Chapter 11
The Epistemological Gap: A Case Study on Networking of APC and IDS

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Abstract The case study of the epistemological gap involves two theoretical approaches, APC and IDS. It describes a networking case that starts from a situation of seemingly contradictory analyses, develops a common methodology, and leads finally to conceptualizing and locally integrating the new concept of the epistemological gap into both theories.

Keywords Networking of theories • Epistemological analysis

11.1 Introduction

In this chapter we present the networking process developed by two teams,[1] namely the APC team using the Space of Action, Production, and Communication theory with its Semiotic Bundle construct (Chap. 3) and the IDS team using the Theory of Interest-Dense Situations (Chap. 7) in which a partial integration of a new theoretical construct, the epistemological gap, took place for both approaches after trying to coordinate seemingly contradictory analysis.

The networking in this case study has its origin and empirical base in a short video excerpt, referred to in Chap. 2 as "extra video" on Task 3 (see Sect. 2.2.3).

[1] A first partial account of this networking process has been presented in CERME6 (Working Group 9: Different theoretical perspectives and approaches in research); see Arzarello et al. (2010).

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It lasts about a minute and a half, and shows Giovanni and Carlo discussing with the teacher what happens to the exponential function for very large *x*.

The need for a second video was raised by the IDS team. In fact, in order to progress with the networking, the IDS-analysis did not need so much the interview data with the teacher that other teams needed (see Sect. 2.2.2), but rather needed additional data about typical social interactions in the class. To be able to reconstruct these typical social interactions, videos from about 20 lessons would be needed in order to shape an appropriate empirical base. However, only one additional video was available from the same classroom in the same school year, and this was the extra video on Task 3 (see Sect. 11.2 below).

The first networking step was to analyze the video from the two perspectives separately. Each team carried out an initial analysis of the episode (as reported in Sect. 11.3). Each of the initial analyses in its own way described a teacher–students interaction that did not lead to a successful outcome, but neither of the two analyses could provide an explanatory account of the empirical phenomenon. On the contrary, the analyses appeared almost contradictory. This surprising result triggered the necessity to carry out a joint analysis that started a coordination process between the two teams. The result was a local integration of the methodologies of the two theories. In Sect. 11.4.1, we describe the process as well as the result of our coordinating strategy.

In a spiral process this coordinated analysis brought about the necessity of further theoretical reflection, especially considering the epistemological dimension. In an interplay between the theoretical reflection and the data analysis, we developed a new concept, which we called the *epistemological gap*, and which could provide a satisfactory explanation (for us) of the empirical phenomenon previously identified (Sect. 11.4.2).

A local integration based on the epistemological dimension was thus realized for both theories. The new tool for analysis, produced in the networking activity, deepened our understanding of the data and opened routes for reflection that were new for each perspective. In the final section (Sect. 11.5) we report our reflections on our networking enterprise.

11.2 The Empirical Base

In the extra video, Giovanni and Carlo discuss with their teacher (T) what happens to the exponential function for very large *x*. This episode occurred immediately after the students had finished Task 3 (see Sect. 2.1.3). The result of the exploration was

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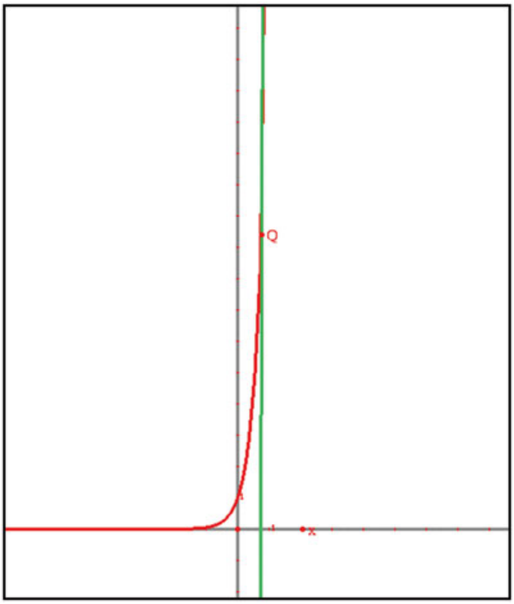


Fig. 11.1 The graph shown on the computer screen

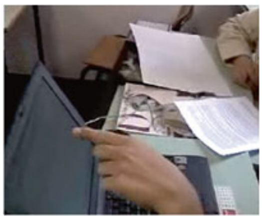
still visible on the computer screen (Fig. 11.1). It shows a secant built by two points very close to each other leading to a quasi-tangent line, which the students and the teacher refer to as the "tangent line."[2]

Due to the specific methodologies of APC and IDS, we report in this chapter the transcript integrated with information about the connotation of the speech, and the occurrence of accompanying gestures. Some screenshots are added from the video, in order to better account for the gestures: they are reported immediately after the corresponding speech line.

In the transcript, underlined words indicate that they are simultaneous with the gestures.

1 G but always for *a* very big this straight line, [Gesture] when they meet each other, there it is again…that is it approximates the, the function very well, because …

Gesture in 1: G is pointing at the line in the screen



2 T what straight line, sorry?

[2]The configuration on the screen is not reported by the video-camera, nor captured in any other way, rather it is reconstructed by the APC team as reported in Fig. 11.1. Detailed information on the line (e.g., exactly which value of *a* is chosen) is therefore not available.

