Chapter 5
Introduction to the Anthropological Theory of the Didactic (ATD)

Marianna Bosch and Josep Gascón

Abstract The chapter briefly introduces the Anthropological Theory of the Didactic (ATD) by referring to the data from Chap. 2. ATD provides a frame for investigating mathematical and didactic activities in terms of *praxeologies*, focusing on their components, dynamics, and the conditions that enable their existence and development in a given institutional setting. The main idea of the concept of praxeologies is that all human activities comprise and link two parts, a practice and a theory one.

Keywords Theories • Anthropological theory of the didactic

5.1 Overview

The Anthropological Theory of the Didactic (ATD) is a program of research in mathematics education initiated by Yves Chevallard in the 1980s with the study of *didactic transposition processes* (Bosch and Gascón 2006; see also Chevallard 1985, 1989, 1992a, b) and which has been evolving continuously for the last 30 years. Nowadays, a community of about one hundred researchers, mainly from Europe, Canada, and Latin America, work on the development of this program, focusing on the current problems of spreading knowledge both at school and outside school, concerning mathematics as well as other fields of knowledge.

M. Bosch (✉)
IQS School of Management, Universitat Ramon Llull,
Via Augusta 390, 08017 Barcelona, Spain
e-mail: marianna.bosch@iqs.edu

J. Gascón
Facultat de Ciències, Departament de Matemàtiques, Universitat Autònoma de Barcelona,
Edifici C – Campus de la UAB, 08193 Bellaterra (Cerdanyola del Vallès), Barcelona, Spain
e-mail: gascon@mat.uab.cat

A. Bikner-Ahsbahs and S. Prediger (eds.), *Networking of Theories as a Research Practice in Mathematics Education*, Advances in Mathematics Education,
DOI 10.1007/978-3-319-05389-9_5,

A good outline of the approached problems and the obtained results within this framework can be found in the proceedings of the four International ATD Conferences held since 2005 in Spain and France (Bosch et al. 2011; Bronner et al. 2010; Estepa et al. 2006).

The meaning and relevance of ATD has to be understood as a development of the project initiated by the Theory of Didactical Situations (TDS) of a science of *didactic phenomena* called *didactics of mathematics* (cf. Chap. 4 on TDS). In the framework proposed by ATD, the institutional dimension of mathematical and didactic[1] activities becomes much more explicit. Doing, teaching, learning, diffusing, creating, and transposing mathematics, as well as any other kind of knowledge, are considered as human activities taking place in institutional settings. The science of *didactics* is thus concerned with the conditions governing these knowledge activities in society, as well as the restrictions hindering their development among social institutions.

5.1.1 Principles and Key Constructs: Praxeologies

ATD postulates that any activity related to the production, diffusion, or acquisition of knowledge should be interpreted as an ordinary human activity, and thus proposes a general model of human activities built on the key notion of *praxeology.* According to Chevallard (2006):

> *A* praxeology is, in some way, the basic unit into which one can analyse human action at large. […] What exactly is *a* praxeology? We can rely on etymology to guide us here – one can analyse any human doing into two main, interrelated components: *praxis*, i.e. the practical part, on the one hand, and *logos*, on the other hand. *"Logos"* is a Greek word which, from pre-Socratic times, has been used steadily to refer to human thinking and reasoning – particularly about the cosmos. […] [According to] one fundamental principle of the ATD – the anthropological theory of the didactic – no human action can exist without being, at least partially, "explained", made "intelligible", "justified", "accounted for", in whatever style of "reasoning" such an explanation or justification may be cast. *Praxis* thus entails *logos*, which, in turn, backs up *praxis*. For *praxis* needs support just because, in the long run, no human doing goes unquestioned. Of course, a praxeology may be a *bad* one, with its "praxis" part being made of an inefficient technique – "technique" is here the official word for a "way of doing" – and its "logos" component consisting almost entirely of sheer nonsense – at least from the praxeologist's point of view! (Chevallard 2006, p. 23)

Both the practical and theoretical components of a praxeology are in turn broken down into two elements. The *praxis* block is made of "types of tasks" and a set of "techniques" (considering this term in a broad sense of "ways of doing") to carry out some of the tasks of the given type (those in the "scope" of the technique). The *logos* block contains two levels of description and justification of the *praxis*. The first level is called a "technology," using here the etymological sense of "discourse" (*logos*) of the technique (*technè*). The second level is simply called the "theory" and its main function is to provide a basis and support of the technological discourse. In

[1] The adjective "didactic" is used to refer to anything related to the teaching, learning, or study of a given content.

general human activities, the "theory" component is generally more difficult to grasp than the others because it is usually taken for granted, unless in times of difficulties, crises, and questioning of the praxeologies. In return, scientific work provides many examples of how these theoretical assumptions can be made explicit in order to provide more control of the techniques carried out and of their description, justification, and validation.

A praxeology is thus an entity formed by four components, usually called the "four Ts": a type of tasks, a set of techniques, a technological discourse, and a theory. As activities and knowledge can be described considering different delimitations or granularities, a distinction is made between a "point praxeology" (containing a single type of task), a "local praxeology" (containing a set of types of task organized around a common technological discourse) and a "regional praxeology" (which contains all point and local praxeologies sharing a common theory). We will see an example of this distinction in the analysis of the episode of Carlo and Giovanni (see also García et al. 2006; Barbé et al. 2005).

