Lagrangian Fibrations of Holomorphic-Symplectic Varieties of $K3^{[n]}$-Type

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Dedicated to Klaus Hulek on the occasion of his sixtieth birthday.

Abstract Let X be a compact Kähler holomorphic-symplectic manifold, which is deformation equivalent to the Hilbert scheme of length n subschemes of a $K3$ surface. Let $\mathscr{L}$ be a nef line-bundle on X, such that the top power $c_1(\mathscr{L})^{2n}$ vanishes and $c_1(\mathscr{L})$ is primitive. Assume that the two dimensional subspace $H^{2,0}(X) \oplus H^{0,2}(X)$ of $H^2(X,\mathbb{C})$ intersects $H^2(X,\mathbb{Z})$ trivially. We prove that the linear system of $\mathscr{L}$ is base point free and it induces a Lagrangian fibration on X. In particular, the line-bundle $\mathscr{L}$ is effective. A determination of the semi-group of effective divisor classes on X follows, when X is projective. For a generic such pair $(X,\mathscr{L})$, not necessarily projective, we show that X is bimeromorphic to a Tate-Shafarevich twist of a moduli space of stable torsion sheaves, each with pure one dimensional support, on a *projective* $K3$ surface.

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Partially supported by Simons Foundation Collaboration Grant 245840 and by NSA grant H98230-13-1-0239.

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A. Frühbis-Krüger et al. (eds.), *Algebraic and Complex Geometry*, Springer Proceedings in Mathematics & Statistics 71, DOI 10.1007/978-3-319-05404-9_10,

1 Introduction

An *irreducible holomorphic symplectic manifold* is a simply connected compact Kähler manifold such that $H^0(X, \wedge^2 T^*X)$ is generated by an everywhere non-degenerate holomorphic 2-form [4]. A compact Kähler manifold X is said to be of $K3^{[n]}$*-type*, if it is deformation equivalent to the Hilbert scheme $S^{[n]}$ of length n subschemes of a $K3$ surface S. Any manifold of $K3^{[n]}$-type is irreducible holomorphic symplectic [4]. The second integral cohomology of an irreducible holomorphic symplectic manifold X admits a natural symmetric non-degenerate integral bilinear pairing $(\bullet, \bullet)$ of signature $(3, b_2(X) - 3)$, called the *Beauville-Bogomolov-Fujiki pairing*. The Beauville-Bogomolov-Fujiki pairing is monodromy invariant, and is thus an invariant of the deformation class of X.

Definition 1.1. An irreducible holomorphic symplectic manifold X is said to be *special*, if the intersection in $H^2(X, \mathbb{C})$ of $H^2(X, \mathbb{Z})$ and $H^{2,0}(X) \oplus H^{0,2}(X)$ is a non-zero subgroup.

The locus of special periods forms a countable union of real analytic subvarieties of half the dimension in the corresponding moduli space.

Definition 1.2. Let X be a $2n$-dimensional irreducible holomorphic symplectic manifold and $\mathscr{L}$ a line bundle on X. We say that $\mathscr{L}$ *induces a Lagrangian fibration,* if it satisfies the following two conditions.

1. $h^0(X, \mathscr{L}) = n + 1$.
2. The linear system $|\mathscr{L}|$ is base point free, and the generic fiber of the morphism $\pi : X \to |\mathscr{L}|^*$ is a connected Lagrangian subvariety.

A line bundle $\mathscr{L}$ on a holomorphic symplectic manifold X is said to be *nef*, if $c_1(\mathscr{L})$ belongs to the closure in $H^{1,1}(X, \mathbb{R})$ of the Kähler cone of X.

Theorem 1.3. *Let X be an irreducible holomorphic symplectic manifold of $K3^{[n]}$-type and $\mathscr{L}$ a nef line-bundle, such that $c_1(\mathscr{L})$ is primitive and isotropic with respect* *to the Beauville-Bogomolov-Fujiki pairing. Assume that X is non-special. Then the line bundle $\mathscr{L}$ induces a Lagrangian fibration $\pi : X \to |\mathscr{L}|^*$.*

See Theorem 6.3 for a variant of Theorem 1.3 dropping the assumption that $\mathscr{L}$ is nef. Theorem 1.3 is proven in Sect. 6. The proof relies on Verbitsky's Global Torelli Theorem [14, 40], on the determination of the monodromy group of X [21, 22], and on a result of Matsushita that Lagrangian fibrations form an *open* subset in the moduli space of pairs $(X, \mathscr{L})$ [27]. Let us sketch the three main new ingredients in the proof of Theorem 1.3.

(1) We associate to the pair $(X, \mathscr{L})$ in Theorem 1.3 a projective $K3$ surface S with a nef line bundle $\mathscr{B}$ of degree $\frac{2n-2}{d^2}$, where $d := \gcd\{(c_1(\mathscr{L}), \lambda) : \lambda \in H^2(X, \mathbb{Z})\}$. The sub-lattice $c_1(\mathscr{B})^\perp$ orthogonal to $c_1(\mathscr{B})$ in $H^2(S, \mathbb{Z})$ is Hodge-isometric to $c_1(\mathscr{L})^\perp / \mathbb{Z}c_1(\mathscr{L})$. The construction realizes the period domain Ω_{20} of the pairs $(X, \mathscr{L})$ as an affine line bundle over a period domain Ω_{19} of semi-polarized $K3$ surfaces (Sect. 4).
(2) The bundle map $q : \Omega_{20} \to \Omega_{19}$ is invariant with respect to a subgroup Q of the monodromy group (Lemma 5.3). The group Q is isomorphic to $c_1(\mathscr{B})^\perp$. Q acts on the fiber of q over the period of a semi-polarized $K3$ surface $(S, \mathscr{B})$. Similarly, the lattice $c_1(\mathscr{B})^\perp$ projects to a subgroup of $H^{0,2}(S)$, which acts on $H^{0,2}(S)$ by translations. There exists an isomorphism, of the fiber of q with $H^{0,2}(S)$, which is equivariant with respect to the two actions (Lemma 5.4).
(3) The fiber of q over the period of a semi-polarized $K3$ surface $(S, \mathscr{B})$ contains the period of a moduli space of sheaves on S with pure one-dimensional support in the linear system $|\mathscr{B}^d|$ (Sect. 5.1). Each such moduli space of sheaves is known to be a Lagrangian fibration [34].

The assumption that X is non-special in Theorem 1.3 is probably not necessary. Unfortunately, our proof will rely on it. When X is non-special the Q-orbit, of every point in the fiber of q through the period of X, is a dense subset of the fiber (Lemma 5.4). This density will have a central role in this paper due to the following elementary observation.

Observation 1.4. *Let T be a topological space and Q a group acting on T. Assume that the Q-orbit of every point of T is dense in T. Then any nonempty Q-invariant open subset of T must be the whole of T.*

The above observation will be used in an essential way in three different proofs (Theorem 6.1, Proposition 7.7, and Theorem 7.11).

The statement of the next result requires the notion of a Tate-Shafarevich twist, which we now recall. Let M be a complex manifold and $\pi : M \to B$ a proper map with connected fibers of pure dimension n. Assume that the generic fiber of π is a smooth abelian variety. Let $\{U_i\}$ be an open covering of B in the analytic topology. Set $U_{ij} := U_i \cap U_j$ and $M_{ij} := \pi^{-1}(U_{ij})$. Assume given a 1-co-cycle g_{ij} of automorphisms of M_{ij}, satisfying $\pi \circ g_{ij} = \pi$, and acting by translations on the smooth fibers of π. We can re-glue the open covering $\{M_i\}$ of M using the co-cycle $\{g_{ij}\}$ to get a complex manifold M' and a proper map $\pi' : M' \to B$, whose fibers are isomorphic to those of π. We refer to (M', π') as the *Tate-Shafarevich twist* of (M, π) associated to the co-cycle $\{g_{ij}\}$. Tate-Shafarevich twists are standard in the study of elliptic fibrations [10, 17].

Let $\mathscr{L}$ be a semi-ample line bundle on a $K3$ surface S with an indivisible class $c_1(\mathscr{L})$. Given an ample line bundle H on S and an integer χ, denote by $M_H(0, \mathscr{L}^d, \chi)$ the moduli space of H-stable coherent sheaves on S of rank zero, determinant $\mathscr{L}^d$, and Euler characteristic χ. Assume that d and χ are relatively prime. For a generic polarization H, the moduli space $M_H(0, \mathscr{L}^d, \chi)$ is smooth and projective and it admits a Lagrangian fibration over the linear system $|\mathscr{L}^d|$ [34].

Let X be an irreducible holomorphic symplectic manifold of $K3^{[n]}$-type and $\pi : X \to \mathbb{P}^n$ a Lagrangian fibration. Set $\alpha := \pi^* c_1(\mathscr{O}_{\mathbb{P}^n}(1))$. The *divisibility* of $(\alpha, \bullet)$ is the positive integer $d := \gcd\{(\alpha, \lambda) : \lambda \in H^2(X, \mathbb{Z})\}$. The integer d^2 divides $n - 1$ (Lemma 2.5).

Theorem 1.5. *Assume that X is non-special and the intersection $H^{1,1}(X, \mathbb{Z}) \cap \alpha^{\perp}$ is $\mathbb{Z}\alpha$. There exists a $K3$ surface S, a semi-ample line bundle $\mathscr{L}$ on S of degree $\frac{2n-2}{d^2}$ with an indivisible class $c_1(\mathscr{L})$, an integer χ relatively prime to d, and a polarization H on S, such that X is bimeromorphic to a Tate-Shafarevich twist of the Lagrangian fibration $M_H(0, \mathscr{L}^d, \chi) \to |\mathscr{L}^d|$.*

Theorem 1.5 is proven in Sect. 7. The semi-polarized $K3$ surface $(S, \mathscr{L})$ in Theorem 1.5 is the one mentioned already above, which is associated to (X, α) in Sect. 4.1. The equality $H^{1,1}(X, \mathbb{Z}) \cap \alpha^{\perp} = \mathbb{Z}\alpha$ is equivalent to the statement that $\mathrm{Pic}(S)$ is cyclic generated by $\mathscr{L}$. This condition is relaxed in Theorem 7.13, which strengthens Theorem 1.5.

A reduced and irreducible divisor on X is called *prime exceptional*, if it has negative Beauville-Bogomolov-Fujiki degree. A divisor D on X is called *movable*, if the base locus of the linear system $|D|$ has co-dimension ≥ 2 in X. The *movable cone* $\mathscr{MV}_X$ of X is the cone in $N^1(X) := H^{1,1}(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$ generated by classes of movable divisors. Assume that X is a projective irreducible holomorphic symplectic manifold of $K3^{[n]}$-type and let $h \in N^1(X)$ be an ample class. Denote by $\mathscr{P}ex_X \subset H^{1,1}(X, \mathbb{Z})$ the set of classes of prime exceptional divisors. The set $\mathscr{P}ex_X$ is determined in [24, Theorem 1.11 and Sec. 1.5]. The closure of the movable cone in $N^1(X)$ is determined as follows:

$$\overline{\mathscr{MV}}_X = \{c \in N^1(X) : (c, c) \geq 0, \ (c, h) \geq 0, \ \text{and } (c, e) \geq 0, \ \text{for all } e \in \mathscr{P}ex_X\},$$

by a result of Boucksom [6, 23, Prop. 5.6 and Lemma 6.22].[1]

Corollary 1.6. *Let X be a projective irreducible holomorphic symplectic manifold of $K3^{[n]}$-type. The semi-group of effective divisor classes on X is generated by*

[1] Prop. 5.6 and Lemma 6.22 in the last reference [23]. The same convention will be used throughout the paper for all citations with multiple references.

the classes of prime exceptional divisors and integral points in the closure of the movable cone in $N^1(X)$.

Corollary 1.6 was shown to follow from Theorem 1.3 in [23, Paragraph following Question 10.11].

We classify the deformation types of pairs $(X, \mathscr{L})$, consisting of an irreducible holomorphic symplectic manifold X of $K3^{[n]}$-type, $n \geq 2$, and a line bundle $\mathscr{L}$ on X with a primitive and isotropic first Chern class, such that $(c_1(\mathscr{L}), \kappa) > 0$, for some Kähler class κ. The following proposition is proven in Sect. 4.3, using monodromy invariants introduced in Lemma 2.5.

Proposition 1.7. *Let d be a positive integer, such that d^2 divides $n-1$. If $1 \leq d \leq 4$, then there exists a unique deformation type of pairs $(X, \mathscr{L})$, with $c_1(\mathscr{L})$ primitive and isotropic, such that $(c_1(\mathscr{L}), \bullet)$ has divisibility d. For $d \geq 5$, let $\nu(d)$ be half the number of multiplicative units in the ring $\mathbb{Z}/d\mathbb{Z}$. Then there are $\nu(d)$ deformation types of pairs $(X, \mathscr{L})$ as above, with $(c_1(\mathscr{L}), \bullet)$ of divisibility d.*

A *generalized Kummer variety* of dimension $2n$ is the fiber of the Albanese map $S^{[n+1]} \to S$ from the Hilbert scheme of length n subschemes of an abelian surface S to S itself [4]. We expect all of the above results to have analogues for X an irreducible holomorphic-symplectic manifold deformation equivalent to a generalized Kummer variety. Yoshioka proved Theorem 1.3 for those X associated to a moduli space of sheaves on an abelian surface [43]. Let the pair $(X, \mathscr{L})$ consist of X, deformation equivalent to a generalized Kummer, and a line bundle $\mathscr{L}$ with a primitive and isotropic first Chern class. The basic construction of Sect. 4.1 associates to the pair $(X, \mathscr{L})$, with $\dim(X) = 2n$, $n \geq 2$, and with $(c_1(\mathscr{L}), \bullet)$ of divisibility d, two dual pairs (S_1, α_1) and (S_2, α_2), each consisting of an abelian surface S_i and a class α_i in the Neron-Severi group of S_i of self intersection $\frac{2n+2}{d^2}$, such that $S_2 \cong S_1^*$ and the natural isometry $H^2(S_1, \mathbb{Z}) \cong H^2(S_2, \mathbb{Z})$ maps α_1 to α_2. A conjectural determination of the monodromy group of generalized Kummer varieties was suggested in the comment after [25, Prop. 4.8]. Assuming that the monodromy group is as conjectured, we expect that the proofs of all the results above can be adapted to this deformation type.

A version of Theorem 1.3 has been conjectured for irreducible holomorphic symplectic manifolds of all deformation types [5, 26, 39, Conjecture 2]. Markushevich, Sawon, and Yoshioka proved a version of Theorem 1.3, when X is the Hilbert scheme of n points on a $K3$ surface and $(c_1(\mathscr{L}), \bullet)$ has divisibility 1 [26, Cor. 4.4] and [39] (the regularity of the fibration, in Sect. 5 of [39], is due to Yoshioka). Bayer and Macri recently proved a strong version of Theorem 1.3 for moduli spaces of sheaves on a projective $K3$ surface [3].

Remark 1.8 (Added in the final revision). Let X_0 be an irreducible holomorphic symplectic manifold and $\mathscr{L}_0$ a nef line bundle on X_0, such that $c_1(\mathscr{L}_0)$ is primitive and isotropic with respect to the Beauville-Bogomolov-Fujiki pairing. Matsushita proved that if $\mathscr{L}_0$ induces a Lagrangian fibration, then so does $\mathscr{L}$ for every pair $(X, \mathscr{L})$ deformation equivalent to $(X_0, \mathscr{L}_0)$, with X irreducible holomorpic symplectic and $\mathscr{L}$ nef (preprint posted very recently [28], announced earlier in his talk [31]). It follows that Theorem 1.3 above holds also without the assumption that X is non-special, since a pair $(X, \mathscr{L})$ with X special is a deformation of a pair $(X_0, \mathscr{L}_0)$ with X_0 non-special. In fact, this stronger version of Theorem 1.3, dropping the non-speciality, follows already from the combination of Matsushita's result and Example 3.1 below, since Example 3.1 exhibits a pair $(X_0, \mathscr{L}_0)$, with a line bundle $\mathscr{L}_0$ inducing a Lagrangian fibration, in each deformation class of pairs $(X, \mathscr{L})$ with X of $K3^{[n]}$-type and $c_1(\mathscr{L})$ primitive, isotropic, and on the boundary of the positive cone. Matsushita's result does not seem to provide an alternative proof of Theorem 1.5 and the only proof we know is presented in Sect. 7 and relies on the preceding sections.

2 Classification of Primitive-Isotropic Classes

A lattice, in this note, is a finitely generated free abelian group with a symmetric bilinear pairing $(\bullet, \bullet) : L \otimes_{\mathbb{Z}} L \to \mathbb{Z}$. The pairing may be degenerate. The isometry group $O(L)$ is the group of automorphisms of L preserving the bilinear pairing.

Definition 2.1. Two pairs (L_i, v_i), $i = 1, 2$, each consisting of a lattice L_i and an element $v_i \in L_i$, are said to be *isometric*, if there exists an isometry $g : L_1 \to L_2$, such that $g(v_1) = v_2$.

Let X be an irreducible holomorphic symplectic manifold of $K3^{[n]}$-type, $n \geq 2$. Set $\Lambda := H^2(X, \mathbb{Z})$. We will refer to Λ as the $K3^{[n]}$*-lattice.* Let $\tilde{\Lambda}$ be the Mukai lattice, i.e., the orthogonal direct sum of two copies of the negative definite $E_8(-1)$ lattice and four copies of the even unimodular rank two lattice with signature $(1, -1)$.

Theorem 2.2 ([22], Theorem 1.10). *X comes with a natural $O(\tilde{\Lambda})$-orbit ι_X of primitive isometric embeddings $\iota : H^2(X, \mathbb{Z}) \hookrightarrow \tilde{\Lambda}$.*

Choose a primitive isometric embedding $\iota : \Lambda \hookrightarrow \tilde{\Lambda}$ in the canonical $O(\tilde{\Lambda})$-orbit ι_X provided by Theorem 2.2. Choose a generator $v \in \tilde{\Lambda}$ of the rank 1 sub-lattice orthogonal to $\iota(\Lambda)$. We say that an isometry $g \in O(\Lambda)$ *stabilizes* the $O(\tilde{\Lambda})$-orbit ι_X, if given a representative isometric embedding ι in the orbit ι_X, there exists an isometry $\tilde{g} \in O(\tilde{\Lambda})$ satisfying $\iota \circ g = \tilde{g} \circ \iota$. Note that $\tilde{g}$ necessarily maps v to $\pm v$.

Set $\Lambda_{\mathbb{R}} := \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$. Let $\tilde{\mathscr{C}} \subset \Lambda_{\mathbb{R}}$ be the positive cone $\{x \in \Lambda_{\mathbb{R}} \ : \ (x, x) > 0\}$. Then $H^2(\tilde{\mathscr{C}}, \mathbb{Z})$ is isomorphic to $\mathbb{Z}$ and is a natural character of the isometry group $O(\Lambda)$ [23, Lemma 4.1]. Denote by $O^+(\Lambda)$ the kernel of this orientation character. Isometries in $O^+(\Lambda)$ are said to be *orientation preserving*.

Definition 2.3. Let X, X_1, and X_2 be irreducible holomorphic symplectic manifolds. An isometry $g : H^2(X_1, \mathbb{Z}) \to H^2(X_2, \mathbb{Z})$ is a *parallel transport operator*, if there exists a family $\pi : \mathscr{X} \to B$ (which may depend on g) of irreducible holomorphic symplectic manifolds, points b_1 and b_2 in B, isomorphisms $X_i \cong \mathscr{X}_{b_i}$, where $\mathscr{X}_{b_i}$ is the fiber over b_i, $i = 1, 2$, and a continuous path γ from b_1 to b_2, such that parallel transport along γ in the local system $R^2\pi_*\mathbb{Z}$ induces the isometry g. When $X = X_1 = X_2$, we call g a *monodromy operator*. The *monodromy group* $Mon^2(X)$ of X is the subgroup, of the isometry group of $H^2(X,\mathbb{Z})$, generated by monodromy operators.

Theorem 2.4 ([22], Theorem 1.2 and Lemma 4.2). *The subgroup $Mon^2(X)$ of $O(\Lambda)$ consists of orientation preserving isometries stabilizing the orbit ι_X.*

Given a lattice L, let $I_n(L) \subset L$ be the subset of primitive classes v with $(v, v) = 2n-2$. Notice that the orbit set $I_n(L)/O(L)$ parametrizes the set of isometry classes of pairs (L', v'), such that L' is isometric to L and v' is a primitive class in L' with $(v', v') = 2n - 2$ [23, Lemma 9.14].

Let n be an integer ≥ 2, let Λ be the $K3^{[n]}$-lattice, and let $\alpha \in \Lambda$ be a primitive isotropic class. Let $\mathrm{div}(\alpha, \bullet)$ be the largest positive integer, such that $(\alpha, \bullet)/\mathrm{div}(\alpha, \bullet)$ is an integral class of Λ^*. Set $d := \mathrm{div}(\alpha, \bullet)$ and

$$\beta := \iota(\alpha).$$

Let $L \subset \tilde{\Lambda}$ be the saturation[2] of $\mathrm{span}_{\mathbb{Z}}\{\beta, v\}$. Clearly, the isometry class of (L, v) depends only on α and the $O(\tilde{\Lambda})$-orbit of ι. Consequently, the isometry class of (L, v) depends only on α, as the $O(\tilde{\Lambda})$-orbit ι_X of ι is natural, by Theorem 2.2. We denote by $[L, v](\alpha)$ the isometry class of the pair (L, v) associated to α.

