Chapter 10
Context, Milieu, and Media-Milieus Dialectic: A Case Study on Networking of AiC, TDS, and ATD

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Abstract The case study of context, milieu, and media-milieu dialectic is a networking case that links TDS, AiC, and ATD. It compares and contrasts three concepts and their status within each theory, in order to learn how concepts which at a first glance seem to have a similar role in the understanding of teaching and learning in each theory differ.

Keywords Networking of theories • Context • Milieu • Media-milieu

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10.1 Introduction

In this chapter we analyze a case study of networking between TDS, ATD, and AiC. As observed in the previous chapters, the foci of the three theoretical approaches are different. In particular, AiC focuses on the learner and his or her cognitive development, while TDS and ATD focus on didactical systems. The three theoretical approaches are sensitive to issues of context but, due to these differences in focus, context is not theorized and treated in the same way. In the next sections, we explain how context is theorized in each of the three theoretical approaches and show some consequences for the analysis of the video episode. We might expect some complexity in the effort of creating a dialogue between the three theories in relation to constructs such as context, milieu, and media-milieus dialectic. However, this case study has its own characteristics. We will observe how the dialogue between the three theories appears as a progressive enlargement of the focus, showing the complementarity of the approaches and the reciprocal enrichment, without losing what is specific for each one.

In Sect. 10.2, we explain the notions of context, milieu, and media-milieus dialectic. Section 10.3 offers a first classification of similarities and differences between the three theories. Separate analyses are presented in Sect. 10.4. A dialogue between AiC, TDS, and ATD with regard to "context," "milieu," and "media-milieus dialectic" is described in Sect. 10.5 and concluding remarks are discussed in the last section of the chapter.

10.2 The Notions of Context, Milieu, and Media-Milieus Dialectic

In this section we explain the meanings of the terms context (for AiC), milieu (for TDS) and media-milieus dialectic (for ATD), each of them being a cornerstone for the theory.

10.2.1 What Is "Context" for AiC?

As explained by Dreyfus and Kidron in Chap. 6, the nested epistemic actions model for abstraction in context is apt for describing processes of abstraction in their specific context. The contextual factors that may influence a process of abstraction include the physical setting, the tasks on which learners work, and the tools (such as paper and pencil or computers and software) that are available to them. They also include students' personal histories and previous constructions of knowledge.

Furthermore, any process of abstraction takes place in a particular social setting and thus the context also includes social relationships among students and between students and teachers. As a consequence, context becomes an inseparable component of the process of constructing knowledge because students act in a manner that

seems appropriate and relevant to them in the given context. The role of context is crucial in learning processes and the complexity of learning processes goes back, at least in part, to the contextual influences on the learner's construction of knowledge. Hence, we believe that a better understanding of the role of context is likely to lead to a better understanding of learning processes. Some parts of the context have a *dynamic nature*: the learner interacts with the context. This may be the case for social interactions or interaction with the computer. The influence of contextual factors on the process of construction of knowledge is an object of analysis with the AiC lenses, especially the influence of context on the epistemic actions (see Chap. 6). For example, the relations between the learner and the computer as a dynamic partner were analyzed in Kidron and Dreyfus (2010). The study describes how the integration of knowledge structures was facilitated by the potential offered by the computer and the learner's ability to make sense of the resources offered by the computer.

The influence of social interactions on processes of abstraction has already been analyzed by Dreyfus et al. (2001). The authors have considered processes of abstraction in pairs of collaborating peers and investigated the distribution of the process of abstraction in the context of peer interaction. This was done by carrying out two parallel analyses of the protocols of the work of the student pairs, an analysis of the epistemic actions of abstraction, as well as an analysis of the peer interaction. The parallel analyses led to the identification of types of social interaction that support processes of abstraction.

10.2.2 What Is "Milieu" for TDS? How Is It Related to A-Didactical and Didactical Situations?

As explained in Chap. 4, the notion of *milieu* is attached to the vision of learning as an adaptation process and to the ambition of optimizing such a process. The milieu is defined as the system separate from any didactical intentionality with which the students interact in the *a-didactical situation.* In line with the idea of learning through adaptation, it should be a source of contradictions, or at least disequilibria, what is captured through the idea of *antagonist milieu*. However, the possibilities of action and feedback it offers should also make possible an evolution towards winning strategies, which lead to the construction of new knowledge. The milieu includes material and symbolic artifacts, and possibly other learners, depending on the social organization of the situation. Note that for interacting with the milieu, learners always need to mobilize some of their already constructed knowledge. Some but not all authors include this knowledge into the milieu.

One essential role of the teacher is to organize this milieu, but in TDS she is not considered herself as a component of the milieu. Organizing the milieu can mean: selecting the problems the learners will have to solve and fixing the values of their *didactical variables*, the way these problems are introduced and managed, the tools and means at students' disposal, and the social organization of the classroom.

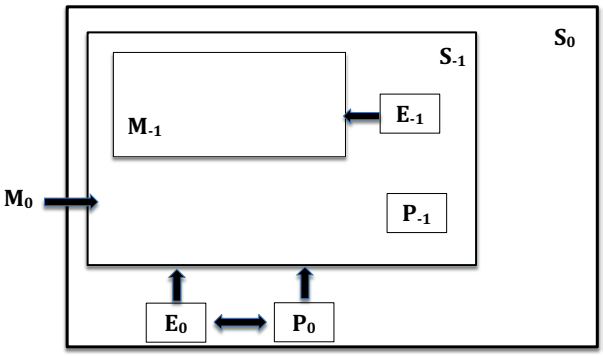

Fig. 10.1 Simplified schema of nested milieus

Milieu is a dynamic object. As long as students' interaction with it develops, new constructions emerge, new representations are built, and the milieu progressively enriches. Quite often, in classrooms, the teacher too contributes to this evolution of the milieu (called semiogenesis) by providing additional information and tools. The reason for this may be that the students do not mobilize previous knowledge as they are supposed to do, that they need some help in interpreting and benefiting from the interactions with the milieu, or that classroom constraints create the necessity of accelerating the dynamics of the situation.

The construct of milieu has been regularly discussed and reworked since its introduction in the 1980s (see for instance Dorier et al. 2002; Perrin-Glorian 1999; Brousseau 1990, 1997; Margolinas 1995), and one essential development has been its vertical and nested structuring. The networking process obliged us to enter into this vertical structure, in fact its negative levels, and more particularly into the levels S_0 and S_{-1} corresponding respectively to the didactical and a-didactical situations (see Fig. 10.1).

