Chapter 2
Coupled Oscillators

This chapter deals with small vibrations of mechanical systems with many degrees of freedom. The effective method of solution for conservative systems is the linear transformation leading to uncoupled single oscillators. For dissipative systems the effective method of solution is the Laplace transform based on the linear superposition principle.

2.1 Conservative Oscillators

Differential Equations of Motion. Just as for systems with one degree of freedom, we can use either the force method or the energy method to derive the equations of motion for systems with two or several degrees of freedom. In the force method we must free each part of the system from the surrounding, then draw the free-body diagram with all acting forces, and finally apply Newton's law. In the energy method based on Hamilton's variational principle, we find the Lagrange function in terms of generalized coordinates and velocities and write down Lagrange's equations. We will see that, although both methods are equivalent, the energy method turns out to be more succinct for systems with many degrees of freedom and with various constraints. Let us begin with conservative systems having two degrees of freedom.

EXAMPLE 2.1. Coupled mass-spring oscillators. Two point-masses m_1 and m_2 move horizontally under the action of two massless springs of stiffnesses k_1 and k_2 (see Fig. 2.1). Derive the equations of motion for these coupled oscillators.

Let x_1 and x_2 be the displacements from the equilibrium positions of the point-masses m_1 and m_2, respectively. In the force method we first free the point-mass m_1 from the springs, then draw the free-body diagram (see Fig. 2.1), and finally apply Newton's law for m_1 in the x-direction

$$m_1\ddot{x}_1 = \sum F_x = -k_1x_1 + k_2(x_2 - x_1).$$

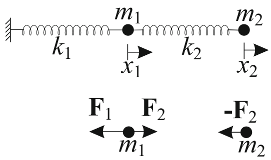

Fig. 2.1 Coupled mass-spring oscillators

Repeating the same procedure for m_2 (see Fig. 2.1), we obtain

$$m_2\ddot{x}_2 = \sum F_x = -k_2(x_2 - x_1).$$

Bringing the spring forces to the left-hand sides, we arrive at the system of equations of motion

$$\begin{aligned} m_1\ddot{x}_1 + k_1x_1 - k_2(x_2 - x_1) &= 0, \\ m_2\ddot{x}_2 + k_2(x_2 - x_1) &= 0. \end{aligned} \tag{2.1}$$

To use the energy method we write down the kinetic energy

$$K(\dot{x}) = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2,$$

and the potential energy

$$U(x) = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2(x_2 - x_1)^2,$$

where $x = (x_1, x_2)$, $\dot{x} = (\dot{x}_1, \dot{x}_2)$. With the Lagrange function $L(x, \dot{x}) = K(\dot{x}) - U(x)$, we derive from Lagrange's equations (see the derivation of these equations from Hamilton's variational principle in Section 2.4)

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_j} - \frac{\partial L}{\partial x_j} = 0, \quad j = 1, 2$$

the equations of motion (2.1).

EXAMPLE 2.2. Coupled pendulums. Two pendulums are connected with each other by a spring of stiffness k (see Fig. 2.2). Derive the equations of motion for this system.

In the force method we must free the first pendulum and add the spring force to the free-body diagram drawn for the mathematical pendulum in example 1.2. Because of the smallness of φ_1 and φ_2, the magnitude of the spring force is equal to $kl(\varphi_2 - \varphi_1)/2$, so the moment equation about A reads

$$m_1l^2\ddot{\varphi}_1 = \sum M_z = -m_1gl\varphi_1 + k\frac{l^2}{4}(\varphi_2 - \varphi_1).$$

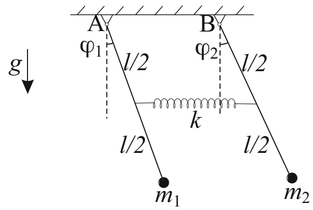

Fig. 2.2 Coupled pendulums

Applying the same procedure to the second pendulum, we obtain

$$m_2 l^2 \ddot{\varphi}_2 = \sum M_z = -m_2 g l \varphi_2 - k\frac{l^2}{4}(\varphi_2 - \varphi_1).$$

To use the energy method we write down the kinetic energy

$$K(\dot{\varphi}) = \frac{1}{2} m_1 l^2 \dot{\varphi}_1^2 + \frac{1}{2} m_2 l^2 \dot{\varphi}_2^2,$$

and, taking into account the smallness of φ_1 and φ_2, the potential energy

$$U(\varphi) = \frac{1}{2} m_1 g l \varphi_1^2 + \frac{1}{2} m_2 g l \varphi_2^2 + \frac{1}{2} k(l(\varphi_2 - \varphi_1)/2)^2,$$

where $\varphi = (\varphi_1, \varphi_2)$, $\dot{\varphi} = (\dot{\varphi}_1, \dot{\varphi}_2)$. The last term corresponds to the energy of the spring. With $L(\varphi, \dot{\varphi}) = K(\dot{\varphi}) - U(\varphi)$ we derive from Lagrange's equations

$$\begin{aligned} m_1 l^2 \ddot{\varphi}_1 + m_1 g l \varphi_1 - k\frac{l^2}{4}(\varphi_2 - \varphi_1) &= 0, \\ m_2 l^2 \ddot{\varphi}_2 + m_2 g l \varphi_2 + k\frac{l^2}{4}(\varphi_2 - \varphi_1) &= 0, \end{aligned} \qquad (2.2)$$

which are equivalent to the above equations.

EXAMPLE 2.3. Primitive model of a vehicle. A rigid bar, supported by two springs of stiffnesses k_1 and k_2, carries out a translational motion of its center of mass S in the vertical direction and a rotation in the plane about S (see Fig. 2.3). Derive the equations of motion for this system.

We see again the typical "engineering" approach to the problem: instead of dealing with a real vehicle with thousands of details and degrees of freedom, we try to select the most important of them.[1] In this simplified model the bar is constrained to have only two degrees of freedom: the vertical motion of S and the rotation in the plane about S. Let the vertical displacement of S from the equilibrium position be x and

[1] This selection depends of course on the aim of our simulations. See also a primitive model of an airplane with three degrees of freedom in exercise 2.12.

the angle of rotation be φ. In the force method we free the bar from the springs and apply Newton's law to it in the x-direction

$$m\ddot{x} = -k_1(x + l_1\varphi) - k_2(x - l_2\varphi).$$

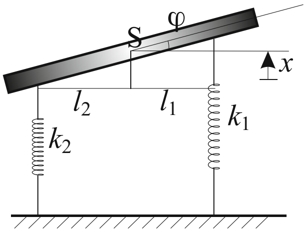

Fig. 2.3 Primitive model of vehicle

Note that the weight of the bar does not contribute to this equation because it is compensated with the static spring forces. In addition, the moment equation about S for the bar reads

$$J_S\ddot{\varphi} = -k_1 l_1(x + l_1\varphi) + k_2 l_2(x - l_2\varphi),$$

with J_S being the moment of inertia of the bar about S. The static spring forces do not contribute to this moment equation by the same reason.

To use the energy method we denote by $q = (x, \varphi)$ and $\dot{q} = (\dot{x}, \dot{\varphi})$ and write down the kinetic energy

$$K(\dot{q}) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}J_S\dot{\varphi}^2,$$

and the potential energy

$$U(q) = \frac{1}{2}k_1(x_{st} + x + l_1\varphi)^2 + \frac{1}{2}k_2(x_{st} + x - l_2\varphi)^2 + mgx,$$

where x_{st} corresponds to the change of length of the springs in the horizontal equilibrium state compared to that in the stress-free state. Expanding the spring energies and taking into account the equilibrium conditions, we see that the linear terms in x and φ are canceled out, so up to a constant,

$$U(q) = \frac{1}{2}k_1(x + l_1\varphi)^2 + \frac{1}{2}k_2(x - l_2\varphi)^2.$$

Thus, we derive again from Lagrange's equations the equations of motion.

Solution. We illustrate the method of solution on example 2.2. To simplify the analysis we consider the special case $m_1 = m_2 = m$. Dividing equations (2.2) by ml^2, we get

$$\begin{aligned} \ddot{\varphi}_1 + \omega_0^2\varphi_1 - \alpha(\varphi_2 - \varphi_1) &= 0, \\ \ddot{\varphi}_2 + \omega_0^2\varphi_2 + \alpha(\varphi_2 - \varphi_1) &= 0, \end{aligned} \tag{2.3}$$

where

$$\omega_0 = \sqrt{\frac{g}{l}}, \quad \alpha = \frac{k}{4m},$$

with ω_0 being the eigenfrequency of the uncoupled pendulum and α the coupling factor. We seek a particular solution of (2.3) in the form

$$\varphi_1 = \hat{\varphi}_1 e^{st}, \quad \varphi_2 = \hat{\varphi}_2 e^{st},$$

where $\hat{\varphi}_1$ and $\hat{\varphi}_2$ are unknown constants. Substituting this Ansatz into (2.3), we obtain

$$[s^2\hat{\varphi}_1 + \omega_0^2\hat{\varphi}_1 - \alpha(\hat{\varphi}_2 - \hat{\varphi}_1)]e^{st} = 0,$$
$$[s^2\hat{\varphi}_2 + \omega_0^2\hat{\varphi}_2 + \alpha(\hat{\varphi}_2 - \hat{\varphi}_1)]e^{st} = 0.$$

Since e^{st} is not equal to zero, the expressions in the square brackets must vanish. We may present these equations in the matrix form as follows

$$\begin{pmatrix} s^2 + \omega_0^2 + \alpha & -\alpha \\ -\alpha & s^2 + \omega_0^2 + \alpha \end{pmatrix} \begin{pmatrix} \hat{\varphi}_1 \\ \hat{\varphi}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{2.4}$$

From linear algebra we know that non-trivial solutions of (2.4) exist if its determinant vanishes

$$\begin{vmatrix} s^2 + \omega_0^2 + \alpha & -\alpha \\ -\alpha & s^2 + \omega_0^2 + \alpha \end{vmatrix} = (s^2 + \omega_0^2 + \alpha)^2 - \alpha^2 = 0. \tag{2.5}$$

Equation (2.5), quadratic with respect to s^2, yields

$$s_1^2 = -\omega_0^2, \quad s_2^2 = -(\omega_0^2 + 2\alpha).$$

Thus, the roots of (2.5) are imaginary numbers given by

$$s_1 = \pm i\omega_1, \quad s_2 = \pm i\omega_2, \tag{2.6}$$

with $\omega_1 = \omega_0$ and $\omega_2 = \sqrt{\omega_0^2 + 2\alpha}$ being called the eigenfrequencies. Note that the amplitudes $\hat{\varphi}_1$ and $\hat{\varphi}_2$ cannot be arbitrary. For example, if $s = s_1$, then (2.4) implies that

$$\hat{\varphi}_1 = \hat{\varphi}_2,$$

or, in the vector form,

$$\hat{\boldsymbol{\varphi}} = \begin{pmatrix} \hat{\varphi}_1 \\ \hat{\varphi}_2 \end{pmatrix} = C_1\mathbf{q}_1, \quad \mathbf{q}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Thus, the vector $\hat{\boldsymbol{\varphi}}$ is proportional to the eigenvector $\mathbf{q}_1$ which is normalized to have the length 1. Likewise, for $s = s_2$ we have from (2.4) $\hat{\varphi}_1 = -\hat{\varphi}_2$, or

$$\hat{\boldsymbol{\varphi}} = C_2\mathbf{q}_2, \quad \mathbf{q}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Note that $\mathbf{q}_2$ is orthogonal to $\mathbf{q}_1$. Because $\mathbf{q}_j e^{st} = \mathbf{q}_j e^{\pm i\omega t}$ satisfy (2.3) which are the differential equations with real coefficients, their real and imaginary parts

$$\mathbf{q}_j \cos\omega t \quad \text{and} \quad \mathbf{q}_j \sin\omega t$$

must satisfy also these equations. The general solution can now be constructed using the linear superposition principle

$$\boldsymbol{\varphi} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \mathbf{q}_1(A_1 \cos \omega_1 t + B_1 \sin \omega_1 t) + \mathbf{q}_2(A_2 \cos \omega_2 t + B_2 \sin \omega_2 t).$$

The four unknown coefficients A_1, B_1 and A_2, B_2 must be found from the initial conditions

$$\boldsymbol{\varphi}(0) = \boldsymbol{\varphi}_0, \quad \dot{\boldsymbol{\varphi}}(0) = \dot{\boldsymbol{\varphi}}_0$$

giving

$$\begin{aligned} A_1\mathbf{q}_1 + A_2\mathbf{q}_2 &= \boldsymbol{\varphi}_0, \\ B_1\omega_1\mathbf{q}_1 + B_2\omega_2\mathbf{q}_2 &= \dot{\boldsymbol{\varphi}}_0. \end{aligned}$$

Then, using the orthogonality of $\mathbf{q}_1$ and $\mathbf{q}_2$, we obtain from here

$$A_j = \boldsymbol{\varphi}_0 \cdot \mathbf{q}_j, \quad B_j = \frac{1}{\omega_j}\dot{\boldsymbol{\varphi}}_0 \cdot \mathbf{q}_j, \quad j = 1,2,$$

with the dot denoting the scalar product of two vectors. Alternatively, we can present the solution in the form

$$\boldsymbol{\varphi} = \mathbf{q}_1 a_1 \cos(\omega_1 t - \phi_1) + \mathbf{q}_2 a_2 \cos(\omega_2 t - \phi_2). \tag{2.7}$$

Recalling the addition theorem for $\cos(\omega t - \phi)$, we find

$$a_j = \sqrt{A_j^2 + B_j^2}, \quad \tan \phi_j = \frac{B_j}{A_j}, \quad j = 1,2.$$

For $\alpha \ll 1$ (weak coupling) solution (2.7) exhibits an interesting phenomenon called beating or amplitude modulation (see exercise 2.4).

Normal Modes and Coordinates. As we see from (2.7) the solution is the superposition of two harmonic cosine functions with different frequencies. If the frequency ratio is not a rational number, the motion is no longer periodic in general.[2] However, for the initial conditions of the special form

$$\varphi_1(0) = \varphi_2(0), \quad \dot{\varphi}_1(0) = \dot{\varphi}_2(0),$$

or

$$\varphi_1(0) = -\varphi_2(0), \quad \dot{\varphi}_1(0) = -\dot{\varphi}_2(0),$$

the motion is purely harmonic with the frequency ω_1 or ω_2. We call such the special periodic motion normal mode. Fig. 2.4 shows the normal modes corresponding to $\omega = \omega_1$ and $\omega = \omega_2$, respectively. For mode 1 (symmetric mode) the pendulums oscillate in phase, consequently the spring does not change its length and has no

[2] It is in general *quasi-periodic* (see exercise 2.5).

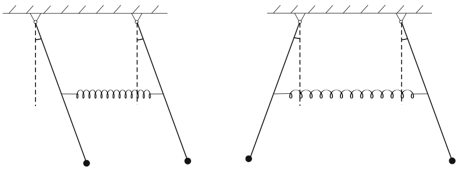

Fig. 2.4 Modes of vibration: 1) $\omega = \omega_1$, 2) $\omega = \omega_2$

influence on the frequency ($\omega = \omega_1 = \omega_0$). For mode 2 (antisymmetric mode) the pendulums oscillate in counter-phases, and the spring stiffness makes the frequency ω_2 higher than ω_1.

The question now arises: can we find the coordinates in which the normal modes become independent? The first observation is that this holds true if the kinetic and potential energies of the system, in terms of the new coordinates ξ_1 and ξ_2, take the form

$$K(\dot{\xi}) = \frac{1}{2}(\dot{\xi}_1^2 + \dot{\xi}_2^2), \quad U(\xi) = \frac{1}{2}(\omega_1^2 \xi_1^2 + \omega_2^2 \xi_2^2).$$

Indeed, in this case Lagrange's equations of the system become uncoupled

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\xi}_j} - \frac{\partial L}{\partial \xi_j} = \ddot{\xi}_j + \omega_j^2 \xi_j = 0, \quad j = 1,2,$$

yielding two independent modes of vibrations with the frequencies ω_1 and ω_2. Thus, the answer must be found by the well-known procedure in linear algebra of simultaneously diagonalizing two positive definite quadratic forms [37]. In our simple example we may divide both the kinetic and potential energies by ml^2 to get

$$K(\dot{\varphi}) = \frac{1}{2}(\dot{\varphi}_1^2 + \dot{\varphi}_2^2),$$

and

$$U(\varphi) = \frac{1}{2}\omega_0^2 \varphi_1^2 + \frac{1}{2}\omega_0^2 \varphi_2^2 + \frac{1}{2}\alpha(\varphi_2 - \varphi_1)^2.$$

These formulas suggest the following obvious choice of normal coordinates

$$\xi_1 = \frac{1}{\sqrt{2}}(\varphi_1 + \varphi_2), \quad \xi_2 = \frac{1}{\sqrt{2}}(\varphi_2 - \varphi_1).$$

In terms of the new coordinates we have

$$K(\dot{\xi}) = \frac{1}{2}(\dot{\xi}_1^2 + \dot{\xi}_2^2), \quad U(\xi) = \frac{1}{2}[\omega_0^2 \xi_1^2 + (\omega_0^2 + 2\alpha)\xi_2^2],$$

so this is the Lagrange function of two independent oscillators with the frequencies ω_1 and ω_2. We will see later that the reduction of a general conservative oscillator with n degrees of freedom to n uncoupled single oscillators is possible and realized by a linear transformation which simultaneously diagonalize the kinetic and potential energies as quadratic forms.

