = 1$ and assume the initial conditions as follows

$$u(x,0) = u_0(x) = 2\pi e^{-x^2}, \quad u_{,t}(x,0) = v_0(x) = 0.$$

Then $\psi_1(k) = e^{-k^2/4}/\sqrt{2}$ and the asymptotic solution takes the form

$$u(x,t) \simeq \psi_1(\frac{x}{2t})\sqrt{\frac{\pi}{t}}\cos(\frac{x^2}{4t} - \frac{\pi}{4}).$$

Fig. 4.6 plots the exact solution in terms of Fourier's integrals computed numerically at time $t = 100$ and the above asymptotic solution at the same time, where the results are nearly identical.

The other important role of the group velocity appears in studying the distribution of amplitude $A(x,t)$. It turns out that $|A|^2$ propagates with the group velocity. To show this let us compute the integral of $|A|^2$ between two points $x_2 > x_1 > 0$. From the above formula for A we have

$$Q(t) = \int_{x_1}^{x_2} AA^* \, dx = 8\pi \int_{x_1}^{x_2} \frac{\psi_1(k_s)\psi_1^*(k_s)}{t|\Omega''(k_s)|} \, dx.$$

In this integral k_s is the root of (4.12). Using the transformation $x = \Omega'(k)t$ as a change of variable $x \to k$, we rewrite $Q(t)$ in the form

$$Q(t) = 8\pi \int_{k_1}^{k_2} \psi_1(k)\psi_1^*(k) \, dk,$$

provided $\Omega''(k) > 0$, where k_1 and k_2 are defined by

$$x_1 = \Omega'(k_1)t, \quad x_2 = \Omega'(k_2)t.$$

If $\Omega''(k) < 0$, the order of the limits must be reversed. Now, if k_1 and k_2 are held fixed as t varies, $Q(t)$ remains constant. But for the fixed k_1 and k_2 the points x_1

and x_2 are moving with the group velocities. Thus, the total amount of $|A|^2$ between any pairs of group lines remains constant, and in this sense, $|A|^2$ propagates with the group velocity. Moreover, we will show in Section 4.4 that the energy also propagates with the group velocity. This puts the question to the radiation conditions for the dispersive waves.

Kinematic Derivation of Group Velocity. We see from the previous paragraph that the concept of group velocity is quite crucial in understanding the phenomenon of wave propagation. This concept must appear and be equally important for inhomogeneous media as well as for non-linear problems, where Fourier's analysis are not directly applicable. Therefore we try to develop below the direct kinematic approach based on the more intuitive arguments rather than using Fourier's integrals and the method of stationary phase. We assume that a wave packet under consideration possesses a phase function $\theta(x,t)$, and that the wave number and frequency defined by

$$k = \theta_{,x}, \quad \omega = -\theta_{,t}, \tag{4.15}$$

are *slowly* changing functions of x and t. If, further, we know or can derive for them a dispersion relation

$$\omega = \Omega(k), \tag{4.16}$$

then we have an equation for θ and we could proceed to solve it to determine the geometry of the wave pattern. The convenient way is to use the kinematic relation

$$k_{,t} + \omega_{,x} = 0,$$

which follows from (4.15). This equation can be regarded as the conservation of waves, with k being the density of waves and ω the flux of waves. Combining it with (4.16), we get a non-linear partial differential equation to determine $k(x,t)$

$$k_{,t} + C(k)k_{,x} = 0, \quad C(k) = \Omega'(k). \tag{4.17}$$

We see that the group velocity $C(k)$ is the propagation velocity for the wave number k. This equation can be solved by the method of characteristics. For an initial distribution $k = f(x)$ at $t = 0$ the solution is

$$k = f(\xi), \quad x = \xi + v_g(\xi)t,$$

where $v_g(\xi) = C(f(\xi))$. Thus, the observer moving with the group velocity sees always the same local wave number k. It is interesting that the above equation for k is non-linear and hyperbolic, even though the original problem is linear and in general non-hyperbolic as in example $(4.8)_2$. In this sense one can preserve the association of wave propagation with hyperbolic equations, but there is a considerable non-hyperbolic background.

Extensions to 2- and 3-D Cases. It is not difficult to extend the obtained results to 2- or 3-D problems. Since the exact solution is expressed in terms of multiple Fourier's integrals, the asymptotically leading terms in the limit $t \to \infty$ with $\mathbf{x}/t$ being held fixed can be obtained by the method of stationary phase. For d-dimensional space we can show that

$$u(\mathbf{x},t) = \int \psi(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x} - i\Omega(\mathbf{k})t} dk$$
$$\simeq \psi(\mathbf{k}_s) \left(\frac{2\pi}{t}\right)^{d/2} \left(\det\left|\frac{\partial\Omega}{\partial k_\alpha \partial k_\beta}\right|\right)^{-1/2} e^{i\mathbf{k}_s\cdot\mathbf{x} - i\Omega(\mathbf{k}_s)t + i\zeta},$$

where $\mathbf{k}_s$ satisfies the equation

$$\frac{\partial\Omega(\mathbf{k})}{\partial k_\alpha} = \frac{x_\alpha}{t},$$

and ζ depends on the number of factors $i\pi/4$ arising from the path rotation. We could use this asymptotic solution to study the group velocity in 2- or 3-D cases. However, it is simpler to develop the direct kinematic approach which may also be applied to weakly inhomogeneous media.

We consider the slowly varying wave packet in the form

$$u(\mathbf{x},t) = a\cos\theta,$$

where the amplitude a and the phase θ are functions of $\mathbf{x}$ and t. We define the wave vector $\mathbf{k}$ and frequency ω by

$$k_\alpha = \theta_{,\alpha}, \quad \omega = -\theta_{,t}. \tag{4.18}$$

We assume that a dispersion relation is known and can be written as

$$\omega = \Omega(\mathbf{x},\mathbf{k}). \tag{4.19}$$

For homogeneous media the dispersion relation does not depend on $\mathbf{x}$ and can be obtained from the monochromatic plane waves. For weakly inhomogeneous media it would appear reasonable to find the dispersion relations first for constant parameters of the media and then reinsert their dependence on $\mathbf{x}$. This will be justified by the variational-asymptotic method in Section 4.4.

Now, by eliminating θ from (4.18), we have

$$k_{\alpha,t} + \omega_{,\alpha} = 0, \quad k_{\alpha,\beta} - k_{\beta,\alpha} = 0.$$

Then, if $\omega = \Omega(\mathbf{x},\mathbf{k})$ is inserted into the first of these equations,

$$k_{\alpha,t} + \frac{\partial\Omega}{\partial k_\beta} k_{\beta,\alpha} = -\frac{\partial\Omega}{\partial x_\alpha}.$$

Since $k_{\alpha,\beta} = k_{\beta,\alpha}$, this may be modified to

$$k_{\alpha,t} + C_\beta k_{\alpha,\beta} = -\frac{\partial \Omega}{\partial x_\alpha}, \tag{4.20}$$

where

$$C_\beta = \frac{\partial \Omega}{\partial k_\beta}.$$

The group velocity **C** defined in this way is the propagation velocity in (4.20) for the determination of **k**. Equation (4.20) may be written in the characteristic form as

$$\frac{dk_\alpha}{dt} = -\frac{\partial \Omega}{\partial x_\alpha} \quad \text{on} \quad \frac{dx_\alpha}{dt} = \frac{\partial \Omega}{\partial k_\alpha}. \tag{4.21}$$

Note that **k** is constant on each characteristic when the medium is homogeneous in **x**, and then the characteristics are straight lines in the $(\mathbf{x}, t)$-space. Each value of **k** propagates with the corresponding constant group velocity $\mathbf{C}(\mathbf{k})$. For inhomogeneous media this is no longer valid: the values of **k** change as they propagate along the characteristics and the characteristics themselves become curves. However, since the medium is time-independent

$$\frac{d\omega}{dt} = \omega_{,t} + C_\beta \omega_{,\beta} = \frac{\partial \Omega}{\partial t} = 0,$$

the frequency remains constant along the characteristics.

It is interesting that equations (4.21) are identical with Hamilton's equations in mechanics if **x** and **k** are interpreted as coordinates and impulses while $\Omega(\mathbf{x}, \mathbf{k})$ is taken to be the Hamilton function (cf. Section 7.1). If instead of eliminating θ, we substitute for ω and **k** in the dispersion relation $-\partial\theta/\partial t$ and $\partial\theta/\partial\mathbf{x}$, respectively, then the following equation holds true

$$\frac{\partial \theta}{\partial t} + \Omega(\mathbf{x}, \frac{\partial \theta}{\partial \mathbf{x}}) = 0.$$

This is nothing else but the Hamilton-Jacobi equation, with θ being regarded as the action [5] (see also exercise 7.2).

4.3 Elastic Waveguide

In signal processing it is often necessary to delay signals by sending them through an elastic waveguide which serves as the delay line. Due to the interaction of waves with the free boundaries, this device exhibits dispersive waves with infinite number of branches. The other interesting property of elastic waveguides is that the phase and group velocities may have different signs for some high-frequency thickness branches. Such waves are called "backward waves". Their presence plays a decisive role in posing the radiation conditions. Guided wave propagation is used intensively also in nondestructive testing as well as in seismology.

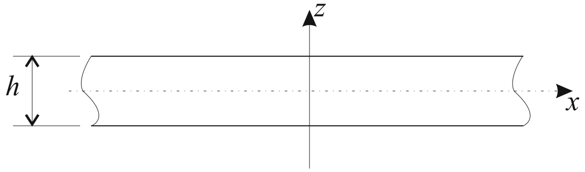

Fig. 4.7 Strip of thickness h

Equation of Motion. For simplicity, let us consider the most simple example of waveguide, namely an elastic strip of thickness h, as shown in Figure 4.7. The cartesian coordinate system is selected, with (x,y)-plane coinciding with the middle surface of the strip. The face surfaces of the strip are given by $z=\pm h/2$. Assuming that the strip is made of a homogeneous isotropic elastic material, we write down the three-dimensional equations of its motion in terms of the displacements u_α

$$\rho u_{\alpha,tt} = (\lambda+\mu)u_{\beta,\beta\alpha} + \mu u_{\alpha,\beta\beta},$$

where λ and μ are Lamé constants. The traction-free boundary conditions on the face surfaces $z=\pm h/2$ read

$$\sigma_{\alpha z}|_{z=\pm h/2} = [\lambda u_{\beta,\beta}\delta_{\alpha z} + \mu(u_{\alpha,z}+u_{z,\alpha})]|_{z=\pm h/2} = 0.$$

We non-dimensionalize these equations by introducing the following variables

$$\bar{t} = \frac{tc_s}{h}, \quad (\bar{x},\bar{y},\bar{z}) = \frac{1}{h}(x,y,z),$$

where $c_s=\sqrt{\mu/\rho}$ is the speed of shear wave in an infinite solid. The equations of motion and the boundary conditions then take the dimensionless form

$$\begin{aligned} u_{\alpha,tt} &= (1+\gamma)u_{\beta,\beta\alpha} + u_{\alpha,\beta\beta}, \\ [\gamma u_{\beta,\beta}\delta_{\alpha z} &+ (u_{\alpha,z}+u_{z,\alpha})]|_{z=\pm 1/2} = 0, \end{aligned} \tag{4.22}$$

where $\gamma=\lambda/\mu$ and the bars are dropped for short.

Rayleigh-Lamb Dispersion Relation. Let us look for particular solutions of the boundary-value problem (4.22) in the form

$$u_\alpha = f_\alpha(z)e^{i(kx-\omega t)}.$$

Substituting this into the equations (4.22), we obtain two uncoupled systems.

For the shear waves (SH-waves)[3] with

$$u_y = f_y(z)e^{i(kx-\omega t)}, \quad u_x = u_z = 0,$$

[3] This terminology arose in seismology where the boundary surface is usually horizontal.

we have

$$f_y'' + p_2^2 f_y = 0,\qquad f_y'|_{z=\pm 1/2} = 0, \tag{4.23}$$

where the prime denotes the derivative with respect to z and

$$p_2^2 = \omega^2 - k^2.$$

The eigenvalue problem (4.23) yields the following eigenfunctions:

$$\begin{aligned} f_y &= a\cos 2\pi n z, \quad p_2 = 2\pi n, \quad \text{for SS-waves,} \\ f_y &= a\sin \pi(2n+1)z, \quad p_2 = \pi(2n+1), \quad \text{for AS-waves,} \end{aligned} \tag{4.24}$$

where SS stands for the symmetric shear waves, while AS for the antisymmetric shear waves.

