A Coevolutionary Model of Strategic Network Formation

Ibrahim Al-Shyoukh and Jeff S. Shamma

Abstract. In foundational models of network formation, the mechanisms for link formation are based solely on network topology. For example, preferential attachment uses degree distributions, whereas a strategic connections model uses internode distances. These dynamics implicitly presume that such benefits and costs are instantaneous functions of the network topology. A more detailed model would include that benefits and costs are themselves derived through a dynamic process, which, in the absence of time-scale separation, necessitates a coevolutionary analysis. This paper introduces a new coevolutionary model of strategic network formation. In this model, network formation evolves along with the flow of benefits from one node to another. We examine the emergent equilibria of this combined dynamics of network formation and benefit flow. We show that the class of strict equilibria is stable (or robust to small perturbations in the benefits flows).

1 Introduction

Networks involving benefit exchanges between the different nodes are ubiquitous. Examples include information exchange in social networks, goods exchange in economic markets, and scientific collaboration networks. The abundance and importance of such networks have manifested a growing area of research that looks into the theory of network formation and the relevance of emerging structures. A number of different models for the network formation in multiple disciplines have been proposed that encompass a range of ideas [1–9]. A common feature of these models is that there is no interdependence or feedback between the network formation dynamics, and the dynamics on

Ibrahim Al-Shyoukh · Jeff S. Shamma
School of Electrical & Computer Engineering
Georgia Institute of Technology
Atlanta, GA 30308 USA

P. Contucci et al. (eds.), *Complex Networks V*, 167
Studies in Computational Intelligence 549,
DOI: 10.1007/978-3-319-05401-8_16,

the network. Recent work has begun to investigate models with endogenously formed (i.e., coevolutionary) network topologies in a wide class of systems including opinion dynamics [10–19].

The work in this paper concerns the study of strategic coevolutionary network formation. In foundational models of network formation, the mechanisms for link formation are based solely on network topology. For example, preferential attachment [20] uses degree distributions, whereas a strategic connections model [21] uses internode distances. These dynamics implicitly presume that such benefits and costs are instantaneous functions of the network topology. A more detailed model would include that benefits and costs are themselves derived through a dynamic process, which, in the absence of time-scale separation, necessitates a coevolutionary analysis.

Here we present a model that captures the dynamic flow of benefits in a network. The model is inspired by and builds upon the strategic network formation model of Bala & Goyal [21]. In the model, and upon link establishment, benefits flow from one node to another over time. The amount of benefit and speed of flow are distance dependent. As the distance between nodes increase, the total attainable benefits becomes smaller and it takes longer for the benefits to be attained. Another feature of the model is that when links are severed, then benefits are not immediately lost. Rather, they are dissipated over time.

By allowing time to propagate, then a node can realize the full benefits from an established link, and nodes can seek to maximize such asymptotically realized benefits. However, this analysis presumes a separation of time scales. Instead, we consider the case when nodes are myopic decision makers that seek to maximize the one time step flow of benefits. We examine the conditions for equilibria of this model and the stability of such equilibria. We also show that this model admits equilibria that can only be realized at a higher cost in the case of immediate benefit availability. This formulation gives rise to a richer set of network topologies without additional cost constraints. Section 2 introduces some preliminaries, and Section 3 presents the model and relevant analysis.

2 Preliminaries

Let us recall the strategic network formation model of Bala & Goyal [21]. The model represents the flow of benefits in a network of $N \geq 3$ nodes. Consider for example the network shown in Fig. 1. A directed edge $i \leftarrow j$ indicates flow of benefits from j to i, e.g, node 8 is an immediate (one-hop) beneficiary from node 5, and an indirect (two-hop) beneficiary from nodes 2, 6, and 7. Nodes dynamically form and sever links based on the rewards of benefit flow and costs of link formation and maintenance.

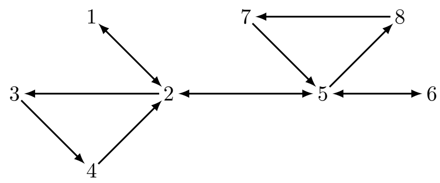


Fig. 1 An example of a directed network of information flow. The arrow direction indicates the direction of flow.

2.1 The Static Game

First, we will consider the static network formation game. Let $\mathcal{N} = \{1, 2, \ldots, N\}$ be the set of all nodes of the network. Given that a node can connect to $N-1$ nodes, a node's strategy can be represented by the binary valued vector $g_i = (g_{i,1}, \ldots, g_{i,i-1}, g_{i,i+1}, \ldots, g_{i,N})$,, where $g_{i,j} = 1$ whenever node i has a link with node j, $g_{i,j} = 0$ otherwise. A network g can be represented by the joint strategies of all nodes as $g = (g_1, g_2, \ldots, g_N)$. We shall use g_{-i} to refer to the network constructed from g by excluding node i's links, i.e., $g_{-i} = (g_1, \ldots, g_{i-1}, g_{i+1}, \ldots, g_N)$. A path from node j to node i is denoted by $\overline{ij}$. Let $|\overline{ij}|$ denote the length of path $\overline{ij}$. Define $d_{ij}(g_i, g_{-i}) = \min_{\overline{ij} \in g} |\overline{ij}|$ as the length of the shortest path from j to i. For compactness, in the remainder of this paper, we will write d_{ij} instead of $d_{ij}(g_i, g_{-i})$ whenever the arguments are clear.