3 G this here [*pointing at the screen*], for *x* very, very [*Gesture*] big

Gesture in 3: G's hand goes upwards

4 T [*Gesture a*] will they meet each other [*Gesture b*]? [*challenging connotation*]

Gestures in 4:

(a) T pointing two forefingers

(b) T crossing the two pointed forefingers

5 G that is [cioè[3]], yes, yes they meet each other [*gesture*]

6 T but after their meeting, what happens? [*continuing to keep the hands in the same configuration as in line 5*]

Gesture in 5: G's two forefingers touching each other

7 G eh…eh, eh no…, it makes so

8 T ah, ok, this then continues [*gesture a*], this, the vertical straight line [*gesture b*], has a well fixed *x*, hasn't it? The exponential function later goes on increasing the *x*, doesn't it [*gesture c*]? Do you agree? Or not?

Gesture in 7: G crosses the left hand over the right one; T is keeping the previous gesture

[3]The expression "cioè" in Italian means literally "that is." Over-used by teenagers, it introduces a reformulation of what was just said. As is likely in this case, it can have the connotation of "I am sorry but."

Gestures in line 8:

(a) T moving rightwards his left hand

(b) T's right hand vertically raised

(c) T moving rightwards his right hand

9 G yes […]

10 T [*addressing C*]: He [*G*] was saying that this vertical straight line [*pointing at the line in the screen*] approximates very well [*gesture*] the exponential functions

Gesture in 10: T raises both hands

11 G that is, but for *x* that are very…very big

Gesture in 11: G moves his left hand upwards

12 T and for how big *x*? 100 billions?
x = 100 billions?

13 G because at a certain point…, that is, if the function [*gesture 13a*] increases more and more, more and more [*gesture 13b*], then it also becomes almost a vertical straight line [*gesture 13b*]

Gesture in 12: T raises his hand at his right and keeps it fixed

Gestures in line 13:

(a) G raises his left hand

(b) G moves his hand upwards

(c) final position of G's hand after moving upwards

14 T eh, this is what it seems to you by looking at; but imagine that if you have $x = 100$ billions [*gesture*], there is this barrier…is it overcome sooner or later, or not? [*connotation: suggesting the answer yes*]

Gesture in 14: T keeps his right hand in the vertical position

15 G yes

16 T and so when it is overcome [*gesture 16a*], this x 100 billions [*gesture 16b*], how many x do you still have at disposal, after 100 billions? [*gesture 16c*]

Gestures in 16:

(a) T crosses left forefinger over right hand

(b) T raises his right hand

(c) T moves right hand rightwards, repeatedly

17 G infinite

18 T infinite… and how much can you go ahead after 100 billion [*repeating the gesture 16c*]?

19 G infinite points

20 T then the exponential function goes ahead on its own, doesn't it?

11.3 Initial Analysis from IDS and APC Perspectives

Initially, the two teams carried out the analysis of the episode separately according to the two perspectives. The result is summarized in the following sections.

11.3.1 Initial Analysis According to the Theory of Interest-Dense Situations

To work out the IDS-analysis (cf. Chap. 7), we have to consider the central question: How is the emergence of an interest-dense situation supported or hindered? (cf. Bikner-Ahsbahs 2005). To answer it, an analysis of speech acts at three levels is conducted. The locutionary level considers the direct meaning of what is said, while the non-locutionary level considers, on the one hand, the information given by saying something and the way it is said (illocutionary level) and, on the other hand, the intended and actual effect on the listener (perlocutionary level) (Austin 1975; cf. Beck and Maier 1994).

In line 1, Giovanni begins to construct mathematical meanings about the growth of the exponential function in broken language as described above. The teacher interrupts him: by apologizing he indicates illocutionarily that he normally would not interrupt the student, but in this case an interruption seems necessary to him. By saying "sorry," he also might want Giovanni to feel accepted. Asking "what straight line" (line 2) indicates either that there is something problematic with the straight line or that the teacher wants to clarify which straight line exactly is meant. Pointing at the line on the screen, Giovanni refers to an answer on the locutionary level, but also adds the condition for his explanation given in the task in line 1: "for very big *x*." The teacher's question "will they meet each other?" locutionarily requires information whether the graph and the line meet, but illocutionarily questions the truth of the condition of Giovanni's beginning explanation in line 1 "when they meet each other…." Therefore, and through connotation, the teacher's question is challenging to Giovanni. It is not clear whether the teacher suggests a negative or positive answer, but the teacher's finger crossing gesture (screenshot 4b) might support the latter, as does the intonation. Giovanni follows the teacher's crossing gesture and answers that "they meet" (line 5), indicating through intonation and by doubling the word "yes" (illocutionarily) that he has no doubts about the fact that they meet. Since Giovanni has perceived the line to be constructed as a secant his certainty is based on his previous experience. On the locutionary level, we would see only the questions and the answers. On the non-locutionary levels there is negotiation underneath.

Looking only at lines 1–5, an interest-dense situation is about to emerge because Giovanni is deeply involved in the mathematical problem and he starts to construct further-reaching mathematical meanings. Based on the theory of interest-dense situations we can predict how the teacher could support or hinder the emergence of interest density. By focusing on the student's ideas and supporting their further construction, the teacher would support emergence of interest density; but by acting according to his own thinking process or his expected answers, he would interrupt its emergence. Going further in our analysis, we will show that there is a negotiation which proceeds underneath the locutionary level.

