Chapter 3
Continuous Oscillators

This Chapter deals with small vibrations of mechanical systems having infinite number of degrees of freedom. It begins with the discrete models of linear chain of oscillators and then moves to the continuum models of strings, beams, membranes, and plates. The last Section is devoted to the most general continuous oscillators. The vibrations of these oscillators can be found in form of the linear superposition of the standing waves leading to the eigenvalue problems in infinite dimensional spaces.

3.1 Chain of Oscillators

Differential Equations of Motion. Crystals having periodic lattice structures with atoms vibrating about the lattice sites can be regarded as mechanical systems with countable number of degrees of freedom. Our aim is to construct mathematical models for such discrete systems with countable number of degrees of freedom by means of the continuum mechanics. Let us first begin with two simple examples.

EXAMPLE 3.1. 1-D chain of mass-spring oscillators. A linear 1-D chain of points of equal mass m connected by springs of equal stiffness k_1 is constrained to move in the longitudinal direction (see Fig. 3.1). Derive the equations of motion.

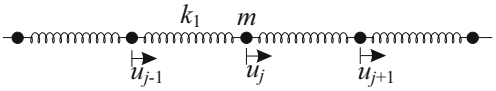

Fig. 3.1 A linear chain of mass-spring-oscillators

In this example the point-masses model atoms, while the springs their nearest neighbor interaction. In equilibrium the distances between neighboring atoms are equal to a lattice constant b. Denoting the displacement of the atom j from its equilibrium position jb by $u_j(t)$, we write down the kinetic energy of the chain

$$K(\dot{u}) = \frac{1}{2} m \sum_j \dot{u}_j^2. \tag{3.1}$$

The potential energy of the chain is the sum of energies of the springs

$$U(u) = \frac{1}{2} k_1 \sum_j (u_j - u_{j-1})^2. \tag{3.2}$$

Thus, Lagrange's equations of this chain read

$$m\ddot{u}_j + k_1(u_j - u_{j-1}) - k_1(u_{j+1} - u_j) = 0$$

for all $j = 1, \ldots, n-1$ except the end points of the chain.

EXAMPLE 3.2. 1-D chain of atoms with next-to-nearest-neighbor interaction. Consider the similar 1-D chain of atoms as in the previous example. But now in addition to the springs of stiffness k_1 there are springs of stiffness k_2 connecting the next to nearest neighboring atoms as well (see Fig. 3.2). Derive the equations of longitudinal motion.

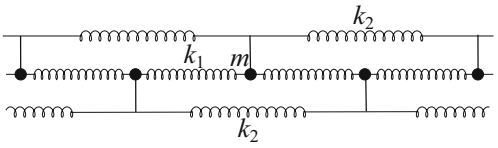

Fig. 3.2 A chain of oscillators with next-to-nearest-neighbor interaction

As before, the kinetic energy of this chain remains the same as (3.1). But its potential energy changes. Because of the presence of the next-to-nearest-neighbor interaction, we have to include energies of the springs of stiffness k_2

$$U(u) = \frac{1}{2} k_1 \sum_j (u_j - u_{j-1})^2 + \frac{1}{2} k_2 \sum_j (u_j - u_{j-2})^2. \tag{3.3}$$

It is interesting to mention that, in some physical situation, we may even assume the negative spring stiffness k_2. Lagrange's equations of this chain become

$$m\ddot{u}_j + \sum_{l=1}^{2} [k_l(u_j - u_{j-l}) - k_l(u_{j+l} - u_j)] = 0 \tag{3.4}$$

for all $j = 2, \ldots, n-2$ except the end points of the chain. It is easy to write down the equations of motion for the chain, where each atom interacts with m neighbors to the left as well as with m neighbors to the right (see exercise 3.1).

The derived systems of coupled differential equations are quite difficult to study. However, it turns out that, as $n \to \infty$, they can be reduced in the long wave limit to one partial differential equation which is easier to solve.

Quasicontinuum. The idea is to set up one-to-one correspondences between functions of discrete argument and functions of continuous argument and between operations on them. Consider the case $n = \infty$ (infinite chain) and let $u(jb) = u_j$ be a function of the discrete argument j defined at the lattice sites. At present, the dependence of u_j on t is suppressed for short; it will be restored in the final stage. We are going to interpolate this function to a smooth function $u(x)$ defined on the whole x-axis such that all wave lengths shorter than b are filtered out. The precise meaning of this can be given in terms of the Fourier transform of $u(x)$ which we denote[1] by $u(k)$

$$u(k) = \mathscr{F}[u(x)] = \int_{-\infty}^{\infty} e^{-ikx} u(x)\, dx, \quad u(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} u(k)\, dk. \tag{3.5}$$

Namely, we require that the Fourier image $u(k)$ differs from zero only on the segment $B = [-\pi/b, \pi/b]$, called Brillouin zone. Then $u(k)$ can be expanded in a Fourier series on this segment

$$u(k) = \chi_B(k) \sum_j c_j e^{-ijbk}, \tag{3.6}$$

where $\chi_B(k)$ is a characteristic function of the segment B, i.e., $\chi_B(k) = 1$ when $k \in B$ and $\chi_B(k) = 0$ otherwise. Substituting this equation in (3.5) for $u(x)$, we find

$$u(x) = \sum_j c_j \delta_B(x - jb),$$

where

$$\delta_B(x) = \frac{1}{2\pi} \int_B e^{ikx}\, dk = \frac{\sin(\pi x/b)}{\pi x}.$$

It is easy to see that $\delta_B(x) = \delta_B(-x)$ and

$$\delta_B(0) = \frac{1}{b}, \quad \delta_B(jb) = 0 \quad \text{for } j \neq 0.$$

Thus, if we set $c_j = bu(jb)$, then

$$u(x) = b \sum_j u(jb) \delta_B(x - jb) \tag{3.7}$$

is the required interpolating function, since it is equal to $u(jb)$ at the lattice sites and its Fourier image has the compact support in the Brillouin zone B. It can be proved that (3.7) is a unique and one-to-one correspondence between functions of discrete and continuous argument satisfying these two requirements [27].

[1] This notation involves no risk of confusion as we can see the difference in arguments of $u(x)$ and $u(k)$. Besides, it emphasizes that $u(x)$ and the image $u(k)$ are the same function in x and k spaces.

Based on this one-to-one correspondence, we can now present the Lagrange function of our chain in terms of the continuous function (3.7). First of all, let us show that the following identity

$$\sum_j u^2(jb) = \frac{1}{b}\int u^2(x)\,dx$$

holds true for an arbitrary function $u(jb)$. Indeed, according to Parseval's identity [47] we have for any real function

$$\int_{-\infty}^{\infty} u^2(x)\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} u^*(k)u(k)\,dk,$$

where $u^*(k)$ is the complex conjugate of $u(k)$. With $u(k)$ from (3.6) (where $c_j = bu(jb)$) and with the identities

$$\frac{1}{2\pi}\int_B e^{i(l-j)bk}\,dk = \frac{1}{b}\delta_{jl},$$

we can easily check the required formula. Thus, the kinetic energy of the chain (3.1) can be expressed in terms of $u_{,t}(x,t)$ as

$$K(u_{,t}) = \int \frac{1}{2}\mu u_{,t}^2\,dx, \tag{3.8}$$

with the comma in indices denoting the partial derivative and $\mu = m/b$ the mass density per unit length. Let us turn now to the potential energy of the chain and rewrite it in the form

$$U(u) = \frac{1}{2}\sum_{j,l} u_j \Phi(j-l)u_l,$$

where $\Phi(-j) = \Phi(j)$ and for $U(u)$ from (3.2)

$$\Phi(0) = 2k_1, \quad \Phi(1) = -k_1, \quad \Phi(j) = 0 \quad \text{for } |j| > 1.$$

For $U(u)$ from (3.3) we have

$$\Phi(0) = 2(k_1+k_2), \quad \Phi(1) = -k_1, \quad \Phi(2) = -k_2, \quad \Phi(j) = 0 \quad \text{for } |j| > 2,$$

and this can easily be generalized for chains with m interacting neighbors, $m > 2$. Applying the convolution theorem and Parseval's identity, we have for any real function $u(x)$ and $\Phi(x)$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(x)\Phi(x-y)u(y)dxdy = \frac{1}{2\pi}\int_{-\infty}^{\infty} u^*(k)\Phi(k)u(k)dk, \tag{3.9}$$

where

$$\Phi(k) = \int_{-\infty}^{\infty} \Phi(x)e^{-ikx}dx.$$

Now, if we set

$$\Phi(k) = \frac{1}{b}\sum_j \Phi(j)e^{-ijbk}, \quad k \in B, \tag{3.10}$$

and substitute it into the right-hand side of (3.9), we obtain

$$U(u) = \frac{1}{2}\sum_{j,l} u_j \Phi(j-l)u_l = \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(x)\Phi(x-y)u(y)\,dx\,dy.$$

Thus, the action functional can be presented in terms of the function $u(x,t)$ as

$$I[u(x,t)] = \int_{t_0}^{t_1} dt \int_{-\infty}^{\infty}\left[\frac{1}{2}\mu u_{,t}^2 - \frac{1}{2}\int_{-\infty}^{\infty} u(x,t)\Phi(x-y)u(y,t)dy\right]dx \tag{3.11}$$

Varying the action functional, we easily obtain the following integral equation

$$\mu u_{,tt} + \int_{-\infty}^{\infty} \Phi(x-y)u(y,t)dy = 0, \tag{3.12}$$

which is equivalent to the system of equations (3.4). Note that equation (3.12) can also be directly obtained from (3.4).

Dispersion Curve and Long-Wave Approximation. Since equation (3.12) is linear, we seek its solution in the form

$$u(x,t) = ae^{i(kx-\omega t)}. \tag{3.13}$$

This solution corresponds to the wave propagating along the x-axis, with a being the amplitude, k the wave number, and ω the frequency of vibration. Substitution of (3.13) into (3.12) leads to the dispersion relation between k and ω

$$-\mu\omega^2 + \Phi(k) = 0. \tag{3.14}$$

For the chain with the nearest neighbor interaction we have from (3.10)

$$\Phi(k) = \frac{2k_1}{b}(1-\cos bk) = \frac{4k_1}{b}\sin^2\frac{bk}{2}, \quad k \in B.$$

Denoting by $\omega_0 = \sqrt{k_1/m}$, we present the dimensionless dispersion curve $\nu = \omega/\omega_0$ versus $\kappa = bk$ in Fig. 3.3.

For the propagating wave (3.13) the characteristic wavelength is $l = 2\pi/k = 2\pi b/\kappa$. If this characteristic wavelength is much larger than the lattice constant b, then $\kappa \ll 1$ and function $\Phi(k)$ can be approximated by

$$\Phi(k) = \frac{4k_1}{b}\sin^2\frac{bk}{2} \approx \frac{k_1}{b}(bk)^2. \tag{3.15}$$

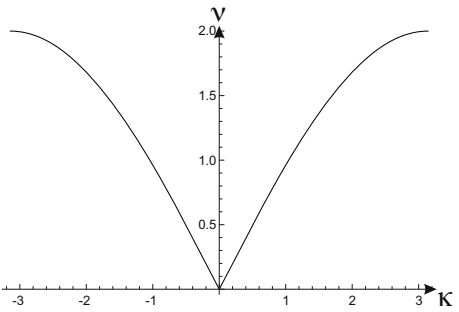

Fig. 3.3 Dispersion curve for a chain with nearest neighbor interaction

This approximation simplifies considerably the potential energy in the action functional and makes the theory local. Indeed, using the property of the Fourier transform

$$\frac{1}{2\pi}\int k^2 u^*(k)u(k)\,dk = \int_{-\infty}^{\infty} u_{,x}^2\,dx,$$

we write the potential energy in the form

$$U(u) = \frac{k_1 b}{2}\int_{-\infty}^{\infty} u_{,x}^2\,dx.$$

Thus, the action functional becomes

$$I[u(x,t)] = \int_{t_0}^{t_1} dt \int_{-\infty}^{\infty} (\frac{1}{2}\mu u_{,t}^2 - \frac{1}{2}k_1 b u_{,x}^2)dx, \tag{3.16}$$

which yields the following Lagrange's equation

$$\mu u_{,tt} - k_1 b u_{,xx} = 0. \tag{3.17}$$

This type of partial differential equations will be studied in the next Sections within the continuum mechanics.

3.2 String

Differential Equation of Motion. In previous Section the transition from discrete to continuum descriptions has been demonstrated. Let us now derive the equation of motion directly within the framework of continuum mechanics. We start with simple one-dimensional continua.

EXAMPLE 3.3. Flexural vibration of string. Derive the equation of small flexural vibration of a pre-stretched string.

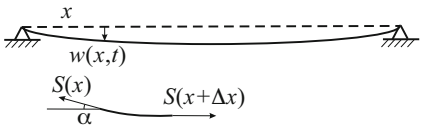

Fig. 3.4 Flexural vibration of string

Under string we mean a thin pre-stretched elastic body with negligible bending stiffness whose diameter of the cross-section is much smaller than its length l. We shall model the string by a one-dimensional continuum. We show first the derivation based on the force method. Let x be the coordinate along the string axis, $x \in (0,l)$, and $w(x,t)$ the transverse displacement of the string. We also denote by $\alpha(x,t)$ the slope of the curve $w(x,t)$. For small vibration both $w(x,t)$ and $\alpha(x,t)$ are small so that $\alpha = w_{,x}$. The tension is assumed to be large, and the change of stress along the string during the vibration is negligibly small compared with this tension. We cut a part of the deformed string from x to $x+\Delta x$ and free it from the rest. Keeping in mind the free body diagram shown in Fig. 3.4, we apply Newton's law in the transverse direction,

$$\mu(x)\Delta x w_{,tt} = S(x+\Delta x)\alpha(x+\Delta x,t) - S(x)\alpha(x,t),$$

where $S(x)$ and $S(x+\Delta x)$ are the forces from the surrounding exerted on the cut part of the string, $\mu(x)$ the mass per unit length, and, due to the smallness of α, $\sin\alpha$ is approximately replaced by α in this equation. Dividing both sides by Δx and letting $\Delta x \to 0$, we obtain in the limit

$$\mu(x)w_{,tt} = \frac{\partial}{\partial x}[S(x)w_{,x}(x,t)].$$

Here the slope α is replaced by $w_{,x}$, with comma denoting the partial derivative with respect to x. For the homogeneous string with constant cross-section area A the mass density per unit length does not depend on x: $\mu(x) = \rho A = \mu$. We also assume that $S(x) = S$, where S/A is the tension in the equilibrium state. Dividing the above equation by μ, we reduce it to the standard form

$$w_{,tt} = c^2 w_{,xx}(x,t), \quad c = \sqrt{\frac{S}{\mu}}. \tag{3.18}$$

This equation is subject to the boundary conditions

$$w(0,t) = w(l,t) = 0. \tag{3.19}$$

The energy method is based on Hamilton's variational principle of least action: among all admissible motions $w(x,t)$ satisfying the initial and end conditions

$$w(x,t_0) = w_0(x), \quad w(x,t_1) = w_1(x)$$

and the boundary conditions (3.19) the true motion is the extremal of the action functional

$$I[w(x,t)] = \int_{t_0}^{t_1} \int_0^l L(w, w_{,x}, w_{,t})\, dx\, dt.$$

The consequence of Hamilton's variational principle is Euler-Lagrange's equation (see the derivation in Section 3.6)

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial w_{,t}} + \frac{\partial}{\partial x}\frac{\partial L}{\partial w_{,x}} - \frac{\partial L}{\partial w} = 0. \tag{3.20}$$

Thus, the motion of the conservative one-dimensional continuum is governed by a single function $L(w, w_{,x}, w_{,t})$, called Lagrangian, which is given by

$$L(w, w_{,x}, w_{,t}) = K(w_{,t}) - U(w, w_{,x}),$$

where $K(w_{,t})$ is the kinetic and $U(w, w_{,x})$ the internal energy densities. In our example the kinetic energy density is equal to

$$K(w_{,t}) = \frac{1}{2}\mu w_{,t}^2.$$

The internal energy density (per unit length) must be a function of the strain ε: $U = U(\varepsilon)$. Denoting the strain in the pre-stretched state as $\varepsilon = \varepsilon_0$, we expand the energy density in Taylor's series near this state:

$$U(\varepsilon) = U(\varepsilon_0) + U'(\varepsilon_0)(\varepsilon - \varepsilon_0) + \ldots.$$

Neglecting the unessential constant $U(\varepsilon_0)$ as well as terms of higher orders and taking into account that $U'(\varepsilon_0) = S$, we obtain

$$U(\varepsilon) = S(\varepsilon - \varepsilon_0).$$

Now, for the transverse displacement considered above

$$\varepsilon - \varepsilon_0 = \lim_{\Delta x \to 0} \frac{\sqrt{(\Delta x)^2 + (w(x+\Delta x) - w(x))^2} - \Delta x}{\Delta x} \approx \frac{1}{2} w_{,x}^2.$$

Thus, the internal energy density depends only on $w_{,x}$, $U(w, w_{,x}) = \frac{1}{2} S w_{,x}^2$, and

$$L(w, w_{,x}, w_{,t}) = \frac{1}{2}\mu w_{,t}^2 - \frac{1}{2} S w_{,x}^2.$$

Substitution of this Lagrangian into (3.20) leads again to the equation of motion (3.18).