Immediate Benefit Availability. Whenever node i establishes a connection with node j, benefits become accessible to i. In the existing models of strategic network formation, the benefits are fully transferred from node j and its neighbors to i immediately upon link establishment. The amount of benefits transferred can be distance dependent. Thus, the value of benefits from a direct connection can generally be assumed to be $\delta \in (0, 1]$. Whereas if j is an indirect connection of i, then the value of benefits is $\delta^{d_{ij}}$. Additionally, let c denote the cost of establishing a connection with another node. The cost is only incurred by the node establishing/maintaining the link. In the directed flow network, the benefits will only flow to the node establishing the connection.

For a given network g, let $\mathcal{N}_i^+(g) = \{k \in \{1, \ldots, N\} : \overline{ik} \in g\}$ denote the set of all nodes that have a path to node i. This set defines all the neighbors of i, direct or indirect. As such, benefits can flow from these nodes to i. We shall define $\mu_i(g_i) = \sum_k g_{i,k}$ as the number of links, or direct neighbors, of node i. The utility of a given strategy can be defined as the net value of the benefits available through the connections established by the strategy minus the cost of establishing these connections.

$$u_i(g_i, g_{-i}) = \sum_{j \in \mathcal{N}_i^+(g)} \delta^{d_{ij}(g_i, g_{-i})} - c\mu_i(g_i). \tag{1}$$

A best response strategy of node i to g_{-i}, hereafter denoted by $\mathrm{BR}(g_{-i})$, is a strategy g_i such that

$$\mathrm{BR}(g_{-i}) \in \arg\max_{g_i \in \mathcal{G}_i} u_i(g_i, g_{-i}), \tag{2}$$

where $\mathcal{G}_i$ is the set of all possible pure strategies of node i[1]. Hence for any best response g_i,

$$u(g_i, g_{-i}) \geq u(g_i', g_{-i}) \qquad \forall g_i' \in \mathcal{G}_i.$$

Definition 1. A network g is said to be a *Nash network* if $g_i = \mathrm{BR}(g_{-i})$, $\forall i \in \mathcal{N}$.

2.2 Repeated Myopic Play

Consider the case when the network formation game described above is played repeatedly at time steps $t = 1, 2, \ldots$. At the beginning of every time step[2], every node plays the same strategy it used in the last time step with probability p_i. That is, the nodes' strategies exhibit inertia from one time step to another. With probability $1 - p_i$, the nodes update their strategies based on myopic best response to the observed network structure from the previous time. In the case that the best response strategy is not unique, the node randomizes its decision over the set of best response strategies. As a result, a node playing a best response to the same network observed in the previous time step might switch strategies.

Let g^{t-1} denote the network at time $t-1$, then the dynamics of network formation for agent i are

$$g_i^t = \mathrm{BR}(g_{-i}^{t-1}). \tag{3}$$

As an example, consider the 3-node networks shown in Fig. 2. Starting with the network in (a), the networks in (b) and (c) can be constructed by having node 1 switch its connection from 2 to 3, or by adding a connection to node 3 respectively. Therefore, $g_1^{(a)} = (1,0)$, $g_1^{(b)} = (0,1)$, $g_1^{(c)} = (1,1)$ and $g_{-1} = (g_2, g_3) = ((1,1),(0,1))$. For node 1, the utility for the different strategies are, $u_1(g_1^{(a)}, g_{-1}) = \delta + \delta^2 - c$, $u_1(g_1^{(b)}, g_{-1}) = \delta + \delta^2 - c$, and $u_1(g_1^{(c)}, g_{-1}) = 2\delta - 2c$. For node 2, if $c \leq \delta$, then $u_2((1,1), g_{-2}) \geq u_2(g_2', g_{-2})$ for any other strategy $g_2' \in \mathcal{G}_2$ of node 2. Because of the symmetry between nodes 1 and 3, then the network in (a) is a Nash network if $c < \delta$ and

[1] Here we are restricting our attention to the set of pure strategies.

[2] Except at $t = 1$.

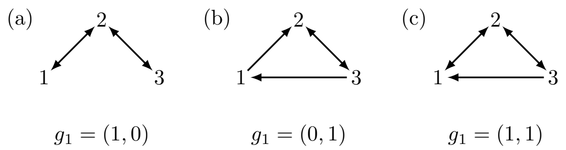


Fig. 2 A 3-node network showing the three strategies for node 1 with nodes 2, and 3 using the strategies $g_2 = (1, 1)$ and $g_3 = (0, 1)$

$$\begin{aligned} u_1(g_1^{(a)}, g_{-1}) &\geq u_1(g_1^{(c)}, g_{-1}) \\ \delta + \delta^2 - c &\geq 2\delta - 2c \\ c &\geq \delta - \delta^2. \end{aligned} \tag{4}$$

Notice that node 1 would be indifferent between the strategies $g_1 = (1, 0)$ and $g_1 = (0, 1)$. Hence, networks (a) and (b) would be Nash networks, and node 1 would switch between these two configurations provided the other two nodes do not change their strategies.