2.2 Dissipative Oscillators

Differential Equations of Motion. We have seen from the previous Sections that, although both the force and the energy methods are equivalent, the latter turns out to be more advantageous for systems with many degrees of freedom. Since we are now familiar with the energy method and convinced in its equivalence with the force method, we shall use exclusively the former to derive the equations of motion.

EXAMPLE 2.4. Mass-spring-damper oscillators. Two masses m_1 and m_2 move horizontally under the action of two massless springs of stiffnesses k_1 and k_2 and two dampers of damping constants c_1, c_2 (see Fig. 2.5). Derive the equations of motion for these coupled oscillators.

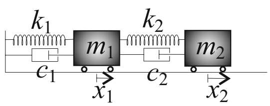

Fig. 2.5 Mass-spring-damper oscillators with two degrees of freedom

Let x_1 and x_2 be the displacements from the equilibrium positions of the masses m_1 and m_2, respectively. Similar to example 2.1 the Lagrange function reads

$$L(x,\dot{x}) = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 - \frac{1}{2}k_1x_1^2 - \frac{1}{2}k_2(x_2-x_1)^2.$$

With the dissipation function

$$D(\dot{x}) = \frac{1}{2}c_1\dot{x}_1^2 + \frac{1}{2}c_2(\dot{x}_2-\dot{x}_1)^2,$$

we derive from modified Lagrange's equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_j} - \frac{\partial L}{\partial x_j} + \frac{\partial D}{\partial \dot{x}_j} = 0, \quad j = 1,2$$

the equations of motion

$$\begin{aligned} m_1\ddot{x}_1 + c_1\dot{x}_1 - c_2(\dot{x}_2-\dot{x}_1) + k_1x_1 - k_2(x_2-x_1) &= 0, \\ m_2\ddot{x}_2 + c_2(\dot{x}_2-\dot{x}_1) + k_2(x_2-x_1) &= 0. \end{aligned} \tag{2.8}$$

EXAMPLE 2.5. Coupled pendulums with spring and damper. Two pendulums are connected with each other by a spring of stiffness k and a damper of damping constant c (see Fig. 2.6). Derive the equations of small vibration for this system.

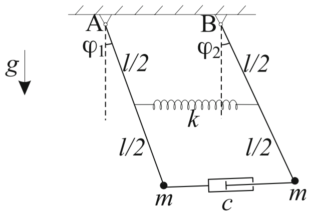

Fig. 2.6 Coupled damped pendulums

Similar to example 2.2 the Lagrange function is given by

$$L(\varphi,\dot{\varphi}) = \frac{1}{2}ml^2\dot{\varphi}_1^2 + \frac{1}{2}ml^2\dot{\varphi}_2^2 - \frac{1}{2}mgl\varphi_1^2 - \frac{1}{2}mgl\varphi_2^2 - \frac{1}{2}k(l(\varphi_2-\varphi_1)/2)^2. \quad (2.9)$$

The dissipation function reads

$$D(\dot{\varphi}) = \frac{1}{2}cl^2(\dot{\varphi}_2 - \dot{\varphi}_1)^2. \quad (2.10)$$

From modified Lagrange's equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}_j} - \frac{\partial L}{\partial \varphi_j} + \frac{\partial D}{\partial \dot{\varphi}_j} = 0, \quad j = 1,2,$$

we derive the equations of motion

$$\begin{aligned} ml^2\ddot{\varphi}_1 - cl^2(\dot{\varphi}_2-\dot{\varphi}_1) + mgl\varphi_1 - k\frac{l^2}{4}(\varphi_2-\varphi_1) &= 0,\\ ml^2\ddot{\varphi}_2 + cl^2(\dot{\varphi}_2-\dot{\varphi}_1) + mgl\varphi_2 + k\frac{l^2}{4}(\varphi_2-\varphi_1) &= 0. \end{aligned} \quad (2.11)$$

EXAMPLE 2.6. Damped vehicle. A rigid bar, connected with two springs of stiffnesses k_1 and k_2 and a damper with the damping force acting in the center of mass S, performs a translational motion of S in the vertical direction and a rotation in the plane about S (see Fig. 2.7). Derive the equations of motion for this damped vehicle.

Let $q = (x,\varphi)$ and $\dot{q} = (\dot{x},\dot{\varphi})$. We write down the Lagrange function as in example 2.3

$$L(q,\dot{q}) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}J_S\dot{\varphi}^2 - \frac{1}{2}k_1(x+l_1\varphi)^2 - \frac{1}{2}k_2(x-l_2\varphi)^2.$$

Furthermore, the dissipation function reads

$$D(\dot{q}) = \frac{1}{2}c\dot{x}^2.$$

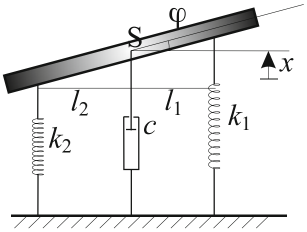

Fig. 2.7 Damped vehicle

Now, from modified Lagrange's equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} + \frac{\partial D}{\partial \dot{q}_j} = 0, \quad j = 1, 2,$$

we derive the equations of motion

$$\begin{aligned} &m\ddot{x} + c\dot{x} + k_1(x + l_1\varphi) + k_2(x - l_2\varphi) = 0, \\ &J_S\ddot{\varphi} + k_1 l_1(x + l_1\varphi) - k_2 l_2(x - l_2\varphi) = 0. \end{aligned} \tag{2.12}$$

Classification of Damping. Let $\mathbf{q}$ be the column vector whose components are the generalized coordinates. In our examples 2.4, 2.5, and 2.6 it is

$$\mathbf{q} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \mathbf{q} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \quad \mathbf{q} = \begin{pmatrix} x \\ \varphi \end{pmatrix},$$

respectively. The equations of motion derived above can be written in the matrix form as follows

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0}, \tag{2.13}$$

where the matrices $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ are called mass, damping, and stiffness matrices, respectively. For instance, in example 2.6 we have

$$\mathbf{M} = \begin{pmatrix} m & 0 \\ 0 & J_S \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{K} = \begin{pmatrix} k_1 + k_2 & k_1 l_1 - k_2 l_2 \\ k_1 l_1 - k_2 l_2 & k_1 l_1^2 + k_2 l_2^2 \end{pmatrix}.$$

In general, all three matrices $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ are symmetric. The symmetry of $\mathbf{C}$ follows from the formula for the damping forces

$$Q_j = -\frac{\partial D}{\partial \dot{q}_j},$$

and from the fact that D is quadratic with respect to $\dot{\mathbf{q}}$. In thermodynamics of irreversible processes this symmetry property is the consequence of Onsager's principle. The mass matrix $\mathbf{M}$ is always positive definite in the sense that there exists a positive constant m such that the inequality

$$\mathbf{q}\cdot\mathbf{Mq} \geq m\mathbf{q}\cdot\mathbf{q}$$

holds true for arbitrary $\mathbf{q}$. If the system does not permit rigid-body motions, then the stiffness matrix $\mathbf{K}$ is also positive definite. Concerning the damping matrix $\mathbf{C}$ we may merely assume in general its non-negative definiteness in the sense that

$$\mathbf{q}\cdot\mathbf{Cq} \geq 0$$

for arbitrary $\mathbf{q}$. Note, however, that in reality, if the resistance to motion through the viscous damping of the air or through the internal damping affecting all degrees of freedom is taken into account, then $\mathbf{C}$ must also be positive definite.

We call the damping exhaustive if the damping matrix $\mathbf{C}$ is positive definite. In this case all motions decay exponentially. If there exists some $\mathbf{q}$ such that $\mathbf{q}\cdot\mathbf{Cq} = 0$, but nevertheless all motions of the system decay exponentially, the damping is called permeating. If there exists some vibration mode which does not decay with time, the damping is called non-permeating. The damping is called proportional if

$$\mathbf{C} = \alpha\mathbf{M} + \beta\mathbf{K}. \tag{2.14}$$

According to this classification the damping in example 2.4 is exhaustive, and if $c_1 = \beta k_1$, $c_2 = \beta k_2$, then it is proportional. In example 2.5 the damping is obviously proportional, but non-exhaustive and non-permeating: the dissipation vanishes for $\varphi_1 = \varphi_2$, and this mode of vibration does not decay with time. In example 2.6 the damping is non-exhaustive but permeating as long as the coupling factor $k_1 l_1 - k_2 l_2$ is not equal to zero. Indeed, if $x(t)$ decays exponentially with time, then it follows from $(2.12)_1$ that $\varphi(t)$ should also decay exponentially if $k_1 l_1 - k_2 l_2$ is not equal to zero.

Solution. We analyze two cases.

Proportional damping. In this case we may choose the normal coordinates which diagonalize all three matrices $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ simultaneously and by this reduce the system to two independent damped oscillators. We illustrate this on example 2.5. Dividing the Lagrange function and the dissipation function by ml^2 and choosing the normal coordinates

$$\xi_1 = \frac{1}{\sqrt{2}}(\varphi_1 + \varphi_2), \quad \xi_2 = \frac{1}{\sqrt{2}}(\varphi_2 - \varphi_1),$$

we obtain

$$L(\xi, \dot{\xi}) = \frac{1}{2}(\dot{\xi}_1^2 + \dot{\xi}_2^2) - \frac{1}{2}(\omega_1^2 \xi_1^2 + \omega_2^2 \xi_2^2), \tag{2.15}$$

and

$$D(\dot{\xi}) = \frac{c}{m}\dot{\xi}_2^2. \tag{2.16}$$

In terms of the new coordinates modified Lagrange's equations become

$$\ddot{\xi}_1 + \omega_1^2 \xi_1 = 0,$$
$$\ddot{\xi}_2 + \frac{2c}{m}\dot{\xi}_2 + \omega_2^2 \xi_2 = 0.$$

Thus, we see that the motions $\xi_1(t)$ and $\xi_2(t)$ are independent, and the motion $\xi_1(t)$ is harmonic confirming that the damping in this example is non-permeating. The obtained uncoupled equations can be solved by the method discussed in Section 1.2.

Non-proportional damping. We illustrate the method of solution on example 2.6. Dividing the equations of motion (2.12) by m and J_s, respectively, and introducing the notations

$$\frac{k_1 + k_2}{m} = \omega_x^2, \quad \frac{k_1 l_1^2 + k_2 l_2^2}{J_S} = \omega_\varphi^2, \quad \frac{c}{m} = \chi,$$
$$\frac{k_1 l_1 - k_2 l_2}{m} = \alpha_1^2, \quad \frac{k_1 l_1 - k_2 l_2}{J_S} = \alpha_2^2, \quad \alpha_1^2 \alpha_2^2 = \alpha^4,$$

with ω_x and ω_φ being the frequencies of uncoupled vibrations and α the coupling factor, we transform (2.12) to

$$\begin{aligned} \ddot{x} + \chi \dot{x} + \omega_x^2 x + \alpha_1^2 \varphi = 0, \\ \ddot{\varphi} + \omega_\varphi^2 \varphi + \alpha_2^2 x = 0. \end{aligned} \tag{2.17}$$

We seek a particular solution of (2.17) in the form

$$x = \hat{x} e^{st}, \quad \varphi = \hat{\varphi} e^{st}.$$

Substituting this Ansatz into (2.17) and eliminating the factor e^{st} we obtain the linear equations

$$\begin{pmatrix} s^2 + \chi s + \omega_x^2 & \alpha_1^2 \\ \alpha_2^2 & s^2 + \omega_\varphi^2 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{\varphi} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{2.18}$$

Non-trivial solutions of this system exist if the determinant vanishes

$$\begin{vmatrix} s^2 + \chi s + \omega_x^2 & \alpha_1^2 \\ \alpha_2^2 & s^2 + \omega_\varphi^2 \end{vmatrix} = 0.$$

This yields the characteristic equation

$$s^4 + \chi s^3 + (\omega_x^2 + \omega_\varphi^2) s^2 + \chi \omega_\varphi^2 s + \omega_x^2 \omega_\varphi^2 - \alpha^4 = 0, \tag{2.19}$$

which is the algebraic equation of fourth order with respect to s.

Since (2.19) is the equation with real coefficients, the complex roots occur in pairs of complex conjugates. We want first to show that all roots have negative real parts. According to the Routh-Hurwitz criterion [19] this is the case if

$$T_0 = a_0 > 0, \quad T_1 = a_1 > 0, \quad T_2 = \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix} > 0,$$

$$T_3 = \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ 0 & a_4 & a_3 \end{vmatrix} > 0, \quad T_4 = \begin{vmatrix} a_1 & a_0 & 0 & 0 \\ a_3 & a_2 & a_1 & a_0 \\ 0 & a_4 & a_3 & a_2 \\ 0 & 0 & 0 & a_4 \end{vmatrix} = a_4 T_3 > 0,$$

where

$$a_0 = 1, \quad a_1 = \chi, \quad a_2 = \omega_x^2 + \omega_\varphi^2, \quad a_3 = \chi \omega_\varphi^2, \quad a_4 = \omega_x^2 \omega_\varphi^2 - \alpha^4$$

are the coefficients of the characteristic equation. Elementary calculations give

$$T_0 = 1 > 0, \quad T_1 = \chi > 0, \quad T_2 = \chi \omega_x^2 > 0,$$
$$T_3 = \chi^2(\omega_x^2 + \omega_\varphi^2 + \omega_x^2 \omega_\varphi^2 + \alpha^4) > 0, \quad T_4 = (\omega_x^2 \omega_\varphi^2 - \alpha^4) T_3 > 0,$$

so the Routh-Hurwitz criterion is fulfilled.

Although the characteristic equation can be solved in closed analytical form, the analysis of exact solution is rather tedious. We therefore consider the case of small damping $\chi \ll \omega_x$, $\chi \ll \omega_\varphi$ and seek s in the form $s = (-\kappa \pm i)\omega$, where $\kappa \ll 1$. Then to the first approximation

$$s^2 \approx -(1 \pm 2\kappa i)\omega^2, \quad s^3 \approx (3\kappa \mp i)\omega^3, \quad s^4 \approx (1 \pm 4\kappa i)\omega^4.$$

Substituting this into (2.19) and neglecting the powers of χ and κ higher than one, we obtain in the first approximation

$$\omega^4 - (\omega_x^2 + \omega_\varphi^2)\omega^2 + \omega_x^2 \omega_\varphi^2 - \alpha^4$$
$$\pm i[4\kappa\omega^4 - \chi\omega^3 - 2\kappa(\omega_x^2 + \omega_\varphi^2)\omega^2 + \chi\omega\omega_\varphi^2] = 0.$$

This complex expression is zero if its real and imaginary parts vanish. So, we obtain two equations determining the eigenfrequencies $\omega_{1,2}$ and the decay rates $\kappa_{1,2}\omega_{1,2}$. Note that the equation for the eigenfrequencies

$$\omega^4 - (\omega_x^2 + \omega_\varphi^2)\omega^2 + \omega_x^2 \omega_\varphi^2 - \alpha^4 = 0$$

is identical with that of the undamped vehicle in example 2.3. Thus, for small damping the eigenfrequencies remain the same as those of the undamped coupled oscillators which are given by

$$\omega_{1,2}^2 = \frac{1}{2}(\omega_x^2 + \omega_\varphi^2) \mp \sqrt{\frac{1}{4}(\omega_x^2 - \omega_\varphi^2)^2 + \alpha^4}.$$

Fig. 2.8 shows the plots of dimensionless frequencies $(\omega_{1,2}/\omega_x)^2$ versus the ratio of frequencies $(\omega_\varphi/\omega_x)^2$ at different coupling ratios $(\alpha/\omega_x)^2$. It can be seen that for the zero coupling $\alpha = 0$ the eigenfrequencies coincide with those of uncoupled

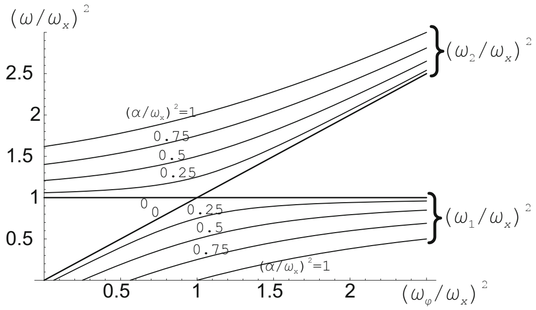

Fig. 2.8 Eigenfrequencies $(\omega_{1,2}/\omega_x)^2$ vs. ratio of uncoupled frequencies $(\omega_\varphi/\omega_x)^2$ at different coupling ratios $(\alpha/\omega_x)^2$

oscillators ω_φ and ω_x. The larger the coupling factor, the farther the eigenfrequencies lie apart. The frequency ω_2 is always larger than the largest from ω_φ and ω_x, while ω_1 is smaller than the smallest from them.

The decay rates $\kappa_{1,2}\omega_{1,2}$ should be determined from the equation

$$4\kappa\omega^4 - \chi\omega^3 - 2\kappa(\omega_x^2 + \omega_\varphi^2)\omega^2 + \chi\omega\omega_\varphi^2 = 0$$

giving

$$\kappa_1\omega_1 = \frac{\chi(\omega_1^2 - \omega_\varphi^2)}{4\omega_1^2 - 2(\omega_x^2 + \omega_\varphi^2)}, \quad \kappa_2\omega_2 = \frac{\chi(\omega_2^2 - \omega_\varphi^2)}{4\omega_2^2 - 2(\omega_x^2 + \omega_\varphi^2)}.$$

Thus, the decay rates are positive and are of the same order as χ.