We turn now to the second case, for which

$$u_x = f_x(z)e^{i(kx-\omega t)}, \quad u_z = f_z(z)e^{i(kx-\omega t)}, \quad u_y = 0.$$

Substitution of these formulas into equations $(4.22)_1$ gives

$$\begin{aligned} f_x'' + (1+\gamma)ikf_z' + (\omega^2 - \eta^{-2}k^2)f_x &= 0, \\ \eta^{-2}f_z'' + (1+\gamma)ikf_x' + (\omega^2 - k^2)f_z &= 0, \end{aligned} \tag{4.25}$$

where

$$\eta^{-2} = \gamma + 2 = \frac{\lambda + 2\mu}{\mu}, \quad \eta = \sqrt{\frac{\mu}{\lambda + 2\mu}} = \sqrt{\frac{1-2\nu}{2-2\nu}},$$

with ν being Poisson's ratio. The boundary conditions $(4.22)_2$ become

$$\begin{aligned} \eta^{-2}f_z' + \gamma ikf_x &= 0, \\ f_x' + ikf_z &= 0. \end{aligned} \tag{4.26}$$

The eigenvalue problem (4.25) and (4.26) admits the symmetric and antisymmetric solutions of the type

$$\begin{aligned} &f_x(z) - \text{even}, \quad f_z(z) - \text{odd} \quad \text{(L-waves)}, \\ &f_x(z) - \text{odd}, \quad f_z(z) - \text{even} \quad \text{(F-waves)}. \end{aligned}$$

The characteristic equation of the system (4.25)

$$\det \begin{vmatrix} s^2 + \omega^2 - \eta^{-2}k^2 & (1+\gamma)iks \\ (1+\gamma)iks & \eta^{-2}s^2 + \omega^2 - k^2 \end{vmatrix} = 0$$

has four roots given by

$$s_{1,2} = \pm i p_1, \quad p_1 = \sqrt{\eta^2 \omega^2 - k^2},$$
$$s_{3,4} = \pm i p_2, \quad p_2 = \sqrt{\omega^2 - k^2}.$$

Therefore the symmetric solutions corresponding to longitudinal waves (L-waves) read

$$\begin{aligned} f_x &= i(Ak \cos p_1 z + B p_2 \cos p_2 z), \\ f_z &= -A p_1 \sin p_1 z + Bk \sin p_2 z, \end{aligned} \tag{4.27}$$

where A and B are still unknown constants. The four boundary conditions on $z = \pm 1/2$ reduce to two equations in A and B

$$\begin{aligned} (k^2 - p_2^2) \cos \frac{p_1}{2} A + 2k p_2 \cos \frac{p_2}{2} B &= 0, \\ -2k p_1 \sin \frac{p_1}{2} A + (k^2 - p_2^2) \sin \frac{p_2}{2} B &= 0. \end{aligned} \tag{4.28}$$

Equating the determinant to zero, we obtain from (4.28) the dispersion relation

$$(k^2 - p_2^2)^2 \sin(p_2/2) \cos(p_1/2) + 4k^2 p_1 p_2 \sin(p_1/2) \cos(p_2/2) = 0. \tag{4.29}$$

This is the Rayleigh-Lamb dispersion relation for the propagation of the L-waves in this waveguide. From (4.28) we also obtain the amplitude ratio

$$\frac{A}{B} = -\frac{2k p_2 \cos(p_2/2)}{(k^2 - p_2^2) \cos(p_1/2)} = \frac{(k^2 - p_2^2) \sin(p_2/2)}{2k p_1 \sin(p_1/2)}.$$

Next, we consider the antisymmetric solutions corresponding to flexural waves (F-waves), which are given by

$$\begin{aligned} f_x &= i(Ck \sin p_1 z - D p_2 \sin p_2 z), \\ f_z &= C p_1 \cos p_1 z + Dk \cos p_2 z, \end{aligned} \tag{4.30}$$

where C and D are unknown constants. The traction-free boundary conditions at $z = \pm 1/2$ reduce also in this case to two equations for C and D

$$\begin{aligned} (k^2 - p_2^2) \sin \frac{p_1}{2} C - 2k p_2 \sin \frac{p_2}{2} D &= 0, \\ 2k p_1 \cos \frac{p_1}{2} C + (k^2 - p_2^2) \cos \frac{p_2}{2} D &= 0. \end{aligned}$$

Since the determinant should vanish to guarantee nontrivial solutions, we derive from here the following dispersion relation for the F-waves:

$$(k^2 - p_2^2)^2 \cos(p_2/2) \sin(p_1/2) + 4k^2 p_1 p_2 \cos(p_1/2) \sin(p_2/2) = 0. \tag{4.31}$$

This is the Rayleigh-Lamb dispersion relation for F-waves. We also obtain the equation for the ratio C/D

$$\frac{C}{D} = \frac{2kp_2 \sin(p_2/2)}{(k^2 - p_2^2)\sin(p_1/2)} = -\frac{(k^2 - p_2^2)\cos(p_2/2)}{2kp_1 \cos(p_1/2)}.$$

Mention that both equations (4.29) and (4.31) can be combined in a single equation

$$\frac{\tan(p_2/2)}{\tan(p_1/2)} = -\left[\frac{4p_1 p_2 k^2}{(k^2 - p_2^2)^2}\right]^{\pm 1}, \quad \begin{cases} + & \text{for L-waves,} \\ - & \text{for F-waves.} \end{cases} \tag{4.32}$$

Dispersion Curves. The dispersion relations (4.24), (4.29) and (4.31) were obtained independently by Rayleigh and Lamb [28, 44]. However, due to their complexity, the full analysis of branches of the dispersion curves in the (k, ω)-plane, as well as branches with imaginary and complex wave number k, was completed much later (see, for instance, [31]). We provide here the detailed asymptotic analysis and numerical simulations of these equations.

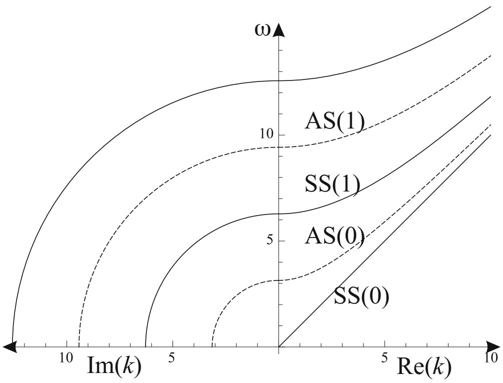

Fig. 4.8 Dispersion curves of shear waves

For SH-waves the dispersion relation (4.24) shows that for each number $n = 0, 1, 2, \ldots$ there are two branches

$$\omega = \pm\sqrt{\pi^2 (2n)^2 + k^2}, \quad \text{for SS(n)-waves,}$$
$$\omega = \pm\sqrt{\pi^2 (2n+1)^2 + k^2}, \quad \text{for AS(n)-waves.}$$

The plus or minus sign indicates the direction of wave propagation. All SS- and AS-waves, except SS(0), are dispersive. At some real and fixed wave number k the eigenfunctions (4.24) form a complete orthogonal basis in the space of regular functions of z. Thus, the series of Fourier's integrals over all branches solves the initial value problem for the infinite strip with arbitrary regular initial displacement $u_{y0}(x,z)$ and velocity $v_{y0}(x,z)$. The solvability of signaling problem for a semi-infinite strip requires the inclusion of solutions with imaginary k. We observe that the wave number k becomes imaginary for $\omega < \omega_c$, where $\omega_c = 2n\pi$ for SS(n) and $\omega_c = (2n+1)\pi$ for AS(n). The frequency ω_c at which the group velocity becomes zero is called a cutoff frequency. Thus, the free propagation of the corresponding branch does not occur at frequencies lower than the cutoff frequency. Several branches of the dispersion curves are plotted in Fig. 4.8. Since the dispersion curves for real k are symmetrical about the ω-axis, the (k,ω)-half-plane with negative real k can be replaced by the (k,ω)-half-plane with positive imaginary k. Looking at the dispersion curves we recognize that at a given fixed frequency there are only a finite number of real k for SH-waves. Thus, we have only a finite number of propagating waves. To satisfy arbitrary boundary conditions for a semi-infinite strip at $x = 0$ in the signaling problem, we have to combine these propagating waves with an infinite number of solutions having imaginary k and corresponding to non-propagating modes. These modes describe vibrations which are localized near the edge of the strip.

We turn now to the longitudinal and flexural waves characterized by the dispersion relation (4.32) and consider the case of real k. Depending on whether (k,ω) is found in the regions I, II, or III, as shown in Figure 4.9, we may have p_1, p_2 being both imaginary, one imaginary and one real, or both real, respectively. The dispersion relations will alter their forms accordingly. In the region I $p_1 = iq_1$, $p_2 = iq_2$, where $q_1 = \sqrt{k^2 - \eta^2\omega^2}$, $q_2 = \sqrt{k^2 - \omega^2}$. The dispersion relations take the form

$$\frac{\tanh(q_2/2)}{\tanh(q_1/2)} = \left[\frac{4q_1q_2k^2}{(k^2+q_2^2)^2}\right]^{\pm 1}, \quad \begin{cases} + & \text{for L-waves,} \\ - & \text{for F-waves.} \end{cases}$$

To find the asymptote of the first F-branch for small k and ω we expand the hyperbolic tangent

$$\tanh x = x(1 - \frac{1}{3}x^2 + \ldots).$$

Retaining the first two terms, we reduce the dispersion relation for F-waves to

$$\frac{q_2(1-\frac{1}{3}(q_2/2)^2)}{q_1(1-\frac{1}{3}(q_1/2)^2)} = \frac{(k^2+q_2^2)^2}{4q_1q_2k^2}.$$

We put this in the form

$$-(k^2-q_2^2)^2 = \frac{1}{3}k^2q_2^4 - \frac{1}{12}q_1^2(k^2+q_2^2)^2.$$

Expanding this and keeping the terms according to Newton's rule, we obtain the asymptotic formula

$$\omega^2 = \frac{1}{6(1-\nu)}k^4 + O(k^6).$$

which agrees with that of the plate theory.

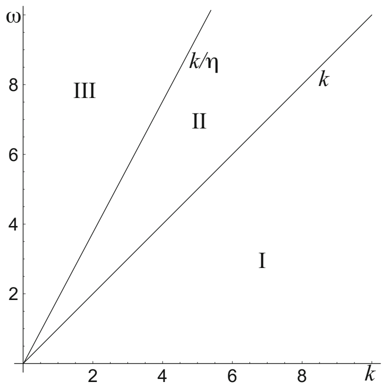

Fig. 4.9 Three regions of the (k,ω)-plane

In the region II $p_1 = iq_1$, and equation (4.32) becomes

$$\frac{\tan(p_2/2)}{\tanh(q_1/2)} = \pm\left[\frac{4q_1p_2k^2}{(k^2-p_2^2)^2}\right]^{\pm1}, \quad \begin{cases} + & \text{for L-waves,} \\ - & \text{for F-waves.} \end{cases}$$

The lowest F-branch has no roots in this region. For the lowest L-branch we replace $\tan x \sim x$ and $\tanh x \sim x$ giving

$$\frac{p_2}{q_1} = \frac{4q_1p_2k^2}{(k^2-p_2^2)^2}, \quad \text{or} \quad (k^2-p_2^2)^2 = 4k^2q_1^2.$$

Keeping the main terms in this equation we find that

$$\omega^2 = \frac{2}{1-\nu}k^2 + O(k^4),$$

which agrees again with the plate theory [31].

Let us consider now the high-frequency branches of L- and F-waves. We are interested in the asymptotic behavior of the dispersion curves near the cutoff frequencies in the long-wave range $k \ll 1$. Since the dispersion curves are in the range $\omega \sim 1$ and $k \ll 1$, we have to analyze (4.29) and (4.31) in the region III of the (k,ω)-plane (see Figure 4.9). Setting $k=0$ in (4.29), we see that the cutoff frequencies ω_c of L-waves are the roots of the equation

$$\sin\frac{\omega_c}{2}\cos\frac{\eta\omega_c}{2} = 0.$$

It implies that

$$\omega_c = 2\pi n, \quad \text{or} \quad \omega_c = \frac{\pi(2n+1)}{\eta}.$$

The first family of roots corresponds to the cutoff frequencies of the series $L_{\|}$, the second one to the cutoff frequencies of the series $L_{\perp}$.