EXAMPLE 3.4. Longitudinal vibration of bar. Derive the equation of small longitudinal vibration of an elastic bar.

Under bar we mean a thin elastic body whose diameter of the cross-section is much smaller than the length l of the bar. Let x be the coordinate along the bar axis, $x \in (0, l)$, and $u(x,t)$ the longitudinal displacement of the bar. The kinetic energy

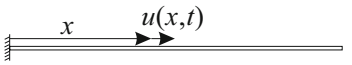

Fig. 3.5 Longitudinal vibration of bar

per unit length of the bar is given by

$$K(u_{,t}) = \frac{1}{2}\rho A(x)u_{,t}^2,$$

while the internal energy per unit length by

$$U(u_{,x}) = \frac{1}{2}EA(x)u_{,x}^2.$$

Here we denote by ρ the mass density, $A(x)$ the cross-section area which may vary along the bar axis, and E the Young modulus. With the Lagrangian $L(u_{,x}, u_{,t}) = K(u_{,t}) - U(u_{,x})$ we derive from (3.20)

$$\rho A(x)u_{,tt} = \frac{\partial}{\partial x}[EA(x)u_{,x}(x,t)],$$

and, if $A(x) = \text{const}$, reduce it to the standard form (3.18) with $c = \sqrt{E/\rho}$ and w being replaced by u.

EXAMPLE 3.5. Torsional vibration of bar. Derive the equation of small torsional vibration of an elastic bar.

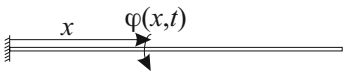

Fig. 3.6 Torsional vibration of bar

Let $\varphi(x,t)$ be the rotation angle of the cross-section in its own plane. The kinetic energy per unit length of the bar is given by

$$K(\varphi_{,t}) = \frac{1}{2}\rho J_p(x)\varphi_{,t}^2,$$

while the internal energy per unit length by

$$U(\varphi_{,x}) = \frac{1}{2}GJ_p(x)\varphi_{,x}^2.$$

Here $J_p(x)$ corresponds to the polar moment of inertia of the cross-section, and G the shear modulus. With $L(\varphi_{,x}, \varphi_{,t}) = K(\varphi_{,t}) - U(\varphi_{,x})$ we derive from Euler-Lagrange's equation

$$\rho J_p(x)\varphi_{,tt} = \frac{\partial}{\partial x}[GJ_p(x)\varphi_{,x}(x,t)],$$

which, for $J_p(x) = \text{const}$, can again be reduced to the standard form (3.18) with $c = \sqrt{G/\rho}$ and w being replaced by φ.

Solution. We first seek particular solutions of equation (3.18) and then construct the general solution using the linear superposition principle. We assume the particular solution of the form

$$w(x,t) = q(x)u(t).$$

As this particular solution is the product of two functions depending separately on x and t, the corresponding method of solution is called separation of variables. Plugging this Ansatz into equation (3.18) and assuming that neither $q(x)$ nor $u(t)$ is identically zero, we divide the obtained equation by $q(x)u(t)$ to get

$$\frac{\ddot{u}}{u} = c^2\frac{q''}{q}.$$

Since the left-hand side expression depends on t while its right-hand side counterpart depends only on x, the equation implies that both must be a constant which we denote by $-\omega^2$. Thus, we obtain two ordinary differential equations

$$\begin{aligned} \ddot{u} + \omega^2 u &= 0, \\ q'' + \left(\frac{\omega}{c}\right)^2 q &= 0. \end{aligned} \tag{3.21}$$

The solution of the second equation reads

$$q(x) = A\cos\frac{\omega}{c}x + B\sin\frac{\omega}{c}x.$$

The boundary conditions (3.19) require that $q(0) = q(l) = 0$, so $A = 0$ and the nontrivial solution exists if

$$\sin\frac{\omega}{c}l = 0 \quad \Rightarrow \quad \omega = \omega_j = j\frac{\pi c}{l}, \quad j = 1,2,\ldots. \tag{3.22}$$

Thus,

$$q_j(x) = B_j\sin\frac{j\pi}{l}x.$$

We may fix the coefficients B_j by choosing some normalization condition. As such we propose

$$\int_0^l q_j^2(x)\,dx = 1 \quad \Rightarrow \quad q_j(x) = \sqrt{\frac{2}{l}}\sin\frac{j\pi}{l}x.$$

Functions $q_j(x)$ describe the shapes of normal modes and are called eigenfunctions (or standing waves). Function $q_1(x)$ corresponds to the shape of mode with the

lowest frequency (or the fundamental tone). Functions $q_j(x)$ with $j > 1$ describe the shape of modes with higher frequencies (called overtones). The eigenfunction $q_j(x)$ has $j-1$ fix points which do not move (one speaks of the vibration nodes). Fig. 3.7 shows three first eigenfunctions. It turns out that the functions $q_j(x)$ are orthogonal in the following sense

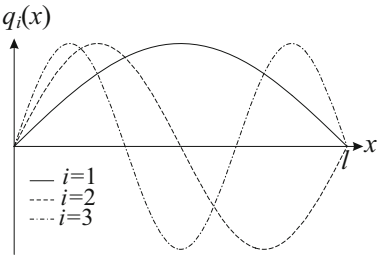

Fig. 3.7 Three first eigenfunctions

$$\int_0^l q_j(x)q_k(x)\,dx = 0 \quad \text{for } j \neq k.$$

This can easily be checked by using the well-known trigonometric formula for sine function.

The first equation of (3.21) for $\omega = \omega_j$ has the solution

$$u(t) = a_j \cos \omega_j t + b_j \sin \omega_j t.$$

Now, the general solution of equation (3.18) satisfying the boundary conditions (3.19) is obtained in form of the Fourier series

$$w(x,t) = \sqrt{\frac{2}{l}} \sum_{j=1}^{\infty} \sin \frac{j\pi}{l} x (a_j \cos \omega_j t + b_j \sin \omega_j t).$$

We have to satisfy also the initial conditions

$$w(x,0) = w_0(x), \quad w_{,t}(x,0) = v_0(x).$$

With the above solution we obtain from the initial conditions

$$\begin{aligned} \sqrt{\frac{2}{l}} \sum_{j=1}^{\infty} a_j \sin \frac{j\pi}{l} x &= w_0(x), \\ \sqrt{\frac{2}{l}} \sum_{j=1}^{\infty} \omega_j b_j \sin \frac{j\pi}{l} x &= v_0(x). \end{aligned} \tag{3.23}$$

To determine the unknown coefficients a_j and b_j we multiply these equations with the eigenfunction $q_k(x) = \sqrt{\frac{2}{l}} \sin \frac{k\pi}{l} x$ and integrate over x from 0 to l. Using the orthogonality and normalization conditions, we obtain

$$a_j = \sqrt{\frac{2}{l}} \int_0^l w_0(x) \sin \frac{j\pi}{l} x\,dx, \quad b_j = \sqrt{\frac{2}{l}} \frac{1}{\omega_j} \int_0^l v_0(x) \sin \frac{j\pi}{l} x\,dx.$$

In the harmonic analysis [12, 56] it is proved that if functions $w_0(x)$ and $v_0(x)$ are continuous and piecewise continuously differentiable, then the Fourier series (3.23) converge uniformly to $w_0(x)$ and $v_0(x)$ in the interval $(0,l)$. Thus, the eigenfunctions found above form a complete orthogonal basis for this class of initial data.

3.3 Beam

Bernoulli-Euler's Beam Theory. Under beam we mean a thin elastic body whose undeformed axis is a straight segment of length l. The thickness h of the beam is assumed to be much smaller than its length l. Let x be the coordinate along the beam axis, $x \in (0,l)$, and $w(x,t)$ the transverse displacement of the beam axis in the (x,y)-plane (see Fig. 3.8).

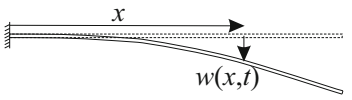

Fig. 3.8 Flexural vibration of beam

According to Bernoulli-Euler's beam theory the kinetic energy density of the beam is equal to

$$K(w_{,t}) = \frac{1}{2}\mu w_{,t}^2,$$

where μ is the mass per unit length. The internal energy density of the beam must be a quadratic function of the curvature of the deformed beam axis $w_{,xx}$

$$U(w_{,xx}) = \frac{1}{2}EJ(w_{,xx})^2,$$

with EJ being the bending stiffness. Thus, the action functional reads

$$I[w(x,t)] = \int_{t_0}^{t_1}\int_0^l L(w_{,xx}, w_{,t})\,dx\,dt,$$

where

$$L(w_{,xx}, w_{,t}) = \frac{1}{2}\mu w_{,t}^2 - \frac{1}{2}EJ(w_{,xx})^2. \tag{3.24}$$

The difference between this Lagrangian and those in the previous Section is the presence of the second derivative in the internal energy density.

We first consider the case of clamped edges such that

$$w(0,t) = w(l,t) = 0, \quad w_{,x}(0,t) = w_{,x}(l,t) = 0. \tag{3.25}$$

The second condition of (3.25) means the vanishing rotation angle of the beam about the clamped edge. Hamilton's variational principle of least action for the beam states

that among all admissible motions $w(x,t)$ satisfying the initial and end conditions

$$w(x,t_0) = w_0(x), \quad w(x,t_1) = w_1(x)$$

and the boundary conditions (3.25) the true motion is the extremal of the action functional

$$\delta I[w(x,t)] = \delta \int_{t_0}^{t_1} \int_0^l L(w_{,xx}, w_{,t})\, dx\, dt = 0.$$

To derive the equation of flexural vibration from this variational principle we calculate the first variation of the action functional

$$\delta I = \int_{t_0}^{t_1} \int_0^l (\mu w_{,t} \delta w_{,t} - EJ w_{,xx} \delta w_{,xx})\, dx\, dt = 0. \tag{3.26}$$

Integrating by parts over x and t and using the initial and end conditions as well as the clamped boundary conditions, we obtain

$$\delta I = \int_{t_0}^{t_1} \int_0^l (-\mu w_{,tt} - EJ w_{,xxxx})\, \delta w\, dx\, dt = 0.$$

Since δw can be chosen arbitrarily inside the region $(0,l) \times (t_0,t_1)$, the variational equation implies that

$$\mu w_{,tt} + EJ w_{,xxxx} = 0. \tag{3.27}$$

Note that Hamilton's principle applies to other boundary conditions as well. Practically, there are three types of boundary conditions at x_* ($x_* = 0$ or $x_* = l$) corresponding to:

a) Clamped edge: $w(x_*,t) = w_{,x}(x_*,t) = 0$.
b) Simply supported edge: $w(x_*,t) = 0$, but $w_{,x}(x_*,t)$ may be varied arbitrarily.
c) Free edge: both $w(x_*,t)$ and $w_{,x}(x_*,t)$ may be varied arbitrarily.

Provided equation (3.27) is fulfilled, we reduce equation (3.26) for the variations not vanishing at the boundaries to

$$\delta I = \int_{t_0}^{t_1} (-EJ w_{,xx} \delta w_{,x} + EJ w_{,xxx} \delta w)\big|_{x=0}^{x=l}\, dt = 0.$$

Thus, from the last equation we see that, in case b) the additional boundary condition obtained from Hamilton's principle is

$$w_{,xx}(x_*,t) = 0,$$

which means the vanishing bending moment. In case c) the boundary conditions read

$$w_{,xx}(x_*,t) = 0, \quad w_{,xxx}(x_*,t) = 0,$$

which mean the vanishing bending moment and shear force. Contrary to the kinematic (or essential) boundary conditions of the type $w(x_*,t) = 0$ or $w_{,x}(x_*,t) = 0$, the

additional boundary conditions derived from Hamilton's principle are called natural boundary conditions.

Solution. The method of solution is quite similar to that considered in previous Section. We first seek a particular solution of the form

$$w(x,t) = q(x)u(t).$$

The separation of variables in (3.27) leads to

$$\frac{\ddot{u}}{u} = -\frac{EJ}{\mu}\frac{q''''}{q}.$$

The left-hand side expression depends on t while its right-hand side counterpart depends only on x, therefore two ordinary differential equations follow

$$\ddot{u} + \omega^2 u = 0,$$
$$q'''' - \kappa^4 q = 0,$$

with $\kappa^4 = \omega^2\mu/EJ$. The solution of the second equation reads

$$q(x) = C_1 \sin\kappa x + C_2 \cos\kappa x + C_3 \sinh\kappa x + C_4 \cosh\kappa x.$$

Consider, for example, the beam which is simply supported at $x = 0$ and $x = l$. In this case the boundary conditions require that

$$q(0) = q''(0) = 0, \quad q(l) = q''(l) = 0.$$

So we get four homogeneous equations for four coefficients C_1, C_2, C_3, and C_4. Dividing the equations $q''(0) = 0$ and $q''(l) = 0$ by κ^2 and rewriting them in one matrix equation, we have

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ \sin\lambda & \cos\lambda & \sinh\lambda & \cosh\lambda \\ -\sin\lambda & -\cos\lambda & \sinh\lambda & \cosh\lambda \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

where $\lambda = \kappa l$. The non-trivial solution exists if the determinant of the matrix vanishes giving

$$\sin\lambda \sinh\lambda = 0 \quad \Rightarrow \quad \lambda_j = j\pi, \quad j = 1,2,\ldots. \tag{3.28}$$

For $\lambda_j = j\pi$ the above equation implies that $C_1 \neq 0$, $C_2 = C_3 = C_4 = 0$. We fix C_1 by the normalization condition

$$\int_0^l q_j^2(x)\,dx = 1 \quad \Rightarrow \quad q_j(x) = \sqrt{\frac{2}{l}} \sin\frac{j\pi}{l}x.$$

Thus, the eigenfunctions $q_j(x)$ for the beam with the supported edges remain the same as the shapes of normal modes of the string.

With $\lambda_j = j\pi$ we obtain for the eigenfrequencies

$$\omega_j = \lambda_j^2 \sqrt{\frac{EJ}{\mu l^4}} = (j\pi)^2 \sqrt{\frac{EJ}{\mu l^4}}, \quad j = 1,2,\ldots. \tag{3.29}$$

Note that the frequencies are proportional to j^2. The first equation for $u(t)$ has the solution

$$u(t) = a_j \cos \omega_j t + b_j \sin \omega_j t.$$

Now, the general solution is obtained in form of the Fourier series

$$w(x,t) = \sqrt{\frac{2}{l}} \sum_{j=1}^{\infty} \sin \frac{j\pi}{l} x (a_j \cos \omega_j t + b_j \sin \omega_j t),$$

where the coefficients a_j and b_j should be determined from the initial data $w_0(x)$ and $v_0(x)$ by using the orthogonality condition.