Praxeologies is a useful term when talking about knowledge, mathematics, or any other teaching and learning content, and also about teaching and learning practices, as it provides a unitary vision of these different activities, without considering some of them as more "intellectual," "abstract," "difficult," or theoretically based than the others, and thus without assuming the scale of values usually given to them (mathematics appearing as something related to "thinking" while teaching is more seen as a "practice" than as a "theory").

Praxeologies do not emerge suddenly and never acquire a final shape. They are the result of ongoing activities, with complex dynamics, that in their turn have to be modeled. We will use the term *didactic praxeologies* to refer to any activity related to "setting up praxeologies" (Chevallard 1999). A didactic praxeology is thus a praxeology that aims at making other praxeologies start living in and migrating within human groups. They are an essential part of the functioning and evolution of our societies, indispensable to keeping institutions running, to modifying them, and also to habilitating people to make them work and progress. They are also essential, of course, for the personal development of human beings, to improving their capacity of action and comprehension.

Here appears a sensible point about the relation between institutional and personal praxeologies. In order to answer the question of why people do what they do, what makes it possible for them to do what they do, etc., ATD postulates that what explains the behavior of people are not only their personal idiosyncrasies but also the existence (or availability) of institutional constructions that each person adapts, adopts, and develops either individually or collectively. An ATD analysis therefore starts by approaching *institutional praxeologies* and then referring individual behavior to them, talking in terms of the "praxeological equipment" of a given person. Observable behavior obviously consists of a mixture of personal and institutional ingredients. This dialectic between the personal and the institutional makes it possible to explain both the regularities of our behavior and its personal "footprint." People evolve as they enter different institutions and, at the same time, these individual participations enable institutions to appear, run, and change.

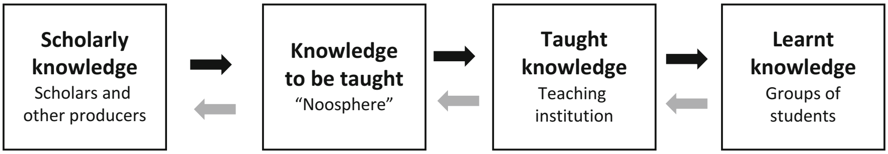

Fig. 5.1 Diagram of the process of didactic transposition

Concerning the dynamics of praxeologies, the ATD assumes an important postulate of the TDS: the fact that any piece of knowledge (i.e., any praxeology) can be considered as an answer provided – explicitly or *de facto* – to a question Q (a problem or a difficulty) arising in an institutional setting (or a "situation"). Question Q then becomes the "raison d'être" of the praxeology constructed, a rationale evolving as the praxeology develops and integrates into other kinds of activities, for instance to provide answers to other kinds of questions. It often occurs that the raisons d'être at the origin of most praxeologies disappear with time, and people end up doing things out of inertia or habit, without questioning their way of doing nor considering the possibility of changing them. Therefore, an important "research gesture" in didactics is to analyze praxeologies to find out their possible raisons d'être (the historic as well as the contemporary ones) and study the conditions that can make them appear – give them sense – in different institutional settings.

5.1.2 *Methodologies and Questions*

5.1.2.1 The Praxeological Analysis

One of the first contributions of ATD through the notion of *didactic transposition process* was to make clear that it is not possible to interpret school mathematics properly without taking into account the phenomena related to the way mathematics is introduced and reconstructed at school. What mathematical praxeologies are proposed to be studied at school and why? What are they made of? Where do they come from? Do they live outside school? Where and under what shapes? Didactic transposition processes underline the *institutional relativity of knowledge* and situate didactic problems at an institutional level, beyond individual characteristics of the subjects of the considered institutions (Fig. 5.1).

The process of didactic transposition (Fig. 5.1) refers to the transformations applied to a "content" or a body of knowledge since it is produced and put into use, until it is actually taught and learned in a given educational institution. This notion is not just the description of a phenomenon, but a tool to emancipate the didactic analysis from the dominant vision of educational content. Teaching and learning processes always include some content or piece of knowledge to be taught and

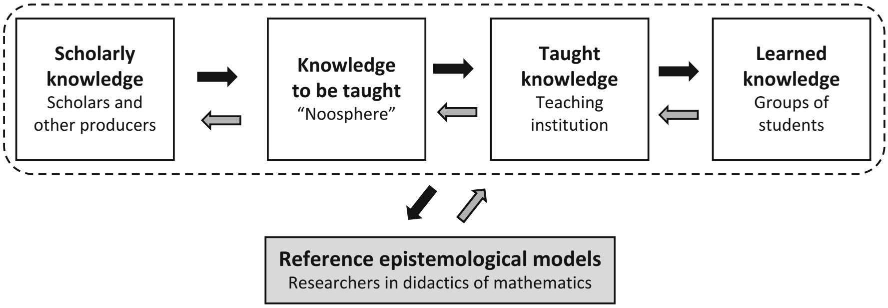

Fig. 5.2 The external position of researchers

learnt. One can take this content as given data or, on the contrary, question its nature and function, considering its *formation* as "knowledge to be taught" through the productions of the *noosphere*-that is, the sphere of those who "think" (*noos*) about teaching-, its relationship to "scholarly knowledge" which usually legitimates its introduction in educational institutions, and the specific form it takes when arriving in the classroom as "taught knowledge," activated by both the teacher and the students. The "knowledge to be taught" can be accessed through official programs, textbooks, recommendations to teachers, didactic materials, etc., which may help in considering also the conditions under which it is constituted and evolves (or remains fixed) in time.

