Chapter 1
Single Oscillator

This chapter deals with small vibrations of the simplest mechanical systems, namely of oscillators having only one degree of freedom. The most general and effective method of solution is the Laplace transform which is based entirely on the linear superposition principle.

1.1 Harmonic Oscillator

Differential Equation of Motion. The derivation of the equation of motion is the first, and at the same time, most responsible step toward solution of a real problem. Having derived the right equation, we have won already half the battle. In contrary, having arrived at some wrong equation, all of our further efforts in solving it will end in nothing but disaster. To derive the equation of motion we must

- idealize the real physical problem,
- apply the first principles of dynamics.

There are two equivalent methods of deriving the equation of motion based on the first principles of dynamics: the force method and the energy method. In the force method, we first free parts of the system under consideration from the surrounding, then draw the free-body diagram with all acting forces, and finally apply Newton's law to each degree of freedom. The energy method is based on Hamilton's variational principle leading to Lagrange's equations. Since we are dealing then with only one function, the energy method turns out to be simpler and much more effective, especially for systems with many degrees of freedom and with various constraints. In order to demonstrate their equivalence, let us begin with simple examples.

EXAMPLE 1.1. Mass-spring oscillator. A point-mass m moves horizontally under the action of a massless spring of stiffness k (see Fig. 1.1). Derive the equation of motion for this oscillator.

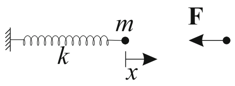

Fig. 1.1 Mass-spring oscillator

We see already in the formulation of the problem various idealizations of the real situation: the point-mass is considered instead of a body of finite size, this mass is constrained to move horizontally, the spring is regarded as massless and linearly elastic, the air resistance to motion through viscous damping is neglected etc. How close this simple mathematical model can describe the real physical problem is the matter of experimental verification.

To use the force method we must first free the point-mass from the spring, then draw the free-body diagram (see Fig. 1.1, right), and finally apply Newton's second law (mass times acceleration = force) in the x-direction

$$m\ddot{x} = \sum F_x = -kx,$$

where the dot denotes the time derivative. Bringing the spring force $-kx$ to the left-hand side and dividing by m, we transform the equation of motion to the standard form

$$\ddot{x} + \omega_0^2 x = 0, \quad \omega_0 = \sqrt{\frac{k}{m}}. \tag{1.1}$$

The energy method is based on Hamilton's variational principle of least action[1] which states that, among all admissible motions $x(t)$ of the point-mass satisfying the initial and end conditions

$$x(t_0) = x_0, \quad x(t_1) = x_1,$$

the true motion is the extremal of the action functional

$$I[x(t)] = \int_{t_0}^{t_1} L(x,\dot{x})\,dt.$$

The direct consequence of Hamilton's variational principle is Lagrange's equation (see the derivation in Section 2.4)

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0.$$

Thus, all we need is a single function $L(x,\dot{x})$, called Lagrange function, which is given by

$$L(x,\dot{x}) = K(\dot{x}) - U(x),$$

where $K(\dot{x})$ is the kinetic energy and $U(x)$ the potential energy. As soon as we have it, the job is done, provided one knows how to differentiate functions. In our example

[1] See [21] and the detailed discussion in [29]. One may also read a curious and fascinating story of Feynman about how he learned Hamilton's principle of least action and tried later to *explain* it from the quantum mechanics and path integral in The Feynman Lectures on Physics [14].

$$K(\dot{x}) = \frac{1}{2}m\dot{x}^2, \quad U(x) = \frac{1}{2}kx^2.$$

Computing the partial derivatives of this Lagrange function

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}, \quad \frac{\partial L}{\partial x} = -kx,$$

and substituting them into Lagrange's equation, we obtain

$$m\ddot{x} + kx = 0,$$

which can again be reduced to the normal form (1.1).

EXAMPLE 1.2. Mathematical pendulum. A point-mass m, connected with a fixed support O by a rigid and massless bar of length l, rotates in the (x,y)-plane about O under the action of gravity (see Fig. 1.2). Derive the equation of motion for this pendulum.

We see that, again, several idealizations are made to simplify the real physical pendulum: the whole mass is concentrated in the point, the carrying bar is rigid and massless, the air resistance to motion through viscous damping is neglected.

In the force method we free the point-mass from the carrying bar, draw the free-body diagram and apply Newton's law in the tangential direction

$$ma_\tau = \sum F_\tau = -mg\sin\varphi.$$

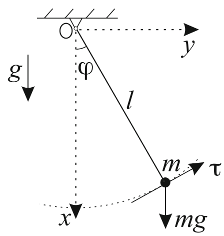

Fig. 1.2 Mathematical pendulum

The force of the bar acting on the point-mass does not contribute to this equation because it is in the radial direction. Substituting the tangential acceleration $a_\tau = l\ddot{\varphi}$ into this equation, bringing the force term $-mg\sin\varphi$ to the left-hand side, and dividing by ml, we obtain

$$\ddot{\varphi} + \omega_0^2 \sin\varphi = 0, \quad \omega_0 = \sqrt{\frac{g}{l}}.$$

For small vibrations the angle φ (measured in radian) is small compared with 1, so we can linearize this equation by approximating $\sin\varphi \approx \varphi$ to obtain

$$\ddot{\varphi} + \omega_0^2 \varphi = 0, \tag{1.2}$$

which is identical in form with equation (1.1).

Alternatively, one can free the point-mass together with the rigid bar from the support and apply the moment equation about the z-axis (which is the consequence of Newton's law) to this system

$$\frac{d}{dt}(ml^2\dot{\varphi}) = \sum M_z = -mgl\sin\varphi.$$

Since the mass is concentrated in the point, its moment of inertia about O is ml^2. In case of a real physical pendulum (rotation of a body about O) the moment of inertia about O is given by $J_O = J_S + mr^2$, where J_S is the moment of inertia about the center of mass S, and r the distance between O and S (see exercise 1.2). The support reaction in O does not contribute to the moment equation, because its line of action goes through O. For small vibrations we obtain from here equation (1.2).

To use the energy method we write down the kinetic energy

$$K(\dot{\varphi}) = \frac{1}{2}ml^2\dot{\varphi}^2,$$

and the potential energy

$$U(\varphi) = mgh = mgl(1-\cos\varphi).$$

Note that the zero level of potential energy (which can be chosen arbitrarily) corresponds to the equilibrium state $\varphi = 0$. Thus, the Lagrange function is

$$L(\varphi,\dot{\varphi}) = \frac{1}{2}ml^2\dot{\varphi}^2 - mgl(1-\cos\varphi).$$

For small vibrations $\varphi \ll 1$, therefore we can approximate $1-\cos\varphi \approx \varphi^2/2$ and write

$$L(\varphi,\dot{\varphi}) = \frac{1}{2}ml^2\dot{\varphi}^2 - \frac{1}{2}mgl\varphi^2.$$

Substituting this into Lagrange's equation

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\varphi}} - \frac{\partial L}{\partial\varphi} = 0,$$

we obtain again the equation of motion of mathematical pendulum.

EXAMPLE 1.3. Rotating disk. A rigid disk rotates about the z-axis under the action of a spiral spring of stiffness k (see Fig. 1.3). Derive the equation of motion of the disk.

This example represents a primitive model of a mechanical clock. In the force method we free the disk and the rotation axis from the supports and the spiral spring, draw the free-body diagram, and apply the moment equation about the z-axis

$$\frac{d}{dt}(J_z\dot{\varphi}) = \sum M_z = -k\varphi, \tag{1.3}$$

where J_z is the moment of inertia of the system disk plus rotation axis about the z-axis. The reaction forces from the supports do not contribute to this moment equation because their lines of action cut the z-axis. Bringing the spring moment $-k\varphi$ to the left-hand side and dividing by J_z, we obtain equation (1.2), where $\omega_0^2 = k/J_z$.

To use the energy method we write

$$K(\dot{\varphi}) = \frac{1}{2}J_z\dot{\varphi}^2, \quad U(\varphi) = \frac{1}{2}k\varphi^2$$

for the kinetic and potential energies, respectively. This leads again to (1.3).

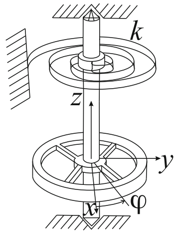

Fig. 1.3 Rotating disk

Solution. Note that the equation of motion of harmonic oscillator

$$\ddot{x} + \omega_0^2 x = 0 \tag{1.4}$$

is linear. So, if we know two linearly independent particular solutions of this equation, then we can construct the general solution by their linear combination in accordance with the superposition principle. It is easy to check that

$$\cos\omega_0 t \quad \text{and} \quad \sin\omega_0 t$$

are the particular solutions of (1.4). Therefore the general solution reads

$$x(t) = A\cos\omega_0 t + B\sin\omega_0 t. \tag{1.5}$$

The unknown coefficients A and B must be found from the initial conditions

$$x(0) = x_0, \quad \dot{x}(0) = v_0.$$

Thus,

$$A = x_0, \quad B = \frac{v_0}{\omega_0}.$$

Alternatively, we can present the solution in form of one harmonic cosine function

$$x(t) = a\cos(\omega_0 t - \phi). \tag{1.6}$$

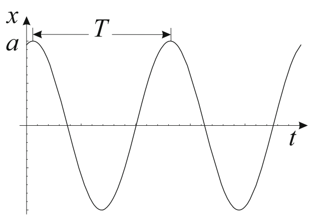

Fig. 1.4 Harmonic motion

In this form a has the meaning of the amplitude of vibration, ω_0 the eigenfrequency, and ϕ the initial phase. Using the addition formula for $\cos(\omega_0 t - \phi)$ we write

$$x(t) = a(\cos\phi\cos\omega_0 t + \sin\phi\sin\omega_0 t).$$

Comparing this with (1.5), we find the relations between a, ϕ and A, B

$$a = \sqrt{A^2 + B^2} = \sqrt{x_0^2 + \frac{v_0^2}{\omega_0^2}}, \quad \tan\phi = \frac{B}{A} = \frac{v_0}{x_0\omega_0}.$$

Fig. 1.4 shows the graph of $x(t)$. The distance between two neighboring maxima (or minima) of this periodic function is called a period T of vibration. Since the period of cosine is 2π,

$$T = \frac{2\pi}{\omega_0}.$$

Phase Portrait. Let the velocity $\dot{x}$ be denoted by

$$y = \dot{x}.$$

Then each state of a single oscillator at fixed t corresponds to one point (x, y) of the so-called phase plane. As t changes this point moves in the phase plane along the curve called a phase curve. For the free vibration of harmonic oscillator we have from (1.6)

$$x = a\cos(\omega_0 t - \phi), \quad y = \dot{x} = -a\omega_0 \sin(\omega_0 t - \phi). \tag{1.7}$$

Consequently, the phase curves satisfy the equation

$$\frac{x^2}{a^2} + \frac{y^2}{a^2\omega_0^2} = 1, \tag{1.8}$$

which describes ellipses with the aspect ratio $1 : \omega_0$. Note that (1.8) can also be obtained from the conservation of energy

$$K(\dot{x}) + U(x) = E_0, \tag{1.9}$$

which is the consequence of Lagrange's equation (see Section 2.4). Indeed, consider for example the mass-spring oscillator for which the energy conservation takes the form

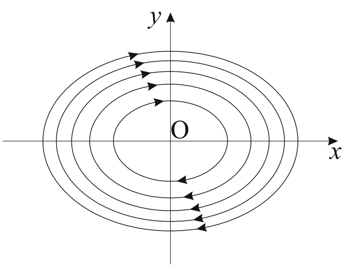

Fig. 1.5 Phase portrait of harmonic oscillator

$$\frac{1}{2}my^2 + \frac{1}{2}kx^2 = \frac{1}{2}mv_0^2 + \frac{1}{2}kx_0^2 = \frac{ka^2}{2}.$$

Dividing this equation by the constant on the right-hand side, we arrive again at (1.8). With (1.8) we can express $y = \dot{x}$ in terms of x and integrate it to obtain the solution (1.6).