We briefly introduce this part of the structure. The simplified definition that we gave of the milieu above corresponds in fact to the level S_{-1}, and to the a-didactical milieu. In the a-didactical situation S_{-1}, students (E_{-1}) are modelled as learners interacting with the a-didactical milieu (M_{-1}), and the teacher (P_{-1}) is outside the system, in the position of an observer. In the nested structure, this a-didactical situation is itself the milieu of the didactical situation S_0. In S_0, the didactical intentionality re-emerges, the teacher acts as a teacher (P_0) and the student as a student (E_0), their interaction being regulated by the rules, mainly implicit, of the didactical contract. S_0 is the level of the structure where the knowledge developed through a-didactical interaction with the milieu is made explicit, partially decontextualized, and connected to the institutional forms of knowledge aimed at. As proved by many studies, this structure is very helpful for understanding the complex relationships between a-didactical and didactical processes in teaching and learning, resulting from the fact that learning is both an adaptation and an acculturation process as pointed out

in Chap. 4. In fact, the nested structure of the milieu includes more levels, both positive and negative, the situation at level n being systematically the milieu for the situation at level $n+1$. For more details, see Perrin-Glorian (1999) and Margolinas (1995).

10.2.3 What Are "Media," "Milieu," and "Media-Milieus Dialectic" for ATD?

In the Anthropological Theory of the Didactic, teaching and learning processes or, more generally, processes of study and inquiry, are described using a very broad frame, the Herbartian formula (Chevallard 2008, 2012), named after Johann Friedrich Herbart (1776–1841), a German philosopher and the founder of pedagogy as an academic discipline. The starting point of the process requires a question Q (not be mixed up with the research questions of a theoretical approach as used in Chaps. 4, 5, 6, and 7), a group of persons X with the project to study question Q, and a team of "study aides" Y which can be eventually empty. This induces the formation of a didactic system around question Q: $S(X; Y; Q)$ the functioning of which must lead to the production of an answer $A^{\heartsuit}$ (where the heart ♥ indicates that it is the answer given by X and Y; see Chap. 5) to question Q, a process represented as:

$$S(X; Y; Q) \hookrightarrow A^{\heartsuit}$$

This is the *reduced* Herbartian schema. To produce $A^{\heartsuit}$, however, $S(X; Y; Q)$ needs "materials"; these materials make up the *didactic milieu* M established by $S(X; Y; Q)$ and represented as follows in the *semi-developed* Herbartian schema:

$$[S(X; Y; Q) \rightharpoonup M] \hookrightarrow A^{\heartsuit}$$

In the didactic milieu M, it is customary to distinguish two main categories: on the one hand, M accommodates already existing and "labelled" *answers* $A^{\diamondsuit}$ drawn from available "resources" (including members of $X \cup Y$); and on the other hand, it may contain *other works* O which, among other things, can be theories, experiments, questions, brought into the milieu M by members of $X \cup Y$. The didactic milieu is represented generically as $\left[M = \left\{A_1^{\diamondsuit}, A_2^{\diamondsuit}, \ldots, A_k^{\diamondsuit}, O_{k+1}, \ldots, O_m\right\}\right]$; hence the *developed* Herbartian schema:

$$[S(X; Y; Q) \rightharpoonup \{A_1^{\diamondsuit}, A_2^{\diamondsuit}, \ldots, A_k^{\diamondsuit}, O_{k+1}, \ldots, O_m\}] \hookrightarrow A^{\heartsuit}.$$

Sometimes, the didactic system does not seem to be formed around a question Q but around a given praxeology P (usually designated as a "content" or a "piece of knowledge") that students X have to "learn" or "appropriate": it is what was called a *didactic stake* in Chap. 5. However, even if praxeologies can be considered on their own and in a decontextualized way, they always originally appear as the result of the inquiry of some questions arising in institutional settings, which give

praxeologies their rationale or raisons d'être. We can thus consider that the didactic system is formed around these questions, even if they are initially unseen by X (and even by Y), or around the questions: What is P? What is it for? How to use it? Etc.

The *didactic milieu* of the Herbartian formula can include an *a-didactic milieu* in the sense given by TDS, that is, a system of objects acting as a fragment of "nature" for Q, able to produce objective feedback about its possible answers without any didactic intention towards X. According to Brousseau (1997), there is no construction of knowledge without such an a-didactic milieu; there can only be *imitation*, that is, the reproduction of somebody else's answer. As in TDS, and even if the a-didactic milieu is usually given or produced by the teacher Y, this is not necessarily the case: the production and organization of an appropriate a-didactic milieu for Q is an essential aspect of the study process carried out by both X and Y. For instance, in scientific work (where X is a team of researchers and Y the leader(s) or supervisor(s) of the investigations), finding or creating an appropriate experimental milieu for the study of a given phenomenon can be one of the most challenging issues to tackle.

Related to the notion of milieu introduced by TDS, the main development brought by ATD is the following. Usually, in the construction of answer $A^{\heartsuit}$, using a set of objects without any didactic intention $\{O_j\}$ is not enough; it is also necessary to use other answers $A^{\diamondsuit}$(also called "cultural works") produced outside the didactic system as the results of other study processes carried out in different institutions to propose answers to different questions Q' more or less related to Q. In these cases, X and Y need to access these already-produced answers and they will do that through some *media*. In a larger sense, media refers to any means addressed to a certain type of audience presenting information about the world or a part of it: any media production in the usual sense (journal, paper, video, etc.); an essay or treatise; a lecture; an informal report or just a system of rumors; etc. Use of media can be considered as carrying out a *didactic intention* towards a given issue or question. To answer a question Q, one of the first things that can be done is to look around, in the media available, for possible existing answers to Q. The aim of the media is to present knowledge or information to others.

The *media-milieus dialectic* appears when considering the different kinds of general *didactic gestures* performed by X and Y in the interaction with M to produce $A^{\heartsuit}$. Both elements $A_j^{\diamondsuit}$ and O_k intervene in the media-milieus dialectic, which can be initially presented as follows. When starting the study of a question Q, a basic action is to look for already available answers $A_j^{\diamondsuit}$ either to Q or to a question Q' that seems related to Q. These answers are produced in different institutions which identified them by a "label" (this is the reason for the upper index $\diamondsuit$) and which are the last respondents of their validity. The answers are made available through different kinds of media: books, treatises, articles in journals or encyclopedias, videos, online resources, etc. However, $A_j^{\diamondsuit}$ are still not the definitive answer $A^{\heartsuit}$ to Q. For the moment, they can only play the role of "conjectures" or "elements of answer" to Q, needing to be patched together and validated in relation to Q. This is the role of the other objects O_k. The media-milieus dialectic corresponds to this continuous interaction between available (partial) answers given by the media and their testing through the interaction with an a-didactic milieu. Of course, already available

answers $A_1^{\diamond}$ … $A_{j-1}^{\diamond}$ are also part of this milieu once they have been tested and can be used as a fragment of nature for X and Y. It may be said that the media-milieus dialectic consists in contrasting previously available answers from the media to convert them into an experimental milieu (as "sure knowledge") and to work with the objects of the milieu so as to get new information from them (new knowledge to be tested), that is, to convert them into media. As we can see, the notions of "media" and "milieu" do not refer to a property of objects but to their use in the process of study of a given question Q. For example, a computer program can be used as a medium to get information about a given issue, or as a milieu to test a conjecture obtained by other means. When a person P asks another person Z a question, P can be using Z as a medium to obtain new information about the issue asked, or as a milieu, for instance to check that Z already knows the answer.