This study should take into account the "scholarly knowledge" produced by mathematicians or other scientists who are recognized as the "experts of the matter" and appears as a source of legitimation of the knowledge to be taught. However, scholarly knowledge should not be considered as the unique reference to which all school mathematical praxeologies are referred to. In order to avoid adopting a particular and "scholarly biased" viewpoint, researchers in didactics need to elaborate their own "reference models" (Fig. 5.2) from which to consider the empirical data of the three corresponding institutions: the mathematical community, the educational system, and the classroom.

5.1.2.2 The Didactic Analysis

A social situation is said to be a *didactic situation* whenever one of its actors (Y) does something to help a person (x) or a group of persons (X) learn something (indicated by a heart ♥). A *didactic system* $S(X; Y; ♥)$ is then formed. The thing that is to be learned is called a *didactic stake* ♥ and is made of questions and/or praxeological components. X is the group of "students of ♥" and Y is the team of "study assistants" (or "study helpers"). The most obvious didactic systems are those formed at school, where Y is ordinarily a "singleton" whose unique member is "the teacher" y. However, there are a multitude of different kinds of didactic systems.

For instance, the authors of this chapter are acting as *Y* to help the reader, *x*, learn something about ATD research.

Given a didactic system *S*(*X*; *Y*; ♥), the praxeological analysis tries to provide answers about the praxeologies the didactic stake ♥ is made of. By contrast, the didactic analysis approaches questions including: What is *X*? What is *Y*? What are the didactic praxeologies put to use by *X* and *Y* and what didactic means have proved necessary to do so? What praxeological equipment can be engendered in *X* as a short-term and as a long-term result of the functioning of *S*(*X*; *Y*; ♥)? To answer these and other questions, ATD provides two different general didactic models. The first one, in terms of six dimensions or *didactic moments*, concerns the case where ♥ is a given local praxeology $\mathscr{P}$ and presents a structure of the construction of the different components of $\mathscr{P}$: the *first encounter* with the praxeology, the *exploration* of the type of tasks and the emergence of a technique, the "*work of the technique*" and the study of its scope, the elaboration of a *theoretical environment*, the *institutionalization*, and the *evaluation* of the work done (Chevallard 1999; see also Barbé et al. 2005; García et al. 2006). The second didactic model is more general and aims to include any process of study and research starting from a problematic question *Q*. It is presented and used in the case study on context-milieu-media (Chap. 10).

5.1.2.3 The Ecological Analysis and the Levels of Didactic Codetermination

The study of the *ecology* of mathematical and didactic praxeologies states that, when the teacher and the students meet around an issue at stake ♥, what can happen is mainly determined by conditions that cannot be reduced to those immediately identifiable in the classroom, such as the teacher's and students' praxeological equipment, the teaching material available, the temporal organization of activities, etc. Even if these conditions play an important role, Chevallard (2002) proposed to consider a "scale of levels of didactic codetermination" (see Fig. 5.3).

General educational research usually focuses on restrictions coming from the *generic levels* (above the discipline), while research in specific subject didactics (such as didactics of mathematics, sciences, language, etc.) hardly take them as an object of study, even if they strongly affect the "specific praxeologies" that can exist in the classroom and the way they can evolve. Moreover, even at the *specific levels* (within the discipline), what is commonly considered in didactics research tends to be reduced to phenomena occurring at the *thematic level*, that is, those concerned by the teaching and learning of a specific topic. Consequently, it becomes very difficult for researchers – and even more for the teacher – to question the cultural vision of mathematics and its teaching as proposed by both school and "scholarly" institutions. The way the levels of didactic codetermination are used to analyze the ecology of mathematical and didactic praxeologies is illustrated at the end of the chapter (see also Artigue and Winsløw 2010).

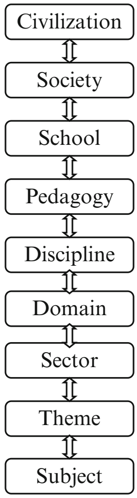

Fig. 5.3 Scale of levels of didactic codetermination

5.2 Illustrating the Theory Through Analysis of the Video of Carlo, Giovanni, and the Exponential Function

5.2.1 Mathematical Praxeologies in the Considered Episode

The description of praxeologies can be carried out at different levels of detail, depending on the kind of problem posed by the researcher. In this case, given the fact that the piece of reality considered does not respond to any specific problem proposed by ATD, we will limit our presentation to an overall illustrative analysis of the mathematical praxeologies involved in the considered episode. We will start by inferring the ingredients of the *praxis* of the mathematical praxeologies and then look for the *logos* used to describe, explain, and justify this praxis.