We turn to the branch $L_{\|}(n)$. To study the asymptotics of the dispersion curve near the cutoff frequency $\omega_c = 2\pi n$ we introduce the notation

$$\omega^2 = \omega_c^2 + y, \quad k^2 = x,$$

with x and y being small quantities. Expanding the left-hand side of equation (4.29) in the Taylor series of x and y and keeping only the principal terms in accordance with Newton's rule, we obtain

$$\omega_c^4 \cos\frac{\omega_c}{2}\frac{1}{4\omega_c}(y-x)\cos\frac{\eta\omega_c}{2} + 4x\eta\omega_c^2 \sin\frac{\eta\omega_c}{2}\cos\frac{\omega_c}{2} = 0.$$

Solving this with respect to y we get finally

$$\omega^2 = \omega_c^2 + (1 - \frac{16\eta\tan(\eta\omega_c/2)}{\omega_c})k^2.$$

For the branch $L_{\perp}(n)$ with $\omega_c = \pi(2n+1)/\eta$ we obtain after performing the same operations

$$\omega^2 = \omega_c^2 + (\frac{1}{\eta^2} + \frac{16\cot(\omega_c/2)}{\omega_c})k^2.$$

Analogously, the asymptotic analysis of the Rayleigh-Lamb equation for F-waves leads to the following cutoff frequencies

$$\omega_c = \frac{2\pi n}{\eta}, \quad \text{or} \quad \omega_c = \pi(2n+1).$$

The first family of roots corresponds to the cutoff frequencies of the series $F_{\perp}$, the second one to the cutoff frequencies of the series $F_{\|}$. Similar asymptotic formulas for the corresponding dispersion curves in the long-wave range can also be obtained (see exercise 4.9).

In the above consideration we implicitly assume the value of η such that $\cos(\eta\pi n) \neq 0$. In the opposite case the coefficient at y in the approximate dispersion equation vanishes, and the above equation fails to provide the true asymptotics for long waves. Consider, for definiteness, the branch $L_{\|}(n)$ and introduce the new variables

$$\omega = \omega_c + y, \quad k^2 = x.$$

Expanding (4.29) in x and y and keeping their principal terms, we arrive at

$$-\omega_c^4 \cos\frac{\omega_c}{2}\sin\frac{\eta\omega_c}{2}\frac{\eta}{4}y^2 + 4x\eta\omega^2\sin\frac{\eta\omega_c}{2}\cos\frac{\omega_c}{2} = 0,$$

yielding

$$\omega = \omega_c \pm \frac{4}{\omega_c} k.$$

One can see from the last equation that the group velocity $v_g = d\omega/dk$ of $L_{\|}(1)$ does not vanish at $k = 0$, but is equal to $\pm 2/\pi$, and consequently, the wave packet moves without deformation in the long-wave range. It is also interesting to observe that, for $\nu = 1/3$, the cutoff frequency of the branch $L_{\|}(1)$ coincides with that of the branch $L_{\perp}(0)$.

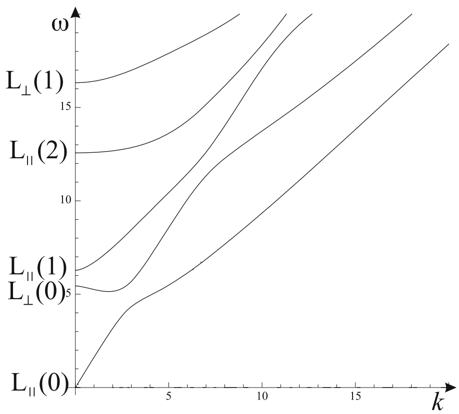

Fig. 4.10 Dispersion curves of L-waves

Figs. 4.10 and 4.11 show the dispersion curves of L- and F-waves, respectively, for $\nu = 0.25$. Since the dispersion curves are symmetrical about the ω-axis, it is enough to show them in the first quadrant $k > 0$, $\omega > 0$. The lowest branches of these waves, $L_{\|}(0)$ and $F_{\perp}(0)$, begin from the origin and approach asymptotically the straight line $\omega = v_r k$ as $k \to \infty$, where $v_r = c_r/c_s$ is the dimensionless Rayleigh's wave speed which may be obtained as the positive real root of the equation

$$v_r^6 - 8v_r^4 + (24 - 16\eta^2)v_r^2 + 16(\eta^2 - 1) = 0.$$

All other branches are high-frequency thickness branches which begin at the corresponding cutoff frequencies and approach the straight line $\omega = k$ as $k \to \infty$. This means that the wave speed of these branches approaches that of the shear waves in an infinite solid, $c_s = \sqrt{\mu/\rho}$, as $k \to \infty$. It is interesting that the dispersion curves of some branches, say $L_{\perp}(0)$ or $F_{\|}(1)$, have negative curvatures and slopes near $k = 0$. We can recognize this also from the asymptotic formulas in the long-wave range derived previously for these branches. Indeed, let us consider, for example, the branch $L_{\perp}(0)$ for which $\omega_c = \pi/\eta$ and

$$\omega^2 = (\pi/\eta)^2 + (\frac{1}{\eta^2} + \frac{16\cot(\omega_c/2)}{\omega_c})k^2$$

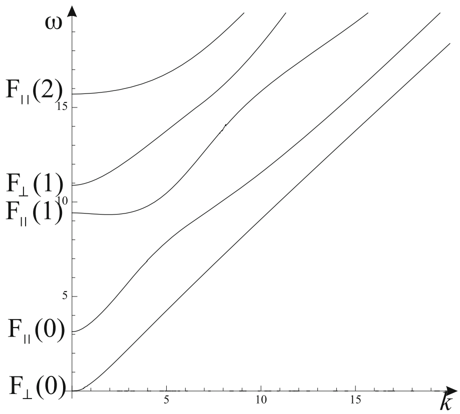

Fig. 4.11 Dispersion curves of F-waves

for small k. If $\nu = 0.25$, then the coefficient at k^2 is negative and equal to

$$\frac{1}{\eta^2} + \frac{16\cot(\omega_c/2)}{\omega_c} = -3.56865.$$

Consequently, the phase and group velocities have different signs in the long-wave range. Such waves carry energy in one direction but their phase surfaces appear to propagate in the opposite direction. Because of this property they are called "backward waves".

Now let us consider the solvability of the initial value problem for an infinite waveguide and for arbitrary initial conditions. This solvability is guaranteed if the eigenfunctions found in (4.27) and (4.30) form a complete orthonormal basis in the space of vector-valued functions of z. To show that this is the case we rewrite the equations (4.25) in the operator form

$$\mathbf{Lf} = \lambda \mathbf{f},$$

where $\lambda = \omega^2$ and

$$\mathbf{f} = \begin{pmatrix} f_x(z) \\ f_z(z) \end{pmatrix}, \quad \mathbf{Lf} = \begin{pmatrix} -f_{x,zz} - (1+\gamma)ikf_{z,z} + \eta^{-2}k^2 f_x \\ -\eta^{-2} f_{z,zz} - (1-\gamma)ikf_{x,z} + k^2 f_z \end{pmatrix}.$$

It is easy to check the following property: if k is real, then the operator $\mathbf{L}$ is Hermitian in the sense that

$$\langle \mathbf{g}, \mathbf{Lf} \rangle - \langle \mathbf{Lg}, \mathbf{f} \rangle = \int_{-1/2}^{1/2} [\mathbf{g}^* \cdot \mathbf{Lf} - \mathbf{f}^* \cdot \mathbf{Lg}] \, dz = 0$$

for arbitrary two vector-valued functions $\mathbf{f}(z)$ and $\mathbf{g}(z)$ satisfying the boundary conditions (4.26). Therefore, the eigenvalue problem (4.25) and (4.26) has a discrete spectrum and the eigenfunctions form a complete orthonormal basis in this function space [23]. Thus, the series of Fourier's integrals over all branches solves the initial value problem for the infinite strip with arbitrary regular initial displacements $\mathbf{u}_0(x,z)$ and velocities $\mathbf{v}_0(x,z)$.

The signaling problem is much more challenging, where many questions remain still open.[4] Similar to the SH-waves, we have at a given fixed frequency only a finite number of real k for L- or F-waves as seen from Figs. 4.10 and 4.11. But in contrast to the SH-waves, the number of solutions with imaginary k is also finite. It can be shown, however, that there exists a countable number of solutions with the complex conjugate k. To satisfy arbitrary boundary conditions for a semi-infinite strip at $x=0$ in the signaling problem, we have to combine the propagating waves with those solutions having imaginary and complex conjugate k. These modes describe vibrations which are localized near the edge of the strip. The other issue is the radiation conditions. Since the backward waves are present, we propose to select among propagating waves only those with positive group velocities which transport the energy from the edge of the strip to infinity. In Fig. 4.12 presenting the dispersion curves of branches $\mathrm{L}_{\|}(0)$ and $\mathrm{L}_{\perp}(0)$ near the cutoff frequency $\omega_c=\pi/\eta$ the only waves corresponding to points A, B, C are selected if the frequency of the sent signal is fixed at the level indicated by the horizontal line.

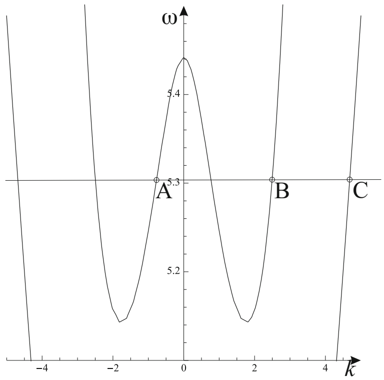

Fig. 4.12 Selected waves

4.4 Energy Method

Energy Balance Equation. Since waves transport energy from one part of the medium to another, the energy balance of its fixed part must involve energy flux entering the boundary. We want to derive the energy balance equation from Euler-Lagrange's equations of motion (3.51) in the general case. Multiplying equations (3.51) with $u_{i,t}$ and summing up over i yields

$$u_{i,t}\frac{\partial}{\partial t}\frac{\partial L}{\partial u_{i,t}}+u_{i,t}\frac{\partial}{\partial x_\alpha}\frac{\partial L}{\partial u_{i,\alpha}}-u_{i,t}\frac{\partial L}{\partial u_i}=0.$$

This equation can be transformed to

$$\frac{\partial}{\partial t}(u_{i,t}\frac{\partial L}{\partial u_{i,t}})+\frac{\partial}{\partial x_\alpha}(u_{i,t}\frac{\partial L}{\partial u_{i,\alpha}})-\underline{u_{i,tt}\frac{\partial L}{\partial u_{i,t}}}-\underline{u_{i,\alpha t}\frac{\partial L}{\partial u_{i,\alpha}}}-\underline{u_{i,t}\frac{\partial L}{\partial u_i}}=0.$$

[4] For instance, the generalization of Saint-Venant's principle to dynamics [15].

Since $u_{i,t}$ enter only the kinetic energy density which is quadratic with respect to $u_{i,t}$, the first term gives

$$\frac{\partial}{\partial t}(u_{i,t}\frac{\partial L}{\partial u_{i,t}}) = \frac{\partial}{\partial t}(u_{i,t}\frac{\partial K}{\partial u_{i,t}}) = \frac{\partial}{\partial t}(2K).$$

It is easy to see that the last three underlined terms lead to

$$-u_{i,tt}\frac{\partial L}{\partial u_{i,t}} - u_{i,\alpha t}\frac{\partial L}{\partial u_{i,\alpha}} - u_{i,t}\frac{\partial L}{\partial u_i} = -\frac{\partial}{\partial t}(L).$$

So, we obtain the energy balance equation in the form

$$\frac{\partial}{\partial t}(K+U) + \frac{\partial}{\partial x_\alpha}(u_{i,t}\frac{\partial L}{\partial u_{i,\alpha}}) = 0. \tag{4.33}$$

The first term of (4.33) corresponds to the local change of the total energy density $E = K + U$, while its gradient term describes the energy transported by the wave motion. We therefore call $J_\alpha = u_{i,t}\frac{\partial L}{\partial u_{i,\alpha}}$ an energy flux.