Table 3.1 The frequency equations

Boundary conditions	Frequency equation
free-free	$1 - \cos\lambda \cosh\lambda = 0$
supported-free	$\tan\lambda - \tanh\lambda = 0$
clamped-free	$1 + \cos\lambda \cosh\lambda = 0$
supported-supported	$\sin\lambda = 0$
clamped-supported	$\tan\lambda - \tanh\lambda = 0$
clamped-clamped	$1 - \cos\lambda \cosh\lambda = 0$

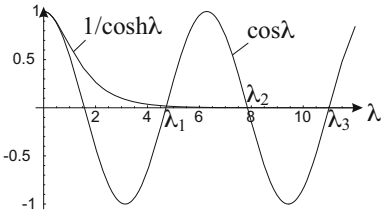

Fig. 3.9 Roots of the frequency equation $1/\cosh\lambda = \cos\lambda$

The modal analysis of the beam with other boundary conditions is similar and the frequency equations are summarized in Table 3.1. Knowing the roots of the frequency equations, one finds the eigenfrequencies in accordance with (3.29). Consider for example the free-free or clamped-clamped edges, for which the frequency

equation is $1/\cosh\lambda = \cos\lambda$. The roots of this transcendental equation correspond to the λ-coordinates of the intersection points of the curves $1/\cosh\lambda$ and $\cos\lambda$ shown in Fig. 3.9, from which it is seen that for $j > 3$ we have approximately $\lambda_j \approx (2j+1)\pi/2$. Note that the beams with the free edges or with one simply supported edge and one free edge possess also zero frequencies as these types of boundary conditions admit rigid-body motions.

3.4 Membrane

Differential Equation of Motion. Under membrane we mean a thin pre-stretched elastic body with negligible bending stiffness whose thickness is much smaller than other characteristic lengths. The pre-stress is assumed to be large, and the change of stresses in the membrane during its vibration is negligibly small compared with the pre-stress. We shall model the membrane by a two-dimensional continuum occupying the area A in its plane. Let x_1 and x_2 be the Cartesian coordinates in this plane, $\mathbf{x} = (x_1, x_2) \in A$, and $w(\mathbf{x},t)$ the small transverse displacement of the membrane in the x_3-direction. Hamilton's variational principle of least action states that, among all admissible motions $w(\mathbf{x},t)$ satisfying the initial and end conditions

$$w(\mathbf{x},t_0) = w_0(\mathbf{x}), \quad w(\mathbf{x},t_1) = w_1(\mathbf{x}),$$

as well as the boundary conditions

$$w(\mathbf{x},t) = 0 \quad \text{for } \mathbf{x} \in \partial A, \tag{3.30}$$

the true motion is the extremal of the action functional

$$I[w(\mathbf{x},t)] = \int_{t_0}^{t_1} \int_A L(w, w_{,\alpha}, w_{,t})\, dx\, dt.$$

Here and in what follows, Greek indices numerating the coordinates run from 1 to 2, the comma in indices denotes partial derivatives with respect to the corresponding coordinates, and $dx = dx_1 dx_2$ is the area element. From Hamilton's variational principle we derive Euler-Lagrange's equation (see Section 3.6)

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial w_{,t}} + \frac{\partial}{\partial x_\alpha}\frac{\partial L}{\partial w_{,\alpha}} - \frac{\partial L}{\partial w} = 0. \tag{3.31}$$

We use for short Einstein's summation convention according to which all the terms with repeated indices will be summed up over these indices from 1 to 2. For example, the second term in the above equation reads

$$\frac{\partial}{\partial x_\alpha}\frac{\partial L}{\partial w_{,\alpha}} = \frac{\partial}{\partial x_1}\frac{\partial L}{\partial w_{,1}} + \frac{\partial}{\partial x_2}\frac{\partial L}{\partial w_{,2}}.$$

The motion of membrane is thus governed by the Lagrangian

$$L(w, w_{,\alpha}, w_{,t}) = K(w_{,t}) - U(w, w_{,\alpha}),$$

where $K(w_{,t})$ is the kinetic and $U(w, w_{,\alpha})$ the internal energy densities. The kinetic energy density of the membrane is equal to

$$K(w_{,t}) = \frac{1}{2}\mu w_{,t}^2,$$

with μ being the mass per unit area. The internal energy density (per unit area) must be a function of the strains $\varepsilon_{\alpha\beta}$: $U = U(\varepsilon_{\alpha\beta})$. Without restricting generality we may regard the strains in the pre-stretched state as $\varepsilon = 0$, and expand the energy density in Taylor's series near this state:

$$U(\varepsilon_{\alpha\beta}) = U(0) + \left.\frac{\partial U}{\partial \varepsilon_{\alpha\beta}}\right|_{\varepsilon_{\alpha\beta}=0} \varepsilon_{\alpha\beta} + \dots.$$

Neglecting the unessential constant $U(0)$ as well as terms of higher orders and assuming that $\partial U/\partial \varepsilon_{\alpha\beta}|_0 = S\delta_{\alpha\beta}$, with S being the pre-stress, we obtain

$$U(\varepsilon) = S\varepsilon_{\alpha\alpha}.$$

Since $\varepsilon_{\alpha\alpha}$ describes the increase in area of the membrane, which, for the small transverse displacement with $w_{,\alpha} \ll 1$, is equal to

$$\varepsilon_{\alpha\alpha} = \sqrt{1 + w_{,\alpha}w_{,\alpha}} - 1 \approx \frac{1}{2}w_{,\alpha}w_{,\alpha},$$

the internal energy density does not depend on w, $U(w, w_{,\alpha}) = \frac{1}{2}Sw_{,\alpha}w_{,\alpha}$. Thus, the Lagrangian reads

$$L(w, w_{,\alpha}, w_{,t}) = \frac{1}{2}\mu w_{,t}^2 - \frac{1}{2}Sw_{,\alpha}w_{,\alpha}.$$

Plugging this into the equation of motion (3.31), we get finally

$$\mu w_{,tt} - S\Delta w = 0,$$

where $\Delta w = w_{,\alpha\alpha}$ is Laplace's operator applied to w. Bringing the second term to the right-hand side and dividing by μ, we reduce the above equation to the standard form

$$w_{,tt} = c^2\Delta w, \quad c = \sqrt{\frac{S}{\mu}}. \tag{3.32}$$

Solution. We separate the variables $\mathbf{x}$ and t by seeking the particular solution in the form

$$w(\mathbf{x}, t) = q(\mathbf{x})u(t).$$

Substitution in (3.32) together with the standard arguments lead to

$$\begin{aligned} \ddot{u} + \omega^2 u &= 0, \\ \Delta q + \lambda q &= 0, \end{aligned} \tag{3.33}$$

where $\lambda = \omega^2/c^2$. The second equation of (3.33) is subject to the boundary condition

$$q(\mathbf{x}) = 0 \quad \text{at } \mathbf{x} \in \partial A.$$

The solution of this eigenvalue problem in closed analytical form is possible for rectangular and circular membranes. We consider these special cases.

Rectangular membrane. Let a and b denote the width and the height of the rectangle so that $A = (0,a) \times (0,b)$. We look for the solution of $(3.33)_2$ in the form

$$q(\mathbf{x}) = X_1(x_1)X_2(x_2).$$

The standard separation of variables leads to

$$\frac{X_1''}{X_1} = -\frac{X_2''}{X_2} - \lambda = -\alpha^2.$$

The equation for $X_1(x_1)$,

$$X_1'' + \alpha^2 X_1 = 0,$$

together with the boundary conditions

$$X_1(0) = X_1(a) = 0,$$

yields

$$X_1(x_1) = C_1 \sin \alpha x_1,$$

where

$$\alpha = j_1 \frac{\pi}{a}, \quad j_1 = 1,2,\ldots.$$

Similarly, the equation for $X_2(x_2)$,

$$X_2'' + \beta^2 X_2 = 0,$$

with $\beta^2 = \lambda - \alpha^2$, together with the boundary conditions

$$X_2(0) = X_2(b) = 0,$$

implies that

$$X_2(x_2) = C_2 \sin \beta x_2,$$

where

$$\beta = j_2 \frac{\pi}{b}, \quad j_2 = 1,2,\ldots.$$

Denoting by j the vector (j_1, j_2), we present the eigenfunctions in the form

$$q_j(\mathbf{x}) = C \sin \frac{j_1 \pi x_1}{a} \sin \frac{j_2 \pi x_2}{b}, \quad j_1, j_2 = 1, 2, \ldots.$$

The corresponding eigenvalues belonging to the spectrum of the rectangular membrane are[2]

$$\lambda_j = \left(\frac{\omega_j}{c}\right)^2 = \alpha^2 + \beta^2 = \pi^2 \left(\frac{j_1^2}{a^2} + \frac{j_2^2}{b^2}\right).$$

We may fix the constant C by the normalization condition

$$\int_A q_j^2(\mathbf{x})\, dx = 1 \quad \Rightarrow \quad C = \frac{2}{\sqrt{ab}}.$$

One can also easily prove that

$$\int_A q_j(\mathbf{x}) q_k(\mathbf{x})\, dx = 0 \quad \text{for } j \neq k.$$

Mention that multiple eigenvalues may occur. For rectangles with rational ratios $a : b$ this is always the case. For instant, if $a = b$ (the square membrane), then $j' = (j_2, j_1)$ gives the same eigenvalue as $j = (j_1, j_2)$. Together with the solution of $(3.33)_1$

$$u_j(t) = a_j \cos \omega_j t + b_j \sin \omega_j t,$$

we construct the general solution of (3.32) by the linear superposition principle

$$w(\mathbf{x}, t) = \frac{2}{\sqrt{ab}} \sum_{j_1, j_2 = 1}^{\infty} \sin \frac{j_1 \pi x_1}{a} \sin \frac{j_2 \pi x_2}{b} (a_j \cos \omega_j t + b_j \sin \omega_j t).$$

Taking into account the initial conditions

$$w(\mathbf{x}, 0) = w_0(\mathbf{x}), \quad w_{,t}(\mathbf{x}, 0) = v_0(\mathbf{x}),$$

we obtain

$$\frac{2}{\sqrt{ab}} \sum_{j_1, j_2 = 1}^{\infty} a_j \sin \frac{j_1 \pi x_1}{a} \sin \frac{j_2 \pi x_2}{b} = w_0(x_1, x_2),$$

$$\frac{2}{\sqrt{ab}} \sum_{j_1, j_2 = 1}^{\infty} \omega_j b_j \sin \frac{j_1 \pi x_1}{a} \sin \frac{j_2 \pi x_2}{b} = v_0(x_1, x_2).$$

The orthogonality property can be used to find the coefficients a_j and b_j from the initial data $w_0(\mathbf{x})$ and $v_0(\mathbf{x})$. In the harmonic analysis [12, 56] it is proved that if

[2] Let us mention in this connection one interesting and still not completely solved mathematical problem: are there two distinct shapes of membrane having the same spectrum? This question was originally posed by Hermann Weyl and rephrased later by Mark Kac in the following way: "Can one hear the shape of a drum?" (see [22]).

functions $w_0(\mathbf{x})$ and $v_0(\mathbf{x})$ are continuous and piecewise continuously differentiable, then the double Fourier series converge uniformly to $w_0(\mathbf{x})$ and $v_0(\mathbf{x})$ in the square $(0,a)\times(0,b)$. Thus, the eigenfunctions found above form a complete orthogonal basis for this class of initial data.

Circular membrane. For the circular membrane we choose the polar coordinates r and ϑ (see Fig. 3.10) in which equation $(3.33)_2$ takes the form

$$q_{,rr}+\frac{1}{r}q_{,r}+\frac{1}{r^2}q_{,\vartheta\vartheta}+\lambda q=0.$$

Looking for the solution of this equation as the product

$$q(r,\vartheta)=R(r)\Theta(\vartheta)$$

and separating the variables, we obtain

$$\frac{R''+R'/r+\lambda R}{R/r^2}=-\frac{\Theta''}{\Theta}=\kappa^2.$$

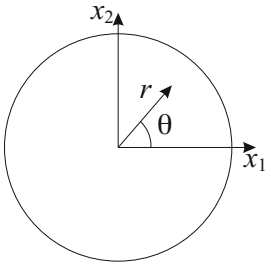

Fig. 3.10 Circular membrane and polar coordinates

Consider first the equation for $\Theta(\vartheta)$

$$\Theta''+\kappa^2\Theta=0,$$

which yields the solution

$$\Theta(\vartheta)=\alpha\cos\kappa\vartheta+\beta\sin\kappa\vartheta.$$

It is obvious that function $\Theta(\vartheta)$ must be periodic with the period 2π

$$\Theta(2\pi)=\Theta(0),\quad \Theta'(2\pi)=\Theta'(0).$$

The periodicity implies that $\kappa=j$, where j is a nonnegative integer $j=0,1,2,\ldots$. Then the equation for $R(r)$ becomes Bessel's equation [51]

$$R'' + \frac{1}{r}R' + \left(\frac{\omega^2}{c^2} - \frac{j^2}{r^2}\right)R = 0.$$

Its solution is expressed in terms of Bessel's function of order j

$$R = J_j(\frac{\omega r}{c}).$$

The boundary condition (3.30) at $r = r_m$, with r_m being the radius of the membrane, requires that

$$J_j(\frac{\omega r_m}{c}) = 0.$$

Thus, the eigenfrequencies are obtained from the zeros of Bessel's function $J_j(x)$, which we denote by ξ_{jk}, $k = 1, 2, \ldots$. The eigenfunctions now read

$$q_{jk}(r, \vartheta) = J_j(\xi_{jk} r/r_m)(\alpha_{jk} \cos j\vartheta + \beta_{jk} \sin j\vartheta).$$

The constants α_{jk} and β_{jk} may still be arbitrary, indicating that ξ_{jk} with $j \neq 0$ are at least double eigenvalues.

Together with the solution of $(3.33)_1$

$$u_{jk}(t) = a_{jk} \cos \omega_{jk} t + b_{jk} \sin \omega_{jk} t,$$

we construct the general solution of (3.32) by the linear superposition principle

$$w = \sum_{j=0,k=1}^{\infty} J_j(\xi_{jk} r/r_m)(\alpha_{jk} \cos j\vartheta + \beta_{jk} \sin j\vartheta)(a_{jk} \cos \omega_{jk} t + b_{jk} \sin \omega_{jk} t).$$

Then the initial conditions

$$w(r, \vartheta, 0) = w_0(r, \vartheta), \quad w_{,t}(r, \vartheta, 0) = v_0(r, \vartheta),$$

yield

$$\sum_{j=0,k=1}^{\infty} a_{jk} J_j(\xi_{jk} r/r_m)(\alpha_{jk} \cos j\vartheta + \beta_{jk} \sin j\vartheta) = w_0(r, \vartheta),$$

$$\sum_{j=0,k=1}^{\infty} b_{jk} \omega_{jk} J_j(\xi_{jk} r/r_m)(\alpha_{jk} \cos j\vartheta + \beta_{jk} \sin j\vartheta) = v_0(r, \vartheta).$$

One can again prove that if functions $w_0(r, \vartheta)$ and $v_0(r, \vartheta)$ are continuous, piecewise continuously differentiable, and periodic in ϑ with the period 2π, then the above double series converge uniformly to them in the domain $(0, r_m) \times (0, 2\pi)$. Thus, the eigenfunctions found above form a complete orthogonal basis for this class of initial data.