If nodes 1 and 3 are allowed to change strategies simultaneously based on a best response to the previous network, then it is conceivable that both nodes would switch strategies where they switch to connections from nodes 3 and 1 instead of the existing connections to node 2. Hence, the network becomes $g = ((0, 1), (1, 1), (1, 0))$. As such the utility of this network for either node becomes $u_1 = u_2 = \delta - c$, which is less than the current utility of $u_1 = u_2 = \delta + \delta^2 - c$. Therefore, in the event that players are allowed to switch strategies simultaneously, the network in (a) is not stable.

3 Coevolutionary Model

3.1 Dynamic Flow of Benefits

This work is concerned with the case of dynamic flow of the benefits. In this case, the benefits flow over time from one node to another. If the timescale for flow is fast compared to the network formation dynamics, then there is a separation of time scales, and this situation would closely resemble the above mentioned case of immediate benefit availability. However, if the time scales for benefit flow and network formation are comparable, then this presents a coevolutionary process through which benefit flows and network formation occur simultaneously and the emergent behavior can be different. We consider the case where the benefits obtained are derived through a dynamic process. Upon establishing a link, a node will realize a portion of the direct benefit of the connected node, and with time, the benefits are asymptotically realized. This model represents delay in the flow of benefits from a node another. The same applies to benefits from non-direct connections, and the delay is distance dependent, i.e., the further away two nodes are from each other, then the slower is the flow of benefits from one to another. The distance dependence is very relevant to a number of systems including physical transfer of goods, and information or knowledge transfer.

Additionally, when a path between two nodes is severed, then the benefits available from a node to another are not lost immediately, but are forgotten over time. Here, the rate is also distance dependent. Formally, we define the benefit flow model to be

$$b_{ij}^t = f(b_{ij}^{t-1}, g_i, g_{-i}) = \begin{cases} \alpha_{d_{ij}} b_{ij}^{t-1} + (1-\alpha_{d_{ij}})\delta^{d_{ij}}, & \delta^{d_{ij}} \geq b_{ij}^{t-1} \quad (5a) \\ \beta_{d_{ij}} b_{ij}^{t-1} + (1-\beta_{d_{ij}})\delta^{d_{ij}}, & \delta^{d_{ij}} < b_{ij}^{t-1} \quad (5b) \end{cases}$$

such that $\beta_i, \alpha_i \in [0,1]$, $\alpha_1 \leq \alpha_2 \leq \ldots$ and $\beta_1 \geq \beta_2 \geq \ldots$.

Here, b_{ij}^t is the benefit available to node i from node j at time t. Let B denote the matrix whose elements are b_{ij}, and b_i be the ith row of the matrix B[3].

Examining Eq. (5a) closely, notice that the benefits are increasing since $\delta^{d_{ij}} \geq b_{ij}$. That is, when the attainable benefit is higher that the current flow of benefits from a given node, then the benefits will increase. The rates for increase $\alpha_{d_{ij}}$ are distance dependent and hence the subscript. Note that the higher the value of $\alpha_{d_{ij}}$, then the slower that benefits flow. As such, the benefits flow slower as the distances between nodes increase.

When the attainable benefit from a given node, $\delta^{d_{ij}}$ is less that the current flow of benefits, then the benefits will decrease according to Eq. (5b). The rates of decrease are distance dependent. The lower the value of $\beta_{d_{ij}}$, the higher the decrease in benefits. When there is no path between two nodes i and j, then d_{ij} is infinite and $\delta^{d_{ij}} = 0$, and benefits will decrease at a rate of β_∞. In the case when $\alpha_{d_{ij}} = 1$, then no benefits will flow. Similarly, when $\beta_{d_{ij}} = 1$, benefits will not decrease. Furthermore, when $\alpha_{d_{ij}} = 0$, $\beta_{d_{ij}} = 0$, then the dynamics in (5) become equivalent to the instantaneous benefit availability model.

Compared to the instantaneous model of benefit availability, for a fixed network, the attainable benefits of both models are the same. However, when the network is fixed, the dynamic flow model reaches that benefit in the limit, as the distances between nodes d_{ij} do not change for a fixed network $\lim_{t\to\infty} b_{ij}^t = \delta^{d_{ij}}$.

For a given network $g = (g_i, g_{-i})$, and given the benefits b_i, the utility for node i can be given by

$$u_i(b_i, g_i) = \sum_j b_{ij} - c\mu(g_i).$$

In the instantaneous benefit flow model, the feedback law to select strategies assumed an instantaneous realization of the full benefits from other nodes. In the dynamic benefit flow case, a similar model can be obtained by allowing

[3] b_{ii} will be assumed to be equal to 1.

the dynamics to propagate to infinite time and and then selecting a new strategy based on the limit of the average utility over time. This model, however, introduces a separation of time scales where the dynamics of benefit flow have no effect on the outcome of repeated play of the strategic network formation game.