Lemma 2.5. *(1) d^2 divides $n - 1$.*

(2) L is isometric to the lattice $L_{n,d}$ with Gram matrix $\frac{2n-2}{d^2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

(3) Let $d \geq 1$ be an integer, such that d^2 divides $n - 1$. The map $\alpha \mapsto [L, v](\alpha)$ induces a one-to-one correspondence between the set of $Mon^2(X)$-orbits, of primitive isotropic classes α with $\mathrm{div}(\alpha, \bullet) = d$*, and the set of isometry classes $I_n(L_{n,d})/O(L_{n,d})$.*

(4) There exists an integer b, such that $(\beta - bv)/d$ is an integral class of L. The isometry class $[L, v](\alpha)$ is represented by $(L_{n,d}, (d, b))$, for any such integer b.

Proof. Part (1): There exists a class $\delta \in \Lambda$, such that $(\delta, \delta) = 2 - 2n$ and the sub-lattice $\delta^\perp_\Lambda$ of Λ, orthogonal to δ, is a unimodular lattice isometric to the $K3$-lattice. The sub-lattice $[\iota(\delta^\perp_\Lambda)]^\perp_{\tilde{\Lambda}}$ of $\tilde{\Lambda}$, which is the saturation of $\mathrm{span}\{\iota(\delta), v\}$, is unimodular, hence isometric to the unimodular hyperbolic plane U with Gram matrix $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$. We may further assume that $v = (1, 1-n)$ and $\iota(\delta) = (1, n-1)$, under this isomorphism. If X is the Hilbert scheme $S^{[n]}$ of a $K3$-surface and δ is half the class of the big diagonal, then δ satisfies the above properties. Write

[2] The *saturation* of a sublattice L' of Λ is the maximal sublattice L of Λ, of the same rank as L', which contains L'.

$\alpha = a\xi + b\delta$, where ξ is a primitive class of the $K3$-lattice $\delta^{\perp}_{\Lambda}$, $a > 0$, and $\gcd(a, b) = 1$. We get

$$0 = (\alpha, \alpha) = a^2(\xi, \xi) - (2n - 2)b^2,$$

and (ξ, ξ) is even. Hence, a^2 divides $n - 1$. Furthermore, $\mathrm{div}(\delta, \bullet) = 2n - 2$, $\mathrm{div}(\xi, \bullet) = 1$, since $\delta^{\perp}_{\Lambda}$ is unimodular, and $\mathrm{div}(\alpha, \bullet) = \gcd(\mathrm{div}(a\xi, \bullet), \mathrm{div}(b\delta, \bullet)) = \gcd(a, (2n-2)b) = a$. Thus, $a = d := \mathrm{div}(\alpha, \bullet)$.

Part (2): Note that $\iota(\delta) - v = (2n - 2)e$, where e is a primitive isotropic class of $\tilde{\Lambda}$. Set $\gamma := \frac{1}{d}(\beta - bv) = \iota(\xi) + \frac{b(2n-2)}{d}e$. We claim that the lattice $L := \mathrm{span}_{\mathbb{Z}}\{v, \gamma\}$ is saturated in $\tilde{\Lambda}$. Indeed, choose $\eta \in \delta^{\perp}_{\Lambda}$, such that $(\xi, \eta) = 1$. Then $\begin{pmatrix} (v, e) & (v, \eta) \\ (\gamma, e) & (\gamma, \eta) \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

Let G be the Gram matrix of L in the basis $\{v, \gamma\}$. Then

$$G = \frac{2n-2}{d^2}\begin{pmatrix} d^2 & -bd \\ -bd & b^2 \end{pmatrix} = \frac{2n-2}{d^2}\begin{pmatrix} d \\ -b \end{pmatrix}\begin{pmatrix} d & -b \end{pmatrix}.$$

Choose a 2×2 invertible matrix A, with integer coefficients, such that $A\begin{pmatrix} d \\ -b \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Then AGA^t is the Gram matrix of $L_{n,d}$.

Part (3): Assume given two primitive isotropic classes α_1 and α_2 in $\Lambda := H^2(X, \mathbb{Z})$ and let (L_i, v_i) be the pair associated to α_i as above, for $i = 1, 2$. In other words, $\iota_i : \Lambda \hookrightarrow \tilde{\Lambda}$ is a primitive embedding in the orbit ι_X, v_i generates the sub-lattice of $\tilde{\Lambda}$ orthogonal to the image of ι_i, and L_i is the saturation of $\mathrm{span}_{\mathbb{Z}}\{\iota(\alpha_i), v_i\}$.

Let us check that the map $\alpha \mapsto [L, v](\alpha)$ is constant on $Mon^2(X)$-orbits. Assume that there exists an element $\mu \in Mon^2(X)$, such that $\mu(\alpha_1) = \alpha_2$. Then there exists an isometry $\tilde{\mu} \in O(\tilde{\Lambda})$, satisfying $\tilde{\mu} \circ \iota_1 = \iota_2 \circ \mu$, by Theorem 2.4. We get that $\tilde{\mu}(L_1) = L_2$ and $\tilde{\mu}(v_1) = v_2$, or $\tilde{\mu}(v_1) = -v_2$. So, the isometry $\tilde{\mu}$ or $-\tilde{\mu}$ from L_1 onto L_2 provides an isometry of the pairs (L_i, v_i), $i = 1, 2$.

We show next that the map $\alpha \mapsto [L, v](\alpha)$ is injective, i.e., that the isometry class of the pair (L, v) determines the $Mon^2(X)$-orbit of α. Assume that there exists as isometry $f : L_1 \to L_2$, such that $f(v_1) = v_2$. Then there exists an isometry $\tilde{f} \in O(\tilde{\Lambda})$, such that $\tilde{f}(L_1) = L_2$ and the restriction of $\tilde{f}$ to L_1 is f, by ([36], Proposition 1.17.1 and Theorem 1.14.4, see also [21], Lemma 8.1 for more details). In particular, $\tilde{f}(v_1) = v_2$. There exists a unique isometry $h \in O(\Lambda)$ satisfying $\iota_2 \circ h = \tilde{f} \circ \iota_1$. There exists an isometry $\phi \in O(\tilde{\Lambda})$, such that $\phi \circ \iota_2 = \iota_1$, since both ι_i belong to the same $O(\tilde{\Lambda})$-orbit ι_X. We get the equality $\iota_1 \circ h = \phi \circ \iota_2 \circ h = (\phi \circ \tilde{f}) \circ \iota_1$. If h is orientation preserving, then h belongs to $Mon^2(X)$, otherwise, $-h$ does, by Theorem 2.4. Let $\mu = h$, if it is orientation preserving. Otherwise, set $\mu := -h$. Then μ is a monodromy operator and $\iota_2(\mu(\alpha_1)) = \pm\iota_2(h(\alpha_1)) = \pm\tilde{f}(\iota_1(\alpha_1))$. The class $\iota_1(\alpha_1)$ spans the null space of L_1, and $\tilde{f}$ restricts to an isometry from L_1 to L_2. Hence, $\iota_2(\mu(\alpha_1))$ spans the null space of L_2. Hence, $\mu(\alpha_1) = \pm\alpha_2$.

Finally we show that α_2 and $-\alpha_2$ belong to the same $Mon^2(X)$-orbit. There exists an element $\tau \in \Lambda$ satisfying $(\tau, \tau) = 2$, and $(\tau, \alpha_2) = 0$. The isometry $\rho_\tau \in O(\Lambda)$, given by $\rho_\tau(\lambda) = -\lambda + (\lambda, \tau)\tau$, belongs to $Mon^2(X)$, by ([21], Corollary 1.8), and it sends α_2 to $-\alpha_2$.

It remains to prove that the map $\alpha \mapsto [L, v](\alpha)$ is surjective. Assume given a primitive class $v \in L_{n,d}$ with $(v, v) = 2n - 2$. There exists a primitive isometric embedding $f : L_{n,d} \hookrightarrow \tilde{\Lambda}$, by ([36], Proposition 1.17.1). The lattice $f(v)^{\perp}_{\tilde{\Lambda}}$, orthogonal to $f(v)$ in $\tilde{\Lambda}$, is isometric to the $K3^{[n]}$-lattice Λ. Choose such an isometry $h : f(v)^{\perp}_{\tilde{\Lambda}} \to \Lambda$, with the property that $h^{-1} : \Lambda \hookrightarrow \tilde{\Lambda}$ belongs to the $O(\tilde{\Lambda})$-orbit ι_X. Such a choice exists, since $O(\Lambda)$ acts transitively on the orbit space $O(\Lambda, \tilde{\Lambda})/O(\tilde{\Lambda})$, by ([22], Lemma 4.3). Above, $O(\Lambda, \tilde{\Lambda})$ denotes the set of primitive isometric embeddings of Λ in $\tilde{\Lambda}$. Denote by $\beta \in L_{n,d}$ a generator of the null space of $L_{n,d}$. Set $\alpha := h(f(\beta))$. Then α is a class in Λ, such that $[L, v](\alpha)$ is represented by $(L_{n,d}, v)$.

Part (4): The existence of such an integer b was established in the course of proving part (1). The rest of the statement follows from Lemma 2.6. □

If $d = 2$, set $\nu(d) := 1$. If $d > 2$, let $\nu(d)$ be half the number of multiplicative units in the ring $\mathbb{Z}/d\mathbb{Z}$.

Lemma 2.6. *A vector $(x, y) \in L_{n,d}$ is primitive of degree $2n - 2$, if and only if $|x| = d$ and $\gcd(d, y) = 1$. Two primitive vectors (d, y), (d, z) belong to the same $O(L_{n,d})$-orbit, if and only if $y \equiv z$ modulo d, or $y \equiv -z$ modulo d. Consequently, $\nu(d)$ is equal to the number of $O(L_{n,d})$-orbits of primitive vectors in $L_{n,d}$ of degree $2n - 2$.*

Proof. The isometry group of $L_{n,d}$ consists of matrices of the form $\begin{pmatrix} \pm 1 & 0 \\ c & \pm 1 \end{pmatrix}$. The orbit $O(L_{n,d})(d, y)$ consists of vectors of the form $(\pm d, cd \pm y)$. Consequently, the number of $O(L_{n,d})$-orbits of primitive vectors in $L_{n,d}$ of degree $2n-2$ is equal to the number of orbits in $\{y : 0 < y < d \text{ and } \gcd(y, d) = 1\}$ under the action $y \mapsto d - y$. The latter number is $\nu(d)$. □

3 An Example of a Lagrangian Fibration for Each Value of the Monodromy Invariants

Let S be a projective $K3$ surface, $K(S)$ its topological K-group, generated by classes of complex vector bundles, and $H^*(S, \mathbb{Z})$ its integral cohomology ring. Let $td_S := 1 + \frac{c_2(S)}{12}$ be the Todd class of S and $\sqrt{td_S} := 1 + \frac{c_2(S)}{24}$ its square root. The homomorphism $v : K(S) \to H^*(S, \mathbb{Z})$, given by $v(x) = ch(x)\sqrt{td_S}$ is an isomorphism of free abelian groups. Given a coherent sheaf E on S, the class $v(E)$ is called the *Mukai vector* of E. Given integers r and s and a class $c \in H^2(S, \mathbb{Z})$, we will denote by (r, c, s) the class of $H^*(S, \mathbb{Z})$, whose graded summand in $H^0(S, \mathbb{Z})$ is r times the class Poincare dual to S, its graded summand in $H^2(S, \mathbb{Z})$ is c, and its graded summand in $H^4(S,\mathbb{Z})$ is s times the class Poincare dual to a point. We endow $H^*(S,\mathbb{Z})$ with the *Mukai pairing*

$$((r,c,s),(r',c',s')) \ := \ (c,c') - rs' - r's,$$

where $(c,c') := \int_S c \cup c'$. Then $(v(x),v(y)) = -\chi(x \otimes y)$, where $\chi : K(S) \to \mathbb{Z}$ is the Euler characteristic [35]. $H^*(S,\mathbb{Z})$, endowed with the Mukai pairing, is called the *Mukai lattice*. The Mukai lattice is an even unimodular lattice of rank 24, which is isometric to the orthogonal direct sum of two copies of the negative definite $E_8(-1)$ lattice and four copies of the even unimodular rank 2 hyperbolic lattice U.

Let $v \in K(S)$ be the class with Mukai vector $(0, d\xi, s)$ in $H^*(S,\mathbb{Z})$, such that ξ a primitive effective class in $H^{1,1}(S,\mathbb{Z})$, $(\xi,\xi) > 0$, d is a positive integer, and $\gcd(d,s) = 1$. There is a system of hyperplanes in the ample cone of S, called v-walls, that is countable but locally finite [15, Ch. 4C]. An ample class is called *v-generic*, if it does not belong to any v-wall. Choose a v-generic ample class H. Let $M_H(v)$ be the moduli space of H-stable sheaves on the $K3$ surface S with class v. $M_H(v)$ is a smooth projective irreducible holomorphic symplectic variety of $K3^{[n]}$-type, with $n = \frac{(v,v)+2}{2} = \frac{d^2(\xi,\xi)+2}{2}$. This is a special case of a result, which is due to several people, including Huybrechts, Mukai, O'Grady [38], and Yoshioka [44]. It can be found in its final form in [44].

Over $S \times M_H(v)$ there exists a universal sheaf $\mathscr{F}$, possibly twisted with respect to a non-trivial Brauer class pulled-back from $M_H(v)$. Associated to $\mathscr{F}$ is a class $[\mathscr{F}]$ in $K(S \times M_H(v))$ ([20], Definition 26). Let π_i be the projection from $S \times M_H(v)$ onto the i-th factor. Denote by $v^\perp$ the sub-lattice in $H^*(S,\mathbb{Z})$ orthogonal to v. The second integral cohomology $H^2(M_H(v),\mathbb{Z})$, its Hodge structure, and its Beauville-Bogomolov-Fujiki pairing, are all described by Mukai's Hodge-isometry

$$\theta : v^\perp \longrightarrow H^2(M_H(v),\mathbb{Z}), \tag{3.1}$$

given by $\theta(x) := c_1\left(\pi_{2_!}\{\pi_1^!(x^\vee) \otimes [\mathscr{F}]\}\right)$ (see [44]).

We provide next an example of a moduli space $M_H(v)$ and a primitive isotropic class $\alpha \in H^{1,1}(M_H(v),\mathbb{Z})$, such that $[L,v](\alpha)$ is represented by $(L_{n,d},(d,b))$, for every integer $n \geq 2$, for every positive integer d, such that d^2 divides $n-1$, and for every integer b satisfying $\gcd(b,d) = 1$.

Example 3.1. Let d be a positive integer, such that d^2 divides $n-1$. Let S be a $K3$ surface with a nef line bundle $\mathscr{L}$ of degree $\frac{2n-2}{d^2}$. Let λ be the class $c_1(\mathscr{L})$ in $H^2(S,\mathbb{Z})$. Fix an integer b satisfying $\gcd(b,d) = 1$. Set $v := (0, d\lambda, s)$, where s is an integer satisfying $sb = 1$ (modulo d). Then v is a primitive Mukai vector and $(v,v) = 2n-2$. Choose a v-generic ample line bundle H. A sheaf F of class v is H-stable, if and only if it is H-semi-stable. The moduli space $M_H(v)$, of H-stable sheaves of class v, is smooth, projective, holomorphic symplectic, and of $K3^{[n]}$-type. Set $\alpha := \theta((0,0,1))$. Let $\iota : H^2(M_H(v),\mathbb{Z}) \to H^*(S,\mathbb{Z})$ be the composition of θ^{-1} with the inclusion of $v^\perp$ into $H^*(S,\mathbb{Z})$. A Mukai vector (r,c,t) belongs to $v^\perp$, if and only if $rs = d(c, \lambda)$. It follows that d divides r, since $\gcd(d, s) = 1$. Thus, $\mathrm{div}(\alpha, \bullet) = d$. Now

$$\iota(\alpha) - bv = (0, -bd\lambda, 1 - bs)$$

is divisible by d, by our assumption on s. Hence, the monodromy invariant $[L, v](\alpha)$ is equal to the isometry class of $(L_{n,d}, (d, b))$, by Lemma 2.5. The cohomology $H^1(S, \mathscr{L}^d)$ vanishes, since $\mathscr{L}$ is a nef divisor of positive degree [32, Prop. 1]. Thus, the vector space $H^0(S, \mathscr{L}^d)$ has dimension $\chi(\mathscr{L}^d) = n + 1$. The support morphism $\pi : M_H(v) \to |\mathscr{L}^d|$ realizes $M_H(v)$ as a completely integrable system. The equality $\pi^* c_1(\mathscr{O}_{|\mathscr{L}^d|}(1)) = \alpha$ is easily verified.

4 Period Domains and Period Maps

4.1 A Projective $K3$ Surface Associated to an Isotropic Class

Let X be an irreducible holomorphic symplectic manifold of $K3^{[n]}$-type, $n \geq 2$. Assume that there exists a non-zero primitive isotropic class $\alpha \in H^{1,1}(X, \mathbb{Z})$. Let $\tilde{\Lambda}$ be the Mukai lattice. Choose a primitive isometric embedding $\iota : H^2(X, \mathbb{Z}) \to \tilde{\Lambda}$ in the canonical $O(\tilde{\Lambda})$-orbit ι_X of Theorem 2.2. Set $\tilde{\Lambda}_{\mathbb{C}} := \tilde{\Lambda} \otimes_{\mathbb{Z}} \mathbb{C}$. Endow $\tilde{\Lambda}_{\mathbb{C}}$ with the weight 2 Hodge structure, so that $\tilde{\Lambda}_{\mathbb{C}}^{2,0} = \iota(H^{2,0}(X))$. Set $\beta := \iota(\alpha)$. Then β belongs to $\tilde{\Lambda}_{\mathbb{C}}^{1,1}$. Set

$$\Lambda_{k3} \; := \; \beta^{\perp}_{\tilde{\Lambda}} / \mathbb{Z}\beta$$

and endow Λ_{k3} with the induced Hodge structure. Let U be the even unimodular rank 2 lattice of signature $(1, 1)$, and $E_8(-1)$ the negative definite E_8 lattice. Then Λ_{k3} is isometric to the $K3$ lattice, which is the orthogonal direct sum of two copies of $E_8(-1)$ and three copies of U. Indeed, this is clear if β is a class in a direct summand of $\tilde{\Lambda}$ isometric to U. It follows in general, since the isometry group of $\tilde{\Lambda}$ acts transitively on the set of primitive isotropic classes in $\tilde{\Lambda}$. The induced Hodge structure on Λ_{k3} is the weight 2 Hodge structure of some $K3$ surface $S(\alpha)$, by the surjectivity of the period map.

Let v be a generator of the rank 1 sub-lattice of $\tilde{\Lambda}$ orthogonal to the image of ι. Then v is of Hodge-type $(1, 1)$. Set $\Lambda := H^2(X, \mathbb{Z})$. Then $v^{\perp}$ is isometric to Λ. We claim that $(v, v) = 2n - 2$. Indeed, the pairing induces an isomorphism of the two discriminant groups $(\mathbb{Z}v)^*/\mathbb{Z}v$ and Λ^*/Λ, since $\mathbb{Z}v$ and Λ are a pair of primitive sublattices, which are orthogonal complements in the unimodular lattice $\tilde{\Lambda}$. We conclude that the order $|(v, v)|$ of $(\mathbb{Z}v)^*/\mathbb{Z}v$ is equal to the order $2n - 2$ of Λ^*/Λ. Finally, $(v, v) > 0$, by comparing the signatures of Λ and $\tilde{\Lambda}$.

Let $\bar{v}$ be the coset $v + \mathbb{Z}\beta$ in Λ_{k3}. Then $\bar{v}$ is of Hodge-type $(1, 1)$ and $(\bar{v}, \bar{v}) = 2n - 2$. Hence $S(\alpha)$ is a projective $K3$ surface (even if X is not projective). We may further choose the Hodge isometry $\eta : H^2(S(\alpha),\mathbb{Z}) \to \Lambda_{k3}$, so that that $\bar{v}$ corresponds to a class in the positive cone of $S(\alpha)$, possibly after replacing v by $-v$. We may further assume that $\bar{v}$ corresponds to a nef class of $S(\alpha)$, possibly after replacing η with $\eta \circ w$, where w is an element of the subgroup $W \subset O^+(H^2(S(\alpha),\mathbb{Z}))$, generated by reflections by classes of smooth rational curves on $S(\alpha)$ [19, Prop. 1.9].

4.2 A Period Domain as an Affine Line Bundle Over Another

Keep the notation of Sect. 4.1. Set $\Lambda := H^2(X,\mathbb{Z})$. Set $d := \mathrm{div}(\alpha,\bullet)$. Let $\alpha^{\perp}_{\Lambda}$ be the (degenerate) lattice orthogonal to α in Λ. Set $Q_\alpha := \alpha^{\perp}_{\Lambda}/\mathbb{Z}\alpha$.