Fig. 1.5 shows the phase curves of the harmonic oscillator. In general there is no more than one phase curve passing through a given point of the phase plane. Since $y = \dot{x} > 0$ in the upper half-plane and $y = \dot{x} < 0$ in the lower half-plane, the phase curves must run from left to right in the upper half-plane and from right to left in the lower half-plane as time increases. All phase curves cut the x-axis at right angle, with points of intersection corresponding to maxima or minima of $x(t)$ which are the turning points. The origin O of the phase plane is the fixed point corresponding to the stable equilibrium state. For the harmonic oscillator this fixed point is called a center.

Energy Balance. As we see from (1.9), the total energy of the harmonic oscillator is conserved. Let us analyze the change of its kinetic and potential energies during the vibration. Substituting $x(t)$ from (1.7) into the potential energy $U(x)$, we get

$$U(x) = \frac{1}{2}kx^2 = \frac{ka^2}{2}\cos^2(\omega_0 t - \phi) = \frac{ka^2}{4}[1 + \cos(2\omega_0 t - 2\phi)].$$

Similarly, with $\dot{x}(t)$ from (1.7) we obtain

$$K(\dot{x}) = \frac{1}{2}m\dot{x}^2 = \frac{ma^2\omega_0^2}{2}\sin^2(\omega_0 t - \phi) = \frac{ka^2}{4}[1 - \cos(2\omega_0 t - 2\phi)].$$

Thus, the kinetic and potential energies oscillate with the same amplitude which is equal to the total energy $E_0 = ka^2/2$, but with the double frequency $2\omega_0$. Fig. 1.6 shows the change of kinetic and potential energies from which it is seen that they oscillate in counter-phases so that their sum remains constant, in full agreement with the conservation of energy.

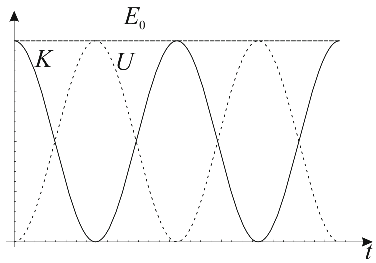

Fig. 1.6 Energy change: a) bold line: kinetic energy, b) dashed line: potential energy, c) horizontal line: total energy

1.2 Damped Oscillator

Differential Equation of Motion. Both the force and the energy methods can again be applied to derive the equation of motion for damped oscillators. However, in the energy method a new function describing the dissipation potential of the damper has to be introduced. We consider two examples.

EXAMPLE 1.4. Mass-spring-damper oscillator. A mass m moves horizontally under the action of a spring of stiffness k and a damper with a damping constant c (see Fig. 1.7). Derive the equation of motion for this oscillator.

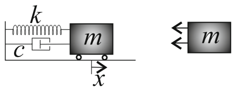

Fig. 1.7 Mass-spring-damper oscillator

To apply the force method we note that the only difference compared with example 1.1 is the additional force from the damper which is proportional to the velocity $\dot{x}$ (see the free-body diagram in Fig. 1.7, right). Thus, Newton's law now reads

$$m\ddot{x} = \sum F_x = -kx - c\dot{x}.$$

Bringing the two terms on the right-hand side to the left-hand side, we obtain

$$m\ddot{x} + c\dot{x} + kx = 0. \tag{1.10}$$

The energy method is based on the following variational principle for dissipative systems: among all admissible motions $x(t)$ constrained by the initial and end conditions

$$x(t_0) = x_0, \quad x(t_1) = x_1,$$

the true motion satisfies the variational equation[2]

$$\delta \int_{t_0}^{t_1} L(x,\dot{x})\,dt - \int_{t_0}^{t_1} \frac{\partial D}{\partial \dot{x}} \delta x\,dt = 0. \tag{1.11}$$

Thus, a new function $D(x,\dot{x})$, called dissipation function, appears such that the damping force f_r is expressed by

$$f_r = -c\dot{x} = -\frac{\partial D}{\partial \dot{x}}.$$

The direct consequence of (1.11) is modified Langrange's equation for dissipative systems (see Section 2.4)

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} + \frac{\partial D}{\partial \dot{x}} = 0. \tag{1.12}$$

We see that the behavior of any dissipative mechanical system is governed by two functions, namely, the Lagrange function $L(x,\dot{x})$ and the dissipation function $D(x,\dot{x})$. In our example

[2] This variational equation originates from d'Alembert's principle in dynamics [8,29], where the last term corresponds to the virtual work done by the damping force expressed in terms of the dissipation function [45].

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2, \quad D = \frac{1}{2}c\dot{x}^2,$$

so, substituting this into (1.12), we derive again the equation of motion (1.10).

EXAMPLE 1.5. Mathematical pendulum with spring and damper. Derive the equation of small vibration for the mathematical pendulum connected with a spring and a damper (see Fig. 1.8).

This model is equivalent to that of the pendulum with the spring and with the air resistance since, in reality, the air acts as a damper with viscous damping. In the force method we must add the forces of spring and damper to the free-body diagram compared with that of the mathematical pendulum in example 1.2. Taking into account the smallness of φ, the moment equation about the z-axis reads

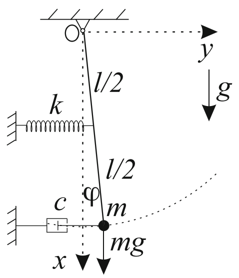

Fig. 1.8 Spring-damper-pendulum

$$ml^2\ddot{\varphi} = \sum M_z = -mgl\varphi - k\frac{l^2}{4}\varphi - cl^2\dot{\varphi}.$$

Bringing all terms to the left-hand side and dividing by l^2, we get

$$m\ddot{\varphi} + c\dot{\varphi} + (\frac{mg}{l} + \frac{k}{4})\varphi = 0. \tag{1.13}$$

This is identical in form with equation (1.10).

To use the energy method we must include in the Lagrange function already found in example 1.2 for small vibrations an additional term associated with the energy of the spring

$$L(\varphi, \dot{\varphi}) = \frac{1}{2}ml^2\dot{\varphi}^2 - \frac{1}{2}mgl\varphi^2 - \frac{1}{2}k\left(\frac{l\varphi}{2}\right)^2.$$

Here, the change in length of the spring, due to the smallness of φ, is approximated by $l\varphi/2$ (see Fig. 1.8). The dissipation function must be a quadratic function of the velocity $l\dot{\varphi}$

$$D(\dot{\varphi}) = \frac{1}{2}c(l\dot{\varphi})^2 = \frac{1}{2}cl^2(\dot{\varphi})^2.$$

Substituting these formulas into modified Langrange's equation for dissipative systems

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} - \frac{\partial L}{\partial \varphi} + \frac{\partial D}{\partial \dot{\varphi}} = 0,$$

we derive the equation

$$ml^2\ddot{\varphi} + cl^2\dot{\varphi} + (mgl + k\frac{l^2}{4})\varphi = 0,$$

which, after division by l^2, takes the form (1.13).

Reduction to the Standard Form. Let us divide the equation of motion (1.10) by k

$$\frac{1}{\omega_0^2}\ddot{x}+\frac{c}{k}\dot{x}+x=0, \quad \omega_0=\sqrt{\frac{k}{m}}. \tag{1.14}$$

We introduce now the dimensionless time $\tau=\omega_0 t$, in terms of which the first and second derivatives of x become

$$\dot{x}=\frac{dx}{dt}=\frac{dx}{d\tau}\frac{d\tau}{dt}=\omega_0 x',$$
$$\ddot{x}=\frac{d\dot{x}}{dt}=\frac{d\dot{x}}{d\tau}\frac{d\tau}{dt}=\omega_0^2 x''.$$

Here the prime denotes the derivative with respect to τ. Substituting these formulas into (1.14), we obtain the equation of motion in standard form

$$x''+2\delta x'+x=0, \tag{1.15}$$

where the positive coefficient

$$\delta=\frac{c\omega_0}{2k}=\frac{c}{2m\omega_0}=\frac{c}{2\sqrt{km}}$$

is called Lehr's damping ratio.

Solution. We seek a particular solution of (1.15) in the form

$$x=e^{s\tau}.$$

Substituting this Ansatz into (1.15)

$$(s^2+2\delta s+1)e^{s\tau}=0,$$

we see that, since the factor $e^{s\tau}$ is not equal to zero, s must satisfy the characteristic equation

$$s^2+2\delta s+1=0. \tag{1.16}$$

The quadratic equation (1.16) has two roots

$$s_{1,2}=-\delta\pm\sqrt{\delta^2-1}.$$

The character of roots and consequently of the solutions depends on whether a) $0<\delta<1$, b) $\delta>1$, or c) $\delta=1$. We analyze now these special cases.

Case a. Since $0<\delta<1$, we set $1-\delta^2=\nu^2>0$. In this case the roots are complex-conjugate

$$s_{1,2}=-\delta\pm i\nu.$$

Because $e^{s\tau}=e^{-\delta\tau}e^{i\nu\tau}$ satisfies (1.15) which is the differential equation with real coefficients, its real and imaginary parts

$$e^{-\delta\tau}\cos\nu\tau \quad \text{and} \quad e^{-\delta\tau}\sin\nu\tau$$

also satisfy this equation. The general solution can now be constructed using the linear superposition principle

$$x = e^{-\delta\tau}(A\cos\nu\tau + B\sin\nu\tau).$$

The unknown coefficients A and B must be found from the initial conditions

$$x(0) = x_0, \quad x'(0) = x'_0. \tag{1.17}$$

Thus,

$$A = x_0, \quad B = \frac{x'_0 + \delta x_0}{\nu}.$$

Alternatively, we can present the solution in the form

$$x = a_0 e^{-\delta\tau}\cos(\nu\tau - \phi). \tag{1.18}$$

Using the addition theorem for $\cos(\nu\tau - \phi)$, we then find that

$$a_0 = \sqrt{A^2 + B^2} = \sqrt{x_0^2 + \frac{(x'_0 + \delta x_0)^2}{\nu^2}}, \quad \tan\phi = \frac{B}{A} = \frac{x'_0 + \delta x_0}{\nu x_0}.$$

Case b. Because now $\delta > 1$, we set $\delta^2 - 1 = \kappa^2 > 0$. Thus, there are two real roots of (1.16)

$$s_1 = -\delta + \kappa = -q_1, \quad s_2 = -\delta - \kappa = -q_2,$$

where $q_2 > q_1 > 0$. The corresponding particular solutions of (1.15) are

$$e^{-q_1\tau} \quad \text{and} \quad e^{-q_2\tau}.$$

The general solution reads

$$x = Ae^{-q_1\tau} + Be^{-q_2\tau}.$$

Then the initial conditions (1.17) lead to

$$A = \frac{1}{2\kappa}(x'_0 + q_2 x_0), \quad B = -\frac{1}{2\kappa}(x'_0 + q_1 x_0).$$

Thus,

$$x = \frac{1}{2\kappa}[(x'_0 + q_2 x_0)e^{-q_1\tau} - (x'_0 + q_1 x_0)e^{-q_2\tau}]. \tag{1.19}$$

Case c. This is the degenerate case, where the real roots are equal (the double real root): $s_1 = s_2 = -\delta = -1$. According to the theory of ordinary differential equations [11] the particular solutions should be $e^{-\tau}$ and $\tau e^{-\tau}$. Combining them, we obtain the general solution in the form

$$x = e^{-\tau}(A\tau + B).$$

The initial conditions (1.17) yield

$$A = x_0 + x_0', \quad B = x_0.$$

Thus,

$$x = e^{-\tau}[x_0(1+\tau) + x_0'\tau].$$

Behavior. Having found the solutions in these cases, we can now study their behaviors.