3.5 Plate

Kirchhoff's Plate Theory. Under plate we mean a thin elastic body occupying in its undeformed state a region $A \times (-h/2, h/2)$ of the three-dimensional Euclidean point space, where A is an area in the (x_1, x_2)-plane. The thickness h of the plate is assumed to be much smaller than the characteristic sizes of A. We shall model the plate by a two-dimensional continuum. Let $\mathbf{x} = (x_1, x_2) \in A$ and $w(\mathbf{x}, t)$ be the small transverse displacement of the middle plane in the x_3-direction (see Fig. 3.11).

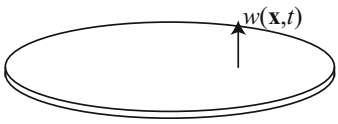

Fig. 3.11 A thin plate and transverse displacement of the middle plane

According to Kirchhoff's plate theory [24] the kinetic energy density of the plate is equal to

$$K(w_{,t}) = \frac{1}{2}\rho h w_{,t}^2,$$

where ρ is the mass density. The internal energy density of the plate must be a quadratic function of the curvature of the deformed middle surface $w_{,\alpha\beta}$

$$U(w_{,\alpha\beta}) = \frac{\mu h^3}{12}(\sigma w_{,\alpha\alpha}^2 + w_{,\alpha\beta} w_{,\alpha\beta}),$$

with $\sigma = \frac{\lambda}{\lambda + 2\mu}$, λ and μ being the Lamé constants of the elastic material. Thus, the action functional reads

$$I[w(\mathbf{x}, t)] = \int_{t_0}^{t_1} \int_A L(w_{,\alpha\beta}, w_{,t})\, dx\, dt,$$

where

$$L(w_{,\alpha\beta}, w_{,t}) = \frac{1}{2}\rho h w_{,t}^2 - \frac{\mu h^3}{12}(\sigma w_{,\alpha\alpha}^2 + w_{,\alpha\beta} w_{,\alpha\beta}). \tag{3.34}$$

The difference between this action functional and that of the membrane is the presence of the second partial derivatives in the internal energy density. The derivation of (3.34) from the three-dimensional elasticity theory based on the variational-asymptotic method is given in [31].

Since L depends on $w_{,\alpha\beta}$, the action functional "feels" the change of the derivative of w at the boundary ∂A. If the edge of the plate is free, then it is natural to assume that no constraints are imposed on w at the boundary. If the edge of the plate is clamped, we let $I[w]$ be defined on the space of admissible displacement fields $w(\mathbf{x}, t)$ satisfying the kinematic boundary conditions

$$w(\mathbf{x},t)=0, \quad w_{,\alpha}\nu_\alpha=0 \quad \text{at } \mathbf{x}\in\partial A, \tag{3.35}$$

where ν_α denotes the components of the outward unit vector normal to the curve ∂A. The last condition of (3.35) expresses the fact that the rotation angle of the plate about the clamped edge vanishes. Finally, if the edge is simply supported, then only the displacement should vanish

$$w(\mathbf{x},t)=0 \quad \text{at } \mathbf{x}\in\partial A. \tag{3.36}$$

Hamilton's variational principle states that, among all admissible motions $w(\mathbf{x},t)$ satisfying the initial and end conditions

$$w(\mathbf{x},t_0)=w_0(\mathbf{x}), \quad w(\mathbf{x},t_1)=w_1(\mathbf{x}),$$

as well as the kinematic boundary conditions, the true motion is the extremal of the action functional

$$\delta I=0.$$

In order to derive the equations of motion of the plate let us calculate the variation of the action functional

$$\delta I=\int_{t_0}^{t_1}\int_A(\rho h w_{,t}\delta w_{,t}-m_{\alpha\beta}\delta w_{,\alpha\beta})\,dx\,dt, \tag{3.37}$$

where

$$m_{\alpha\beta}=\frac{\partial U}{\partial w_{,\alpha\beta}}=\mu\frac{h^3}{6}(\sigma w_{,\gamma\gamma}\delta_{\alpha\beta}+w_{,\alpha\beta}).$$

From equation (3.37) one can see that $m_{\alpha\beta}$ "works" on the bending (or change of the curvature) of the plate. Therefore it is natural to call $m_{\alpha\beta}$ bending moments.

Integrating (3.37) by parts with the help of Gauss' theorem, we obtain for the variations vanishing at the boundary ∂A

$$\delta I=\int_{t_0}^{t_1}\int_A(-\rho h w_{,tt}-m_{\alpha\beta,\alpha\beta})\delta w\,dx\,dt=0.$$

Since δw is arbitrary inside the region $A\times(t_0,t_1)$, we conclude that

$$\rho h w_{,tt}+m_{\alpha\beta,\alpha\beta}=0 \quad \text{or} \quad \rho h w_{,tt}+D\Delta\Delta w=0, \tag{3.38}$$

where

$$D=\mu(\sigma+1)\frac{h^3}{6}=\frac{Eh^3}{12(1-\nu^2)}$$

is the bending stiffness of the plate. This is the two-dimensional equation of flexural vibration of the thin plate.

Provided the equation (3.38) is fulfilled, we reduce the equation $\delta I=0$ for the variations not vanishing at the boundary to

$$\int_{t_0}^{t_1}\int_{\partial A}(m_{\alpha\beta,\beta}\nu_\alpha\delta w - m_{\alpha\beta}\delta w_{,\alpha}\nu_\beta)\,ds\,dt = 0, \tag{3.39}$$

where ds is the length element. Now we need to select independent variations at the boundary. The gradient $\delta w_{,\alpha}$ may be resolved in the normal and tangential directions to the boundary as follows:

$$\delta w_{,\alpha} = \nu_\alpha\nu_\gamma\delta w_{,\gamma} + \tau_\alpha\tau_\gamma\delta w_{,\gamma}.$$

This is due to the identity $\delta_{\alpha\gamma} = \nu_\alpha\nu_\gamma + \tau_\alpha\tau_\gamma$, with τ_α being the components of the vector tangential to the curve ∂A. Since $\tau_\gamma\delta w_{,\gamma} = d\delta w/ds$, we can integrate by parts that term in (3.39) containing it to get

$$\int_{t_0}^{t_1}\int_{\partial A}[m_{\alpha\beta,\beta}\nu_\alpha\delta w + \frac{d}{ds}(m_{\alpha\beta}\tau_\alpha\nu_\beta)\delta w - m_{\alpha\beta}\nu_\alpha\nu_\beta\nu_\gamma\delta w_{,\gamma}]\,ds\,dt = 0.$$

For the free edge of the plate the variations δw and $\nu_\gamma\delta w_{,\gamma}$ are arbitrary at ∂A; hence

$$\begin{aligned} m_{\alpha\beta,\beta}\nu_\alpha + \frac{d}{ds}(m_{\alpha\beta}\tau_\alpha\nu_\beta) &= 0,\\ m_{\alpha\beta}\nu_\alpha\nu_\beta &= 0. \end{aligned} \tag{3.40}$$

These are the free-edge boundary conditions. For the clamped edge, the kinematic boundary conditions (3.35) should be fulfilled. If the edge is simply supported, (3.36) and $(3.40)_2$ are the boundary conditions at ∂A.

Frequency Spectra of Circular Plate. We investigate the free vibrations of a circular plate of radius r_m. We look for solutions of the form

$$w(\mathbf{x},t) = q(\mathbf{x})u(t).$$

The standard separation of variables $\mathbf{x}$ and t leads to

$$D\Delta\Delta q - \rho h\omega^2 q = 0, \tag{3.41}$$

where ω is the frequency of vibration. We introduce the dimensionless variables

$$\zeta_\alpha = \frac{x_\alpha}{r_m}, \quad \beta^4 = \omega^2\rho h r_m^4/D = \frac{6\rho\omega^2 r_m^4}{\mu(\sigma+1)h^2}.$$

Now we transform (3.41) to the dimensionless form

$$(\Delta\Delta - \beta^4)q = (\Delta+\beta^2)(\Delta-\beta^2)q = 0. \tag{3.42}$$

Therefore the solution of (3.42) may be written as

$$q = q_1 + q_2,$$

where functions q_1, q_2 satisfy respectively

$$\begin{aligned} \Delta q_1 + \beta^2 q_1 &= 0, \\ \Delta q_2 - \beta^2 q_2 &= 0. \end{aligned} \tag{3.43}$$

These equations can be solved again by the separation of variables. In the polar coordinates r, ϑ (see Fig. 3.10) we have

$$\Delta q = \frac{\partial^2 q}{\partial r^2} + \frac{1}{r}\frac{\partial q}{\partial r} + \frac{1}{r^2}\frac{\partial^2 q}{\partial \vartheta^2}.$$

Assuming $q_1 = R_1(r)\Theta_1(\vartheta)$, from $(3.43)_1$ we obtain the following differential equation for $\Theta_1(\vartheta)$

$$\Theta_1'' = -\kappa^2 \Theta_1.$$

Since $\Theta_1(\vartheta)$ must be periodic with the period 2π, we find that $\kappa = j$, where j is a nonnegative integer, $j = 0, 1, 2, \ldots$, and

$$\Theta_1(\vartheta) = \cos j\vartheta \quad \text{or} \quad \Theta_1(\vartheta) = \sin j\vartheta.$$

Then the equation for $R_1(r)$ reads

$$R_1'' + \frac{1}{r}R_1' + (\beta^2 - \frac{j^2}{r^2})R_1 = 0.$$

This is Bessel's equation of order j, which has the following non-singular solution

$$R_1 = aJ_j(\beta r).$$

Combination of R_1 and Θ_1 yields

$$q_1 = aJ_j(\beta r)\begin{cases} \cos j\vartheta \\ \sin j\vartheta \end{cases}.$$

For $q_2 = R_2(r)\Theta_2(\vartheta)$ the same results are obtained for $\Theta_2(\vartheta)$, while for $R_2(r)$ the modified Bessel's equation holds true

$$R_2'' + \frac{1}{r}R_2' - (\beta^2 + \frac{j^2}{r^2})R_2 = 0.$$

The non-singular solution of this equation is given by

$$R_2 = bI_j(\beta r),$$

where $I_j(x)$ is the modified Bessel's function of order j. Combining q_1 and q_2, we get finally

$$q = [aJ_j(\beta r) + bI_j(\beta r)] \begin{cases} \cos j\vartheta \\ \sin j\vartheta \end{cases} . \tag{3.44}$$

Consider the simplest case of the clamped edge, for which the boundary conditions

$$q|_{r=1} = \frac{\partial}{\partial r} q|_{r=1} = 0 \tag{3.45}$$

hold true. Substituting (3.44) into (3.45) and equating the determinant to zero, we obtain the following frequency equation:

$$J_j(\beta) I'_j(\beta) - I_j(\beta) J'_j(\beta) = 0. \tag{3.46}$$

The three lowest roots β_{jk} of (3.46) for $j = 0, 1, 2$ are given in the following table:

k	$j=0$	$j=1$	$j=2$
1	3.196	4.611	5.906
2	6.306	7.799	9.197
3	9.439	10.958	12.402

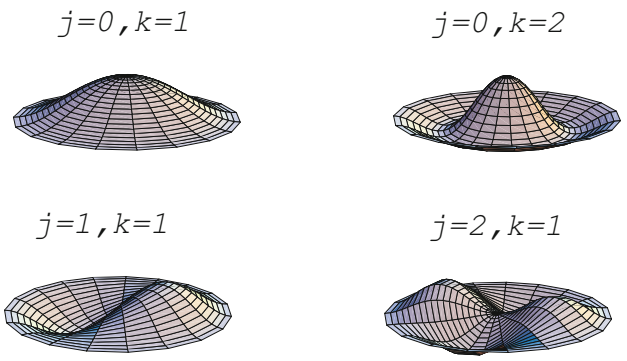

Fig. 3.12 Few normal modes of clamped circular plate

The frequencies of vibrations should be calculated by the formula

$$\omega_{jk} = \frac{\beta_{jk}^2 h}{r_m^2} \sqrt{\frac{\mu}{6(1-\nu)\rho}},$$

while the corresponding eigenfunctions are given by

$$q_{jk} = [J_j(\beta_{jk} r) - \frac{J_j(\beta_{jk})}{I_j(\beta_{jk})} I_j(\beta_{jk} r)] \begin{cases} \cos j\vartheta \\ \sin j\vartheta \end{cases} .$$

A few of the deformed shapes of the plate are shown in Fig. 3.12. Chladni invented an experimental method of observing the nodal lines of vibrations of a plate by pouring a thin layer of sand on its upper face: during the vibration the sand-grains will move to the nodal lines. Such nodal patterns are called Chladni's figures.[3]

Let us turn to the case of the free edge. The displacement w should then satisfy the boundary conditions (3.40). In the polar coordinates r, ϑ we have

$$\begin{aligned} m_{\alpha\beta,\beta}\nu_\alpha &= \mu\frac{h^3}{6}(\sigma+1)\frac{\partial}{\partial r}\Delta q,\\ m_{\alpha\beta}\tau_\alpha\nu_\beta &= \mu\frac{h^3}{6}\frac{1}{r}(\frac{\partial^2 q}{\partial r\partial\vartheta}-\frac{1}{r}\frac{\partial q}{\partial\vartheta}),\\ m_{\alpha\beta}\nu_\alpha\nu_\beta &= \mu\frac{h^3}{6}(\sigma\Delta q+\frac{\partial^2 q}{\partial r^2}). \end{aligned}$$

Substituting these into (3.40), we obtain the following conditions at the boundary $r=1$:

$$\begin{aligned} [(\sigma+1)\frac{\partial}{\partial r}\Delta q+\frac{\partial^3 q}{\partial r\partial\vartheta^2}-\frac{\partial^2 q}{\partial\vartheta^2}]|_{r=1}=0,\\ (\sigma\Delta q+\frac{\partial^2 q}{\partial r^2})|_{r=1}=0. \end{aligned} \tag{3.47}$$

With q from (3.44) we transform (3.47) to

$$\begin{aligned} a\{-\beta^3 J_j'(\beta)+(1-\nu)j^2[J_j(\beta)-\beta J_j'(\beta)]\}\\ +b\{\beta^3 I_j'(\beta)+(1-\nu)j^2[I_j(\beta)-\beta I_j'(\beta)]\}=0,\\ a\{-\beta^2 J_j(\beta)+(1-\nu)[j^2 J_j(\beta)-\beta J_j'(\beta)]\}\\ +b\{\beta^2 I_j(\beta)+(1-\nu)[j^2 I_j(\beta)-\beta I_j'(\beta)]\}=0. \end{aligned} \tag{3.48}$$

The frequency equation is obtained by the condition of vanishing determinant of (3.48). When $j=0$, the frequency equation can be presented in a simple form

$$2(1-\nu)+\beta\frac{J_0(\beta)}{J_0'(\beta)}-\beta\frac{I_0(\beta)}{I_0'(\beta)}=0.$$

The three lowest roots β_{jk} of the frequency equation for $j=0,1,2$ and $\nu=0.31$ have the following values:

k	$j=0$	$j=1$	$j=2$
1	0.0	0.0	2.308
2	3.004	4.526	5.938
3	6.202	7.735	9.185

[3] One can see such experiments as well as various Chladni's figures, for instance, on the following website: `http://www.youtube.com/watch?v=Qf0t4qIVWF4`

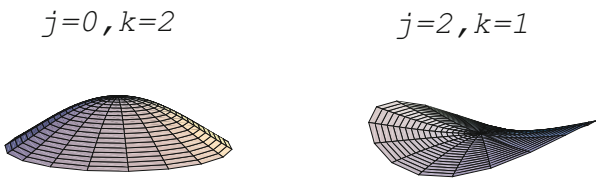

Fig. 3.13 Two normal modes of free circular plate

The first two zero frequencies correspond to the translation $q = a$ and small rotation $q = ar\sin\vartheta$ of the plate without deformation. The frequency according to β_{21} is the lowest one that is positive. Two deformed shapes of the plate for $j = 0$, $k = 2$ and $j = 2$, $k = 1$ are shown in Figure 3.13. If we sum up all these particular solutions, the obtained Fourier series converges again uniformly to the solution satisfying arbitrary regular initial conditions. The approximate solution for the rectangular plate can be found in [48].