5.2.1.1 The Technical-Practical Block of Mathematical Praxeologies

In the episode in which Carlo and Giovanni solve Task 1 and 2 (see Sect. 2.1.3 of this book), the mathematical praxeology at stake consists of two related tasks (or point praxeologies), the second one constituting a development of the first. They both integrate into a broader (local) praxeology that we will comment on later, based on the extra material we asked the teacher to provide (see Sect. 2.2.2). Due to lack of space, we will not carry out a detailed analysis of the three point praxeologies that appear in the episode and will only highlight the aspects they have in common. We may consider that Task 1 and 2 (see Figs. 2.1 and 2.2 in Sect. 2.1.3) stem from the same generating question Q, which can be formulated in the following terms: How to describe the variation of exponential functions $y = a^x$, both from a global and a local viewpoint?

In the observed episodes, this question is divided into three sub-questions:

How to *describe* the global variation of $y = 2.7^x$ when varying x? (case $a = 2.7$)
How to *describe* the global variation of $y = a^x$ for different values of a ($a > 0$)?
How to *quantify the local variation* of $y = a^x$ from the study of Δy for different values of Δx and considering the slope of the tangent line of the function at point x?

Obviously, this first task description is being done in terms related to our own mathematical experience and trying to remain close to the considered institution (the Italian secondary school, in this case). Our main empirical material is the worksheet the teacher hands out to the students as a guide to carry out the work. The questions in the worksheet (see Figs. 2.1 and 2.2 in Sect. 2.1.3) are divided into sub-questions that need to be answered. There is no introduction to the tasks proposed, nor are there any references to a more general framework (for instance to study the variation of the exponential function) in which the study takes place. It is possible that the introduction was done before the considered episode, but we do not know.

What are the techniques used to elaborate an answer to the three previous questions? In the case we are working on, we consider the global techniques used, and not only those the students are asked to carry out. In other words, we will consider the techniques useful to provide answers to the previous questions and that appear in the episode as activities carried out both by the teacher and the students, according to a precise distribution of responsibilities into which we will look in Sect. 5.2.2.

In the three considered tasks, the technique contains a specific device: a Cabri Geometry file with interactive graphs elaborated by the teacher, which the students are asked to manipulate and interpret. A certain manipulation of the devices – which the teacher has specified in the tasks – leads them to conjecture some of the "visible" properties of the functions considered from interpreting what is observed on the computer screen (graphic and numeric information). We are faced with a kind of *exploratory techniques* of specific mathematical objects which do not have a standard mathematical denomination (for example "calculating the derivative of a function"). Some of the "gestures" performed when carrying out those techniques are not visible in the video: the part of choosing and providing the experimental device, which the teacher did beforehand. What does appear in the Cabri file (and is observed in the activity the students carry out) is the detail of some of the manipulations of the device, which in some cases figure in relative detail in the task instructions: "*Open… file…*", "*Move the point x on the x-axis*", "*modify also the measure unit on the y-axis*", "*Move the point x towards the left until arriving nearly at the end of the field of variation of the negative x's*", etc. (see Fig. 2.1 in Sect. 2.1.3). The students' participation in carrying out the three tasks consists of performing the indications provided by the teacher and taking charge of the "gestures" that are not indicated: relate the graphic variation of x to the graphic variation of y; interpret it in terms of functional relations; formulate those relations in graphic and functional terms, both verbally and in writing; discuss and reach an agreement about how to draw up the observations; etc. This type of technique may be portrayed as "ostensive" in the sense that it is mainly based on the description of facts (numerical and graphical) which may be observed on a screen, both verbally (orally and written) and graphically (sketches).

An important part of the development of these techniques is the preparation of the computer devices carried out by the teacher. The students intervene at a specific moment of the development of the technique, but only the teacher is in charge of its global use. This situation is different from other mathematical techniques in which the students are fully in charge of generating the device and the gestures (for instance in the case of drawing the graph of a function and interpreting some of its elements, or carrying out a numerical simulation). The students are only asked to prepare a final statement, first orally and then a written version including graphs, so as to provide answers to the questions posed by means of provisional conjectures. They will also need to choose the known elements of the exponential functions in order to partially contrast some of the conjectures formulated (for instance that curve $y=a^x$ is a horizontal line when $a=1$). In the exchange between students, we can observe the functioning of mathematical objects that are essential to the formulation of conjectures and that have previously been integrated in their praxeological equipment: "tangent line", "slope", "effect of the change of units", "to grow more and more", etc.

Given the fact that the episode is situated at an initial stage of the study of the variation of exponential functions, what is observed in the work done by the students is the use of scattered technical elements which, we suppose, will gradually be integrated so as to form more powerful and systematic exploratory techniques. We thus see the emergence of new technical elements such as identifying the secant line with the tangent line (and with the curve itself of the function) when Δx gets close to 0, or the sudden change of behavior of the function when going from the case $0<a<1$ to the case $a>1$. Undoubtedly, more exhaustive technical and theoretical work will be necessary to systematize and institutionalize those elements in further lessons, which are still incipient in the observed episodes.