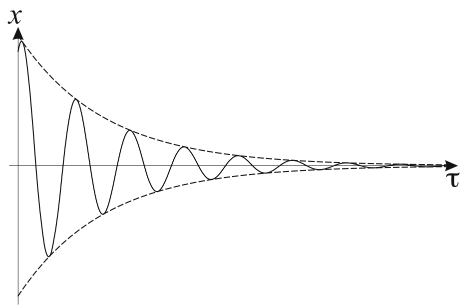

Fig. 1.9 Solid line: motion $x = a_0 e^{-\delta\tau}\cos(\nu\tau - \phi)$, dashed lines: envelopes $x = \pm a_0 e^{-\delta\tau}$

Case a. The motion is classified as damped vibration. Fig. 1.9 shows the plot of $x(\tau)$ (the solid line). Since $|\cos(\nu\tau - \phi)| \le 1$, the motion oscillates between two envelopes $x = \pm a_0 e^{-\delta\tau}$ drawn by the dashed lines in this Figure. Looking at this motion we can recognize two characteristic dimensionless time scales

$$\tau_d = \frac{1}{\delta} \quad \text{and} \quad \tau_c = \frac{2\pi}{\nu} = \frac{2\pi}{\sqrt{1-\delta^2}},$$

or, in the dimension of real time

$$T_d = \frac{1}{\delta\omega_0} = \frac{2m}{c} \quad \text{and} \quad T_c = \frac{2\pi}{\omega_c} = \frac{2\pi}{\omega_0\sqrt{1-\delta^2}} = \frac{T_0}{\sqrt{1-\delta^2}}.$$

The time scale τ_d characterizes the decay rate of amplitude due to damping: the exponent function $e^{-\delta\tau}$ decays after τ_d by the factor $1/e \approx 0,368$, the amplitude of vibration diminishes by 63%. The time scale τ_c tells us about the so-called conditional period T_c of vibration, which is larger (by the factor $1/\sqrt{1-\delta^2}$) than the period T_0 of the corresponding harmonic oscillator.

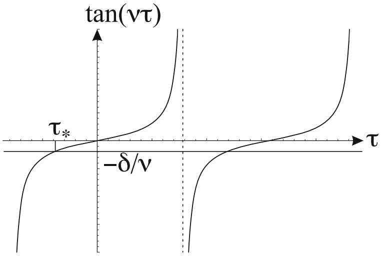

Fig. 1.10 Roots of equation $\tan \nu\tau = -\delta/\nu$

The distance between zeros of $x(\tau)$ (the roots of $\cos(\nu\tau - \phi) = 0$) is π/ν. The points at which $x(\tau)$ touches the envelopes correspond to the roots of the equations

$$\cos(\nu\tau - \phi) = \pm 1.$$

Thus, they lie in the middle between zeros. However these points are not identical with the points at which maxima or minima of $x(\tau)$ are achieved. Its maxima or minima are achieved at time instants satisfying the equation

$$x'(\tau) = -a_0\delta e^{-\delta\tau}\cos(\nu\tau - \phi) - a_0\nu e^{-\delta\tau}\sin(\nu\tau - \phi) = 0,$$

so, they are roots of the equation $\tan(\nu\tau - \phi) = -\delta/\nu$. Assuming for simplicity $\phi = 0$, we find that these roots are displaced from the zeros of the function $\tan \nu\tau$ to the left by the constant amount

$$\tau_* = \arctan(\delta/\nu)/\nu \tag{1.20}$$

on the τ-axis (see Fig. 1.10). Thus, the conditional period of vibration can be read off also from the distance between two maxima or minima.

There is another important characteristic of amplitude decay which can easily be measured by the oscillograph. To introduce it we denote by

$$x_1, x_2, \dots, x_n, \dots$$

the maxima of $x(\tau)$, and by

$$\tau_1, \tau_2, \dots, \tau_n, \dots$$

the corresponding time instants, at which these maxima are achieved. From the behavior of solution we know that

$$x_n = a_0 e^{-\delta\tau_n}\cos(\nu\tau_n - \phi),$$
$$x_{n+1} = a_0 e^{-\delta(\tau_n + \tau_c)}\cos[\nu(\tau_n + \tau_c) - \phi].$$

Dividing x_n by x_{n+1} and using the periodicity of cosine function, we get

$$\frac{x_n}{x_{n+1}} = e^{\delta\tau_c}.$$

We define

$$\vartheta = \ln\frac{x_n}{x_{n+1}} = \delta\tau_c = \frac{2\pi\delta}{\sqrt{1-\delta^2}}$$

as a logarithmic decrement of vibration. Knowing ϑ from measurements we can restore the damping ratio δ according to

$$\delta = \frac{\vartheta}{\sqrt{4\pi^2+\vartheta^2}}.$$

Case b. The motion is overdamped and loses the oscillatory character (it is called therefore an aperiodic motion). The decay rates of exponential functions $e^{-q_1\tau}$ and $e^{-q_2\tau}$ to zero are characterized by two time scales

$$\tau_{d1} = \frac{1}{q_1} = \frac{1}{\delta-\kappa} \quad \text{and} \quad \tau_{d2} = \frac{1}{q_2} = \frac{1}{\delta+\kappa}.$$

To recognize the aperiodic character of motion let us find the instants of time, τ_1 and τ_2, at which $x(\tau_1) = 0$ and $x'(\tau_2) = 0$, respectively. Using (1.19), we derive the following equations for τ_1 and τ_2:

$$e^{2\kappa\tau_1} = 1 - \frac{2\kappa x_0}{x_0' + q_2 x_0},$$
$$e^{2\kappa\tau_2} = 1 + \frac{2\kappa x_0'}{q_1(x_0' + q_2 x_0)}.$$

Since the exponent is a monotonic function, we see that each equation has no more than one root. Thus, oscillatory motion is impossible. If we fix the initial coordinate x_0 and variate the initial velocity x_0', then the solution may have different behaviors depending on the initial velocity as shown in Fig. 1.11.

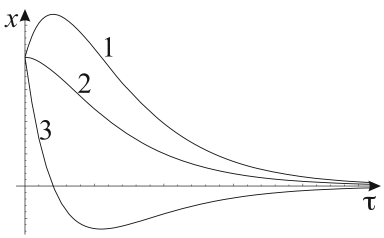

Fig. 1.11 Different aperiodic motions: 1) one root τ_2, 2) no roots for τ_1 and τ_2 (monotone decreasing function $x(\tau)$), 3) one root for τ_1 and τ_2

Case c. This is the limiting case of aperiodic motion. The behavior is similar to the previous case.

Phase Portrait. The phase portraits exhibit different characters in the above cases.

Case a. To find the phase curves in the phase plane (x,y), $y = x'$, we transform the equation of motion in standard form (1.15) to the system of first-order differential equations

$$x' = y, \quad y' = -x - 2\delta y.$$

Thus, the tangent vector to the phase curve at point (x,y) is $(y, -x-2\delta y)$. Fig. 1.12 shows the vector field $(y, -x-2\delta y)$ and one phase curve in the phase plane for $\delta = 0.1$.

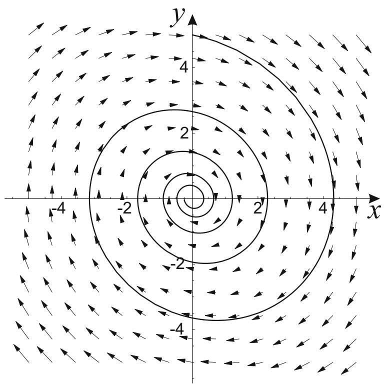

Fig. 1.12 Vector field $(y, -x-2\delta y)$ and a phase curve

We derive the equation for the phase curves from the solution

$$x = a_0 e^{-\delta\tau}\cos(\nu\tau - \phi),$$
$$x' = -a_0 e^{-\delta\tau}[\delta\cos(\nu\tau - \phi) + \nu\sin(\nu\tau - \phi)].$$

Introducing

$$u = \nu x, \quad v = x' + \delta x, \tag{1.21}$$

we obtain

$$u = a_1 e^{-\delta\tau}\cos(\nu\tau - \phi), \quad v = -a_1 e^{-\delta\tau}\sin(\nu\tau - \phi),$$

where $a_1 = \nu a_0$. In terms of the polar coordinates ρ and ϑ

$$u = \rho\cos\vartheta, \quad v = \rho\sin\vartheta,$$

these equations become

$$\rho = a_1 e^{-\delta\tau}, \quad \vartheta = -\nu\tau + \phi.$$

Expressing τ through ϑ by $\tau = -\frac{1}{\nu}(\vartheta - \phi)$, we obtain finally

$$\rho = a_1 e^{\delta\vartheta/\nu} e^{-\delta\phi/\nu} = a_2 e^{\delta\vartheta/\nu}, \tag{1.22}$$

where $a_2 = a_1 e^{-\delta\phi/\nu}$. Equation (1.22) describes the family of logarithmic spirals in the (u,v)-plane. As τ increases, ϑ decreases and the spirals approach the origin.

Coming back to the original coordinates x and y, we have

$$\rho^2 = u^2 + v^2 = \nu^2 x^2 + (y + \delta x)^2 = y^2 + 2\delta xy + x^2,$$
$$\vartheta = \arctan\frac{v}{u} = \arctan\frac{y + \delta x}{\nu x}.$$

Thus, the equation of phase curves in terms of x and y reads

$$y^2 + 2\delta xy + x^2 = a_2^2 e^{2\frac{\delta}{\nu}\arctan\frac{y+\delta x}{\nu x}}.$$

Since the transformation (1.21) from (u,v) to (x,y) is linear, this equation also describes the logarithmic spirals in the (x,y)-plane. All spirals approach the origin as τ goes to infinity. The origin is a fixed point called a (stable) focus.

Case b. To derive the equation of phase curves in the phase plane we use the solution

$$x = Ae^{-q_1\tau} + Be^{-q_2\tau},$$
$$y = x' = -q_1 Ae^{-q_1\tau} - q_2 Be^{-q_2\tau}.$$

Thus, their linear combinations give

$$y + q_1 x = (q_1 - q_2)Be^{-q_2\tau}, \quad y + q_2 x = (q_2 - q_1)Ae^{-q_1\tau}.$$

Raising the first equation to the power q_1 and the second to the power q_2 and comparing them, we obtain

$$(y + q_1 x)^{q_1} = C(y + q_2 x)^{q_2}.$$

This is the equation of the phase curves in the (x,y)-plane. Introducing the new variables

$$u = y + q_2 x, \quad v = y + q_1 x, \tag{1.23}$$

we can rewrite the equation of the phase curves in the form

$$v = Cu^\alpha, \quad \alpha = \frac{q_2}{q_1} > 1.$$

This equation describes the family of power functions Cu^α (with $\alpha > 1$) in the (u,v)-plane (see Fig. 1.13 on the left). The linear transformation (1.23) transforms the u- and v-axis to the straight lines $y + q_1 x = 0$ and $y + q_2 x = 0$ in the (x,y)-plane.