By substituting s found above into (2.18) we may establish the relations between the amplitudes of vibrations. For $s = (-\kappa_i \pm i)\omega_i$ we have

$$\hat{\mathbf{q}} = \begin{pmatrix} \hat{x} \\ \hat{\varphi} \end{pmatrix} = C \begin{pmatrix} (1 \pm 2i\kappa_i)\omega_i^2 - \omega_\varphi^2 \\ \alpha_2^2 \end{pmatrix}, \quad i = 1, 2.$$

Denoting by $\mathbf{q}_1$ and $\mathbf{q}_2$ the complex-valued vectors

$$\mathbf{q}_1 = \begin{pmatrix} (1 + 2i\kappa_1)\omega_1^2 - \omega_\varphi^2 \\ \alpha_2^2 \end{pmatrix}, \quad \mathbf{q}_2 = \begin{pmatrix} (1 + 2i\kappa_2)\omega_2^2 - \omega_\varphi^2 \\ \alpha_2^2 \end{pmatrix},$$

we may present the general solution of (2.17) in the form

$$\mathbf{q} = e^{-\kappa_1\omega_1 t}(A_1\mathbf{q}_1 e^{i\omega_1 t} + B_1\mathbf{q}_1^* e^{-i\omega_1 t}) + e^{-\kappa_2\omega_2 t}(A_2\mathbf{q}_2 e^{i\omega_2 t} + B_2\mathbf{q}_2^* e^{-i\omega_2 t}),$$

where asterisks denote complex conjugates. The four unknown real constants A_1, B_1, A_2, and B_2 must be determined from the initial conditions.

2.3 Forced Oscillators and Vibration Control

Differential Equations of Motion. We illustrate the derivation of the equations of forced vibrations for systems with two degrees of freedom.

EXAMPLE 2.7. Mass-spring forced oscillators. The mass-spring oscillators with two degrees of freedom are excited by the motion of the end-point $x_e(t)$. Derive the equations of motion for these forced oscillators.

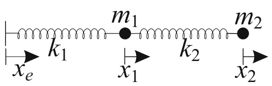

Fig. 2.9 Mass-spring forced oscillators

Since the change in length of the first spring is $x_1 - x_e$, we write for the Lagrange function

$$L(x,\dot{x}) = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 - \frac{1}{2}k_1(x_1 - x_e)^2 - \frac{1}{2}k_2(x_2 - x_1)^2.$$

This Lagrange function differs from that of example 2.1 only by the third term corresponding to the energy of the first spring. From Lagrange's equations we derive

$$m_1\ddot{x}_1 + k_1(x_1 - x_e) - k_2(x_2 - x_1) = 0,$$
$$m_2\ddot{x}_2 + k_2(x_2 - x_1) = 0.$$

Bringing the term $-k_1 x_e$ to the right-hand side we obtain

$$\begin{aligned} m_1\ddot{x}_1 + k_1x_1 - k_2(x_2 - x_1) &= k_1x_e(t), \\ m_2\ddot{x}_2 + k_2(x_2 - x_1) &= 0. \end{aligned} \tag{2.20}$$

EXAMPLE 2.8. Mass-spring-damper forced oscillators. The mass-spring-damper oscillators with two degrees of freedom are excited by the force $f(t)$ acting on the mass m_1 (see Fig. 2.8). Derive the equations of motion for these forced oscillators.

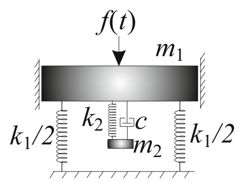

Fig. 2.10 Mass-spring-damper forced oscillators

We denote by x_1 and x_2 the displacements of m_1 and m_2 in the vertical direction from their equilibrium positions, respectively. Then the Lagrange function equals

$$L(x,\dot{x}) = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 - \frac{1}{2}k_1x_1^2 - \frac{1}{2}k_2(x_2-x_1)^2,$$

while the dissipation function is

$$D(\dot{x}) = \frac{1}{2}c(\dot{x}_2-\dot{x}_1)^2.$$

From modified Lagrange's equations for the forced vibrations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_j} - \frac{\partial L}{\partial x_j} + \frac{\partial D}{\partial \dot{x}_j} = f_j(t), \quad j=1,2,$$

with $f_j(t)$ being the external forces acting on the masses m_j, we derive

$$\begin{aligned} m_1\ddot{x}_1 - c(\dot{x}_2-\dot{x}_1) + k_1x_1 - k_2(x_2-x_1) &= f(t), \\ m_2\ddot{x}_2 + c(\dot{x}_2-\dot{x}_1) + k_2(x_2-x_1) &= 0. \end{aligned} \tag{2.21}$$

EXAMPLE 2.9. Coupled forced pendulums. The coupled pendulums as in example 2.5 are excited by a force $p(t)$ acting on the second mass (see Fig. 2.11). Derive the equations of motion for these coupled forced pendulums.

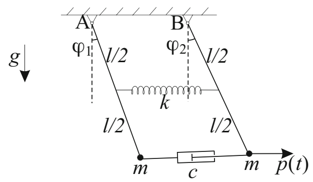

Fig. 2.11 Coupled forced pendulums

Similar to example 2.5 the Lagrange function is given by (2.9), while the dissipation function by (2.10). The virtual work done by the external force $p(t)$ is

$$\delta A = \int_{t_0}^{t_1} p(t)l\delta\varphi_2\,dt.$$

From modified Lagrange's equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}_j} - \frac{\partial L}{\partial \varphi_j} + \frac{\partial D}{\partial \dot{\varphi}_j} = f_j(t), \quad j=1,2,$$

we derive the equations of motion

$$\begin{aligned} ml^2\ddot{\varphi}_1 - cl^2(\dot{\varphi}_2 - \dot{\varphi}_1) + mgl\varphi_1 - k\frac{l^2}{4}(\varphi_2 - \varphi_1) &= 0, \\ ml^2\ddot{\varphi}_2 + cl^2(\dot{\varphi}_2 - \dot{\varphi}_1) + mgl\varphi_2 + k\frac{l^2}{4}(\varphi_2 - \varphi_1) &= p(t)l. \end{aligned} \tag{2.22}$$

Harmonic Excitations. Equations of motion derived above are the inhomogeneous linear differential equations of second order. The solution of these linear equations is the sum of any particular solution of the inhomogeneous equations and the general solution of the homogeneous equations which has been found in previous Section. Thus, it is enough to find any particular solution of the inhomogeneous equations. For the harmonic excitations this can be done directly. We consider two cases.

Conservative oscillators. We illustrate the method of solution on example 2.7, where the excitation is assumed in the form: $x_e(t) = \hat{x}_e \cos(\omega t)$. Dividing the first and the second equations of (2.20) by m_1 and m_2, respectively, we rewrite them in the form

$$\begin{aligned} \ddot{x}_1 + \nu_1^2 x_1 - \mu\nu_2^2 x_2 &= \nu_{10}^2 \hat{x}_e \cos(\omega t), \\ \ddot{x}_2 + \nu_2^2 x_2 - \nu_2^2 x_1 &= 0, \end{aligned}$$

where

$$\nu_1^2 = \frac{k_1 + k_2}{m_1}, \quad \nu_2^2 = \frac{k_2}{m_2}, \quad \mu = \frac{m_2}{m_1}, \quad \nu_{10}^2 = \frac{k_1}{m_1}.$$

Since the first derivatives $\dot{x}_1$ and $\dot{x}_2$ do not enter the equations of motion, we seek a particular solution of these inhomogeneous differential equations in the form

$$x_1 = \hat{x}_1 \cos\omega t, \quad x_2 = \hat{x}_2 \cos\omega t.$$

Substituting this Ansatz into the above equations and eliminating the common factor $\cos\omega t$ on both sides, we obtain

$$\begin{aligned} (\nu_1^2 - \omega^2)\hat{x}_1 - \mu\nu_2^2\hat{x}_2 &= \nu_{10}^2\hat{x}_e, \\ -\nu_2^2\hat{x}_1 + (\nu_2^2 - \omega^2)\hat{x}_2 &= 0. \end{aligned} \tag{2.23}$$

Thus, the amplitudes of forced vibration are given by

$$\begin{aligned} \hat{x}_1 &= \frac{\nu_{10}^2(\nu_2^2 - \omega^2)\hat{x}_e}{(\nu_1^2 - \omega^2)(\nu_2^2 - \omega^2) - \mu\nu_2^4}, \\ \hat{x}_2 &= \frac{\nu_{10}^2\nu_2^2\hat{x}_e}{(\nu_1^2 - \omega^2)(\nu_2^2 - \omega^2) - \mu\nu_2^4}. \end{aligned}$$

The behavior of the amplitudes, as functions of the frequency ω, is characterized by the zeros of the denominator and the numerator. The denominator vanishes for

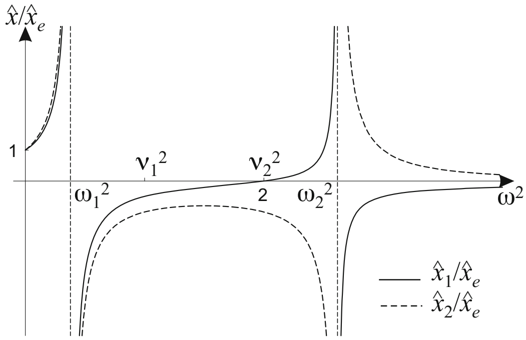

Fig. 2.12 Resonance curves of mass-spring forced oscillators

$$\omega_{1,2}^2 = \frac{1}{2}(\nu_1^2 + \nu_2^2) \mp \sqrt{\frac{1}{4}(\nu_1^2 - \nu_2^2)^2 + \mu \nu_2^4},$$

which correspond to the eigenfrequencies of free vibration of this system. We see that the eigenfrequencies ω_1 and ω_2 always lie outside the frequency range (ν_1, ν_2). The plot of resonance functions $\hat{x}_i/\hat{x}_e$ versus ω^2 is shown in Fig. 2.12. These resonance functions tend to infinity as ω approaches one of the frequencies ω_1 and ω_2, corresponding to the resonances, and to zero as $\omega \to \infty$. While $\hat{x}_2/\hat{x}_e \neq 0$ for all frequencies, the amplitude $\hat{x}_1$ vanishes at $\omega = \nu_2$. This phenomenon is called anti-resonance (or vibration elimination) and the mass m_2 together with the spring k_2 a vibration eliminator. The elimination of forced vibration can be explained physically as follows. At the frequency $\omega = \nu_2$ the eliminator and the excitation vibrate in counter-phases such that the spring force acting on m_1 from the eliminator is equal and opposite to the exciting force $k_1 \hat{x}_e \cos \omega t$. Indeed, equations (2.23) at $\omega = \nu_2$ yield

$$\hat{x}_2 = -\frac{\nu_{10}^2 \hat{x}_e}{\mu \nu_2^2} = -\frac{k_1}{k_2} \hat{x}_e.$$

Thus, the resultant force acting on m_1 is zero and therefore that mass does not vibrate. To eliminate the unwanted forced vibration of m_1 caused by some excitation source with the fixed frequency ω we must therefore choose the mass and the spring of the eliminator in such a relation that $\sqrt{k_2/m_2} = \omega$. However, if the excitation source has a wider range of frequencies, this choice is no longer effective because, as it is seen from Fig. 2.12, the resonance function $\hat{x}_1/\hat{x}_e$ increases rapidly as ω deviates from ν_2.

Damped oscillators. We see from the previous example that the elimination of forced vibration for the conservative oscillators is effective only if the excitation source has a constant frequency. In the case of non-zero damping the situation changes. We illustrate the method of solution on example 2.8, where the external force is assumed in the form $f(t) = \hat{f} \cos(\omega t)$. We rewrite equations (2.21) in the matrix form

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \hat{\mathbf{f}}\cos(\omega t), \tag{2.24}$$

where

$$\mathbf{M} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} c & -c \\ -c & c \end{pmatrix}, \mathbf{K} = \begin{pmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix}, \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \hat{\mathbf{f}} = \begin{pmatrix} \hat{f} \\ 0 \end{pmatrix}.$$

Now, the Ansatz $\mathbf{x} = \hat{\mathbf{x}}\cos(\omega t)$ with real $\hat{\mathbf{x}}$ does not work because the first derivative in (2.24) brings terms with the factor $\sin(\omega t)$. However, we may do the following "trick" to get the solution quickly. We regard the right-hand side of (2.24) as $\hat{\mathbf{f}}\cos(\omega t) = \mathrm{Re}(\hat{\mathbf{f}}e^{i\omega t})$ and consider instead the following auxiliary equation

$$\mathbf{M}\ddot{\mathbf{z}} + \mathbf{C}\dot{\mathbf{z}} + \mathbf{K}\mathbf{z} = \hat{\mathbf{f}}e^{i\omega t}. \tag{2.25}$$

Now $\mathbf{z}$ may be complex-valued. Then we substitute the Ansatz $\mathbf{z} = \hat{\mathbf{z}}e^{i\omega t}$ into this equation and eliminate the common factor $e^{i\omega t}$ to obtain

$$(-\omega^2\mathbf{M} + i\omega\mathbf{C} + \mathbf{K})\hat{\mathbf{z}} = \hat{\mathbf{f}}.$$

Provided the matrix on the left-hand side has an inverse, this equation yields

$$\hat{\mathbf{z}} = (-\omega^2\mathbf{M} + i\omega\mathbf{C} + \mathbf{K})^{-1}\hat{\mathbf{f}} = \mathbf{G}(\omega)\hat{\mathbf{f}}.$$

Matrix $\mathbf{G}(\omega)$ is called a transmittance matrix of the system. Since $\hat{\mathbf{z}}e^{i\omega t}$ is the solution of (2.25) which is the equation with real matrices, its real part must satisfy equation (2.24). So, the trick works!

Thus, the particular solution of (2.24) is

$$\mathbf{x}(t) = \mathrm{Re}(\hat{\mathbf{z}}e^{i\omega t}),$$

or, in components,

$$x_j(t) = \mathrm{Re}(\hat{z}_j e^{i\omega t}), \quad j = 1,2.$$

Since each complex number z can be presented as $z = |z|e^{-i\phi}$, we obtain from here

$$x_j = |\hat{z}_j|\cos(\omega t - \phi_j), \quad j = 1,2.$$

With the matrices given above we may calculate the amplitude of x_1

$$|x_1| = |\hat{z}_1| = \left| \frac{\hat{f}(-m_2\omega^2 + ic\omega + k_2)}{(-m_1\omega^2 + ic\omega + k_1 + k_2)(-m_2\omega^2 + ic\omega + k_2) - (ic\omega + k_2)^2} \right|.$$

Dividing both the numerator and the denominator by k_1^2 and introducing

$$x_{10} = \frac{\hat{f}}{k_1}, \quad \omega_0 = \sqrt{\frac{k_1}{m_1}}, \quad \kappa = \frac{k_2}{k_1}, \quad \mu = \frac{m_2}{m_1}, \quad \eta = \frac{\omega}{\omega_0}, \quad \delta = \frac{c}{m_1\omega_0},$$

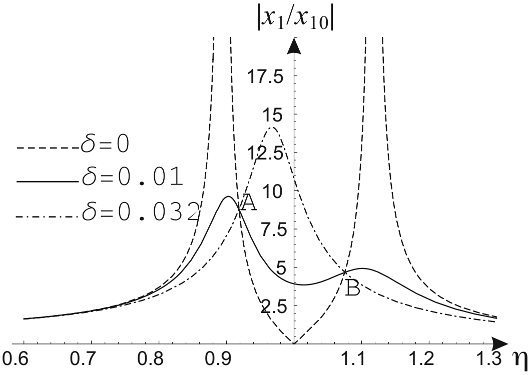

Fig. 2.13 Resonance curves of mass-spring-damper forced oscillators

we present the previous equation in the dimensionless form as follows

$$\left|\frac{x_1}{x_{10}}\right| = \left|\frac{-\mu\eta^2 + i\delta\eta + \kappa}{(-\eta^2 + i\delta\eta + 1 + \kappa)(-\mu\eta^2 + i\delta\eta + \kappa) - (i\delta\eta + \kappa)^2}\right|,$$

where the right-hand side is called a resonance function.

Fig. 2.13 shows the resonance curves $|x_1/x_{10}|$ against the frequency ratio $\eta = \omega/\omega_0$ for $\mu = \kappa = 0.05$ and for three damping ratios $\delta = 0$ (dashed line), $\delta = 0.01$ (bold line), $\delta = 0.032$ (dotted line) [48]. In contrast to the conservative oscillators ($\delta = 0$), neither resonance nor vibration elimination is observed for the damped forced oscillators with the finite damping. We call therefore the mass m_2 together with the spring k_2 and the damper c a vibration absorber. It turns out that all resonance curves corresponding to different damping ratios intersect at the two fixed points A and B (see exercise 2.8). If we want to reduce the maxima of the resonance curve in equal way, then an optimal choice of the parameters of absorber is achieved when points A and B are at equal level. This takes place when

$$\kappa = \frac{\mu}{(1+\mu)^2}.$$

Arbitrary Excitations. We illustrate the method of solution on example 2.9 for which the proportional damping holds true. The more general non-proportional damping case will be considered in Section 2.5. The coupled forced oscillators with the proportional damping can always be reduced to the uncoupled single forced oscillators. Indeed, in this example we choose the normal coordinates as

$$\xi_1 = \frac{1}{\sqrt{2}}(\varphi_1 + \varphi_2), \quad \xi_2 = \frac{1}{\sqrt{2}}(\varphi_2 - \varphi_1),$$

and present the normalized virtual work in the form

$$\frac{1}{ml^2}\delta A = \int_{t_0}^{t_1} \frac{p(t)}{ml}\delta\varphi_2\,dt = \int_{t_0}^{t_1} \frac{p(t)}{ml}\frac{1}{\sqrt{2}}(\delta\xi_1+\delta\xi_2)\,dt.$$

Together with the Lagrange function (2.15) and the dissipation function (2.16) we derive modified Lagrange's equations

$$\ddot{\xi}_1 + \omega_1^2\xi_1 = \frac{1}{\sqrt{2}}\frac{p(t)}{ml},$$
$$\ddot{\xi}_2 + \frac{2c}{m}\dot{\xi}_2 + \omega_2^2\xi_2 = \frac{1}{\sqrt{2}}\frac{p(t)}{ml},$$

which can be solved by the Laplace transform as shown in Section 1.3.