5.2.1.2 The Technological-Theoretical Block of Mathematical Praxeologies

After having proposed a possible description of the *praxis* of the mathematical activity partially appearing in the episodes, we can turn the attention to the *logos* block, that is, the elements used to "talk about" the work done, to describe and justify it. Some elements make it possible for the practice to be understandable and allow interaction between the students (each one understanding what the other does or says) as well as between the students and the teacher: they are part of the *technology* of the technique. We can mention, for instance, the interpretation of the elements of the Cabri files in functional terms: the correspondence between the graph and the values of the function; the fact that the values of the function are obtained by moving point x on the x-axis; the relationship between the slope of the tangent line and its "growth," etc. Other technological elements, maybe of a less mathematical nature, also contribute to justifying the functioning of the technique of manipulating the graph (correspondence between segment Δx and point P; between segment a and the base of the exponential function; etc.) and to the use of Cabri. Usually the elements of the technological discourse (basically implicit) are built at the same time as the tasks are explored and only rise to the surface in case of difficulty. In fact, the aim of the task partially consists in formulating some of those elements, those of most "mathematical nature" related to the observed variations of the functions.

The second level of justification considered in any praxeology, the level of the theory, corresponds to those suppositions that explain and validate the technological discourse. It contains some aspects of the development and justification of the techniques that are usually taken for granted and, therefore, rarely specified. In this case, two implicit principles seem to "support" the activated praxeologies. The first one – which we could call the *empiricist principle* – consists in assuming that the answers to the questions related to the behavior of an exponential function can be deduced from the simple observation of the images on the screen, using the graphical and numerical information provided. They thus appear as self-justified verifications or, at the most, provisional conjectures that require a subsequent justification. Students say what they say "because it is what they see on the screen" and it seems that "everything that appears on the screen is true." This is the theoretical foundation of ostensive techniques based on the observation of empirical objects.

The second theoretical principle that seems to act (although not always in the same way) is what we could call the *principle of coherence*, which is also essential to the experimental work. We indeed see that some of the affirmations of the students are algebraically validated (for example that $2.7^0 = 1$ or that $1^x = 1$) following the principle of "what is observed has to be compatible with what one already knows." However, this principle does not always function in the same way. For instance, students conjecture that $y = 0$ when the values of x are lower than –5.3, stating what they see on the screen. (Given the fact that numerical values appear to two significant figures and $2.7^{-5.3} \approx 0.0052$ while $2.7^{-5.4} \approx 0.0047$, the Cabri file shows $2.7^x = 0.00$ for all $x < -5.3$.) Here, we see how the two aforementioned principles clash with each other in a certain way, without posing any difficulties to the students, certainly because the teacher remains ultimately responsible for the validity of the activated praxeology.

Finally, we would like to comment that the praxeologies observed "in action" in the video seem to be oriented towards drawing up a global *technological discourse* on how exponential functions vary. In other words, despite having highlighted the practical and theoretical elements of the praxeology involved in the episode, the final result of its setting up basically consists of generating technological elements of a broader praxeology that exceeds the observed work. This special situation makes it difficult to distinguish between the elements of the praxeology at stake (carrying it out consists in producing technological ingredients of a broader praxeology) and the technological and theoretical elements that correspond to those technical elements.

5.2.2 *Didactic Praxeologies*

Besides the description of the mathematical praxeology at stake, the second kind of question that guides the analysis consists of asking: What are the didactic praxeologies put to use by X and Y and what didactic means have proved necessary to do so? In the considered episodes, two types of didactic praxeologies (or two positions in a cooperative didactic praxeology) can be distinguished, depending on whether we consider the teacher or the students to be the main character. We will here focus on describing some of the elements of the didactic praxeology of the teacher (which we may also call the "teaching praxeology") because in general they contribute more to explaining what students do and why they do what they do. It is, however, obvious that, considering that the didactic process is based on cooperation between teachers and students, the praxeologies of both types are always mutually influenced.

In the episode considered, and through the actions of the subjects observed – two students working on a computer in class under the supervision of the teacher – we will try to describe in the first place the (regular) institutional praxeologies that are "activated" by the people observed, or in which they "enter." Given the fact that all praxeologies contain a descriptive and justificatory discourse, their analysis needs to be carried out from an external position in order to grasp this discourse from a critical point of view.

If we respect the chronology of the episode and stick to the point mathematical praxeologies described in the previous section, a first element of the teaching practice is precisely the choice and formulation of the concrete tasks proposed to the students. A second element of this practice is the election of the type of "materials" proposed to provide and validate the answers to the questions posed. And, finally, there is a set of types of didactic task and techniques carried out in order to help students elaborate those answers until turning them into something that may be used again later on.

5.2.2.1 The Practical-Technical Block of Didactic Praxeologies

We assume that the didactic process is centered on the study of a local praxeology about exponential functions and, more precisely, on the variations of exponential functions of the type $y = a^x$. The whole didactic process, which goes from considering the initial question Q until constructing a validated and potentially reusable praxeology, may be described in terms of six *didactic moments*: the *first encounter* with the praxeology and the formulation of the tasks to be carried out, the *exploration* of the tasks and the emergence of a technique to carry out, the *work of the technique*, the elaboration of a *theoretical environment*, the *institutionalization*, and the *validation* of the work done. Even if they can be considered chronologically, the "moments" constitute dimensions of the process of study: they can take place simultaneously and can be repeated at different periods of time. In the case here considered, we may think that the episode corresponds to the moment of the elaboration of the technological-theoretical block of the praxeology.