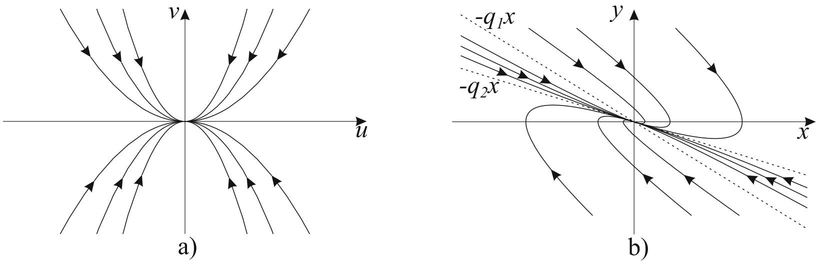

Fig. 1.13 Phase portrait of overdamped oscillator in: a) (u,v)-plane, b) (x,y)-plane

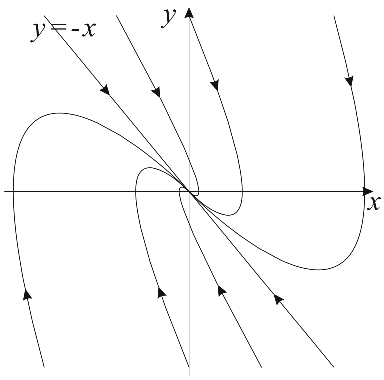

Fig. 1.14 Phase portrait of critically damped oscillator in (x,y)-plane

The phase curves in the (x,y)-plane are shown in Fig. 1.13 on the right. Similar to the previous case all phase curves approach the origin as τ tends to infinity. The origin is a fixed point called a (stable) node.

Case c. This is the degenerate case of aperiodic motion. Since $q_1 = q_2 = 1$, the two axes $y + q_2 x = 0$ and $y + q_1 x = 0$ coincide with the bisector $y = -x$. The phase curves in the (x,y)-plane are shown in Fig. 1.14. Similar to the previous case all phase curves approach the origin as τ tends to infinity.

Since Lehr's damping ratio δ is given by $\delta = c/2m\omega_0$, equation $\delta = 1$ describes the parabola $c/m = 2\sqrt{k/m}$ in the $(k/m, c/m)$-plane of parameters. The latter is the boundary between different types of motion considered above as shown in Fig. 1.15.

Energy Balance. Because of the presence of damper in the system, the energy is no longer conserved. The initial energy will be dissipated gradually by the damper into heat, and the motion decays as time increases. As time goes to infinity, the initial energy will be dissipated completely, and the system approaches equilibrium. To find the rate of decay of the total energy we multiply modified Langrange's equation (1.12) by $\dot{x}$

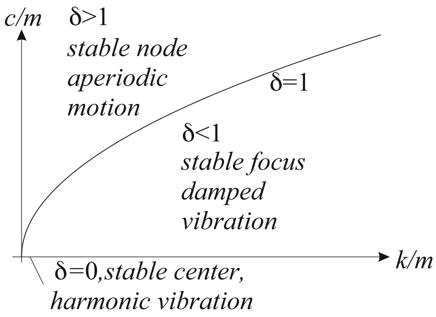

Fig. 1.15 Classification of motion in the $(k/m, c/m)$-plane

$$\dot{x}\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \dot{x}\frac{\partial L}{\partial x} = -\frac{\partial D}{\partial \dot{x}}\dot{x}.$$

Observing that

$$\dot{x}\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \dot{x}\frac{d}{dt}\frac{\partial K}{\partial \dot{x}} = m\dot{x}\ddot{x} = \frac{d}{dt}(\frac{1}{2}m\dot{x}^2) = \frac{dK}{dt},$$
$$-\dot{x}\frac{\partial L}{\partial x} = \dot{x}\frac{\partial U}{\partial x} = \frac{dU}{dt},$$
$$-\frac{\partial D}{\partial \dot{x}}\dot{x} = -c\dot{x}^2 = -2D(\dot{x}),$$

we obtain the energy dissipation rate in the form

$$\frac{d}{dt}(K+U) = -2D(\dot{x}). \tag{1.24}$$

A similar equation also holds true for oscillators with many degrees of freedom (see Section 2.4). Integrating equation (1.24) from t_0 to t, we find the energy change at time t

$$K+U-E_0 = -2\int_{t_0}^{t} D(\dot{x}(s))ds = -E_d(t),$$

where E_0 is the total energy at $t = t_0$ and $E_d(t)$ the amount of energy dissipated by the damper at time t.

1.3 Forced Oscillator

Differential Equation of Motion. If there is an additional external force (excitation) acting on the oscillator, the latter is called a forced oscillator. Also in this case both the force and the energy methods can be used to derive the equation of motion. In the energy method the virtual work done by the external force must be taken into account. We consider an example.

EXAMPLE 1.6. Mass-spring-damper forced oscillator. A mass m, connected with a spring of stiffness k and a damper of damping constant c, moves horizontally under the action of an external force $f(t)$ (see Fig. 1.16). Derive the equation of motion for this forced oscillator.

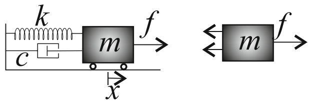

Fig. 1.16 Mass-spring-damper forced oscillator

In the force method the only difference compared with example 1.4 is the external force $f(t)$ (see the free-body diagram in Fig. 1.16). Thus, Newton's law in the x-direction reads

$$m\ddot{x} = \sum F_x = -kx - c\dot{x} + f(t).$$

Bringing the spring and damping forces to the left-hand side, we obtain

$$m\ddot{x} + c\dot{x} + kx = f(t). \tag{1.25}$$

To use the energy method we must add to the left-hand side of variational equation (1.11) the virtual work done by the external force. The variational principle becomes: among all admissible motions $x(t)$ constrained by the conditions

$$x(t_0) = x_0, \quad x(t_1) = x_1,$$

the true motion satisfies the variational equation (see the footnote on page 10)

$$\delta \int_{t_0}^{t_1} L(x,\dot{x})\,dt - \int_{t_0}^{t_1} \frac{\partial D}{\partial \dot{x}} \delta x\,dt + \int_{t_0}^{t_1} f(t)\delta x\,dt = 0. \tag{1.26}$$

Note that the last integral representing the virtual work of external force can also be included in the first integral as follows

$$\delta \int_{t_0}^{t_1} [L(x,\dot{x}) + f(t)x]\,dt - \int_{t_0}^{t_1} \frac{\partial D}{\partial \dot{x}} \delta x\,dt = 0.$$

From (1.26) we can derive modified Lagrange's equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} + \frac{\partial D}{\partial \dot{x}} = f(t). \tag{1.27}$$

With

$$L(x,\dot{x}) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2, \quad D(\dot{x}) = \frac{1}{2}c\dot{x}^2,$$

we arrive again at the equation of motion (1.25).

Reduction to the Standard Form. Let us divide equation (1.25) by k

$$\frac{1}{\omega_0^2}\ddot{x}+\frac{c}{k}\dot{x}+x=\frac{f(t)}{k}.$$

Introducing the dimensionless time $\tau=\omega_0 t$ as in the previous Section, we transform this equation to the standard form

$$x''+2\delta x'+x=g(\tau), \tag{1.28}$$

where the prime denotes as before the derivative with respect to τ and $g(\tau)=f(\tau/\omega_0)/k$. Equation (1.28) is the inhomogeneous linear differential equation of second order. According to the theory of ordinary differential equations [11] the solution of this linear equation is the sum of any particular solution of the inhomogeneous equation and the general solution of the homogeneous equation which has been found in the previous Section. Thus, the problem reduces to finding any particular solution of the inhomogeneous equation (1.28).

Particular Solution for a Step Function. Consider first a special excitation in form of the unit step (Heaviside) function

$$g(\tau)=h(\tau)=\begin{cases}0 & \text{for } \tau\le 0,\\ 1 & \text{for } \tau>0.\end{cases}$$

We seek the solution of equation (1.28) satisfying the initial conditions

$$x(0)=0,\quad x'(0)=0.$$

Such the solution is called a unit step response. For an underdamped oscillator ($\delta<1$) the solution has obviously the form

$$x=1+Ce^{-\delta\tau}\cos(\nu\tau-\phi).$$

The initial conditions will be satisfied if

$$x(0)=1+C\cos\phi=0,$$

and

$$x'(0)=-C(\delta\cos\phi-\nu\sin\phi)=0. \tag{1.29}$$

It follows from the last equation and (1.20) that

$$\tan\phi=\frac{\delta}{\nu}=\frac{\delta}{\sqrt{1-\delta^2}}\Rightarrow\phi=\nu\tau_*=\sqrt{1-\delta^2}\tau_*,$$

where τ_* is given by (1.20). From $x(0)=0$ we find the coefficient C

$$C = -\frac{1}{\cos\phi}.$$

Since $\tan\phi = \delta/\sqrt{1-\delta^2}$, it is easy to show that $\cos\phi = \sqrt{1-\delta^2}$, so

$$C = -\frac{1}{\sqrt{1-\delta^2}}.$$

Thus, the unit step response for the underdamped oscillator is given by

$$x_r(\tau) = 1 - \frac{e^{-\delta\tau}}{\sqrt{1-\delta^2}} \cos[\sqrt{1-\delta^2}(\tau - \tau_*)].$$

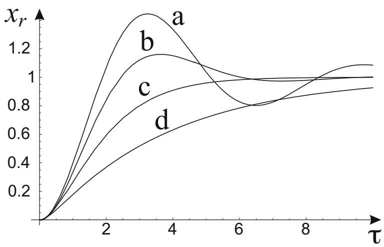

Fig. 1.17 Unit step response: a) $\delta = 0.25$, b) $\delta = 0.5$, c) $\delta = 1$, d) $\delta = 2$

Doing similar calculations, we can obtain the unit step responses also for the overdamped oscillator ($\delta > 1$)

$$x_r(\tau) = 1 - \frac{\delta+\kappa}{2\kappa} e^{-(\delta-\kappa)\tau} + \frac{\delta-\kappa}{2\kappa} e^{-(\delta+\kappa)\tau},$$

as well as for the critically damped oscillator ($\delta = 1$)

$$x_r(\tau) = 1 - (1+\tau)e^{-\tau},$$

(see exercise 1.5). The graphs of these unit step responses are plotted in Fig. 1.17 for different values of damping ratio δ.