3.6 General Continuous Oscillators

We present in this Section the variational principles for general continuous oscillators and the method of solution.

Conservative Systems. We model an arbitrary continuous oscillator by a continuum occupying the region V in the d-dimensional space, where $\mathbf{x} = (x_1, \ldots, x_d)$ denotes any point of V. We have in all applications $d = 1, 2, 3$. Suppose that each configuration of this continuum is uniquely determined by a set of functions (fields) $u_1(\mathbf{x}), \ldots, u_n(\mathbf{x})$. If $u_1(\mathbf{x}), \ldots, u_n(\mathbf{x})$ can vary independently and arbitrarily at each point $\mathbf{x}$ of the continuum, they are called generalized displacements, and n a number of degrees of freedom at that point. Motion of the system is described by the time dependent fields $u_i(\mathbf{x}, t)$. We denote by $u_{i,t} = (u_{1,t}, \ldots, u_{n,t})$ the corresponding velocities.

Hamilton's variational principle. Hamilton's variational principle of least action states that among all admissible motions of the conservative system satisfying the initial and end conditions

$$u_i(\mathbf{x}, t_0) = u_{i0}(\mathbf{x}), \quad u_i(\mathbf{x}, t_1) = u_{i1}(\mathbf{x}),$$

as well as the boundary conditions

$$u_i(\mathbf{x}, t) = 0 \quad \text{at } \mathbf{x} \in \partial V, \tag{3.49}$$

the true motion is the extremal of the action functional

$$I[u_i(\mathbf{x}, t)] = \int_{t_0}^{t_1} \int_V L(\mathbf{x}, u_i, u_{i,\alpha}, u_{i,t})\, dx\, dt.$$

Here and in what follows, Latin indices numerating the degrees of freedom run from 1 to n, while Greek indices numerating the coordinates run from 1 to d, the comma in indices denotes partial derivatives with respect to the corresponding coordinates, and $dx = dx_1 \dots dx_d$ is the volume element. Before deriving the equations of motion let us consider briefly, without detailed derivations, some further examples of Lagrangian.

EXAMPLE 3.6. Timoshenko's beam theory.

In this one-dimensional model ($d = 1$) we have two degrees of freedom at each point x: the displacement of the beam axis $w(x)$ and function $\psi(x)$ describing the first branch of the thickness-shear vibration of the cross-section. The Lagrangian of the Timoshenko beam is given by

$$L = \frac{1}{2}\rho h^2(w_{,t}^2 + \alpha h^2 \psi_{,t}^2) - \frac{1}{2}\mu h^2[sh^2\psi_{,x}^2 + \beta^2\alpha(\psi + w_{,x})^2], \tag{3.50}$$

with ρ being the mass density, h the thickness of the cross-section, μ the shear modulus, and α, s, and β the constants depending on the geometry of the cross-section (see the derivation of this formula in [31]).

EXAMPLE 3.7. Reissner-Mindlin's plate theory.

This is the two-dimensional model ($d = 2$) of the plate with three degrees of freedom at each point: $w(\mathbf{x})$ corresponding to the mean transverse displacement of the plate, $\psi_1(\mathbf{x})$ and $\psi_2(\mathbf{x})$ describing the first branches of the thickness-shear vibration. The Lagrangian of Reissner-Mindlin's plate theory reads

$$\begin{aligned} L = {} & \frac{1}{2}\rho h(w_{,t}^2 + \alpha h^2 \psi_{1,t}^2 + \alpha h^2 \psi_{2,t}^2) \\ & - \frac{1}{2}\mu h[\frac{h^2}{6}(\sigma\psi_{\alpha,\alpha}^2 + \psi_{(\alpha,\beta)}\psi_{(\alpha,\beta)}) + \alpha\pi^2(\psi_\alpha + w_{,\alpha})(\psi_\alpha + w_{,\alpha})], \end{aligned}$$

where

$$\psi_{(\alpha,\beta)} = \frac{1}{2}(\psi_{\alpha,\beta} + \psi_{\beta,\alpha}), \quad \alpha = \frac{1}{2}\left(\frac{\pi^2}{24}\right)^2$$

and all other notations remain the same as for Kirchhoff's plate theory (see [31] for the derivation of this theory as well as many other shell and rod theories).

EXAMPLE 3.8. Acoustic vibrations (sound waves).

Small amplitude vibrations of ideal compressible fluids (or ideal gases) are governed by the equations of motion of these media linearized about their equilibrium state [8,45]. The velocity potential $\varphi(\mathbf{x})$ is regarded as the only generalized displacement, so in 3-D case we have $d = 3$ and $n = 1$. The Lagrangian reads

$$L = \frac{1}{2}\rho_0(\varphi_{,\alpha}\varphi_{,\alpha} - \frac{1}{c^2}\varphi_{,t}^2),$$

where ρ_0 is the equilibrium mass density and $c = \sqrt{\partial p/\partial \rho|_{\rho_0}}$ the speed of sound, with p being the pressure. The interesting feature of this Lagrangian is that its first term corresponds to the kinetic energy, while the second term describes the deviation of the internal energy density from that of the equilibrium state. However, the governing equation does not change if we change the sign of L and interpret the second term as the kinetic energy, while the first term as the potential energy.

EXAMPLE 3.9. Vibrations of a three-dimensional elastic body.

Vibrations of a three-dimensional elastic body are best described within the 3-D elasticity theory [33], for which $d = n = 3$. At each point $\mathbf{x}$ of the body we have three displacements $u_\alpha(\mathbf{x})$, $\alpha = 1,2,3$. The Lagrangian reads

$$L = \frac{1}{2}\rho(\mathbf{x})u_{\alpha,t}u_{\alpha,t} - \frac{1}{2}E_{\alpha\beta\gamma\delta}(\mathbf{x})\varepsilon_{\alpha\beta}\varepsilon_{\gamma\delta},$$

where $\rho(\mathbf{x})$ is the mass density,

$$\varepsilon_{\alpha\beta} = \frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha})$$

are the components of the strain tensor and $E_{\alpha\beta\gamma\delta}(\mathbf{x})$ the elastic moduli. For homogeneous bodies ρ and $E_{\alpha\beta\gamma\delta}$ do not depend on $\mathbf{x}$.

EXAMPLE 3.10. Vibrations of a three-dimensional piezoelectric body.

Piezoelectric crystals and ceramics are widely used as sensors and actuators for the active vibration control [31, 42]. Their vibrations are described within the 3-D dynamic theory of piezoelectricity, for which $d = 3$ and $n = 4$. At each point $\mathbf{x}$ of the body we have three displacements $u_\alpha(\mathbf{x})$, $\alpha = 1,2,3$ and the electric potential $\varphi(\mathbf{x})$. The Lagrangian reads

$$L = \frac{1}{2}\rho u_{\alpha,t}u_{\alpha,t} - (\frac{1}{2}c^E_{\alpha\beta\gamma\delta}\varepsilon_{\alpha\beta}\varepsilon_{\gamma\delta} - e_{\alpha\beta\gamma}\varepsilon_{\alpha\beta}E_\gamma - \frac{1}{2}\varepsilon^S_{\alpha\beta}E_\alpha E_\beta),$$

where

$$\varepsilon_{\alpha\beta} = \frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}) \quad \text{and} \quad E_\alpha = -\varphi_{,\alpha}$$

are the components of the strain tensor and the electric field, respectively.

Let us derive the equations of motion from Hamilton's variational principle. To this end we calculate the variation of the action functional

$$\delta I = \int_{t_0}^{t_1}\int_V \left(\frac{\partial L}{\partial u_i}\delta u_i + \frac{\partial L}{\partial u_{i,\alpha}}\delta u_{i,\alpha} + \frac{\partial L}{\partial u_{i,t}}\delta u_{i,t}\right) dx\,dt.$$

As before we employ Einstein's summation convention according to which all terms with repeated indices will be summed up within their ranges. For example, the second term in the above equation reads

$$\frac{\partial L}{\partial u_{i,\alpha}}\delta u_{i,\alpha} = \sum_{i=1}^{n}\sum_{\alpha=1}^{d}\frac{\partial L}{\partial u_{i,\alpha}}\delta u_{i,\alpha}.$$

Integrating by parts over $\mathbf{x}$ and t with the help of Gauss' theorem and using the initial and end conditions as well as the boundary conditions, we obtain

$$\delta I = \int_{t_0}^{t_1}\int_V \left(\frac{\partial L}{\partial u_i} - \frac{\partial}{\partial x_\alpha}\frac{\partial L}{\partial u_{i,\alpha}} - \frac{\partial}{\partial t}\frac{\partial L}{\partial u_{i,t}}\right)\delta u_i\, dx\, dt = 0.$$

Since δu_i can be chosen independently and arbitrarily inside the region $V\times(t_0,t_1)$, the variational equation implies Euler-Lagrange's equations

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial u_{i,t}} + \frac{\partial}{\partial x_\alpha}\frac{\partial L}{\partial u_{i,\alpha}} - \frac{\partial L}{\partial u_i} = 0, \quad i = 1,\ldots,n. \tag{3.51}$$

These equations are subject to the kinematic boundary conditions (3.49) for the fixed boundary. For the free boundary δu_i may vary arbitrarily at ∂V, so the natural boundary conditions

$$\frac{\partial L}{\partial u_{i,\alpha}}\nu_\alpha = 0 \quad \text{at } \partial V, \quad i = 1,\ldots,n,$$

must be used instead.

Thus, the motion of any conservative mechanical system is governed by a single function, the Lagrangian, which is of the form

$$L(\mathbf{x}, u_i, u_{i,\alpha}, u_{i,t}) = K(\mathbf{x}, u_{i,t}) - U(\mathbf{x}, u_i, u_{i,\alpha}),$$

where $K(\mathbf{x}, u_{i,t})$ is the kinetic energy density and $U(\mathbf{x}, u_i, u_{i,\alpha})$ the internal energy density. The kinetic energy density $K(\mathbf{x}, u_{i,t})$ is a positive definite quadratic form with respect to $u_{i,t}$

$$K(\mathbf{x}, u_{i,t}) = \frac{1}{2}\rho_{ij}(\mathbf{x})u_{i,t}u_{j,t}, \tag{3.52}$$

where $\rho_{ij}(\mathbf{x})$ is $n\times n$ symmetric matrix called a mass density matrix. Thus,

$$\frac{\partial K}{\partial u_{i,t}}u_{i,t} = 2K(u_{i,t}),$$

showing that K is a homogeneous function of order two with respect to $u_{i,t}$. We now prove that the conservation of energy follows from equations (3.51). Indeed, multiplying (3.51) with $u_{i,t}$ and integrating over V, we obtain

$$\int_V \left(\frac{\partial}{\partial t}\frac{\partial K}{\partial u_{i,t}}u_{i,t} + \frac{\partial}{\partial x_\alpha}\frac{\partial L}{\partial u_{i,\alpha}}u_{i,t} - \frac{\partial L}{\partial u_i}u_{i,t}\right) dx = 0.$$

Integrating the second term by parts using the boundary conditions and keeping in mind the property of K, we get

$$\int_V \left[\frac{\partial}{\partial t}(2K) - \frac{\partial L}{\partial u_{i,t}} u_{i,tt} - \frac{\partial L}{\partial u_{i,\alpha}} u_{i,\alpha t} - \frac{\partial L}{\partial u_i} u_{i,t} \right] dx = 0.$$

The last three terms in the integrand give $-\partial L/\partial t$. Thus,

$$\frac{d}{dt}E = \frac{d}{dt}\int_V (K+U)dx = 0,$$

and the total energy E is conserved.

For small vibrations we may assume that both u_i and $u_{i,\alpha}$ are small and present the internal energy density U as the quadratic form

$$U(\mathbf{x}, u_i, u_{i,\alpha}) = \frac{1}{2}E_{i\alpha j\beta}(\mathbf{x})u_{i,\alpha}u_{j,\beta} + \frac{1}{2}A_{ij}(\mathbf{x})(u_i - B_\alpha(\mathbf{x})u_{i,\alpha})(u_j - B_\beta(\mathbf{x})u_{j,\beta}). \tag{3.53}$$

The examples considered above show that the internal energy density $U(\mathbf{x}, u_i, u_{i,\alpha})$ must be non-negative definite[4] with respect to its arguments u_i and $u_{i,\alpha}$, but not necessarily positive definite. The internal energy density may vanish for example at rigid-body motions if the boundary conditions admit such motions. Concerning the coefficients of this quadratic form we require the following symmetry properties

$$A_{ij}(\mathbf{x}) = A_{ji}(\mathbf{x}), \quad E_{i\alpha j\beta}(\mathbf{x}) = E_{j\beta i\alpha}(\mathbf{x}).$$

Taking these symmetry properties into account, we get for the partial derivatives of the Lagrangian with respect to u_i and $u_{i,\alpha}$ the following formulas

$$\begin{aligned}
\frac{\partial L}{\partial u_i} &= -\frac{\partial U}{\partial u_i} = -A_{ij}(u_j - B_\beta u_{j,\beta}),\\
\frac{\partial L}{\partial u_{i,\alpha}} &= -\frac{\partial U}{\partial u_{i,\alpha}} = -E_{i\alpha j\beta}u_{j,\beta} + A_{ij}B_\alpha(u_j - B_\beta u_{j,\beta}),
\end{aligned}$$

where the argument $\mathbf{x}$ of the coefficients is suppressed for short. Substituting these formulas into Euler-Lagrange's equations, we obtain the equations of motion

$$\rho_{ij}u_{j,tt} - (E_{i\alpha j\beta}u_{j,\beta})_{,\alpha} + (A_{ij}B_\alpha(u_j - B_\beta u_{j,\beta}))_{,\alpha} + A_{ij}(u_j - B_\beta u_{j,\beta}) = 0.$$

These equations can be presented also in the operator form as follows

$$\mathbf{M}\mathbf{u}_{,tt} + \mathbf{K}\mathbf{u} = \mathbf{0}, \tag{3.54}$$

where $\mathbf{M}$ is the mass density matrix and $\mathbf{K}$ the differential operator, called a stiffness operator, which maps the vector-valued function $\mathbf{u}(\mathbf{x}, t)$ (having n components) into the vector-valued function according to

[4] Except piezoelectric bodies considered in example 3.10 where U represents the electric enthalpy. Using Legendre transformation we can obtain the non-negative definite internal energy in terms of the strain tensor and the electric induction field [31].

$$(\mathbf{Ku})_i = -(E_{i\alpha j\beta}u_{j,\beta})_{,\alpha} + (A_{ij}B_\alpha(u_j - B_\beta u_{j,\beta}))_{,\alpha} + A_{ij}(u_j - B_\beta u_{j,\beta}).$$

We have to find the solution of (3.54) satisfying the initial conditions

$$\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}), \quad \mathbf{u}_{,t}(\mathbf{x},0) = \mathbf{v}_0(\mathbf{x}). \tag{3.55}$$

and the boundary conditions (3.49) (or the natural boundary conditions for the free boundary).