2.4 Variational Principles

We present in this Section the variational principles for general systems having n degrees of freedom [29]. For small vibrations about equilibrium states the energy and dissipation become quadratic with respect to the generalized coordinates and velocities, so that generalized Lagrange's equations become linear.

Conservative Systems. Suppose that each configuration of a mechanical system is uniquely determined by a point $q=(q_1,\ldots,q_n)$ in an n-dimensional space. If $q_1,\ldots,q_n$ can vary independently and arbitrarily, they are called generalized coordinates, and n a number of degrees of freedom. Motion of the system is described by a function $q(t)$. We denote by $\dot{q}=(\dot{q}_1,\ldots,\dot{q}_n)$ the corresponding generalized velocities. Hamilton's variational principle states that among all admissible motions of the conservative system satisfying the initial and end conditions

$$q(t_0)=\hat{q}_0, \quad q(t_1)=\hat{q}_1,$$

the true motion is the extremal of the action functional

$$I[q(t)] = \int_{t_0}^{t_1} L(q,\dot{q})\,dt.$$

Let us derive the equations of motion from Hamilton's variational principle. To this end we calculate the variation of the action functional (see also [17])

$$\delta I = \int_{t_0}^{t_1}\sum_{j=1}^{n}\left(\frac{\partial L}{\partial q_j}\delta q_j + \frac{\partial L}{\partial \dot{q}_j}\delta\dot{q}_j\right)dt.$$

Integrating the second term by parts and taking into account that $\delta q_j(t_0)=\delta q_j(t_1)=0$ due to the initial and end conditions, we get

$$\delta I = \int_{t_0}^{t_1} \sum_{j=1}^{n} \left(\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} \right) \delta q_j \, dt = 0. \tag{2.26}$$

Since the variations δq_j can be chosen independently and arbitrarily inside the interval (t_0, t_1), (2.26) implies Lagrange's equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0, \quad j = 1, \ldots, n. \tag{2.27}$$

For any conservative mechanical system the Lagrange function equals

$$L(q, \dot{q}) = K(q, \dot{q}) - U(q),$$

where $K(q, \dot{q})$ is the kinetic energy and $U(q)$ the potential energy. The kinetic energy $K(q, \dot{q})$ is assumed to be a positive definite quadratic form[3] with respect to $\dot{q}$

$$K(q, \dot{q}) = \frac{1}{2} \sum_{j,k=1}^{n} m_{jk}(q) \dot{q}_j \dot{q}_k.$$

Thus,

$$\sum_{j=1}^{n} \frac{\partial K}{\partial \dot{q}_j} \dot{q}_j = 2K(q, \dot{q}).$$

Any function possessing this property is called homogeneous function of order two with respect to $\dot{q}$. We want to show now that the conservation of energy follows from Lagrange's equations (2.27). Indeed, multiplying (2.27) by $\dot{q}_j$ and summing up over j from 1 to n, we obtain

$$\sum_{j=1}^{n} \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} \dot{q}_j - \frac{\partial L}{\partial q_j} \dot{q}_j \right) = 0.$$

Using the product and chain rules of differentiation, we get

$$\sum_{j=1}^{n} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \dot{q}_j \right) - \sum_{j=1}^{n} \left(\frac{\partial L}{\partial \dot{q}_j} \ddot{q}_j + \frac{\partial L}{\partial q_j} \dot{q}_j \right) = \frac{d}{dt} (\sum_{j=1}^{n} \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j - L) = 0. \tag{2.28}$$

Taking into account the property of K, we see that the expression in parentheses is equal to $2K - L = K + U$. Thus, the total energy $E = K + U = E_0$ is conserved. Alternatively, the conservation of energy can also be obtained directly from (2.26) by replacing the variations δq_j with the real velocities $\dot{q}_j$. Indeed, the same procedure transforms (2.26) to

$$\int_{t_0}^{t_1} \frac{d}{dt}(K + U) \, dt = 0 \quad \Rightarrow \quad K + U = E_0.$$

[3] In some cases rearrangement of terms between the kinetic and potential energies is required to achieve this property (see exercise 5.1).

Assume that $U(q)$ has a local minimum at some point q_0 corresponding to a stable equilibrium state and consider small vibrations of our mechanical system about this stable equilibrium state. For small q we may expand $U(q)$ and $K(q,\dot{q})$ in Taylor's series with respect to q near q_0 to get

$$U(q) = U(q_0) + \frac{1}{2}\sum_{j,k=1}^{n} \left.\frac{\partial^2 U}{\partial q_j \partial q_k}\right|_{q_0} q_j q_k + \dots,$$

$$K(q,\dot{q}) = K(q_0,\dot{q}) + \dots = \frac{1}{2}\sum_{j,k=1}^{n} m_{jk}(q_0)\dot{q}_j\dot{q}_k + \dots.$$

Due to the smallness of q_j and $\dot{q}_j$, we keep only the quadratic terms in these series. Thus, neglecting the unessential constant $U(q_0)$ in the potential energy, we may present both kinetic and potential energies as follows

$$K(\dot{q}) = \frac{1}{2}\sum_{j,k=1}^{n} m_{jk}\dot{q}_j\dot{q}_k, \quad U(q) = \frac{1}{2}\sum_{j,k=1}^{n} k_{jk}q_jq_k. \tag{2.29}$$

Thus, for small vibrations near the stable equilibrium state the kinetic energy $K(\dot{q})$ and the potential energy $U(q)$ are the quadratic forms with respect to $\dot{q}$ and q, respectively. We call the matrix $\mathbf{M}$ with the elements m_{jk} mass matrix, while $\mathbf{K}$, with the elements k_{jk}, stiffness matrix. Both matrices are symmetric and positive definite. The positive definiteness of $\mathbf{K}$ is due to the fact that $U(q)$ has a local minimum at q_0. Lagrange's equations of small vibrations near the equilibrium state become linear equations

$$\sum_{k=1}^{n}(m_{jk}\ddot{q}_k + k_{jk}q_k) = 0, \quad j = 1,\dots,n.$$

Let $\mathbf{q}$ be the column vector $\mathbf{q} = (q_1,\dots,q_n)^T$. We may present these equations also in the matrix form as follows

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0}.$$

Dissipative Systems. In this case the following variational principle holds true: among all admissible motions of a dissipative system constrained by the initial and end conditions

$$q(t_0) = \hat{q}_0, \quad q(t_1) = \hat{q}_1,$$

the true motion satisfies the variational equation[4]

$$\delta\int_{t_0}^{t_1} L(q,\dot{q})\,dt - \int_{t_0}^{t_1}\sum_{j=1}^{n}\frac{\partial D}{\partial \dot{q}_j}\delta q_j\,dt = 0. \tag{2.30}$$

[4] See page 10, *loc. cit.*

Here $D(q,\dot{q})$ is the dissipation function introduced first by Rayleigh [45]. Calculating the variation of the first term of (2.30) in exactly the same manner as in the previous case leads to

$$\int_{t_0}^{t_1} \sum_{j=1}^{n} \left(\frac{\partial L}{\partial q_j} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} \right) \delta q_j \, dt - \int_{t_0}^{t_1} \sum_{j=1}^{n} \frac{\partial D}{\partial \dot{q}_j} \delta q_j \, dt = 0.$$

Due to the arbitrariness of δq_j inside the time interval (t_0, t_1) the following equations are obtained

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} + \frac{\partial D}{\partial \dot{q}_j} = 0, \quad j = 1, \ldots, n. \tag{2.31}$$

For dissipative systems vibrating near the equilibrium states the dissipation function can be assumed as a non-negative definite quadratic form with respect to $\dot{q}$

$$D(q,\dot{q}) = \frac{1}{2} \sum_{j,k=1}^{n} c_{jk}(q) \dot{q}_j \dot{q}_k \geq 0 \quad \text{for all } \dot{q},$$

where c_{jk} is a symmetric matrix (Onsager's principle [38]). In this case $D(q,\dot{q})$ is also the homogeneous function of order two with respect to $\dot{q}$. We now derive the balance equation of energy from modified Lagrange's equations (2.31). Multiplying (2.31) by $\dot{q}_j$ and summing up over j from 1 to n, we obtain

$$\sum_{j=1}^{n} \left(\frac{d}{dt} \frac{\partial K}{\partial \dot{q}_j} \dot{q}_j - \frac{\partial L}{\partial q_j} \dot{q}_j \right) = - \sum_{j=1}^{n} \frac{\partial D}{\partial \dot{q}_j} \dot{q}_j.$$

The expression on the right-hand side is nothing else but the power of the damping forces. Making the same observations as in the previous case and using the property of D we get

$$\frac{d}{dt}(K+U) = -2D(q,\dot{q}).$$

Thus, the rate of change of energy is equal to $-2D(q,\dot{q})$. Since $-2D(q,\dot{q})$ is the energy loss per unit time, we call $2D(q,\dot{q})$ energy dissipation rate. We see that the energy dissipation rate is non-negative.[5] Integrating this equation from t_0 to t, we find the energy change at time t

$$K + U - E_0 = -2 \int_{t_0}^{t} D(q(s), \dot{q}(s)) ds = -E_d(t), \tag{2.32}$$

where E_0 is the total energy at $t = t_0$ and $E_d(t)$ the amount of energy dissipated by the dampers at time t. Note that this balance equation can also be directly obtained from the variational equation (2.30) by replacing the variations δq_j by the real velocities $\dot{q}_j$.

[5] It is interesting to mention that, if the system does not vibrate about the equilibrium states, this property is no longer valid (see Section 5.3).

For small vibrations near the stable equilibrium state q_0 we may, to the first approximation, assume the kinetic and potential energies in the form (2.29). The dissipation function can also be expanded in Taylor's series near this state. Neglecting all small terms of higher orders we write

$$D(q,\dot{q}) = D(q_0,\dot{q}) = \frac{1}{2}\sum_{j,k=1}^{n} c_{jk}(q_0)\dot{q}_j\dot{q}_k,$$

where the matrix **C** with the elements $c_{jk}(q_0)$ is called the damping matrix. Modified Lagrange's equations of small vibrations near the equilibrium state take the form

$$\sum_{k=1}^{n}(m_{jk}\ddot{q}_k + c_{jk}\dot{q}_k + k_{jk}q_k) = 0, \quad j = 1,\ldots,n.$$

We may present these equations also in the matrix form as follows

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0}.$$

Systems with External Forces. If there are external generalized forces $f_j(t)$ acting on q_j, we must add to the left-hand side of variational equation (2.30) the virtual work done by the external forces. The variational principle becomes: among all admissible motions constrained by the initial and end conditions

$$q(t_0) = \hat{q}_0, \quad q(t_1) = \hat{q}_1,$$

the true motion satisfies the variational equation

$$\delta\int_{t_0}^{t_1} L(q,\dot{q})\,dt - \int_{t_0}^{t_1}\sum_{j=1}^{n}\frac{\partial D}{\partial \dot{q}_j}\delta q_j\,dt + \delta A = 0, \tag{2.33}$$

where δA is the virtual work done by the generalized forces $f_j(t)$

$$\delta A = \int_{t_0}^{t_1}\sum_{j=1}^{n} f_j(t)\delta q_j\,dt.$$

We can also take the Lagrange function in the form

$$L(q,\dot{q},t) = K(q,\dot{q}) - U(q) + \sum_{j=1}^{n} f_j(t)q_j,$$

and reformulate the variational equation as follows

$$\delta\int_{t_0}^{t_1} L(q,\dot{q},t)\,dt - \int_{t_0}^{t_1}\sum_{j=1}^{n}\frac{\partial D}{\partial \dot{q}_j}\delta q_j\,dt = 0.$$

Since time enters the Lagrange function explicitly, such systems are called non-autonomous.

From (2.33) one can derive modified Lagrange's equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j}-\frac{\partial L}{\partial q_j}+\frac{\partial D}{\partial \dot{q}_j}=f_j(t), \quad j=1,\dots,n.$$

Replacing in the variational equation (2.33) the variations δq_i by the real velocities $\dot{q}_i$ and repeating the transformations as in the previous paragraph, we obtain the balance of energy in the form

$$K+U-E_0=-2\int_{t_0}^{t}D(q(s),\dot{q}(s))ds+\int_{t_0}^{t}\sum_{j=1}^{n}f_j(s)\dot{q}_j(s)ds=-E_d(t)+W(t), \tag{2.34}$$

where E_0 is the total energy at $t=t_0$. The last term $W(t)$ is the work done by the external forces which is stored in the energy of the system except that part $E_d(t)$ dissipated by the dampers.

For small vibrations near the stable equilibrium state Lagrange's equations can be presented in the matrix form as follows

$$\mathbf{M}\ddot{\mathbf{q}}+\mathbf{C}\dot{\mathbf{q}}+\mathbf{K}\mathbf{q}=\mathbf{f}(t),$$

with $\mathbf{f}(t)=(f_1(t),\dots,f_n(t))^T$ being the column vector of external forces.

2.5 Oscillators with *n* Degrees of Freedom

We present in this Section the method of solution and some general properties for systems with n degrees of freedom, where n is an arbitrary natural number.

Conservative Oscillators. The motion is described by the equation

$$\mathbf{M}\ddot{\mathbf{q}}+\mathbf{K}\mathbf{q}=\mathbf{0}, \tag{2.35}$$

where $\mathbf{M}$ and $\mathbf{K}$ are symmetric and positive definite matrices. We have to find the solution of this equation satisfying the initial conditions

$$\mathbf{q}(0)=\mathbf{q}_0, \quad \dot{\mathbf{q}}(0)=\mathbf{v}_0. \tag{2.36}$$

Solution. Let us first seek a particular solution of (2.35) in the form

$$\mathbf{q}=\hat{\mathbf{q}}e^{st},$$

where $\hat{\mathbf{q}}$ is a constant vector. Substituting this Ansatz into (2.35) and eliminating the non-vanishing factor e^{st}, we reduce the latter to the eigenvalue problem

$$(\mathbf{M}s^2+\mathbf{K})\hat{\mathbf{q}}=\mathbf{0}. \tag{2.37}$$

The related characteristic equation

$$\det(\mathbf{M}s^2 + \mathbf{K}) = 0$$

is the algebraic equation of order n with respect to s^2 yielding n eigenvalues. It is easy to see that all eigenvalues are real and negative. Indeed, if s_j^2 is an eigenvalue and $\mathbf{q}_j$ a corresponding eigenvector, then, multiplying (2.37) by the vector $\mathbf{q}_j$, we have

$$\mathbf{q}_j \cdot \mathbf{M}\mathbf{q}_j s_j^2 + \mathbf{q}_j \cdot \mathbf{K}\mathbf{q}_j = 0.$$

Thus,

$$s_j^2 = -\frac{\mathbf{q}_j \cdot \mathbf{K}\mathbf{q}_j}{\mathbf{q}_j \cdot \mathbf{M}\mathbf{q}_j} < 0, \tag{2.38}$$

since both the numerator and denominator are positive. Therefore the roots of the characteristic equation are imaginary numbers given by

$$s_j = \pm i\omega_j, \quad j = 1, \ldots, n,$$

where ω_j are called eigenfrequencies of vibrations. We will order them in such a way that

$$0 < \omega_1 \le \omega_2 \le \ldots \le \omega_n.$$

Let $\mathbf{q}_j$ be the eigenvector (the solution of (2.37)) corresponding to the j-th eigenvalue. It is defined uniquely up to a constant factor. We can fix this constant by some normalization condition. As such we choose

$$\mathbf{q}_j \cdot \mathbf{M}\mathbf{q}_j = 1.$$

Note that two eigenvectors $\mathbf{q}_j$ and $\mathbf{q}_k$ corresponding to two different eigenvalues s_j^2 and s_k^2 are orthogonal in the sense that

$$\mathbf{q}_j \cdot \mathbf{M}\mathbf{q}_k = 0.$$

To show this we multiply equation (2.37) for $s = s_j$ by $\mathbf{q}_k$ to get

$$\mathbf{q}_k \cdot \mathbf{M}\mathbf{q}_j s_j^2 = -\mathbf{q}_k \cdot \mathbf{K}\mathbf{q}_j. \tag{2.39}$$

Similar procedure applied to the equation for $s = s_k$ gives

$$\mathbf{q}_j \cdot \mathbf{M}\mathbf{q}_k s_k^2 = -\mathbf{q}_j \cdot \mathbf{K}\mathbf{q}_k.$$

Subtracting these equations from each other and taking into account that $\mathbf{M}$ and $\mathbf{K}$ are symmetric, we obtain

$$(s_j^2 - s_k^2)\mathbf{q}_j \cdot \mathbf{M}\mathbf{q}_k = 0,$$

which implies the orthogonality. The orthogonality and normalization conditions can be presented in one equation

$$\mathbf{q}_j \cdot \mathbf{M}\mathbf{q}_k = \delta_{jk} = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{otherwise.} \end{cases} \tag{2.40}$$

If there is a multiple eigenvalue, then the corresponding eigenvectors span a subspace of dimension equal to the multiplicity of the eigenvalue. Therefore, it is always possible to find a set of vectors in this subspace satisfying the orthogonality and normalization conditions.