When looking at traditional teaching systems, the dialectic of the media and the milieus is very weakly balanced. There exist some activities where students overuse a small number of media (the teacher and the textbook, for instance) without feeling the need to test the validity of the information they get from them; while other activities (like a session of practical exercises, for instance) seem not to allow access to any extra media (other students' or other people's answers). This is a situation very different from specialist work, when any partial answer given is acutely searched and also continuously checked by as many means as possible.

10.3 Similarities and Differences: A First Classification

Definitively, "context," "media-milieus," and "milieu" do not mean the same thing in the approaches considered. A look at the AiC components of context shows that a computer program, a teacher, the web, as well as peers can all act as either media or milieus. Furthermore, there is an important difference in the way the questions concerning the contextual influences are formulated in the different theoretical approaches. The different ways in which the three theoretical approaches take the interactions with the context/milieu into consideration are mirrored by the different questions asked by the researchers. For example, in relation to the role of the teacher, TDS researchers might ask what milieu the teacher is making available to the students and how she is managing its evolution in order to establish a meaningful connection with the mathematical knowledge aimed at. AiC researchers might ask how the teacher's intervention influences the students' construction process as described by means of the RBC epistemic actions. ATD researchers in their turn might ask what responsibilities the teacher and the students are assuming in the media-milieus dialectics and what conditions enable them to manage it.

The three theories share some similarities in the central role assigned to the construction of mathematical knowledge in the analysis. For example, TDS and ATD look for conditions for a study process not to be reduced to a simple copy of previously elaborated answers. Also in AiC, abstraction is defined as an activity of vertically reorganizing previous mathematical constructs within mathematics and

by mathematical means so as to lead to a construct that is new to the learner (see Chap. 6). Nevertheless, the research questions posed in the three theories are different: AiC researchers are interested in finding out how a given path of thinking works, while TDS and ATD researchers will rather ask: what produces the "path" (the answer given by the students to the question Q), what makes it possible to happen, is it something from the milieu, allowing the students to work autonomously (in an a-didactic situation) or, on the contrary, is it something coming from the teacher or from some other media proposed by her? The three theories will also consider the possible paths that, even if virtually possible, do not really happen, and ask about the reasons for that. Nevertheless, there is a crucial difference: for AiC the emphasis is on the way students develop their answer, strategy, or "thinking"; for TDS and ATD it is on the conditions making this development possible and the restrictions hindering other possible answers or strategies.

10.4 Separate Analyses

As a first step, the networking efforts start by analyzing the episodes in line with the way each theory views the role of context: for AiC, the focus is on the influence of the context on the learners' process of constructing knowledge; for ATD the focus is on the media-milieus dialectic; while TDS researchers are interested on the potential and limitation of the milieu.

10.4.1 AiC: The Contextual Influences on the Construction of Knowledge

For the purpose of illustrating the contextual influences on the construction of knowledge, the AiC team decided to analyze the episode in which Carlo and Giovanni treat Task 3, investigating how the slope of the line tangent to the graph of the function $x \to a^x$ at the point of abscissa x changes with x. They focus on lines 249–379 of the transcript (see Appendix). The reason for this is discussed in the chapter on gestures (Chap. 9), namely, that this is the most complex situation and that one may expect constructing actions here as opposed to situations on Task 1 and 2, where mostly descriptive expressions of how the quantities under discussion behave might be expected. The worksheet which includes the task given to the students is described in Chap. 2 and a narrative summary of the students' work on Task 3 is described in Chap. 9 on gestures.

The first part of the AiC analysis consists of an a priori analysis of the learning situation. The aim of the a priori analysis is to identify the constructs that the students are expected to construct, while considering the task and the context. This a priori analysis will be followed by an a posteriori analysis of the learning event, using the RBC-model.

10.4.1.1 A Priori Analysis

The TDS researchers have proposed in Chap. 4 an a priori analysis in terms of constructs for the episodes on Task 1 and 2. We would probably largely accept their analysis into nine constructs C_1–C_9; however, they have not done a similar analysis for the episode on Task 3. As already mentioned in Chap. 6 on Abstraction in Context, the AiC team proposed the following intended constructs as being those intended by the designer/teacher to be constructed. We repeat them here for the convenience of the readers:

C_{10} For any given P, that is, locally, as Δx tends to zero, the slope of the secant tends to the slope of the tangent; the slope of the secants and the tangent are all positive (for $a > 1$).

C_{11} As P moves on the graph, the slopes of the corresponding secants (and hence the slope of the tangent) vary. As x grows (P moves to the right), the slope of the tangent grows (for $a > 1$). As x decreases (P moves to the left), the slope of the (secants and the) tangent decreases to zero (for $a > 1$).

C_{12} As a increases, the slope of the secant (for given x, P) increases (and consequently the slope of the tangent increases as well). As a decreases towards 1, the slope of the secant decreases towards 0. As a becomes smaller than 1, the slope of the secant (and consequently of the tangent) becomes negative; the function is decreasing rather than increasing. The parts of C_{10} that depend on $a > 1$ have to be adapted for $a < 1$.

While these intended constructs have been formulated on the basis of the tasks given to the students, they are compatible with the declarations of the teacher and they are also within reach of the students, given the previous knowledge of the class and the socio-mathematical norms that are characteristic of the class.

10.4.1.2 A Posteriori Analysis

In the a posteriori analysis the AiC team gives an account of constructions by the students, Carlo and Giovanni. As mentioned in Chap. 6 on AiC and in Chap. 9 on gestures, in addition to the (partial) constructing of C_{10}, C_{11}, and C_{12}, we expected the students to develop other constructs. We observed the following, which might be considered as an enlargement of the intended constructs observed in the a priori analysis:

C_{11}' As P gets closer to $y = 0$, the function can be approximated by the secant line.

C* The exponential function can be approximated by many small lines with an increasing slope that join together.

C* is a transition from a local to a global view.

The AiC analysis in the previous chapters on AiC and gestures did not explicitly focus on the different components of the context and their role in the process of constructing knowledge. This is what we propose to do in the following. We discuss the students' constructing actions, focusing on *the role of different contextual*

factors in the construction of knowledge: (1) the task, (2) the learners' personal history, (3) the computer software, (4) the teacher, and (5) the teachers' learning goals.