Lemma 4.1. *Q_α is isometric to the sub-lattice $\bar{v}^{\perp}$ of Λ_{k3} and both are isometric to the orthogonal direct sum*

$$E_8(-1) \oplus E_8(-1) \oplus U \oplus U \oplus \mathbb{Z}\lambda,$$

where $(\lambda,\lambda) = \frac{2-2n}{d^2}$.

Proof. The $K3$ lattice $\Lambda_{k3} := [\beta^{\perp}_{\tilde{\Lambda}}]/\mathbb{Z}\beta$ is isometric to $E_8(-1) \oplus E_8(-1) \oplus U \oplus U \oplus U$. Let L be the saturation of $\mathrm{span}_{\mathbb{Z}}\{v,\beta\}$ in $\tilde{\Lambda}$. Then L is contained in $\beta^{\perp}_{\tilde{\Lambda}}$ and the image of L in Λ_{k3} is spanned by a class ξ of self-intersection $\frac{2n-2}{d^2}$, such that $\bar{v} = d\xi$, by Lemma 2.5.

It remains to prove that Q_α is isometric to $\xi^{\perp}_{\Lambda_{k3}}$. Consider the following commutative diagram.

$$\begin{array}{ccccccccc}
0 \to & \mathbb{Z}\beta & \to & \beta^{\perp}_{\tilde{\Lambda}} & \to & \Lambda_{k3} & \to 0 \\
 & = \uparrow & & \uparrow & & \uparrow\ j & \\
0 \to & \mathbb{Z}\beta & \to & L^{\perp}_{\tilde{\Lambda}} & \to & L^{\perp}_{\tilde{\Lambda}}/\mathbb{Z}\beta & \to 0 \\
 & \cong \uparrow & & \cong \uparrow\ \iota & & \uparrow\ \bar{\iota} & \\
0 \to & \mathbb{Z}\alpha & \to & \alpha^{\perp}_{\Lambda} & \to & Q_\alpha & \to 0.
\end{array}$$

The lower vertical arrow $\bar{\iota}$ in the rightmost column is evidently an isomorphism. The image of the upper one j is precisely $\xi^{\perp}_{\Lambda_{k3}}$. □

Let Ω_Λ be the period domain

$$\Omega_\Lambda := \{\ell \in \mathbb{P}[H^2(X,\mathbb{C})] \ :\ (\ell,\ell) = 0 \text{ and } (\ell,\bar{\ell}) > 0\}. \quad (4.1)$$

Set

$$\Omega_{\alpha^\perp} := \{\ell \in \Omega_\Lambda \ :\ (\ell,\alpha) = 0\}. \quad (4.2)$$

Then $\Omega_{\alpha^\perp}$ is an affine line-bundle over the period domain

$$\Omega_{Q_\alpha} \ := \ \{\ell \in \mathbb{P}[Q_\alpha \otimes_{\mathbb{Z}} \mathbb{C}] \ : \ (\ell,\ell)=0 \text{ and } (\ell,\bar{\ell})>0\}.$$

Given a point of Ω_{Q_α}, corresponding to a one-dimensional subspace ℓ of $Q_\alpha \otimes_{\mathbb{Z}} \mathbb{C}$, we get a two dimensional subspace V_ℓ of $H^2(X,\mathbb{C})$ orthogonal to α and containing α. The line in $\Omega_{\alpha^\perp}$, over the point ℓ of Ω_{Q_α}, is $\mathbb{P}[V_\ell]\setminus\{\mathbb{P}[\mathbb{C}\alpha]\}$. Denote by

$$q \ : \ \Omega_{\alpha^\perp} \ \to \ \Omega_{Q_\alpha} \tag{4.3}$$

the bundle map. A *semi-polarized* $K3$ surface of degree k is a pair consisting of a $K3$ surface together with a nef line bundle of degree k (also known as weak algebraic polarization of degree k in [33, Section 5]). Note that each component of Ω_{Q_α} is isomorphic to the period domain of the moduli space of semi-polarized $K3$ surfaces of degree $\frac{2n-2}{d^2}$.

Definition 4.2. Fibers of q will be called *Tate-Shafarevich lines* for reasons that will become apparent in Sect. 7.

Tate-Shafarevich lines are limits of twistor lines, as will be explained in Remark 4.6.

4.3 The Period Map

Given a period $\ell \in \Omega_\Lambda$, set $\Lambda^{1,1}(\ell,\mathbb{Z}) := \{\lambda \in \Lambda \ : \ (\lambda,\ell)=0\}$. Define $Q_\alpha^{1,1}(q(\ell),\mathbb{Z})$ similarly. We get the short exact sequence

$$0 \to \mathbb{Z}\alpha \to \left[\alpha^\perp \cap \Lambda^{1,1}(\ell,\mathbb{Z})\right] \to Q_\alpha^{1,1}(q(\ell),\mathbb{Z}) \to 0.$$

$\Omega_{\alpha^\perp}$ has two connected components, since Ω_{Q_α} has two connected components. Indeed, Q_α has signature $(2, b_2(X)-4)$, and a period ℓ comes with an oriented positive definite plane $[\ell \oplus \bar{\ell}] \cap [\Lambda_{\mathbb{R}}]$, which, in turn, determines the orientation of the positive cone in $Q_\alpha \otimes_{\mathbb{Z}} \mathbb{R}$.

The positive cone $\tilde{\mathscr{C}}_\Lambda$ in $\Lambda_{\mathbb{R}}$ is the cone

$$\tilde{\mathscr{C}}_\Lambda := \{x \in \Lambda_{\mathbb{R}} \ : \ (x,x)>0\}. \tag{4.4}$$

The cohomology group $H^2(\tilde{\mathscr{C}}_\Lambda,\mathbb{Z})$ is isomorphic to $\mathbb{Z}$ and an *orientation* of $\tilde{\mathscr{C}}_\Lambda$ is the choice of one of the two generator of $H^2(\tilde{\mathscr{C}}_\Lambda,\mathbb{Z})$. An orientation of $\tilde{\mathscr{C}}_\Lambda$ determines an orientation of every positive definite three dimensional subspace of $\Lambda_{\mathbb{R}}$ [23, Lemma 4.1]. A choice of an orientation of $\tilde{\mathscr{C}}_\Lambda$ determines a choice of a component of $\Omega_{\alpha^\perp}$ as follows. A period $\ell \in \Omega_\Lambda$ determines the subspace $\Lambda^{1,1}(\ell,\mathbb{R})$ and the cone $\mathscr{C}'_\ell := \{x \in \Lambda^{1,1}(\ell,\mathbb{R}) \ : \ (x,x)>0\}$ in $\Lambda^{1,1}(\ell,\mathbb{R})$ has two connected components. A choice of a connected component of $\mathscr{C}'_\ell$ is equivalent to a choice of an orientation of the positive cone of $\tilde{\mathscr{C}}_\Lambda$. Indeed, a non-zero element $\sigma \in \ell$ and an element $\omega \in \mathscr{C}'_\ell$ determine a basis $\{\mathrm{R}e(\sigma), \mathrm{I}m(\sigma), \omega\}$, hence an orientation, of a positive definite three dimensional subspace of $\Lambda_{\mathbb{R}}$, and the corresponding orientation of $\tilde{\mathscr{C}}_\Lambda$ is independent of the choice of σ and ω. Thus, the choice of the orientation of the positive cone $\tilde{\mathscr{C}}_\Lambda$ determines a connected component $\mathscr{C}_\ell$ of $\mathscr{C}'_\ell$, called the *positive cone* (for the orientation). If ℓ belongs to $\Omega_{\alpha^\perp}$, then the class α belongs to $\Lambda^{1,1}(\ell, \mathbb{R})$ and α is in the closure of precisely one of the two connected components of $\mathscr{C}'_\ell$. The connected component of $\Omega_{\alpha^\perp}$, compatible with the chosen orientation of $\tilde{\mathscr{C}}_\Lambda$, is the one for which α belongs to the boundary of the positive cone $\mathscr{C}_\ell$ for the chosen orientation.

A *marked pair* (Y, ψ) consists of an irreducible holomorphic symplectic manifold Y and an isometry ψ from $H^2(Y, \mathbb{Z})$ onto a fixed lattice. The moduli space of isomorphism classes of marked pairs is a non-Hausdorff complex manifold [13]. Let $\mathfrak{M}^0_\Lambda$ be a connected component of the moduli space of marked pairs of $K3^{[n]}$-type, where the fixed lattice is Λ. The *period map*

$$P_0 : \mathfrak{M}^0_\Lambda \to \Omega_\Lambda$$

sends a marked pair (Y, ψ) to the point $\psi(H^{2,0}(Y))$ of Ω_Λ. P_0 is a holomorphic map and a local homeomorphism [4]. The positive cone $\mathscr{C}_Y$ is the connected component of the cone $\{x \in H^{1,1}(Y, \mathbb{R}) \ : \ (x, x) > 0\}$ containing the Kähler cone. Hence, the positive cone in $H^2(Y, \mathbb{R})$ comes with a canonical orientation and the marking ψ determines an orientation of the positive cone in $\tilde{\mathscr{C}}_\Lambda$. We conclude that $\mathfrak{M}^0_\Lambda$ determines an orientation of the positive cone $\tilde{\mathscr{C}}_\Lambda$ [23, Sec. 4]. Let

$$\Omega^+_{\alpha^\perp} \tag{4.5}$$

be the connected component of $\Omega_{\alpha^\perp}$, inducing the same orientation of $\tilde{\mathscr{C}}_\Lambda$ as $\mathfrak{M}^0_\Lambda$. Let

$$\mathfrak{M}^0_{\alpha^\perp} \tag{4.6}$$

be the inverse image $P_0^{-1}(\Omega^+_{\alpha^\perp})$.

Theorem 4.3 (The Global Torelli Theorem [14, 40]). *The period map $P_0 : \mathfrak{M}^0_\Lambda \to \Omega_\Lambda$ is surjective. Any two points in the same fiber of P_0 are inseparable. If (X_1, η_1) and (X_2, η_2) correspond to two inseparable points in $\mathfrak{M}^0_\Lambda$, then X_1 and X_2 are bimeromorphic. If the Kähler cone of X is equal to its positive cone and (X, η) corresponds to a point of $\mathfrak{M}^0_\Lambda$, then this point is separated.*

Lemma 4.4. *$\mathfrak{M}^0_{\alpha^\perp}$ is path-connected.*

Proof. The statement follows from the Global Torelli Theorem 4.3 and the fact that $\Omega^+_{\alpha^\perp}$ is connected. The proof is similar to that of [24, Proposition 5.11]. □

Proposition 4.5. *Let X_1 and X_2 be two irreducible holomorphic symplectic manifolds of $K3^{[n]}$-type and $\eta_j : H^2(X_j,\mathbb{Z}) \to \Lambda$, $j = 1,2$, isometries. The marked pairs (X_1,η_1) and (X_2,η_2) belong to the same connected moduli space $\mathfrak{M}^0_{\alpha^\perp}$, provided the following conditions hold.*

(1) The $O(\tilde{\Lambda})$ orbits $\iota_{X_j} \circ \eta_j^{-1}$, $j = 1,2$, are equal. Above ι_{X_j} is the canonical $O(\tilde{\Lambda})$-orbit of primitive isometric embeddings of $H^2(X_j,\mathbb{Z})$ into $\tilde{\Lambda}$ mentioned in Theorem 2.2.
(2) $\eta_2^{-1} \circ \eta_1 : H^2(X_1,\mathbb{Z}) \to H^2(X_2,\mathbb{Z})$ is orientation preserving.
(3) $\eta_j^{-1}(\alpha)$ is of Hodge type $(1,1)$ and it belongs to the boundary of the positive cone $\mathscr{C}_{X_j}$ in $H^{1,1}(X_j,\mathbb{R})$, for $j = 1,2$.

Proof. Conditions 1 and 2 imply that $\eta_2^{-1} \circ \eta_1$ is a parallel-transport operator, by Theorem 2.4. Hence, the two marked pairs belong to the same connected component $\mathfrak{M}^0_\Lambda$ of $\mathfrak{M}_\Lambda$. Condition 3 implies that both belong to $\mathfrak{M}^0_{\alpha^\perp}$, and the latter is connected, by Lemma 4.4. □

Proof (of Proposition 1.7). Lemma 2.5 introduced the monodromy invariant $[L,v](c_1(\mathscr{L}))$ of the pair $(X,\mathscr{L})$. The claimed number of deformation types in the statement of the proposition is equal to the number of values of the monodromy invariant $[L,v](\bullet)$ for fixed n and d, by Lemma 2.6. Assume given another pair $(X',\mathscr{L}')$ as above, such that the monodromy invariants $[L,v](c_1(\mathscr{L}'))$ and $[L,v](c_1(\mathscr{L}))$ are equal. Choose a parallel transport operator $g : H^2(X',\mathbb{Z}) \to H^2(X,\mathbb{Z})$. We do not assume that $g(c_1(\mathscr{L}'))$ is of Hodge type $(1,1)$. Set $\alpha := c_1(\mathscr{L})$ and $\alpha' := c_1(\mathscr{L}')$. The monodromy invariant $[L,v](g(\alpha'))$ is equal to $[L,v](\alpha')$ and hence also to $[L,v](\alpha)$. Hence, there exists a monodromy operator $f \in Mon^2(X)$, such that $fg(\alpha') = \alpha$, by Lemma 2.5. Choose a marking $\eta : H^2(X,\mathbb{Z}) \to \Lambda$. Then $\eta' := \eta \circ f \circ g$ is a marking of X' satisfying $\eta(\alpha) = \eta'(\alpha')$. Hence, the triples (X,α,η) and (X',α',η') both belong to the moduli space $\mathfrak{M}^0_{\eta(\alpha)^\perp}$, by Proposition 4.5. $\mathfrak{M}^0_{\eta(\alpha)^\perp}$ is connected, by Lemma 4.4. Hence, $(X,\mathscr{L})$ and $(X',\mathscr{L}')$ are deformation equivalent. □

Remark 4.6. Tate-Shafarevich lines (Definition 4.2) are limits of twistor lines in the following sense. Let ℓ be a point of Ω_Λ and ω a class in the positive cone $\mathscr{C}_\ell$ in $\Lambda^{1,1}(\ell,\mathbb{R})$. Assume that ω is not orthogonal to any class in $\Lambda^{1,1}(\ell,\mathbb{Z})$. Then there exists a marked pair (X,η) in each connected component $\mathfrak{M}^0_\Lambda$ of the moduli space of marked pairs, such that $P(X,\eta) = \ell$ and $\eta^{-1}(\omega)$ is a Kähler class of X [13, Cor. 5.7]. Set $W' := \ell \oplus \bar{\ell} \oplus \mathbb{C}\omega$. $\mathbb{P}(W') \cap \Omega_\Lambda$ is a *twistor line* for (X,η); it admits a canonical lift to a smooth rational curve in $\mathfrak{M}^0_\Lambda$ containing the point (X,η) [13, Cor. 5.8]. This lift corresponds to an action of the quaternions $\mathbb{H}$ on the real tangent bundle of the differentiable manifold X, such that the unit quaternions act as integrable complex structures, one of which is the complex structure of X. Let $\alpha \in \Lambda$ be the primitive isotropic class as above. Assume that ℓ belongs to $\Omega^+_{\alpha^\perp}$. Consider the three dimensional subspace $W := \ell \oplus \bar{\ell} \oplus \mathbb{C}\alpha$ of $H^2(X,\mathbb{C})$. Then W is a limit of a sequence of three dimensional subspaces W'_i, associated to some sequence of classes ω_i as above, since α belongs to the boundary of the positive cone $\mathscr{C}_\ell$. Now W is contained in $\alpha^\perp$, and so $\mathbb{P}(W) \cap \Omega_{\alpha^\perp} = \mathbb{P}(W) \cap \Omega_\Lambda$. In this degenerate case, the conic $\mathbb{P}(W) \cap \Omega_\Lambda$ consists of two irreducible components, the Tate-Shafarevich line $\mathbb{P}[\ell \oplus \mathbb{C}\alpha] \setminus \{\mathbb{P}[\mathbb{C}\alpha]\}$ in $\Omega^+_{\alpha^\perp}$ and the line $\mathbb{P}[\bar{\ell} \oplus \mathbb{C}\alpha] \setminus \{\mathbb{P}[\mathbb{C}\alpha]\}$ in the other connected component $\Omega^-_{\alpha^\perp}$ of $\Omega_{\alpha^\perp}$. Theorem 7.11 will provide a lift of a generic Tate-Shafarevich line in the period domain to a line in the moduli space of marked pairs.

A summary of notation related to lattices and period domains

U	The rank 2 even unimodular lattice of signature $(1,1)$
$E_8(-1)$	The root lattice of type E_8 with a negative definite pairing
$\widetilde{\Lambda}$	The Mukai lattice; the orthogonal direct sum $U^{\oplus 4} \oplus E_8(-1)^{\oplus 2}$
Λ	The $K3^{[n]}$-lattice; the orthogonal direct sum $U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus \langle 2-2n \rangle$, where $\langle 2-2n \rangle$ is the rank 1 lattice generated by a class of self-intersection $2-2n$
α	A primitive isotropic class in Λ
Q_α	The subquotient $\alpha^\perp/\mathbb{Z}\alpha$
ι	A primitive embedding of Λ in $\widetilde{\Lambda}$
β	The primitive isotropic class $\iota(\alpha)$ in $\widetilde{\Lambda}$
Λ_{k3}	The subquotient $\beta^\perp/\mathbb{Z}\beta$, which is isomorphic to the $K3$ lattice $U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$
v	A generator of the rank 1 sublattice of $\widetilde{\Lambda}$ orthogonal to $\iota(\Lambda)$
$\bar{v}$	The coset $v + \mathbb{Z}\beta$ in Λ_{k3}
d	The divisibility of $(\alpha, \bullet)$ in Λ^*; $d := \gcd\{(\alpha,\lambda) : \lambda \in \Lambda\}$
ξ	The integral element $(1/d)\bar{v}$ of Λ_{k3}. We have $(\xi,\xi) = \frac{2n-2}{d^2}$
Ω_Λ	The period domain given in (4.1)
$\widetilde{\mathscr{C}}_\Lambda$	The positive cone given in (4.4)
Ω^+_Λ	The connected component of Ω_Λ determined by the orientation of $\widetilde{\mathscr{C}}_\Lambda$
$\Omega_{\alpha^\perp}$	The hyperplane section of Ω_Λ given in (4.2)
$\Omega^+_{\alpha^\perp}$	The connected component of $\Omega_{\alpha^\perp}$ given in (4.5)
Ω_{Q_α}	The period domain of the lattice Q_α
q	The fibration $q : \Omega_{\alpha^\perp} \to \Omega_{Q_\alpha}$ by Tate-Shafarevich lines given in (4.3)
$\mathfrak{M}^0_\Lambda$	A connected component of the moduli space of marked pairs
P_0	The period map $P_0 : \mathfrak{M}^0_\Lambda \to \Omega^+_\Lambda$
$\mathfrak{M}^0_{\alpha^\perp}$	The inverse image of $\Omega^+_{\alpha^\perp}$ in $\mathfrak{M}^0_\Lambda$ via P_0
$[L,v](\alpha)$	The monodromy invariant associated to the class α in Lemma 2.5 (4)

5 Density of Periods of Relative Compactified Jacobians

We keep the notation of Sect. 4. In Sect. 5.1 we construct a section $\tau : \Omega^+_{Q_\alpha} \to \Omega^+_{\alpha^\perp}$, given in (5.2), of the fibration $q : \Omega^+_{\alpha^\perp} \to \Omega^+_{Q_\alpha}$ by Tate-Shafarevich lines. We then show that τ maps a period $\underline{\ell}$, of a semi-polarized $K3$ surface $(S, \mathscr{B})$ in the period domain $\Omega^+_{Q_\alpha}$, to the period $\tau(\underline{\ell})$ of a moduli space M of sheaves on S

with pure one-dimensional support in the linear system $|\mathscr{B}^d|$. The moduli space M admits a Lagrangian fibration over $|\mathscr{B}^d|$. In Sect. 5.2 we construct an injective homomorphism $g : Q_\alpha \to O(\Lambda)$, whose image is contained in the subgroup of the monodromy group which stabilizes α. We get an action of Q_α on the period domain $\Omega^+_{\alpha^\perp}$, which lifts to an action on connected components $\mathfrak{M}^0_{\alpha^\perp}$ of the moduli space of marked pairs given in Eq. (4.6). We then show that the fibration q by Tate-Shafarevich lines is $g(Q_\alpha)$-invariant. In Sect. 5.3 we prove that the $g(Q_\alpha)$-orbit of every point in a non-special Tate-Shafarevich line is dense in that line. Consequently, the non-special Tate-Shafarevich line $q^{-1}(\underline{\ell})$ contains the dense orbit $g(Q_\alpha)\tau(\underline{\ell})$ of periods of marked pairs in $\mathfrak{M}^0_{\alpha^\perp}$ admitting a Lagrangian fibration.