In line 6, the teacher builds on the student's utterance by asking what happens after the two lines meet. Imitating the teacher's finger crossing (that he gives as a hint, perlocutionarily), by his hands Giovanni shows how the two graphs cross each other. However, the teacher does not seem to be content because he explains his view of the situation in line 8 and gives a more elaborate answer to his own question. Giovanni does not fulfill the teacher's expectations. In this way the teacher establishes an argumentation as a proof by contradiction, following his own train of thought and not that of the student. In line 8, he constitutes the basis of his argument. In order to include Giovanni into the process, his rhetorical questions "Do you agree? Or not?" demand Giovanni's agreement (line 9). Summarizing Giovanni's statement addressed to Carlo (line 10), the teacher puts forward the statement that he wants to prove false. However, Giovanni modifies and restricts the range of the statement's validity by "but for x that are very…very big." This utterance (locutionarily) adds a condition, but illocutionarily Giovanni corrects the teacher. Hence, he only partially agrees, because his description was based on "very…very big x" (line 11). Again, Giovanni indicates that his train of thought is a bit different. Perlocutionarily Giovanni succeeds at this moment because the teacher changes his focus, by locutionarily taking up the student's idea in the question: "for how big x?" (line 12). Giovanni seems to feel encouraged to explain: "because at a certain point…, that is, if the function increases more and more, more and more, then it also becomes almost a vertical straight line" (line 13). Because of Giovanni's deep involvement and the dynamic of the epistemic process, the situation has the potential to lead to an interest-dense situation. It is the attribution of mathematical value, which is not yet expressed. On the illocutionary level, the teacher indicates understanding of Giovanni's point of view (line 14) by saying "this is what it seems to you by looking at," but he also implies that the student's way of arguing is not the correct way. In this way, he divests Giovanni of his argumentation base, that is, the diagram on the screen. Through the word "imagine" he refers to another argumentation base but asks whether $x = 100$ billion will be overcome, suggesting that the answer is positive. Giovanni agrees. The teacher now finishes proving the statement (line 10) to be wrong by a proof of contradiction that he orchestrates through social interaction. The teacher works his argument out. "there is this barrier… is it overcome sooner or later, or not?" he closes (line 14), demanding agreement. Lacking an argumentation base, Giovanni now gives up following his own train of thought (lines 15–20). The emergence of interest-density dries up.

11.3.2 Initial Analysis According to the APC Model

The APC team analyzed the episode by focusing attention on the semiotic resources shown in the teacher–students interaction, that is, on the *semiotic bundle* that is the combination of words, gestures, and representations in the Cabri file. As discussed in

Chap. 3, the semiotic bundle construct is the main analysis tool of micro-processes within the APC approach.

The basic point of discussion between students and teacher concerns the behavior of the exponential function for a large base *a* and large values of *x*. In his first utterances (lines 1–3) Giovanni claims that in such conditions the straight line that appears on the screen, that is, the quasi-tangent line, can be a good approximation of the exponential function. Such a conjecture is fostered by the image from the Dynamic Geometry Software the students are using (see Fig. 11.1): the quasi-tangent line appears almost vertical, and the exponential function comes to be perceptually confounded in it.

However, the teacher misinterprets Giovanni's words: whereas Giovanni is referring to the tangent line (he points also to it on the screen, lines 1 and 3), the teacher appears to interpret the student's words as referring to a vertical asymptote (lines 4–6). There are two hints for this misinterpretation:

- Giovanni says "when[4] they meet each other" (line 1): he seems to refer to the fact that the tangent approximates the function well near the tangent point.
- The teacher starts speaking (line 4) without giving Giovanni time to complete the sentence.

Hence there is a conflict between Giovanni's gesture, pointing to the tangent on the screen, and the teacher's gesture, which shows the vertical asymptote. A possible origin of this misinterpretation can be traced to the teacher's professional knowledge regarding the exponential functions and teaching–learning processes about it – what in literature has been called "specialized content knowledge" (Ball et al. 2008) of the teacher.

Asking about a hypothetic meeting of the function with the straight line, the teacher is representing the graphs by means of an iconic gesture (screenshots 4a, b and subsequent pictures): his right forefinger stands for a straight line, and his left forefinger moves in an upwards inclined way to represent the exponential function graph. In his subsequent interventions, Giovanni (lines 5–7, and corresponding gestures) is tuning with the teacher's semiotic resources, both speech and gesture. With his hands, he represents the graph of the exponential crossing the straight line (gesture in line 7): he is answering the teacher's question by showing the behavior of the function through the gesture. The teacher (line 8) accepts such an answer and endeavors to make explicit the idea that the domain of the exponential function is not limited, and therefore its graph intersects any vertical line. To do so, he uses both speech and gestures (see lines 8–20, and the related pictures).

We can now focus more closely on the dynamics using the semiotic bundle lens. In order to include Carlo in the discussion, the teacher reports Giovanni's observation (line 10). By repeating and rephrasing Giovanni's words (line 10) he is tuning with the student's speech. But through gestures (screenshots 10, 16b, c), he is trying to make apparent a specific feature of the graph of the exponential function, that is, the fact that it *crosses* any vertical line. The teacher is demonstrating what we call a *semiotic game* (see Chap. 3), in that he is tuning with one semiotic resource, and is

[4]In English "when" may have the meanings of "if" and of "where." In this case, the sense is "where."

using another resource used by a student to make meanings evolve in order to align them with the culturally established mathematical ones.