What is the didactic strategy used by the teacher to make the students experience this moment? To propose two mathematical tasks to be carried out using some Cabri file previously prepared by the teacher and, eventually, other technological means. The way students deal with the tasks proposed shows that this kind of activity is not strange to them. They read the statement and start working without any trouble. We can thus suppose that the didactic technique used by the teacher is common practice in the class. We do not know if it has a specific name or how the authors interpret it (aspects that are part of the *technology* of the didactic praxeology). From our position of external observers, we could classify this didactic technique as the one of "filling gaps": when facing the initial question of describing the properties of the variation of the exponential function, the distribution of responsibilities between the teacher and the students consists of the teacher carrying out an important part of the work (formulating the question, elaborating the Cabri files, giving exact indications of certain gestures to carry out, etc.) and leaving some substantial gaps as gestures for the students to do and questions to answer in writing. The teacher here assumes the why of the questions he formulates, their sequencing and motivation, as well as their functionality (the fact that they will lead somewhere). The students follow the indications of the teacher and have the responsibility of providing a first written formulation, discussing, and drawing up valuable observations, comments, and conjectures on aspects about the functions that are new to the students. The teacher occasionally intervenes during those critical moments to help the students elaborate their answers: gestures concerning the secant lines; the idea of zoom, the fact that with the function graph "[…] *you can approximate it with many small lines*" (53:29); verbal expressions such as "*the growth percentage of the y*'*s*" (54:22); or make the groups of students share some answers as in "*the other group have used a very good example*" (55:32).

As we only see a limited part of this didactic praxeology of the teacher, we are not totally aware of the kind of didactic tasks he feels responsible for, what the destiny of the technical and technological elements activated by the students will be, how these elements are being institutionalized and validated to conform to the final praxeology at stake. Neither do we know the motivation that surrounds this construction, that is, its *raison d'être*.

5.2.2.2 The Technological-Theoretical Block of Didactic Praxeologies

What does the technology and the theory of a didactic praxeology consist of? Just as in any praxeology, it is made up of elements of different natures, well or poorly articulated depending on the case and on the degree of development of the praxeology. In this case, it seems that the didactic praxeology set up by the teacher is not spontaneous, but comes from previous preparation and experimentation supported by elaborated technological-theoretical elements. Some of these elements may be deduced from the details of the episode (the students do not seem astonished by the tasks proposed), others are clarified from the teacher's answers to our questions (see Sect. 2.2.2). However, some aspects will remain blurred. We will infer them as a conjecture from the analysis.

The *technological level* of the didactic praxeology consists in a descriptive and justifying discourse close to the teaching and learning practice. For instance, with respect to the mathematical praxeology at stake, Domingo specifies what kind of answer he wishes to obtain at the end of the study process:

> I wanted the students to understand that exponential functions are functions for which the growth is proportional to the function itself. In other terms, the derivative of an exponential function is proportional to the function itself. This consideration, in my opinion, should allow students to understand why the exponential function a^x with a greater than 1 grows with x faster than any power of x. (Answer to question 8, Sect. 2.2.2)

In fact, he describes this local mathematical praxeology at stake accurately and even proposes an analysis of it in terms of three levels of complexity:

> A first level is that of perceiving the different velocity of variation that exists between x and a^x. […] A second level is that of the understanding of how the graph of an exponential function varies when the base varies. A third level, as in the third worksheet of Cabri, is relative to the understanding that the incremental ratio is a function of two variables (the x and the increment h). […] A fourth level is the passage from the local to the global aspects of the derivative. From the gradient to the gradient function. (Answer to question 12, Sect. 2.2.2)

He even places this local praxeology in a broader one around exponential functions:

> [I follow] two paths. In the first one I pose some problematic situations which, to be solved, ask for exponential models. In the second one I present the properties of exponentials and I introduce the logarithmic function as the inverse function of an exponential. […] Finally I propose some techniques to solve exponential and logarithmic equations and inequations […]. (Answer to question 10, Sect. 2.2.2)

As far as the selected order of the tasks is concerned, he justifies it with the argument of complexity and justifies the necessity of the experimental work with Cabri in terms of the construction of a "cognitive root" for the later "formal" work.

With regard to the criteria to intervene in the independent work of the students, the teacher argues:

> Sometimes I enter in a working group if I realize that students are stuck. Other times I enter because I realize that students are working very well and they have very good ideas that need to be treated more deeply. […] (Answer to question 4, Sect. 2.2.2)

However, in order to justify his interventions in the teamwork, the teacher refers to a broader explanatory framework around the notions of "zone of proximal development" and "semiotic game":

> […] a constant is that I try to work in a zone of proximal development. The analysis of video and the attention we paid to gestures made me aware of the so-called "semiotic game" that consists in using the same gestures as students but accompanying them with more specific and precise language […] (Answer to question 4, Sect. 2.2.2)

Here is where the *didactic theory* shows up. It also includes a certain conception of mathematics, the rationale of teaching it and the mission of schools in society:

> The main aim of the posed activity was to allow students to develop an understanding of the concept of exponential growth […]. This consideration, in my opinion, should allow students to understand why the exponential function […] grows with x faster than any power of x. (Answer to question 8, Sect. 2.2.2)

> I try to assess in the students the competence to observe and explore situations; to produce and to support conjectures; to understand what they are doing and to reflect on it […] (Answer to question 11, Sect. 2.2.2)
>
> the main function of teaching, not only of math, is to help students to exercise critical thought, to acquire the necessary competences for an informed and aware citizenship. (Answer to question 11, Sect. 2.2.2)

We can add another theoretical element the teacher does not explicitly formulate but that seems to support his practice with respect to the mathematical knowledge at stake: the fact that it is not necessary for the teacher to explain to the students why the properties of exponential functions are worthwhile to identify and what is the main purpose of the tasks given to them.