A first contextual element is *the task* itself and a first question is: why can the task make sense for the students? The AiC researchers were missing for their analysis a more specific task design. In Task 3, the task design does not require the students to answer specific questions but encourages them to explore, and to report on the exploration. On the other hand, especially in Task 3 the students appear to act with mathematical purpose; they vary quantities in a way that allows them to learn and construct new knowledge.

Secondly, the task also interacts with the *learners' personal history* and this part of the context is related to socio-mathematical norms. Giovanni and Carlo are used to exploring mathematical objects and situations, even when they are not given any specific tasks to carry out. In our experience, most students find it difficult to explore and even students who are used to exploring, and whose teachers have stressed exploration activities, might be lost in the situation as given by the teacher because there are only very general instructions; there are no tasks with clear goals, no questions. Trying to answer the question of how the task makes sense for the students, our first thinking was that this kind of situation in which students are asked to "explore and describe what is happening" could be "normal" when working with the Dynamic Geometry Software, at least in this classroom. Indeed, Giovanni and Carlo knew the software. Besides, they have earlier worked on the concept of function, as regards the numerical, graphical, and symbolic aspects. In other words, the reason why the task can make sense for the students is that appropriate conditions have been previously instituted.

One of these appropriate conditions concerns *the computer software* as the third contextual element. Interacting with the computer, the students vary quantities in a way that allows them to construct new knowledge. It enables them to use multiple graphical representations: specifically, this enabled them to carry out the transition from a static graphical view to a more dynamic graphical view. This transition is expressed in the following utterances:

287	C	look it… slowly… slowly it seems that… I do not know, like, saying, tangent
288	G	eh… yes
289	C	it seems that it touches it, let's go, let's go, let's go
290	G	eh yes… here
291	C	slowly… slowly
292	G	it's tangent

The students vary Δx and as a consequence of their interaction with the computer, a dynamic view of the secants turned out to be the emergent tangent. This is expressed in line 287. At this first stage, the students do not refer to the notion of slope, but rather to the geometrical objects that they can see, that is, the secants which become tangent. However, their view might be connected to the following part of C_{10}:

C_{10} For any given P, that is, locally, as Δx tends to zero, the slope of the secant tends to the slope of the tangent […]

We might define this construction as "the geometric representation of the derivative as a tangent." We see it as a local view and we will refer later to the transition from this local view to a more global view. In line 293, we observe the potential offered by the computer allowing the learners to check a new idea:

293 C if instead you make the contrary, increasing, increasing the differences

In line 316 the interaction with the computer facilitates the construction of

C_{11}' As P gets closer to $y=0$, the function can be approximated by the secant line.

316 G Eh, ok, when the P it's very close to the 0, the line that passes through Q and H represents more and more the function… the smaller it is [moves the right hand on the table, first showing something like a decreasing interval and then circling]

Construct C_{11}' is still fragile but continuing the work and interacting with the computer enables the students to consolidate fragile knowledge by means of checking.

By interacting with the computer, a previous fragile construct is consolidated. This was the case concerning the process of construction of C_{11}' as quoted by one of the students, Giovanni: "…if the point P is very near to zero, this line approximates very much the exponential function" (in line 368). The following utterances demonstrate how the language becomes progressively more and more precise and this fact shows the consolidation of C_{11}':

331 G the nearer P is to y equal to zero, the more this line approximates the function
368.2 G Then, if the point P is very near to zero, this line approximates very much the exponential function. Also here even if numbers are very small, it increases not so much, hence like a line…

It might be of interest to note that the consolidation of C_{11}' has been discussed in the gestures chapter (Chap. 9) in terms of catchment. On the other hand, an analysis is never complete; specifically, the AiC analysis in Chap. 9 ignored, to a large extent, the influence of context. In the present discussion, we refer to the influence of the computer context on the consolidation process.

In line 349, we observe the construction of C* – a transition from the previous local view observed in lines 287–289 ("the geometric representation of the derivative as a tangent") to a global view: the exponential function can be approximated by many small lines which have an increasing slope. This is expressed in the following utterances:

345 G ah, one can say… one can say that the exponential function becomes very little lines…
346 T uh… it could be approximated to some small lines, which however…
347 G that is with increasing slopes that join together in a…, that touch each other in a point
348 T therefore you are imagining to approximate with many small segments
349 G well, if you take it… I don't know, if you take it with a very large zoom… you can approximate it with many small lines

Since the teacher was involved in this part of the transcript, we will analyze *the role of the teacher* as the fourth contextual element influencing the students' constructing process. AiC analyzes the role of the teacher as a part of the context in order to identify how the RBC actions might have been influenced by the role of the teacher. AiC researchers would see semiotic games as an integral part of the teacher's actions and the teacher's actions as an integral part of the context. AiC researchers ask how the teacher's intervention influences the students' construction and consolidation process as described by means of the RBC epistemic actions. RBC lenses reveal the influence of the teacher on the students' construction of knowledge.

The interview with the teacher supports this view. In the interview, the teacher was asked how he decides to get involved with a pair of students:

> I enter in a working group if the students call me. Sometimes I enter in a working group if I realize that students are stuck. Other times I enter because I realize that students are working very well and they have very good ideas that need to be treated more deeply. Obviously the type of things that I do vary with the situations, but a constant is that I try to work in a zone of proximal development. The analysis of video and the attention we paid to gestures made me aware of the so-called "semiotic game" that consists in using the same gestures as students but accompanying them with more specific and precise language compared with the language used by students. The semiotic game, if it is used with awareness, may be a very good tool to introduce students to institutional knowledge. (Answer to question 4, Sect. 2.2.2)

In order to better analyze the role of the teacher as a contextual influence, we will take into account, as the fifth contextual element, *the teacher's learning goals* which we consider as a part of the context as well.

Knowing the intended constructs makes a difference in the analysis. The interview with the teacher shed new light on the contextual influence on the different modes of thinking. It offered a kind of a priori analysis exposing the expectations of the teacher at the different stages of the teaching learning experience. The teacher (Chap. 2) explained the "raison d'être" of the entire project:

> Engaging students in knowledge building, settlement, reorganizing and communicating, thus providing the teacher tools for obtaining information not only on the products, but also on the cognitive processes, necessary for any serious evaluation escaping the chimera [i.e. wrong idea] of objectivity. (Answer to question 17, Sect. 2.2.2)

This view is appropriate to the RBC+C lenses which focus on process aspects of construction of the knowledge constructs rather than on outcomes. The teacher focuses on the cognitive processes of the learner and this is appropriate to RBC analysis which focuses on the learner. Nevertheless, we will point to some differences between the teacher's expectation of the use of the semiotic game and the RBC analysis of the cognitive process of construction of knowledge.