The gesture appears as a powerful resource in the teacher's hands, in order to prompt the students' imagination. In fact, the gesture allows the teacher to refer to what cannot be seen in the representation on the screen and may be difficult for the students to understand from a purely verbal description (the graph for very large x). In particular, gestures seem to be a suitable means to refer to very large values and to evoke their infinite quantity (screenshot 16c).

As to Giovanni, we see that he does not appear to have profited from the teacher's semiotic game. In fact, considering lines 11–13 (and the related pictures), we can see that in his words he is expressing the idea that the function will become "almost a vertical straight line"; and his gestures appear very different from the teacher's. Whereas the teacher's gestures place large values of x in the correct location with respect to his gesture space[5] (hand moving rightwards: screenshots 8c, 12, and 16c), the student's gestures place large values of x to a high location in space (hand moving upwards, screenshots 11, 13a, b, c): he is probably referring to the values taken by the exponential functions, rather than to the abscissas. As a result of the analysis, we can conclude that the episode shows an example of a non-successful semiotic game.

11.4 Networking of the Approaches

The two analyses were exchanged between the teams. Through this exchange, the researchers made a strong effort *to make themselves understandable and to understand the other's perspective*, which constituted an important networking strategy (see Chap. 8).

This initial step led to *contrasting* the two analyses: we acknowledged their complementarity, but also felt that the two results had feeble explanatory power. Thus, we were led to a new common question: what is the deeper reason why the epistemic process (socially and semiotically) is not successful in this episode? Guided by this common question, the two teams jointly carried out a common analysis, through the *coordination* of the two theories. The resulting coordinated analysis is presented in the next Section.

11.4.1 Coordinating the Two Analyses

Based on the theoretical account and the two initial analyses, the IDS and APC teams considered the two perspectives as providing *complementary analytical tools and, thus, complementary interpretations* of the data. In fact, each view shed light on different aspects of the teacher–students interaction. The strength of the

[5]The gesture space (McNeill 1992) is the area in front of the speaker's body, in which he performs the majority of his gestures.

IDS-perspective is in the possibility to predict the emergence of interest-dense situations according to the type of social interactions that hinder or foster it. It includes the analysis of the locutionary and non-locutionary levels of speech, reconstructs epistemic processes within social interactions, and shows negotiations underneath the content level. According to this perspective, the student and the teacher are not able to merge their argumentations in the episode although there is a lot of negotiation about whose train of thought will be followed. Neither the teacher nor the student is able to engage with the other's perspective. The analysis shows a gap that cannot be overcome. However, the IDS-approach is, from an epistemic point of view, unable to provide tools to find out why this is the case.

On the other hand, by looking at a wide range of signs (in Peirce's sense), the APC-analysis identifies phenomena that could go unnoticed under standard linguistic-based analysis, such as the semiotic game between teacher and student. The analysis with the semiotic bundle provides the means to observe and to properly describe this game. From our classroom observations at several school levels, we can say that students did succeed to learn by means of semiotic games in other cases (see, e.g., Arzarello et al. 2009). This episode was one of the first in the APC team's research in which things appeared to go wrong. It was therefore a good occasion to investigate the *scope* and the *limits* of the semiotic game construct. Using the APC-frame one could observe that in the above episode the semiotic game shows the gesture–speech resources in the opposite direction with respect to semiotic games previously analyzed as "successful" (called "standard semiotic games" for the moment). In standard semiotic games, in fact, the teacher tunes with the students' gestures and uses speech to foster meaning development; in the above episode it was the other way round: tuning with speech (line 10) and fostering meaning through gestures. Using the semiotic bundle lens one can identify this difference, but within the theory it is not possible to say why this semiotic game is not working.[6]

The discussion so far led us to argue that the simple juxtaposition of the two perspectives was not enough to deeply understand what went wrong in the interaction. Since the student and the teacher referred to different resources in their argumentation, we conjectured that the reason for this gap might be located in the epistemological viewpoints of the student and the teacher, that is, in their views on relevant knowledge and ways of knowing. The idea of epistemological viewpoints was elaborated during the networking process.

The coordinated analysis of this episode was accomplished by re-analyzing the entire transcript line by line. The analysis in line with the theory of IDS was complemented with attention to the gestures. The speech–gesture in line with APC was complemented with attention to the non-locutionary levels. Figure 11.2 depicts the resulting lens of the analysis, from the coordination of the two previous ones which connects two levels of meaning-making with speech and gestures.

[6] Realizing that the semiotic game had not worked properly led us to suspect that the episode showed a case of a "Topaze effect," as described in the TDS theory. This was the prompt for further elaborating the networking process, and the outcome is reported in Chap. 12. The reader will see that, finally, the episode is considered to be neither a case of a genuine semiotic game, nor of a genuine Topaze effect.

	Speech	Gestures
Locutionary level		
Non-locutionary level		

Fig. 11.2 Two-level analysis of semiotic resources, deriving from a coordination of the two perspectives

There is no space here to report on the complete analysis. However, we provide the main parts to illustrate how our networking strategy was implemented.