Alternatively, the strategy of a node can dynamically depend on the available benefits at a given time. Here, a node can be selecting a strategy at a given point in time such that it maximizes a utility dependent cost function, for example the total discounted utility

$$J = \sum_t \rho^t u(b_i^t, g_i).$$

The complexity introduced by the dynamic interdependence of benefits on the strategies of other nodes, renders the computation of strategies a challenging task. Alternatively, we will consider the case when nodes are myopic decision makers, whose interest is to maximize the projected utility based on the benefit flow in the next time step, and assuming the strategies of other nodes remain unchanged from the currently observed topology.

Hence, for a given strategy g_i, strategies of other nodes g_{-i}^{t-1}, and benefits vector at time $t-1$, the utility is

$$u_i(g_i, g_{-i}^{t-1}, b_i^{t-1}) = \sum_j f(b_{ij}^{t-1}, g_i, g_{-i}^{t-1}) - c\mu(g_i). \tag{6}$$

As such, at time steps $t = 1, 2, \ldots$, a randomly selected node plays a best response to the currently observed benefit flow and network topology,

$$g_i^t = \mathrm{BR}(g_{-i}^{t-1}, b_i^{t-1}) \in \arg\max_{g_i \in \mathcal{G}_i} u_i(g_i, g_{-i}^{t-1}, b_i^{t-1}). \tag{7}$$

To that end, at time $t-1$ a node i evaluates, for every possible strategy, the benefits available using (5). With the selected strategy, the benefit flow dynamics are propagated one time step and a new node is selected randomly to update its strategy.

3.2 Equilibria of the Coupled Dynamics

We shall consider the limiting behavior of the interconnection of the dynamic benefit flow model (5) and myopic best response network formation (7).

Definition 2. The pair (B^*, g^*) is an equilibrium of the coupled dynamics in (5) and (7) if $\forall i \in \mathcal{N}$, $g_i^* = \mathrm{BR}(g_{-i}^*, b_i^*)$ and $\forall j$ $b_{ij}^* = f(b_{ij}^*, g_i^*, g_{-i}^*)$.

Here the equilibrium involves both network topology g^* and a steady-state benefit flow B^*. One class of equilibria that can emerge is when the topology of the network remains unchanged, i.e., $g^t = g^*$, $\forall t \geq t_0$ for some network topology g^*. As a consequence, the shortest distances between nodes remain unchanged, i.e, $d_{ij}(g_i^t, g_{-i}^t) = d_{ij}(g_i^*, g_{-i}^*)$, $\forall i, j$ and $\forall t \geq t_0$. Therefore, the benefits for each node will correspond to $b_{ij}^* = \delta^{d_{ij}(g_i^*, g_{-i}^*)}$.

Definition 3. The pair (B^*, g^*) is a strict equilibrium of the coupled dynamics in (5) and (7) if and only if $u(g_i^*, g_{-i}^*, b_i^*) - u(g_i', g_{-i}^*, b_i^*) > 0$, $\forall g_i' \in \mathcal{G}_i \backslash g_i^*$, $\forall i \in \mathcal{N}$.

Now let $d_{ij}^* = d_{ij}(g_i^*, g_{-i}^*)$ and $d_{ij}' = d_{ij}(g_i', g_{-i}^*)$ for some strategy $g_i' \in \mathcal{G}_i \backslash g_i^*$. Also define $\mathcal{S}_1 = \{j : d_{ij}' \leq d_{ij}^*\}$ and $\mathcal{S}_2 = \{j : d_{ij}' > d_{ij}^*\}$, these are the sets of nodes whose distance to node i given strategy g_i' are respectively smaller than and greater than their distances given the equilibrium strategy g_i^*. In retrospect, the sets would correspond to those nodes whose benefit dynamics will be updated using Equations (5a) and (5b) respectively.

Proposition 1. *An equilibrium (B^*, g^*) is strict if $\forall i \in \mathcal{N}$, and $\forall g_i' \in \mathcal{G}_i \backslash g_i^*$*

$$\sum_{j \in \mathcal{S}_1} (1 - \alpha_{d_{ij}'})(\delta^{d_{ij}^*} - \delta^{d_{ij}'}) + \sum_{j \in \mathcal{S}_2} (1 - \beta_{d_{ij}'})(\delta^{d_{ij}^*} - \delta^{d_{ij}'}) + c(\mu(g_i') - \mu(g_i^*)) > 0. \tag{8}$$