Energy Propagation. To see how the energy is transported by the traveling waves let us first consider the 1-D Klein-Gordon equation $(4.8)_1$ which can be obtained from the following Lagrangian

$$L = \frac{1}{2}u_{,t}^2 - \frac{1}{2}(\omega_0^2 u^2 + c^2 u_{,x}^2).$$

According to the energy balance equation (4.33) we have the energy density

$$E = \frac{1}{2}u_{,t}^2 + \frac{1}{2}(\omega_0^2 u^2 + c^2 u_{,x}^2),$$

and the energy flux

$$J = -c^2 u_{,t} u_{,x}.$$

As we know, the asymptotically leading term of solution can be written in form of wave packet

$$u \simeq \mathrm{Re}(Ae^{i\theta}) = a\cos(\theta + \phi),$$

where $a = |A|$, $\phi = \arg A$. The wave number $k = \theta_{,x}$, the frequency $\omega = -\theta_{,t}$, the initial phase ϕ, and the amplitude a are slowly changing functions of x and t. We use this asymptotic formula to compute the energy density and energy flux.

First we compute the term $\frac{1}{2}u_{,t}^2$ in the kinetic energy density

$$\frac{1}{2}u_{,t}^2 \simeq \frac{1}{2}\omega^2 a^2 \sin^2(\theta + \phi)$$

together with terms involving $a_{,t}$ and $\phi_{,t}$. Since a and ϕ are slowly changing functions of t, these terms can be neglected in the first approximation. Treating the other terms in the same way, we obtain for the energy density

$$E = \frac{1}{2}(\omega^2 + c^2k^2)a^2 \sin^2(\theta + \phi) + \frac{1}{2}\omega_0^2 a^2 \cos^2(\theta + \phi).$$

Similarly, the energy flux becomes

$$J = c^2 \omega k a^2 \sin^2(\theta + \phi).$$

Now let us take the average of these quantities over one period. Since the average values of $\cos^2(\theta + \phi)$ and $\sin^2(\theta + \phi)$ over one period are equal to $1/2$, we get

$$\bar{E} = \frac{1}{4}(\omega^2 + c^2k^2 + \omega_0^2)a^2, \quad \bar{J} = \frac{1}{2}c^2 \omega k a^2,$$

where bars over quantities denote their averaged values over one period. For Klein-Gordon equation the dispersion relation of waves propagating to the right reads

$$\omega = \sqrt{\omega_0^2 + c^2k^2}.$$

Therefore

$$\bar{E} = \frac{1}{2}(c^2k^2 + \omega_0^2)a^2, \quad \bar{J} = \frac{1}{2}c^2 k \sqrt{\omega_0^2 + c^2k^2}a^2.$$

As we remember, the group velocity is

$$C(k) = \frac{d\Omega(k)}{dk} = \frac{c^2 k}{\sqrt{\omega_0^2 + c^2k^2}},$$

so we get the following relation

$$\bar{J} = C(k)\bar{E}.$$

This relation turns out to be general.

Based on the above relation we are going to derive now the average energy balance equation

$$\bar{E}_{,t} + (C\bar{E})_{,x} = 0, \tag{4.34}$$

which can be interpreted as follows: the total average energy between any two group lines remains constant, or, in other words, energy propagates with the group velocity. For, if we consider the energy

$$E(t) = \int_{x_1(t)}^{x_2(t)} \bar{E}\, dx$$

between two points $x_1(t)$ and $x_2(t)$ moving with the group velocities $C(k_1)$, $C(k_2)$, respectively, then

$$\frac{dE}{dt} = \int_{x_1}^{x_2} \frac{\partial \bar{E}}{\partial t}\, dx + C_2\bar{E}_2 - C_1\bar{E}_1 = 0$$

if (4.34) is valid. Conversely, (4.34) is just the limit of the last equation as $x_2 - x_1 \to 0$.

To prove (4.34) we use the above formula for the average energy $\bar{E} = f(k)a^2$. Substituting it into the left-hand side of (4.34), we obtain

$$\bar{E}_{,t} + (C\bar{E})_{,x} = f(k)[(a^2)_{,t} + (Ca^2)_{,x}] + f'(k)a^2(k_{,t} + Ck_{,x}).$$

The last term on the right-hand side vanishes due to (4.17). By the same arguments given for $\bar{E}$, the first term must vanish too since the expression in the square brackets is the differential form of the result found in Section 4.2 that

$$Q(t) = \int_{x_1(t)}^{x_2(t)} a^2\, dx$$

remains constant between any two group lines.

The established equations of energy propagation can easily be extended to the cases involving more unknown functions and to higher dimension. Consider, for example, the scalar Klein-Gordon equation in 3-D case corresponding to the Lagrangian

$$L = \frac{1}{2}u_{,t}^2 - \frac{1}{2}(\omega_0^2 u^2 + c^2 u_{,\alpha}u_{,\alpha}).$$

From (4.33) follows the energy balance equation

$$E_{,t} + J_{\alpha,\alpha} = 0,$$

where

$$E = \frac{1}{2}u_{,t}^2 + \frac{1}{2}\omega_0^2 u^2 + \frac{1}{2}c^2 u_{,\alpha}u_{,\alpha}, \quad J_\alpha = -c^2 u_{,t}u_{,\alpha}.$$

For a slowly varying wave packet $u = a\cos(\theta + \phi)$ the average values of E and J_α over one period are

$$\bar{E} = \frac{1}{4}(\omega^2 + c^2 k_\alpha k_\alpha + \omega_0^2)a^2, \quad \bar{J}_\alpha = \frac{1}{2}c^2 \omega k_\alpha a^2,$$

with $\mathbf{k} = \nabla\theta$ being the wave vector and $\omega = -\theta_{,t}$ the frequency. Since the dispersion relation for the first branch reads $\omega = \Omega(\mathbf{k}) = \sqrt{\omega_0^2 + c^2|\mathbf{k}|^2}$, we see that

$$\bar{J}_\alpha = C_\alpha \bar{E},$$

where C_α is the group velocity

$$C_\alpha = \frac{\partial \Omega}{\partial k_\alpha} = \frac{c^2 k_\alpha}{\sqrt{\omega_0^2 + c^2|\mathbf{k}|^2}}.$$

The average energy balance equation becomes

$$\bar{E}_{,t} + (C_\alpha \bar{E})_{,\alpha} = 0. \tag{4.35}$$

Equivalently, the total energy in any volume $V(t)$ moving with the group lines remains constant

$$\frac{d}{dt}\int_{V(t)} \bar{E}dx = \int_{V(t)} \bar{E}_{,t}\,dx + \int_{S(t)} \bar{E}C_\alpha n_\alpha da = 0,$$

where $S(t)$ is the boundary of $V(t)$, $\mathbf{n}$ is the outward normal vector to $S(t)$, and $C_\alpha n_\alpha$ is its normal velocity. The last equation is obtained from (4.35) by integrating it over $V(t)$ and applying Gauss' theorem. Note that equation (4.35) can be presented in the characteristic form

$$\frac{d\bar{E}}{dt} = -\frac{\partial C_\alpha}{\partial x_\alpha}\bar{E} \quad \text{on} \quad \frac{dx_\alpha}{dt} = C_\alpha(\mathbf{k}).$$

So, the energy decays due to the divergence $C_{\alpha,\alpha}$ of the group lines. This effect is due lonely to the dispersion as there is no energy loss in this case.

It seems clear that these results should be established once and for all by general arguments without pursuing the detailed derivation each time. Such arguments are provided by the variational-asymptotic method.

Variational-Asymptotic Method. In this paragraph we are going to apply the variational-asymptotic method to quadratic functionals only. The generalization to non-linear problems will be given in Chapter 8.

Consider the variational problem in form of Hamilton's variational principle: find the extremal of the action functional

$$I[u_i(\mathbf{x},t)] = \iint\limits_R L(u_i, u_{i,\alpha}, u_{i,t})\,dx\,dt, \tag{4.36}$$

where $R = V \times (t_0, t_1)$ is any finite and fixed region in $(d+1)$-dimensional space-time. We assume that u_i are prescribed at the boundary ∂R. We look for the extremal of this variational problem in form of a slowly varying wave packet[5]

$$u_i = \psi_i(\theta, \mathbf{x}, t), \tag{4.37}$$

where θ is a function of $\mathbf{x}$ and t, ψ_i are periodic functions (with the period 2π) with respect to θ. Function θ plays the role of the phase, while $\theta_{,\alpha}$ and $-\theta_{,t}$ correspond to the wave vector k_α and the frequency ω, respectively. We assume that the characteristic scales Λ and T of changes of the functions $\theta_{,\alpha}$, $\theta_{,t}$ and $\psi_i(\theta,\mathbf{x},t)|_{\theta=\text{const}}$

[5] The amplitudes a_i appear later.

are considerably larger than the characteristic wavelength λ and period τ. The latter are defined as the best constants in the inequalities

$$|\theta_{,\alpha}| \le \frac{2\pi}{\lambda}, \quad |\theta_{,t}| \le \frac{2\pi}{\tau}, \tag{4.38}$$

while the former are the best constants in the inequalities

$$|\theta_{,\alpha\beta}| \le \frac{2\pi}{\lambda\Lambda}, \quad |\theta_{,\alpha t}| \le \frac{2\pi}{\lambda T}, \quad |\theta_{,\alpha t}| \le \frac{2\pi}{\tau\Lambda}, \quad |\theta_{,tt}| \le \frac{2\pi}{\tau T},$$
$$|\partial_\alpha \psi_i| \le \frac{\bar{\psi}_i}{\Lambda}, \quad |\partial_t \psi_i| \le \frac{\bar{\psi}_i}{T}, \quad |\psi_{i,\theta}| \le \bar{\psi}_i, \tag{4.39}$$

where $\partial_\alpha \psi_i = \partial \psi_i / \partial x_\alpha$ with $\theta = \text{const}$, and $\partial_t \psi_i = \partial \psi_i / \partial t$ with $\theta = \text{const}$. In other words, the wave vector $k_\alpha = \theta_{,\alpha}$, the frequency $\omega = -\theta_{,t}$, and functions ψ_i change little in one wavelength and one period. Therefore it makes sense to call θ "fast" variable as opposed to the "slow" variables x_α and t. Thus, in this variational problem we have two small parameters λ/Λ and τ/T.

We now calculate the derivatives $u_{i,\alpha}$ and $u_{i,t}$. According to (4.37) we have

$$u_{i,\alpha} = \partial_\alpha \psi_i + \psi_{i,\theta}\theta_{,\alpha}, \quad u_{i,t} = \partial_t \psi_i + \psi_{i,\theta}\theta_{,t}.$$

Because of (4.38) and (4.39) they can be approximately replaced by

$$u_{i,\alpha} = \psi_{i,\theta}\theta_{,\alpha}, \quad u_{i,t} = \psi_{i,\theta}\theta_{,t}.$$

Keeping in the action functional (4.36) the asymptotically principal terms, we obtain in the first approximation

$$I_0[\psi_i] = \iint_R L(\psi_i, \psi_{i,\theta}\theta_{,\alpha}, \psi_{i,\theta}\theta_{,t})\, dx\, dt.$$

Let us decompose the domain R into the $(d+1)$-dimensional strips bounded by the d-dimensional phase surfaces $\theta = 2\pi n$, $n = 0, \pm 1, \pm 2, \ldots$. The integral over R can then be replaced by the sum of the integrals over the strips

$$\iint_R L\, dx\, dt = \sum \iint L(\psi_i, \psi_{i,\theta}\theta_{,\alpha}, \psi_{i,\theta}\theta_{,t})\kappa\, d\theta\, d\zeta, \tag{4.40}$$

where ζ_α are the coordinates along the phase surface $\theta = \text{const}$, and κ is the Jacobian of transformation from x_α, t to θ, ζ_α. In the first approximation we may regard κ, $\theta_{,\alpha}$ and $\theta_{,t}$ in each strip as independent from θ. Therefore we obtain the same problem in each strip at the first step of the variational-asymptotic procedure: find the extremal of the functional

$$\bar{I}_0[\psi_i] = \int_0^{2\pi} L(\psi_i, \psi_{i,\theta}\theta_{,\alpha}, \psi_{i,\theta}\theta_{,t})\, d\theta \tag{4.41}$$

among periodic functions $\psi_i(\theta)$ with the period 2π. Since the quantities $k_\alpha = \theta_{,\alpha}$ and $-\omega = \theta_{,t}$ change little within one strip, they are regarded as constants in the functional (4.41). The Euler-Lagrange's equation of this functional is a system of n second-order ordinary differential equations. Its solutions contain $2n$ arbitrary constants: n of them are determined from the conditions that $\psi_i(\theta)$ are 2π-periodic functions, the other n conditions can be chosen by fixing the amplitudes a_i as follows: $\max \psi_i = |a_i|$, where a_i are arbitrary real constants. We call this variational problem strip problem.