At the beginning of the episode, Giovanni's words and gestures convey information at the locutionary level. Giovanni is trying to express his ideas on the behavior of the function for large *x*: his gestures and his words complement each other. The pointing gesture (line 1) specifies the reference of his words, making clear that he is referring to the straight line represented in the Cabri file. His gesture (line 3) complements the information given in words, by showing the behavior of the function when *x* is very large.

However, the gesture–speech analysis at the locutionary level has already been carried out by the APC team. The substantial novelty brought about by the coordinated analysis is constituted by considering the non-locutionary aspects. In line 4, the teacher is asking a question at the locutionary level, that is, he is asking whether the line and the function will meet. But at a non-locutionary level:

– his words have a challenging connotation,
– his gesture illocutionarily suggests that he is thinking about a vertical line (right forefinger in screenshot 4a).

line 4 | Speech | Gestures

Locutionary level | 4 T: will they meet each other?

(a)

(b)

Non-locutionary level

The sentence is spoken with challenging connotation (illocutionary).

The vertical right forefinger suggests that the teacher is thinking about a vertical line, and referring to it in his question illocutionarily.

line 5	Speech	Gestures
Locutionary level	G: that is, yes, yes <u>they meet each other</u>	
Non-locutionary level	*The sentence is uttered with a connotation like: "Of course, they will meet" (illocutionary).* *The teacher's gesture (showing how the crossing point will look, directing Giovanni's answer) more or less forces Giovanni to approve the teacher's gesture (perlocutionary level).*	

If we consider the prosody in Giovanni's answer (line 5), we can hear that he feels very sure of his words (illocutionary level). At the same time, his gesture (two forefingers touching each other, screenshot 5) is completing (locutionarily) the verbal answer by expressing how the line and the function will meet: they will have a tangent point.

In line 5, while Giovanni is answering, the teacher keeps his gesture (screenshot 5, introduced in screenshot 4b). The gesture shows a configuration in which the function is crossing the line, thus suggesting an answer to the question he has just asked (perlocutionary level, Chap. 7).

The teacher continues keeping his gesture (line 6), until the student indicates agreement with him. Giovanni changes his gesture from touching the forefingers to crossing hands (gesture in line 7), deictically saying "it makes so." Locutionarily he shows how the graph of the function and the line meet. At the non-locutionary level his speech and gesture show that the student is trying to agree with or to follow the teacher's perspective.

This is the only case in the entire episode in which Giovanni shows a gesture similar to the teacher's. In all the other cases, Giovanni's gestures have very different configurations.

In the following lines (8–12), the teacher's gestures illustrate the graphical situation that he is speaking about, thus complementing at the locutionary level his verbal utterances. However, the constant presence of the right finger or hand kept vertical constitutes a catchment (in the sense of McNeill 2005) and at the illocutionary level it tells us that the vertical "barrier" is crucial in his argumentation all the time. Note that the barrier is mentioned locutionarily in an explicit way in the speech only later (line 14).

Finally, the teacher's last gesture consists of his right hand moving repeatedly rightwards (screenshot 16c). This movement is not only depicting a graphical situation in an iconic way, but also at a non-locutionary level it is suggesting the answer ("infinite") to the student, which Giovanni takes up (line 17).

11.4.2 A Local Integration Based on the Epistemological Dimension

The joint analysis process produced an *integration at a methodological level* between the two theories, as shown in the tables above.

Furthermore, during this process, a new idea arose, consisting in hypothesizing the existence of an *epistemological gap* between teacher and student in the episode. The idea at this stage was just a sensitizing idea for the different epistemological viewpoints, and at the beginning was not clearly defined (rather it was quite fuzzy!), yet we felt that it helped to deepen our understanding. In order to clarify whether it could provide a suitable means for understanding the episode, we started on the one hand to apply it to the data, and on the other hand to frame it theoretically. Indeed, by applying it to the data und theorizing about it, we elaborated the epistemological gap concept and began to see it as being valuable. Data analysis and theoretical reflection mutually enriched each other in a nonlinear process, until a satisfactory understanding of the episode was reached and the epistemological dimension was theorized through two new constructs: the *epistemological view* and the *epistemological gap*. Their integration into both theoretical approaches provided a new, symmetrical case of local integration of common new theoretical constructs (see Chap. 8).

As a starting point, we elaborated a working definition for the epistemological gap – which was new to both of the theories – and through a spiral process we checked it against the data and theorized about that. Since space is insufficient here to present the entire process, we now present our final definition, and apply it to the data analysis. The notion of epistemological gap is based on two domain-specific concepts: the "personal epistemology" and the "epistemological view" of mathematics.

Personal epistemology has been described in the literature as a theory-like background view that an individual holds about the nature of knowledge and the nature of knowing (Feucht and Bendixen 2010, p. 10 ff.; Lising and Elby 2004). The *nature of knowledge* encompasses aspects of certainty (stable–fragile) and simplicity (simple–complex); and the *nature of knowing* specifies the kind of justification and sources that are taken as legitimate in the specific domain (Hofer and Pintrich 1997). Someone's personal epistemology can be different in different domains (Lising and Elby 2004) and can be regarded as part of the belief system of an individual that influences learning processes. Hofer and Pintrich (1997) have shown that personal epistemology is not stable over one's lifetime, and that it develops in a domain-specific way.