Proof. For any equilibrium such that $g_i^t = g_i^{t-1}$, $\forall t \geq t_0$, we know that $b_{ij}^* = \delta^{d_{ij}(g_i^*, g_{-i}^*)} = \delta^{d_{ij}^*}$. For a strict equilibrium we have

$$\begin{aligned}
u(g_i^*, g_{-i}^*, b_i^*) - u(g_i', g_{-i}^*, b_i^*) &> 0 \\
\sum_j f(b_{ij}^*, g_i^*, g_{-i}^*) - c\mu(g_i^*) - \sum_j f(b_{ij}^*, g_i', g_{-i}^*) - c\mu(g_i') &> 0 \\
\sum_{j \in \mathcal{S}_1 \cup \mathcal{S}_2} \delta^{d_{ij}^*} - \sum_{j \in \mathcal{S}_1} \alpha_{d_{ij}'} \delta^{d_{ij}^*} + (1 - \alpha_{d_{ij}'}) \delta^{d_{ij}'} & \\
- \sum_{j \in \mathcal{S}_2} \beta_{d_{ij}'} \delta^{d_{ij}^*} + (1 - \beta_{d_{ij}'}) \delta^{d_{ij}'} + c(\mu(g_i') - \mu(g_i^*)) &> 0 \\
\sum_{j \in \mathcal{S}_1} (1 - \alpha_{d_{ij}'})(\delta^{d_{ij}^*} - \delta^{d_{ij}'}) + \sum_{j \in \mathcal{S}_2} (1 - \beta_{d_{ij}'})(\delta^{d_{ij}^*} - \delta^{d_{ij}'}) + c(\mu(g_i') - \mu(g_i^*)) &> 0
\end{aligned}$$

□

Notice that for $j \in \mathcal{S}_1$, $\delta^{d_{ij}(g_i^*, g_{-i}^*)} \leq \delta^{d_{ij}(g_i', g_{-i}^*)}$, and for $j \in \mathcal{S}_2$, $\delta^{d_{ij}(g_i^*, g_{-i}^*)} > \delta^{d_{ij}(g_i', g_{-i}^*)}$. Therefore, the first term on the left in (8) is nonpositive and the second term is positive.

An equilibrium of the combined dynamics of network formation and benefit flow involves both a network topology and benefit flow matrix. The equilibria here concern the limiting behavior of the coupled dynamics and not just a best response network like a Nash network of the static game. One of the questions to consider is whether the coevolutionary dynamics can induce some network topologies to become equilibria while they are not equilibria of the non-coevolutionary network formation (static game). In the following we show through an example that given a common set of parameters, such topologies exist.

Proposition 2. *For $N = 3$, if $(1-\beta_\infty)\delta \geq c \geq (1-\alpha_1)(\delta-\delta^2)$, and $\beta_2 < \alpha_1$, then the pair (B^*, g^*) given by*

$$g^* = ((1,0),(1,1),(0,1)), \quad B_g^* = \begin{bmatrix} 1 & \delta & \delta^2 \\ \delta & 1 & \delta \\ \delta^2 & \delta & 1 \end{bmatrix},$$

is an equilibrium of the coupled dynamics in (5) and (7), and g^* is not a Nash equilibrium of the static game when $\alpha_1 > 0$.

Proof. Consider the node utilities of the network in Fig. 2(a) and the associated benefits matrix B^*. By symmetry of nodes 1 and 3, we will only consider the utilities of nodes 1 and 2. For node 1, comparing strategies (1,0) and (0,1), using (8), we have

$$\begin{aligned} (1-\alpha_{d_{13}})(\delta^{d^*_{13}} - \delta^{d'_{13}}) + (1-\beta_{d'_{12}})(\delta^{d^*_{12}} - \delta^{d'_{12}}) &> 0 \\ (1-\alpha_1)(\delta^2 - \delta) - (1-\beta_2)(\delta^2 - \delta) &> 0 \\ \alpha_1 &> \beta_2. \end{aligned}$$

Moreover, comparing strategies (1,0) and (1,1) using (8) we have

$$\begin{aligned} (1-\alpha_{d_{13}})(\delta^{d^*_{13}} - \delta^{d'_{13}}) + c &> 0 \\ (1-\alpha_1)(\delta - \delta^2) &< c. \end{aligned}$$

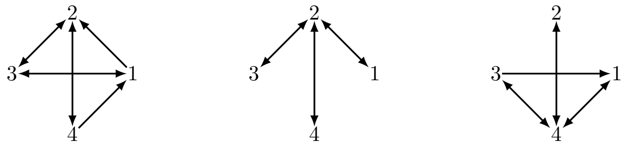
For node 2, comparing strategies (1,1) and (1,0) or equivalently (0,1) we have

$$\begin{aligned} (1-\beta_{d'_{23}})(\delta^{d^*_{23}} - \delta^{d'_{23}}) - c &> 0 \\ (1-\beta_\infty)\delta &> c. \end{aligned}$$

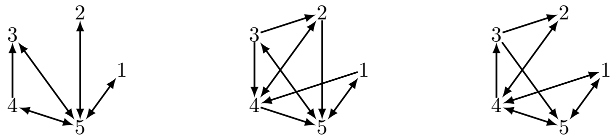
Notice that when $\alpha_1 = 0$, then g^* is an equilibrium if $c \geq \delta - \delta^2$ which retrieves the conditions for the static game shown before in (4). $\square$

The above shows that the coupled coevolutionary dynamics can create equilibria that can only be possible at higher costs of link formation in the non-coevolutionary case. Additionally, the equilibrium also highlights the requirement that nodes need to forget benefits of distant or severed nodes faster than receiving benefits.