Since $\mathbf{q}_j e^{i\omega_j t}$ are the solutions of (2.35) which is the differential equation with real matrices, their real and imaginary parts

$$\mathbf{q}_j \cos \omega_j t \quad \text{and} \quad \mathbf{q}_j \sin \omega_j t$$

must also satisfy this equation. The general solution can now be constructed as the linear superposition

$$\mathbf{q}(t) = \sum_{j=1}^{n} \mathbf{q}_j (A_j \cos \omega_j t + B_j \sin \omega_j t).$$

The unknown coefficients A_j and B_j must be found from the initial conditions (2.36) giving

$$\sum_{j=1}^{n} A_j \mathbf{q}_j = \mathbf{q}_0, \quad \sum_{j=1}^{n} B_j \omega_j \mathbf{q}_j = \mathbf{v}_0.$$

Multiplying these equations from the left by $\mathbf{M}$ and then by $\mathbf{q}_i$ and making use of the orthogonality and normalization conditions, we obtain from here

$$A_i = \mathbf{q}_i \cdot \mathbf{M}\mathbf{q}_0, \quad B_i = \frac{1}{\omega_i} \mathbf{q}_i \cdot \mathbf{M}\mathbf{v}_0, \quad i = 1, \ldots, n.$$

Alternatively, we can present the solution in the form

$$\mathbf{q}(t) = \sum_{j=1}^{n} \mathbf{q}_j a_j \cos(\omega_j t - \phi_j),$$

where

$$a_j = \sqrt{A_j^2 + B_j^2}, \quad \tan \phi_j = \frac{B_j}{A_j}, \quad j = 1, \ldots, n.$$

Normal modes and coordinates. The above solution is the sum of n harmonic motions, so it is in general non-periodic if the frequency ratios are not rational numbers. However, for the initial conditions of the special form

$$\mathbf{q}_0 = q_0 \mathbf{q}_j, \quad \mathbf{v}_0 = v_0 \mathbf{q}_j,$$

the motion is purely harmonic with the frequency ω_j. We call such motion normal mode.

The question now arises: can we find the coordinates in which the normal modes become independent? The similar consideration as that provided in example 2.2 shows that this is possible if the kinetic and potential energies of the system, in terms of the new coordinates ξ_j, take the form

$$K(\dot{\xi}) = \frac{1}{2}\sum_{j=1}^{n} \dot{\xi}_j^2, \quad U(\xi) = \frac{1}{2}\sum_{j=1}^{n} \omega_j^2 \xi_j^2.$$

Thus, the problem reduces to finding a linear transformation which simultaneously diagonalizes two quadratic forms. As we know from linear algebra [37], the required transformation is given by

$$\mathbf{q} = \mathbf{Q}\boldsymbol{\xi},$$

with

$$\mathbf{Q} = (\mathbf{q}_1\, \mathbf{q}_2 \ldots \mathbf{q}_n)$$

being the $n \times n$ matrix, whose j-th column is the j-th eigenvector found above. We shall call $\mathbf{Q}$ modal matrix. In terms of the vector of normal coordinates $\boldsymbol{\xi}$ we have

$$K(\dot{\xi}) = \frac{1}{2}\dot{\mathbf{q}} \cdot \mathbf{M}\dot{\mathbf{q}} = \frac{1}{2}\dot{\boldsymbol{\xi}} \cdot \mathbf{Q}^T\mathbf{M}\mathbf{Q}\dot{\boldsymbol{\xi}} = \frac{1}{2}\dot{\boldsymbol{\xi}} \cdot \dot{\boldsymbol{\xi}},$$
$$U(\xi) = \frac{1}{2}\mathbf{q} \cdot \mathbf{K}\mathbf{q} = \frac{1}{2}\boldsymbol{\xi} \cdot \mathbf{Q}^T\mathbf{K}\mathbf{Q}\boldsymbol{\xi} = \frac{1}{2}\boldsymbol{\xi} \cdot \boldsymbol{\Omega}^2\boldsymbol{\xi},$$

where $\mathbf{Q}^T$ denotes the transpose of $\mathbf{Q}$, and $\boldsymbol{\Omega}^2$ is the diagonal matrix with the elements ω_j^2 on the diagonal. The last identities in these formulas are obtained by the orthogonality conditions (2.39) and (2.40). So, the corresponding Lagrange function describes the motion of n uncoupled single oscillators with the frequencies ω_j, $j = 1, \ldots, n$.

Extremal properties. If ω is an eigenfrequency and $\hat{\mathbf{q}}$ a corresponding eigenvector, then it follows from (2.38) that

$$\omega^2 = \frac{\hat{\mathbf{q}} \cdot \mathbf{K}\hat{\mathbf{q}}}{\hat{\mathbf{q}} \cdot \mathbf{M}\hat{\mathbf{q}}} = r(\hat{\mathbf{q}}).$$

The right-hand side of this equation is called Rayleigh's quotient [45]. It turns out that the following extremal properties hold true.

1. The square of smallest eigenfrequency ω_1^2 is the minimum of $r(\mathbf{q})$ among all $\mathbf{q} \neq 0$. The easiest way to prove this is to rewrite Rayleigh's quotient in terms of the vector of normal coordinates

$$r(\boldsymbol{\xi}) = \frac{\omega_1^2\xi_1^2 + \ldots + \omega_n^2\xi_n^2}{\xi_1^2 + \ldots + \xi_n^2}.$$

Since $\omega_n \geq \ldots \geq \omega_1$, Rayleigh's quotient is always larger than or equal to ω_1^2. From the other side $r(\boldsymbol{\xi}) = \omega_1^2$ if $\xi_1 = 1$ and $\xi_2 = \ldots = \xi_n = 0$. So the statement is proved.

2. The square of j-th eigenfrequency ω_j^2 is equal to the minimum of Rayleigh's quotient

$$\omega_j^2 = \min_{\mathbf{q}} r(\mathbf{q})$$

among all $\mathbf{q} \neq 0$ satisfying $j-1$ constraints

$$\mathbf{q}_1 \cdot \mathbf{M}\mathbf{q} = 0, \ldots, \mathbf{q}_{j-1} \cdot \mathbf{M}\mathbf{q} = 0.$$

Indeed, in terms of the normal coordinates the above constraints become

$$\xi_1 = \ldots = \xi_{j-1} = 0.$$

Thus, Rayleigh's quotient under these constraints reduces to

$$r(\boldsymbol{\xi}) = \frac{\omega_j^2 \xi_j^2 + \ldots + \omega_n^2 \xi_n^2}{\xi_j^2 + \ldots + \xi_n^2},$$

and the proof can be provided in a similar manner.

The extremal properties of Rayleigh's quotient are quite useful in approximate calculations of the eigenfrequencies [45].

Damped Oscillators. The motion is described by the equation

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0}, \tag{2.41}$$

subject to the initial conditions

$$\mathbf{q}(0) = \mathbf{q}_0, \quad \dot{\mathbf{q}}(0) = \mathbf{v}_0, \tag{2.42}$$

where $\mathbf{M}$ and $\mathbf{K}$ are symmetric and positive definite matrices, while $\mathbf{C}$ is symmetric and non-negative definite.

Solution. A particular solution of (2.41) is sought in the form

$$\mathbf{q} = \hat{\mathbf{q}} e^{st},$$

where $\hat{\mathbf{q}}$ is a constant vector. Equation (2.41) reduces then to the algebraic equation

$$(\mathbf{M}s^2 + \mathbf{C}s + \mathbf{K})\hat{\mathbf{q}} = \mathbf{0}. \tag{2.43}$$

Non-trivial solutions of (2.43) exist if

$$\det(\mathbf{M}s^2 + \mathbf{C}s + \mathbf{K}) = 0.$$

This is the algebraic equation of order $2n$ with respect to s having $2n$ roots. Since the matrices $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ are real, the complex roots must occur in pairs of complex conjugates. Moreover, if s_j^* is the complex conjugate root with respect to s_j, then the corresponding eigenvector $\mathbf{q}_j^*$ must be complex conjugate to the eigenvector $\mathbf{q}_j$

of s_j. It turns out that all roots of the characteristic equation have non-positive real parts. To show this one can apply the Routh-Hurwitz criterion although the proof is not elementary. The more elementary proof is based on the balance of energy (2.32) for dissipative systems. To this end let us assume that there is a root of the characteristic equation with the positive real part $s = \delta + i\omega$, where $\delta > 0$. Then a free vibration of the form

$$\mathbf{q} = e^{\delta t}\mathrm{Re}(\hat{\mathbf{q}}e^{i\omega t})$$

exists, with $\hat{\mathbf{q}}$ being the eigenvector corresponding to s. Substituting this particular solution into the energy balance equation (2.32) and using the positive definiteness of the dissipation function, we see that the amount of energy dissipation goes to $-\infty$ as t tends to infinity, what contradicts the positiveness of the total energy.

The general solution of (2.41) is given in the form

$$\mathbf{q} = \sum_{j=1}^{2n} A_j \mathbf{q}_j e^{s_j t}.$$

Using the initial conditions (2.42), we obtain the system of $2n$ linear equations

$$\sum_{j=1}^{2n} A_j \mathbf{q}_j = \mathbf{q}_0, \quad \sum_{j=1}^{2n} A_j s_j \mathbf{q}_j = \mathbf{v}_0,$$

for the determination of $2n$ coefficients A_j.

Modal decomposition. The coupled oscillators with n degrees of freedom and with the proportional damping can be reduced to n uncoupled damped oscillators. To show this let us introduce the vector of normal coordinates $\boldsymbol{\xi}$ such that $\mathbf{q} = \mathbf{Q}\boldsymbol{\xi}$, with $\mathbf{Q}$ being the modal matrix, into the equation of motion (2.41). Multiplying this equation from the left by $\mathbf{Q}^T$, we obtain

$$\mathbf{Q}^T\mathbf{M}\mathbf{Q}\ddot{\boldsymbol{\xi}} + \mathbf{Q}^T\mathbf{C}\mathbf{Q}\dot{\boldsymbol{\xi}} + \mathbf{Q}^T\mathbf{K}\mathbf{Q}\boldsymbol{\xi} = \mathbf{0}.$$

The modal matrix $\mathbf{Q}$ diagonalizes simultaneously $\mathbf{M}$ and $\mathbf{K}$, so

$$\mathbf{Q}^T\mathbf{M}\mathbf{Q} = \mathbf{I}, \quad \mathbf{Q}^T\mathbf{K}\mathbf{Q} = \boldsymbol{\Omega}^2 = \mathrm{diag}(\omega_j^2),$$

where $\mathbf{I}$ is the identity matrix and $\boldsymbol{\Omega}^2$ the diagonal matrix with the elements ω_j^2. Because of the proportional damping $\mathbf{C} = \alpha\mathbf{M} + \beta\mathbf{K}$ we have

$$\boldsymbol{\Delta} = \mathbf{Q}^T\mathbf{C}\mathbf{Q} = \mathbf{Q}^T(\alpha\mathbf{M} + \beta\mathbf{K})\mathbf{Q} = \alpha\mathbf{I} + \beta\boldsymbol{\Omega}^2 = \mathrm{diag}(2\delta_j\omega_j),$$

where $\delta_j\omega_j = (\alpha + \beta\omega_j^2)/2$ are the decay rates. Thus, the damping matrix $\boldsymbol{\Delta}$ becomes also diagonal in terms of the normal coordinates. The equation of motion is decomposed into n uncoupled equations

$$\ddot{\xi}_j + 2\delta_j\omega_j\dot{\xi}_j + \omega_j^2\xi_j = 0, \quad j = 1, \ldots, n,$$

which can be solved by the method discussed in Section 1.2.

Alternatively, we can also realize the modal decomposition by diagonalizing the kinetic and potential energies together with the dissipation function as the quadratic forms.

Forced Oscillators. The motion is described by the equation

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{f}(t). \tag{2.44}$$

Since this equation is linear, its solution is the sum of any particular solution and the general solution of the homogeneous equation which has been found previously. Thus, the problem reduces to finding any particular solution of (2.44). Besides, if the damping is permeating, then all solutions of the homogeneous equation decay with time, so only the particular solution of (2.44) persists at large time.

Harmonic excitations. For the harmonic excitations of the form $\mathbf{f}(t) = \hat{\mathbf{f}}\cos\omega t$ which is the real part of $\hat{\mathbf{f}}e^{i\omega t}$ we consider the auxiliary equation

$$\mathbf{M}\ddot{\mathbf{z}} + \mathbf{C}\dot{\mathbf{z}} + \mathbf{K}\mathbf{z} = \hat{\mathbf{f}}e^{i\omega t},$$

where $\mathbf{z}(t)$ may be complex-valued. We look for the solution of the form $\mathbf{z}(t) = \hat{\mathbf{z}}e^{i\omega t}$. Substituting this into the above equation and eliminating the factor $e^{i\omega t}$, we obtain

$$(-\omega^2\mathbf{M} + i\omega\mathbf{C} + \mathbf{K})\hat{\mathbf{z}} = \hat{\mathbf{f}}.$$

Provided the matrix on the left-hand side has an inverse, this equation yields

$$\hat{\mathbf{z}} = (-\omega^2\mathbf{M} + i\omega\mathbf{C} + \mathbf{K})^{-1}\hat{\mathbf{f}} = \mathbf{G}(\omega)\hat{\mathbf{f}}.$$

Matrix $\mathbf{G}(\omega)$ is called a transmittance matrix of the system. The particular solution of (2.44) is the real part of $\mathbf{z}(t)$, so

$$\mathbf{q}(t) = \mathrm{Re}(\mathbf{G}(\omega)\hat{\mathbf{f}}e^{i\omega t}).$$

The analysis of forced vibrations simplifies considerably for the conservative oscillators with $\mathbf{C} = \mathbf{0}$. In this case the solution also has the form $\mathbf{q}(t) = \hat{\mathbf{q}}\cos\omega t$, where $\hat{\mathbf{q}}$ satisfies the linear equation

$$(-\omega^2\mathbf{M} + \mathbf{K})\hat{\mathbf{q}} = \hat{\mathbf{f}}.$$

If the determinant $\Delta(\omega)$ of $-\omega^2\mathbf{M} + \mathbf{K}$ differs from zero, we use Cramer's rule to present the solution in the form

$$\hat{q}_j = \frac{\Delta_j(\omega)}{\Delta(\omega)}, \quad j = 1, \ldots, n, \tag{2.45}$$

where $\Delta_j(\omega)$ is the determinant obtained on replacing the j-th column of Δ by the vector $\hat{\mathbf{f}}$. The following interesting cases may occur:

a) $\Delta(\omega) = 0$, $\Delta_j(\omega) \neq 0$: the frequency of excitation coincides with one of the eigenfrequency and the oscillators are in resonance,
b) $\Delta(\omega) = 0$, $\Delta_j(\omega) = 0$ for all j so that $\lim_{\tilde{\omega}\to\omega} \Delta_j(\tilde{\omega})/\Delta(\tilde{\omega}) < \infty$: this situation is classified as pseudo-resonance,
c) $\Delta(\omega) \neq 0$, $\Delta_j(\omega) = 0$: the forced vibration corresponding to the j-th degree of freedom is eliminated (anti-resonance).

For the dissipative oscillators with small but finite damping coefficients neither resonance nor anti-resonance occurs. The problem of vibration control reduces then to finding optimal parameters of vibration absorbers to effectively absorb energy of the unwanted forced vibration.

Arbitrary excitations. For forced oscillators with proportional damping the problem can be solved by the modal decomposition as shown in example 2.9. For forced oscillators with non-proportional damping, the Laplace transform should be used instead. Not restricting the generality, we look for the particular solution of (2.44) satisfying the initial conditions

$$\mathbf{q}(0) = \mathbf{0}, \quad \dot{\mathbf{q}}(0) = \mathbf{0}.$$

Applying the Laplace transform to both sides of equation (2.44), we obtain

$$\int_0^\infty (\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q})e^{-st}\,dt = \int_0^\infty \mathbf{f}(t)e^{-st}\,dt.$$

Using the properties of the Laplace transform and the vanishing initial conditions, we reduce this to the algebraic equation

$$(\mathbf{M}s^2 + \mathbf{C}s + \mathbf{K})\mathbf{X}(s) = \mathbf{F}(s),$$

where $\mathbf{X}(s)$ and $\mathbf{F}(s)$ are the Laplace images of $\mathbf{q}(t)$ and $\mathbf{f}(t)$, respectively. This yields

$$\mathbf{X}(s) = (\mathbf{M}s^2 + \mathbf{C}s + \mathbf{K})^{-1}\mathbf{F}(s).$$

Applying the inverse Laplace transform, we get

$$\mathbf{q}(t) = \mathscr{L}^{-1}[\mathbf{X}(s)] = \frac{1}{2\pi i}\int_{\alpha - i\infty}^{\alpha + i\infty} (\mathbf{M}s^2 + \mathbf{C}s + \mathbf{K})^{-1}\mathbf{F}(s)e^{st}\,ds,$$

where α is any positive number. Since all roots of the characteristic equation lie in the left half-plane or on the imaginary axis, the integrand is an analytic function in the right half-plane of the complex s-plane. Thus, for an arbitrary regular excitation $\mathbf{f}(t)$ which remains finite as t goes to infinity the integral converges. The line of integration $(\alpha - i\infty, \alpha + i\infty)$ can be moved arbitrarily in the right half-plane.