Conventions: The discussion in the current Sect. 5 concerns only period domains, so we are free to choose the embedding ι. When we consider in subsequent sections a component $\mathfrak{M}^0_\Lambda$ of the moduli space of marked pairs (X,η) of $K3^{[n]}$-type, together with such an embedding $\iota : \Lambda \to \tilde{\Lambda}$, we will always assume that ι is chosen so that $\iota \circ \eta$ belongs to the canonical $O(\tilde{\Lambda})$-orbit ι_X of Theorem 2.2, for all (X,η) in $\mathfrak{M}^0_\Lambda$. We choose the orientation of the positive cone $\tilde{\mathscr{C}}_\Lambda$ of Λ, so that α belongs to the boundary of the positive cone in $\Lambda^{1,1}(\ell,\mathbb{R})$, for every $\ell \in \Omega^+_{\alpha^\perp}$. We choose the orientation of the positive cone $\tilde{\mathscr{C}}_{\Lambda_{k3}}$, so that $\bar{v}$ belongs to the positive cone in $\Lambda^{1,1}_{k3}(\underline{\ell},\mathbb{R})$, for every $\underline{\ell} \in \Omega^+_{\bar{v}^\perp}$. Note that the composition $\alpha^\perp_\Lambda \xrightarrow{\iota} \beta^\perp_{\tilde{\Lambda}} \to \Lambda_{k3}$ induces an isometry from $Q_\alpha := \alpha^\perp_\Lambda/\mathbb{Z}\alpha$ onto $\bar{v}^\perp_{\Lambda_{k3}}$, by Lemma 4.1. The choice of orientation of the positive cone of Λ_{k3} determines an orientation of the positive cone of Q_α.

5.1 A Period of a Lagrangian Fibration in Each Tate-Shafarevich Line

Choose a class γ in $\tilde{\Lambda}$ satisfying $(\gamma,\beta) = -1$ and $(\gamma,\gamma) = 0$. Note that β and γ span a unimodular sub-lattice of $\tilde{\Lambda}$ of signature $(1,1)$. We construct next a section of the affine bundle $q : \Omega_{\alpha^\perp} \to \Omega_{Q_\alpha}$, given in Eq. (4.3), in terms of γ. We have the following split short exact sequence.

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}\beta \arrow[r] & \beta^\perp_{\tilde{\Lambda}} \arrow[r, "j"] \arrow[l, bend left, "\sigma_\gamma"] & \Lambda_{k3} \arrow[r] \arrow[l, bend left, "\tilde{\tau}_\gamma"] & 0. \end{tikzcd} \tag{5.1}$$

Above, $\sigma_\gamma(x) = -(x,\gamma)\beta$, and $\tilde{\tau}_\gamma(y) = \tilde{y} + (\tilde{y},\gamma)\beta$, where $\tilde{y}$ is any element of $\beta^\perp_{\tilde{\Lambda}}$ satisfying $j(\tilde{y}) = y$. One sees that $\tilde{\tau}_\gamma$ is well defined as follows. If $\tilde{y}_1$ and $\tilde{y}_2$ satisfy $j(\tilde{y}_k) = y$, then the difference $[\tilde{y}_1 + (\tilde{y}_1,\gamma)\beta] - [\tilde{y}_2 + (\tilde{y}_2,\gamma)\beta]$ belongs to the kernel of j and is sent to 0 via σ_γ, so the difference is equal to 0. Note that $\tilde{\tau}_\gamma$ is an isometric embedding and its image is precisely $\{\beta,\gamma\}^\perp_{\tilde{\Lambda}}$.

We regard $\Omega^+_{Q_\alpha}$ as the period domain for semi-polarized $K3$ surfaces, with a nef line bundle of degree $\frac{2n-2}{d^2}$, via the isomorphism $\bar{v}^\perp_{\Lambda_{k3}} \cong Q_\alpha$ of Lemma 4.1. The homomorphism $\iota^{-1} \circ \tilde{\tau}_\gamma$ induces an isometric embedding of Q_α in $\alpha^\perp_\Lambda$. We get a section

$$\tau_\gamma \ : \ \Omega^+_{Q_\alpha} \to \Omega^+_{\alpha^\perp} \tag{5.2}$$

of $q : \Omega^+_{\alpha^\perp} \to \Omega^+_{Q_\alpha}$. Following is an explicit description of τ_γ. Let $\underline{\ell}$ be a period in $\Omega^+_{Q_\alpha}$. Choose a period ℓ in $\Omega^+_{\alpha^\perp}$ satisfying $q(\ell) = \underline{\ell}$. Let x be a non-zero element of the line ℓ in $\alpha^\perp_\Lambda \otimes_{\mathbb{Z}} \mathbb{C}$. Then

$$\tau_\gamma(\underline{\ell}) = \mathrm{span}_{\mathbb{C}}\{x + (\iota(x), \gamma)\alpha\}. \tag{5.3}$$

We see that γ belongs to $\tilde{\Lambda}^{1,1}(\tau_\gamma(\underline{\ell}))$, for every $\underline{\ell}$ in $\Omega^+_{Q_\alpha}$.

Fix a period $\underline{\ell}$ in $\Omega^+_{Q_\alpha}$. We construct next a marked pair $(M_H(u), \eta_1)$ with period $\tau_\gamma(\underline{\ell})$, such that $\eta_1^{-1}(\alpha)$ induces a Lagrangian fibration. Let S be a $K3$ surface and $\eta : H^2(S, \mathbb{Z}) \to \Lambda_{k3}$ a marking, such that the period $\eta(H^{2,0}(S))$ is $\underline{\ell}$. Such a marked pair (S, η) exists, by the surjectivity of the period map. Extend η to the Hodge isometry

$$\tilde{\eta} \ : \ H^*(S, \mathbb{Z}) \to \tilde{\Lambda},$$

given by $\tilde{\eta}((0, 0, 1)) = \beta$, $\tilde{\eta}((1, 0, 0)) = \gamma$, and $\tilde{\eta}$ restricts to $H^2(S, \mathbb{Z})$ as $\tilde{\tau}_\gamma \circ \eta$. We have the equality $v = \sigma_\gamma(v) + \tilde{\tau}_\gamma(\bar{v}) = -(\gamma, v)\beta + \tilde{\tau}_\gamma(\bar{v})$. Set $a := -(\gamma, v)$ and $u := (0, \eta^{-1}(\bar{v}), a)$. Then $\tilde{\eta}(u) = v$. We may choose the marking η so that the class $\eta^{-1}(\bar{v})$ is nef, possibly after replacing η by $\pm\eta \circ w$, where w is an element of the group of isometries of $H^2(S, \mathbb{Z})$, generated by reflections by -2 curves [2, Ch. VIII Prop. 3.9]. Choose a u-generic polarization H of S. Then $M_H(u)$ is a projective irreducible holomorphic symplectic manifold. Let

$$\theta : u^\perp \to H^2(M_H(u), \mathbb{Z})$$

be Mukai's isometry, given in Eq. (3.1). We get the commutative diagram:

$$\begin{tikzcd}
\Lambda \arrow[r, "\iota", "\cong"'] & v^\perp \arrow[r, "\subset"] & \widetilde{\Lambda} \\
H^2(M_H(u), \mathbb{Z}) \arrow[u, "\eta_1"] \arrow[r, "\theta^{-1}"] & u^\perp \arrow[u, "\eta_2"] \arrow[r, "\subset"] & H^*(S, \mathbb{Z}), \arrow[u, "\tilde{\eta}"]
\end{tikzcd} \tag{5.4}$$

where η_2 is the restriction of $\tilde{\eta}$ and $\eta_1 = \iota^{-1} \circ \eta_2 \circ \theta^{-1}$. Note that $\eta_1(\theta(0, 0, 1)) = \alpha$. Let L be the saturation in $H^*(S, \mathbb{Z})$ of the sub-lattice spanned by $(0, 0, 1)$ and u. Let b be an integer satisfying $ab \equiv 1$ (modulo d). The monodromy invariant

$[L,u](\theta(0,0,1))$ of Lemma 2.5 is the isometry class of the pair $(L_{n,d},(d,b))$, by the commutativity of the above diagram. Furthermore, η_1 is a Hodge isometry with respect to the Hodge structure on Λ induced by $\tau_\gamma(\underline{\ell})$. In particular, $(M_H(u),\eta_1)$ is a marked pair with period $\tau_\gamma(\underline{\ell})$. Example 3.1 exhibits $\theta(0,0,1)$ as the class $\pi^*c_1(\mathscr{O}_{|\mathscr{L}^d|}(1))$, for a Lagrangian fibration $\pi: M_H(u)\to|\mathscr{L}^d|$, where $\mathscr{L}$ is the line bundle over S with class $\eta^{-1}(\xi)$.

Remark 5.1. The isometry η_1 is compatible with the orientations of the positive cones, the canonical one of $H^2(M_H(u),\mathbb{Z})$ and the chosen one of Λ. Indeed, it maps the class $\theta(0,0,1)$, on the boundary of the positive cone of $H^{1,1}(M_H(u),\mathbb{R})$, to the class α on the boundary of the positive cone of $\Lambda^{1,1}(\tau_\gamma(\underline{\ell}),\mathbb{R})$. The composition $\tilde{\eta}\circ\theta^{-1}$ in Diagram (5.4) belongs to the canonical orbit $\iota_{M_H(u)}$ of Theorem 2.2, by [22, Theorem 1.14]. The commutativity of the Diagram implies that the isometric embedding $\iota\circ\eta_1$ also belongs to the orbit $\iota_{M_H(u)}$.

5.2 Monodromy Equivariance of the Fibration by Tate-Shafarevich Lines

Denote by $O(\tilde{\Lambda})^+_{\beta,v}$ the subgroup of $O(\tilde{\Lambda})^+$ stabilizing both β and v. Following is a natural homomorphism

$$h: O(\tilde{\Lambda})^+_{\beta,v}\to O(\Lambda_{k3})_{\bar{v}}. \tag{5.5}$$

If ψ belongs to $O(\tilde{\Lambda})^+_{\beta,v}$, then $\psi(\beta)=\beta$ and $\beta^\perp_{\tilde{\Lambda}}$ is ψ-invariant. Thus ψ induces an isometry $h(\psi)$ of $\Lambda_{k3}:=\beta^\perp_{\tilde{\Lambda}}/\mathbb{Z}\beta$. We construct next a large subgroup in the kernel of h.

Given an element z of $\tilde{\Lambda}$, orthogonal to β and v, define the map $\tilde{g}_z:\tilde{\Lambda}\to\tilde{\Lambda}$ by

$$\tilde{g}_z(x) \;:=\; x-(x,\beta)z+\left[(x,z)-\frac{1}{2}(x,\beta)(z,z)\right]\beta.$$

Lemma 5.2. *The map $\tilde{g}_z$ is the unique isometry in $O(\tilde{\Lambda})_{\beta,v}$, which sends γ to an element of $\tilde{\Lambda}$ congruent to $\gamma+z$ modulo $\mathbb{Z}\beta$ and belongs to the kernel of h. The isometry $\tilde{g}_z$ is orientation preserving.*

Proof. We first define an isometry f with the above property, then prove its uniqueness, and finally prove that it is equal to $\tilde{g}_z$. Set $\gamma_1:=\gamma+z+\left[(\gamma,z)+\frac{1}{2}(z,z)\right]\beta$. Then $(\gamma_1,\gamma_1)=0$, $(\gamma_1,\beta)=-1$, and γ_1 is the unique element of $\tilde{\Lambda}$ satisfying the above equalities and congruent to $\gamma+z$ modulo $\mathbb{Z}\beta$. Define $\tilde{\sigma}_\gamma:\tilde{\Lambda}\to\mathbb{Z}\beta+\mathbb{Z}\gamma$ by $\tilde{\sigma}_\gamma(x):=-(x,\beta)\gamma-(x,\gamma)\beta$. We get the commutative diagram with split short exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}\beta + \mathbb{Z}\gamma \arrow[r] \arrow[d] & \widetilde{\Lambda} \arrow[r, "\tilde{j}"] \arrow[d, "f"] \arrow[l, bend right, "\tilde{\sigma}_\gamma"'] & \Lambda_{k3} \arrow[r] \arrow[d, "id"] \arrow[l, bend right, "\tilde{\tau}_\gamma"'] & 0 \\
0 \arrow[r] & \mathbb{Z}\beta + \mathbb{Z}\gamma_1 \arrow[r] & \widetilde{\Lambda} \arrow[r, "\tilde{j}_1"] \arrow[l, bend left, "\tilde{\sigma}_{\gamma_1}"] & \Lambda_{k3} \arrow[r] \arrow[l, bend left, "\tilde{\tau}_{\gamma_1}"] & 0
\end{tikzcd}$$

Above $\tilde{\tau}_\gamma$ and j are the homomorphisms given in Eq. (5.1), $\tilde{j}(x) = j(x + (x, \beta)\gamma)$, and $\tilde{\sigma}_{\gamma_1}$, $\tilde{\tau}_{\gamma_1}$, and $\tilde{j}_1$ are defined similarly, replacing γ by γ_1. The map f is defined by $f(\beta) = \beta$, $f(\gamma) = \gamma_1$, and $f(\tilde{\tau}_\gamma(y)) = \tilde{\tau}_{\gamma_1}(y)$. Then f is clearly an isometry.

The isometry f can be extended to an isometry of $\tilde{\Lambda}_{\mathbb{R}}$ and we can continuously deform z to 0 in $\{\beta, v\}^\perp \otimes_{\mathbb{Z}} \mathbb{R}$, resulting in a continuous deformation of f to the identity. Hence, f is orientation preserving.

Note the equalities $\tilde{\sigma}_\gamma(v) = -(v, \gamma)\beta = -(v, \gamma_1)\beta = \tilde{\sigma}_{\gamma_1}(v)$, where the middle one follows from that fact that both z and β are orthogonal to v. We get the equality

$$\tilde{\tau}_\gamma(\bar{v}) = v - \tilde{\sigma}_\gamma(v) = v - \tilde{\sigma}_{\gamma_1}(v) = \tilde{\tau}_{\gamma_1}(\bar{v}).$$

Thus $f(v) = v$ and f belongs to $O(\tilde{\Lambda})^+_{\beta,v}$. Let x be an element of $\beta^\perp$. Then $\tilde{j}(x) = j(x) = \tilde{j}_1(x)$. Set $y := j(x)$. Now $\tilde{\tau}_\gamma(y) \equiv \tilde{\tau}_{\gamma_1}(y)$ modulo $\mathbb{Z}\beta$, by definition of both. Hence, $h(f)$ is the identity isometry of Λ_{k3}.

Let f' be another isometry of $\tilde{\Lambda}$ satisfying the assumptions of the Lemma. Then $f'(\gamma) = \gamma_1$, by the characterization of γ_1 mentioned above. Set $e := f^{-1} \circ f'$. Then $e(\beta) = \beta$, $e(\gamma) = \gamma$, $e(v) = v$, and $h(e) = id$. Given $x \in \beta^\perp$, we get that $e(x) \equiv x$ modulo $\mathbb{Z}\beta$. Now $(e(x), \gamma) = (e(x), e(\gamma)) = (x, \gamma)$. Thus, e restricts to the identity on $\beta^\perp$. We conclude that e is the identity of $\tilde{\Lambda}$, as the latter is spanned by γ and $\beta^\perp$. Thus $f' = f$.

It remains to prove the equality $f = \tilde{g}_z$. We already know that $f(\gamma) = \gamma_1 = \tilde{g}_z(\gamma)$ and $f(\beta) = \beta = \tilde{g}_z(\beta)$. Given $y \in \Lambda_{k3}$, we have

$$\tilde{g}_z(\tilde{\tau}_\gamma(y)) = \tilde{\tau}_\gamma(y) + (\tilde{\tau}_\gamma(y), z)\beta = \tilde{\tau}_{\gamma_1}(y) = f(\tilde{\tau}_\gamma(y)).$$

Hence, $\tilde{g}_z = f$. □

Let

$$\tilde{g} : \alpha^\perp_\Lambda \to O(\tilde{\Lambda})^+_{\beta,v}$$

be the map sending z to $\tilde{g}_{\iota(z)}$. Denote by $Mon^2(\Lambda, \iota)$ the subgroup of $O^+(\Lambda)$ of isometries stabilizing the orbit $O(\tilde{\Lambda})\iota$. Note that $O(\tilde{\Lambda})^+_v$ is conjugated via ι onto $Mon^2(\Lambda, \iota)$, if $n = 2$, and to an index 2 subgroup of $Mon^2(\Lambda, \iota)$, if $n \geq 2$ [21, Lemma 4.10]. Let $Mon^2(\Lambda, \iota)_\alpha$ be the subgroup of $Mon^2(\Lambda, \iota)$ stabilizing α.

Lemma 5.3. *(1) The map $\tilde{g}$ is a group homomorphism with kernel $\mathbb{Z}\alpha$. It thus factors through an injective homomorphism*

$$g : Q_\alpha \to Mon^2(\Lambda, \iota)_\alpha.$$

(2) Let z be an element of $\alpha^\perp_\Lambda$ and $[z]$ its coset in Q_α. Then $g_{[z]} : \alpha^\perp \to \alpha^\perp$ sends $x \in \alpha^\perp$ to $x + (x, z)\alpha$.

(3) The map $q : \Omega^+_{\alpha^\perp} \to \Omega^+_{Q_\alpha}$ is $Mon^2(\Lambda, \iota)_\alpha$-equivariant and it is invariant with respect to the image $g(Q_\alpha) \subset Mon^2(\Lambda, \iota)_\alpha$ of g.

(4) The image of $\tilde{g}$ is equal to the kernel of the homomorphism h, given in Eq. (5.5).

Proof. Part (1) follows from the characterization of $\tilde{g}_z$ in Lemma 5.2. Part (2) is straightforward as is the $Mon^2(\Lambda, \iota)_\alpha$-equivariance of q. The $g(Q_\alpha)$-invariance of q follows from part (2). Part (3) is thus proven.

Part (4): The image of $\tilde{g}$ is contained in the kernel of h, by Lemma 5.2. Let $f \in O(\tilde{\Lambda})_{\beta,v}$ belong to the kernel of h. Set $\gamma_1 := f(\gamma)$ and $z := \gamma_1 - \gamma$. Then $(\gamma_1, \beta) = (f(\gamma), \beta) = (f(\gamma), f(\beta)) = (\gamma, \beta)$ and similarly $(\gamma_1, v) = (\gamma, v)$. Hence, $(z, \beta) = 0$ and $(z, v) = 0$. The isometry $\tilde{g}_z$ is thus well defined and it is equal to f, by Lemma 5.2. □

5.3 Density

A period $\underline{\ell}$ in $\Omega_{\Lambda_{k3}}$ is said to be *special*, if it satisfies the condition analogous to the one in Definition 1.1. We identify Ω_{Q_α} as a submanifold of $\Omega_{\Lambda_{k3}}$, via Lemma 4.1. Note that a period $\ell \in \Omega_{\alpha^\perp}$ is special, if and only if the period $q(\ell)$ is.

Lemma 5.4. *1. $g(Q_\alpha)$ has a dense orbit in $q^{-1}(\underline{\ell})$, if and only if $\underline{\ell}$ is non-special.*
2. If $g(Q_\alpha)$ has a dense orbit in $q^{-1}(\underline{\ell})$, then every $g(Q_\alpha)$-orbit in $q^{-1}(\underline{\ell})$ is dense.

Proof. Part 2 follows from the description of the action in Lemma 5.3 part 2. We prove part 1. Fix a period ℓ such that $q(\ell) = \underline{\ell}$ and choose a non-zero element t of the line ℓ in $\alpha^\perp_\Lambda \otimes_\mathbb{Z} \mathbb{C}$. Then $q^{-1}(\underline{\ell}) = \mathbb{P}[\mathbb{C}\alpha + \mathbb{C}t] \setminus \{\mathbb{P}[\mathbb{C}\alpha]\}$ and $g_{[z]}(a\alpha + t) = (a + (t, z))\alpha + t$, by Lemma 5.3 part 2. The fiber $q^{-1}(\underline{\ell})$ has a dense $g(Q_\alpha)$-orbit, if and only if the image of

$$(t, \bullet) : Q_\alpha \to \mathbb{C} \tag{5.6}$$

is dense in $\mathbb{C}$.

Suppose first that $\underline{\ell}$ is special. Set $V := [\underline{\ell} \oplus \bar{\underline{\ell}}] \cap [Q_\alpha \otimes_\mathbb{Z} \mathbb{R}]$. Let λ be a non-zero element in $V \cap Q_\alpha$. There exists an element $t \in \underline{\ell}$, such that $\lambda = t + \bar{t}$. Given an element $z \in Q_\alpha$, then $2Re(z, t) = (z, t) + (z, \bar{t}) = (z, \lambda)$ is an integer. Thus, $Re(z, t)$ belongs to the discrete subgroup $\frac{1}{2}\mathbb{Z}$ of $\mathbb{R}$. Hence, the image of the homomorphism (5.6) is not dense in $\mathbb{C}$.