We mentioned earlier that, in line 349, we observe the construction of C*: a transition from the previous local view observed in lines 287–289 ("the geometric representation of the derivative as a tangent") to a global view: the exponential

function can be approximated by many small lines which have an increasing slope. This transition is mentioned in the teacher's answer to question 9 where he was asked how he planned the lesson:

> …the third worksheet gives a local and a global approach to the exponential function thanks to the construction of the derivative of an exponential function. (Answer to question 9, Sect. 2.2.2)

The teacher's expectation of the use of semiotic games is directed towards a future opportunity for the students to better understand the formal view:

> When I tell about the formal aspects of the derivative I often make some reference to these experiences and activities. It seems to me that also a lot of students are able to make these connections to give meaning to formal aspects. (Answer to question 13, Sect. 2.2.2)

The previous local view observed in lines 287–289 ("the geometric representation of the derivative as a tangent") corresponds to what the teacher calls LOCAL – a little segment that approximates the function locally.

Let us consider the GLOBAL view which is expressed in C* – a transition from a local view to a global view: the exponential function can be approximated by many small lines which have an increasing slope. This view as expressed in the RBC analysis is different from the teacher's view. For AiC, GLOBAL was the global approach to the exponential function – the envelope – but for the teacher LOCAL–GLOBAL is the transition from the local to the global aspects of the derivative. More precisely, for the teacher, the global is the function derivative compared with the derivative at a point as a local approximation. The transition from the local to the global aspects of the derivative will be reached by recognizing the characteristics of the derivative function (itself an exponential). The teacher expressed his aim:

> …to pay attention to the slope of the little segments, because their slope gives information on the growth of the function. (Answer to question 5, Sect. 2.2.2)

The AiC researchers observed the students' construction of the view: an exponential function can be approximated by a sequence of tangent line elements… "well, if you take it… with a very large zoom… you can approximate it with many small lines…" as expressed by Giovanni in line 349. The AiC researchers observed a global view of an envelope in addition to an idea of how the slope increases. There is no expression by the students of the expected construction by the teacher that the slope has an exponential growth. In the following the teacher expressed this expected construction:

> My aim is to induce the students to reflect on the fact that it is important to pay attention to the slope of the little segments, because their slope gives information on the growth of the function. Giovanni says "it is twice the previous slope…" I, using his same gesture, say more precisely that "the slope has an exponential growth." (Answer to question 5, Sect. 2.2.2)

This view was expressed by the teacher in his intervention, as observed in the transcript, BUT not by the two students. The AiC researchers did not find any indication of such a construction by the students.

10.4.2 TDS Analysis of the Potential and Limitation of the Milieu

10.4.2.1 A Priori Analysis

The a priori analysis of the potential of the a-didactical milieu for this episode on Task 3 is coherent with the a priori analysis piloted by AiC: the interaction with the software file guided by the text defining the task can reasonably lead to the conjectures mentioned in this a priori analysis. We complement it below by some elements which explain why, in our a posteriori analysis of this situation, we will focus on the part during which the teacher directly interacts with Carlo and Giovanni.

It is indeed interesting to note that, compared with the Tasks 1 and 2 guiding the work with the two first files, the text of Task 3 is longer and provides substantial information. For instance, it explicitly mentions the notion of tangent (recalling its status of best linear approximation around a given point) and explicitly associates it with the decreasing of Δx towards 0. The transition from a local to a global perspective on the derivative is carefully detailed and supported by the introduction of the specific functional notation $m=m(x)$. The task description also mentions the possibility of using other software and it is reasonable to think that the teacher has in mind the software Graphic Calculus which has already been used in this class for exploring polynomial functions, their tangents, and their derivatives in a similar way. We interpret this as a sign that the teacher thinks this help is necessary in order for the students to engage in a productive interaction with the material milieu (here the file), and especially use it for moving from a local vision to a global vision on the derivative.

In fact, in his answer to Question 12, the teacher shows that he is perfectly aware of the difficulty of such a move, and of the limitation of the a-didactical interaction with the milieu for achieving it. In the interview, he structures his expectations into four different levels. The transition from a local to a global perspective on the derivative corresponds to the fourth and last level and he points out that:

> Generally, from the third level, the understanding happens only thanks to the direct intervention of the teacher in the small groups and this understanding is consolidated in the mathematical discussions guided by the teacher with the whole class. (Answer to question 12, Sect. 2.2.2)

The teacher also expresses his expectations that the interaction with the file will lead students to conjecturing that the derivative of an exponential function is also an exponential function:

> The aim of the DGS file was to make the students understand that an exponential growth is directly proportional to the value of the function itself. This is an important step in understanding why the derivative of an exponential function is still an exponential function of the same base. (Answer to question 7, Sect. 2.2.2)
>
> With this activity, with the help of Cabri, I wanted the students to understand that exponential functions are functions for which the growth is proportional to the function itself. In other terms, the derivative of an exponential function is proportional to the function itself. This consideration, in my opinion, should allow students to understand why the exponential function a^x with a greater than 1 grows with x faster than any power of x. (Answer to question 8, Sect. 2.2.2)

This is of course the fundamental characteristic of exponential functions, but examining the file and the precise questions posed to the students in the task, we consider that such conjectures, contrary to those mentioned above, are quite unlikely to result from mere interaction with the a-didactical milieu in this episode.

Our a priori analysis leads us thus to conjecture the risk of a gap between the teacher's expectations and the potential of the a-didactical interaction with the milieu, and thus to pay specific attention in the a posteriori analysis on the strategies that the teacher uses for going beyond these limits, if this conjecture turns out to be true.

10.4.2.2 A Posteriori Analysis

The exchanges before the intervention of the teacher show the students undertaking exploration, following the guiding text, observing increasing lengths of HQ as P moves to the right and that the line (PQ) better approximates the function when P moves to the left. They try to make sense of their observations, but the expression and argumentation is rather fuzzy, and utterances are not so easy to interpret. Carlo also seems to conjecture that PQ is constant, which of course is not mathematically the case but can appear so when HQ is small with respect to HP, that is to say when P is close to the x-axis, and this is the first question he asks the teacher when he joins the group. But Giovanni contradicts him, moves to the approximation result mentioned above, and then to what happens when Δx decreases to 0, articulating that the line becomes a tangent. Carlo adds that PQ is decreasing, showing that his attention is still on PQ, but the teacher tries to orient the discussion in a more productive direction:

342	T	oh sure, it is almost trivial, isn't it? Therefore he was saying that this line tends to become…
343	G	tangent
344	T	and then what kind of information will it give you in this case?