Considering that we are analyzing a teaching–learning situation with social interactions constituting an epistemic process towards solution of a mathematical problem, we can adapt the given definition in order to define a student's or teacher's *personal epistemology of mathematics* as a theory-like background view that the student or teacher holds about the nature of mathematical knowledge, and the nature of knowing it. In the mathematics classroom, the personal epistemologies of students and teacher are influential. In addition, the teacher's *personal epistemology towards mathematics teaching and learning* plays a crucial role. The importance of recognizing the professional knowledge of mathematics teachers has been highlighted for example by Ball and colleagues (Ball et al. 2008); our focus is more specifically on the epistemological dimension of professional knowledge as part of the teacher's personal epistemology.

Boaler and Greeno (2000) have shown that personal epistemologies of learners depend on the kind of epistemic climate they experience in the class. Because of this influence, we may assume that personal epistemology becomes partly visible in processes of knowledge construction. What is taken as legitimate knowledge and knowing in a specific task is determined by one's personal epistemology and at the same time by the affordances and aims of the task, the social and instructional environment, the tools available, and the development of the current learning process. Thus, when faced with a mathematical task, students base their actions on their personal epistemologies towards mathematics, and through the process of working with the task they build and develop their *epistemological view*. In other words, we call the collection of aspects of the nature of what is taken as mathematical knowledge and as legitimate knowing in mathematics the *epistemological view* in a specific mathematical task. These epistemological views develop over time in the learning process and have an impact back on the personal epistemologies, which change more slowly.

Hence, the epistemological view of a *student* is individual, locally dependent on the current mathematical task situation, and is shown through *epistemic and semiotic actions* and within social interactions. It is not static; on the contrary, it can be enriched and widened within the process of working with the task.

Due to his professional knowledge about teaching and learning mathematics, the *teacher* may build several possible epistemological views on the same situation within the task. These views may anticipate the students' views and they are dependent on the teacher's personal epistemology as well as on many didactical variables, such as the students' age, knowledge, ability, the curriculum, the tools available, the processes the students are familiar with, and so on.

After this necessary theoretical digression, we now come back to the episode, apply the introduced notions, and define the *epistemological gap*.

From the point of view of a researcher, the student's and teacher's epistemological views are only accessible through their semiotic productions and epistemic actions (see, e.g., Chap. 12). They are revealed by including the non-locutionary dimension of the semiotic resources, which we developed in our coordinated analysis (see Sect. 11.4.1). Through this analysis of the video data, we identified that there was a gap between the teacher's and the students' epistemological views. We will call it an *epistemological gap* and now explain this notion in greater detail.

In the first part of the excerpt, Giovanni is trying to express his interpretation of the exponential function in the case of large x (line 1). The teacher interrupts the student, prompts him with questions, and does not allow him to properly complete his argument (lines 2–7). Then the teacher performs a semiotic game articulated in a tuning in words and a dissonance in gestures (lines 6–8): the teacher is using gestures to focus on the possibility of vertical asymptotes to the exponential function. This semiotic game is different from "standard" ones. In particular, there are two main differences:

1. The teacher tunes with the student's words and uses gestures to express further meanings (whereas, usually, it is the other way round: the tuning is with gestures, and words are used to better articulate meanings).
2. The teacher does not repeat the words exactly, but rephrases them, by inserting the word "vertical" (whereas, usually, one semiotic resource is repeated as it is expressed by the student).

This refined analysis suggests that, in the teacher's interpretation, Giovanni is referring to a vertical line that the exponential function is crossing. However, from the student's semiotic resources we get no hints that Giovanni is thinking of a vertical line as an asymptote. In fact, he is deictically referring (with both speech and gesture) to the screen images, which show the exponential function and its secant line (line 3). Using the notions discussed above, we can say that the student and the teacher are showing two different epistemological views of the same situation, and therefore an epistemological gap is apparent:

- Giovanni is concerned with what happens for "very big x" (line 11), but he is relying on the *visual perception* of the exponential function graph, which is provided by the software.
- The teacher is focusing on *mathematical properties* of the exponential function, in particular to its lack of a vertical asymptote.

Giovanni is building his epistemological view about what happens to exponential functions for large x on the experience of exploring them with the digital learning environment. The nature of Giovanni's *knowledge* is therefore strongly experiential and empirical, since the computer shows empirical facts and images. This background knowledge is enriched by experiential knowledge developed through exploring the graph of exponential functions with the computer, extrapolating what he observes, and using what he knows about the construction of the objects on the screen. In the specific environment, the tangent was constructed by approaching a specific point on the graph using secants, and results can be seen on the computer screen. They are gained by extrapolating what is observed for very large x. The task is interpreted as getting a description about what can happen for very large x based on what can be seen. Also the source of *knowing* is based on experiences with the slope of the exponential function: the justification refers to what is explored and is visible on the screen. Terms such as "approximating" are used by the student intuitively and informally.

On the other hand, the teacher refers to more theoretical knowledge from logic and limits as analytical concepts of calculus. He is basing his argument on his mathematical knowledge of exponential functions, and more generally of functions and limits. Therefore when he speaks of "approaching," he is using it in a way coherent with the theory about formal limits. Since "almost" does not exist in this theory, he overlooks it. Also the teacher's nature of *knowing* differs from that of the student, since the teacher is referring to a proof by contradiction, which is part of the official mathematical epistemology.