Let $\mathbf{x}_{rj}(t)$ be the solution of (2.44) with zero initial condition, where

$$\mathbf{f}(t) = \mathbf{h}_j(t) = (0, \ldots, h(t), \ldots, 0)^T,$$

$h(t)$ being Heaviside's step function. Thus, $\mathbf{h}_j(t)$ is the column vector whose components are zero except the j-th component which is the Heaviside's step function. The $n \times n$ matrix

$$\mathbf{X}_r(t) = (\mathbf{x}_{r1}(t) \dots \mathbf{x}_{rn}(t)),$$

with j-th column being the vector $\mathbf{x}_{rj}(t)$, is called a unit step response matrix of the system. It is easy to see that

$$(\mathbf{M}s^2 + \mathbf{C}s + \mathbf{K})^{-1}\frac{1}{s} = \mathscr{L}(\mathbf{X}_r(t)).$$

Using the convolution theorem for the Laplace transform, we obtain finally

$$\mathbf{q}(\tau) = \int_0^\tau \mathbf{X}_r(\tau - t)\dot{\mathbf{f}}(t)dt. \tag{2.46}$$

This is generalized Duhamel's formula which solves the problem if the unit step response matrix of the system is known.

Mention that the Laplace transform can also be used to solve the initial value problem similar to that analyzed in Section 1.3.

2.6 Exercises

EXERCISE 2.1. Two point-masses m_1 and m_2 are connected with a fixed support O and with each other by two rigid and massless bars of lengths l_1 and l_2 (see Fig. 2.14). Derive the equations of small vibration of this double pendulum under the action of gravity. Determine the eigenfrequencies of vibrations.

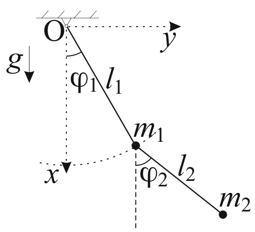

Fig. 2.14 Double pendulum

Solution. This system has two degrees of freedom described by the angles φ_1 and φ_2. Let us write down the kinetic and potential energies of this double pendulum. For the kinetic energy we have

$$K(\dot{\varphi}) = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2.$$

As the first point-mass m_1 rotates about O with the angular velocity $\dot{\varphi}_1$, the magnitude of its velocity is $v_1 = l_1\dot{\varphi}_1$. The velocity of m_2 is the superposition of the velocity of m_1 and the relative velocity of m_2 with respect to m_1, so

$$\mathbf{v}_2 = \mathbf{v}_1 + \mathbf{v}_{21}.$$

Since both angles φ_1 and φ_2 are small, these two vectors are nearly parallel. Taking into account that $v_{21} = l_2\dot{\varphi}_2$, we can write

$$v_2^2 = v_1^2 + v_{21}^2 + 2\mathbf{v}_1\cdot\mathbf{v}_{21} \approx l_1^2\dot{\varphi}_1^2 + l_2^2\dot{\varphi}_2^2 + 2l_1l_2\dot{\varphi}_1\dot{\varphi}_2.$$

Thus, the kinetic energy is equal to

$$K(\dot{\varphi}) = \frac{1}{2}(m_1+m_2)l_1^2\dot{\varphi}_1^2 + \frac{1}{2}m_2l_2^2\dot{\varphi}_2^2 + m_2l_1l_2\dot{\varphi}_1\dot{\varphi}_2.$$

Let us choose the zero level of the potential energy at $x = 0$. Then the potential energy of the point-masses in the gravitational field is given by

$$U(\varphi) = -m_1gx_1 - m_2gx_2 = -m_1gl_1\cos\varphi_1 - m_2g(l_1\cos\varphi_1 + l_2\cos\varphi_2).$$

For small angles φ_1 and φ_2 we may replace $\cos\varphi_j \approx 1-\varphi_j^2/2$, so up to an unessential constant,

$$U(\varphi) = \frac{1}{2}m_1gl_1\varphi_1^2 + \frac{1}{2}m_2gl_1\varphi_1^2 + \frac{1}{2}m_2gl_2\varphi_2^2.$$

Thus, the Lagrange function reads

$$L = \frac{1}{2}(m_1+m_2)l_1^2\dot{\varphi}_1^2 + \frac{1}{2}m_2l_2^2\dot{\varphi}_2^2 + m_2l_1l_2\dot{\varphi}_1\dot{\varphi}_2 - \frac{1}{2}(m_1+m_2)gl_1\varphi_1^2 - \frac{1}{2}m_2gl_2\varphi_2^2.$$

From Lagrange's equations

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\varphi}_j} - \frac{\partial L}{\partial\varphi_j} = 0, \quad j = 1,2,$$

we derive the equations of motion

$$\begin{aligned} \frac{d}{dt}((m_1+m_2)l_1^2\dot{\varphi}_1 + m_2l_1l_2\dot{\varphi}_2) + (m_1+m_2)gl_1\varphi_1 &= 0,\\ \frac{d}{dt}(m_2l_2^2\dot{\varphi}_2 + m_2l_1l_2\dot{\varphi}_1) + m_2gl_2\varphi_2 &= 0. \end{aligned}$$

Dividing the first equation by l_1 and the second one by m_2l_2, we reduce this system to

$$\begin{aligned} (m_1+m_2)l_1\ddot{\varphi}_1 + m_2l_2\ddot{\varphi}_2 + (m_1+m_2)g\varphi_1 &= 0,\\ l_1\ddot{\varphi}_1 + l_2\ddot{\varphi}_2 + g\varphi_2 &= 0. \end{aligned}$$

To determine the eigenfrequencies of vibrations we seek for the solution in the form

$$\varphi_j = \hat{\varphi}_j e^{i\omega t}.$$

Substituting this into the equations of motion, we get

$$\begin{pmatrix} (m_1+m_2)(g-l_1\omega^2) & -m_2 l_2\omega^2 \\ -l_1\omega^2 & g-l_2\omega^2 \end{pmatrix} \begin{pmatrix} \hat{\varphi}_1 \\ \hat{\varphi}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Non-trivial solutions of this equation exist if its determinant vanishes

$$\begin{vmatrix} (m_1+m_2)(g-l_1\omega^2) & -m_2 l_2\omega^2 \\ -l_1\omega^2 & g-l_2\omega^2 \end{vmatrix} = 0.$$

Computing the determinant, we get the following characteristic equation

$$m_1 l_1 l_2 \omega^4 - (m_1+m_2)g(l_1+l_2)\omega^2 + (m_1+m_2)g^2 = 0.$$

Solving this quadratic equation (with respect to ω^2), we obtain two roots $\omega_{1,2}^2$ given by

$$\frac{g}{2m_1 l_1 l_2}\left[(m_1+m_2)(l_1+l_2) \mp \sqrt{(m_1+m_2)[(m_1+m_2)(l_1+l_2)^2 - 4m_1 l_1 l_2]}\right].$$

EXERCISE 2.2. A body of mass m is connected with the wall through a spring of stiffness k and with a bar of length l and equal mass m which rotates in the plane about O (see Fig. 2.15). Derive the equations of small vibration of this system. Determine the eigenfrequencies of vibrations.

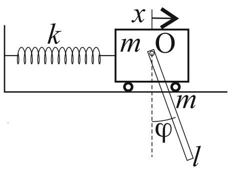

Fig. 2.15 Body connected with spring and bar

Solution. Let $q = (x, \varphi)$ be the generalized coordinates and S be the center of mass of the bar. We write down the kinetic energy of this system

$$K(\dot{q}) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}mv_S^2 + \frac{1}{2}J_S\dot{\varphi}^2,$$

where the last two terms represent the kinetic energy of the bar, with v_S being the velocity of the center of mass and $J_S = ml^2/12$ the moment of inertia of the bar about S. For small angle $\varphi \ll 1$

$$v_S = \dot{x} + \frac{l}{2}\dot{\varphi}.$$

So, the kinetic energy of this system reads

$$K(\dot{q}) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m(\dot{x} + \frac{l}{2}\dot{\varphi})^2 + \frac{1}{24}ml^2\dot{\varphi}^2.$$

Concerning the potential energy, we have for small angle

$$U(q) = \frac{1}{2}kx^2 + mg\frac{l}{2}(1 - \cos\varphi) \approx \frac{1}{2}kx^2 + mg\frac{l}{4}\varphi^2.$$

Thus,

$$L(q,\dot{q}) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m(\dot{x} + \frac{l}{2}\dot{\varphi})^2 + \frac{1}{24}ml^2\dot{\varphi}^2 - \frac{1}{2}kx^2 - mg\frac{l}{4}\varphi^2.$$

From Lagrange's equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0, \quad j = 1,2,$$

we derive the equations of motion

$$m\ddot{x} + m(\ddot{x} + \frac{l}{2}\ddot{\varphi}) + kx = 0,$$
$$m\frac{l}{2}(\ddot{x} + \frac{l}{2}\ddot{\varphi}) + \frac{1}{12}ml^2\ddot{\varphi} + \frac{1}{2}mgl\varphi = 0.$$

These equations can be simplified to

$$2m\ddot{x} + m\frac{l}{2}\ddot{\varphi} + kx = 0,$$
$$\frac{1}{3}ml^2\ddot{\varphi} + m\frac{l}{2}\ddot{x} + \frac{1}{2}mgl\varphi = 0.$$

Dividing the first equation by $2m$ and the second one by $ml^2/3$, respectively, we rewrite them in the form

$$\ddot{x} + \frac{l}{4}\ddot{\varphi} + \omega_x^2 x = 0,$$
$$\frac{3}{2l}\ddot{x} + \ddot{\varphi} + \omega_\varphi^2\varphi = 0,$$

where

$$\omega_x^2 = \frac{k}{2m}, \quad \omega_\varphi^2 = \frac{3g}{2l}.$$

To determine the eigenfrequencies of vibrations we seek for the solution in the form

$$\begin{pmatrix} x \\ \varphi \end{pmatrix} = \begin{pmatrix} \hat{x} \\ \hat{\varphi} \end{pmatrix} e^{i\omega t}.$$

Substituting this into the equations of motion, we get

$$\begin{pmatrix} -\omega^2 + \omega_x^2 & -\frac{l}{4}\omega^2 \\ -\frac{3}{2l}\omega^2 & -\omega^2 + \omega_\varphi^2 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{\varphi} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Non-trivial solutions of this equation exist if its determinant vanishes

$$\begin{vmatrix} -\omega^2 + \omega_x^2 & -\frac{l}{4}\omega^2 \\ -\frac{3}{2l}\omega^2 & -\omega^2 + \omega_\varphi^2 \end{vmatrix} = 0.$$

Computing the determinant, we get the following characteristic equation

$$(-\omega^2 + \omega_x^2)(-\omega^2 + \omega_\varphi^2) - \frac{3}{8}\omega^4 = 0,$$

yielding two roots

$$\omega_{1,2}^2 = \frac{4}{5}\left(\omega_x^2 + \omega_\varphi^2 \mp \sqrt{(\omega_x^2 + \omega_\varphi^2)^2 - \frac{5}{2}\omega_x^2\omega_\varphi^2}\right).$$

EXERCISE 2.3. A rigid bar of mass m and moment of inertia $J_S = m\rho^2$ is hung on two massless and unstretchable ropes of equal length l (this is the primitive mechanical model of the swing). The distance between the ropes in the equilibrium state is s. The distances between the attachment points and the center of mass of the bar are s_1 and s_2, respectively. Under the assumption $\varphi_1 \ll 1$, $\varphi_2 \ll 1$ derive the equations of out-of-plane vibration of the bar, neglecting its in-plane motion. Determine the eigenfrequencies of vibrations.

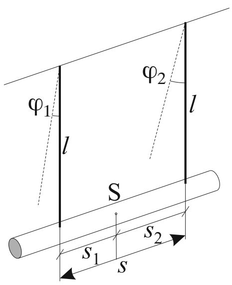

Fig. 2.16 Bar hung on two ropes

Solution. The motion of the bar as rigid body is the superposition of the translation of the center of mass S and the rotation about S. Accordingly, the kinetic energy of the bar equals

$$K = \frac{1}{2} m v_S^2 + \frac{1}{2} J_S \omega^2,$$

where ω is the angular velocity and J_S the moment of inertia of the bar about S. This motion can also be regarded as the pure rotation about the instantaneous center of rotation P with the same angular velocity ω (see Fig. 2.17).

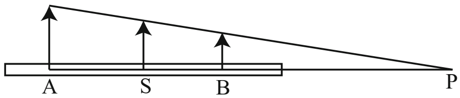

Fig. 2.17 Pure rotation of the bar about P

The velocities of the attachment points A and B are $l\dot{\varphi}_1$ and $l\dot{\varphi}_2$, respectively. Let the distance between A and P be x, then the distance between B and P is $x - s_1 - s_2$, so

$$\begin{aligned} x\omega &= l\dot{\varphi}_1, \\ (x - s_1 - s_2)\omega &= l\dot{\varphi}_2. \end{aligned}$$

From here we find that

$$\omega = \frac{l}{s_1 + s_2}(\dot{\varphi}_1 - \dot{\varphi}_2), \quad x = \frac{(s_1 + s_2)\dot{\varphi}_1}{\dot{\varphi}_1 - \dot{\varphi}_2}.$$

The velocity of the center of mass, v_S, can also be easily found as

$$v_S = (x - s_1)\omega = l\left(\frac{s_2}{s_1 + s_2}\dot{\varphi}_1 + \frac{s_1}{s_1 + s_2}\dot{\varphi}_2\right).$$

Thus, the kinetic energy of the bar reads

$$K = \frac{1}{2} m l^2 \left(\frac{s_2}{s_1 + s_2}\dot{\varphi}_1 + \frac{s_1}{s_1 + s_2}\dot{\varphi}_2\right)^2 + \frac{1}{2} m \rho^2 \frac{l^2}{(s_1 + s_2)^2}(\dot{\varphi}_1 - \dot{\varphi}_2)^2.$$

To write down the potential energy of the bar we find out the change of height of the center of mass. The changes of height of the attachment points A and B are

$$w_1 = l(1 - \cos\varphi_1) \approx l\frac{\varphi_1^2}{2}, \quad w_2 = l(1 - \cos\varphi_2) \approx l\frac{\varphi_2^2}{2}.$$

For the bar, the change of height must be a linear function of x:

$$w(x) = ax + b,$$

where x is the coordinate along the bar axis. Choosing $x = 0$ at A, we find that $b = w_1$. For $x = s_1 + s_2$ at B we have $a(s_1 + s_2) + w_1 = w_2$, so

$$a = \frac{w_2 - w_1}{s_1 + s_2}.$$

Consequently, the change of height of the center of mass equals

$$w_S = \frac{l}{2(s_1 + s_2)}(s_2\varphi_1^2 + s_1\varphi_2^2),$$

and the potential energy reads

$$U = \frac{mgl}{2(s_1 + s_2)}(s_2\varphi_1^2 + s_1\varphi_2^2).$$

Combining the kinetic and potential energies, we obtain the Lagrange function in the form

$$L = \frac{ml^2}{2(s_1 + s_2)^2}(s_2\dot{\varphi}_1 + s_1\dot{\varphi}_2)^2 + \frac{m\rho^2 l^2}{2(s_1 + s_2)^2}(\dot{\varphi}_1 - \dot{\varphi}_2)^2 - \frac{mgl}{2(s_1 + s_2)}(s_2\varphi_1^2 + s_1\varphi_2^2).$$

Lagrange's equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}_j} - \frac{\partial L}{\partial \varphi_j} = 0, \quad j = 1,2$$

lead to

$$\frac{d}{dt}\Big[\frac{ml^2 s_2}{(s_1 + s_2)^2}(s_2\dot{\varphi}_1 + s_1\dot{\varphi}_2) + \frac{m\rho^2 l^2}{(s_1 + s_2)^2}(\dot{\varphi}_1 - \dot{\varphi}_2)\Big] + \frac{mgls_2}{s_1 + s_2}\varphi_1 = 0,$$
$$\frac{d}{dt}\Big[\frac{ml^2 s_1}{(s_1 + s_2)^2}(s_2\dot{\varphi}_1 + s_1\dot{\varphi}_2) - \frac{m\rho^2 l^2}{(s_1 + s_2)^2}(\dot{\varphi}_1 - \dot{\varphi}_2)\Big] + \frac{mgls_1}{s_1 + s_2}\varphi_2 = 0.$$

Dividing both equations by $ml^2/(s_1 + s_2)^2$, we reduce them to

$$(s_2^2 + \rho^2)\ddot{\varphi}_1 + (s_1 s_2 - \rho^2)\ddot{\varphi}_2 + \frac{gss_2}{l}\varphi_1 = 0,$$
$$(s_1 s_2 - \rho^2)\ddot{\varphi}_1 + (s_1^2 + \rho^2)\ddot{\varphi}_2 + \frac{gss_1}{l}\varphi_2 = 0,$$

where $s = s_1 + s_2$.

To determine the eigenfrequencies of vibrations we seek for the solution in the form

$$\varphi_j = \hat{\varphi}_j e^{i\omega t}.$$

Substituting this into the equations of motion, we get

$$\begin{pmatrix} (\frac{gss_2}{l} - (s_2^2 + \rho^2)\omega^2) & -(s_1 s_2 - \rho^2)\omega^2 \\ -(s_1 s_2 - \rho^2)\omega^2 & (\frac{gss_1}{l} - (s_1^2 + \rho^2)\omega^2) \end{pmatrix} \begin{pmatrix} \hat{\varphi}_1 \\ \hat{\varphi}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

From the condition of vanishing determinant, we get the following characteristic equation

$$\left(\frac{gss_2}{l}-(s_2^2+\rho^2)\omega^2\right)\left(\frac{gss_1}{l}-(s_1^2+\rho^2)\omega^2\right)-(s_1s_2-\rho^2)^2\omega^4=0,$$

which can be reduced to

$$\rho^2\omega^4-\frac{g}{l}(s_1s_2+\rho^2)\omega^2+\frac{g^2}{l^2}s_1s_2=0.$$

Solving this quadratic equation (with respect to ω^2), we obtain two roots

$$\omega_1^2=\frac{g}{l},\quad \omega_2^2=\frac{gs_1s_2}{l\rho^2}.$$

EXERCISE 2.4. Beating phenomenon. Find solution of (2.7) for the coupled pendulums satisfying the initial conditions: $\varphi_1(0)=1$, $\varphi_2(0)=\dot{\varphi}_1(0)=\dot{\varphi}_2(0)=0$. Plot $\varphi_1(t)$ and $\varphi_2(t)$ for $\alpha=0.1$ and analyze their behaviors.