5.3 New Questions Enlarging the Empirical Unit of Analysis

Until now we have just proposed a description in terms of praxeologies of the activities observed (or deduced) from the video and from the extra empirical data gathered. However, the aim of ATD is not just to describe teaching and learning realities, but to *explain* and *question* it from different perspectives, confronting the observed facts with those that could happen and did not, also analyzing the conditions that enable teaching and learning processes to happen in the way they happen, while hindering or impeding other kinds of activities from taking place. As in the case of mathematical praxeologies, when dealing with the description of the didactic praxeologies, the analysis of the observed situation depends on the type of questions we wish to answer as researchers.

For instance, if we consider the mathematic praxeologies described in the previous section as if they make sense on their own, then we would be assuming the didactic project of the teacher without further analysis and we would only be questioning what the students do, what they learn, and how they learn it. However, if we make a step aside and look at the teacher's whole didactic project, numerous questions arise related, for instance, to the didactic transposition process and the elaboration of the mathematical praxeologies to be taught:

> Where do the proposed tasks come from? What questions could they contribute to answering? What broader praxeology are they supposed to integrate? Why is it important to describe the properties of variation of the exponential function? What is being done with those properties? In which broader praxeology and at what level (practical or theoretical) will the obtained technological elements on the exponential function integrate?

With the extra information gathered in Sect. 2.2, some of those questions can be partially answered. For instance, the broader mathematical praxeology at stake is basically generated by problematic situations modeled by *discrete* exponential functions, a previous work that can motivate the study of the properties of the graphs of continuous exponential functions. At the moment considered in the episode, this work can only be carried out with ostensive techniques in order to conclude that the function depending on the tangent line is proportional to the corresponding exponential function. A first approximation to the notion of derivative and some of the praxeological elements that will be necessary later on for its formal construction are thus obtained. Finally, logarithmic functions are defined as the inverse of exponential functions and, as the teacher indicates, the properties of both are used to propose some techniques of solving exponential and logarithmic equations and inequalities. Given this, the crucial question of the criteria used to choose the structure and dynamics of the mathematical praxeology to be taught should be asked, as well as the conditions needed to make this choice and the restrictions that hinder it. This is part of the analysis of the didactic transposition process that is not being developed here. It requires the elaboration of a *reference epistemological model* about the *theme* of exponential functions and its relationship with the different *sectors* and *domains* of school mathematics to provide researchers with an alternative point of view.

This praxeological analysis about didactic stake ♥ (the characterization of exponential functions through their point and global variation) and the description of the didactic praxeologies used by both the teacher and the students should be completed by an *ecological* analysis about their conditions of possibility. It starts by asking questions such as:

> Where does the didactic praxeology enacted by the teacher and the students come from? How is it built? Is it a common organization in the educational system considered? What institutional conditions, at what level of the scale of didactic codetermination, make it possible to appear? What other alternative organizations exist or could exist?

If we stick at *the level of the discipline*, that is, the teaching of mathematics in grade 10, the teaching strategy followed by the teacher does not seem to correspond to a "standard" content organization, where topics usually have a more classical structure generally imposed by official curricula: the discipline divided into domains or sectors (sometimes called "blocks of content") with a given list of themes or topics in each. Teachers organize, sequence, and program the themes their own way, but they rarely question or, much less, modify the given structure. This curriculum constraint tends to confine the teacher's didactic praxeology at the level of the theme and makes it difficult to draw attention to the *rationale* of the taught mathematical praxeologies because they often appear to be beyond the themes (and even beyond the sectors or domains) where they take place. This is not the case in the teaching process considered here, since the teacher seems to be responsible for the whole organization of the content. It is interesting to ask what kind of institutional as well as personal conditions are necessary to do so. Certainly the teacher's involvement with research in didactics is one of the conditions for this didactic praxeology to exist, since the technological and theoretical discourses underlying it are far from being spontaneous or professionally shared.

The level of the *pedagogy* corresponds to the conditions that are common to the teaching and learning of any discipline in a school institution. In this respect, the considered episode is a good illustration of another phenomenon related to the usual distribution of responsibilities between the teacher and the students in traditional didactic praxeologies. Current curricula tend to refer the main goal of teaching and learning projects to a list of predetermined praxeological elements ("topics," "concepts," "competences," etc.) teachers should teach and students learn. The way these elements are organized, motivated and made available to the students, as well as the reasons for the choices made, are part of the teacher's responsibilities. Students do not participate in this kind of decision, which is even often hidden to them. They are just asked to do things and they usually do them heedfully and obediently. Even if the teaching strategy in the analyzed episode is not a common one and seems modern and innovative, it still contains some remains of the classic "authoritarian" pedagogical gestures: the teacher presents some tasks and an experimental tool and gives instructions to the students without explaining where they come from nor where they lead to; we cannot see any information about the map of the trip students are invited to follow; they do not seem either to be asked to participate in its configuration. The teacher proposes, the students accomplish.