This analysis has shown a local integration of the new concept of epistemological gap which is the product of a long period of networking efforts. However, there are two further steps of the networking methodology still to be carried out: elaborating the status of the new concept within the two theories and reflecting on the networking process itself.

11.4.3 Including the Epistemological Dimension into APC and IDS

By introducing the new epistemological dimensions, we could explain the phenomena that we identified within the APC-approach and the IDS-theory: the failure of the semiotic game, and the drying up of the interest-dense situation.

How can the epistemological dimension be related to the semiotic game? Let us consider the final part of the episode. The teacher and the student are performing different gestures: for example, in screenshot 11 the student's hand is moving upwards, to indicate large values of the function, whereas in the gesture in screenshot 12, the teacher's hand is moving rightwards. We interpret the dissonance – we could say this semiotic gap – in gesture as a signal that the teacher and the student have different epistemological views: the teacher's hand goes *to the right*, based on the fact that, being defined for every x, the exponential function cannot stop (screenshot 16c). Giovanni moves his hand *upwards* (gestures in screenshots 11, 13a, b, c): these gestures (and in particular their location) suggest that he is considering the points on the graph, without stressing the distinction between x-values and y-values, as the teacher does.

The teacher's reference to a vertical line, which is a key part of his argument, is firstly introduced through his extended forefinger (screenshot 4a), and then made more explicit through words ("vertical straight line," line 8) and a whole hand gesture showing a "barrier" (screenshots 10–12). The word "barrier" is finally uttered in line 14.

Line 14 was crucial for us in identifying the epistemological gap: in this line the teacher is starting his argumentation as a proof by contradiction, and at the same time he is telling the student (both at a locutionary and an illocutionary level) that he should not trust completely the images on the screen, and rather should follow his argumentation by imagining.

Basing on his *personal epistemology towards mathematics teaching and learning*, the teacher has to contradict the student's epistemological view. In fact, we know from the teacher's interview (Sect. 2.2.2) that as a teacher he tries to work within a zone of proximal development for the students (Vygotsky 1978). To do so, he uses different kinds of semiotic resources, including speech and gesture. Sometimes he tries to tune with those of the students in order to support them (as in the case of semiotic games); other times he introduces new words or gestures to offer the students means to enter into his epistemological view, as he is doing in line 14 with the word "imagine" and the configuration he represents with gestures. However, to be successful these didactic actions require that the teacher's and the student's epistemological views are close. If there is an *epistemological gap* between teacher and student, we can hypothesize that this gap prevents the semiotic game and more generally the semiotic interaction from working successfully. And, this way, the teacher is not addressing the zone of proximal development of the students.

Taking the concept of epistemological view into the IDS-analysis, we are able to explain why an interest-dense situation does not emerge. In the episodes before this extra video, the students have built some knowledge through gathering and connecting mathematical meanings and structure-seeing based on their interpretations of what they have experienced with the computer. Since the tasks previous to this one have been implemented by the teacher, Giovanni takes the experiences and visual representations on the computer screen as a legitimate source for argumentation, as has been accepted before. This interpretation can be supported by Giovanni's behavior in sticking to his own train of thought and referring to the image on the screen (lines 3, 5, 7, and 11). Thus, Giovanni shows an experiential epistemological view based on visual representations. The teacher seems to be aware of this view because he explicitly rejects the students' epistemological view as a legitimate source for argumentation (line 14).

Already in line 10, the teacher has started an argumentation process that is not based on visual experiences but on a proposition that the teacher imputes to Giovanni (line 10). The teacher takes this proposition as a hypothesis that he disproves within social interactions by the use of theoretical knowledge about exponential functions. The term "approximation" is interpreted differently by the two. In Giovanni's view, approximation means coming near (line 11), but the teacher takes this term as a theoretical part of the proposition that he starts to disprove by a proof by contradiction. As the teacher deprives Giovanni of his visual argumentation base, he says "imagine that if you have $x = 100$ billions…" (line 14); in this way he offers Giovanni imagination as a source for a legitimate argumentation that is different from the visualizations on the screen. However, for Giovanni, imagination separated from visual perception does not provide suitable arguments. We observe an epistemic situation with an epistemological gap that the social interactions do not bridge. The student adheres to his epistemological view and refuses to follow the teacher. The teacher does not accept the student's epistemological view as valid. Since the two epistemological views are not compatible and the student does not yet have access to the teacher's view, only the teacher could interact based on an epistemological view much closer to the student's one. Since this does not happen, in order to fulfil the teacher's expectations the student can only either stop interacting at all or reduce his participation.

In fact, Giovanni only partly fulfils the teacher's expectations. He does not fully give up his epistemological view because in line 19 he paraphrases the expected answer "infinite" into "infinite points". This is an indicator that Giovanni thinks of the points on the graph that grow towards infinity but not of the x-values as the teacher refers to them. Since the teacher's personal epistemology does not allow acceptance of the student's epistemological view as legitimate in the task at hand, and the student does not have enough means to enter into the teacher's epistemological view, he cannot get involved in the social interaction about the task deeply enough. Therefore the interest-dense situation cannot develop suitably.

Theoretically, we can say that epistemic actions emerging in social interactions are based on the epistemological view of the interlocutors. When an epistemological gap between students and teacher occurs, the epistemic process can only proceed and the emergence of interest density can only be supported if this gap is bridged, either by the teacher or by the students.