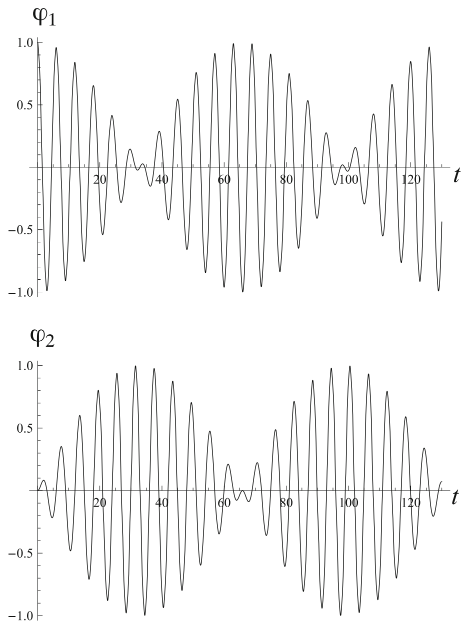

Fig. 2.18 Free vibrations of the coupled pendulums ($\omega_0=1$, $\alpha=0.1$)

Solution. As shown in Section 2.1, the solution to equations (2.3) describing the vibration of the coupled pendulums is given by

$$\boldsymbol{\varphi} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \mathbf{q}_1(A_1 \cos\omega_1 t + B_1 \sin\omega_1 t) + \mathbf{q}_2(A_2 \cos\omega_2 t + B_2 \sin\omega_2 t),$$

with $\omega_1 = \omega_0 = \sqrt{g/l}$, $\omega_2 = \sqrt{\omega_0^2 + 2\alpha}$, and

$$\mathbf{q}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \mathbf{q}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

To compute the coefficients we use the initial conditions

$$\boldsymbol{\varphi}(0) = \boldsymbol{\varphi}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \dot{\boldsymbol{\varphi}}(0) = \dot{\boldsymbol{\varphi}}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

together with the orthogonality and normalization conditions. We easily find that

$$A_1 = \frac{1}{\sqrt{2}}, \quad A_2 = -\frac{1}{\sqrt{2}}, \quad B_1 = B_2 = 0.$$

Thus, the solution is

$$\varphi_1(t) = \frac{1}{2}(\cos\omega_1 t + \cos\omega_2 t), \quad \varphi_2(t) = \frac{1}{2}(\cos\omega_1 t - \cos\omega_2 t).$$

According to the addition formulas

$$\varphi_1(t) = \cos\omega' t \cos\varepsilon t, \quad \varphi_2(t) = \sin\omega' t \sin\varepsilon t,$$

where, for small α,

$$\omega' = \frac{1}{2}(\omega_1 + \omega_2) \approx \omega_0, \quad \varepsilon = \frac{1}{2}(\omega_2 - \omega_1) \approx \frac{1}{2}\frac{\alpha}{\omega_0}.$$

Thus, $\varphi_1(t)$ and $\varphi_2(t)$ oscillate with the frequency ω' but with slowly changing amplitude $\cos\varepsilon t$ and $\sin\varepsilon t$, respectively. This is the so called beating phenomenon (or amplitude modulation) typical for the oscillation with two nearly equal frequencies.

To simulate this solution numerically we put $\omega_1 = \omega_0 = 1$ and $\alpha = 0.1$ so that $\omega_2 = \sqrt{1+2\alpha} \approx 1.095$. The plots of $\varphi_1(t)$ and $\varphi_2(t)$ are shown in Fig. 2.18, from which it is seen that the second pendulum begins to oscillate when the first comes to rest and vice versa. Thus, the energy is transferred from the first to the second pendulum and back.

EXERCISE 2.5. Consider a pair of uncoupled harmonic oscillators described by the equations $\ddot{x} + x = 0$ and $\ddot{y} + \omega^2 y = 0$. Using t as parameter, plot the trajectory of the motion in the (x,y)-plane given by $x(t) = \cos t$ and $y(t) = \cos\omega t$ for $t \in (0, 1000)$ in two cases: i) $\omega = 3$ and ii) $\omega = \pi$. The curves of this type are called Lissajous

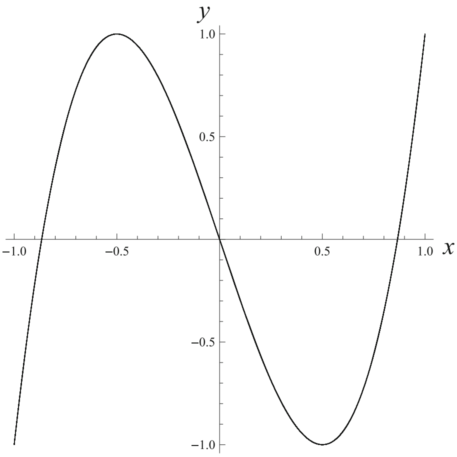

Fig. 2.19 Lissajous figure for $\omega = 3$

figures, and due to the periodicity in x and y the trajectories can be regarded as moving on a two-dimensional torus. Observe the difference in cases i) and ii).

Solution. In case i) the frequency ratio is equal to 3 which is a rational number. In case ii) the frequency ratio is π which is an irrational number. The plots of the trajectories of motion in the (x,y)-plane, made with the help of ParametricPlot command in *Mathematica*, are shown in Fig. 2.19 and 2.20 for the case i) and ii), respectively. In case i) the trajectory is periodic, with the period 2π. In case ii) the trajectory is non-periodic and for an infinitely large interval of time it is dense on the whole domain $(-1,1)\times(-1,1)$. Such motion is classified as quasi-periodic. Note that, due to the periodicity in x and y, one can wrap the square $(-1,1)\times(-1,1)$ onto the cylinder along the lines $x=\pm 1$ and then onto the torus along the lines $y=\pm 1$. Thus, the trajectories can be regarded as moving on a two-dimensional torus. The difference between cases i) and ii) is:

i) the frequency ratio is a rational number, and each trajectory is a closed periodic orbit on the torus;

ii) the frequency ratio is an irrational number, and each trajectory winds around endlessly on the torus and corresponds to the quasi-periodic motion.

EXERCISE 2.6. Determine the vibration modes and the normal coordinates of the double pendulum with $m_1 = m_2 = m$ and $l_1 = l_2 = l$.

Solution. Under the conditions $m_1 = m_2 = m$ and $l_1 = l_2 = l$ the Lagrange function, as seen from the solution of the exercise 2.1, is given by

$$L = \frac{1}{2}ml^2\dot{\varphi}_1^2 + \frac{1}{2}ml^2(\dot{\varphi}_1+\dot{\varphi}_2)^2 - mgl\varphi_1^2 - \frac{1}{2}mgl\varphi_2^2.$$

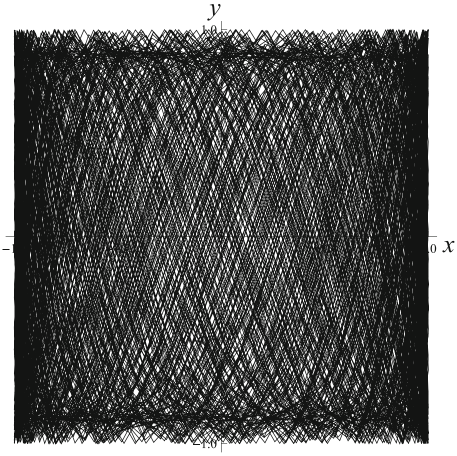

Fig. 2.20 Lissajous figure for $\omega = \pi$

The division of this Lagrange function by ml^2 does not influence the equations of motion, so we can write

$$L = \frac{1}{2}\dot{\varphi}_1^2 + \frac{1}{2}(\dot{\varphi}_1 + \dot{\varphi}_2)^2 - \omega_0^2\varphi_1^2 - \frac{1}{2}\omega_0^2\varphi_2^2,$$

where $\omega_0^2 = g/l$. The equation of motion in the matrix form reads

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0},$$

where

$$\mathbf{M} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad \mathbf{K} = \begin{pmatrix} 2\omega_0^2 & 0 \\ 0 & \omega_0^2 \end{pmatrix}.$$

The problem is to bring both matrices to the diagonal form. This can be realized by solving the eigenvalue problem

$$(-\omega^2\mathbf{M} + \mathbf{K})\hat{\mathbf{q}} = \mathbf{0},$$

or

$$\begin{pmatrix} -2\omega^2 + 2\omega_0^2 & -\omega^2 \\ -\omega^2 & -\omega^2 + \omega_0^2 \end{pmatrix} \begin{pmatrix} \hat{q}_1 \\ \hat{q}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation

$$\det(-\omega^2\mathbf{M} + \mathbf{K}) = 2(\omega^2 - \omega_0^2)^2 - \omega^4 = 0$$

yields two eigenfrequencies

$$\omega_{1,2}^2 = \omega_0^2(2 \mp \sqrt{2}).$$

The corresponding normalized eigenvectors of these two modes of vibrations are

$$\mathbf{q}_1 = \frac{1}{\sqrt{2(2+\sqrt{2})}} \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} \approx \begin{pmatrix} 0.382683 \\ 0.541196 \end{pmatrix},$$

$$\mathbf{q}_2 = \frac{1}{\sqrt{2(2-\sqrt{2})}} \begin{pmatrix} -1 \\ \sqrt{2} \end{pmatrix} \approx \begin{pmatrix} -0.92388 \\ 1.30656 \end{pmatrix}.$$

With these eigenvectors we form the modal matrix

$$\mathbf{Q} = \begin{pmatrix} 0.382683 & -0.92388 \\ 0.541196 & 1.30656 \end{pmatrix},$$

which has the inverse

$$\mathbf{Q}^{-1} = \begin{pmatrix} 1.30656 & 0.92388 \\ -0.541196 & 0.382683 \end{pmatrix}.$$

Since the normal coordinates are $\boldsymbol{\xi} = \mathbf{Q}^{-1}\mathbf{q}$, we obtain

$$\xi_1 = 1.30656\varphi_1 + 0.92388\varphi_2, \quad \xi_2 = -0.541196\varphi_1 + 0.382683\varphi_2.$$

EXERCISE 2.7. Determine the vibration modes and the normal coordinates in exercise 2.3.

Solution. From the solution of exercise 2.3 we see that there are two eigenfrequencies of vibrations

$$\omega_1^2 = \frac{g}{l}, \quad \omega_2^2 = \frac{g s_1 s_2}{l\rho^2}.$$

Let us find out the corresponding eigenvectors. For mode 1 with $\omega_1^2 = \frac{g}{l}$ we have

$$\frac{g}{l} \begin{pmatrix} s_1 s_2 - \rho^2 & -(s_1 s_2 - \rho^2) \\ -(s_1 s_2 - \rho^2) & s_1 s_2 - \rho^2 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Together with the normalization condition $\mathbf{q}_1 \cdot \mathbf{M}\mathbf{q}_1 = 1$ we find that

$$\mathbf{q}_1 = \begin{pmatrix} 1/s \\ 1/s \end{pmatrix}.$$

Thus, this mode of vibration corresponds to the synchronized parallel motion of the bar with $\varphi_1 = \varphi_2$ (the swing mode). For mode 2 with $\omega_2^2 = \frac{g s_1 s_2}{l\rho^2}$ we have

$$\frac{g}{l} \begin{pmatrix} s s_2 - (s_2^2 + \rho^2)\frac{s_1 s_2}{\rho^2} & -(s_1 s_2 - \rho^2)\frac{s_1 s_2}{\rho^2} \\ -(s_1 s_2 - \rho^2)\frac{s_1 s_2}{\rho^2} & s s_1 - (s_1^2 + \rho^2)\frac{s_1 s_2}{\rho^2} \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Consequently,

$$\frac{q_2}{q_1} = \frac{ss_2 - (s_2^2+\rho^2)\frac{s_1 s_2}{\rho^2}}{(s_1 s_2 - \rho^2)\frac{s_1 s_2}{\rho^2}} = -\frac{s_2}{s_1}.$$

Together with the normalization condition $\mathbf{q}_2 \cdot \mathbf{M}\mathbf{q}_2 = 1$ we find that

$$\mathbf{q}_2 = \frac{1}{\rho s}\begin{pmatrix} -s_1 \\ s_2 \end{pmatrix}.$$

This mode of vibration describes the rotation of the bar about the center of mass (antisymmetric mode). Thus, the modal matrix equals

$$\mathbf{Q} = \begin{pmatrix} 1/s & -s_1/(\rho s) \\ 1/s & s_2/(\rho s) \end{pmatrix}.$$

and the normal coordinates are $\boldsymbol{\xi} = \mathbf{Q}^{-1}\mathbf{q}$.

EXERCISE 2.8. Find the coordinates of the fixed points A and B of the resonance curves in example 2.8. Show that A and B are at equal level when

$$\kappa = \frac{\mu}{(1+\mu)^2}.$$

Solution. Let us analyze the resonance function

$$\left|\frac{x_1}{x_{10}}\right| = \left|\frac{-\mu\eta^2 + i\delta\eta + \kappa}{(-\eta^2 + i\delta\eta + 1 + \kappa)(-\mu\eta^2 + i\delta\eta + \kappa) - (i\delta\eta + \kappa)^2}\right|.$$

Expanding the nominator and denominator on the right-hand side, we obtain

$$\left|\frac{x_1}{x_{10}}\right| = \left|\frac{-\mu\eta^2 + \kappa + i\delta\eta}{(-\eta^2 + 1 + \kappa)(-\mu\eta^2 + \kappa) - \kappa^2 + i\delta\eta(-\mu\eta^2 - \eta^2 + 1)}\right|.$$

Thus,

$$\left|\frac{x_1}{x_{10}}\right|^2 = \frac{\delta^2\eta^2 + (\kappa - \mu\eta^2)^2}{\delta^2\eta^2(-\mu\eta^2 - \eta^2 + 1)^2 + [(-\eta^2 + 1 + \kappa)(-\mu\eta^2 + \kappa) - \kappa^2]^2}.$$

Let us first consider the limiting case of vanishing damping: $\delta = 0$. In this case the resonance function becomes

$$\left|\frac{x_1}{x_{10}}\right| = \frac{|\kappa - \mu\eta^2|}{|(-\eta^2 + 1 + \kappa)(-\mu\eta^2 + \kappa) - \kappa^2|}.$$

In the other extreme case with $\delta \to \infty$ we have

$$\left|\frac{x_1}{x_{10}}\right| = \frac{1}{|-\eta^2(\mu+1) + 1|}.$$

Since x_1/x_{10} near the fixed points has different signs in these two cases, the η-coordinates of the fixed points A and B satisfy the equation

$$\frac{\kappa-\mu\eta^2}{(-\eta^2+1+\kappa)(-\mu\eta^2+\kappa)-\kappa^2}=\frac{1}{\eta^2(\mu+1)-1},$$

or

$$(\mu^2+2\mu)\eta^4-2(\kappa+\kappa\mu+\mu)\eta^2+2\kappa=0.$$

The resonance function for other δ can be presented in the form

$$\left|\frac{x_1}{x_{10}}\right|^2=\frac{M\delta^2+N}{P\delta^2+Q}$$

so that it will be independent of δ^2 only if $M/P=N/Q$ which is again identical with the above equation. From this equation two roots η_1^2 and η_2^2 can be found, which determine the coordinates of A and B. The ordinates of points A and B are obtained by substituting these roots in the resonance function for the case $\delta\to\infty$. Since the signs of this function is positive at A and negative at B, the ordinates are

$$\frac{1}{-\mu\eta_1^2-\eta_1^2+1}\quad\text{and}\quad\frac{1}{\mu\eta_2^2+\eta_2^2-1}.$$

We want to choose the parameters of absorber in such a way that points A and B are at equal level. This requires that

$$\frac{1}{-\mu\eta_1^2-\eta_1^2+1}=\frac{1}{\mu\eta_2^2+\eta_2^2-1},$$

or

$$\eta_1^2+\eta_2^2=\frac{2}{1+\mu}.$$

Taking into account that η_1^2 and η_2^2 are the roots of the quadratic equation, we obtain

$$\frac{2(\kappa+\kappa\mu+\mu)}{\mu^2+2\mu}=\frac{2}{1+\mu},$$

which implies that

$$\kappa=\frac{\mu}{(1+\mu)^2}.$$

EXERCISE 2.9. Find the solution of example 2.9 by the Laplace transform and show that it is equal to the solution found by the modal decomposition.