Solution. We seek a particular solution of (3.54) in the form

$$\mathbf{u}(\mathbf{x},t) = \mathbf{q}(\mathbf{x})w(t).$$

Separating the variables $\mathbf{x}$ and t as usual, we arrive at the eigenvalue problem

$$(\mathbf{K} - \lambda\mathbf{M})\mathbf{q} = \mathbf{0}, \tag{3.56}$$

subject to the boundary condition

$$\mathbf{q} = \mathbf{0} \quad \text{at } \mathbf{x} \in \partial V,$$

where $\lambda = \omega^2$, ω being the eigenfrequency. It turns out that all eigenvalues are real and non-negative. Indeed, if λ is an eigenvalue and $\mathbf{q}(\mathbf{x})$ a corresponding eigenfunction, then, by taking the scalar product of (3.56) with $\mathbf{q}(\mathbf{x})$ we have

$$\langle \mathbf{q}, \mathbf{Kq} \rangle - \lambda \langle \mathbf{q}, \mathbf{Mq} \rangle = 0.$$

Here and in what follows the scalar product between two vector-valued functions $\mathbf{u}(\mathbf{x})$ and $\mathbf{v}(\mathbf{x})$ is defined by

$$\langle \mathbf{u}, \mathbf{v} \rangle \equiv \int_V u_i(\mathbf{x})v_i(\mathbf{x})\,dx.$$

Thus,

$$\lambda = \frac{\langle \mathbf{q}, \mathbf{Kq} \rangle}{\langle \mathbf{q}, \mathbf{Mq} \rangle}.$$

The numerator can be transformed by the integration by parts using the boundary conditions giving

$$\begin{aligned}\langle \mathbf{q}, \mathbf{Kq} \rangle &= \int_V q_i[-(E_{i\alpha j\beta}q_{j,\beta})_{,\alpha} + (A_{ij}B_\alpha(q_j - B_\beta q_{j,\beta}))_{,\alpha} + A_{ij}(q_j - B_\beta q_{j,\beta})]dx \\ &= \int_V [E_{i\alpha j\beta}q_{i,\alpha}q_{j,\beta} + A_{ij}(q_i - B_\alpha q_{i,\alpha})(q_j - B_\beta q_{j,\beta})]dx = 2\int_V U(\mathbf{x}, q_i, q_{i,\alpha})dx,\end{aligned}$$

so $\langle \mathbf{q}, \mathbf{Kq} \rangle \geq 0$. Since the denominator $\langle \mathbf{q}, \mathbf{Mq} \rangle = 2\int_V K(\mathbf{x}, q_i)dx$ is positive, the eigenvalue λ is non-negative. The expression for λ is Rayleigh's quotient, for which extremal properties of eigenfrequencies of a continuous oscillator can be established (see exercise 3.11).

Note that the stiffness operator $\mathbf{K}$ is self-adjoint in the following sense: for arbitrary two functions $\mathbf{u}(x)$ and $\mathbf{v}(x)$

$$\langle \mathbf{u}, \mathbf{K}\mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{K}\mathbf{u} \rangle.$$

Indeed, integrating the expression on the left-hand side by parts using the boundary conditions and the symmetry properties of $A_{ij}(\mathbf{x})$ and $E_{i\alpha j\beta}(\mathbf{x})$, we have

$$\begin{aligned}\langle \mathbf{u}, \mathbf{K}\mathbf{v} \rangle &= \int_V u_i[-(E_{i\alpha j\beta} v_{j,\beta})_{,\alpha} + (A_{ij} B_\alpha (v_j - B_\beta v_{j,\beta}))_{,\alpha} + A_{ij}(v_j - B_\beta v_{j,\beta})]dx \\ &= \int_V [E_{i\alpha j\beta} u_{i,\alpha} v_{j,\beta} + A_{ij}(u_i - B_\alpha u_{i,\alpha})(v_j - B_\beta v_{j,\beta})]dx \\ &= \int_V v_i[-(E_{i\alpha j\beta} u_{j,\beta})_{,\alpha} + (A_{ij} B_\alpha (u_j - B_\beta u_{j,\beta}))_{,\alpha} + A_{ij}(u_j - B_\beta u_{j,\beta})]dx,\end{aligned}$$

and thus $\langle \mathbf{u}, \mathbf{K}\mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{K}\mathbf{u} \rangle$. The self-adjointness of $\mathbf{K}$ implies the following orthogonality property: two eigenfunctions $\mathbf{q}_1, \mathbf{q}_2$ corresponding to two different eigenvalues $\lambda_1 \neq \lambda_2$ are orthogonal in the following sense

$$\langle \mathbf{q}_1, \mathbf{M}\mathbf{q}_2 \rangle = 0, \quad \langle \mathbf{q}_1, \mathbf{K}\mathbf{q}_2 \rangle = 0.$$

To show this we multiply equation (3.56) for λ_1 with $\mathbf{q}_2$ to get

$$\langle \mathbf{q}_2, \mathbf{K}\mathbf{q}_1 \rangle = \lambda_1 \langle \mathbf{q}_2, \mathbf{M}\mathbf{q}_1 \rangle.$$

Similar procedure applied to the equation for λ_2 gives

$$\langle \mathbf{q}_1, \mathbf{K}\mathbf{q}_2 \rangle = \lambda_2 \langle \mathbf{q}_1, \mathbf{M}\mathbf{q}_2 \rangle.$$

Subtracting these equations from each other and using the symmetry of $\mathbf{M}$ and the self-adjointness of $\mathbf{K}$, we obtain

$$(\lambda_1 - \lambda_2)\langle \mathbf{q}_1, \mathbf{M}\mathbf{q}_2 \rangle = 0,$$

which implies the orthogonality. We choose the following normalization condition for the eigenfunctions

$$\langle \mathbf{q}_j, \mathbf{M}\mathbf{q}_j \rangle = 1.$$

Provided the region V is compact and the operator $\mathbf{K}$ is self-adjoint and nonnegative definite, one can show that the problem (3.56) has a countable set of eigenvalues

$$0 \leq \lambda_1 \leq \lambda_2 \leq \ldots, \quad \lim_{j\to\infty} \lambda_j = +\infty,$$

called a spectrum of the continuous oscillator. Based on this result we can now solve the initial boundary-value problem by combining the eigenfunctions with the solutions for $w_j(t)$

$$w_j(t) = a_j \cos \omega_j t + b_j \sin \omega_j t$$

to present the general solution of (3.54) in form of the series

$$\mathbf{u}(\mathbf{x},t) = \sum_{j=1}^{\infty} \mathbf{q}_j(\mathbf{x})(a_j \cos \omega_j t + b_j \sin \omega_j t).$$

The initial conditions (3.55) lead to

$$\sum_{j=1}^{\infty} a_j \mathbf{q}_j(\mathbf{x}) = \mathbf{u}_0(\mathbf{x}), \quad \sum_{j=1}^{\infty} b_j \omega_j \mathbf{q}_j(\mathbf{x}) = \mathbf{v}_0(\mathbf{x}).$$

Multiplying these equations with $\mathbf{M}\mathbf{q}_i$ and making use of the orthogonality and normalization conditions, we get

$$a_i = \langle \mathbf{u}_0, \mathbf{M}\mathbf{q}_i \rangle, \quad b_i = \frac{1}{\omega_i} \langle \mathbf{v}_0, \mathbf{M}\mathbf{q}_i \rangle.$$

For the compact region V one can prove that the eigenfunctions form an orthogonal basis in the space of continuous and piecewise continuously differentiable functions (see, e.g., [12]) and the Fourier series converges uniformly to the solution if its initial data $\mathbf{u}_0$ and $\mathbf{v}_0$ belong to this function space.

Dissipative Systems. For dissipative continuous oscillators the following variational principle holds true: among all admissible motions of an arbitrary dissipative system constrained by the initial and end conditions

$$\mathbf{u}(\mathbf{x},t_0) = \mathbf{u}_0(\mathbf{x}), \quad \mathbf{u}(\mathbf{x},t_1) = \mathbf{u}_1(\mathbf{x}),$$

as well as the boundary conditions (3.49), the true motion satisfies the variational equation

$$\delta \int_{t_0}^{t_1} \int_V L(\mathbf{x},\mathbf{u},\mathbf{u}_{,\alpha},\mathbf{u}_{,t}) dx dt - \int_{t_0}^{t_1} \int_V \left(\frac{\partial D}{\partial u_{i,t}} \delta u_i + \frac{\partial D}{\partial u_{i,\alpha t}} \delta u_{i,\alpha} \right) dx dt = 0. \quad (3.57)$$

Here $D(\mathbf{x}, u_{i,t}, u_{i,\alpha t})$ is the dissipation function. Calculation of variation in exactly the same manner as in the previous case leads to

$$\int_{t_0}^{t_1} \int_V \left(\frac{\partial L}{\partial u_i} - \frac{\partial}{\partial x_\alpha} \frac{\partial L}{\partial u_{i,\alpha}} - \frac{\partial}{\partial t} \frac{\partial L}{\partial u_{i,t}} - \frac{\partial D}{\partial u_{i,t}} + \frac{\partial}{\partial x_\alpha} \frac{\partial D}{\partial u_{i,\alpha t}} \right) \delta u_i \, dx dt = 0.$$

Due to the arbitrariness of δu_i inside the region $V \times (t_0, t_1)$ the following equations are obtained

$$\frac{\partial}{\partial t} \frac{\partial L}{\partial u_{i,t}} + \frac{\partial}{\partial x_\alpha} \frac{\partial L}{\partial u_{i,\alpha}} - \frac{\partial L}{\partial u_i} + \frac{\partial D}{\partial u_{i,t}} - \frac{\partial}{\partial x_\alpha} \frac{\partial D}{\partial u_{i,\alpha t}} = 0, \quad i = 1, \ldots, n.$$

These equations are subject to the kinematic boundary conditions (3.49) for the fixed boundary. For the free boundary the natural boundary conditions

$$\left(\frac{\partial L}{\partial u_{i,\alpha}} - \frac{\partial D}{\partial u_{i,\alpha t}}\right)\nu_\alpha = 0 \quad \text{at } \partial V, \quad i = 1,\ldots,n,$$

must be used instead.

Thus, the motion of continuous dissipative mechanical systems is governed by two functions, the Lagrangian and the dissipation function. We take the Lagrangian in the form

$$L(\mathbf{x}, u_i, u_{i,\alpha}, u_{i,t}) = K(\mathbf{x}, u_{i,t}) - U(\mathbf{x}, u_i, u_{i,\alpha}),$$

where the kinetic and potential energy densities are given by (3.52) and (3.53), respectively. Concerning the dissipation function the most simple assumption is that of proportional damping, for which

$$D(\mathbf{x}, u_{i,t}, u_{i,\alpha t}) = \alpha K(\mathbf{x}, u_{i,t}) + \beta U(\mathbf{x}, u_{i,t}, u_{i,\alpha t}),$$

with α and β being two constants. The first term in the right-hand side of this equation is thought of as the external damping due to the resistance to motion by the surrounding medium (say, the air resistance), while its second term is normally referred to as the internal damping which must be proportional to the relative motion of parts of the system. In this case, it is easy to show that the equation of motion can be presented in the form

$$\mathbf{M}\mathbf{u}_{,tt} + \mathbf{C}\mathbf{u}_{,t} + \mathbf{K}\mathbf{u} = \mathbf{0}, \tag{3.58}$$

where $\mathbf{M}$ is the mass density matrix and $\mathbf{K}$ the stiffness operator obtained previously. The operator $\mathbf{C}$, called a damping operator, is given by

$$\mathbf{C} = \alpha\mathbf{M} + \beta\mathbf{K}.$$

Equation (3.58) is subject to the boundary conditions (3.49) and the initial conditions (3.55).

Solution. We seek a particular solution of (3.58) in the form

$$\mathbf{u}(\mathbf{x},t) = \mathbf{q}_j(\mathbf{x})w_j(t),$$

where $\mathbf{q}_j(\mathbf{x})$ is an eigenvector found from the eigenvalue problem (3.56) and $w_j(t)$ an unknown scalar function. Substituting this Ansatz into (3.58) and taking into account the proportionality property, we reduce this equation to

$$\mathbf{M}\mathbf{q}_j[\ddot{w}_j + (\alpha + \beta\omega_j^2)\dot{w}_j + \omega_j^2 w_j] = \mathbf{0}.$$

Since $\mathbf{M}\mathbf{q}_j \neq \mathbf{0}$, the expression in the square brackets must vanish giving

$$\ddot{w}_j + (\alpha + \beta\omega_j^2)\dot{w}_j + \omega_j^2 w_j = 0.$$

This ordinary differential equation can be solved by the method discussed in Section 1.2. The general solution of (3.58) is obtained as the linear superposition of these particular solutions. Thus, the problem reduces to solving a countable set of uncoupled differential equations. Since the damping coefficients of these equations increase with the frequencies, the amplitudes of high frequency modes decay much faster than those of low-frequency modes. Thus, the truncation of the mass, stiffness, and damping matrices makes sense and its error can be controlled if α and β are known.

3.7 Exercises

EXERCISE 3.1. Derive the equations of motion for an infinite chain of atoms, where each atom interacts with m nearest neighbors on the left as well as m nearest neighbors on the right. Show the transition to the continuum.

Solution. Let u_j be the displacements of atoms from their equilibrium positions. Then the kinetic energy of this chain reads

$$K(\dot{u}) = \frac{1}{2} m \sum_j \dot{u}_j^2.$$

The potential energy of the springs is

$$U(u) = \frac{1}{2} \sum_j \sum_{\alpha=1}^{m} k_\alpha (u_j - u_{j-\alpha})^2.$$

Lagrange's equations of this chain lead to

$$m\ddot{u}_j + \sum_{\alpha=1}^{m} [k_\alpha (u_j - u_{j-\alpha}) - k_\alpha (u_{j+\alpha} - u_j)] = 0.$$

We can represent these equations in the equivalent quasi-continuum form by using the one-to-one correspondence between functions of discrete and continuous arguments

$$u(x) = b \sum_j u(jb) \delta_B(x - jb),$$

where b is the lattice constant. As shown in Section 3.1, the kinetic energy of the chain becomes

$$K(u_{,t}) = \int \frac{1}{2} \mu u_{,t}^2 \, dx,$$

where $\mu = m/b$. Let us rewrite the potential energy of the discrete chain in the form

$$U(u) = \frac{1}{2} \sum_{j,l} u_j \Phi(j - l) u_l,$$

where $\Phi(-j) = \Phi(j)$ and

$$\Phi(0) = 2\sum_{\alpha=1}^{m} k_\alpha, \quad \Phi(j) = -k_j \quad \text{for } 0 < j \le m, \quad \Phi(j) = 0 \quad \text{for } |j| > m.$$

In terms of $u(x)$ the potential energy is given by

$$U(u) = \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(x)\Phi(x-y)u(y)\,dx\,dy,$$

where

$$\Phi(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi(k)e^{ikx}dk,$$

and

$$\Phi(k) = \frac{1}{b}\sum_{|j|\le m} \Phi(j)e^{-ijbk}, \quad k \in B.$$

Thus, the action functional can be presented in terms of the function $u(x,t)$ as follows

$$I[u(x,t)] = \int_{t_0}^{t_1} dt \int_{-\infty}^{\infty} \left[\frac{1}{2}\mu u_{,t}^2 - \frac{1}{2}\int_{-\infty}^{\infty} u(x,t)\Phi(x-y)u(y,t)dy\right] dx.$$

Varying this action functional, we obtain the integral equation

$$\mu u_{,tt} + \int_{-\infty}^{\infty} \Phi(x-y)u(y,t)dy = 0.$$

The continuum limit can be obtained by approximating function $\Phi(k)$ in the long wave range $|k| \ll 1$:

$$\Phi(k) = \frac{2}{b}\sum_{\alpha=1}^{m} k_\alpha(1-\cos\alpha bk) \approx \sum_{\alpha=1}^{m} \frac{k_\alpha\alpha^2}{b}(bk)^2.$$

With this approximation the equation of motion reduces to

$$\mu u_{,tt} - Su_{,xx} = 0,$$

where $S = b\sum_{\alpha=1}^{m} k_\alpha\alpha^2$.

EXERCISE 3.2. A string of length l is released from a position shown in Fig. 3.14. Determine its motion.