Finally, at the level of the *society*, the episode also illustrates how didactic praxeologies – even the most "elaborated" ones – are always permeable, vulnerable even, to practices with a high cultural value, independently of their didactic "utility" or "productivity." According to the task instructions, students are required to "observe" the properties of the graphs of the functions they see on the screen, "discuss their observations" and then deduce some of the "features" of the graphs. Therefore the teaching strategy seems to be taking advantage of the current fascination for visual representations in our western culture. It thus appears here as a strong condition to facilitate the use of Cabri files as a means for the students' main exploration work. The situation would certainly be more difficult if the experimental work was organized around the observation and manipulation of numerical tables or algebraic formulae, since they tend to appear as meaningless to our common culture. The tasks prepared by the teacher in the sessions following the episode include these kinds of alternative experimental means, but they seem to play a less central role in the whole teaching and learning process.

Because of the loss of its social leadership, school encounters more and more difficulties in giving sense to some didactic practices that are not easily recognized by common culture. In the other sense, school is permeable to some social practices that are easily adopted as didactic ones, while remaining resistant to others. Little is known about the specific ecology of didactic praxeologies at school and how this ecology is related to their existence in other social institutions. This is the reason why researchers are interested in tracking data coming from outside school and in looking into school as *outsiders*, that is, without assuming that anything that happens there is normal or necessary. The theoretical and methodological framework provided by ATD, throughout the delimitation of a unit of analysis that goes far beyond the limits of the classroom activities appears to be a useful tool to emancipate researchers from the "transparency" of didactic facts and from the cultural values about the social and human phenomena they have to approach.

References

Artigue, M., & Winsløw, C. (2010). International comparative studies on mathematics education: A viewpoint from the Anthropological Theory of Didactics. *Recherches en Didactique des Mathématiques, 31*(1), 47–82.

Barbé, Q., Bosch, M., Espinoza, L., & Gascón, J. (2005). Didactic restrictions on the teacher's practice. The case of limits of functions. *Educational Studies in Mathematics, 59*, 235–268.

Bosch, M., & Gascón, J. (2006). Twenty-five years of the didactic transposition. *ICMI Bulletin, 58*, 51–63.

Bosch, M., Gascón, J., Ruiz Olarría, A., Artaud, M., Bronner, A., Chevallard, Y., Cirade, G., Ladage, C., & Larguier, M. (2011). *Un panorama de la TAD. An overview on ATD* (CRM Documents, Vol. 10). Bellaterra (Barcelona): Centre de Recerca Matemàtica.

Bronner, A., Larguier, M., Artaud, M., Bosch, M., Chevallard, Y., Cirade, G., & Ladage, C. (2010). *Diffuser les mathématiques (et les autres savoirs) comme outils de connaissance et d'action*. Montpellier: IUFM.

Chevallard, Y. (1985). *La Transposition Didactique. Du savoir savant au savoir enseigné* (2nd ed., 1991). Grenoble: La Pensée Sauvage.

Chevallard, Y. (1989). On didactic transposition theory: Some introductory notes. In *Proceedings of the international symposium on selected domains of research and development in mathematics education*, Bratislava (pp. 51–62). http://yves.chevallard.free.fr/spip/spip/IMG/pdf/On_Didactic_Transposition_Theory.pdf. Accessed 30 Oct 2013.

Chevallard, Y. (1992a). Fundamental concepts in didactics: Perspectives provided by an anthropological approach. In R. Douady & A. Mercier (Eds.), *Research in Didactique of mathematics, selected papers* (pp. 131–167). Grenoble: La Pensée Sauvage.

Chevallard, Y. (1992b). A theoretical approach to curricula. *Journal für Mathematikdidaktik*, *13*(2/3), 215–230. http://yves.chevallard.free.fr/spip/spip/IMG/pdf/A_Theoretical_Approach_to_Curricula.pdf. Accessed 30 Oct 2013.

Chevallard, Y. (1999). L'analyse des pratiques enseignantes en théorie anthropologique du didactique. *Recherches en Didactique des Mathématiques, 19*(2), 221–266.

Chevallard, Y. (2002). Organiser l'étude. 3. Écologie regulation. In J.-L. Dorier, M. Artaud, M. Artigue, R. Berthelot, & R. Floris (Eds.), *Actes de la 11e École d'Été de didactique des mathématiques* (pp. 41–56). Grenoble: La Pensée sauvage.

Chevallard, Y. (2006). Steps towards a new epistemology in mathematics education. In M. Bosch (Ed.), *Proceedings of the IV Congress of the European Society for Research in Mathematics Education* (pp. 21–30). Barcelona: FUNDEMI-IQS.

Estepa, A., García, F. J., & Ruiz-Higueras, L. (2006). *Sociedad, escuela y matemáticas. Aportaciones de la Teoría Antropológica de lo Didáctico (TAD)*. Jaén: Publicaciones de la Universidad de Jaén.

García, F. J., Gascón, J., Ruiz Higueras, L., & Bosch, M. (2006). Mathematical modelling as a tool for the connection of school mathematics. *ZDM – The International Journal on Mathematics Education, 38*(3), 226–246.