Let us denote by $2\pi\bar{L}$ the value of the functional (4.41) at its extremal. The quantity $\bar{L}$ is a function of $a_i, \theta_{,\alpha}$ and $\theta_{,t}$. The sum (4.40), as $\lambda/\Lambda \to 0$ and $\tau/T \to 0$, can again be replaced by the integral

$$\iint_R \bar{L}(a_i, \theta_{,\alpha}, \theta_{,t})\, dx\, dt. \tag{4.42}$$

Euler-Lagrange's equations of the average functional (4.42) read

$$\frac{\partial \bar{L}}{\partial a_i} = 0, \quad \frac{\partial}{\partial t}\frac{\partial \bar{L}}{\partial \theta_{,t}} + \frac{\partial}{\partial x_\alpha}\frac{\partial \bar{L}}{\partial \theta_{,\alpha}} = 0. \tag{4.43}$$

We will see that equations $(4.43)_1$ express the solvability condition for the strip problem leading to the dispersion relation, while $(4.43)_2$ is equivalent to the equation of energy propagation.

Notice that the variational approach described here was initiated by Whitham [53]. His arguments were based on some heuristic reasoning. The variational-asymptotic method in its most general formulation was proposed a little later by Berdichevsky [7]. It has then been applied to a wide class of variational problems having small parameters, including the homogenization of periodic and random structures leading to the cell problems, as well as approximate theories of shells and rods resulting in the thickness and cross-section problems (see [8, 31]). In all problems the variational-asymptotic method yielded the same results as the traditional asymptotic analysis of differential equations. But the former has some advantages compared with the latter. First, as we have to deal only with the variational equation, neglecting a small term in this equation means neglecting terms in several differential equations which are not always easy to be recognized as small ones. Second, no *ad hoc* assumptions about the order of smallness are needed. The order of smallness of terms in the asymptotic expansion is determined exclusively by minimizing the action functional. Thus, the more degrees of freedom and the more complicated the energy and dissipation we have to deal with, the more effective we may expect from the variational-asymptotic method compared with other traditional asymptotic methods as will be seen in the subsequent chapters.

Applications. Let us investigate the strip problem and the average variational problem on some concrete examples. As the first example we consider 1-D Klein-Gordon equation $(4.8)_1$ corresponding to the Lagrangian

$$L=\frac{1}{2}u_{,t}^2-\frac{1}{2}(\omega_0^2u^2+c^2u_{,x}^2).$$

Then the strip problem becomes: find the extremal of the functional

$$\bar{I}_0[\psi]=\int_0^{2\pi}[\frac{1}{2}(\omega^2-c^2k^2)\psi_{,\theta}^2-\frac{1}{2}\omega_0^2\psi^2]\,d\theta$$

among 2π-periodic functions $\psi(\theta)$ satisfying the constraint $\max\psi=a$. The quantities $\omega=-\theta_{,t}$ and $k=\theta_{,x}$ are regarded as constants in this variational problem. Lagrange's equation implies that the 2π-periodic extremal can only be of the form

$$\psi(\theta)=a\cos(\theta+\phi),$$

provided $\omega^2-c^2k^2=\omega_0^2$. The latter is the solvability condition for the strip problem. Substituting this back to $\bar{I}_0$, we obtain the average Lagrangian

$$\bar{L}(a,\theta_{,x},\theta_{,t})=\frac{1}{4}(\theta_{,t}^2-\omega_0^2-c^2\theta_{,x}^2)a^2.$$

Thus, the average Lagrangian does not depend on the initial phase ϕ. Let us analyze now Euler-Lagrange's equations of the average variational problem. Once these equations have been obtained, it is convenient to work with them in terms of a, k, ω:

$$\frac{\partial\bar{L}}{\partial a}=0,\quad \frac{\partial}{\partial t}\frac{\partial\bar{L}}{\partial\omega}-\frac{\partial}{\partial x}\frac{\partial\bar{L}}{\partial k}=0, \tag{4.44}$$

where

$$\bar{L}=G(\omega,k)a^2,\quad G(\omega,k)=\frac{1}{4}(\omega^2-\omega_0^2-c^2k^2).$$

We see that the equation $\bar{L}_{,a}=0$ is nothing else but the solvability condition for the strip problem which leads to the dispersion relation $G(\omega,k)=0$. We can solve this relation with respect to ω to have the explicit form $\omega=\pm\Omega(k)=\pm\sqrt{\omega_0^2+c^2k^2}$. The second equation of (4.44) can be written as

$$\frac{\partial}{\partial t}(G_{,\omega}a^2)-\frac{\partial}{\partial x}(G_{,k}a^2)=0.$$

Since $G(\Omega(k),k)=0$, we have

$$G_{,\omega}\Omega'(k)+G_{,k}=0,$$

and consequently,

$$C=\Omega'(k)=-\frac{G_{,k}}{G_{,\omega}}.$$

If we denote $G_{,\omega}(\Omega,k)$ by $g(k)$, then $(4.44)_2$ takes the form

$$(g(k)a^2)_{,t}+(g(k)C(k)a^2)_{,x}=0.$$

It follows from the consistency condition $k_{,t}+\omega_{,x}=0$ that

$$k_{,t}+Ck_{,x}=0.$$

By using this kinematic relation, the factor $g(k)$ can be removed so that

$$(a^2)_{,t}+(Ca^2)_{,x}=0.$$

This is nothing else but the equation of amplitude modulations. The equation governing energy propagation can easily be derived from here. We can also obtain the energy equation directly from balance equation (4.33) for the average variational problem.

Let us turn now to waves propagating in Timoshenko's beam with the Lagrangian given by (3.50). Introducing the unknown function u and the dimensionless variables according to

$$u=h\psi,\quad \bar{t}=tc_s/h,\quad \bar{x}=x/h,$$

we present the Lagrangian in the form (the bar is dropped for short)

$$L=\frac{1}{2}(w_{,t}^2+\alpha u_{,t}^2)-\frac{1}{2}[su_{,x}^2+\beta^2\alpha(u+w_{,x})^2].$$

The strip problem becomes: find the extremal of the functional

$$\bar{I}_0[\psi_1,\psi_2]=\int_0^{2\pi}[\frac{1}{2}\omega^2(\psi_{1,\theta}^2+\alpha\psi_{2,\theta}^2)-\frac{1}{2}sk^2\psi_{2,\theta}^2-\frac{1}{2}\beta^2\alpha(\psi_2+k\psi_{1,\theta})^2]\,d\theta$$

among 2π-periodic functions $\psi_1(\theta)$, $\psi_2(\theta)$ satisfying the constraints $\max\psi_i=|a_i|$. In this functional $\omega=-\theta_{,t}$ and $k=\theta_{,x}$ are treated as constants. Lagrange's equations of this problem imply that the 2π-periodic extremal can only be of the form

$$\psi_1(\theta)=a_1\cos(\theta+\phi),\quad \psi_2(\theta)=a_2\sin(\theta+\phi).$$

The average Lagrangian becomes

$$\bar{L}(a_1,a_2,\theta_{,x},\theta_{,t})=\frac{1}{4}\theta_{,t}^2(a_1^2+\alpha a_2^2)-\frac{1}{4}s\theta_{,x}^2a_2^2-\frac{1}{4}\beta^2\alpha(a_2-\theta_{,x}a_1)^2.$$

The Euler-Lagrange's equations $\partial\bar{L}/\partial a_i=0$ yield the system of two linear equations

$$(\omega^2-\beta^2\alpha k^2)a_1+\beta^2\alpha ka_2=0,$$
$$\beta^2\alpha ka_1+(\omega^2\alpha-sk^2-\beta^2\alpha)a_2=0,$$

which has non-trivial solutions only if the determinant vanishes. This is the solvability condition for the strip problem which leads also to the dispersion relation

$$(\omega^2-\beta^2\alpha k^2)(\omega^2\alpha-sk^2-\beta^2\alpha)-\beta^4\alpha^2k^2=0.$$

One can check that this equation coincides with the dispersion relation obtained by assuming the harmonic wave form. One can also find the amplitude ratio a_1/a_2 from this system. Finally, one can verify that the other Euler-Lagrange's equation implies the equation of energy propagation in this Timoshenko's beam (see exercise (4.10)).

It is not difficult now to rederive the geometrical optics considered in Section 4.1 from the variational-asymptotic method. The same can be said about weakly inhomogeneous media. This would be the case, for example, if the parameters ω_0 and c in the Klein-Gordon equation were functions of $\mathbf{x}$. The derivation of the strip problem remains unchanged. If the characteristic length of change of material parameters is much larger than the characteristic wavelength, then we can again regard them as constant in this strip problem. After finding the average Lagrangian the slow dependence of the material parameters on $\mathbf{x}$ can be reinserted. The method can also be applied for the case of external forces which change slowly in time. In this case the Lagrangian depends explicitly on time, but this dependence can be ignored in the strip problem. However, the energy is no longer conserved. But notice that wave action is conserved in all cases.

4.5 Exercises

EXERCISE 4.1. Solve the 1-D wave equation with $c = 1$ and with the following initial conditions

$$u(x,0) = 0, \quad u_{,t}(x,0) = \begin{cases} x+1 & \text{for } x \in (-1,0), \\ 1-x & \text{for } x \in (0,1), \\ 0 & \text{otherwise.} \end{cases}$$

Plot the solution at $t = 0.5$ and at $t = 10$.

Solution. According to the d'Alembert solution with $c = 1$

$$u(x,t) = f(x-t) + g(x+t).$$

Functions $f(x)$ and $g(x)$ should be found from the initial conditions

$$u(x,0) = f(x) + g(x) = 0, \quad u_{,t}(x,0) = -f'(x) + g'(x) = v_0(x).$$

Thus, $f(x) = -g(x)$, while $g(x)$ satisfies the equation

$$g'(x) = \frac{1}{2}v_0(x) = \begin{cases} (x+1)/2 & \text{for } x \in (-1,0), \\ (1-x)/2 & \text{for } x \in (0,1), \\ 0 & \text{otherwise.} \end{cases}$$

Integrating this equation, we obtain for $g(x)$

$$g(x) = \begin{cases} 0 & \text{for } x < -1, \\ \frac{1}{4}(x+1)^2 & \text{for } x \in (-1,0), \\ -\frac{1}{4}(x-1)^2 + \frac{1}{2} & \text{for } x \in (0,1), \\ 1/2 & \text{for } x > 1. \end{cases}$$

The plot of this function is shown in Fig. 4.13.

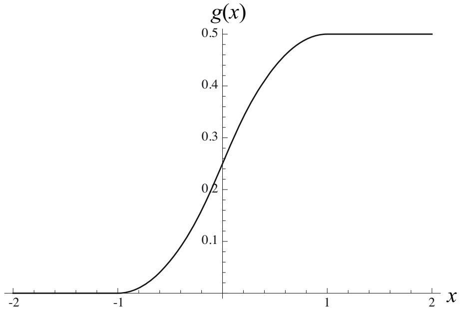

Fig. 4.13 Function $g(x)$

Substituting $g(x)$ and $f(x) = -g(x)$ into the d'Alembert solution, we can evaluate $u(x,t)$ and plot it at different instants of time. Figs. 4.14 and 4.15 show the solution at $t = 0.5$ and $t = 10$, respectively. We observe that, at large t, the solution is constant and equal to $1/2$ inside the interval $x \in (-t+1, t-1)$. Besides, there are two wave fronts of the width 2 propagating to the left and to the right with the velocity 1.