Particular Solution for General Excitations. Let us consider now an arbitrary excitation $g(\tau)$ which is zero for $\tau \le 0$ and remains finite as τ goes to infinity. Since the initial conditions can later be satisfied by the solution of the corresponding homogeneous equation, we seek a particular solution of (1.28) satisfying the initial conditions

$$x(0) = 0, \quad x'(0) = 0.$$

The effective way of finding the solution is the Laplace transform (see, for example [13]). For any function $x(\tau)$ we define its Laplace transform according to

$$X(s) = \mathscr{L}[x(\tau)] = \int_0^\infty x(\tau)e^{-s\tau}\,d\tau,$$

with $X(s)$ being called the Laplace image of $x(\tau)$. We assume that the Laplace transforms of $g(\tau), x(\tau)$ and its derivatives are defined for any complex number s with the positive real part. Applying the Laplace transform to both sides of equation (1.28), we obtain

$$\int_0^\infty (x'' + 2\delta x' + x)e^{-s\tau} d\tau = \int_0^\infty g(\tau)e^{-s\tau} d\tau.$$

Performing the partial integration, we have

$$\mathscr{L}[x'] = \int_0^\infty x' e^{-s\tau} d\tau = xe^{-s\tau}\Big|_0^\infty + \int_0^\infty sxe^{-s\tau} d\tau = s\mathscr{L}[x] = sX(s).$$

The initial condition $x(0) = 0$ as well as the behavior of $x(\tau)$ at infinity have been taken into account. Similarly,

$$\mathscr{L}[x''] = \int_0^\infty x'' e^{-s\tau} d\tau = x' e^{-s\tau}\Big|_0^\infty + \int_0^\infty sx' e^{-s\tau} d\tau = s\mathscr{L}[x'] = s^2 X(s).$$

Thus, the differential equation (1.28) is transformed into an algebraic equation

$$(s^2 + 2\delta s + 1)X(s) = G(s),$$

yielding immediately

$$X(s) = \frac{G(s)}{s^2 + 2\delta s + 1}. \tag{1.30}$$

To find the original function from its image function we apply the inverse Laplace transform to (1.30)

$$x(\tau) = \mathscr{L}^{-1}[X(s)] = \frac{1}{2\pi i}\int_{\alpha - i\infty}^{\alpha + i\infty} \frac{G(s)}{s^2 + 2\delta s + 1} e^{s\tau} ds, \tag{1.31}$$

where α is any real and positive number. Integral (1.31) is taken along the line $(\alpha - i\infty, \alpha + i\infty)$ in the complex plane of s. Since the roots of the characteristic equation have non-positive real parts, the integrand of (1.31) is regular along this line and thus, the integral converges. The problem reduces then to computing the inverse Laplace transform of the product $G(s)/(s^2 + 2\delta s + 1)$. Observe that the inverse Laplace transform of $sG(s)$ is $g'(\tau)$, while the inverse Laplace transform of $1/s(s^2 + 2\delta s + 1)$ is the unit step response $x_r(\tau)$ found previously. Indeed, the Laplace transform of the Heaviside function is

$$\mathscr{L}[h(\tau)] = \int_0^\infty e^{-s\tau} d\tau = \frac{1}{s},$$

so, by substituting this in (1.30), we obtain $1/s(s^2 + 2\delta s + 1)$ as the image function of the unit step response.

To compute the inverse Laplace transform of the product we use the following property of the Laplace transform. Consider two arbitrary functions $f(\tau)$ and $g(\tau)$, with $f(\tau) = g(\tau) = 0$ for $\tau \leq 0$. Denote the convolution of two functions $f(\tau)$ and $g(\tau)$ by

$$(f * g)(\tau) = \int_0^\infty f(\tau - t)g(t)dt = \int_0^\tau f(\tau - t)g(t)dt = (g * f)(\tau).$$

We compute the Laplace transform of the convolution

$$\mathscr{L}[f * g] = \int_0^\infty \left(\int_0^\infty f(\tau - t)g(t)\, dt \right) e^{-s\tau} d\tau.$$

Changing the order of integration with respect to τ and t, we have

$$\mathscr{L}[f * g] = \int_0^\infty \int_0^\infty f(\tau - t)e^{-s\tau} d\tau\, g(t)\, dt.$$

Changing the variable of integration from τ to $u = \tau - t$, we obtain finally

$$\mathscr{L}[f * g] = \int_0^\infty f(u)e^{-su} du \int_0^\infty g(t)e^{-st} dt = F(s)G(s).$$

Thus, the Laplace transform of the convolution $f * g$ is equal to the product $F(s)G(s)$ and vice versa. Consequently, the inverse Laplace transform of (1.31) yields

$$x(\tau) = \int_0^\tau g'(t)x_r(\tau - t)dt. \tag{1.32}$$

This is Duhamel's formula for the particular solution of (1.28).

Solution of Initial-Value Problem. It turns out that the Laplace transform can also be used to find the solution of the initial-value problem

$$x'' + 2\delta x' + x = 0,$$
$$x(0) = x_0, \quad x'(0) = x_0'.$$

Indeed, applying the Laplace transform to this equation and observing that, due to the initial conditions,

$$\mathscr{L}[x'] = \int_0^\infty x' e^{-s\tau} d\tau = -x_0 + sX(s),$$
$$\mathscr{L}[x''] = \int_0^\infty x'' e^{-s\tau} d\tau = -x_0' - sx_0 + s^2 X(s),$$

we obtain

$$(s^2 + 2\delta s + 1)X(s) = x_0' + sx_0 + 2\delta x_0.$$

Thus,

$$X(s)=\frac{x_0'+sx_0+2\delta x_0}{s^2+2\delta s+1},$$

and the problem reduces to computing the inverse Laplace transform of the rational function (see exercise 1.6).

Energy Balance. We calculate the rate of change of the total energy due to the work done by the external force and the dissipation. Multiplying modified Langrange's equation (1.27) by $\dot{x}$, we obtain

$$\dot{x}\frac{d}{dt}\frac{\partial L}{\partial \dot{x}}-\dot{x}\frac{\partial L}{\partial x}=-\frac{\partial D}{\partial \dot{x}}\dot{x}+f(t)\dot{x}.$$

Making the same observation as in previous Section, we obtain the rate of change of energy in the form

$$\frac{d}{dt}(K+U)=-2D(\dot{x})+f(t)\dot{x}. \tag{1.33}$$

Integrating equation (1.33) from t_0 to t, we find the energy change at time t

$$K+U-E_0=-2\int_{t_0}^{t}D(\dot{x}(s))ds+\int_{t_0}^{t}f(s)\dot{x}(s)ds=-E_d(t)+W(t),$$

where E_0 is the total energy at $t=t_0$. The last term $W(t)$ is the work done by the external force which is stored in the energy of the system except that part $E_d(t)$ dissipated by the damper.

1.4 Harmonic Excitations and Resonance

As we know from Section 1.2, any solution of the homogeneous equation approaches zero as τ becomes large if the damping ratio δ is positive. Therefore only the particular solution of inhomogeneous equation which persists at large time (called forced vibration) presents interest in most applications. The forced vibration has been found in the previous Section for an arbitrary excitation through the Laplace transform leading to Duhamel's formula. In spite of this general method of solution we consider in this Section the special case of harmonic excitations for which the forced vibration can be determined directly and in a simple way, without using the Laplace transform technique. The results of this Section are also important for the variational-asymptotic method in non-linear vibrations.

Type of Excitations. We consider three cases of harmonic excitations.

Case a. Harmonic force excitation or excitation through the spring.

EXAMPLE 1.7. The damper-mass-spring oscillator is excited by the harmonic motion of the spring hanger: $x_e=x_0\cos\omega t$ (see Fig. 1.18).

Since the spring force is proportional to the change of length $x-x_e(t)$, the equation of motion reads

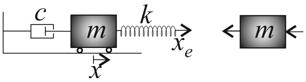

Fig. 1.18 Oscillator excited through spring hanger

$$m\ddot{x} = -c\dot{x} - k(x - x_e).$$

On the other side, the same equation can also be derived from the Lagrange function

$$L(x,\dot{x}) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}k(x_e(t) - x)^2$$

and the dissipation function $D(\dot{x}) = \frac{1}{2}c\dot{x}^2$. Bringing the two terms $-c\dot{x}$ and $-kx$ to the left-hand side and transforming the obtained equation to the dimensionless form as in Section 1.2, we get

$$x'' + 2\delta x' + x = x_0 \cos \eta \tau,$$

where $\eta = \omega/\omega_0$ is the frequency ratio. Note that the same equation of motion holds true for the forced oscillator in example 1.6 if we set $f(t) = f_0 \cos \omega t$ and $x_0 = f_0/k$.

Case b. Harmonic excitation through the damper.

EXAMPLE 1.8. The spring-mass-damper oscillator is excited by the harmonic motion of the damper piston: $x_e = x_0 \sin \omega t$ (see Fig. 1.19).

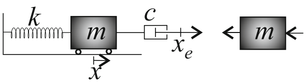

Fig. 1.19 Oscillator excited through damper piston

In this case the damping force as well as the dissipation function depend on the relative velocity $\dot{x} - \dot{x}_e$. Thus, the equation of motion takes the form

$$m\ddot{x} = -kx - c(\dot{x} - \dot{x}_e).$$

Bringing the two terms $-kx$ and $-c\dot{x}$ to the left-hand side and transforming the obtained equation to the dimensionless form, we get

$$x'' + 2\delta x' + x = 2\delta\eta x_0 \cos \eta \tau.$$

Case c. Harmonic excitation through the motion of the frame.

EXAMPLE 1.9. The spring-mass-damper oscillator is excited by the harmonic motion of the support frame: $x_e = x_0 \cos \omega t$ (see Fig. 1.20).

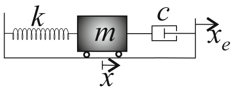

Fig. 1.20 Oscillator excited by motion of frame

We write the equation of motion in terms of the relative displacement of the mass with respect to the moving frame, $x_r = x - x_e$. Since the acceleration in the fixed inertial frame is $\ddot{x} = \ddot{x}_r + \ddot{x}_e$, we have

$$m(\ddot{x}_r + \ddot{x}_e) = -c\dot{x}_r - kx_r.$$

Bringing all terms in the right-hand side to the left-hand side, the term $m\ddot{x}_e = -mx_0\omega^2 \cos\omega t$ to the right-hand side and transforming the obtained equation to the dimensionless form, we get

$$x_r'' + 2\delta x_r' + x_r = \eta^2 x_0 \cos\eta\tau.$$

One can of course derive the equations of motion in examples 1.8 and 1.9 also by the energy method (see exercise 1.8).

Thus, in all three cases we may present the equations of motion in the form

$$x'' + 2\delta x' + x = x_0 \alpha \cos\eta\tau, \tag{1.34}$$

where the factor α is equal to

$$\alpha = \begin{cases} 1 & \text{in case a,} \\ 2\delta\eta & \text{in case b,} \\ \eta^2 & \text{in case c.} \end{cases}$$

Amplitude and Phase of Forced Vibration. We seek the particular solution of equation (1.34) in form of harmonic motion

$$x = x_0 M \cos(\eta\tau - \psi) = x_0 M(\cos\eta\tau\cos\psi + \sin\eta\tau\sin\psi),$$

where M is called a magnification factor and ψ the phase of forced vibration. Differentiating this equation with respect to τ, we have

$$x' = -x_0 M\eta(\sin\eta\tau\cos\psi - \cos\eta\tau\sin\psi),$$
$$x'' = -x_0 M\eta^2(\cos\eta\tau\cos\psi + \sin\eta\tau\sin\psi).$$

Substitution of these formulas into the equation of motion (1.34) gives

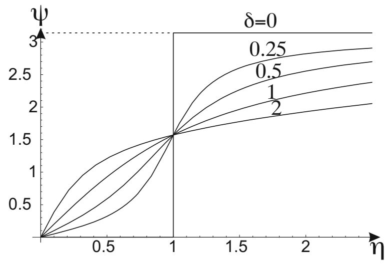

Fig. 1.21 Phase ψ versus frequency ratio η at different damping ratios δ

$$\cos\eta\tau[x_0M(1-\eta^2)\cos\psi+2\delta\eta x_0M\sin\psi-x_0\alpha]$$
$$+\sin\eta\tau[x_0M(1-\eta^2)\sin\psi-2\delta\eta x_0M\cos\psi]=0.$$

Since $\cos\eta\tau$ and $\sin\eta\tau$ are independent and are not identically zero, the expressions in the square brackets must vanish yielding

$$\tan\psi=\frac{2\delta\eta}{1-\eta^2}, \tag{1.35}$$

and

$$M=\frac{\alpha}{(1-\eta^2)\cos\psi+2\delta\eta\sin\psi}.$$

We see from (1.35) that the phase ψ is independent of α, so it remains the same in all three cases. However, one should keep in mind that ψ in case b) is the phase difference between the response x and the velocity $\dot{x}_e$. The plot of $\psi(\eta)$ for different values of δ is shown in Fig. 1.21.