Solution. The initial conditions of the string are

$$w(x,0) = w_0(x) = \begin{cases} \frac{w_0}{a}x & \text{for } x < a, \\ \frac{w_0}{a-l}(x-l) & \text{otherwise,} \end{cases} \qquad w_{,t}(x,0) = 0.$$

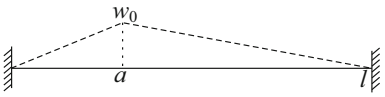

Fig. 3.14 Initial position of string

The solution to the equation of motion $w_{,tt} = c^2 w_{,xx}$ satisfying the boundary conditions $w(0,t) = w(l,t) = 0$ reads

$$w(x,t) = \sqrt{\frac{2}{l}} \sum_{j=1}^{\infty} \sin \frac{j\pi}{l} x (a_j \cos \omega_j t + b_j \sin \omega_j t),$$

where $\omega_j = j\frac{\pi c}{l}$. The initial conditions yield

$$\sqrt{\frac{2}{l}} \sum_{j=1}^{\infty} a_j \sin \frac{j\pi}{l} x = w_0(x),$$

$$\sqrt{\frac{2}{l}} \sum_{j=1}^{\infty} \omega_j b_j \sin \frac{j\pi}{l} x = v_0(x) = 0.$$

Thus, the coefficients b_j vanish. To determine the coefficients a_j, we use the orthogonality and normalization conditions to get

$$\begin{aligned}
a_j &= \sqrt{\frac{2}{l}} \int_0^l w_0(x) \sin \frac{j\pi}{l} x\, dx \\
&= \sqrt{\frac{2}{l}} \left[\int_0^a \frac{w_0}{a} x \sin \frac{j\pi}{l} x\, dx + \int_a^l \frac{w_0}{a-l} (x-l) \sin \frac{j\pi}{l} x\, dx \right] \\
&= \sqrt{\frac{2}{l}} \left[\frac{l w_0 \left(l \sin\left(\frac{\pi a j}{l}\right) - \pi a j \cos\left(\frac{\pi a j}{l}\right) \right)}{\pi^2 a j^2} \right. \\
&\left. + \frac{l w_0 \left(l \left(\sin(\pi j) - \sin\left(\frac{\pi a j}{l}\right) \right) + \pi j (a-l) \cos\left(\frac{\pi a j}{l}\right) \right)}{\pi^2 j^2 (a-l)} \right] \\
&= \sqrt{\frac{2}{l}} \frac{l^3 w_0 \sin\left(\frac{\pi a j}{l}\right)}{\pi^2 j^2 a (l-a)}.
\end{aligned}$$

Finally, the solution takes the form

$$w(x,t) = \sum_{j=1}^{\infty} \frac{2 l^2 w_0 \sin\left(\frac{\pi a j}{l}\right)}{\pi^2 j^2 a (l-a)} \sin \frac{j\pi}{l} x \cos j \frac{\pi c}{l} t.$$

EXERCISE 3.3. An elastic bar of length l has its free end stretched uniformly so that its length becomes $l+u_0$, and then is released from that position (see Fig. 3.15). Determine its motion.

Fig. 3.15 Uniformly stretched bar

Solution. Let $u(x,t)$ be the longitudinal displacement of the bar. The equation of longitudinal vibration of the bar reads

$$u_{,tt} = c^2 u_{,xx}.$$

This equation is subject to the boundary conditions

$$u(0,t) = 0, \quad u_{,x}(l,t) = 0.$$

The last condition is the traction free boundary condition.

Let us first find the particular solutions by separating the variables with the Ansatz $u(x,t) = q(x)p(t)$. It is easy to show that $q(x)$ satisfies the equation

$$q'' + \left(\frac{\omega}{c}\right)^2 q = 0$$

and the boundary conditions

$$q(0) = 0, \quad q'(l) = 0.$$

The solution reads

$$q(x) = A\cos\frac{\omega}{c}x + B\sin\frac{\omega}{c}x.$$

The first boundary condition implies that $A = 0$. From the second boundary condition we get

$$\cos\frac{\omega}{c}l = 0 \quad \Rightarrow \quad \omega = \omega_j = (2j-1)\frac{\pi c}{2l}, \quad j = 1,2,\ldots.$$

As the normalization condition we choose again

$$\int_0^l q_j^2(x)\,dx = 1 \quad \Rightarrow \quad q_j(x) = \sqrt{\frac{2}{l}}\sin\frac{(2j-1)\pi}{2l}x.$$

Combining this with the solution for $p(t)$

$$p(t) = a_j\cos\omega_j t + b_j\sin\omega_j t,$$

we represent the general solution in form of the Fourier series

$$u(x,t) = \sqrt{\frac{2}{l}} \sum_{j=1}^{\infty} \sin \frac{(2j-1)\pi}{2l} x (a_j \cos \omega_j t + b_j \sin \omega_j t).$$

The initial conditions of the bar

$$u(x,0) = u_0(x) = \frac{u_0}{l} x, \quad u_{,t}(x,0) = 0,$$

yield

$$\sqrt{\frac{2}{l}} \sum_{j=1}^{\infty} a_j \sin \frac{(2j-1)\pi}{2l} x = u_0(x),$$

$$\sqrt{\frac{2}{l}} \sum_{j=1}^{\infty} \omega_j b_j \sin \frac{(2j-1)\pi}{2l} x = v_0(x) = 0.$$

Thus, the coefficients b_j vanish. To determine the coefficients a_j we use the orthogonality and normalization conditions to get

$$a_j = \sqrt{\frac{2}{l}} \int_0^l \frac{u_0}{l} x \sin \frac{(2j-1)\pi}{2l} x \, dx = \sqrt{\frac{2}{l}} \frac{4l u_0 (-1)^{j+1}}{\pi^2 (2j-1)^2}.$$

Finally, the solution takes the form

$$u(x,t) = \sum_{j=1}^{\infty} \frac{8u_0(-1)^{j+1}}{\pi^2(2j-1)^2} \sin \frac{(2j-1)\pi}{2l} x \cos(2j-1) \frac{\pi c}{2l} t.$$

EXERCISE 3.4. An elastic shaft having a rigid disk attached at its free end performs torsional vibrations. The disk has a moment of inertia J_D (see Fig. 3.16). Derive the equation of small vibrations and the boundary conditions from Hamilton's variational principle. Determine the eigenfrequencies.

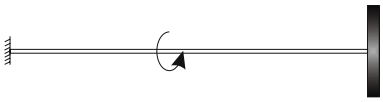

Fig. 3.16 Shaft with rigid disk attached at its end

Solution. We write down the action functional of this system

$$I[\varphi(x,t)] = \int_{t_0}^{t_1} \int_0^l (\frac{1}{2} \rho J_p \varphi_{,t}^2 - \frac{1}{2} G J_p \varphi_{,x}^2) \, dx \, dt + \int_{t_0}^{t_1} \frac{1}{2} J_D \varphi_{,t}(l,t)^2 \, dt.$$

The last term corresponds to the action functional of the disk. Varying this action functional, we have

$$\delta I = \int_{t_0}^{t_1} \int_0^l (\rho J_p \varphi_{,t} \delta\varphi_{,t} - G J_p \varphi_{,x} \delta\varphi_{,x})\, dx\, dt + \int_{t_0}^{t_1} J_D \varphi_{,t}(l,t) \delta\varphi_{,t}(l,t)\, dt$$
$$= \int_{t_0}^{t_1} \int_0^l (-\rho J_p \varphi_{,tt} + G J_p \varphi_{,xx}) \delta\varphi\, dx\, dt - \int_{t_0}^{t_1} (G J_p \varphi_{,x} + J_D \varphi_{,tt}) \delta\varphi\big|_{x=l}\, dt = 0.$$

Since $\delta\varphi$ can be chosen arbitrarily in the interval $(0,l)$ and at the end point $x = l$, this equation implies that

$$\rho J_p \varphi_{,tt} - G J_p \varphi_{,xx} = 0 \quad \Rightarrow \quad \varphi_{,tt} - c^2 \varphi_{,xx} = 0$$

inside $(0,l)$, with $c^2 = G/\rho$, and

$$G J_p \varphi_{,x} + J_D \varphi_{,tt} = 0$$

at $x = l$. Together with the boundary condition at $x = 0$

$$\varphi(0,t) = 0,$$

these constitute the eigenvalue problem. To determine the spectrum of this system we seek for the solution in the form

$$\varphi(x,t) = q(x) e^{i\omega t}.$$

Substituting this Ansatz into the equation of motion and the boundary conditions, we obtain

$$\omega^2 q + c^2 q'' = 0,$$

and

$$q(0) = 0, \quad G J_p q'(l) - J_D \omega^2 q(l) = 0.$$

From the equation for $q(x)$ we find that

$$q(x) = A \cos\frac{\omega}{c} x + B \sin\frac{\omega}{c} x.$$

The boundary condition $q(0) = 0$ yields $A = 0$. The other boundary condition at $x = l$ leads to the transcendental equation to determine the eigenfrequencies

$$G J_p \frac{\omega}{c} \cos\frac{\omega}{c} l - J_D \omega^2 \sin\frac{\omega}{c} l = 0,$$

or

$$\tan\frac{\omega}{c} l = \frac{G J_p}{J_D} \frac{1}{\omega c}.$$

EXERCISE 3.5. Find the eigenfrequencies of flexural vibrations of a beam with one clamped edge and one free edge. Plot the shapes of the first three modes of vibrations.

Solution. For the beam with one clamped edge at $x = 0$ and one free edge at $x = l$ the boundary conditions read

$$w(0,t) = w_{,x}(0,t) = 0, \quad w_{,xx}(l,t) = w_{,xxx}(l,t) = 0.$$

The standard separation of variables leads to the following eigenvalue problem

$$q'''' - \kappa^4 q = 0,$$

with $\kappa^4 = \omega^2 \mu / EJ$, together with the boundary conditions

$$q(0) = q'(0) = 0, \quad q''(l) = q'''(l) = 0.$$

The solution reads

$$q(x) = C_1 \sin \kappa x + C_2 \cos \kappa x + C_3 \sinh \kappa x + C_4 \cosh \kappa x.$$

Substituting this solution into the boundary conditions, we obtain

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ -\sin\lambda & -\cos\lambda & \sinh\lambda & \cosh\lambda \\ -\cos\lambda & \sin\lambda & \cosh\lambda & \sinh\lambda \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

where $\lambda = \kappa l$. The non-trivial solution exists if the determinant of the matrix vanishes giving

$$1 + \cos\lambda \cosh\lambda = 0.$$

The first three roots of this equation are

$$\lambda_1 = 1.8751, \quad \lambda_2 = 4.6941, \quad \lambda_3 = 7.8548.$$

The roots λ_j with $j > 3$ are approximately given by $(2j-1)\pi/2$. The shapes of the first three modes of vibrations are plotted in Fig. 3.17.

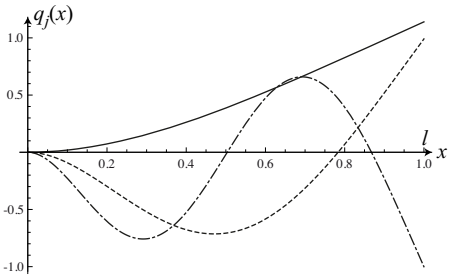

Fig. 3.17 Three first eigenfunctions

EXERCISE 3.6. The beam of length l and mass m sketched in Fig. 3.18 is released and latches upon impact onto the support B. Provided there is no rebound and no

loss of energy, determine the flexural vibration of the beam after impact. How to proceed if there is a rebound?

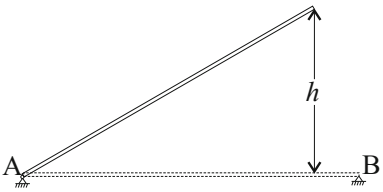

Fig. 3.18 Falling beam

Solution. Before impact the beam experiences a free falling. The conservation of energy yields

$$\frac{1}{2}J_A\dot{\varphi}_0^2 = mg\frac{h}{2},$$

where $\dot{\varphi}_0$ is the angular velocity of the beam immediately before impact, and

$$J_A = J_S + m(l/2)^2 = m\frac{l^2}{12} + m\frac{l^2}{4} = m\frac{l^2}{3}$$

is the moment of inertia of the beam about A. Thus, the angular velocity $\dot{\varphi}_0$ is equal to

$$\dot{\varphi}_0 = \sqrt{\frac{3gh}{l^2}}.$$

Knowing this angular velocity before impact, we find the initial conditions of the beam upon impact

$$w(x,0) = w_0(x) = 0, \quad w_{,t}(x,0) = \dot{\varphi}_0 x.$$

The solution to the equation of motion $\mu w_{,tt} = EJw_{,xxxx}$ satisfying the simply supported boundary conditions reads

$$w(x,t) = \sqrt{\frac{2}{l}}\sum_{j=1}^{\infty}\sin\frac{j\pi}{l}x(a_j\cos\omega_j t + b_j\sin\omega_j t),$$

where $\omega_j = (j\pi)^2\sqrt{\frac{EJ}{\mu l^4}}$. The initial conditions yield

$$\sqrt{\frac{2}{l}}\sum_{j=1}^{\infty}a_j\sin\frac{j\pi}{l}x = 0,$$

$$\sqrt{\frac{2}{l}}\sum_{j=1}^{\infty}\omega_j b_j\sin\frac{j\pi}{l}x = v_0(x) = \dot{\varphi}_0 x.$$

Thus, the coefficients a_j vanish. To determine the coefficients b_j we use the orthogonality and normalization condition to get

$$b_j = \frac{1}{\omega_j}\sqrt{\frac{2}{l}}\int_0^l \dot{\varphi}_0 x \sin\frac{j\pi}{l}x\,dx = \frac{1}{\omega_j}\sqrt{\frac{2}{l}}\frac{\dot{\varphi}_0 l^2(-1)^{j+1}}{\pi j}.$$

Finally, the solution takes the form

$$w(x,t) = \sum_{j=1}^{\infty}\frac{2\dot{\varphi}_0 l(-1)^{j+1}}{\omega_j \pi j}\sin\frac{j\pi}{l}x\sin\omega_j t.$$

In case of rebound the simply supported boundary conditions at $x = l$ should be replaced by the free boundary conditions

$$w_{,xx}(l,t) = w_{,xxx}(l,t) = 0,$$

while the initial conditions are obtained from the previous solution by the continuity of displacement and velocity.

EXERCISE 3.7. Derive the boundary conditions for a beam connected with a spring shown in Fig. 3.19. Find the eigenfrequencies.

Fig. 3.19 Beam with spring

Solution. To derive the boundary conditions we write down the action functional

$$I[w(x,t)] = \int_{t_0}^{t_1}\int_0^l [\frac{1}{2}\mu w_{,t}^2 - \frac{1}{2}EJ(w_{,xx})^2]\,dx\,dt - \int_{t_0}^{t_1}\frac{1}{2}k(w(l,t))^2\,dt,$$

where the last term is associated with the potential energy of the spring. Varying this action functional, we obtain

$$\delta I = \int_{t_0}^{t_1}\int_0^l (\mu w_{,t}\delta w_{,t} - EJw_{,xx}\delta w_{,xx})\,dx\,dt - \int_{t_0}^{t_1} kw(l,t)\delta w(l,t)\,dt.$$

Integrating the first two terms by parts and using the initial and end conditions as well as the kinematic boundary conditions at $x = 0$

$$w(0,t) = w_{,x}(0,t) = 0,$$

we reduce the first variation to

$$\delta I = \int_{t_0}^{t_1} \int_0^l (-\mu w_{,tt} - EJw_{,xxxx})\delta w\, dx\, dt - \int_{t_0}^{t_1} EJw_{,xx}\delta w_{,x}|_{x=l}\, dt$$
$$+ \int_{t_0}^{t_1} (EJw_{,xxx} - kw)\delta w|_{x=l}\, dt.$$

Thus, the first variation vanishes for arbitrary variations of w when

$$\mu w_{,tt} + EJw_{,xxxx} = 0$$

inside the interval $(0, l)$ and

$$w = w_{,x} = 0 \quad \text{at } x = 0,$$
$$w_{,xx} = 0, \quad EJw_{,xxx} - kw = 0 \quad \text{at } x = l.$$

The last boundary condition means that the resultant force acting on the beam is equal to the spring force. This condition can also be written in the form

$$w_{,xxx} - \alpha w = 0 \quad \text{at } x = l, \quad \alpha = \frac{k}{EJ}.$$

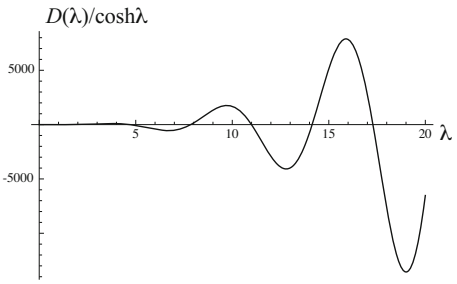

Fig. 3.20 Function $D(\lambda)/\cosh\lambda$

The standard separation of variables leads to the following equation for the shape function

$$q'''' - \kappa^4 q = 0,$$

with $\kappa^4 = \omega^2\mu/EJ$. The solution of this equation reads

$$q(x) = C_1 \sin\kappa x + C_2 \cos\kappa x + C_3 \sinh\kappa x + C_4 \cosh\kappa x.$$

Substituting this solution into the boundary condition we get four linear equations to determine four coefficients C_1, C_2, C_3, C_4. The determinant of this system, $D(\lambda)$, reads

$$\begin{vmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ -\sin\lambda & -\cos\lambda & \sinh\lambda & \cosh\lambda \\ -\lambda^3\cos\lambda - \beta\sin\lambda & \lambda^3\sin\lambda - \beta\cos\lambda & \lambda^3\cosh\lambda - \beta\sinh\lambda & \lambda^3\sinh\lambda - \beta\cosh\lambda \end{vmatrix}$$

where $\lambda = \kappa l$ and $\beta = l^3 k/EJ$. Fig. 3.20 shows the plot of function $D(\lambda)/\cosh\lambda$ against λ at $\beta = 1$. The roots of this function correspond to the dimensionless frequencies.