Thanks to the answer provided by Giovanni in line 345 ("one can say that the exponential function becomes very little lines") and the gesture accompanying it, an episode can start in which the teacher–students interaction allows the students to increase the cognitive benefit of their a-didactical interaction with the milieu. The teacher rephrases Giovanni's utterance in line 346 ("it could be approximated to some small lines, which however…"), and Giovanni follows in line 347: "that is, that… with increasing slopes, that join together in a, that touch each other in a point." After consolidating this first achievement, the teacher asks in line 350: "and such lines which features have they?" Once again, Giovanni's answer offers an opportunity for going further as it introduces the idea of constant ratio between the slopes of successive segments which, appropriately worked out, could lead to the property that the derivative is also an exponential function as aimed at by the teacher in line 351: "they have… well, they may have a function, a slope are, possibly always

twice than before." But, this time, the teacher does not jump in it with something like: "well, I don't know if the slope is twice, but… in any case… their slope increases, does it?" (line 352), but rather comes back to the growth ratio of the exponential function itself and to the compatibility of this property with the observation that the function crushes on the *x*-axis. His interaction with other groups mentioned in line 365 may have contributed to this orientation of the interaction.

It is interesting to point out that in his answer to Question 5, the teacher mentions this episode as especially interesting, explaining:

> I use a gesture used before by Giovanni. This gesture is towards a little segment that approximates locally the function and I ask: "What is the characteristic of this segment?" My aim is to induce the students to reflect on the fact that it is important to pay attention to the slope of the little segments, because their slope gives information on the growth of the function. Giovanni says "it is twice the previous slope …" I, using his same gesture, say more precisely that "the slope has an exponential growth." At the minute 54 and 24 seconds, I help the students to remember that the characteristic of the exponential successions is that of having the ratio of two consecutive terms constant. Immediately after, I ask the students: "Are you surprised that the graph of the function is so close to zero for small x?" Giovanni, at the minute 55 and 28 seconds says something like "with number smaller and smaller, I have number smaller and smaller." I reword this idea with a more precise language. In the following dialogue, Giovanni and Carlo are able to explain in a comprehensible way the reason why the graph of an exponential function of base greater than 1 is so close to the x-axis for x less than 0 and explodes for high values of x. (Answer to question 5, Sect. 2.2.2)

Thanks to this answer, we access the didactical technique (semiotic game) he consciously uses in this episode. From the perspective of TDS, this episode is quite interesting. The a posteriori analysis confirms the limitation of the a-didactical milieu anticipated in the a priori analysis. However, it also shows a specific technique used by the teacher for compensating this limitation. In the didactical situation S_0, through the technique of semiotic game, the teacher succeeds in extending the outcomes of the students' a-didactical interaction with the milieu in S_{-1}.

In actual classroom situations, even when tasks are carefully designed for fostering learning through adaptation, limitations such as those observed here are frequent. Research shows that in such cases, teachers' actions are not necessarily as productive as is the case here. On the contrary, they often degenerate into didactical phenomena such as Topaze effects which just maintain the fiction that students have learnt what they were supposed to learn (see Chap. 12). Semiotic games thus appear as a didactical technique which can be used for linking in a productive way the a-didactical and didactical levels of classroom situations, and extend in a didactical phase the potential of a-didactical interaction.

10.4.3 ATD Analysis of the Media-Milieus Dialectic

Let us analyze the episode under consideration using the Herbartian formula and the media-milieus dialectic provided by ATD. We can consider that the question Q in the Herbartian formula – according to the instructions in Task 3 – is the study of the "features of the graph of the function $y=m(x)$, where m is the slope of the line tangent

to the function $y=a^x$ at the point of abscissa x" or, in other words, "how the slope of the line tangent to the function $y=a^x$ at the point of abscissa x changes as x changes" (quoting Task 3 from Fig. 2.3). In this episode we can observe the functioning of a didactic system formed around this question Q and where X is a couple of students (Giovanni and Carlo) and Y is the teacher. The Herbartian formula now leads to the question: what are the elements $A_j^{\diamond}$ and O_k intervening in the study process and how is the media-milieus dialectics managed by both the students and the teacher?

First of all, let us notice that the answer $A^{\heartsuit}$ the students are asked to provide – an explanation about the behavior of $y=m(x)$, that is, a piece of a "technology" in the ATD sense – is rather difficult to validate experimentally. Thus, what is the status we can give to the file prepared by the teacher in a way that, according to him, it may help the students with their explorations? The students are asked to interact with the files and extract some information about the "features" of the observed graphs. In a sense, they have to *read* or *interpret* what they see on the screen through some manipulations tightly specified by the teacher. We can thus consider that the files act as media, presenting – even if it is done in a quite hidden way – some previously elaborated answers $A_j^{\diamond}$ and which deliver partial information about the question Q at stake. Conjectures $C_{10,}$ C_{11}, and C_{12} indicated above are part of the answers that could be extracted, even if, as shown by TDS analysis, the media provided seem to have some limitations if the students are left alone with it. However, these limitations do not constitute any constraint to the didactic process since this is carried out by the students *and the teacher*. The interventions of the teacher and his use of the semiotic games can thus be considered as part of the didactic gestures necessary to deal with the media provided by him.

Of course, even if it occasionally requires some help from the teacher, students need to know how to "read" the files in order to obtain the information requested. It is during this "reading" that the media-milieus dialectic starts running, as the students contrast the information on the screen with some other previous knowledge they have about function graphs, exponential functions, and growth variation. These are the objects O_k of the *a-didactic milieu*, the objects that are already available and the existence of which is stable enough to act as a fragment of nature. The following sequences show how new observed properties are contrasted with some previously known features, that are thus acting as an a-didactic milieu:

255	C	well… P moves on the graph
256	G	yes, and also a
257	C	a is the rate of growth
258	G	perfect
…		
303	C	but like… try to put it $a=1$, it must result
304	G	a line…
305	C	$a=1$ we know it already… than you must do less than 1
…		
311	C	ok, so ok, ok, so ok, because if it means that they increase, the more you move them over there, it increases very very much
312	G	yes
313	C	because it's an exponential function

When the media-milieus dialectic stops being productive, the students search another medium. The classic didactic gesture in this case is to ask the teacher, to use him as one of the media. Then a new media-milieus dialectic starts running as the teacher does not limit himself to providing new information; he also intervenes to validate or question the answers found by the students, refusing to act as a medium and returning the question to the students, as in the following interactions:

327 C we wished, practically, is there always the same distance between P and Q?
328 T always the same distance?
329 G no no, it decreases
…
351 G they have… well, they may have a function, a slope are, possibly always twice than before
352 T well, I don't know if the slope is twice, but… in any case… their slope increases, does it? In this case, when this function increases

Another form of the media-milieus dialectic takes place when the teacher mentions the result obtained by another group:

365 T yes, the other group have used a very good example: if we take 10 % of 5 cents it is 0,5… it doesn't exist, isn't it? It is as it did not exist; if we take 10 % of 5 million euro on the contrary thing start changing, isn't it? It is a considerable amount of money… here the hypothesis are the same… and it is ok; now you go on in this way. Where have you arrived?

Here the teacher is using the answer provided by another group as a way to validate Giovanni and Carlo's proposal: when there is no experimental milieu, as is the case here with the conjecture provided by the students (the *explanation* of how $y=m(x)$ grows), there is always the possibility of contrasting the conjecture with different media and seeing if the different answers provided are coherent with each other. This is what we can call the "contrast between media" strategy, a very usual form of the media-milieus dialectic, currently used by scientists, journalists, and also students, to check their results.