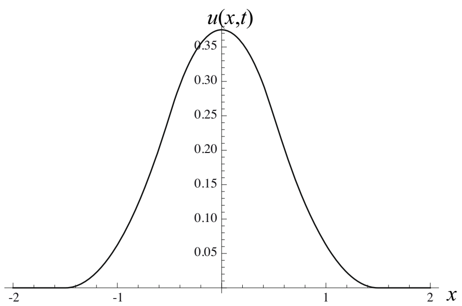

Fig. 4.14 Solution $u(x,t)$ at $t = 0.5$

EXERCISE 4.2. For waves propagating in an infinite elastic material which is homogeneous and isotropic we seek particular solutions in form of plane waves $\mathbf{u} = \mathbf{a}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$. Show that there are two velocities of propagation given by

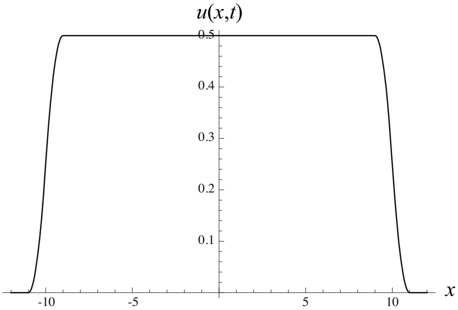

Fig. 4.15 Solution $u(x,t)$ at $t = 10$

$$c_d = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \quad c_s = \sqrt{\frac{\mu}{\rho}},$$

corresponding to dilatational waves (**a** is parallel to **k**) and shear waves (**a** is orthogonal to **k**). Generalize this to homogeneous anisotropic materials.

Solution. Consider first the general case of infinite elastic material which is homogeneous and anisotropic. Then it is easy to show that the extremal of the action functional of this elastic material (see example 3.9) satisfies the Euler-Lagrange's equations

$$\rho u_{\alpha,tt} - E_{\alpha\beta\gamma\delta} u_{\gamma,\delta\beta} = 0.$$

We look for the particular solutions of these equations in form of the plane wave

$$u_\alpha = a_\alpha e^{i(k_\mu x_\mu - \omega t)},$$

where **a** and **k** are constant vectors. Substituting this formula into the equations of motion and removing the non-zero factor $e^{i(k_\mu x_\mu - \omega t)}$, we get the eigenvalue problem

$$(-\rho\omega^2 \delta_{\alpha\gamma} + E_{\alpha\beta\gamma\delta} k_\delta k_\beta) a_\gamma = 0,$$

with $K_{\alpha\gamma} = E_{\alpha\beta\gamma\delta} k_\delta k_\beta$ being called the acoustic tensor.

We solve this eigenvalue problem for the case of isotropic material, for which

$$E_{\alpha\beta\gamma\delta} = \lambda \delta_{\alpha\beta} \delta_{\gamma\delta} + \mu(\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}).$$

The acoustic tensor becomes

$$K_{\alpha\gamma} = (\lambda + \mu) k_\alpha k_\gamma + \mu k^2 \delta_{\alpha\gamma},$$

where k is the magnitude of vector **k**, that is, $k = |\mathbf{k}|$. Without limiting generality we may choose the x_1-axis to be in the direction of vector **k**, i.e., $\mathbf{k} = (k, 0, 0)$. Then the eigenvalue problem can be written in the matrix form

$$\begin{pmatrix} -\rho c^2+\lambda+2\mu & 0 & 0 \\ 0 & -\rho c^2+\mu & 0 \\ 0 & 0 & -\rho c^2+\mu \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

with $c^2 = \omega^2/k^2$ being the phase velocity of wave propagation. There are one single eigenvalue and one double eigenvalue

$$c_1 = c_d = \sqrt{\frac{\lambda+2\mu}{\rho}}, \quad c_2 = c_3 = c_s = \sqrt{\frac{\mu}{\rho}},$$

corresponding to three eigenvectors

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{a}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{a}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Thus, the first eigenvector $\mathbf{a}_1$ points in the direction of $\mathbf{k}$ and describes dilatational waves propagating with the velocity c_d. Two other eigenvectors $\mathbf{a}_2$ and $\mathbf{a}_3$ are orthogonal to $\mathbf{k}$ and correspond to shear waves propagating with the velocity c_s which is the double eigenvalue.

EXERCISE 4.3. Consider the "balloon problem" in acoustics: the pressure inside a sphere of radius R_0 is p_0+P while the pressure outside is p_0. The gas is initially at rest, and the balloon is burst at $t=0$. The initial conditions for the velocity potential read

$$\varphi(\mathbf{x},0) = 0, \quad \varphi_{,t}(\mathbf{x},0) = \begin{cases} -P/\rho_0 & 0<r<R_0, \\ 0 & R_0<r. \end{cases}$$

Find the change of pressure with time.

Solution. Due to the spherical symmetry, the velocity potential depends only on r and t, so

$$\varphi(r,t) = \frac{f(r-ct)}{r} + \frac{g(r+ct)}{r}.$$

Substituting this into the initial conditions, we have

$$f(r)+g(r) = 0, \quad f'(r)-g'(r) = \begin{cases} \frac{P}{\rho_0 c} r & 0<r<R_0, \\ 0 & R_0<r. \end{cases}$$

These conditions determine f and g for positive values of their arguments. However, it is also necessary to evaluate function f with negative argument in the solution. The condition for that is obtained by requiring the absence of source at the origin

$$\lim_{r\to 0} r^2 \frac{\partial\varphi}{\partial r} = 0,$$

which implies

$$f(-ct)+g(ct)=0 \quad \text{for } t>0.$$

The last condition determines f for negative argument in terms of g for positive argument.

Solving the equations for f and g, we obtain

$$f(x)=\begin{cases}\frac{1}{4}\frac{P}{\rho_0 c}(x^2-R_0^2) & |x|<R_0,\\ 0 & \text{otherwise,}\end{cases}$$

and

$$g(x)=\begin{cases}-\frac{1}{4}\frac{P}{\rho_0 c}(x^2-R_0^2) & 0<x<R_0,\\ 0 & R_0<x.\end{cases}$$

With these functions we find the pressure difference

$$p-p_0=-\rho_0\varphi_{,t}=\frac{P}{2r}[(r-ct)F(r-ct)+(r+ct)G(r+ct)],$$

where

$$F(x)=\begin{cases}1 & -R_0<x<R_0,\\ 0 & \text{otherwise,}\end{cases}$$

and

$$G(x)=\begin{cases}1 & 0<x<R_0,\\ 0 & R_0<x.\end{cases}$$

EXERCISE 4.4. Search for particular solutions in form of plane waves and derive the dispersion relation for 1-D waves propagating in Timoshenko's beam, the dimensionless Lagrangian of which is

$$L=\frac{1}{2}(w_{,t}^2+\alpha u_{,t}^2)-\frac{1}{2}[su_{,x}^2+\beta^2\alpha(u+w_{,x})^2].$$

Plot the dispersion curves and study their asymptotic behavior as $k\to 0$ and $k\to\infty$.

Solution. Let $u_1=w$ and $u_2=u$. From the Euler-Lagrange's equations

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial u_{i,t}}+\frac{\partial}{\partial x}\frac{\partial L}{\partial u_{i,x}}-\frac{\partial L}{\partial u_i}=0, \quad i=1,2,$$

we derive

$$\begin{aligned} w_{,tt}-\beta^2\alpha(u+w_{,x})_{,x}&=0,\\ \alpha u_{,tt}-su_{,xx}+\beta^2\alpha(u+w_{,x})&=0.\end{aligned}$$

We look for the particular solutions of these equations in form of plane waves

$$\begin{pmatrix}w(x,t)\\u(x,t)\end{pmatrix}=\begin{pmatrix}a_1\\a_2\end{pmatrix}e^{i(kx-\omega t)},$$

where a_1 and a_2 are constants. Substituting this Ansatz into the equations of motion and removing the common non-zero factor $e^{i(kx-\omega t)}$, we obtain

$$\begin{pmatrix} -\omega^2+\beta^2\alpha k^2 & -\beta^2\alpha ik \\ \beta^2\alpha ik & -\alpha\omega^2+sk^2+\beta^2\alpha \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Nontrivial solutions exist if the determinant of the matrix vanishes yielding the dispersion relation

$$(-\omega^2+\beta^2\alpha k^2)(-\alpha\omega^2+sk^2+\beta^2\alpha)-\beta^4\alpha^2k^2=0.$$

Thus, for each real k there are two real and positive roots of this dispersion relations corresponding to two different branches of the dispersion curves. To plot the dispersion curves we use the following parameters

$$\alpha=\frac{1}{2}\left(\frac{\pi^2}{24}\right)^2,\quad \beta=\pi,\quad s=\frac{1}{6(1-\nu)},$$

with ν being Poisson's ratio (see [31]). The plot of the dispersion curves for $\nu=0.31$ (dashed lines) are shown in Fig. 4.16. We also plot the dispersion curves of the two first branches of F-waves according to Rayleigh-Lamb dispersion relation (solid lines). The comparison shows quite good agreement in the long-wave range.

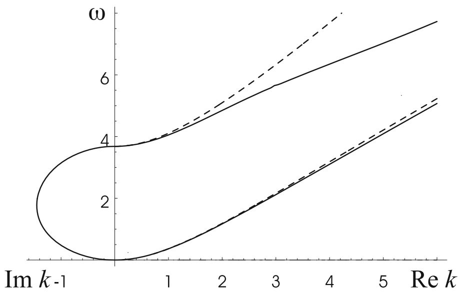

Fig. 4.16 Dispersion curves of flexural waves propagating in a beam: a) 1-D Timoshenko beam theory: dashed line and b) 3-D theory: solid line

In the long-wave range ($k \ll 1$) the asymptotic analysis of dispersion relation yields the following formula

$$\omega^2=sk^4+O(k^6)$$

for the low-frequency branch, and

$$\omega^2=\beta^2+(\alpha\beta^2+\frac{s}{\alpha})k^2+O(k^4)$$

for the high-frequency thickness vibrations.

In the short-wave range ($k \to \infty$) the dispersion curves approach the asymptotes

$$\omega = \beta\sqrt{\alpha}k \quad \text{and} \quad \omega = \sqrt{\frac{s}{\alpha}}k,$$

respectively.

EXERCISE 4.5. Solve the linearized Korteweg-de Vries equation with $\alpha = 0$, $\beta = 1$ and with the initial condition $u(x,0) = e^{-x^2}$. Compute Fourier's integral numerically[6] and plot the solution at $t = 1$.

Solution. Using the Fourier transform, we find that

$$u(x,t) = \int_{-\infty}^{\infty} \psi(k) e^{ikx - i\Omega(k)t}\, dk,$$

where, for the linearized KdV equation with $\alpha = 0$, $\beta = 1$,

$$\Omega(k) = -k^3.$$

Function $\psi(k)$ should be determined from the initial condition

$$\int_{-\infty}^{\infty} \psi(k) e^{ikx} dk = u(x,0) = e^{-x^2}.$$

Applying the Fourier transform to both sides, we obtain

$$\psi(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-x^2} e^{-ikx} dx = \frac{1}{2\sqrt{\pi}} e^{-k^2/4}.$$

Thus, the problem reduces to computing the integral

$$u(x,t) = \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi}} e^{-k^2/4} e^{i(kx+k^3t)}\, dk.$$

This can be done numerically by using the Fourier series package in *Mathematica*. Due to the highly oscillatory integrand, we choose the maximum number of recursive subdivisions to be 100 to achieve the required accuracy. The plot of $u(x,t)$ at $t = 1$ is shown in Fig. 4.17. We see the dispersive waves propagating in the negative direction of the x-axis. For $x > 0$ the solution decays quickly and does not have the oscillatory character. This behavior remains valid also for the later instants of time.

EXERCISE 4.6. Use the method of stationary phase to find the asymptotically leading term of the solution obtained in the previous exercise as $t \to \infty$ at fixed x/t. Compare this asymptotic solution with the exact one at $t = 10$.

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[6] Since the integrand is highly oscillatory, the accuracy is achieved only by increasing the maximum number of recursive subdivisions.