Assume next that $\underline{\ell}$ is non-special. Denote by $\Theta(\underline{\ell}) \subset Q_\alpha$ the lattice orthogonal to the kernel of the homomorphism (5.6). $\Theta(\underline{\ell})$ is the transcendental lattice of the $K3$-surface with period $\underline{\ell}$. We know that $\Theta(\underline{\ell})$ has rank at least two, and if the rank of $\Theta(\underline{\ell})$ is 2, then the Hodge decomposition is defined over $\mathbb{Q}$ and so $\underline{\ell}$ is special. Thus, the rank of $\Theta(\underline{\ell})$ is at least three. Let $G \subset \Theta(\underline{\ell})$ be a co-rank 1 subgroup. We claim that the image (t, G), of G via the homomorphism (5.6), spans $\mathbb{C}$ as a 2-dimensional real vector space. The latter statement is equivalent to the statement that the image of G in V^*, under the map $z \mapsto (z, \bullet)$ which has real values on V, spans V^*. The equivalence is clear considering the following isomorphisms of two dimensional real vector spaces:

$$\mathbb{C} \xleftarrow{ev_t} \mathrm{Hom}_{\mathbb{C}}(\ell, \mathbb{C}) \xrightarrow{Re} \mathrm{Hom}_{\mathbb{R}}(\ell, \mathbb{R}) \xrightarrow{p^*} \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R}) = V^*,$$

where ev_t is evaluation at t, Re takes $(z, \bullet)$ to its real part $Re(z, \bullet)$, and p^* is pullback via the projection $p : V \to \ell$ on the $(2,0)$ part. Assume that the image of G in V^* spans a one-dimensional subspace W. Let U be the subspace of V annihilated by W, and hence also by $(z, \bullet)$, $z \in G$. Then the kernel of the homomorphism $\Lambda_{k3} \to U^*$, given by $z \mapsto (z, \bullet)$, has co-rank 1 in Λ_{k3}. It follows that the decomposition $\Lambda_{k3} \otimes_{\mathbb{Z}} \mathbb{R} = U \oplus U^{\perp}$ is defined over $\mathbb{Q}$. Thus, $U \cap \Lambda_{k3}$ is non-trivial and $\underline{\ell}$ is special. A contradiction. Thus, indeed, the image (t, G) of G spans $\mathbb{C}$. Let $Z \subset \mathbb{C}$ be the image $(t, \Theta(\underline{\ell}))$ of $\Theta(\underline{\ell})$ via the homomorphism (5.6). We have established that Z satisfies the hypothesis of Lemma 5.5 below, which implies that the image of the homomorphism (5.6) is dense in $\mathbb{C}$. □

Lemma 5.5. *Let $Z \subset \mathbb{R}^2$ be a free additive subgroup of rank ≥ 3. Assume that any co-rank 1 subgroup of Z spans $\mathbb{R}^2$ as a real vector space. Then Z is dense in $\mathbb{R}^2$.*

Proof. Let Σ be the set of all bases of $\mathbb{R}^2$, consisting of elements of Z. Given a basis $\beta \in \Sigma$, $\beta = \{z_1, z_2\}$, set $|\beta| = |z_1| + |z_2|$. Set $I := \inf\{|\beta| \ : \ \beta \in \Sigma\}$. Note that the closed parallelogram P_β with vertices $\{0, z_1, z_2, z_1 + z_2\}$ has diameter $< |\beta|$. Furthermore, every point of the plane belongs to a translate of P_β by an element of the subset $\mathrm{span}_{\mathbb{Z}}\{z_1, z_2\}$ of Z. Hence, it suffices to prove that $I = 0$.

The proof is by contradiction. Assume that $I > 0$. Let $\beta = \{z_1, z_2\}$ be a basis satisfying $I \leq |\beta| < \frac{12}{11} I$. We may assume, without loss of generality, that $|z_1| \geq |z_2|$.

We prove next that there exists an element $w \in Z$, such that $w = c_1 z_1 + c_2 z_2$, where the coefficients c_i are irrational. Set $r := \mathrm{rank}(Z)$. Let $z_3, \ldots, z_r$ be elements of Z completing $\{z_1, z_2\}$ to a subset, which is linearly independent over $\mathbb{Q}$. Write $z_j = c_{j,1} z_1 + c_{j,2} z_2$, for $3 \leq j \leq r$. Assume that $c_{j,1}$ are rational, for $3 \leq j \leq r$. Then there exists a positive integer N, such that $N c_{j,1}$ are integers, for all $3 \leq j \leq r$. Then

$$\{z_2, N z_3 - N c_{3,1} z_1, \ldots, N z_r - N c_{r,1} z_1\}$$

spans a co-rank 1 subgroup of Z, which lies on $\mathbb{R} z_2$. This contradicts the assumption on Z. Hence, there exists an element $w \in Z$, such that $w = c_1 z_1 + c_2 z_2$, where the coefficient c_1 is irrational. Repeating the above argument for c_2, we get the desired conclusion.

Choose an element w as above. By adding vectors in $\mathrm{span}_{\mathbb{Z}}\{z_1, z_2\}$, and possibly after changing the signs of z_1 or z_2, we may assume that $w = c_1 z_1 + c_2 z_2$, with $0 < c_1 < \frac{1}{2}$ and $0 < c_2 < \frac{1}{2}$. Then w belongs to the parallelogram $\frac{1}{2} P_\beta$ with vertices $\{0, \frac{z_1}{2}, \frac{z_2}{2}, \frac{z_1+z_2}{2}\}$. If c_1 and c_2 are both larger than $\frac{1}{3}$ replace w by z_1+z_2-2w. We may thus assume further, that at least one c_i is $\leq \frac{1}{3}$. In particular, $|w| \leq c_1 |z_1| + c_2 |z_2| < \frac{5}{6} |z_1|$. Consider the new basis $\tilde{\beta} := \{w, z_2\}$ of $\mathbb{R}^2$. Then $\left|\tilde{\beta}\right| = |w| + |z_2| < \frac{5}{6} |z_1| + |z_2| = |\beta| - \frac{1}{6} |z_1| \leq \frac{11}{12} |\beta| < I$. We obtain the desired contradiction. □

Denote by $J_\alpha \subset \Omega_{\alpha^\perp}$ the union of all the $g(Q_\alpha)$ translates of the section τ_γ constructed in Eq. (5.2) above.

$$J_\alpha := \bigcup_{y \in Q_\alpha} g_y \left[\tau_\gamma \left(\Omega^+_{Q_\alpha} \right) \right].$$

One easily checks that $g_{[z]} \circ \tau_\gamma = \tau_\delta$, where $\delta := \gamma + \iota(z) + (\gamma, \iota(z))\beta + \frac{(z,z)}{2}\beta$, for all $z \in \alpha^\perp_\Lambda$, and so J_α is independent of the choice of γ.

Proposition 5.6. *(1) J_α is a dense subset of $\Omega^+_{\alpha^\perp}$.*

(2) If V is a $g(Q_\alpha)$-invariant open subset of $\Omega^+_{\alpha^\perp}$, which contains J_α, then V contains every non-special period in $\Omega^+_{\alpha^\perp}$.

(3) For every $\ell \in J_\alpha$, there exists a marked pair (M, η), consisting of a smooth projective irreducible holomorphic symplectic manifold M of $K3^{[n]}$-type and a marking $\eta : H^2(M, \mathbb{Z}) \to \Lambda$ with period ℓ satisfying the following properties.

- *(a) The composition $\iota \circ \eta : H^2(M, \mathbb{Z}) \to \tilde{\Lambda}$ belongs to the canonical $O(\tilde{\Lambda})$-orbit ι_M of Theorem 2.2.*
- *(b) There exists a Lagrangian fibration $\pi : M \to \mathbb{P}^n$, such that the class $\eta^{-1}(\alpha)$ is equal to $\pi^* c_1(\mathscr{O}_{\mathbb{P}^n}(1))$.*

Proof.

(1) The density of J_α follows from Lemma 5.4.

(2) V intersects every non-special fiber $q^{-1}(\underline{\ell})$ in a non-empty open $g(Q_\alpha)$-equivariant subset of the latter. The complement $q^{-1}(\underline{\ell}) \setminus V$ is thus a closed $g(Q_\alpha)$-equivariant proper subset of the fiber. But any $g(Q_\alpha)$-orbit in the non-special fiber $q^{-1}(\underline{\ell})$ is dense in $q^{-1}(\underline{\ell})$, by Lemma 5.4. Hence, the complement $q^{-1}(\underline{\ell}) \setminus V$ must be empty.

(3) If ℓ_0 belongs to the section $\tau_\gamma \left(\Omega^+_{Q_\alpha} \right)$, then such a pair $(M, \eta) := (M_H(u), \eta_1)$ was constructed in Diagram (5.4) as mentioned in Remark 5.1. If $\ell = g_z(\ell_0)$, $z \in \alpha^\perp_\Lambda$, set $(M, \eta) = (M_H(u), g_z \circ \eta_1)$. □

6 Primitive Isotropic Classes and Lagrangian Fibrations

We prove Theorem 1.3 in this section using the geometry of the moduli space $\mathfrak{M}^0_{\alpha^\perp}$ given in Eq. (4.6). Recall that $\mathfrak{M}^0_{\alpha^\perp}$ is a connected component of the moduli space of marked pairs (X, η) with X of $K3^{[n]}$-type and such that $\eta^{-1}(\alpha)$ is a primitive isotropic class of Hodge type $(1, 1)$ in the boundary of the positive cone in $H^{1,1}(X, \mathbb{R})$.

Fix a connected moduli space $\mathfrak{M}^0_{\alpha^\perp}$ as in Eq. (4.6). Denote by $\mathscr{L}_{\eta^{-1}(\alpha)}$ the line bundle on X with $c_1(\mathscr{L}) = \eta^{-1}(\alpha)$. Let V be the subset of $\mathfrak{M}^0_{\alpha^\perp}$ consisting of all pairs (X, η), such that $\mathscr{L}_{\eta^{-1}(\alpha)}$ induces a Lagrangian fibration.

Theorem 6.1. *The image of V via the period map contains every non-special period in $\Omega^+_{\alpha^\perp}$.*

Proof. Let (X, η) be a marked pair in $\mathfrak{M}^0_{\alpha^\perp}$. The property that $\eta^{-1}(\alpha)$ is the first Chern class of a line-bundle $\mathscr{L}$ on X, which induces a Lagrangian fibration $X \to |\mathscr{L}|^*$, is an open property in the moduli space of marked pairs, by a result of Matsushita [27]. V is thus an open subset.

Choose a primitive embedding $\iota : \Lambda \to \tilde{\Lambda}$ with the property that $\iota \circ \eta$ belongs to the canonical $O(\tilde{\Lambda})$-orbit ι_X of Theorem 2.2, for all (X, η) in $\mathfrak{M}^0_\Lambda$. Let $Mon^2(\Lambda, \iota)$ and its subgroup $Mon^2(\Lambda, \iota)_\alpha$ be the subgroups of $O^+(\Lambda)$ introduced in Lemma 5.3. The component $\mathfrak{M}^0_\Lambda$ of the moduli space of marked pairs is invariant under $Mon^2(\Lambda, \iota)$, by Theorem 2.4. The subset $\mathfrak{M}^0_{\alpha^\perp}$ of $\mathfrak{M}^0_\Lambda$ is invariant under the subgroup $Mon^2(\Lambda, \iota)_\alpha$. Hence, the subset V is $Mon^2(\Lambda, \iota)_\alpha$ invariant. The construction in Sect. 5.1 yields a marked pair $(M_H(u), \eta_1)$ with period in the image of the section $\tau_\gamma : \Omega^+_{Q_\alpha} \to \Omega^+_{\alpha^\perp}$, given in Eq. (5.2). Furthermore, the class $\eta_1^{-1}(\alpha)$ induces a Lagrangian fibration of $M_H(u)$. The marked pair $(M_H(u), \eta_1)$ belongs to $\mathfrak{M}^0_{\alpha^\perp}$, by Proposition 4.5 (Remark 5.1 verifies the conditions of Proposition 4.5). Hence, $(M_H(u), \eta_1)$ belongs to V and the image of the section $\tau_\gamma : \Omega^+_{Q_\alpha} \to \Omega^+_{\alpha^\perp}$ is thus contained in the image of V via the period map. The period map P_0 is $Mon^2(\Lambda, \iota)_\alpha$ equivariant and a local homeomorphism, by the Local Torelli Theorem [4]. Hence, the image $P_0(V)$ is an open and $Mon^2(\Lambda, \iota)_\alpha$ invariant subset of $\Omega^+_{\alpha^\perp}$. Any $Mon^2(\Lambda, \iota)_\alpha$ invariant subset, which contains the section $\tau_\gamma(\Omega^+_{Q_\alpha})$, contains also the dense subset J_α of Proposition 5.6. $P_0(V)$ thus contains every non-special period in $\Omega^+_{\alpha^\perp}$, by Proposition 5.6 (2). □

We will need the following criterion of Kawamata for a line bundle to be semi-ample. Let X be a smooth projective variety and D a divisor class on X. Set $\nu(X, D) := \max\{e : D^e \not\equiv 0\}$, where $\equiv$ denotes numerical equivalence. If $D \equiv 0$, we set $\nu(X, D) = 0$. Denote by $\Phi_{kD} : X \dashrightarrow |kD|^*$ the rational map, defined whenever the linear system is non-empty. Set $\kappa(X, D) := \max\{\dim \Phi_{kD}(X) : k > 0\}$, if $|kD|$ is non-empty for some positive integer k, and $\kappa(X, D) := -\infty$, otherwise.

Theorem 6.2 (A special case of [16, Theorem 6.1]). *Let X be a smooth projective variety with a trivial canonical bundle and D a nef divisor. Assume that $\nu(X, D) = \kappa(X, D)$ and $\kappa(X, D) \geq 0$. Then D is semi-ample, i.e., there exists a positive integer k such that the linear system $|kD|$ is base point free.*

An alternate proof of Kawamata's Theorem is provided in [11]. A reduced and irreducible divisor E on X is called *prime-exceptional*, if the class $e \in H^2(X,\mathbb{Z})$ of E satisfies $(e,e) < 0$. Consider the reflection $R_E : H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{Z})$, given by

$$R_E(x) = x - \frac{2(x,e)}{(e,e)}e.$$

It is known that the reflection R_E by the class of a prime exceptional divisor E is a monodromy operator, and in particular an integral isometry [24, Cor. 3.6]. Let $W(X) \subset O(H^2(X,\mathbb{Z}))$ be the subgroup generated by reflections R_E by classes of prime exceptional divisors $E \subset X$. Elements of $W(X)$ preserve the Hodge structure, hence $W(X)$ acts on $H^{1,1}(X,\mathbb{Z})$.

Let $\mathscr{P}ex_X \subset H^{1,1}(X,\mathbb{Z})$ be the set of classes of prime exceptional divisors. The *fundamental exceptional chamber* of the positive cone $\mathscr{C}_X$ is the set

$$\mathscr{F}\mathscr{E}_X \ := \ \{a \in \mathscr{C}_X \ : \ (a,e) > 0, \text{ for all } e \in \mathscr{P}ex_X\}.$$

The closure of $\mathscr{F}\mathscr{E}_X$ in $\mathscr{C}_X$ is a fundamental domain for the action of $W(X)$ [23, Theorem 6.18]. Let $f : X \dashrightarrow Y$ be a bimeromorphic map to an irreducible holomorphic symplectic manifold Y and $\mathscr{K}_Y$ the Kähler cone of Y. Then $f^*\mathscr{K}_Y$ is an open subset of $\mathscr{F}\mathscr{E}_X$. Furthermore, the union of $f^*\mathscr{K}_Y$, as f and Y vary over all such pairs, is a dense open subset of $\mathscr{F}\mathscr{E}_X$, by a result of Boucksom [6] (see also [23, Theorem 1.5]).

Proof (of Theorem 1.3). Step 1: Keep the notation in the opening paragraph of Sect. 5. Choose a marking $\eta : H^2(X,\mathbb{Z}) \to \Lambda$, such that $\iota \circ \eta$ belongs to the canonical $O(\tilde{\Lambda})$-orbit ι_X. Set $\alpha := \eta(c_1(\mathscr{L}))$. Then (X,η) belongs to a component $\mathfrak{M}^0_{\alpha^\perp}$ of the moduli space of marked pairs of $K3^{[n]}$-type considered in Theorem 6.1. We use here the assumption that $\mathscr{L}$ is nef in order to deduce that $\eta^{-1}(\alpha)$ belongs to the boundary of the positive cone of X, used in Theorem 6.1.

The period $P_0(X,\eta)$ is non-special, by assumption. There exists a marked pair (Y,ψ) in $\mathfrak{M}^0_{\alpha^\perp}$ satisfying $P_0(Y,\psi) = P_0(X,\eta)$, such that the class $\psi^{-1}(\alpha)$ induces a Lagrangian fibration, by Theorem 6.1. The marked pairs (X,η) and (Y,ψ) correspond to inseparable points in the moduli space $\mathfrak{M}^0_{\alpha^\perp}$, by the Global Torelli Theorem 4.3. Hence, there exists an analytic correspondence $Z \subset X \times Y$, $Z = \sum_{i=0}^k Z_i$ in $X \times Y$, of pure dimension $2n$, with the following properties, by results of Huybrechts [13, Theorem 4.3] (see also [23, Sec. 3.2]).

(1) The homomorphism $Z_* : H^*(X,\mathbb{Z}) \to H^*(Y,\mathbb{Z})$ is a Hodge isometry, which is equal to $\psi^{-1} \circ \eta$. The irreducible component Z_0 of the correspondence is the graph of a bimeromorphic map $f : X \dashrightarrow Y$.
(2) The images in X and Y of all other components Z_i, $i > 0$, are of positive co-dimension.

Step 2: We prove next that the line bundle $\mathscr{L}$ over X is semi-ample. We consider separately the projective and non-algebraic cases.

Step 2.1: Assume that X is not projective.[3] We claim that $f_*(c_1(\mathscr{L})) = \psi^{-1}(\alpha)$. The Neron-Severi group $NS(X)$ does not contain any positive class, by Huybrechts projectivity criterion [13]. Hence, the Beauville-Bogomolov-Fujiki pairing restricts to $NS(X)$ as a non-positive pairing with a rank one null sub-lattice spanned by the class $c_1(\mathscr{L})$. Similarly, the Beauville-Bogomolov-Fujiki pairing restricts to $NS(Y)$ with a rank one null space spanned by $\psi^{-1}(\alpha)$. Hence, $f_*(c_1(\mathscr{L})) = \pm\psi^{-1}(\alpha)$. Now $\psi^{-1}(\alpha)$ is semi-ample and hence belongs to the closure of $\mathscr{F}\mathscr{E}_Y$. The class $c_1(\mathscr{L})$ is assumed nef, and hence belongs to the closure of $\mathscr{F}\mathscr{E}_X$. The bimeromorphic map f induces a Hodge-isometry $f_* : H^2(X,\mathbb{Z}) \to H^2(Y,\mathbb{Z})$, which maps $\mathscr{F}\mathscr{E}_X$ onto $\mathscr{F}\mathscr{E}_Y$ [6]. Hence, $f_*(c_1(\mathscr{L}))$ belongs to $\overline{\mathscr{F}\mathscr{E}}_Y$ as well. We conclude the equality $f_*(c_1(\mathscr{L})) = \psi^{-1}(\alpha)$.

Let $\mathscr{L}_2$ be the line bundle with $c_1(\mathscr{L}_2) = \psi^{-1}(\alpha)$. The bimeromorphic map $f : X \dashrightarrow Y$ is holomorphic in co-dimension one, and so induces an isomorphism $f_1 : |\mathscr{L}| \to |\mathscr{L}_2|$ of the two linear systems. Denote by $\Phi_{\mathscr{L}_2} : Y \to |\mathscr{L}_2|^*$ the Lagrangian fibration induced by $\mathscr{L}_2$. We conclude that $|\mathscr{L}|$ is n dimensional and the meromorphic map $\Phi_{\mathscr{L}} : X \dashrightarrow |\mathscr{L}|^*$ is an algebraic reduction of X (see [8]). By definition, an *algebraic reduction* of X is a dominant meromorphic map $\pi : X \dashrightarrow B$ to a normal projective variety B, such that π^* induces an isomorphism of the function fields of meromorphic functions [8]. Only the birational class of B is determined by X. Fibers of the algebraic reduction π are defined via a resolution of indeterminacy, and are closed connected analytic subsets of X. In our case, the generic fiber of $\Phi_{\mathscr{L}}$ is bimeromorphic to the generic fiber of $\Phi_{\mathscr{L}_2}$. The generic fiber of $\Phi_{\mathscr{L}_2}$ is a complex torus, and hence algebraic, by [7, Prop. 2.1]. Hence, the generic fiber of $\Phi_{\mathscr{L}}$ has algebraic dimension n. It follows that the line bundle $\mathscr{L}$ is semi-ample, it is the pullback of an ample line-bundle over B, via a holomorphic reduction map $\pi : X \to B$ which is a regular morphism, by [8, Theorems 1.5 and 3.1].