Characterizing equilibria for all N is a difficult problem that is yet to be tackled. Instead, we present some equilibria of some small networks to highlight some of the typical topologies of such equilibria. For $c < \delta - \delta^2$, the equilibrium in the model of Bala & Goyal [21] is the complete network. For the coevolutionary model, the equilibria are quite diverse and examples of these equilibria for 4- and 5-node are presented in Fig. 3. A sample run converging to a non Nash-network of the static game is shown in Fig. 4.



$$\begin{bmatrix} 1 & \delta^2 & \delta & \delta \\ \delta & 1 & \delta & \delta \\ \delta & \delta & 1 & \delta^2 \\ \delta^2 & \delta & \delta^2 & \delta^2 \end{bmatrix} \quad \begin{bmatrix} 1 & \delta & \delta^2 & \delta^2 \\ \delta & 1 & \delta & \delta \\ \delta^2 & \delta & 1 & \delta^2 \\ \delta^2 & \delta & \delta^2 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & \delta^2 & \delta & \delta \\ \delta^2 & 1 & \delta^2 & \delta \\ \delta^2 & \delta^2 & 1 & \delta \\ \delta & \delta & \delta & 1 \end{bmatrix}$$



$$\begin{bmatrix} 1 & \delta^2 & \delta^2 & \delta^2 & \delta \\ \delta^2 & 1 & \delta^2 & \delta^2 & \delta \\ \delta^2 & \delta^2 & 1 & \delta & \delta \\ \delta^2 & \delta^2 & \delta^2 & 1 & \delta \\ \delta & \delta & \delta & \delta & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & \delta^2 & \delta^2 & \delta^2 & \delta \\ \delta^3 & 1 & \delta & \delta & \delta^2 \\ \delta^2 & \delta^3 & 1 & \delta^2 & \delta \\ \delta & \delta & \delta & 1 & \delta^2 \\ \delta & \delta & \delta & \delta & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & \delta^2 & \delta^2 & \delta & \delta \\ \delta^2 & 1 & \delta & \delta & \delta^2 \\ \delta^2 & \delta^2 & 1 & \delta^2 & \delta^2 \\ \delta & \delta & \delta^2 & 1 & \delta^2 \\ \delta & \delta & \delta & \delta & 1 \end{bmatrix}$$

Fig. 3 Examples of equilibria of a 4- and 5-node network, when $\delta = 0.9$, $c = 0.05$, $\alpha_1 = 0.6$, $\alpha_2 = 0.7$, $\alpha_3 = 0.8$, $\alpha_1 = 0.9$, $\beta_1 = 0.4$, $\beta_2 = 0.3$, $\beta_3 = 0.2$, $\beta_4 = 0.1$, $\beta_\infty = 0.01$

3.3 Equilibrium Stability

In this section, we will examine the behavior of the coupled dynamics when the network topology is at that of an equilibrium whereas the benefits available are close to the equilibrium values.

Proposition 3. *Let (g^*, B^*), where $g^* = (g_i^*, g_{-i}^*)$, $B^* = [b_{ij}^*]$, be a strict equilibrium such that $\forall i,\ u_i(g_i^*, g_{-i}^*, b_i^*) - u_i(g_i, g_{-i}^*, b_i^*) \geq \gamma,\ \forall g_i \in \mathcal{G}_i \backslash g_i^*$, for some $\gamma > 0$. Also, let $g_{t_0} = g^*$ and $b_{ij}^{t_0} = b_{ij}^* \pm \epsilon\ \forall i, j$, for some time t_0. If ϵ is sufficiently small, then $g_t = g^*,\ \forall t \geq t_0$, and $\lim_{t\to\infty} b_{ij}^t = b_{ij}^*,\ \forall i, j$.*

Proof. We shall examine the utility of a strategy $g_i^t = g_i'$ compared to $g_i^t = g_i^*$. First define,

$$\mathcal{I}_1' = \{j : \delta^{d_{ij}(g_i', g_{-i}^*)} \geq b_{ij}^{t_0}\}, \quad \mathcal{I}_1 = \{j : \delta^{d_{ij}(g_i^*, g_{-i}^*)} \geq b_{ij}^{t_0}\},$$
$$\mathcal{I}_2' = \{j : \delta^{d_{ij}(g_i', g_{-i}^*)} < b_{ij}^{t_0}\}, \quad \mathcal{I}_2 = \{j : \delta^{d_{ij}(g_i^*, g_{-i}^*)} < b_{ij}^{t_0}\}.$$