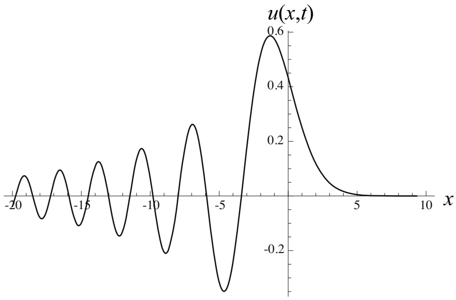

Fig. 4.17 Solution $u(x,t)$ at $t = 1$

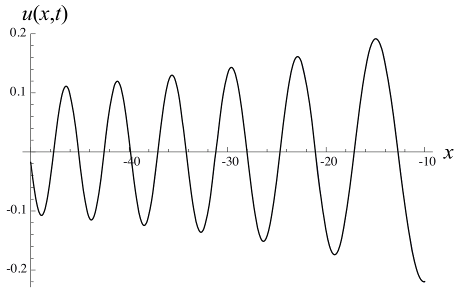

Fig. 4.18 Asymptotic solution $u(x,t)$ at $t = 10$

Solution. For the large time asymptotics the method of stationary phase can be used instead of numerical integration. In our case we rewrite the solution in the form

$$u(x,t) = \int_{-\infty}^{\infty} \psi(k) e^{-i\chi(k)t} dk,$$

where

$$\chi(k) = -k^3 - k\frac{x}{t}.$$

As we know, the main contributions to the integral come from the neighborhoods of stationary points of $\chi(k)$

$$\chi'(k) = -3k^2 - \frac{x}{t} = 0.$$

Thus, for $x > 0$ there is no stationary point, and the solution at fixed x/t and large t must be nearly zero. For $x < 0$ there are two stationary points given by

$$k_1(x,t) = \sqrt{\frac{-x}{3t}} \quad \text{and} \quad k_2(x,t) = -k_1(x,t) = -\sqrt{\frac{-x}{3t}}.$$

At these stationary points

$$\psi(k_s) = \frac{1}{2\sqrt{\pi}} e^{x/12t}.$$

Taking into account that $\chi''(k_s) = -6k_s$, we find that

$$\begin{aligned} u(x,t) &\simeq \frac{1}{2\sqrt{\pi}} e^{x/12t} \sqrt{\frac{2\pi}{6tk_1}} e^{ik_1x + ik_1^3t + \frac{i\pi}{4}} + \frac{1}{2\sqrt{\pi}} e^{x/12t} \sqrt{\frac{2\pi}{6tk_1}} e^{-ik_1x - ik_1^3t - \frac{i\pi}{4}} \\ &\simeq e^{x/12t} \sqrt{\frac{1}{3tk_1}} \cos(k_1x + k_1^3t + \pi/4). \end{aligned}$$

Fig. 4.18 plots the exact solution in terms of Fourier's integrals computed numerically at time $t = 10$ and the above asymptotic solution at the same time, where the results are nearly identical in the region $x < 0$.

EXERCISE 4.7. Show that the lowest branches of the dispersion curves of F- and L-waves in an elastic waveguide approach the straight line $\omega = v_r k$ as $k \to \infty$.

Solution. Consider first the lowest branch of L-waves which must be determined by the dispersion relation

$$\frac{\tanh(q_2/2)}{\tanh(q_1/2)} = \frac{4q_1q_2k^2}{(k^2+q_2^2)^2},$$

where

$$q_1 = \sqrt{k^2 - \eta^2\omega^2}, \quad q_2 = \sqrt{k^2 - \omega^2}, \quad \eta = \sqrt{\frac{\mu}{\lambda + 2\mu}} = \sqrt{\frac{1-2\nu}{2-2\nu}}.$$

Introducing the dimensionless phase velocity $v = \omega/k$, we can represent the above equation in the form

$$\frac{\tanh(k\sqrt{1-v^2}/2)}{\tanh(k\sqrt{1-\eta^2v^2}/2)} = \frac{\sqrt{1-\eta^2v^2}\sqrt{1-v^2}}{(1-v^2/2)^2}.$$

As $k \to \infty$ the left-hand side must go to 1 for any finite and fixed $v \in (0,1)$. Thus, v must be determined from the equation

$$\frac{\sqrt{1-\eta^2v^2}\sqrt{1-v^2}}{(1-v^2/2)^2} = 1.$$

This equation has a unique solution $v_r = c_r/c_s$ in the range $v \in (0,1)$, where c_r is the Rayleigh wave speed (see exercise 4.9).

For the lowest branch of F-waves the dispersion relation is obtained from the above equation by inverting the right-hand side, so $\omega/k \to v_r$ also in the limit $k \to \infty$.

EXERCISE 4.8. Prove that all high-frequency thickness branches of F- and L-waves in an elastic waveguide approach the line $\omega = k$ from above as $k \to \infty$.

Solution. As seen from the Rayleigh-Lamb equation for L-waves, when $v = 1/\eta$ there is an infinite number of roots given by

$$k = \frac{2n\pi}{\sqrt{1/\eta^2 - 1}}, \quad n = 1, 2, \ldots .$$

This means that there is an infinite number of branches of the dispersion curves crossing the straight line $\omega = k/\eta$ and entering the region II as k becomes large. The dispersion relation for L-waves in this region read

$$\frac{\tan(k\sqrt{v^2-1}/2)}{\tanh(k\sqrt{1-\eta^2 v^2}/2)} = \frac{\sqrt{1-\eta^2 v^2}\sqrt{v^2-1}}{(1-v^2/2)^2}.$$

For large k function $\tanh(k\sqrt{1-\eta^2 v^2}/2)$ is close to 1, so the above equation can be replaced by

$$\tan(k\sqrt{v^2-1}/2) = \frac{\sqrt{1-\eta^2 v^2}\sqrt{v^2-1}}{(1-v^2/2)^2}.$$

This equation has an infinite number of roots for each fixed $v \in (1, 1/\eta)$. From this observation it follows that any straight line $\omega = vk$, with $v > 1$, cannot be an asymptote to any of the branches. Indeed, if the n-th branch would have v as the limiting speed, the straight line $\omega = vk$ would intersect at most $n-1$ branches (the dispersion curves cannot intersect each other), but this contradicts the fact that there are infinitely many dispersion curves intersecting this line. Thus, v must approach 1. In this limit the above equation can further be simplified to take the form

$$\tan(k\sqrt{v^2-1}/2) = 4\sqrt{1-\eta^2}\sqrt{v^2-1}.$$

Let $\varepsilon = \sqrt{v^2-1}$. Solving this equation, we find that, to the first order of ε

$$k \sim \frac{2n\pi}{\varepsilon} \sim \frac{2\pi n}{\sqrt{\omega^2/k^2-1}} \quad \Longrightarrow \quad \omega^2 - k^2 = (2\pi n)^2.$$

These equations of hyperbolas describe the asymptotic behavior of the dispersion curves as k goes to infinity. The proof for F-waves can be done in a similar manner.

EXERCISE 4.9. Rayleigh surface wave. Determine the velocity of wave propagating near the free surface of an isotropic elastic half-space.

Solution. Let us choose the coordinate system such that the elastic medium occupies the domain $z \le 0$ with the plane $z = 0$ as its free boundary. We write the dimensionless equations of motion

$$u_{\alpha,tt} = (1+\gamma)u_{\beta,\beta\alpha} + u_{\alpha,\beta\beta}$$

and the traction free boundary conditions

$$[\gamma u_{\beta,\beta}\delta_{\alpha z} + (u_{\alpha,z} + u_{z,\alpha})]|_{z=0} = 0$$

as in (4.22), where h is an arbitrary length. We look for the solution in form of the surface wave propagating in the x-direction

$$u_x = \hat{f}_x e^{sz} e^{i(kx-\omega t)}, \quad u_y = 0, \quad u_z = \hat{f}_z e^{sz} e^{i(kx-\omega t)},$$

with $\hat{f}_x$ and $\hat{f}_z$ being constants. Since the solution must decay exponentially as $z \to -\infty$, we choose s to be real and positive. Substituting this Ansatz into the equations of motion, we obtain the system

$$\begin{pmatrix} s^2+\omega^2-\eta^{-2}k^2 & (1+\gamma)iks \\ (1+\gamma)iks & \eta^{-2}s^2+\omega^2-k^2 \end{pmatrix} \begin{pmatrix} \hat{f}_x \\ \hat{f}_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The condition of vanishing determinant yields two real positive roots

$$s_1 = \sqrt{k^2-\eta^2\omega^2}, \quad s_2 = \sqrt{k^2-\omega^2},$$

corresponding to two eigenvectors

$$\begin{pmatrix} \hat{f}_x \\ \hat{f}_z \end{pmatrix} = \begin{pmatrix} ik \\ s_1 \end{pmatrix}, \quad \begin{pmatrix} \hat{f}_x \\ \hat{f}_z \end{pmatrix} = \begin{pmatrix} is_2 \\ k \end{pmatrix},$$

provided (k,ω) is found in the region I. Thus, the general solution reads

$$u_x = i(Ake^{s_1 z} + Bs_2 e^{s_2 z})e^{i(kx-\omega t)},$$
$$u_z = (As_1 e^{s_1 z} + Bke^{s_2 z})e^{i(kx-\omega t)}.$$

The traction-free boundary conditions at $z=0$ yield two equations for A and B

$$(\eta^{-2}s_1^2 - \gamma k^2)A + ks_2(\eta^{-2}-\gamma)B = 0,$$
$$2ks_1 A + (s_2^2+k^2)B = 0.$$

Equating the determinant to zero, we obtain from here the relation

$$(2k^2-\omega^2)^2 - 4k^2\sqrt{k^2-\eta^2\omega^2}\sqrt{k^2-\omega^2} = 0.$$

Introducing the dimensionless velocity of wave propagation $v = \omega/k$, we bring this relation to the form

$$(2-v^2)^2 - 4\sqrt{1-\eta^2 v^2}\sqrt{1-v^2} = 0,$$

or, equivalently,

$$v^6 - 8v^4 + (24-16\eta^2)v^2 + 16(\eta^2-1) = 0.$$

It is easy to see that this cubic equation with respect to v^2 has a unique root in the range $v \in (0,1)$. The plot of $v = c_r/c_s$ versus Poisson's ratio is shown in Fig. 4.19. As the Poisson ratio changes from zero to $1/2$, c_r/c_s changes from 0.874 to 0.955.

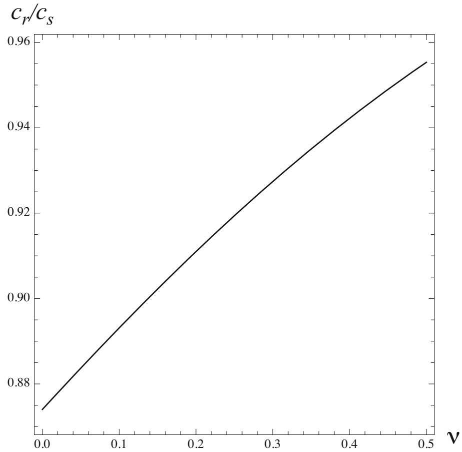

Fig. 4.19 The dimensionless velocity of Rayleigh surface wave c_r/c_s versus Poisson's ratio

EXERCISE 4.10. Derive the equation of energy propagation for Timoshenko's beam using the variational-asymptotic method and compare it with the similar equation obtained via averaging the energy balance equation.

Solution. The Euler-Lagrange's equation of the average variational problem

$$\frac{\partial}{\partial t}\frac{\partial \bar{L}}{\partial \omega} - \frac{\partial}{\partial x}\frac{\partial \bar{L}}{\partial k} = 0$$

implies the equation of energy propagation

$$(\omega \bar{L}_{,\omega} - \bar{L})_{,t} + (-\omega \bar{L}_{,k})_{,x} = 0,$$

where $\omega \bar{L}_{,\omega} - \bar{L}$ is the average total energy density and $-\omega \bar{L}_{,k}$ is the average energy flux (see exercise 4.12). For the Timoshenko beam we have

$$\bar{L}(a_1, a_2, k, \omega) = \frac{1}{4}\omega^2(a_1^2 + \alpha a_2^2) - \frac{1}{4}sk^2a_2^2 - \frac{1}{4}\beta^2\alpha(a_2 - ka_1)^2.$$

Thus, the average total energy density reads

$$\bar{E} = \omega \bar{L}_{,\omega} - \bar{L} = \frac{1}{4}\omega^2(a_1^2 + \alpha a_2^2) + \frac{1}{4}sk^2a_2^2 + \frac{1}{4}\beta^2\alpha(a_2 - ka_1)^2,$$

while the average energy flux equals

$$\bar{J} = -\omega \bar{L}_{,k} = \frac{1}{2}sa_2^2k\omega - \frac{1}{2}\beta^2\alpha(a_2 - ka_1)a_1\omega.$$

Let us show that the same equation can also be derived by averaging the exact energy balance equation (4.33). For Timoshenko's beam theory the total energy density is

$$E = K + U = \frac{1}{2}(w_{,t}^2 + \alpha u_{,t}^2) + \frac{1}{2}[s u_{,x}^2 + \beta^2 \alpha (u + w_{,x})^2],$$

while the energy flux equals

$$J = u_{i,t} \frac{\partial L}{\partial u_{i,x}} = -\beta^2 \alpha w_{,t}(u + w_{,x}) - s u_{,t} u_{,x}.$$

The asymptotically leading terms of solution can be written in form of wave packet

$$w \simeq a_1 \cos(\theta + \phi), \quad u \simeq a_2 \sin(\theta + \phi),$$

where a_1 and a_2 are amplitudes of w and u, respectively. The wave number $k = \theta_{,x}$, the frequency $\omega = -\theta_{,t}$, the initial phase ϕ, and the amplitudes a_1 and a_2 are slowly changing functions of x and t. We use these formulas to compute the asymptotically leading terms of the total energy density and the energy flux.