Step 2.2: When X is projective there exists an element $w \in W(X)$, such that Huybrecht's birational map $f : X \dashrightarrow Y$ satisfies $f^* \circ \psi^{-1} \circ \eta = w$, by [23, Theorem 1.6]. Set $\alpha_X := \eta^{-1}(\alpha)$ and $\alpha_Y := \psi^{-1}(\alpha)$. We get the equality $w(\alpha_X) = f^*(\alpha_Y)$.

Let $\overline{\mathscr{F}\mathscr{E}}_X$ be the closure of the fundamental exceptional chamber $\mathscr{F}\mathscr{E}_X$ in $H^{1,1}(X,\mathbb{R})$. The class α_X is nef, by assumption, and it thus belongs to $\overline{\mathscr{F}\mathscr{E}}_X$. We already know that α_Y is the class of a line bundle, which induces a Lagrangian

[3] I thank K. Oguiso and S. Rollenske for pointing out to me that in the non-algebraic case the result should follow from the above via the results of Ref. [8].

fibration. Hence, $f^*(\alpha_Y)$ belongs to $\overline{\mathscr{F}\mathscr{E}}_X$. The class $w(\alpha_X)$ thus belongs to the intersection $w\left(\overline{\mathscr{F}\mathscr{E}}_X\right) \cap \overline{\mathscr{F}\mathscr{E}}_X$.

Let J be the subset of $\mathscr{P}ex_X$ given by $J = \{e \in \mathscr{P}ex_X \ : \ (e, \alpha_X) = 0\}$. Denote by W_J the subgroup of $W(X)$ generated by reflections R_e, for all $e \in J$. Then W_J is equal to

$$\{w \in W(X) \ : \ w(\alpha_X) \in \overline{\mathscr{F}\mathscr{E}}_X\},$$

by a general property of crystalographic hyperbolic reflection groups [12, Lecture 3, Proposition on page 15]. We conclude that $w(\alpha_X) = \alpha_X$ and

$$\alpha_X = f^*(\alpha_Y). \tag{6.1}$$

We are ready to prove[4] that $\mathscr{L}$ is semi-ample. The rational map f is regular in co-dimension one. The map f thus induces an isomorphism $f_m : |\mathscr{L}^m| \to |\mathscr{L}_2^m|$, for every integer m. Hence, $\kappa(X, \mathscr{L}) = \kappa(Y, \mathscr{L}_2) = n$. Any non-zero isotropic divisor class D on a $2n$ dimensional irreducible holomorphic symplectic manifold satisfies $\nu(X, D) = n$, by a result of Verbitsky [41]. Hence, $\nu(X, \mathscr{L}) = n$. The line bundle $\mathscr{L}$ is assumed to be nef. Hence, $\mathscr{L}$ is semi-ample, by Theorem 6.2.

Step 3: We return to the general case, where X may or may not be projective. In both cases we have seen that there exists a positive integer m, such that the linear system $|\mathscr{L}^m|$ is base point free and $\Phi_{\mathscr{L}^m}$ is a regular morphism. Furthermore, the bimeromorphic map $f : X \dashrightarrow Y$ is regular in co-dimension one and thus induces an isomorphism $f_k : |\mathscr{L}^k| \to |\mathscr{L}_2^k|$, for every positive integer k. Denote by $f_k^* : |\mathscr{L}_2^k|^* \to |\mathscr{L}^k|^*$ the transpose of f_k. We get the equality $\Phi_{\mathscr{L}^k} = f_k^* \circ \Phi_{\mathscr{L}_2^k} \circ f$, for all k. Let $V_m : |\mathscr{L}_2|^* \to |\mathscr{L}_2^m|^*$ be the Veronese embedding. We get the equalities

$$V_m \circ (f_1^*)^{-1} \circ \Phi_{\mathscr{L}} = V_m \circ \Phi_{\mathscr{L}_2} \circ f = \Phi_{\mathscr{L}_2^m} \circ f = (f_m^*)^{-1} \circ \Phi_{\mathscr{L}^m}. \tag{6.2}$$

Now, $V_m \circ (f_1^*)^{-1} : |\mathscr{L}|^* \to |\mathscr{L}_2^m|^*$ is a closed immersion and the morphism on the right hand side of (6.2) is regular. Hence, the rational map $\Phi_{\mathscr{L}}$ is a regular morphism. The base locus of the linear system $|\mathscr{L}|$ is thus either empty, or a divisor. The latter is impossible, since f is regular in co-dimension one and $|\mathscr{L}_2|$ is base point free. Hence, $|\mathscr{L}|$ is base point free. □

Let X and $\mathscr{L}$ be as in Theorem 1.3, except that we drop the assumption that $\mathscr{L}$ is nef and assume only that $c_1(\mathscr{L})$ belongs to the boundary of the positive cone. Assume that X is projective.

[4] I thank C. Lehn for Ref. [18, Prop. 2.4], used in an earlier argument, and T. Peternell and Y. Kawamata for suggesting the current more direct argument.

Theorem 6.3. *There exists an element $w \in W(X)$, a projective irreducible holomorphic symplectic manifold Y, a birational map $f : X \dashrightarrow Y$, and a Lagrangian fibration $\pi : Y \to \mathbb{P}^n$, such that $w(\mathscr{L}) = f^*\pi^*\mathscr{O}_{\mathbb{P}^n}(1)$.*

Proof. Let (Y, ψ) be the marked pair constructed in Step 1 of the proof of Theorem 1.3. Then Y admits a Lagrangian fibration $\pi : Y \to \mathbb{P}^n$ and the class $\pi^* c_1(\mathscr{O}_{\mathbb{P}^n}(1))$ was denoted α_Y in that proof. In step 2.2 of that proof we showed the existence of a birational map $f : X \dashrightarrow Y$ and an element $w \in W(X)$, such that $w(c_1(\mathscr{L})) = f^*(\alpha_Y)$ (see Equality (6.1)). □

7 Tate-Shafarevich Lines and Twists

7.1 The Geometry of the Universal Curve

Let S be a projective $K3$ surface, d a positive integer, and $\mathscr{L}$ a nef line bundle on S of positive degree, such that the class $c_1(\mathscr{L})$ is indivisible. Set $n := 1 + \frac{d^2 \deg(\mathscr{L})}{2}$. Let $\mathscr{C} \subset S \times |\mathscr{L}^d|$ be the universal curve, π_i the projection from $S \times |\mathscr{L}^d|$ to the i-th factor, $i = 1, 2$, and p_i the restriction of π_i to $\mathscr{C}$. We assume in this section the following assumptions about the line bundle $\mathscr{L}$.

Assumption 7.1. *(1) The linear system $|\mathscr{L}^d|$ is base point free.*
(2) The locus in $|\mathscr{L}^d|$, consisting of divisors which are non-reduced, or reducible having a singularity which is not an ordinary double point, has co-dimension at least 2.

Remark 7.2. Assumption 7.1 holds whenever $\mathrm{Pic}(S)$ is cyclic generated by $\mathscr{L}$. The base point freeness Assumption 7.1 (1) follows from [32, Prop. 1]. Assumption 7.1 (2) is verified as follows. If $a+b = d$, $a \geq 1$, $b \geq 1$, then the image of $|\mathscr{L}^a| \times |\mathscr{L}^b|$ in $|\mathscr{L}^d|$ has co-dimension $2ab\left(\frac{n-1}{d^2}\right) - 1$. The co-dimension is at least two, except in the case $(n, d) = (5, 2)$. In the latter case $|\mathscr{L}| \cong \mathbb{P}^2$, $|\mathscr{L}^2| \cong \mathbb{P}^5$ and the generic curve in the image of $|\mathscr{L}| \times |\mathscr{L}|$ in $|\mathscr{L}^2|$ is the union of two smooth curves of genus 2 meeting transversely at two points. Hence, Assumption 7.1 (2) holds in this case as well.

The morphism $p_1 : \mathscr{C} \to S$ is a projective hyperplane sub-bundle of the trivial bundle over S with fiber $|\mathscr{L}^d|$, by the base point freeness Assumption 7.1 (1). Assumption 7.1 (2) will be used in the proof of Lemma 7.9. Consider the exponential short exact sequence over $\mathscr{C}$

$$0 \to \mathbb{Z} \to \mathscr{O}_{\mathscr{C}} \to \mathscr{O}^*_{\mathscr{C}} \to 0.$$

We get the exact sequence of sheaves of abelian groups over $|\mathscr{L}^d|$

$$0 \longrightarrow R^1 p_{2*}\mathbb{Z} \longrightarrow R^1 p_{2*}\mathscr{O}_{\mathscr{C}} \longrightarrow R^1 p_{2*}\mathscr{O}^*_{\mathscr{C}} \xrightarrow{\deg} R^2 p_{2*}\mathbb{Z} \longrightarrow 0, \tag{7.1}$$

where we work in the complex analytic category. Note that deg above is surjective, since $R^2 p_{2_*}\mathcal{O}_{\mathscr{C}}$ vanishes. Set $Ш := H^1(|\mathscr{L}^d|, R^1 p_{2_*}\mathcal{O}^*_{\mathscr{C}})$ and $\widetilde{Ш} := H^1(|\mathscr{L}^d|, R^1 p_{2_*}\mathcal{O}_{\mathscr{C}})$. Set $Br'(S) := H^2(S, \mathcal{O}^*_S)$ and $Br'(\mathscr{C}) := H^2(\mathscr{C}, \mathcal{O}^*_{\mathscr{C}})$.

Lemma 7.3. *(1) There is a natural isomorphism*

$$R^1 p_{2_*}\mathcal{O}_{\mathscr{C}} \cong T^*|\mathscr{L}^d| \otimes_{\mathbb{C}} H^{2,0}(S)^*.$$

(2) $\widetilde{Ш}$ is naturally isomorphic to $H^{0,2}(\mathscr{C})$. Consequently, $\widetilde{Ш}$ is one dimensional.
(3) $H^2(\mathscr{C}, \mathbb{Z})$ decomposes as a direct sum

$$H^2(\mathscr{C}, \mathbb{Z}) = p_1^* H^2(S, \mathbb{Z}) \oplus p_2^* H^2(|\mathscr{L}^d|, \mathbb{Z}).$$

The groups $H^i(\mathscr{C}, \mathbb{Z})$ vanish for odd i. The Dolbeault cohomologies $H^{p,q}(\mathscr{C})$ vanish, if $|p-q| > 2$.
(4) The pullback homomorphism $p_1^* : H^2(S, \mathcal{O}^*_S) \to H^2(\mathscr{C}, \mathcal{O}^*_{\mathscr{C}})$ is an isomorphism. The Leray spectral sequence yields an isomorphism

$$b : H^2(\mathscr{C}, \mathcal{O}^*_{\mathscr{C}}) \to H^1(|\mathscr{L}^d|, R^1 p_{2_*}\mathcal{O}^*_{\mathscr{C}}).$$

Consequently, we have the isomorphisms

$$Br'(S) \xrightarrow[\cong]{p_1^*} Br'(\mathscr{C}) \xrightarrow[\cong]{b} Ш.$$

Let $\mathscr{F}$ be a sheaf of abelian groups over $\mathscr{C}$. Let $F^p H^k(\mathscr{C}, \mathscr{F})$ be the Leray filtration associated to the morphism $p_2 : \mathscr{C} \to |\mathscr{L}^d|$ and $E_\infty^{p,q} := F^p H^{p+q}(\mathscr{C}, \mathscr{F}) / F^{p+1} H^{p+q}(\mathscr{C}, \mathscr{F})$ its graded pieces. Recall that the $E_2^{p,q}$ terms are $E_2^{p,q} := H^p(|\mathscr{L}^d|, R^q p_{2_*}\mathscr{F})$ and the differential at this step is $d_2 : E_2^{p,q} \to E_2^{p+2,q-1}$.

Proof. (1) We have the isomorphism $\mathcal{O}_{S\times|\mathscr{L}^d|}(\mathscr{C}) \cong \pi_1^*\mathscr{L}^d \otimes \pi_2^*\mathcal{O}_{|\mathscr{L}^d|}(1)$. Apply the functor $R\pi_{2_*}$ to the short exact sequence $0 \to \mathcal{O}_{S\times|\mathscr{L}^d|} \to \mathcal{O}_{S\times|\mathscr{L}^d|}(\mathscr{C}) \to \mathcal{O}_{\mathscr{C}}(\mathscr{C}) \to 0$ to obtain the Euler sequence of the tangent bundle.

$$0 \to \mathcal{O}_{|\mathscr{L}^d|} \to H^0(S, \mathscr{L}^d) \otimes_{\mathbb{C}} \mathcal{O}_{|\mathscr{L}^d|}(1) \to T|\mathscr{L}^d| \to 0.$$

Now $\mathcal{O}_{\mathscr{C}}(\mathscr{C}) \otimes_{\mathbb{C}} H^{2,0}(S)$ is isomorphic to the relative dualizing sheaf ω_{p_2}. We get the isomorphisms

$$R^1 p_{2_*}\mathcal{O}_{\mathscr{C}} \cong [R^0 p_{2_*}\mathcal{O}_{\mathscr{C}}(\mathscr{C}) \otimes_{\mathbb{C}} H^{2,0}(S)]^* \cong [R^0 p_{2_*}\mathcal{O}_{\mathscr{C}}(\mathscr{C})]^* \otimes_{\mathbb{C}} H^{2,0}(S)^*$$
$$\cong T^*|\mathscr{L}^d| \otimes_{\mathbb{C}} H^{2,0}(S)^*.$$

(2) $R^2 p_{2_*}\mathcal{O}_{\mathscr{C}}$ vanishes, since p_2 has one-dimensional fibers. $H^2(|\mathscr{L}^d|, p_{2_*}\mathcal{O}_{\mathscr{C}})$ vanishes, since $p_{2_*}\mathcal{O}_{\mathscr{C}} \cong \mathcal{O}_{|\mathscr{L}^d|}$. The latter isomorphism follow from the

fact that p_2 has connected fibers. We conclude that $H^2(\mathscr{C}, \mathscr{O}_{\mathscr{C}})$ is isomorphic to the $E_\infty^{1,1}$ graded summand of its Leray filtration. The differential $d_2 : H^1(|\mathscr{L}^d|, R^1 p_{2_*}\mathscr{O}_{\mathscr{C}}) \to H^3(|\mathscr{L}^d|, p_{2_*}\mathscr{O}_{\mathscr{C}})$ vanishes, since $H^{0,3}(|\mathscr{L}^d|)$ vanishes. Hence, the $E_2^{1,1}$ term $\widetilde{\text{III}} := H^1(|\mathscr{L}^d|, R^1 p_{2_*}\mathscr{O}_{\mathscr{C}})$ is isomorphic to $H^2(\mathscr{C}, \mathscr{O}_{\mathscr{C}})$.

(3) The statement is topological and so it suffices to prove it in the case where $\mathrm{Pic}(S)$ is cyclic generated by $\mathscr{L}$. In this case $\mathscr{L}$ is ample, and so the line bundle $\pi_1^*\mathscr{L}^d \otimes \pi_2^*\mathscr{O}_{|\mathscr{L}^d|}(1)$ is ample. The Lefschetz Theorem on Hyperplane sections implies that the restriction homomorphism $H^2(S \times |\mathscr{L}^d|, \mathbb{Z}) \to H^2(\mathscr{C}, \mathbb{Z})$ is an isomorphism.

$\mathscr{C}$ is the projectivization of a rank n vector bundle F over S. Hence, $H^*(\mathscr{C}, \mathbb{Z})$ is the quotient of $H^*(S, \mathbb{Z})[x]$, with x of degree 2, by the ideal generated by $\sum_{i=0}^{n+1} c_i(F)x^i$. The image of x in $H^*(\mathscr{C}, \mathbb{Z})$ corresponds to the class $\bar{x} := c_1(\mathscr{O}_{\mathscr{C}}(1))$ of Hodge type (1, 1). In particular, $H^*(\mathscr{C}, \mathbb{Z})$ is a free $H^*(S, \mathbb{Z})$-module of rank n generated by $1, \bar{x}, \ldots, \bar{x}^{n-1}$.

(4) The vanishing of $H^3(S, \mathbb{Z})$ and $H^3(\mathscr{C}, \mathbb{Z})$ yields the commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & H^2(S,\mathbb{Z})/NS(S) \arrow[r] \arrow[d, "p_1^*"] & H^2(S,\mathscr{O}_S) \arrow[r] \arrow[d, "p_1^*", "\cong"'] & H^2(S,\mathscr{O}_S^*) \arrow[r] \arrow[d, "p_1^*"] & 0 \\
0 \arrow[r] & H^2(\mathscr{C},\mathbb{Z})/NS(\mathscr{C}) \arrow[r] & H^2(\mathscr{C},\mathscr{O}_{\mathscr{C}}) \arrow[r] & H^2(\mathscr{C},\mathscr{O}_{\mathscr{C}}^*) \arrow[r] & 0
\end{tikzcd}$$

Part (3) of the Lemma implies that the left and middle vertical homomorphism are isomorphisms. It follows that the right vertical homomorphism is an isomorphism as well.

The sheaf $R^2 p_{2_*}\mathscr{O}_{\mathscr{C}}^*$ vanishes, by the exactness of $R^2 p_{2_*}\mathscr{O}_{\mathscr{C}} \to R^2 p_{2_*}\mathscr{O}_{\mathscr{C}}^* \to R^3 p_{2_*}\mathbb{Z}$ and the vanishing of the left and right sheaves due to the fact that p_2 has one-dimensional fibers. The sheaf $p_{2_*}\mathscr{O}_{\mathscr{C}}^*$ is isomorphic to $\mathscr{O}_{|\mathscr{L}^d|}^*$, since p_2 has connected complete fibers. Thus, $H^2(\mathscr{C}, \mathscr{O}_{\mathscr{C}}^*)$ is isomorphic to the kernel of the differential

$$d_2 : E_2^{1,1} := H^1(|\mathscr{L}^d|, R^1 p_{2_*}\mathscr{O}_{\mathscr{C}}^*) \to E_2^{3,0} := H^3(|\mathscr{L}^d|, \mathscr{O}_{|\mathscr{L}^d|}^*). \tag{7.2}$$

We prove next that d_2 vanishes. The co-kernel of d_2 is equal to $F^3H^3(\mathscr{C}, \mathscr{O}_{\mathscr{C}}^*)$. Now $F^3H^3(\mathscr{C}, \mathscr{O}_{\mathscr{C}}^*)$ is equal to the image of $p_2^* : H^3(|\mathscr{L}^d|, \mathscr{O}_{|\mathscr{L}^d|}^*) \to H^3(\mathscr{C}, \mathscr{O}_{\mathscr{C}}^*)$. We have a commutative diagram

$$\begin{tikzcd}
H^3(\mathscr{C},\mathscr{O}_{\mathscr{C}}^*) \arrow[r, "\cong"] & H^4(\mathscr{C},\mathbb{Z}) \\
H^3(|\mathscr{L}^d|,\mathscr{O}_{|\mathscr{L}^d|}^*) \arrow[r, "\cong"] \arrow[u, "p_2^*"] & H^4(|\mathscr{L}^d|,\mathbb{Z}). \arrow[u, "p_2^*"']
\end{tikzcd}$$

The horizontal homomorphisms, induced by the connecting homomorphism of the exponential sequence, are isomorphisms, since $h^{0,3}(\mathscr{C}) = h^{0,3}(|\mathscr{L}^d|) = 0$ and $h^{0,4}(\mathscr{C}) = h^{0,4}(|\mathscr{L}^d|) = 0$. The right vertical homomorphism is injective. We conclude that the left vertical homomorphism is injective. Hence the differential d_2 in (7.2) vanishes and $H^2(\mathscr{C}, \mathscr{O}^*_{\mathscr{C}})$ is isomorphism to $H^1(|\mathscr{L}^d|, R^1p_{2*}\mathscr{O}^*_{\mathscr{C}})$, yielding the isomorphism b. ☐

Let $\Sigma \subset H^2(S,\mathbb{Z})$ be the sub-lattice generated by classes of irreducible components of divisors in the linear system $|\mathscr{L}^d|$. Denote by $\Sigma^\perp$ the sub-lattice of $H^2(S,\mathbb{Z})$ orthogonal to Σ.

Lemma 7.4. *(1) The Leray filtration of $H^2(\mathscr{C},\mathbb{Z})$ associated to p_2 is identified as follows:*

$$F^2H^2(\mathscr{C},\mathbb{Z}) = p_2^*H^2(|\mathscr{L}^d|,\mathbb{Z}),$$
$$F^1H^2(\mathscr{C},\mathbb{Z}) = p_2^*H^2(|\mathscr{L}^d|,\mathbb{Z}) \oplus p_1^*\Sigma^\perp.$$

(2) $E_2^{p,q} = E_\infty^{p,q}$, if $(p,q) = (2,0)$, or $(1,1)$. Consequently, we get the following isomorphisms.

$$E_2^{2,0} := H^2(|\mathscr{L}^d|, p_{2*}\mathbb{Z}) \cong p_2^*H^2(|\mathscr{L}^d|,\mathbb{Z}),$$
$$E_2^{1,1} := H^1(|\mathscr{L}^d|, R^1p_{2*}\mathbb{Z}) \cong p_1^*\Sigma^\perp,$$

(3) If the sub-lattice Σ is saturated in $H^2(S,\mathbb{Z})$, then $H^2(|\mathscr{L}^d|, R^1p_{2*}\mathbb{Z})$ vanishes.