EXERCISE 3.8. An elastic beam is subjected to a harmonic end load as shown in Fig. 3.21. Determine its forced vibration.

Fig. 3.21 Beam under harmonic end load

Solution. The vibration of the beam must be the extremal of the following action functional

$$I[w(x,t)] = \int_{t_0}^{t_1}\int_0^l [\frac{1}{2}\mu w_{,t}^2 - \frac{1}{2}EJ(w_{,xx})^2]\,dx\,dt + \int_{t_0}^{t_1} f(t)w(l,t)\,dt,$$

where the last term describes the virtual work done by the concentrated load. Varying this action functional we have

$$\begin{aligned}\delta I &= \int_{t_0}^{t_1}\int_0^l (\mu w_{,t}\delta w_{,t} - EJw_{,xx}\delta w_{,xx})\,dx\,dt + \int_{t_0}^{t_1} f(t)\delta w(l,t)\,dt \\ &= \int_{t_0}^{t_1}\int_0^l (-\mu w_{,tt} - EJw_{,xxxx})\delta w\,dx\,dt \\ &- \int_{t_0}^{t_1} EJw_{,xx}\delta w_{,x}|_{x=l}\,dt + \int_{t_0}^{t_1} (EJw_{,xxx} + f(t))\delta w|_{x=l}\,dt = 0.\end{aligned}$$

This implies the equation of motion

$$\mu w_{,tt} + EJw_{,xxxx} = 0,$$

and the boundary conditions at $x = l$

$$w_{,xx} = 0, \quad EJw_{,xxx} + f(t) = 0.$$

Together with the kinematic boundary conditions at $x = 0$

$$w(0,t) = 0, \quad w_{,x}(0,t) = 0,$$

these constitute the boundary-value problem to determine the forced vibration. For the harmonic end load $f(t) = -\hat{f}\cos\omega t$ we look for the particular solution (describing the forced vibration) in the form

$$w(x,t) = q(x)\cos\omega t.$$

Substituting into the equation of motion and the boundary conditions, we obtain

$$q'''' - \kappa^4 q = 0,$$

with $\kappa^4 = \omega^2\mu/EJ$, and

$$q(0) = 0, \quad q'(0) = 0,$$

as well as

$$q''(l) = 0, \quad q'''(l) = \frac{\hat{f}}{EJ}.$$

Thus, the solution reads

$$q(x) = C_1\sin\kappa x + C_2\cos\kappa x + C_3\sinh\kappa x + C_4\cosh\kappa x.$$

Substituting this solution into the above boundary conditions, we get four linear equations to determine four coefficients C_1, C_2, C_3, C_4

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ -\sin\lambda & -\cos\lambda & \sinh\lambda & \cosh\lambda \\ -\cos\lambda & \sin\lambda & \cosh\lambda & \sinh\lambda \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{\hat{f}}{EJ\kappa^3} \end{pmatrix},$$

where $\lambda = \kappa l$.

EXERCISE 3.9. A square membrane is subjected to a harmonic load acting at its center. Determine the forced vibration.

Solution. Consider first the case of an arbitrarily distributed load $f(x,t)$. The equation of motion of the membrane can be obtained from Hamilton's variational principle and from the following action functional

$$I[w(\mathbf{x},t)] = \int_{t_0}^{t_1}\int_A \left(\frac{1}{2}\mu w_{,t}^2 - \frac{1}{2}S w_{,\alpha} w_{,\alpha}\right) dx\,dt + \int_{t_0}^{t_1}\int_A f(\mathbf{x},t) w\,dx\,dt.$$

The last term in this action functional represents the work done by the external load. Varying this functional we easily derive the following equation

$$\mu w_{,tt} - S\Delta w = f(\mathbf{x},t).$$

Dividing this equation by μ, we reduce it to the form

$$w_{,tt} - c^2\Delta w = p(\mathbf{x},t),$$

where $p = f/\mu$. For the harmonic load $p = \hat{p}(\mathbf{x})\cos\omega t$ we look for the particular solution (describing the forced vibration) in the form

$$w(\mathbf{x},t) = q(\mathbf{x})\cos\omega t.$$

Substituting this Ansatz into the equation of motion, we obtain the Helmholtz equation

$$-\omega^2 q - c^2 \Delta q = \hat{p}(\mathbf{x})$$

Since the eigenfunctions

$$q_j(\mathbf{x}) = \frac{2}{a}\sin\frac{j_1\pi x_1}{a}\sin\frac{j_2\pi x_2}{a}, \quad j = (j_1, j_2), \quad j_1, j_2 = 1,2,\ldots,$$

form a complete orthogonal basis in the space of continuously differentiable functions, we expand both $\hat{p}(\mathbf{x})$ and $q(\mathbf{x})$ in the double Fourier series

$$\hat{p}(\mathbf{x}) = \sum_j P_j q_j(\mathbf{x}), \quad q(\mathbf{x}) = \sum_j Q_j q_j(\mathbf{x}).$$

Substituting these series into the equation for q, making use of the properties of eigenfunctions

$$\Delta q_j + \lambda_j q_j = 0, \quad \lambda_j = \frac{\omega_j^2}{c^2} = \pi^2\left(\frac{j_1^2}{a^2} + \frac{j_2^2}{b^2}\right),$$

and comparing the terms on the right- and left-hand sides with the same q_j, we obtain for the coefficients Q_j the formulas

$$(\omega_j^2 - \omega^2)Q_j = P_j \quad \Rightarrow \quad Q_j = \frac{P_j}{\omega_j^2 - \omega^2}.$$

For the concentrated load acting in the middle of the membrane $\hat{p}(\mathbf{x}) = \hat{p}_0\delta(\mathbf{x} - \mathbf{x}_0)$, with $\mathbf{x}_0 = (a/2, a/2)$, the coefficients P_j are given by

$$P_j = \int_A \hat{p}_0\delta(\mathbf{x} - \mathbf{x}_0)q_j(\mathbf{x})\,dx = \hat{p}_0 q_j(\mathbf{x}_0) = \hat{p}_0\frac{2}{a}\sin\frac{j_1\pi}{2}\sin\frac{j_2\pi}{2}.$$

Thus, if $j_1 = 2k_1 - 1$ and $j_2 = 2k_2 - 1$, then $P_j = \frac{2\hat{p}_0}{a}(-1)^{k_1+k_2}$, otherwise $P_j = 0$. Consequently, the solution reads

$$w(\mathbf{x},t) = \sum_{k_1,k_2=1}^{\infty} \frac{4\hat{p}_0}{a^2}\frac{1}{\omega_j^2 - \omega^2}(-1)^{k_1+k_2}\sin\frac{(2k_1-1)\pi x_1}{a}\sin\frac{(2k_2-1)\pi x_2}{a}\cos\omega t,$$

where $j = (2k_1 - 1, 2k_2 - 1)$.

EXERCISE 3.10. Determine the eigenfrequencies of a circular plate with a simply supported boundary.

Solution. The equation of motion of the plate as well as the boundary conditions are derived from Hamilton's variational principle with the following action functional

$$I[w(\mathbf{x},t)] = \int_{t_0}^{t_1} \int_A [\frac{1}{2}\rho h w_{,t}^2 - \frac{\mu h^3}{12}(\sigma w_{,\alpha\alpha}^2 + w_{,\alpha\beta} w_{,\alpha\beta})]\, dx\, dt.$$

Computing the first variation of this action functional in exactly the same manner as in Section 3.5, we derive the equation

$$\rho h w_{,tt} + D\Delta\Delta w = 0.$$

Taking into account that $w = 0$ at ∂A but $\nu_\gamma \delta w_{,\gamma}$ may be chosen arbitrarily, we derive from Hamilton's variational principle the following boundary conditions

$$w = 0, \quad m_{\alpha\beta}\nu_\alpha \nu_\beta = 0 \quad \text{at } \partial A.$$

For the circular plate the standard separation of time and of the polar coordinates r and ϑ leads to

$$w(r,\vartheta,t) = q(r,\vartheta)e^{i\omega t},$$

where

$$q = [aJ_j(\beta r) + bI_j(\beta r)] \begin{cases} \cos j\vartheta \\ \sin j\vartheta \end{cases},$$

and $\beta^4 = \omega^2 \rho h r_m^4 / D = \frac{6\rho\omega^2 r_m^4}{\mu(\sigma+1)h^2}$. In the polar coordinates r, ϑ we have

$$m_{\alpha\beta}\nu_\alpha\nu_\beta = \mu\frac{h^3}{6}(\sigma\Delta q + \frac{\partial^2 q}{\partial r^2}) = 0$$

at $r = 1$. Substituting the solution into these boundary conditions, we obtain

$$\begin{aligned} aJ_j(\beta) + bI_j(\beta) &= 0, \\ a\{-\beta^2 J_j(\beta) + (1-\nu)[j^2 J_j(\beta) - \beta J_j'(\beta)]\} & \\ + b\{\beta^2 I_j(\beta) + (1-\nu)[j^2 I_j(\beta) - \beta I_j'(\beta)]\} &= 0. \end{aligned}$$

The frequency equation is obtained by the condition of vanishing determinant. When $j = 0$, the frequency equation can be presented in a simple form

$$J_0(\beta)[\beta I_0(\beta) - (1-\nu)I_0'(\beta)] + I_0(\beta)[\beta J_0(\beta) + (1-\nu)J_0'(\beta)] = 0.$$

The three lowest roots β_k of this frequency equation for $\nu = 0.31$ are

$$\beta_1 = 2.5504, \quad \beta_2 = 5.585, \quad \beta_3 = 8.6948.$$

EXERCISE 3.11. Prove the extremal properties of eigenfrequencies of a continuous oscillator based on the minimization of Rayleigh's quotient.

Solution. Consider the variational problem of minimization of Rayleigh's quotient

$$\min_{\mathbf{q}\neq 0} \frac{\langle \mathbf{q}, \mathbf{K}\mathbf{q}\rangle}{\langle \mathbf{q}, \mathbf{M}\mathbf{q}\rangle},$$

where

$$\langle \mathbf{q}, \mathbf{M}\mathbf{q}\rangle = \int_V q_i(\mathbf{x})\rho_{ij}(\mathbf{x})q_j(\mathbf{x})\,dx,$$

and

$$\langle \mathbf{q}, \mathbf{K}\mathbf{q}\rangle = \int_V [E_{i\alpha j\beta}q_{i,\alpha}q_{j,\beta} + A_{ij}(q_i - B_\alpha q_{i,\alpha})(q_j - B_\beta q_{j,\beta})]dx.$$

Obviously, this minimization problem is equivalent to the following problem: find the minimum of the functional

$$\langle \mathbf{q}, \mathbf{K}\mathbf{q}\rangle$$

among $\mathbf{q}$ satisfying the constraint

$$\langle \mathbf{q}, \mathbf{M}\mathbf{q}\rangle = 1.$$

To get rid of this constraint we introduce the Lagrange multiplier, λ, and consider the following functional

$$\langle \mathbf{q}, \mathbf{K}\mathbf{q}\rangle - \lambda(\langle \mathbf{q}, \mathbf{M}\mathbf{q}\rangle - 1).$$

Varying this functional with respect to λ and $\mathbf{q}$ we obtain the above constraint together with the following equation

$$(\mathbf{K} - \lambda\mathbf{M})\mathbf{q} = \mathbf{0}.$$

Thus, λ is the eigenvalue. By writing Rayleigh's quotient in terms of the normal coordinates we can show that its minimum corresponds to the smallest eigenvalue.

EXERCISE 3.12. Find the spectrum of radial vibrations for an elastic isotropic sphere of radius a.

Solution. We derive the equation of motion in the spherical coordinates. For the radial vibrations the non-zero component of displacement, u_r, depends only on r. In this situation the non-zero components of the strain tensor are

$$\varepsilon_{rr} = u_{r,r}, \quad \varepsilon_{\theta\theta} = \varepsilon_{\varphi\varphi} = \frac{u_r}{r}.$$

Then the action functional reduces to

$$I[u_r(r)] = \int_{t_0}^{t_1}\int_0^a L(r, u_r, u_{r,r}, u_{r,t})\,dr\,dt,$$

where

$$L(r,u_r,u_{r,r},u_{r,t}) = 4\pi\left[\frac{1}{2}\rho(u_{r,t})^2 - \frac{1}{2}\lambda(u_{r,r}+\frac{2u_r}{r})^2 - \mu(u_{r,r})^2 - 2\mu(\frac{u_r}{r})^2\right]r^2.$$

It is easy to show that the Euler-Lagrange equation

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial u_{r,t}} + \frac{\partial}{\partial r}\frac{\partial L}{\partial u_{r,r}} - \frac{\partial L}{\partial u_r} = 0$$

implies

$$\rho u_{r,tt} - (\lambda+2\mu)\left(u_{r,r}+\frac{2u_r}{r}\right)_{,r} = 0.$$

The standard Ansatz with $u_r(r,t) = q(r)e^{i\omega t}$ reduces this equation to

$$\left(q_{,r}+\frac{2q}{r}\right)_{,r} + k^2 q = 0,$$

where $k^2 = \omega^2/c_d^2$, with $c_d = \sqrt{(\lambda+2\mu)/\rho}$ being the velocity of dilatational waves. Let $q = \phi_{,r}$. Then in terms of ϕ we have the equation

$$\phi_{,rr} + \frac{2\phi_{,r}}{r} + k^2\phi = 0.$$

The solution which is regular at $r = 0$ reads

$$\phi = \frac{A}{r}\sin kr.$$

The boundary condition obtained from the above action functional at $r = a$ is

$$\lambda(u_{r,r}+2\frac{u_r}{r}) + 2\mu u_{r,r} = 0.$$

Taking into account the equation for ϕ, this boundary condition can be written in the form

$$c_d^2 k^2\phi + 4c_s^2\frac{\phi_{,r}}{a} = 0,$$

where $c_s = \sqrt{\mu/\rho}$ denotes the velocity of shear waves. Substituting the solution into this boundary condition we obtain the transcendental equation

$$\frac{\tan ka}{ka} = \frac{1}{1-(kac_d/2c_s)^2}.$$

whose roots determine the spectrum of the radial vibrations.