Then,

$$\begin{aligned}
u_i(g_i', g_{-i}^*, b_i^{t_0}) =& \sum_j f(b_{ij}^{t_0}, g_i', g_{-i}^*) - c\mu(g_i') \\
=& \sum_{j\in\mathcal{I}_1'} \alpha_{d_{ij}(g_i', g_{-i}^*)} b_{ij}^{t_0} + (1-\alpha_{d_{ij}(g_i', g_{-i}^*)})\delta^{d_{ij}(g_i', g_{-i}^*)} \\
&+ \sum_{j\in\mathcal{I}_2'} \beta_{d_{ij}(g_i', g_{-i}^*)} b_{ij}^{t_0} + (1-\beta_{d_{ij}(g_i', g_{-i}^*)})\delta^{d_{ij}(g_i', g_{-i}^*)} - c\mu(g_i') \\
=& \sum_{j\in\mathcal{I}_1'} \alpha_{d_{ij}(g_i', g_{-i}^*)} b_{ij}^{*} + (1-\alpha_{d_{ij}(g_i', g_{-i}^*)})\delta^{d_{ij}(g_i', g_{-i}^*)} \\
&+ \sum_{j\in\mathcal{I}_2'} \beta_{d_{ij}(g_i', g_{-i}^*)} b_{ij}^{*} + (1-\beta_{d_{ij}(g_i', g_{-i}^*)})\delta^{d_{ij}(g_i', g_{-i}^*)} \\
&- c\mu(g_i') \pm \sum_{j\in\mathcal{I}_1'} \alpha_{d_{ij}(g_i', g_{-i}^*)}\epsilon \pm \sum_{j\in\mathcal{I}_2'} \beta_{d_{ij}(g_i', g_{-i}^*)}\epsilon \\
=& u_i(g_i', g_{-i}^*, b_i^*) \pm \sum_{j\in\mathcal{I}_1'} \alpha_{d_{ij}(g_i', g_{-i}^*)}\epsilon \pm \sum_{j\in\mathcal{I}_2'} \beta_{d_{ij}(g_i', g_{-i}^*)}\epsilon.
\end{aligned}$$

Similarly, we can write

$$u_i(g_i^*, g_{-i}^*, b_i^{t_0}) = u_i(g_i^*, g_{-i}^*, b_i^*) \pm \sum_{j\in\mathcal{I}_1} \alpha_{d_{ij}(g_i^*, g_{-i}^*)}\epsilon \pm \sum_{j\in\mathcal{I}_2} \beta_{d_{ij}(g_i^*, g_{-i}^*)}\epsilon.$$

Therefore,

$$\begin{aligned}
u_i(g_i^*, g_{-i}^*, b_i^{t_0}) - u_i(g_i', g_{-i}^*, b_i^{t_0}) =& u_i(g_i^*, g_{-i}^*, b_i^*) - u_i(g_i', g_{-i}^*, b_i^*) \\
&\pm \sum_{j\in\mathcal{I}_1} \alpha_{d_{ij}(g_i^*, g_{-i}^*)}\epsilon \pm \sum_{j\in\mathcal{I}_2} \beta_{d_{ij}(g_i^*, g_{-i}^*)}\epsilon \\
&\mp \sum_{j\in\mathcal{I}_1'} \alpha_{d_{ij}(g_i', g_{-i}^*)}\epsilon \mp \sum_{j\in\mathcal{I}_2'} \beta_{d_{ij}(g_i', g_{-i}^*)}\epsilon \\
\geq& \gamma - \epsilon\Big(\sum_{\mathcal{I}_1} \alpha_{d_{ij}(g_i^*, g_{-i}^*)} + \sum_{\mathcal{I}_2} \beta_{d_{ij}(g_i^*, g_{-i}^*)}\Big) \\
&- \epsilon\Big(\sum_{\mathcal{I}_1'} \alpha_{d_{ij}(g_i', g_{-i}^*)} + \sum_{\mathcal{I}_2'} \beta_{d_{ij}(g_i', g_{-i}^*)}\Big).
\end{aligned}$$

Since $\alpha_k, \beta_k \in [0,1]$, then for small ϵ we have

$$u_i(g_i^*, g_{-i}^*, b_i^{t_0}) - u_i(g_i', g_{-i}^*, b_i^{t_0}) > 0.$$

This implies that g_i^* is a best response and that $g_i^{t_0+1} = g_i^*$. Since the topology remains unchanged, then $\delta^{d_{ij}}, \forall i, j$ remain unchanged. Furthermore, the stable dynamics in (5) results in $|b_{ij}^{t_0+1} - b_{ij}^*| \leq \epsilon' < \epsilon$. Using the same arguments recursively, the results follow. $\square$

Here we have shown that local stability is guaranteed, for small deviations in the benefit flows from their equilibrium values, for strict equilibria. Strict equilibria are equilibria such that their utilities are strictly greater than the utilities of other strategies given a unilateral deviation of strategy.