Proof. (1), (2) The sheaf $p_{2*}\mathbb{Z}$ is the constant sheaf $\mathbb{Z}$, since p_2 has connected fibers. Then $E_2^{3,0} = H^3(|\mathscr{L}^d|,\mathbb{Z}) = 0$, and so $E_\infty^{1,1} = E_2^{1,1} = H^1(|\mathscr{L}^d|, R^1p_{2*}\mathbb{Z})$. $E_2^{2,0} := H^2(|\mathscr{L}^d|, p_{2*}\mathbb{Z})$ has rank 1 and it maps injectively into $H^2(\mathscr{C},\mathbb{Z})$, with image equal to $p_2^*H^2(|\mathscr{L}^d|,\mathbb{Z})$. Thus, $E_2^{2,0} = E_\infty^{2,0}$ and $E_\infty^{1,1} := F^1H^2(\mathscr{C},\mathbb{Z})/E_\infty^{2,0}$ is isomorphic to $F^1H^2(\mathscr{C},\mathbb{Z})/p_2^*H^2(|\mathscr{L}^d|,\mathbb{Z})$. Finally, $E_2^{0,2}$ is the kernel of

$$d_2 : H^0(|\mathscr{L}^d|, R^2p_{2*}\mathbb{Z}) \to H^2(|\mathscr{L}^d|, R^1p_{2*}\mathbb{Z}).$$

Thus, $F^1H^2(\mathscr{C},\mathbb{Z})$ is the kernel of the homomorphism $H^2(\mathscr{C},\mathbb{Z}) \to H^0(|\mathscr{L}^d|, R^2p_{2*}\mathbb{Z})$. The latter kernel is equal to $p_1^*\Sigma^\perp \oplus p_2^*H^2(|\mathscr{L}^d|,\mathbb{Z})$, by Lemma 7.3 (3). We conclude that $F^1H^2(\mathscr{C},\mathbb{Z})/p_2^*H^2(|\mathscr{L}^d|,\mathbb{Z})$ is isomorphic to both $H^1(|\mathscr{L}^d|, R^1p_{2*}\mathbb{Z})$ and $p_1^*\Sigma^\perp$.

(3) The composition $H^2(\mathscr{C},\mathbb{Z}) \to H^0(R^2p_{2*}\mathbb{Z}) \hookrightarrow \Sigma^*$ factors through $H^2(S,\mathbb{Z})$. If Σ is saturated, then the composition is surjective, since $H^2(S,\mathbb{Z})$ is unimodular. Thus, $d_2^{0,2} : H^0(R^2p_{2*}\mathbb{Z}) \to H^2(R^1p_{2*}\mathbb{Z})$ vanishes. The sheaf $p_{2*}\mathbb{Z}$ is the trivial local system and the homomorphism $H^4(|\mathscr{L}^d|, p_{2*}\mathbb{Z}) \cong H^4(|\mathscr{L}^d|,\mathbb{Z}) \to H^4(\mathscr{C},\mathbb{Z})$ is the injective pull-back homomorphism p_2^*. Thus the differential $d_2^{2,1} : H^2(R^1p_{2*}\mathbb{Z}) \to H^4(p_{2*}\mathbb{Z})$ vanishes. We conclude that

$E_2^{2,1} := H^2(R^1 p_{2_*}\mathbb{Z})$ is isomorphic to $E_\infty^{2,1}$. Now $E_\infty^{2,1}$ vanishes, since $H^3(\mathscr{C},\mathbb{Z})$ vanishes. □

Let $\mathscr{A}^0$ be the kernel of the homomorphism deg, given in (7.1). Then $\mathscr{A}^0$ is a subsheaf of $R^1 p_{2_*}\mathscr{O}^*_{\mathscr{C}}$ and we get the short exact sequences

$$0 \longrightarrow \mathscr{A}^0 \longrightarrow R^1 p_{2_*}\mathscr{O}^*_{\mathscr{C}} \stackrel{\deg}{\longrightarrow} R^2 p_{2_*}\mathbb{Z} \longrightarrow 0, \tag{7.3}$$

$$0 \longrightarrow R^1 p_{2_*}\mathbb{Z} \longrightarrow R^1 p_{2_*}\mathscr{O}_{\mathscr{C}} \longrightarrow \mathscr{A}^0 \longrightarrow 0, \tag{7.4}$$

and the long exact

$$\cdots \to H^1(|\mathscr{L}^d|, R^1 p_{2_*}\mathbb{Z}) \to H^1(|\mathscr{L}^d|, R^1 p_{2_*}\mathscr{O}_{\mathscr{C}}) \to H^1(|\mathscr{L}^d|, \mathscr{A}^0) \to \cdots$$

Lemma 7.5. *The group $H^0(|\mathscr{L}^d|, \mathscr{A}^0)$ is isomorphic to $NS(S) \cap \Sigma^\perp$. The composite homomorphism*

$$H^2(S,\mathbb{Z}) \to H^{0,2}(S) \stackrel{p_1^*}{\to} H^{0,2}(\mathscr{C}) \cong \widetilde{\text{Ш}} \to H^1(|\mathscr{L}^d|, \mathscr{A}^0)$$

factors through an injective homomorphism from $H^2(S,\mathbb{Z})/[\Sigma^\perp + NS(S)]$ into the kernel of the homomorphism $H^1(|\mathscr{L}^d|, \mathscr{A}^0) \to$ Ш.

Proof. The space $H^0(|\mathscr{L}^d|, R^1 p_{2_*}\mathscr{O}_{\mathscr{C}})$ vanishes, by Lemma 7.3 (1). Hence, $H^0(|\mathscr{L}^d|, \mathscr{A}^0)$ is the kernel of the homomorphism $H^1(|\mathscr{L}^d|, R^1 p_{2_*}\mathbb{Z}) \to \widetilde{\text{Ш}} \cong H^{0,2}(S)$. Compose the above homomorphism with the isomorphism $\Sigma^\perp \cong H^1(|\mathscr{L}^d|, R^1 p_{2_*}\mathbb{Z})$ of Lemma 7.4 in order to get the isomorphism $H^0(|\mathscr{L}^d|, \mathscr{A}^0) \cong NS(S) \cap \Sigma^\perp$.

We have a commutative diagram with short exact rows

$$\begin{tikzcd}
0 \arrow[r] & \frac{\Sigma^\perp}{NS(S)\cap\Sigma^\perp} \arrow[r] \arrow[d] & \widetilde{\text{Ш}} \arrow[r, "j"] \arrow[d, "\cong"'] & \ker[H^1(\mathscr{A}^0) \to H^2(R^1 p_{2_*}\mathbb{Z})] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \frac{H^2(S,\mathbb{Z})}{NS(S)} \arrow[r] & H^2(S,\mathscr{O}_S) \arrow[r] & H^2(S,\mathscr{O}_S^*) \arrow[r] & 0.
\end{tikzcd} \tag{7.5}$$

The top row is obtained from the long exact sequence of sheaf cohomologies associated to the short exact sequence (7.4). The left vertical homomorphism is injective and the right vertical homomorphism is surjective. The co-kernel of the former is isomorphic to the kernel of the latter and both are isomorphic to $H^2(S,\mathbb{Z})/[\Sigma^\perp + NS(S)]$. Setting

$$\text{Ш}^0 \;:=\; \ker[H^1(\mathscr{A}^0) \to H^2(R^1 p_{2_*}\mathbb{Z})], \tag{7.6}$$

we see that the right vertical homomorphism fits in the short exact sequence

$$0 \longrightarrow \frac{H^2(S, \mathbb{Z})}{\Sigma^{\perp} + NS(S)} \longrightarrow Ш^0 \longrightarrow Ш \longrightarrow 0. \tag{7.7}$$

The statement of the Lemma follows. □

Let $Ш^0$ be the group given in Eq. (7.6). Classes of $Ш$ represent torsors for the relative Picard group scheme, while classes of $Ш^0$ represent torsors for the relative Pic^0 group scheme. This comment will be illustrated in Example 7.8 below.

7.2 A Universal Family of Tate-Shafarevich Twists

Let S be the marked $K3$ surface in Diagram (5.4) and $M_H(u)$ the moduli space of H-stable sheaves of pure one-dimensional support on S in that Diagram. Recall that $c_1(u)$ is the first Chern class of $\mathscr{L}^d$, for a nef line-bundle $\mathscr{L}$ on S, and the support map $\pi : M_H(u) \to |\mathscr{L}^d|$ is a Lagrangian fibration.

Let σ be a section of $R^1 p_{2_*}(\mathscr{O}^*_{\mathscr{C}})$ over an open subset U of $|\mathscr{L}^d|$. Assume that σ is the image of a section $\tilde{\sigma}$ of $R^1 p_{2_*}(\mathscr{O}_{\mathscr{C}})$ over U. Then σ lifts to an automorphism of the open subset $\pi^{-1}(U)$ of $M_H(u)$. This is seen as follows. Fix a point $t \in |\mathscr{L}^d|$ and denote by C_t the corresponding divisor in S. Denote by $\sigma(t)$ the image of σ in $H^1(C_t, \mathscr{O}^*_{C_t})$ and by $L_{\sigma(t)}$ the line-bundle over C_t with class $\sigma(t)$. A sheaf F over C_t is H-stable, if and only if $F \otimes L_{\sigma(t)}$ is H-stable, since tensorization by $L_{\sigma(t)}$ induces a one-to-one correspondence between the set of subsheaves, which is slope-preserving, since $L_{\sigma(t)}$ belongs to the identity component of the Picard group of C_t.

Let s be an element of $Ш^0$. We can choose a Čech 1-co-cycle $\sigma := \{\sigma_{ij}\}$ for the sheaf $\mathscr{A}^0$ representing s in $Ш^0$, with respect to an open covering $\{U_i\}$ of $|\mathscr{L}^d|$, such that each σ_{ij} is the image of a section $\tilde{\sigma}_{ij}$ of $R^1 p_{2_*}(\mathscr{O}_{\mathscr{C}})$, since the homomorphism $R^1 p_{2_*}(\mathscr{O}_{\mathscr{C}}) \to \mathscr{A}^0$ is surjective. The co-cycle $\{\sigma_{ij}\}$ may be used to re-glue the open covering $\pi^{-1}(U_i)$ of $M_H(u)$ to obtain a separated complex manifold M_σ together with a proper map $\pi_\sigma : M_\sigma \to |\mathscr{L}^d|$. The latter is independent of the choice of the co-cycle, by the following Lemma, so we denote it by

$$\pi_s : M_s \to |\mathscr{L}^d|. \tag{7.8}$$

Lemma 7.6. *Let $\sigma := \{\sigma_{ij}\}$ and $\sigma' := \{\sigma'_{ij}\}$ be two co-cycles representing the same class in $Ш^0$. Then there exists an isomorphism $h : M_\sigma \to M_{\sigma'}$ satisfying the equation $\pi_{\sigma'} \circ h = \pi_\sigma$. If the lattice Σ of Lemma 7.4 has finite index in $NS(S)$, then h depends canonically on σ and σ'.*

Proof. There exists a co-chain $h := \{h_i\}$ in $C^0(\{U_i\}, \mathscr{A}^0)$, such that $h_i \sigma_{ij} = \sigma'_{ij} h_j$, possibly after refining the covering and restricting the co-cycles σ and σ' to the refinement. Each h_i is the image of a section $\tilde{h}_i$ of $R^1 p_{2_*} \mathscr{O}_{\mathscr{C}}$, possibly after further refinement of the covering, since the sheaf homomorphism $R^1 p_{2_*} \mathscr{O}_{\mathscr{C}} \to \mathscr{A}^0$ is surjective. Hence, h_i lifts canonically to an automorphism of $\pi^{-1}(U_i)$. The co-chain $\{h_i\}$ of automorphisms glues to a global isomorphism from $M_{\sigma'}$ to M_σ, by the equality $h_i\sigma_{ij} = \sigma'_{ij}h_j$.

If $h' := \{h'_i\}$ is another co-chain satisfying the equality $\delta(h) = \sigma(\sigma')^{-1}$, then $h^{-1}h'$ is a global section of $\mathscr{A}^0$. The assumption that Σ has finite index in $NS(S)$ implies that $H^0(\mathscr{A}^0)$ vanishes, by Lemma 7.5. Hence $h = h'$ and the above refinements are not needed. □

In the relative setting the above construction gives rise to a natural proper family

$$\tilde{\pi} : \mathscr{M} \to \widetilde{\text{Ш}} \times |\mathscr{L}^d|,$$

which restricts over $\{0\} \times |\mathscr{L}^d|$ to $\pi : M_H(u) \to |\mathscr{L}^d|$, and over $\tilde{s} \in \widetilde{\text{Ш}}$ to $\pi_{j(\tilde{s})} : M_{j(\tilde{s})} \to |\mathscr{L}^d|$. Indeed, let $(\{U_i\}, \tilde{\sigma}_{ij})$ be a Čech co-cycle representing a non-zero class $\tilde{\sigma}$ in $H^1(|\mathscr{L}^d|, R^1p_*\mathscr{O}_{\mathscr{C}})$. Let

$$\tau : \widetilde{\text{Ш}} \to \mathbb{C} \tag{7.9}$$

be the function satisfying $\tau(x)\tilde{\sigma} = x$. Then $(\{\widetilde{\text{Ш}} \times U_i\}, \exp(\tau\tilde{\sigma}_{ij}))$ is a global co-cycle representing the desired family. Let

$$f : \mathscr{M} \to \widetilde{\text{Ш}}$$

be the composition of $\tilde{\pi}$ with the projection to $\widetilde{\text{Ш}}$.

Proposition 7.7. *If the weight 2 Hodge structure of S is non-special, then M_s is Kähler, for all $s \in \text{Ш}^0$.*

Proof. There is an open neighborhood of the origin in $\widetilde{\text{Ш}}$, over which the fibers of f are Kähler, by the stability of Kähler manifolds [42, Theorem 9.3.3]. Let $j : \widetilde{\text{Ш}} \to \text{Ш}^0$ be the homomorphism given in Eq. (7.5) The kernel $\ker(j)$ is isomorphic to the group $[\Sigma^\perp + NS(S)]/NS(S)$, by Lemma 7.5. As a subgroup of the base $\widetilde{\text{Ш}}$ of the family f, the kernel $\ker(j)$ acts on the base. Let z be an element of $\ker(j)$ and $\tilde{s}$ an element of $\widetilde{\text{Ш}}$. The fibers $M_{\tilde{s}}$ and $M_{\tilde{s}+z}$ of f are both isomorphic to $M_{j(\tilde{s})}$ by Lemma 7.6. Let $V \subset \widetilde{\text{Ш}}$ be the subset consisting of points over which the fiber of f is Kähler. Then V is an open and $\ker(j)$-invariant subset of $\widetilde{\text{Ш}}$. Note that $\ker(j)$ is a finite index subgroup of $H^2(S,\mathbb{Z})/NS(S)$. The kernel $\ker(j)$ is a dense subgroup of $\widetilde{\text{Ш}}$, if and only if the image of $H^2(S,\mathbb{Z})/NS(S)$ is dense in $H^{0,2}(S)$, by Lemma 7.3 (4). This is indeed the case, by the assumption that the weight 2 Hodge structure of S is non-special, and Lemmas 5.4 and 5.5. The complement V^c of V in $\widetilde{\text{Ш}}$ is $\ker(j)$ invariant. If non-empty, then V^c is dense and closed and so equal to $\widetilde{\text{Ш}}$. But we know that V is non-empty. Hence, $V = \widetilde{\text{Ш}}$. □

Example 7.8. Consider the case where $d = 1$ and $\text{Pic}(S)$ is cyclic generated by the line bundle $\mathscr{L}$ of degree $2n-2$, $n \geq 2$. Then $H^2(|\mathscr{L}^d|, R^1p_{2_*}\mathbb{Z})$ vanishes, by Lemma 7.4 (3), and $\text{Ш}^0 = H^1(\mathscr{A}^0)$. The linear system $|\mathscr{L}|$ consists of integral curves, and so we can find an open covering $\{U_i\}$ of $|\mathscr{L}|$, and sections $\zeta_i : U_i \to \mathscr{C}$, such that $p_2 \circ \zeta_i$ is the identity. Set $D_i := \zeta_i(U_i)$. We get the line bundle $\mathscr{O}_{p_2^{-1}(U_i)}(D_i)$, which restricts to a line bundle of degree 1 on fibers of p_2 over points of U_i. Let h_i be the section of $R^1 p_{2_*}\mathscr{O}^*_{\mathscr{C}}$ over U_i corresponding to $\mathscr{O}_{p_2^{-1}(U_i)}(D_i)$ and denote by $h := \{h_i\}$ the corresponding co-chain in $C^0(\{U_i\}, R^1 p_{2_*}\mathscr{O}^*_{\mathscr{C}})$.

Consider the Lagrangian fibrations $\pi_0 : M_{\mathscr{L}}(0, \mathscr{L}, \chi) \to |\mathscr{L}|$ and $\pi_1 : M_{\mathscr{L}}(0, \mathscr{L}, \chi + 1) \to |\mathscr{L}|$, for some integer χ. The push-forward of every rank 1 torsion free sheaf on a curve in the linear system $|\mathscr{L}|$ is an $\mathscr{L}$-stable sheaf on S, since the curve is integral. Hence, the section h_i induces an isomorphism $h_i : \pi_0^{-1}(U_i) \to \pi_1^{-1}(U_i)$. The co-boundary $(\delta h)_{ij} := h_j h_i^{-1}$ is a co-cycle in $Z^1(\{U_i\}, \mathscr{A}^0)$ representing a class $s \in Ш^0$ mapping to the identity in Ш. The Lagrangian fibration $\pi_s : M_s \to |\mathscr{L}|$, associated to the class s in Eq. (7.8) with $u = (0, \mathscr{L}, \chi)$, coincides with $\pi_1 : M_{\mathscr{L}}(0, \mathscr{L}, \chi + 1) \to |\mathscr{L}|$, by the commutativity of the following diagram.

$$\begin{tikzcd}
\pi_0^{-1}(U_j) \arrow[d, "h_j"'] & \pi_0^{-1}(U_{ij}) \arrow[l, "\supset"'] \arrow[r, "h_i^{-1}h_j"] & \pi_0^{-1}(U_{ij}) \arrow[r, "\subset"] & \pi_0^{-1}(U_i) \arrow[d, "h_i"'] \\
\pi_1^{-1}(U_j) & \pi_1^{-1}(U_{ij}) \arrow[l, "\supset"'] \arrow[r, "id"] & \pi_1^{-1}(U_{ij}) \arrow[r, "\subset"] & \pi_1^{-1}(U_i).
\end{tikzcd}$$

The moduli spaces $M_{\mathscr{L}}(0, \mathscr{L}, \chi)$ and $M_{\mathscr{L}}(0, \mathscr{L}, \chi + 1)$ are not isomorphic for generic $(S, \mathscr{L})$, since their weight 2 Hodge structures are not Hodge isometric.

The kernel of $Ш^0 \to Ш$ is cyclic of order $2n - 2$, by the exactness of the sequence (7.7). The class s constructed above generates the kernel. This is seen as follows. The sheaf $R^2 p_{2_*}\mathbb{Z}$ is trivial, in our case, and the homomorphism deg, given in (7.3), maps the 0-co-chain h to a global section of $R^2 p_{2_*}\mathbb{Z}$, which generates $H^0(R^2 p_{2_*}\mathbb{Z})$. Hence, δh generates the image of the connecting homomorphism $H^0(R^2 p_{2_*}\mathbb{Z}) \to H^1(\mathscr{A}^0)$ associated to the short exact sequence (7.3). The latter image is precisely the kernel of $Ш^0 \to Ш$.

7.3 *The Period Map of the Universal Family is Étale*

Denote by $T_{\pi_s} := \ker\left[d\pi_s : TM_s \to \pi_s^* T|\mathscr{L}^d|\right]$ the relative tangent sheaf of $\pi_s : M_s \to |\mathscr{L}^d|$.

Lemma 7.9. *The vertical tangent sheaf T_{π_s} is isomorphic to $\pi_s^* T^*|\mathscr{L}^d|$.*

Proof. Let $\mathrm{sing}(\pi_s)$ be the support of the co-kernel of the differential $d\pi_s : TM_s \to \pi_s^* T|\mathscr{L}^d|$. We use Assumption 7.1 to prove that the co-dimension of $\mathrm{sing}(\pi_s)$ in M_s is ≥ 2. The generic fiber of π_s is smooth, since M_s is smooth. All fibers of π_s have pure dimension n [29]. Hence, the only way $\mathrm{sing}(\pi_s)$ could contain a divisor is if π_s

has fibers with a non-reduced irreducible component over some divisor in $|\mathscr{L}^d|$. The generic divisor in the linear system $|\mathscr{L}^d|$ is a smooth curve, by Assumption 7.1 (1) and [32, Prop. 1]. The fiber of π_s, over a reduced divisor $C \in |\mathscr{L}^d|$, is isomorphic to the compactified Picard of C, consisting of $\mathscr{L}$-stable sheaves of Euler characteristic χ with pure one-dimensional support C, which are the push forward of rank 1 torsion free sheaves over C. If C is an integral curve, then the moduli space of rank 1 torsion free sheaves over C with a fixed Euler characteristic is irreducible and reduced [1]. If C is reduced (possibly reducible) with at worst ordinary double point singularities, then the compactified Picard is reduced, by a result of Oda and Seshadri [37]. Assumption 7.1 (2) thus implies that sing(π_s) has co-dimension ≥ 2 in M_s.