11.5 Reflection and Conclusions

Going from a coordination of two complementary analyses to a local integration of a new theoretical construct into both approaches was possible for this networking process since IDS and APC share many *common features*:

- The view on data: in both cases, the data concerns processes of teaching and learning mathematics in regular classrooms. Even if we investigate them in teaching experiments we assume that the students act according their everyday practices (methodology).
- The unit of analysis: both theories use micro-perspectives taking into account every single utterance or semiotic action (methodology).
- Both approaches are "transformation oriented" in the following sense. Ulich (1976) distinguishes between two paradigms: *stability-oriented* and *transformation-oriented*. In a stability-oriented paradigm, the objects of investigation are of a stable nature and can therefore be investigated separately from their constitution. In a transformation-oriented paradigm, objects are regarded as dependent on their constitution, and they can only be investigated looking at their processes of creation. Results in a transformation-oriented paradigm are, for example, patterns of constitution. Our paradigms are transformation-oriented, since we look at changes and are interested in the patterns of change. The epistemological gap is a pattern that is constituted within the current situation through teacher–student interaction.
- Both approaches focus on the students' actions and interactions with each other and the teacher, with respect to the evolution of their mathematical ideas.

Reflecting on the networking activity carried out in this case study, we can refer to Radford's quadruplet (Radford 2008, 2012) [(P, M, Q), R]. The separate analyses done by means of the IDS- and APC-theories (P1, M1, Q1) and (P2, M2, Q2) brought similar results to the fore: learning was not successful since the emergence of interest density was interrupted (R1: result 1) and the semiotic game was different from those in the successful cases (R2: result 2). By contrasting the two theories in the analyses, the idea of an epistemological gap appeared. The analyses seemed to complement each other. Therefore, we worked out a common coordinated analysis, locally integrating tools of the two theoretical approaches on the methodological level (M), that is, *micro-analysis of the video encompassing both verbal and non-verbal dimensions, and locutionary and non-locutionary ones*. A more consistent understanding

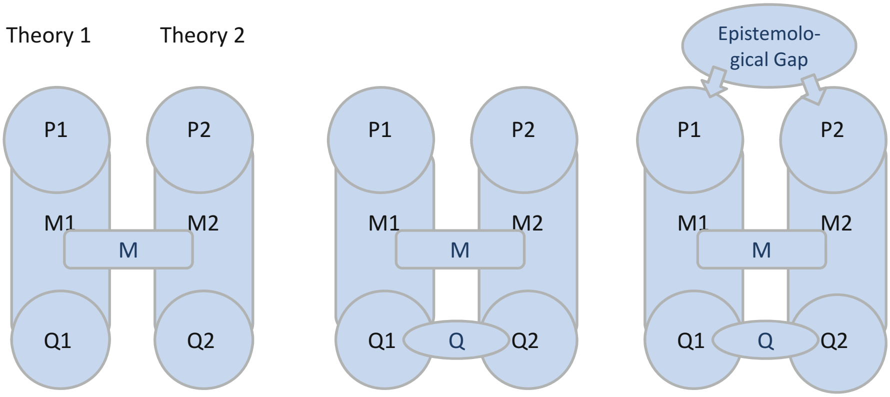


Fig. 11.3 Using Radford's (2008) categories to describe the process from coordination to local integration [*P* = Principles; *M* = Methodology; *Q* = Questions]

of the teacher–student interaction came out of this, indicating that the explanatory power that was lacking in our separate analyses could be provided in the common analysis by answering the common question Q, why the construction of knowledge was not successful. This made us include an epistemological dimension of the activity with the conjecture that there seemed to be an epistemological gap between the teacher's and the students' behavior in their interaction (Fig. 11.3).

Starting from a working definition about what is meant by the epistemological gap, a common re-analysis of the data was worked out that answered our question. The explanatory power of the new concept on the one hand became a common result R (Radford 2012) of the networking process and on the other hand initiated a spiral process of mutually improving the theoretical understanding of the concept of the epistemological gap and the empirical understanding of the extra video.

Finally, it was theoretically investigated how this new concept fits into the two sets of principles. In fact both theories carefully analyze relationships between different aspects of students' and teacher's actions and productions in the classroom: IDS considers mainly the different levels of discourse (locutionary, illocutionary and perlocutionary), while APC studies the relationships between the different semiotic productions in the classroom through the semiotic bundle lens. Both theories consider the dynamical and reciprocal evolution in time of such components and point out their possible convergence in dramatic moments, when they deeply interact possibly producing new knowledge: for example, when the situation becomes highly interest-dense or when a semiotic game is successful.

The joint analysis through both theories underlines that the dynamics of the construction of new knowledge can be successful provided the different discursive and semiotic components synchronize and converge. We can use a metric metaphor to describe this process: it is as though the mutual "distances" between the different semiotic and discursive components diminished more and more. This can happen if the students are in a zone of proximal development with respect to the piece of knowledge to be built. In such cases the actions and productions of teacher and students converge towards a shared knowledge, which is built up through a progressively shared epistemological basis. But sometimes this convergence process does not happen, for example in cases when a common epistemological basis is missing. Then there is an epistemological gap between the actors, and the process of building new knowledge is broken. As already pointed out, such a gap can be properly grasped only through the coordinated analysis of the two approaches, namely through extending the discursive analysis of IDS towards the APC model.

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