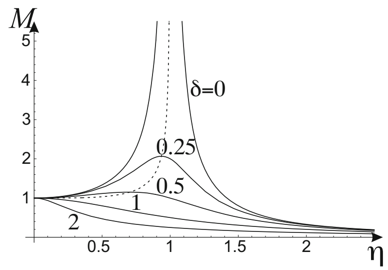

Fig. 1.22 Magnification factor M versus frequency ratio η at different damping ratios δ (case a)

The equation for the magnification factor M can still be reduced to the form independent of the phase ψ. We first note that, due to (1.35),

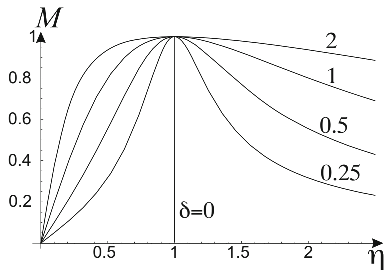

Fig. 1.23 Magnification factor M versus frequency ratio η at different damping ratios δ (case b)

$$(1-\eta^2)\cos\psi + 2\delta\eta\sin\psi = 2\delta\eta(\sin\psi + \frac{\cos\psi}{\tan\psi}) = \frac{2\delta\eta}{\sin\psi}.$$

From the same formula (1.35) we can easily express $\sin\psi$ as

$$\sin\psi = \frac{2\delta\eta}{\sqrt{(1-\eta^2)^2 + 4\delta^2\eta^2}}. \tag{1.36}$$

Thus,

$$M = \frac{\alpha}{\sqrt{(1-\eta^2)^2 + 4\delta^2\eta^2}}. \tag{1.37}$$

The plots of magnification factor M versus the frequency ratio η for different values of damping ratio δ are shown in Figs. 1.22-1.24 in cases a, b, and c, respectively.

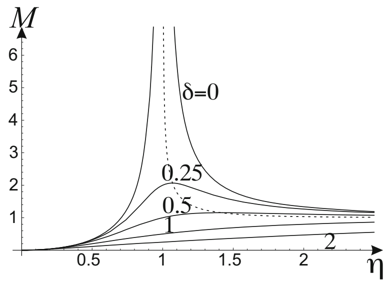

Fig. 1.24 Magnification factor M versus frequency ratio η at different damping ratios δ (case c)

One may be interested in finding the maximum of magnification factor and the frequency ratio at which this maximum is achieved. One speaks then of the resonant vibration. In case a) the maximum of M is achieved at $\eta_m = \sqrt{1-2\delta^2}$ giving

$$M_m = \frac{1}{\sqrt{1-\eta_m^4}} = \frac{1}{2\delta\sqrt{1-\delta^2}}$$

for $\delta < 1/\sqrt{2}$ and at $\eta = 0$ giving $M_m = 1$ otherwise. In case b) the maximum is always achieved at $\eta = 1$ giving $M_m = 1$. In case c) the maximum of M is achieved at $\eta_m = 1/\sqrt{1-2\delta^2}$ giving

$$M_m = \frac{\eta_m^2}{\sqrt{\eta_m^4 - 1}} = \frac{1}{2\delta\sqrt{1-\delta^2}}$$

for $\delta < 1/\sqrt{2}$ and at $\eta = \infty$ giving $M_m = 1$ otherwise. The curves corresponding to the maxima of the magnification factors are drawn in Figs. 1.22 and 1.24 by the dashed lines. It is remarkable that the maxima of M are in general not achieved when the frequency of external excitation coincides with the frequency of free vibration $\omega_c = \omega_0\sqrt{1-\delta^2}$ (which means $\eta = \sqrt{1-\delta^2}$) except the case $\delta = 0$ for which $M_m = \infty$ (strict resonance).[3] Some characteristic values of phase and magnification factors are presented in Table 1.1.

Table 1.1 Characteristic values of ψ and M

η	ψ	M (case a)	M (case b)	M (case c)
0	0	1	0	0
1	$\pi/2$	$\frac{1}{2\delta}$	1	$\frac{1}{2\delta}$
∞	π	0	0	1
η_m	-	$\frac{1}{2\delta\sqrt{1-\delta^2}}$	1	$\frac{1}{2\delta\sqrt{1-\delta^2}}$

Power and Work. Let us find out the power and work done by the external force on the forced vibration. We define the power of the external force as

$$P = f(t)\dot{x}.$$

For the forced oscillator with $f(t) = kx_0 \cos\omega t$ (case a) the forced vibration is described by

$$x = x_0 M \cos(\omega t - \psi), \quad \dot{x} = -x_0 \omega M \sin(\omega t - \psi).$$

[3] Actually the solution in this case is $x_0 M \tau \sin\tau$, and its amplitude tends to infinity only in the limit $\tau \to \infty$ (see exercise 1.10).

Thus,

$$P = f(t)\dot{x} = -kx_0^2\omega M \cos\omega t \sin(\omega t - \psi)$$
$$= \frac{1}{2}kx_0^2\omega M[\sin\psi - \sin(2\omega t - \psi)] = P_a - P_i,$$

where the constant part $P_a = \frac{1}{2}kx_0^2\omega M \sin\psi$ is called an active power, while the oscillating with doubled frequency part $P_i = \frac{1}{2}kx_0^2\omega M \sin(2\omega t - \psi)$ an idle power. Remembering the formulas (1.36) and (1.37), we obtain in case a)

$$P_a = kx_0^2\omega_0 \frac{\delta\eta^2}{(1-\eta^2)^2 + 4\delta^2\eta^2} = kx_0^2\omega_0 M_a,$$
$$P_i = kx_0^2\omega_0 \frac{\eta}{2\sqrt{(1-\eta^2)^2 + 4\delta^2\eta^2}} \sin(2\omega t - \psi) = kx_0^2\omega_0 M_i \sin(2\omega t - \psi).$$

Knowing the power, we can easily calculate the work done by the external force according to

$$W = \int_{t_0}^{t} P\,dt = \frac{1}{2}F_0\hat{x}\omega(t - t_0)\sin\psi + \frac{1}{4}F_0\hat{x}[\cos(2\omega t - \psi) - \cos(2\omega t_0 - \psi)],$$

where $F_0 = kx_0$ and $\hat{x} = x_0 M$. We see that the work done by the external force can be decomposed into two parts: the active work W_a which grows linearly with time, and the idle work W_i which is the periodic function. The work done in one period of vibration is given by

$$W^* = \pi F_0 \hat{x} \sin\psi.$$

We know that, for periodic motions, the kinetic and potential energies are periodic functions, so they do not change in one period. In contrary, the energy dissipation in one period of vibration is positive and equals

$$R^* = 2\int_0^T D(\dot{x})\,dt = c\omega\hat{x}^2 \int_0^{2\pi} \sin^2(\omega t - \psi)\,d(\omega t) = \pi c\hat{x}^2\omega.$$

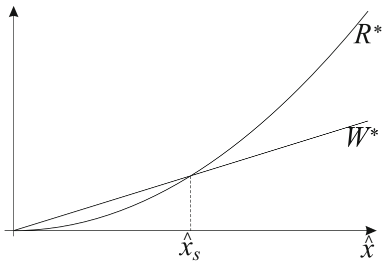

Fig. 1.25 Energy diagram of forced vibration

Diagram 1.25 shows the comparison between the work done W^* and the energy dissipation R^* in one period of forced vibration as function of the amplitude $\hat{x}$ at some fixed frequency. While the work done in one period is a linear function of $\hat{x}$, the energy dissipation is quadratic with respect to $\hat{x}$. The straight line cuts the parabola at a point with coordinate $\hat{x}_s = F_0 \sin\psi/c\omega$. If $\hat{x} < \hat{x}_s$, then the work done by the external force is larger than the energy dissipation, so the amplitude of forced vibration must increase. If $\hat{x} > \hat{x}_s$, then the work done by the external force is smaller than the energy dissipation, so the amplitude decreases. Thus, $\hat{x}_s$ corresponds to the steady-state amplitude of forced vibration.

1.5 Exercises

EXERCISE 1.1. Derive the equation of motion of a roller (mass m, radius r) hung on an unstretchable rope and a spring (see Fig. 1.26) with the help of

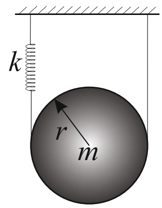

Fig. 1.26 Roller hung on rope and spring

i) the force method,
ii) the energy method.

Determine the eigenfrequency of vibration.

Solution. i) The force method. We free the roller from the rope and the spring (see Fig. 1.27) and apply the moment equation about A

$$\frac{d}{dt}(J_A\dot{\varphi}) = \sum M_z = -F_s 2r.$$

From the kinematics we know that $\varphi = x/2r$. Besides, the spring force is equal to $F_s = kx$, while the moment of inertia of the roller about A is

$$J_A = J_O + mr^2 = \frac{1}{2}mr^2 + mr^2 = \frac{3}{2}mr^2.$$

Thus, the equation of motion reads

$$\frac{3}{2}mr^2\frac{\ddot{x}}{2r} = -kx2r \quad \Rightarrow \quad \ddot{x} + \frac{8k}{3m}x = 0,$$

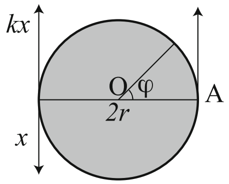

Fig. 1.27 Roller and the forces

and the eigenfrequency is given by

$$\omega_0^2 = \frac{8k}{3m} \quad \Rightarrow \quad \omega_0 = \sqrt{\frac{8k}{3m}}.$$

ii) The energy method. We write down the kinetic and potential energies:

$$K = \frac{1}{2} J_A \dot{\varphi}^2, \quad U = \frac{1}{2} k x^2.$$

Taking into account the kinematic relation $\varphi = x/2r$ and the formula $J_A = \frac{3}{2} m r^2$, we obtain the Lagrange function in the form

$$L = K - U = \frac{3}{16} m \dot{x}^2 - \frac{1}{2} k x^2.$$

Then, from Lagrange's equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0,$$

we derive again the above equation of motion and the formula for ω_0.

EXERCISE 1.2. Derive the equation of motion of a thin circular ring (mass m, radius r) hung on a support O (see Fig. 1.28). Determine the eigenfrequency of its small vibration.