Let U be the complement of sing(π_s) in M_s. The isomorphism $TM_s \to T^*M_s$, induced by a non-degenerate global holomorphic 2-form, maps the restriction of T_{π_s} to U isomorphically onto the restriction of $\pi_s^*T^*|\mathscr{L}^d|$. The isomorphism $TM_s \to T^*M_s$ must map T_{π_s} as a subsheaf of the locally free $\pi_s^*T|\mathscr{L}^d|$, by the fact that sing(π_s) has codimension ≥ 2. But T_{π_s} is a saturated subsheaf of TM_s. Hence, the image of T_{π_s} is also saturated in T^*M_s, and is thus equal to $\pi_s^*T^*|\mathscr{L}^d|$. □

When the $K3$ surface S is non-special, the fibers of the family f are irreducible holomorphic symplectic manifolds, by Proposition 7.7 and the fact that Kähler deformations of an irreducible holomorphic symplectic manifold remain such [4]. Denote by

$$\eta : R^2 f_*\mathbb{Z} \to (\Lambda)_{\widetilde{\text{Ш}}} \tag{7.10}$$

the trivialization, which restricts to the marking η_1 in Diagram (5.4) over the point $0 \in \widetilde{\text{Ш}}$. Let $P_f : \widetilde{\text{Ш}} \to \Omega^+_{\alpha^\perp}$ be the period map of the family f and the marking η. Let $dP_f : T_{\tilde{s}}\widetilde{\text{Ш}} \to H^{2,0}(M_s)^* \otimes H^{1,1}(M_s)$ be the differential at $\tilde{s}$ of the period map.

Lemma 7.10. *The differential dP_f is injective, for all $\tilde{s}$ in $\widetilde{\text{Ш}}$, and its image is equal to $H^{2,0}(M_s)^* \otimes \pi_s^* H^{1,1}(|\mathscr{L}^d|)$.*

Proof. Let $\psi : H^{2,0}(M_s)^* \otimes H^1(|\mathscr{L}^d|, T^*|\mathscr{L}^d|) \to H^1(M_s, T_{\pi_s})$ be the composition of

$$1\otimes\pi_s^*: H^{2,0}(M_s)^*\otimes H^1(|\mathscr{L}^d|, T^*|\mathscr{L}^d|)\to H^0(M_s, \wedge^2 TM_s)\otimes H^1(M_s, \pi_s^*T^*|\mathscr{L}^d|)$$

with the contraction homomorphism $H^0(M_s, \wedge^2 TM_s) \otimes H^1(M_s, \pi_s^*T^*|\mathscr{L}^d|) \to H^1(M_s, T_{\pi_s})$. Let $\kappa_{\tilde{s}} : T_{\tilde{s}}\widetilde{\text{Ш}} \to H^1(M_s, TM_s)$ be the Kodaira-Spencer map. We have the commutative diagram.

$$\begin{tikzcd}
H^{2,0}(M_s)^* \otimes H^1(|\mathscr{L}^d|, T^*|\mathscr{L}^d|) \arrow[rrr, bend left, "1\otimes\pi_s^*"] \arrow[drr, "\psi"] & & T_{\tilde{s}}\widetilde{Ш} \arrow[r, "dP_f"] \arrow[d, "\nu"] \arrow[dr, "\kappa_{\tilde{s}}"] & H^{2,0}(M_s)^* \otimes H^{1,1}(M_s) \\
& & H^1(M_s, T_{\pi_s}) \arrow[r, "\gamma"] & H^1(M_s, TM_s). \arrow[u, "\cong"']
\end{tikzcd}$$

Above, the right vertical homomorphism is induced by the sheaf homomorphism $TM_s \rightarrow T^*M_s$, associated to a holomorphic 2-form, and γ is induced by the inclusion of the relative tangent sheaf T_{π_s} as a subsheaf of TM_s. The homomorphism ν is defined as follows. A tangent vector ξ at a class $\tilde{s}$ of $\widetilde{Ш}$ is represented by a co-cycle of infinitesimal automorphisms – tangent vector fields – which are vertical, being a limit of translations by local sections of the image of $R^1 p_{2_*}\mathscr{O}_{\mathscr{C}}$ in $R^1 p_{2_*}\mathscr{O}^*_{\mathscr{C}}$. So ξ corresponds to an element $\nu(\xi)$ in $H^1(M_s, T_{\pi_s})$.

The top right triangle commutes, by Griffiths' identification of the differential of the period map [9]. The middle triangle commutes, by definition of the family f. The commutativity of the outer polygon is easily verified. The top horizontal homomorphism $1 \otimes \pi_s^*$ is injective, with image equal to the tangent line to the fiber of q. Hence, it suffices to prove that ψ and ν have the same image in $H^1(M_s, T_{\pi_s})$. The latter statement would follow once we prove that ν is an isomorphism.

The homomorphism ν is induced by the pullback

$$\pi_s^* H^1(|\mathscr{L}^d|, R^1 p_{2_*}\mathscr{O}_{\mathscr{C}}) \rightarrow H^1(M_s, \pi_s^* R^1 p_{2_*}\mathscr{O}_{\mathscr{C}}),$$

followed by the homomorphism of sheaf cohomologies induced by an injective sheaf homomorphism

$$\tilde{\nu} \; : \; \pi_s^* R^1 p_{2_*}\mathscr{O}_{\mathscr{C}} \rightarrow T_{\pi_s}.$$

The domain of $\tilde{\nu}$ is isomorphic to $\pi_s^* T^*|\mathscr{L}^d|$, by Lemma 7.3, and its target is isomorphic to $\pi_s^* T^*|\mathscr{L}^d|$, by Lemma 7.9. Hence, $\tilde{\nu}$ is an isomorphism. It remains to prove that $H^1(M_s, \pi_s^* T^*|\mathscr{L}^d|)$ is one dimensional. We have the exact sequence

$$\begin{aligned} 0 \rightarrow H^1(|\mathscr{L}^d|, \pi_{s_*}\pi_s^* T^*|\mathscr{L}^d|) \rightarrow H^1(M_s, \pi_s^* T^*|\mathscr{L}^d|) \\ \rightarrow H^0(|\mathscr{L}^d|, T^*|\mathscr{L}^d| \otimes R^1\pi_{s_*}\mathscr{O}_{M_s}). \end{aligned}$$

The left hand space is one-dimensional. It remains to prove that the right hand one vanishes. It suffices to prove that $R^1\pi_{s_*}\mathscr{O}_{M_s}$ is isomorphic to $T^*|\mathscr{L}^d|$, since $T^*|\mathscr{L}^d| \otimes T^*|\mathscr{L}^d|$ does not have any non-zero global sections.

When $s = 0$ and $M_0 = M_H(u)$, then M_0 is projective and $R^1\pi_{0_*}\mathscr{O}_{M_0}$ is isomorphic to $T^*|\mathscr{L}^d|$, by [30, Theorem 1.3]. Let us show that the sheaves $R^1\pi_{s_*}\mathscr{O}_{M_s}$ are naturally isomorphic to $R^1\pi_{0_*}\mathscr{O}_{M_0}$, for all s in Ш. The fibrations π_s agree, by definition, over the open sets in a Čech covering of $|\mathscr{L}^d|$, and the gluing transformations for the co-cycle representing the class s do not change the induced sheaf transition functions for the sheaves $R^1\pi_{s_*}\mathscr{O}_{M_s}$, as we show next. The gluing transformations glue locally free sheaves, so it suffices to prove that they agree with those of π_0 over a dense open subset of $|\mathscr{L}^d|$. Indeed, if the fiber of $M_H(u)$ over $t \in |\mathscr{L}^d|$ is a smooth and projective $\mathrm{Pic}^d(C_t)$, then an automorphism of an abelian variety $\mathrm{Pic}^d(C_t)$, acting by translation, acts trivially on the fiber $H^1(\mathrm{Pic}^d(C_t), \mathscr{O}_{\mathrm{Pic}^d(C_t)})$ of $R^1\pi_*\mathscr{O}_{M_H(u)}$. □

7.4 The Tate-Shafarevich Line as the Base of the Universal Family

Let $q : \Omega^+_{\alpha^\perp} \to \Omega^+_{Q_\alpha}$ be the morphism given in Eq. (4.3).

Theorem 7.11. *Assume that the weight 2 Hodge structure of S is non-special and Assumption 7.1 holds. Then the period map P_f of the family f maps $\widetilde{\text{Ш}}$ isomorphically onto the fiber of the morphism q through the period of $M_H(u)$.*

Proof. We already know that P_f is non-constant, by Lemma 7.10. The statement implies that P_f is an affine linear isomorphism of one-dimensional complex affine spaces. It suffices to prove the statement for a dense subset in moduli, since the condition of being affine linear is closed. We may thus assume that $\mathrm{Pic}(S)$ is cyclic generated by $\mathscr{L}$. Then $H^0(|\mathscr{L}^d|, \mathscr{A}^0)$ is trivial, by Lemma 7.5.

Set $\Gamma := c_1(\mathscr{L})^\perp$. Note that $NS(S) = \mathbb{Z}c_1(\mathscr{L})$ and Γ has finite index in $H^2(S,\mathbb{Z})/NS(S)$. Let

$$e : \Gamma \to \widetilde{\text{Ш}}$$

be the composition of the projection $\Gamma \to H^{0,2}(S)$ with the isomorphisms $H^{0,2}(S) \cong H^{0,2}(\mathscr{C}) \cong \widetilde{\text{Ш}}$ of Lemma 7.3. Then e is injective and its image is dense in $\widetilde{\text{Ш}}$, by Lemma 5.4.

Given an element $x \in \widetilde{\text{Ш}}$, we get a marked pair (M_x, η_x), as above. $M_H(u)$ will be denoted by M_0, it being the fiber of f over the origin in $\widetilde{\text{Ш}}$. We associate next to an element $\gamma \in \Gamma$ a canonical isomorphism

$$h_\gamma : M_0 \to M_{e(\gamma)}.$$

Let $\tau : \widetilde{\text{Ш}} \to \mathbb{C}$ be the function given in (7.9), which was used in the construction of the family f. Let $\tilde{\sigma} := \{\tilde{\sigma}_{ij}\}$ be the co-cycle used in that construction. Let a be the 1-co-cycle given by $a_{ij} := \exp(\tau(e(\gamma))\tilde{\sigma}_{ij})$. Then $M_{e(\gamma)}$ is the Tate-Shafarevich twist of M_0 with respect to the co-cycle a. The 1-co-cycle a is a co-boundary in $Z^1(\{U_i\}, \mathscr{A}^0)$, by Lemma 7.5 and the definition of Γ. Thus, there exists a 0-co-chain $h := \{h_i\}$ in $C^0(\{U_i\}, \mathscr{A}^0)$, satisfying $\delta h = a$. The co-chain h is unique, since $H^0(\mathscr{A}^0)$ is trivial, by our assumption on S. The co-chain h determines the isomorphism $h_\gamma : M_0 \to M_{e(\gamma)}$ (Lemma 7.6).

We define next a monodromy representation associated to the family f. Denote by $h_{\gamma*} : H^2(M_0, \mathbb{Z}) \to H^2(M_{e(\gamma)}, \mathbb{Z})$ the isomorphism induced by h_γ. Let

$$\mu \ : \ \Gamma \to Mon^2(M_0)$$

be given by the composition $\mu_\gamma := \eta_0^{-1} \circ \eta_{e(\gamma)} \circ h_{\gamma*}$ of the parallel-transport operator $\eta_0^{-1} \circ \eta_{e(\gamma)}$ and the isomorphism $h_{\gamma*}$.

Claim 7.12. The map μ is a group homomorphism.

Proof. Let γ_1, γ_2 be elements of Γ and set $\gamma_3 := \gamma_1 + \gamma_2$. Let the topological space B be the quotient of $\widetilde{\text{Ш}}$ obtained by identifying the four points 0, $e(\gamma_1)$, $e(\gamma_2)$, $e(\gamma_3)$. The family f descends to a family $\bar{f} : \overline{\mathscr{M}} \to B$ by identifying the fiber $M_{e(\gamma_i)}$ with M_0 via the isomorphisms h_{γ_i}, $1 \leq i \leq 3$. Then μ_{γ_i} is the monodromy operator corresponding to any loop in B, which is the image of some continuous path from 0 to $e(\gamma_i)$ in $\widetilde{\text{Ш}}$. Let $\bar{0} \in B$ be the image of $0 \in \widetilde{\text{Ш}}$. The statement now follows from the fact that the monodromy representation of $\pi_1(B, \bar{0})$ in $H^2(M_0, \mathbb{Z})$ is a group homomorphism. □

The image of $\widetilde{\text{Ш}}$ via the period map is contained in the fiber of q, since the differential of the morphism $q \circ P_f$ vanishes, by Lemma 7.10. It follows that the variation of Hodge structures of the local system $R^2 f_* \mathbb{Z}$ over $\widetilde{\text{Ш}}$ is the pullback of the one over the fiber of q via the period map P_f. Let η be the trivialization of $R^2 f_* \mathbb{Z}$ given in Eq. (7.10). Given a point $x \in \widetilde{\text{Ш}}$, set $\alpha_x := \eta_x^{-1}(\alpha)$. Then $\alpha_x = \pi_x^*(c_1(\mathscr{O}_{|\mathscr{L}^d|}(1)))$ and the sub-quotient variation of Hodge structures $\alpha_x^\perp / \mathbb{Z}\alpha_x$ is trivial.

The vertical tangent sheaf T_{π_x} is naturally isomorphic to T_{π_0}, as we saw in the last paragraph of the proof of Lemma 7.10. The 2-form w_x induces an isomorphism $\pi_{x_*} T_{\pi_x} \stackrel{w_x}{\to} T^*|\mathscr{L}^d|$, by Lemma 7.9. We get the composite isomorphism $\pi_{0_*} T_{\pi_0} \cong \pi_{x_*} T_{\pi_x} \stackrel{w_x}{\to} T^*|\mathscr{L}^d|$. Let w_x be the unique holomorphic 2-form, for which the composite isomorphism is equal to $\pi_{0_*} T_{\pi_0} \stackrel{w_0}{\to} T^*|\mathscr{L}^d|$. Such a form w_x exists, since the endomorphism algebra of $T^*|\mathscr{L}^d|$ is one dimensional.

We show next that the class of w_x is the $(2,0)$ part of the flat deformation of the class of w_0 in the local system $R^2 f_* \mathbb{C}$. It suffices to prove the local version of that statement. Let x_0 be a point of $\widetilde{\text{Ш}}$. There is a differentiable trivialization of $f : \mathscr{M} \to \widetilde{\text{Ш}}$, over an open analytic neighborhood U of x_0, and a C^∞ family of complex structures J_x, $x \in U$, such that (M_{x_0}, J_x) is biholomorphic to M_x. Furthermore, the complex structures J_{x_0} and J_x restrict to the same complex structure on each fiber of π_{x_0} and π_{x_0} is holomorphic with respect to both. Both complex structures induce the same complex structure on $\mathrm{Hom}\left(T_{\pi_{x_0}}, \pi_{x_0}^* T_{\mathbb{R}}^* |\mathscr{L}^d|\right)$ and the two forms w_{x_0} and w_x induce the same section in the complexification of that bundle. Hence, the difference $w_{x_0} - w_x$ is a closed 2-form in $\pi_{x_0}^* \wedge^2 T_{\mathbb{R}}^* |\mathscr{L}^d| \otimes_{\mathbb{R}} \mathbb{C}$. Being closed, the latter 2-form must be the pull-back of a closed 2-form θ on $|\mathscr{L}^d|$, since fibers of π_{x_0} are connected. Now the cohomology class of $\pi_{x_0}^*\theta$ is of type $(1,1)$ with respect to all complex structures, since $H^{1,1}(|\mathscr{L}^d|) = H^2(|\mathscr{L}^d|, \mathbb{C})$. Hence, the class of w_x is the $(2,0)$ part of the class of w_{x_0} with respect to the complex structure J_x.

There exists a constant $c_x \in \mathbb{C}$, such that the equality

$$\eta_x(w_x) = \eta_0(w_0) + c_x\alpha$$

holds in $\Lambda_{\mathbb{C}}$, by the characterization of ω_x in the above paragraph. The function $c : \widetilde{Ш} \to \mathbb{C}$ defined above is equivalent to the period map P_f and is thus holomorphic and its derivative is no-where vanishing, by Lemma 7.10. If $x = e(\gamma)$, we get $\eta_0^{-1}\eta_{e(\gamma)}(w_{e(\gamma)}) = w_0 + c_{e(\gamma)}\alpha_0$, Now $h_\gamma(w_0) = w_{e(\gamma)}$, by definition of w_x, $x \in \widetilde{Ш}$, and the construction of h_γ. We get the equality

$$\mu_\gamma(w_0) = w_0 + c_{e(\gamma)}\alpha_0. \tag{7.11}$$

The composition $c \circ e : \Gamma \to \mathbb{C}$ is a group homomorphism,

$$c(e(\gamma_1) + e(\gamma_2)) = c(e(\gamma_1)) + c(e(\gamma_2)),$$

by Eq. (7.11) and Claim 7.12. The image $e(\Gamma)$ is dense in $\widetilde{Ш}$ and so $e(\Gamma) \times e(\Gamma)$ is dense in $\widetilde{Ш} \times \widetilde{Ш}$. We conclude that c is a group homomorphism, $c(x_1 + x_2) = c(x_1) + c(x_2)$, for all $(x_1, x_2) \in \widetilde{Ш} \times \widetilde{Ш}$. Continuity of c implies that it is a linear transformation of real vector spaces. Indeed, given x_1, x_2 in $\widetilde{Ш}$, $c(ax_1 + bx_2) = ac(x_1) + bc(x_2)$, for all $a, b \in \mathbb{Z}$, hence also for all $a, b \in \mathbb{Q}$, and continuity implies that the equality holds also for all $a, b \in \mathbb{R}$. The map c is holomorphic, hence it is a linear transformation of one-dimensional complex vector spaces, which is an isomorphism, since c is non-constant. This completes the proof of Theorem 7.11. □

Let X be an irreducible holomorphic symplectic manifold of $K3^{[n]}$-type and $\pi : X \to \mathbb{P}^n$ a Lagrangian fibration. Set $\alpha := \pi^*c_1(\mathscr{O}_{\mathbb{P}^n}(1))$. Let d be the divisibility of $(\alpha, \bullet)$. Let $(S, \mathscr{L})$ be the semi-polarized $K3$ surface associated to (X, α) in Diagram (5.4) and χ the Euler characteristic of the Mukai vector u in that diagram. Choose a u-generic polarization H on S.

Theorem 7.13. *Assume that X is non-special and $(S, \mathscr{L})$ satisfies Assumption 7.1. Then X is bimeromorphic to a Tate-Shafarevich twist of the Lagrangian fibration $M_H(0, \mathscr{L}^d, \chi) \to |\mathscr{L}^d|$.*

Proof. Fix a marking $\eta : H^2(X, \mathbb{Z}) \to \Lambda$. Starting with the period of (X, η), Theorem 7.11 exhibits a marked triple (X', α', η'), with $\eta'(\alpha') = \eta(\alpha)$, in the same connected component $\mathfrak{M}^+_{\eta(\alpha)^\perp}$ as the triple (X, α, η), such that the class α' is semi-ample as well and the periods $P(X, \eta)$ and $P(X', \eta')$ are equal. Furthermore, the Lagrangian fibration $\pi' : X' \to |\mathscr{L}^d|$ induced by α' is a Tate-Shafarevich twist of $\pi_0 : M_H(0, \mathscr{L}^d, \chi) \to |\mathscr{L}^d|$. Step 1 of the proof of Theorem 1.3 yields a bimeromorphic map $f : X \dashrightarrow X'$, which is shown in Step 2 of that proof to satisfy $f^*(\alpha') = \alpha$ (see Eq. (6.1)). □

Proof (of Theorem 1.5). The condition that $NS(X) \cap \alpha^{\perp}$ is cyclic generated by α implies that the semi-polarized $K3$ surface $(S, \mathscr{L})$, associated to (X, α), has a cyclic Picard group generated by $\mathscr{L}$. Assumption 7.1 thus holds, by Remark 7.2. Theorem 1.5 thus follows from Theorem 7.13. □

Acknowledgements I would like thank Yujiro Kawamata, Christian Lehn, Daisuke Matsushita, Keiji Oguiso, Osamu Fujino, Thomas Peternell, Sönke Rollenske, Justin Sawon, and Kota Yoshioka for helpful communications. I would like to thank the two referees for their careful reading and insightful comments and suggestions.

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