Finally, we can find at the end of the protocol some of the materials that will compose the final answer $A^{\bullet}$ provided by Giovanni and Carlo, a partial result of the didactic system S (*Carlo & Giovanni*; *Teacher*; Q) that will supposedly be later on incorporated into the answer of the larger didactic system S(*Whole class*; *Teacher*; Q):

368.1 G that if the x increases again, the line passes through P and Q and is almost constant, it becomes almost a tangent… this because if we take a very big zoom we can approximate the exponential function with many lines, which have an increasing slope…
368.2 G Then, if the point P is very near to zero, this line approximates very much the exponential function. Also here even if numbers are very small, it increases not so much, hence like a line… and then we can write that we were waiting for it even if the ratios are constant at the beginning… it was almost a line… [not understandable]

369 C hence we write that it is a graph with a constant rate of growth, of a… of a if x is always the same… [not understandable] but the y's…

…

379 C it is ok! Otherwise it had no sense… that maintaining PH constant and therefore also the Δx's constant we notice that.. [not understandable] while P increases, P increases more and more, that is the Δy's increase; they increase more and more

It is interesting to notice that, in the episode observed, the class works in a complete autarchy regarding outside media: the teacher brings the information into the class, through the files and his own knowledge; the students are expected to obtain all the desired information only with the means they are given and their previous knowledge. We do not know if, after the work done during the observed session, some other external media are being consulted (such as mathematical books, encyclopedias, internet files). The traditional functioning of our current mathematical teaching systems shows a tendency to avoid these types of media and limit the work to the information provided by the teacher or "extracted" by the students. It implies an important loss in terms of the elaboration of strategies to validate the answers presented by these media and, as a consequence, a tendency to take the traditional media (teacher, lesson notes, and textbooks) as previously granted and without any need to be contrasted with a milieu.

10.5 Dialogue Between AiC, TDS, and ATD with Regard to "Context," "Milieu," and "Media-Milieus Dialectic"

10.5.1 Different Analyses, Different Priorities

The three analyses provided in the previous sections illustrate the differences between the three theories as observed in our first classification in Sect. 10.2 as well as the shared epistemological sensitivity. The three analyses demonstrate how these similarities and differences are practically expressed in the analysis of the episode and therefore allow the dialogue between the three theories.

In order to better understand the dialogue between AiC, TDS, and ATD with regard to "context", "milieu," and "media-milieus dialectic" we come back to the different priorities of the theories with regard to the focus of analysis. For example, AiC researchers focus on the learner. This is not the case for TDS researchers, as expressed in Chap. 4:

> Even if TDS has the ultimate goal of improving students' mathematics learning, the learner is not at the center of the theory. TDS gives priority to the understanding of how the conditions and constraints of didactical systems enable or hinder learning, and how the functioning of such systems can be improved. (Sect. 4.1.2)

And a similar position is adopted by ATD researchers who consider a larger environment of conditions and constraints for the evolution of didactic systems. Since in AiC the focus of analysis is on the learner, all other factors such as the task,

the computer, the teacher, and the learning goals are considered as contextual factors. Therefore, the notion of context for AiC is especially wide since it includes the external world of the learner and part of his internal world. This might include notions which are not necessarily considered as part of the milieu for TDS or ATD. This reflection is well illustrated in the differences in the a priori analyses of TDS and AiC. The AiC a priori analysis is concerned with the learner's intended constructs. The TDS a priori analysis included already a conjecture on the role of a gap between the teacher's expectation and the potential of the a-didactical interaction with the milieu. It seems that for TDS researchers the "context" is already taken into account and is structured already in the a priori analysis.

As a consequence, the AiC researchers learnt the importance of the TDS a priori analysis but also the fact that some excerpts might add direct knowledge to the analysis of the cognitive processes which might be missed if one focuses first on the cognitive processes and only then analyses the influence of other parts of the context.

In addition, some insight is offered while reflecting on the question: what can another theory (semiotic games) offer to the three existing theories (AiC, ATD, and TDS) in terms of insight regarding relationships between the existing theories? In the previous section we observed that the AiC analysis of the role of the teacher demonstrates that the students' construction of knowledge was not as expected by the teacher. The AiC analysis points to the limitations of the semiotic games. This is in accord with TDS and ATD discussion on the limitation of the milieu.

The three theories agree therefore on the limitation of the semiotic games and the limitation of the milieu. Nevertheless, the insights offered by each theory are different and we may say that they complement each other. We explain this complementarity as follows.

AiC offers a fine-grained analysis of the students' epistemic processes and makes subtle evolutions visible in the process of construction of knowledge. TDS and ATD offer to AiC the benefits of a more systematic engagement in a priori analysis for anticipating the possible effect of contextual characteristics on epistemic actions. TDS and ATD observed the entire situation from the beginning. Both support the AiC a posteriori analysis. This is done by means of the analysis of the role of the teacher, first as an observer in the a-didactical situation and then as an active actor exploiting the milieu provided by the a-didactical situation. In other words:

- TDS complements the AiC analysis in analyzing how the teacher extends the outcomes of the a-didactical interaction. The TDS analysis seems to start where the AiC analysis stops.
- This link between the a-didactical and didactical levels is offered by the ATD analysis as well. For ATD, the limitation of the milieu does not constitute any constraint to the didactic process since this process is carried out by both the students and the teacher. The ATD media-milieus dialectic permits taking into account the different ways one can use the context. This capacity is not offered by AiC nor TDS.
- AiC offers the possibility to discern which element of context leads to the conceptualization and to the construction of new meaning on the part of the students.

10.5.2 The Subtle Interaction Between Contexts and Theoretical Approaches

The complexity being addressed by the notion of context is well known. A first problem is that what is considered as a part of the context in one theory is not necessarily considered as that in another theory. There are different approaches towards context in different didactic cultures.

We can illustrate this comment by taking some different parts of context analyzed by AiC researchers and see how it fits for the TDS or ATD analysis. For example, taking the learner's personal history as part of the context according to AiC, one could ask how TDS and ATD take it into account. Concerning personal history, a distinction may be between the history of a specific individual student (AiC), a student with a typical history for a specific situation (TDS), and taking into account the institutional background of the whole process (ATD). The focus of each of the teams corresponds closely to the elements of the theory within which this team works. For example, in their a priori analysis neither the TDS researchers nor the ATD ones would take into account personal data concerning the learners' personal history. Even in the a posteriori analysis there is no description of each learner's individual trajectory of thinking. By the ATD approach, students are considered as "normal subjects" of the class, that is, of the didactic system $S(X; Teacher; Q)$. The focus is on the functioning of this system and its ecology: the conditions that make the functioning possible and the restrictions that hinder other possible evolutions. This study would need, however, some extra information about the teaching process in which the episode takes place, to know what type of tasks corresponds to the questions proposed, what kind of production is expected from the students, and what praxeological elements are made available to make this production possible.