Solution. Dividing equations (2.22) describing the motion of these coupled pendulums by ml^2, we rewrite them as

$$\ddot{\varphi}_1-\chi(\dot{\varphi}_2-\dot{\varphi}_1)+\omega_0^2\varphi_1-\alpha(\varphi_2-\varphi_1)=0,$$
$$\ddot{\varphi}_2+\chi(\dot{\varphi}_2-\dot{\varphi}_1)+\omega_0^2\varphi_2+\alpha(\varphi_2-\varphi_1)=f(t),$$

where

$$\omega_0 = \sqrt{g/l}, \quad \chi = c/m, \quad \alpha = \frac{k}{4m}, \quad f(t) = \frac{p(t)}{ml}.$$

Let us look for the particular solution of these equations satisfying the homogeneous initial conditions. Applying the Laplace transform to both sides of these equations, we obtain

$$s^2\Phi_1 - \chi s(\Phi_2 - \Phi_1) + \omega_0^2\Phi_1 - \alpha(\Phi_2 - \Phi_1) = 0,$$
$$s^2\Phi_2 + \chi s(\Phi_2 - \Phi_1) + \omega_0^2\Phi_2 + \alpha(\Phi_2 - \Phi_1) = F(s),$$

where $\Phi_1(s)$, $\Phi_2(s)$, and $F(s)$ are the Laplace images of $\varphi_1(t)$, $\varphi_2(t)$, and $f(t)$, respectively. The latter equations can be represented in the matrix form as

$$\begin{pmatrix} s^2 + \chi s + \omega_0^2 + \alpha & -\chi s - \alpha \\ -\chi s - \alpha & s^2 + \chi s + \omega_0^2 + \alpha \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ F(s) \end{pmatrix}.$$

Solving these equation, we find that

$$\Phi_1 = \frac{F(s)(\chi s + \alpha)}{(s^2 + \chi s + \omega_0^2 + \alpha)^2 - (\chi s + \alpha)^2}, \quad \Phi_2 = \frac{F(s)(s^2 + \chi s + \omega_0^2 + \alpha)}{(s^2 + \chi s + \omega_0^2 + \alpha)^2 - (\chi s + \alpha)^2}.$$

To compare with the solution obtained by the modal decomposition let us consider the image functions

$$\Xi_1 = \frac{1}{\sqrt{2}}(\Phi_1 + \Phi_2) = \frac{1}{\sqrt{2}} \frac{F(s)}{s^2 + \omega_0^2},$$

and

$$\Xi_2 = \frac{1}{\sqrt{2}}(\Phi_2 - \Phi_1) = \frac{1}{\sqrt{2}} \frac{F(s)}{s^2 + \omega_0^2 + 2\chi s + 2\alpha} = \frac{1}{\sqrt{2}} \frac{F(s)}{s^2 + \omega_2^2 + 2\chi s}.$$

It is easy to see that the original functions $\xi_1(t)$ and $\xi_2(t)$ corresponding to these image functions satisfy the differential equations

$$\ddot{\xi}_1 + \omega_1^2\xi_1 = \frac{1}{\sqrt{2}} \frac{p(t)}{ml},$$
$$\ddot{\xi}_2 + \frac{2c}{m}\dot{\xi}_2 + \omega_2^2\xi_2 = \frac{1}{\sqrt{2}} \frac{p(t)}{ml}.$$

Thus, the solution obtained by the Laplace transform coincides with the solution obtained by the modal decomposition.

EXERCISE 2.10. A point-mass m moves in the space under the action of three springs of stiffnesses k_1, k_2, and k_3 whose axes do not lie in one plane (see Fig. 2.21). The equilibrium position of the point-mass is chosen as the origin of the coordinate system, while $\mathbf{n}_1$, $\mathbf{n}_2$, and $\mathbf{n}_3$ denote the unit vectors along the spring axes. Derive the equation of small vibrations for this oscillator and determine the eigenfrequencies.

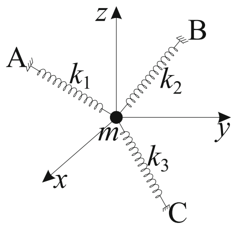

Fig. 2.21 Mass-spring oscillator with 3 degrees of freedom

Solution. Let $\mathbf{r} = (x, y, z)$ be the position vector of the point-mass. We write down its kinetic energy

$$K(\dot{\mathbf{r}}) = \frac{1}{2} m \dot{\mathbf{r}} \cdot \dot{\mathbf{r}} = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2).$$

The potential energy of the springs reads

$$U(\mathbf{r}) = \frac{1}{2}[k_1(\Delta l_1)^2 + k_2(\Delta l_2)^2 + k_3(\Delta l_3)^2],$$

where Δl_i is the change of length of i-th spring. Let l_{0i} be the original length of the springs. Then the position vectors of points A, B, C are $\mathbf{r}_i = l_{0i}\mathbf{n}_i$, $i = 1,2,3$, respectively. The change of length of i-th spring equals

$$\Delta l_i = l_i - l_{0i} = \sqrt{(l_{0i}\mathbf{n}_i - \mathbf{r}) \cdot (l_{0i}\mathbf{n}_i - \mathbf{r})} - l_{0i}.$$

Using the smallness of $\mathbf{r}$, it is easy to see that

$$\sqrt{(l_{0i}\mathbf{n}_i - \mathbf{r}) \cdot (l_{0i}\mathbf{n}_i - \mathbf{r})} \approx l_{0i}\sqrt{1 - \frac{2}{l_{0i}}\mathbf{n}_i \cdot \mathbf{r}} \approx l_{0i} - \mathbf{n}_i \cdot \mathbf{r}.$$

Thus,

$$\Delta l_i = -\mathbf{n}_i \cdot \mathbf{r},$$

and the potential energy of the springs becomes

$$U(\mathbf{r}) = \frac{1}{2}[k_1(\mathbf{n}_1 \cdot \mathbf{r})^2 + k_2(\mathbf{n}_2 \cdot \mathbf{r})^2 + k_3(\mathbf{n}_3 \cdot \mathbf{r})^2].$$

Now Lagrange's equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{r}}} - \frac{\partial L}{\partial \mathbf{r}} = 0$$

yields the equation of motion

$$m\ddot{\mathbf{r}} + k_1\mathbf{n}_1(\mathbf{n}_1 \cdot \mathbf{r}) + k_2\mathbf{n}_2(\mathbf{n}_2 \cdot \mathbf{r}) + k_3\mathbf{n}_3(\mathbf{n}_3 \cdot \mathbf{r}) = 0.$$

To find the eigenfrequencies, we look for the solution in the form

$$\mathbf{r}(t) = \hat{\mathbf{r}}e^{i\omega t},$$

where $\hat{\mathbf{r}}$ is a constant vector. Substituting this solution Ansatz into the equation of motion, we obtain

$$(-m\omega^2\mathbf{I} + \mathbf{K})\hat{\mathbf{r}} = 0.$$

Here $\mathbf{I}$ is the 3×3 identity matrix, and $\mathbf{K}$ is the stiffness matrix with the components

$$K_{ij} = k_1 n_{1i} n_{1j} + k_2 n_{2i} n_{2j} + k_3 n_{3i} n_{3j}.$$

Therefore, the eigenfrequencies should be found from the equation

$$\det(-m\omega^2\mathbf{I} + \mathbf{K}) = 0.$$

This equation can be simplified if the unit vectors $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ are mutually orthogonal. By choosing the coordinate system with these vectors as basis vectors, the stiffness matrix becomes also diagonal

$$\mathbf{K} = \begin{pmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{pmatrix}.$$

Thus, in this case the equations become uncoupled and the eigenfrequencies are given by

$$\omega_j = \sqrt{k_j/m}, \quad j = 1,2,3.$$

Correspondingly, the eigenvectors are $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$.

EXERCISE 2.11. A pre-stretched string contains three equal and equally spaced point-masses m (see Fig. 2.22). The tension in the string is assumed to be large so that for small lateral displacements of the point-masses it does not change appreciably. Derive the equation of small lateral vibration and determine the eigenfrequencies.

Solution. Let the displacements of the point-masses from their equilibrium positions be x_1, x_2, and x_3 (see Fig. 2.22). The kinetic energy of the point-masses is

$$K = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2).$$

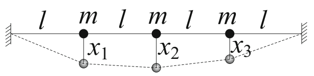

Fig. 2.22 Pre-stretched string with 3 point-masses

We denote the tension in the string by S. Since S is large, the potential energy of the string equals

$$U = S(\Delta l_1 + \Delta l_2 + \Delta l_3 + \Delta l_4),$$

where Δl_i are the changes of lengths of the string segments. Let us express these changes in terms of x_i

$$\begin{aligned}
\Delta l_1 &= \sqrt{l^2 + x_1^2} - l = l(\sqrt{1 + (x_1/l)^2} - 1) \approx \frac{1}{2} l (\frac{x_1}{l})^2, \\
\Delta l_2 &= \sqrt{l^2 + (x_2 - x_1)^2} - l \approx \frac{1}{2} l (\frac{x_2 - x_1}{l})^2, \\
\Delta l_3 &= \sqrt{l^2 + (x_3 - x_2)^2} - l \approx \frac{1}{2} l (\frac{x_3 - x_2}{l})^2, \\
\Delta l_4 &= \sqrt{l^2 + x_3^2} - l \approx \frac{1}{2} l (\frac{x_3}{l})^2.
\end{aligned}$$

Here the smallness of x_i compared with l as well as the formula $\sqrt{1+\varepsilon} \approx 1 + \frac{1}{2}\varepsilon$ are used. Introducing $x_0 = x_4 = 0$, we may present the potential energy in the form

$$U = \frac{S}{2l}[(x_1 - x_0)^2 + (x_2 - x_1)^2 + (x_3 - x_2)^2 + (x_4 - x_3)^2].$$

With the Lagrange function $L = K - U$ it is easy to derive the equations of motion

$$m\ddot{x}_j + k(x_j - x_{j-1}) + k(x_j - x_{j+1}) = 0, \quad j = 1,2,3,$$

where $k = S/l$. We look for the solution of these coupled equations in the form

$$x_j = \hat{x}_j \cos(\omega t - \phi),$$

where $\hat{x}_j$ are the amplitudes of vibrations. With this Ansatz we reduce the differential equations to the algebraic equations

$$(2k - \omega^2 m)\hat{x}_j - k(\hat{x}_{j-1} + \hat{x}_{j+1}) = 0,$$

or to

$$(2 - \eta^2)\hat{x}_j - (\hat{x}_{j-1} + \hat{x}_{j+1}) = 0,$$

where $\eta^2 = \omega^2 m/k$. As the amplitudes of vibrations are determined up to an arbitrary constant factor, we normalize them by

$$\kappa_j = \frac{\hat{x}_j}{\hat{x}_1}, \quad j = 0,1,2,3,4.$$

The above system becomes

$$(2-\eta^2)\kappa_j - (\kappa_{j-1}+\kappa_{j+1}) = 0.$$

Thus, knowing $\kappa_0 = 0$, $\kappa_1 = 1$, we can successively determine other κ_j according to

$$\kappa_j = (2-\eta^2)\kappa_{j-1} - \kappa_{j-2}, \quad j = 2,3,4.$$

So,

$$\kappa_2 = -\eta^2+2, \quad \kappa_3 = \eta^4 - 4\eta^2 + 3, \quad \kappa_4 = -\eta^6 + 6\eta^4 - 10\eta^2 + 4.$$

Since $\kappa_4 = 0$, we obtain the following equation to determine the eigenfrequencies

$$-\eta^6 + 6\eta^4 - 10\eta^2 + 4 = 0.$$

The alternative method of solution is based on the following Ansatz

$$\hat{x}_j = C \sin j\alpha.$$

This Ansatz satisfies the boundary condition $\hat{x}_0 = 0$. The other boundary condition $\hat{x}_4 = 0$ will be satisfied if

$$4\alpha = k\pi, \quad k = 1,2,3 \quad \Rightarrow \quad \alpha = \frac{k\pi}{4}.$$

On the other side, substituting the above Ansatz into the algebraic equations for $\hat{x}_j$, we obtain

$$C \sin j\alpha (2 - \eta^2 - 2\cos\alpha) = 0.$$

Since C cannot be zero, we obtain the equation to determine the eigenfrequencies

$$\eta^2 = 2(1-\cos\alpha) = 4\sin^2\frac{\alpha}{2}.$$

Denoting by $\omega_0 = \sqrt{k/m} = \sqrt{S/ml}$, we can write

$$\omega_k = 2\omega_0 \sin\frac{\alpha}{2} = 2\omega_0 \sin\frac{k\pi}{8}, \quad k = 1,2,3.$$

EXERCISE 2.12. The free vibrations of an airplane can be described in a simplified model with three degrees of freedom representing the motion of the fuselage and the wings which are connected with the fuselage by the spiral springs of stiffnesses k_1 and k_2 (see Fig. 2.23). Derive the equations of small vibrations. Under the assumptions of symmetry $\theta_1 = \theta_2 = \theta$, $m_1 = m_2 = m$, and $k_1 = k_2 = k$, find the eigenfrequencies of vibrations.

Solution. Let the changes in angles of the wings be φ_1 and φ_2. We write down the kinetic energy of this system

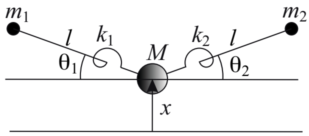

Fig. 2.23 A primitive model of an airplane with 3 degrees of freedom

$$K = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2.$$

The velocity of the point-mass m_1 is $\mathbf{v}_1 = \mathbf{v}_M + \mathbf{v}_{rel}$, where $\mathbf{v}_{rel}$ denotes the relative velocity. Since the vertical and horizontal components of this relative velocity are $l\dot{\varphi}_1 \cos\theta_1$ and $-l\dot{\varphi}_1 \sin\theta_1$, respectively, we have

$$v_1^2 = \dot{x}^2 + 2l\cos\theta_1 \dot{x}\dot{\varphi}_1 + l^2\dot{\varphi}_1^2.$$

Similarly,

$$v_2^2 = \dot{x}^2 + 2l\cos\theta_2 \dot{x}\dot{\varphi}_2 + l^2\dot{\varphi}_2^2.$$

Thus, the kinetic energy of the system becomes

$$K = \frac{1}{2}(M + m_1 + m_2)\dot{x}^2 + m_1 l\cos\theta_1 \dot{x}\dot{\varphi}_1 + \frac{1}{2}m_1 l^2\dot{\varphi}_1^2 + m_2 l\cos\theta_2 \dot{x}\dot{\varphi}_2 + \frac{1}{2}m_2 l^2\dot{\varphi}_2^2.$$

It is easy to show that the static spring forces, the gravitational forces, and the aerodynamic force due to the steady state flow do not contribute to the potential energy. Therefore

$$U = \frac{1}{2}k_1\varphi_1^2 + \frac{1}{2}k_2\varphi_2^2.$$

With $L = K - U$ we derive from Lagrange's equations

$$\begin{aligned}
(M + m_1 + m_2)\ddot{x} + m_1 l\cos\theta_1 \ddot{\varphi}_1 + m_2 l\cos\theta_2 \ddot{\varphi}_2 &= 0,\\
m_1 l^2\ddot{\varphi}_1 + m_1 l\cos\theta_1 \ddot{x} + k_1\varphi_1 &= 0,\\
m_2 l^2\ddot{\varphi}_2 + m_2 l\cos\theta_2 \ddot{x} + k_2\varphi_2 &= 0.
\end{aligned}$$

Let $\mathbf{q}^T = (x, \varphi_1, \varphi_2)$ and $m_T = M + m_1 + m_2$. Then we can represent the equations of free vibrations of this system in the matrix form

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = 0,$$

where

$$\mathbf{M} = \begin{pmatrix} m_T & m_1 l\cos\theta_1 & m_2 l\cos\theta_2 \\ m_1 l\cos\theta_1 & m_1 l^2 & 0 \\ m_2 l\cos\theta_2 & 0 & m_2 l^2 \end{pmatrix}, \quad \mathbf{K} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & k_1 & 0 \\ 0 & 0 & k_2 \end{pmatrix}.$$

Seeking the particular solution in the form

$$\mathbf{q} = \hat{\mathbf{q}} e^{i\omega t},$$

we obtain the eigenvalue problem

$$(-\omega^2 \mathbf{M} + \mathbf{K})\hat{\mathbf{q}} = \mathbf{0}.$$

The related characteristic equation

$$\det(-\omega^2 \mathbf{M} + \mathbf{K}) = \begin{vmatrix} -\omega^2 m_T & -\omega^2 m_1 l \cos\theta_1 & -\omega^2 m_2 l \cos\theta_2 \\ -\omega^2 m_1 l \cos\theta_1 & -\omega^2 m_1 l^2 + k_1 & 0 \\ -\omega^2 m_2 l \cos\theta_2 & 0 & -\omega^2 m_2 l^2 + k_2 \end{vmatrix} = 0$$

can be written in the form

$$\omega^2 \left[l^4 (m_T m_1 m_2 - m_1 m_2^2 \cos^2\theta_2 - m_2 m_1^2 \cos^2\theta_1)\omega^4 - l^2 (m_T (m_1 k_2 + m_2 k_1) \right. \\ \left. - m_2^2 k_1 \cos^2\theta_2 - m_1^2 k_2 \cos^2\theta_1)\omega^2 + m_T k_1 k_2 \right] = 0.$$

Thus, there is always the zero frequency corresponding to the mode of vertical motion of the airplane as a rigid body with $x \neq 0$, $\varphi_1 = \varphi_2 = 0$. In the symmetric case ($\theta_1 = \theta_2 = \theta$, $m_1 = m_2 = m$, and $k_1 = k_2 = k$), the remaining factor in the square brackets reduces to

$$m^2 l^4 (m_T - 2m\cos^2\theta)\omega^4 + 2kml^2(m\cos^2\theta - m_T)\omega^2 + m_T k^2 = 0.$$

This yields the following eigenfrequencies

$$\omega_1^2 = \frac{k}{ml^2}, \quad \omega_2^2 = \frac{km_T}{ml^2(m_T - m - m\cos 2\theta)}.$$