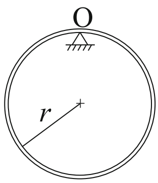

Fig. 1.28 Ring hung on support

Solution. We write down the kinetic and potential energies:

$$K = \frac{1}{2} J_O \dot{\varphi}^2, \quad U = mgr(1 - \cos\varphi).$$

The moment of inertia of the thin ring is equal to

$$J_O = J_S + mr^2 = mr^2 + mr^2 = 2mr^2.$$

For small vibrations with $\varphi \ll 1$ we approximate $1 - \cos\varphi \approx \frac{1}{2}\varphi^2$. Thus,

$$L = mr^2 \dot{\varphi}^2 - \frac{1}{2} mgr\varphi^2.$$

From Lagrange's equation we derive the equation of motion

$$2mr^2 \ddot{\varphi} + mgr\varphi = 0.$$

Consequently, the eigenfrequency is

$$\omega_0 = \sqrt{\frac{g}{2r}}.$$

EXERCISE 1.3. Three turning points are measured from the vibration of a damped oscillator: $x_1 = 8.6$mm, $x_2 = -4.1$mm, $x_3 = 4.3$mm. Determine the middle point of vibration (position of equilibrium). Find the logarithmic decrement ϑ and the damping ratio δ.

Solution. The free vibration of the underdamped oscillator is described by

$$x(\tau) = x_m + a_0 e^{-\delta\tau} \cos(\nu\tau - \phi),$$

where x_m corresponds to the position of equilibrium. As we know, $x(\tau)$ achieves maxima or minima if

$$\tan(\nu\tau - \phi) = -\delta/\nu.$$

So, the turning points (corresponding to maxima or minima) occur at the time instants τ_1, $\tau_1 + \tau_c/2$, $\tau_1 + \tau_c$ and so on, with τ_1 being the time instant of the first turning point and $\tau_c = 2\pi/\nu$ the conditional period of vibration. Taking the periodicity of cosine function into account, we have

$$\begin{aligned} x_1 &= x_m + C, \\ x_2 &= x_m - Ce^{-\delta\pi/\nu}, \\ x_3 &= x_m + Ce^{-\delta 2\pi/\nu}, \end{aligned}$$

where $C = a_0 e^{-\delta\tau_1} \cos(\nu\tau_1 - \phi)$. Forming the differences we easily see that

$$\frac{x_1 - x_2}{x_3 - x_2} = e^{\delta\pi/\nu} = e^{\delta\tau_c/2}.$$

Thus,

$$\ln \frac{x_1 - x_2}{x_3 - x_2} = \delta \tau_c / 2 = \frac{\vartheta}{2},$$

with ϑ being the logarithmic decrement. Substituting the given values of turning points into this formula we obtain

$$\vartheta = 2 \ln \frac{x_1 - x_2}{x_3 - x_2} = 0.827.$$

Knowing the logarithmic decrement, we find Lehr's damping ratio

$$\delta = \frac{\vartheta}{\sqrt{4\pi^2 + \vartheta^2}} = 0.13.$$

Now we form the differences $x_1 - x_m$ and $x_m - x_2$ and consider the quotient

$$\frac{x_1 - x_m}{x_m - x_2} = e^{\delta \tau_c / 2} = e^{\vartheta / 2}.$$

From the last equation it is ready to find x_m

$$x_m = \frac{x_1 + x_2 e^{\vartheta / 2}}{1 + e^{\vartheta / 2}} = 0.956 \text{mm}.$$

EXERCISE 1.4. The time constants are measured from the vibration of a damped oscillator: $T_d = 5$s, $T_c = 2$s. Determine ϑ and δ.

Solution. According to the formulas for the characteristic times T_d and T_c we have

$$T_d = \frac{1}{\delta \omega_0} \quad \text{and} \quad T_c = \frac{2\pi}{\omega_0 \sqrt{1 - \delta^2}}.$$

Thus, it is not difficult to see that

$$\vartheta = \frac{2\pi\delta}{\sqrt{1 - \delta^2}} = \frac{T_c}{T_d} = 0.4.$$

Knowing ϑ, we can restore the damping ratio δ according to

$$\delta = \frac{\vartheta}{\sqrt{4\pi^2 + \vartheta^2}} = 0.0635.$$

EXERCISE 1.5. Determine the unit step responses for the overdamped and the critically damped oscillators.

Solution. For an overdamped oscillator ($\delta > 1$) the unit step response satisfying the initial conditions

$$x_r(0) = 0, \quad x_r'(0) = 0$$

has obviously the form

$$x_r(\tau) = 1 + Ae^{-q_1\tau} + Be^{-q_2\tau},$$

where

$$q_1 = \delta - \kappa, \quad q_2 = \delta + \kappa,$$

and $\kappa = \sqrt{\delta^2 - 1}$. Substituting this solution into the initial conditions, we obtain for A and B the equations

$$x_r(0) = 1 + A + B = 0,$$
$$x_r'(0) = -Aq_1 - Bq_2 = 0.$$

Solving these equations with respect to A and B, we get

$$A = -\frac{\delta + \kappa}{2\kappa}, \quad B = \frac{\delta - \kappa}{2\kappa}.$$

Thus,

$$x_r(\tau) = 1 - \frac{\delta + \kappa}{2\kappa} e^{-(\delta-\kappa)\tau} + \frac{\delta - \kappa}{2\kappa} e^{-(\delta+\kappa)\tau}.$$

For a critically damped oscillator ($\delta = 1$) the unit step response must have the form

$$x_r(\tau) = 1 + e^{-\tau}(A\tau + B).$$

The initial conditions lead to

$$x_r(0) = 1 + B = 0, \quad x_r'(0) = A - B = 0,$$

yielding

$$A = B = -1.$$

Thus,

$$x_r(\tau) = 1 - (1 + \tau)e^{-\tau}.$$

EXERCISE 1.6. Find the solution of the initial-value problem

$$x'' + 2\delta x' + x = 0,$$

satisfying $x(0) = x_0$ and $x'(0) = x_0'$ with the help of the Laplace transform.

Solution. As shown in Section 1.3 the Laplace transform applied to this initial-value problem leads to

$$X(s) = \frac{x_0' + sx_0 + 2\delta x_0}{s^2 + 2\delta s + 1},$$

where $X(s)$ is the Laplace image of $x(\tau)$. Thus, the problem reduces to computing the inverse Laplace transform of the rational function

$$x(\tau) = \mathscr{L}^{-1}[X(s)] = \frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} \frac{x_0' + sx_0 + 2\delta x_0}{s^2 + 2\delta s + 1} e^{s\tau} ds.$$

Since $X(s)$ is the ratio of two polynomials $P(s)/Q(s)$, its inverse Laplace transform equals the sum of the residues of $X(s)e^{s\tau}$ at the singular points (poles) of $X(s)$. For the overdamped oscillator ($\delta > 1$) we have two different real roots of the characteristic equation corresponding to two simple poles of $X(s)$

$$s = s_1 = -q_1, \quad s = s_2 = -q_2,$$

therefore

$$x(\tau) = \frac{P(-q_1)}{Q'(-q_1)} e^{-q_1\tau} + \frac{P(-q_2)}{Q'(-q_2)} e^{-q_2\tau}.$$

Since

$$P(s) = x_0' + sx_0 + 2\delta x_0, \quad Q'(s) = 2s + 2\delta,$$

and

$$q_1 = \delta - \kappa, \quad q_2 = \delta + \kappa,$$

we easily find that

$$\frac{P(-q_1)}{Q'(-q_1)} = \frac{1}{2\kappa}(x_0' + q_2 x_0), \quad \frac{P(-q_2)}{Q'(-q_2)} = -\frac{1}{2\kappa}(x_0' + q_1 x_0).$$

Thus,

$$x(\tau) = \frac{1}{2\kappa}[(x_0' + q_2 x_0)e^{-q_1\tau} - (x_0' + q_1 x_0)e^{-q_2\tau}],$$

which agrees with the formula (1.19).

For the underdamped oscillator with $\delta < 1$ we have two complex conjugate roots, and the computation of the inverse Laplace transform can be done in a similar manner. For the critically damped oscillator ($\delta = 1$) there is one double root $s = -1$ corresponding to the pole of order two, so

$$x(\tau) = \frac{d}{ds}[P(s)e^{s\tau}]\bigg|_{s=-1} = e^{-\tau}[x_0(1+\tau) + x_0'\tau].$$

EXERCISE 1.7. Use Duhamel's formula to compute the response of the critically damped oscillator with $\delta = 1$ to the so-called ramp function

$$g(\tau) = \begin{cases} 0 & \text{for } \tau < 0, \\ \alpha\tau & \text{for } 0 \le \tau \le \tau_0, \\ \alpha\tau_0 & \text{for } \tau > \tau_0. \end{cases}$$

The oscillator was at rest for $\tau \le 0$.

Solution. According to Duhamel's formula

$$x(\tau) = \int_0^\tau g'(t) x_r(\tau - t)\, dt,$$

where $x_r(\tau)$ is the unit step response function. For $\delta = 1$ we have

$$x_r(\tau) = 1 - (1+\tau)e^{-\tau}.$$

We compute the derivative of the ramp function $g(\tau)$ given above

$$g'(\tau) = \begin{cases} 0 & \text{for } \tau < 0, \\ \alpha & \text{for } 0 \le \tau \le \tau_0, \\ 0 & \text{for } \tau > \tau_0. \end{cases}$$

So we need to consider two different cases.

Case a: $\tau \le \tau_0$. In this case

$$\begin{aligned} x(\tau) &= \int_0^\tau \alpha[1-(1+(\tau-t))e^{-(\tau-t)}]\,dt \\ &= \alpha\left[\tau - \int_0^\tau e^{-(\tau-t)}\,dt - \int_0^\tau (\tau-t)e^{-(\tau-t)}\,dt\right] \\ &= \alpha\left[\tau - \int_{-\tau}^0 e^u\,du + \int_{-\tau}^0 ue^u\,du\right] \\ &= \alpha(\tau - 2 + 2e^{-\tau} + \tau e^{-\tau}) = \alpha[\tau - 2 + (\tau+2)e^{-\tau}]. \end{aligned}$$

Case b: $\tau > \tau_0$. Since $g'(\tau) = 0$ for $\tau > \tau_0$, we have

$$\begin{aligned} x(\tau) &= \int_0^{\tau_0} \alpha[1-(1+(\tau-t))e^{-(\tau-t)}]\,dt \\ &= \alpha\left[\tau_0 - \int_0^{\tau_0} e^{-(\tau-t)}\,dt - \int_0^{\tau_0} (\tau-t)e^{-(\tau-t)}\,dt\right] \\ &= \alpha\left[\tau_0 - \int_{-\tau}^{\tau_0-\tau} e^u\,du + \int_{-\tau}^{\tau_0-\tau} ue^u\,du\right] \\ &= \alpha[\tau_0 - (2+\tau-\tau_0)e^{-(\tau-\tau_0)} + (\tau+2)e^{-\tau}]. \end{aligned}$$

EXERCISE 1.8. Derive the equations of motion in examples 1.8 and 1.9 by the energy method.