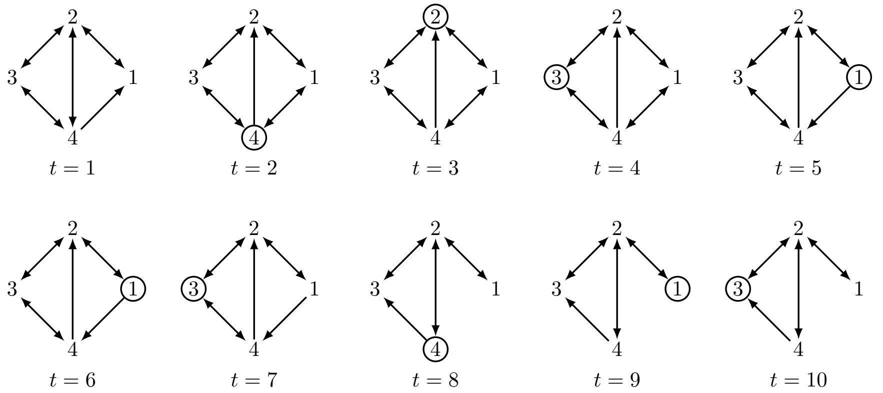


Fig. 4 A sample run of the algorithm converging to a non-Nash network of the static game. Initially $b_{12} = 5.835e^{-1}$, $b_{13} = 8.893e^{-1}$, $b_{14} = 1.893e^{-2}$, $b_{21} = 2.966e^{-4}$, $b_{23} = 1.934e^{-2}$, $b_{24} = 1.299e^{-1}$, $b_{31} = 6.847e^{-1}$, $b_{32} = 5.124e^{-2}$, $b_{34} = 9.707e^{-1}$, $b_{41} = 1.099e^{-3}$, $b_{42} = 6.443e^{-1}$, $b_{43} = 1.711e^{-1}$. A circle around the node number denotes the node updating its network.

4 Conclusions

In this work, we presented a coevolutionary model of network formation based on dynamic flow of benefits between nodes. We showed that the combined dynamics can induce network topologies to be equilibria of the dynamics, whereas these topologies are not Nash networks of the static game. These equilibria can emerge at a lower cost than the non-coevolutionary case. We also showed the stability of a class of equilibria of the combined network formation and benefit flow dynamics. The model can be extended to cases where each edge has a weight that corresponds to the strength of the connection. However, this setup can manifest different behaviors and will be the subject of further studies.

Acknowledgement. Supported by AFOSR project #FA9550-09-1-0420 and ONR project #N00014-09-1-0751.

References

1. Jackson, M.O.: A survey of network formation models: stability and efficiency. In: Group Formation in Economics: Networks, Clubs and Coalitions, pp. 11–57. Cambridge University Press, Cambridge (2005)
2. Tardos, E., Wexler, T.: Network formation games and the potential function method. In: Algorithmic Game Theory, pp. 487–516 (2007)
3. Bloch, F., Jackson, M.O.: The formation of networks with transfers among players. Journal of Economic Theory 133(1), 83–110 (2007)
4. Jackson, M.O.: Social and economic networks. Princeton University Press (2010)
5. Arcaute, E., Johari, R., Mannor, S.: Network formation: Bilateral contracting and myopic dynamics. IEEE Transactions on Automatic Control 54(8), 1765–1778 (2009)
6. Christakis, N.A., Fowler, J.H., Imbens, G.W., Kalyanaraman, K.: An empirical model for strategic network formation. Technical report, National Bureau of Economic Research (2010)
7. Cowan, R.: Network models of innovation and knowledge diffusion. In: Clusters, Networks and Innovation, pp. 29–53 (2005)
8. Young, H.P.: The dynamics of social innovation. Proceedings of the National Academy of Sciences 108(suppl. 4), 21285–21291 (2011)
9. Young, H.P.: Individual strategy and social structure: An evolutionary theory of institutions. Princeton University Press (2001)
10. Ehrhardt, G., Marsili, M., Vega-Redondo, F.: Diffusion and growth in an evolving network. International Journal of Game Theory 34(3), 383–397 (2006)
11. Kozma, B., Barrat, A.: Consensus formation on adaptive networks. Physical Review E 77(1), 016102 (2008)
12. Burda, Z., Krzywicki, A., Martin, O.: Adaptive networks of trading agents. Physical Review E 78(4), 046106 (2008)
13. Blondel, V.D., Hendrickx, J.M., Tsitsiklis, J.N.: On Krause's multi-agent consensus model with state-dependent connectivity. IEEE Transactions on Automatic Control 54(11), 2586–2597 (2009)
14. Skyrms, B., Pemantle, R.: A dynamic model of social network formation. In: Adaptive Networks, pp. 231–251. Springer (2009)
15. Marceau, V., Noël, P.A., Hébert-Dufresne, L., Allard, A., Dubé, L.J.: Adaptive networks: Coevolution of disease and topology. Physical Review E 82(3), 036116 (2010)
16. Acemoglu, D., Mostagir, M., Ozdaglar, A.: State-dependent opinion dynamics. Working paper (2012)
17. Fox, M.J., Piliouras, G., Shamma, J.S.: Medium and long-run properties of linguistic community evolution. In: 9th International Conference on the Evolution of Language (Evolang IX) (2012)
18. Mirtabatabaei, A., Bullo, F.: Opinion dynamics in heterogeneous networks: convergence conjectures and theorems. SIAM Journal on Control and Optimization 50(5), 2763–2785 (2012)
19. Kianercy, A., Galstyan, A.: Coevolutionary networks of reinforcement-learning agents. Physical Review E 88(1), 012815 (2013)

20. Barabási, A.L., Albert, R.: Emergence of scaling in random networks. Science 286(5439), 509–512 (1999)
21. Bala, V., Goyal, S.: A noncooperative model of network formation. Econometrica 68(5), 1181–1229 (2000)