First we compute the kinetic energy density

$$\frac{1}{2}(w_{,t}^2 + \alpha u_{,t}^2) \simeq \frac{1}{2}[\omega^2 a_1^2 \sin^2(\theta + \phi) + \alpha \omega^2 a_2^2 \cos^2(\theta + \phi)]$$

together with terms involving $a_{i,t}$ and $\phi_{,t}$. Since a_i and ϕ are slowly changing functions of t, these terms can be neglected in the first approximation. Treating the other terms in the same way, we obtain for the total energy density

$$\begin{aligned} E = &\frac{1}{2}[\omega^2 a_1^2 \sin^2(\theta + \phi) + \alpha \omega^2 a_2^2 \cos^2(\theta + \phi)] \\ &+ \frac{1}{2}[s k^2 a_2^2 \cos^2(\theta + \phi) + \beta^2 \alpha (a_2 - a_1 k)^2 \sin^2(\theta + \phi)]. \end{aligned}$$

Similarly, the energy flux becomes

$$J = -\beta^2 \alpha \omega a_1 (a_2 - k a_1) \sin^2(\theta + \phi) + s \omega k a_2^2 \cos^2(\theta + \phi).$$

Now let us take the average of these quantities over one period. Since the average values of $\cos^2(\theta + \phi)$ and $\sin^2(\theta + \phi)$ over one period are equal to $1/2$, we get

$$\bar{E} = \frac{1}{4}\omega^2 (a_1^2 + \alpha a_2^2) + \frac{1}{4} s k^2 a_2^2 + \frac{1}{4}\beta^2 \alpha (a_2 - k a_1)^2,$$

and

$$\bar{J} = \frac{1}{2} s a_2^2 k \omega - \frac{1}{2}\beta^2 \alpha (a_2 - k a_1) a_1 \omega,$$

which coincide with the above equations obtained from the variational-asymptotic method.

EXERCISE 4.11. Solve the strip problem for 3-D Klein-Gordon equation

$$u_{,tt} + \omega_0^2 u = c^2 \Delta u$$

to find the average Lagrangian, the dispersion relation, and the equation of energy propagation.

Solution. For 3-D Klein-Gordon equation corresponding to the Lagrangian

$$L = \frac{1}{2}u_{,t}^2 - \frac{1}{2}(\omega_0^2 u^2 + c^2 u_{,\alpha} u_{,\alpha}),$$

the strip problem becomes: find the extremal of the functional

$$\bar{I}_0[\psi] = \int_0^{2\pi} [\frac{1}{2}(\omega^2 - c^2 k_\alpha k_\alpha)\psi_{,\theta}^2 - \frac{1}{2}\omega_0^2 \psi^2]\, d\theta$$

among 2π-periodic functions $\psi(\theta)$ satisfying the constraint $\max \psi = a$. The quantities $\omega = -\theta_{,t}$ and $\mathbf{k} = \nabla\theta$ are regarded as constants in this variational problem. Lagrange's equation implies that the 2π-periodic extremal can only be of the form

$$\psi(\theta) = a\cos(\theta + \phi),$$

provided $\omega^2 - c^2|\mathbf{k}|^2 = \omega_0^2$, where $|\mathbf{k}|^2 = k_\alpha k_\alpha$. The latter is the solvability condition for the strip problem. Using this solution, we compute the average Lagrangian

$$\bar{L}(a, \theta_{,\alpha}, \theta_{,t}) = \frac{1}{4}(\theta_{,t}^2 - \omega_0^2 - c^2\theta_{,\alpha}\theta_{,\alpha})a^2.$$

Euler-Lagrange's equations of the average variational problem, in terms of a, $\mathbf{k}$, and ω, read

$$\frac{\partial \bar{L}}{\partial a} = 0, \quad \frac{\partial}{\partial t}\frac{\partial \bar{L}}{\partial \omega} - \frac{\partial}{\partial x_\alpha}\frac{\partial \bar{L}}{\partial k_\alpha} = 0.$$

Let us write $\bar{L} = G(\omega, \mathbf{k})a^2$, where $G(\omega, \mathbf{k}) = \frac{1}{4}(\omega^2 - \omega_0^2 - c^2|\mathbf{k}|^2)$. We see that the equation $\bar{L}_{,a} = 0$ is nothing else but the solvability condition $G(\omega, \mathbf{k}) = 0$ for the strip problem which is equivalent to the dispersion relation

$$\omega^2 = \omega_0^2 + c^2|\mathbf{k}|^2.$$

We can solve this relation with respect to ω to have the explicit form $\omega = \pm\Omega(\mathbf{k}) = \pm\sqrt{\omega_0^2 + c^2|\mathbf{k}|^2}$. The second equation can be written as

$$\frac{\partial}{\partial t}(G_{,\omega}a^2) - \frac{\partial}{\partial x_\alpha}(G_{,k_\alpha}a^2) = 0.$$

Since $G(\Omega(\mathbf{k}), \mathbf{k}) = 0$, we have

$$G_{,\omega}\Omega_{,k_\alpha} + G_{,k_\alpha} = 0,$$

and consequently,

$$C_\alpha = \Omega_{,k_\alpha} = -\frac{G_{,k_\alpha}}{G_{,\omega}}.$$

If we denote $G_{,\omega}(\Omega(\mathbf{k}),\mathbf{k})$ by $g(\mathbf{k})$, then the second equation takes the form

$$(g(\mathbf{k})a^2)_{,t} + (g(\mathbf{k})C_\alpha a^2)_{,\alpha} = 0.$$

It follows from the consistency condition $k_{\alpha,t} + \omega_{,\alpha} = 0$ that

$$k_{\alpha,t} + C_\beta k_{\beta,\alpha} = 0.$$

By using this kinematic relation, the factor $g(\mathbf{k})$ can be removed so that

$$(a^2)_{,t} + (C_\alpha a^2)_{,\alpha} = 0.$$

This is nothing else but the equation of amplitude modulations. The equation governing energy propagation reads (see the next exercise)

$$(\omega\bar{L}_{,\omega} - \bar{L})_{,t} + (-\omega\bar{L}_{,k_\alpha})_{,\alpha} = 0.$$

With $\bar{L} = G(\omega,\mathbf{k})a^2$ we get

$$\bar{E} = \omega\bar{L}_{,\omega} - \bar{L} = (\omega G_{,\omega} - G)a^2,$$

and

$$\bar{J}_\alpha = -\omega\bar{L}_{,k_\alpha} = -\omega G_{,k_\alpha}a^2 = \omega G_{,\omega}C_\alpha a^2.$$

Substituting these formulas into the equation of energy propagation and taking into account the dispersion relation $G(\Omega(\mathbf{k}),\mathbf{k}) = 0$, we easily see that it is equivalent to the equation of amplitude modulations.

EXERCISE 4.12. Derive the following equations

$$(\omega\bar{L}_{,\omega} - \bar{L})_{,t} + (-\omega\bar{L}_{,k_\alpha})_{,\alpha} = 0,$$
$$(k_\alpha\bar{L}_{,\omega})_{,t} + (-k_\alpha\bar{L}_{,k_\beta} + \bar{L}\delta_{\alpha\beta})_{,\beta} = 0,$$

for homogeneous media, which can be interpreted as the energy and "wave momentum" equations, respectively. What happens if $\bar{L}$ depends on the slow variables x_α and t?

Solution. The derivation of the energy equation is quite similar to that given at the beginning of Section 4.4. Starting from the average Euler-Lagrange's equation

$$\frac{\partial}{\partial t}\frac{\partial\bar{L}}{\partial\omega} - \frac{\partial}{\partial x_\alpha}\frac{\partial\bar{L}}{\partial k_\alpha} = 0$$

and multiplying it with ω, we obtain

$$\omega(\bar{L}_{,\omega})_{,t} - \omega(\bar{L}_{,k_\alpha})_{,\alpha} = 0.$$

Rearrange terms to get

$$(\omega\bar{L}_{,\omega})_{,t} - (\omega\bar{L}_{,k_\alpha})_{,\alpha} - \omega_{,t}\bar{L}_{,\omega} + \omega_{,\alpha}\bar{L}_{,k_\alpha} = 0.$$

Replacing $\omega_{,\alpha}$ in the last term by $-k_{\alpha,t}$ in accordance with the consistency condition and taking into account the dispersion relations $\bar{L}_{,a_i} = 0$, we see that the last two terms give $-\bar{L}_{,t}$ since, according to the chain rule of differentiation,

$$\bar{L}_{,t} = \frac{\partial\bar{L}}{\partial\theta_{,t}}\theta_{,tt} + \frac{\partial\bar{L}}{\partial\theta_{,\alpha}}\theta_{,\alpha t} + \frac{\partial\bar{L}}{\partial a_i}a_{i,t} = \omega_{,t}\bar{L}_{,\omega} + k_{\alpha,t}\bar{L}_{,k_\alpha}.$$

Thus,

$$(\omega\bar{L}_{,\omega} - \bar{L})_{,t} + (-\omega\bar{L}_{,k_\alpha})_{,\alpha} = 0.$$

Since $\omega\bar{L}_{,\omega}$ is the average kinetic energy density $\bar{K}$, the expression in the square brackets of the first term is the average total energy density, while $-\omega\bar{L}_{,k_\alpha}$ is the average energy flux. So, this equation is the equation of energy propagation.

The "wave momentum" equation can be derived by multiplying the average Euler-Lagrange's equation with k_α

$$k_\alpha(\bar{L}_{,\omega})_{,t} - k_\alpha(\bar{L}_{,k_\beta})_{,\beta} = 0.$$

Rearranging terms to get

$$(k_\alpha\bar{L}_{,\omega})_{,t} - (k_\alpha\bar{L}_{,k_\beta})_{,\beta} - k_{\alpha,t}\bar{L}_{,\omega} + k_{\alpha,\beta}\bar{L}_{,k_\beta} = 0.$$

Replacing $k_{\alpha,t}$ by $-\omega_{,\alpha}$ and keeping in mind the dispersion relations, we reduce the last two terms to $\bar{L}_{,\alpha}$, so

$$(k_\alpha\bar{L}_{,\omega})_{,t} + (-k_\alpha\bar{L}_{,k_\beta} + \bar{L}\delta_{\alpha\beta})_{,\beta} = 0.$$

If the average Lagrangian depends on the slow variables x_α (weakly inhomogeneous media) and t (slowly changing external forces), the energy and wave momentum equations change. In the case of slow dependence on t, the energy equation becomes

$$(\omega\bar{L}_{,\omega} - \bar{L})_{,t} + (-\omega\bar{L}_{,k_\alpha})_{,\alpha} = -\partial_t\bar{L},$$

where $\partial_t\bar{L}$ denotes the partial time derivative of $\bar{L}$ at fixed ω and k_α. In case of slow dependence on x_α, the wave momentum equation reads

$$(k_\alpha\bar{L}_{,\omega})_{,t} + (-k_\alpha\bar{L}_{,k_\beta} + \bar{L}\delta_{\alpha\beta})_{,\beta} = \partial_\alpha\bar{L},$$

where $\partial_\alpha\bar{L}$ denotes the partial derivative of $\bar{L}$ with respect to x_α at fixed ω and k_α.