Solution. In example 1.8 (see Fig. 1.19) the Lagrange function reads

$$L(x,\dot{x}) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2.$$

The dissipation function depends however on the relative velocity

$$D = \frac{1}{2}c(\dot{x} - \dot{x}_e)^2.$$

From modified Lagrange's equation for dissipative systems

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} + \frac{\partial D}{\partial \dot{x}} = 0,$$

we derive the equation of motion

$$m\ddot{x} + kx + c(\dot{x} - \dot{x}_e) = 0.$$

For the system in example 1.9 (see Fig. 1.20) the Lagrange function reads

$$L(x,\dot{x}) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}k(x - x_e)^2.$$

We see that the potential energy depends on the relative displacement. The dissipation function depends also on the relative velocity

$$D = \frac{1}{2}c(\dot{x} - \dot{x}_e)^2.$$

From modified Langrange's equation for dissipative systems

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} + \frac{\partial D}{\partial \dot{x}} = 0,$$

we derive the equation of motion

$$m\ddot{x} + k(x - x_e) + c(\dot{x} - \dot{x}_e) = 0.$$

Introducing the relative displacement $x_r = x - x_e$, we can present this equation in the form

$$m\ddot{x}_r + c\dot{x}_r + kx_r = -m\ddot{x}_e.$$

EXERCISE 1.9. Derive the equation of vertical motion of a frame (mass m_f) excited by two rotating unbalanced masses (mass $m_u/2$, frequency of rotation ω, radius of rotation r). The frame is connected with two springs of equal stiffness $k/2$ and a damper with damping constant c (see Fig. 1.29). Determine the magnification factor of forced vibration.

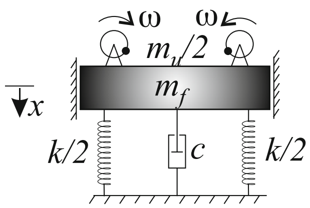

Fig. 1.29 Vertical forced vibration of frame

Solution. We derive the equation of motion by the energy method. Let x be the vertical displacement of the center of mass of the frame from the equilibrium position. The displacements of the unbalanced masses from their equilibrium positions are given by

$$x_u = x + r\cos\omega t, \quad y_u = \pm r\sin\omega t.$$

The plus and minus signs are due to the fact that the unbalanced masses rotate in the opposite directions. Their velocities are

$$\dot{x}_u = \dot{x} - r\omega\sin\omega t, \quad \dot{y}_u = \pm r\omega\cos\omega t.$$

Thus, the kinetic energy of masses equals

$$K(\dot{x}) = \frac{1}{2}m_f\dot{x}^2 + \frac{1}{2}m_u(\dot{x} - r\omega\sin\omega t)^2 + \frac{1}{2}m_u r^2\omega^2\cos^2\omega t.$$

Since the gravitational force and the static spring forces do not contribute to the potential energy, we have

$$U(x) = \frac{1}{2}kx^2.$$

The Lagrange function reads

$$L = K - U = \frac{1}{2}m_f\dot{x}^2 + \frac{1}{2}m_u(\dot{x} - r\omega\sin\omega t)^2 + \frac{1}{2}m_u r^2\omega^2\cos^2\omega t - \frac{1}{2}kx^2.$$

The dissipation function is given by

$$D(\dot{x}) = \frac{1}{2}c\dot{x}^2.$$

From modified Lagrange's equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} + \frac{\partial D}{\partial \dot{x}} = 0,$$

we derive the equation of motion of this system

$$(m_f + m_u)\ddot{x} + c\dot{x} + kx = m_u r\omega^2\cos\omega t.$$

The eigenfrequency of free and undamped vibration is $\omega_0 = \sqrt{\frac{k}{m_f+m_u}}$. Introducing the dimensionless time $\tau = \omega_0 t$, we reduce the equation of motion to the standard form

$$x'' + 2\delta x' + x = x_0\eta^2\cos\eta\tau,$$

where

$$\delta = \frac{c\omega_0}{2k}, \quad x_0 = \frac{m_u}{m_f + m_u}r, \quad \eta = \frac{\omega}{\omega_0}.$$

The forced vibration reads

$$x = x_0 M\cos(\eta\tau - \psi) = x_0 M(\cos\eta\tau\cos\psi + \sin\eta\tau\sin\psi),$$

where M is called a magnification factor and ψ the phase of forced vibration. The magnification factor is equal to (see Section 1.4)

$$M = \frac{\eta^2}{\sqrt{(1-\eta^2)^2 + 4\delta^2\eta^2}}.$$

EXERCISE 1.10. Show that the variational problem

$$\delta \int_0^{2\pi} (\frac{1}{2}x'^2 - \frac{1}{2}x^2 + \cos\tau\, x)\, d\tau = 0$$

has no extremal in the class of periodic functions with $x(0) = x(2\pi)$ and $x'(0) = x'(2\pi)$. Find its extremal. What happens if the last term in the integrand is $\sin\tau\, x$?

Solution. Lagrange's equation for the extremal of this action functional reads

$$x'' + x = \cos\tau.$$

Since the frequency of the external excitation coincides with the eigenfrequency of the system corresponding to the resonance, we look for the particular solution of this equation in the form

$$x(\tau) = A\tau\sin\tau.$$

Computing the derivatives of $x(\tau)$, we have

$$x' = A(\sin\tau + \tau\cos\tau), \quad x'' = A(2\cos\tau - \tau\sin\tau).$$

Substituting these formulas into Lagrange's equation, we find that

$$x'' + x = 2A\cos\tau = \cos\tau \quad \Rightarrow \quad A = \frac{1}{2}.$$

The general solution then follows

$$x(\tau) = \frac{1}{2}\tau\sin\tau + a\cos(\tau - \phi).$$

We see that the extremal is non-periodic, and the amplitude of forced vibration goes to infinity as $\tau \to \infty$.

If the last term in the action functional is replaced by $\sin\tau\, x$, then the Lagrange's equation changes to

$$x'' + x = \sin\tau.$$

Similar calculations show that

$$x(\tau) = -\frac{1}{2}\tau\cos\tau + a\cos(\tau - \phi).$$

This corresponds again to the resonance because the extremal is non-periodic, and the amplitude of forced vibration goes to infinity as $\tau \to \infty$.

EXERCISE 1.11. Find the maxima of the magnification factors M in three cases a, b, and c considered in Section 1.4.

Solution. As shown in Section 1.4, the magnification factor M is given by

$$M = \frac{\alpha}{\sqrt{(1-\eta^2)^2 + 4\delta^2\eta^2}},$$

where

$$\alpha = \begin{cases} 1 & \text{in case a,} \\ 2\delta\eta & \text{in case b,} \\ \eta^2 & \text{in case c.} \end{cases}$$

To find the maximum of M we must differentiate M with respect to η. Consider first case a. In this case

$$\frac{dM_a}{d\eta} = \frac{2\eta(1-\eta^2-2\delta^2)}{[(1-\eta^2)^2 + 4\delta^2\eta^2]^{3/2}}.$$

Equating this derivative with zero, we find that the maximum is achieved either at $\eta = 0$ or at

$$\eta = \sqrt{1-2\delta^2}.$$

Substituting these values into the formula for M, we find that the maximum is either 1 or

$$\max M_a = \frac{1}{2\delta\sqrt{1-\delta^2}}.$$

In case b we have

$$\frac{dM_b}{d\eta} = \frac{2\delta}{\sqrt{(1-\eta^2)^2 + 4\delta^2\eta^2}} - \frac{\delta\eta(-4(1-\eta^2)\eta + 8\delta^2\eta)}{[(1-\eta^2)^2 + 4\delta^2\eta^2]^{3/2}}.$$

Simplifying this expression, we obtain

$$\frac{dM_b}{d\eta} = \frac{2\delta(1-\eta^2)(1+\eta^2)}{[(1-\eta^2)^2 + 4\delta^2\eta^2]^{3/2}}.$$

Thus, the maximum is always achieved at $\eta = 1$ giving $\max M_b = 1$.

In case c the calculations give

$$\frac{dM_c}{d\eta} = \frac{2\eta}{\sqrt{(1-\eta^2)^2 + 4\delta^2\eta^2}} - \frac{\eta^2(-2(1-\eta^2)\eta + 4\delta^2\eta)}{[(1-\eta^2)^2 + 4\delta^2\eta^2]^{3/2}}.$$

Simplifying this expression, we obtain

$$\frac{dM_c}{d\eta} = \frac{2\eta(1-(1-2\delta^2)\eta^2)}{[(1-\eta^2)^2 + 4\delta^2\eta^2]^{3/2}}.$$

Thus, the maximum of M_c is achieved at $\eta_m = 1/\sqrt{1-2\delta^2}$ giving

$$\max M_c = \frac{\eta_m^2}{\sqrt{\eta_m^4 - 1}} = \frac{1}{2\delta\sqrt{1-\delta^2}}$$

for $\delta < 1/\sqrt{2}$, and at $\eta = \infty$ giving $\lim_{\eta\to\infty} M_c = 1$ otherwise.

EXERCISE 1.12. Find the idle and active works done by the external forces in cases b and c considered in Section 1.4.

Solution. In example 1.8 corresponding to case b (see Fig. 1.19) the equation of motion reads

$$m\ddot{x} + kx + c\dot{x} = c\dot{x}_e = c\omega x_0 \cos \omega t.$$

The right-hand side of this equation is the force acting on the damper piston. The power of this force is equal to

$$\begin{aligned} P &= f(t)\dot{x} = -cx_0^2\omega^2 M \cos\omega t \sin(\omega t - \psi) \\ &= \frac{1}{2}cx_0^2\omega^2 M[\sin\psi - \sin(2\omega t - \psi)] = P_a - P_i. \end{aligned}$$

The constant part $P_a = \frac{1}{2}cx_0^2\omega^2 M \sin\psi$ is the active power while the oscillating with doubled frequency part $P_i = \frac{1}{2}cx_0^2\omega^2 M \sin(2\omega t - \psi)$ an idle power. Remembering the formulas for M and $\sin\psi$, we have in case b

$$\begin{aligned} P_a &= cx_0^2\omega_0^2 \frac{2\delta^2\eta^4}{(1-\eta^2)^2 + 4\delta^2\eta^2}, \\ P_i &= cx_0^2\omega_0^2 \frac{\delta\eta^3}{\sqrt{(1-\eta^2)^2 + 4\delta^2\eta^2}} \sin(2\omega t - \psi). \end{aligned}$$

In example 1.9 corresponding to case c (see Fig. 1.20) the equation of motion reads

$$m\ddot{x}_r + kx_r + c\dot{x}_r = -m\ddot{x}_e = m\omega^2 x_0 \cos\omega t.$$

The right-hand side of this equation is the force acting on the frame. The power of this force is equal to

$$\begin{aligned} P &= f(t)\dot{x}_e = -mx_0^2\omega^3 M \cos\omega t \sin(\omega t - \psi) \\ &= \frac{1}{2}mx_0^2\omega^3 M[\sin\psi - \sin(2\omega t - \psi)] = P_a - P_i. \end{aligned}$$

The constant part $P_a = \frac{1}{2}mx_0^2\omega^3 M \sin\psi$ is the active power while the oscillating with doubled frequency part $P_i = \frac{1}{2}mx_0^2\omega^3 M \sin(2\omega t - \psi)$ an idle power. Recalling the formulas for M and $\sin\psi$, we have in case c

$$\begin{aligned} P_a &= mx_0^2\omega_0^3 \frac{\delta\eta^6}{(1-\eta^2)^2 + 4\delta^2\eta^2}, \\ P_i &= \frac{1}{2}mx_0^2\omega_0^3 \frac{\eta^5}{\sqrt{(1-\eta^2)^2 + 4\delta^2\eta^2}} \sin(2\omega t - \psi). \end{aligned}$$

Integrating these formulas over t, we easily find the active and idle works.