In a similar way, some other elements of context could be considered, for example the teacher. Contrary to AiC, the teacher for TDS is neither an element of the context nor a component of the milieu: he is an actor. TDS is interested in relations between systems and the teacher is an element of the system. TDS does not theorize the context in itself; but through the different levels of the notion of milieu, characteristics of the context are progressively taken into account in the analysis, from those which are controlled by the teacher through the organization of the material milieu and selection of appropriate didactical variables and up to the conditions which influence design decisions outside the classroom. This can be linked to the distinction evoked in Chap. 4 between two different perspectives on didactical situations: a restricted vision of these elements as the student's environment organized and piloted by the teacher and a broader vision including further elements such as the teacher and the educational system itself.

Taking context into account supposes an enlargement on the unit of analysis considered at an early stage of development of research in mathematics education, where the focus was essentially put on the students and their knowledge development. From its beginnings, the TDS has gone beyond the simple consideration of the student and the mathematical activity. It started to consider *situations*, that is, problems arising in

institutional settings about something that is already there, the *milieu*. The construction of mathematical knowledge cannot thus be carried out *in a vacuum*, it needs to suppose the existence of something "external": a milieu in a situation. A given piece of reality can then correspond to different possible situations depending on what is considered as the related milieu. The notion of situation thus allows consideration of different types of activities concurring in the teaching and learning process (students doing mathematics, teacher helping students do mathematics, etc.), and different pieces of reality at different levels of generalization.

Also, for ATD researchers, when facing a given "piece of reality" such as an activity carried out in the classroom, the focus is on the institutional conditions making this activity possible, using for instance the different levels of codetermination (see Chap. 5). scale of levels of didactic codetermination emphasizes that the concrete actions of a teacher and a group of students in a classroom may depend on the mathematical activity and on the domain or sector where this activity takes place (a reasoning about functions and their variations does not need the same elements as a reasoning about random variation of a statistical variable, for instance). But it also depends on determinants related to how the activity is organized, for example in a school context, with a specific pedagogy (in the case considered, the students are used to interacting and discussing with the teacher, to using computers, and understanding that they are supposed to give an answer to the questions posed, etc.). And, as was stated in Chap. 5 on ATD, the existence of a task of the kind "look at the computer and tell something about what you see" seems related to a current social practice where new information seems to come from the direct observation of phenomena (and not, for instance, from the study of old books…). The hierarchy of levels of codetermination does not exist in other frames such as AiC or TDS. It is the approach of research problems in terms of the *ecology* of mathematical and didactic praxeologies that drives researchers to look for conditions and restrictions that go beyond the narrow space of the classroom (and even of the school). Compared with other approaches, ATD proposes a huge expansion of the "external" world that should be taken into account to explain why things happen the way they happen and, furthermore, why many other things that could happen never really happen.

10.6 Concluding Remarks About Networking Strategies

10.6.1 Proximity

A dialogue between different approaches can only start when a point of contact is found. In this case, we may talk about a common "epistemological sensibility" of AiC, TDS, and ATD, which can be noticed in the a priori analyses provided by each frame. These analyses are the starting point of the dialogue between the approaches and, in a sense, they seem to answer each other. For instance, AiC researchers refer to an analysis provided by TDS researchers about the episodes on Task 1 and 2 and they use and complete it in the episode on Task 3 in terms of an

analysis of intended constructs. The ATD researchers take a further step when they refer to a (hypothetical) teaching project used as a reference and enlarge the description of students' activities using the media-milieus dialectics. This initial proximity seems essential for a dialogue to start and become productive, showing the complementarity of the approaches and the reciprocal enrichment, without losing what is specific to each one.

10.6.2 TDS as "Mediator"

Another contextual condition which enhances the dialogue is that TDS is in an intermediate position, like a theoretical frame which permits to establish a bridge, a communication between AiC and ATD with their different foci which are very far from each other. The dialogue has then appeared as a progressive enlargement of the focus which, in a sense, is central in the notion of context. For AiC, with its focus on the learner, the context integrates any piece of the present and past environment that can influence the individual epistemic processes. For TDS, with its focus on situations, there is a kind of split between an explicitly theorized part (in terms of milieu and didactical contract) and a part that is not explicitly taken into account by the theory and that could also play the role of "context" as in AiC. With ATD, again the vision is enlarged since the whole teaching project is taken into account and the focus is on the conditions for a given didactic process to exist and evolve in a given direction.

In this theoretical dialogue on questions about "context," we should differentiate between the notion of media-milieus dialectic and the hierarchy of levels of codetermination. ATD's hierarchy of levels of codetermination permits enrichment of the theorization of TDS. Nevertheless, for what concerns the media-milieus dialectic, the situation is different for ATD and TDS even if both theories start their analysis with the same observation of the limitation of the milieu. In TDS analysis, we observe the view of a situation with an a-didactical potential. TDS analysis demonstrates the limitation of this potential and the need for an action from the teacher. The analysis shows how the teacher extends the outcomes of the a-didactical interaction. ATD analysis starts with the same observation of the limitation of the milieu but demonstrates an absence of a media-milieus dialectic which could have been expected and questions the reasons for this absence.

10.6.3 Different "Units of Analysis"

An interesting, and also revealing, point is the fact that, in the analysis, AiC researchers focus on the autonomous work of the students, while TDS researchers pay more attention to the episode where the students interact with the teacher, and ATD approaches the teacher's overall strategy. The AiC analysis shows the

richness of the knowledge constructed by the students during their interaction with the context and the role of the different elements of the context in this construction. As a counterpart, the TDS analysis pays less attention to the students' constructions. The TDS analysis is guided by the conjecture (coming from the a priori analysis) of the gap between the power of the milieu and the teacher's expectations, which is supposed to lead to an impasse where the milieu should be enriched in order to avoid a Topaze effect (see Chap. 12). We should also notice another reason why the TDS analysis pays less attention to the students' constructions: the difficulty for the TDS researchers to achieve a fine grained analysis of the first exchanges between the students and their constructing processes. AiC researchers with their analytical tools have no problem in carrying out this analysis. Nevertheless, their tools, in contrast to TDS tools, are less operational when the teacher interacts with the students. ATD researchers take a further step when they take the whole episode into account and focus on the aim of the didactic process and the strategies used by both the teacher and the students to make the needed praxeological ingredients available. Without this, ATD researchers are not able to give sense to the students' interaction, nor to their interaction with the teacher.

As we see, the theories have different foci and the potential of the analytical tools is different for each theory. These differences can explain the fact that each group of researchers can learn and evolve from the others' analyses.

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