Advances in Mathematics Education

Angelika Bikner-Ahsbahs
Susanne Prediger *Editors*

Networking of Theories as a Research Practice in Mathematics Education

Authored by the Networking Theories Group

Springer

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Series Preface

The present volume of *Advances in Mathematics Education* examines a heavily debated topic in mathematics education, namely that of theories, theoretical frameworks and ways in which they are deployed in existing research. Given the heterogeneity of theoretical frameworks used in mathematics education today compared to the psychometric paradigm of the 1960s, which was firmly anchored in psychology, the current book examines how different theories can be made to network with each other and in particular inform researchers interested in analyzing their data from multiple perspectives.

The Networking Theories Group was initiated and coordinated by Angelika Bikner-Ahsbahs, with founding members Michèle Artigue, Ferdinando Arzarello, Marianna Bosch, Tommy Dreyfus, Ivy Kidron, Susanne Prediger, and Kenneth Ruthven in 2006. There were some forerunners to this group, such as the work of Hans-Georg Steiner in Germany and the PME research forum on Theories of Mathematics Education in Melbourne-2005, which led to the first volume in this series (Sriraman and English 2010). However in spite of these forerunners, the Networking Theories Group has been a consistent focus group in mathematics education, with intense work done on capturing the essence of data through the use of different theoretical lenses. The group formally established itself at CERME 2005 in Spain, and subsequently has held summer research meetings in the following years. A core group of researchers from the Networking Theories Group have also been involved in the working group on theories at the CERME congresses and has run various PME research forums on theories.

Given the substantial work of this group that was reported in a ZDM special issue on Comparing, Combining, Coordinating – Networking Strategies for Connecting Theoretical Approaches (Volume 40, Issue 2, 2008), based on a paper by Bikner-Ahsbahs and Prediger already in (2006), the mathematics education community has been eager to learn of newer developments within this group on how researchers can further utilize theories in advantageous ways. The present book may serve as basis for younger researchers who often indulge in bricolaging theories on an ad-hoc basis to construct theoretical frameworks that inform their work. Moreover the chapters in the book contain a diversity of perspectives that captures the current state of the art of networking theories in mathematics education. We are pleased to have this book in our series and thank the editors for producing what we hope will be a valuable resource for the community.

Hamburg, Germany — Gabriele Kaiser
Missoula, MT, USA — Bharath Sriraman

References

Bikner-Ahsbahs, A., & Prediger, S. (2006). Diversity of theories in mathematics education – How can we deal with it? *ZDM – Zentralblatt für Didaktik der Mathematik*, *38*(1), 52–57.
Sriraman, B., & English, L. (2010). *Theories of mathematics education*. Berlin: Springer.

Preface

How can we deal with the diversity of theories? This was the main question that led the authors of this book to found the Networking Theories Group with members from France, Germany, Israel, Italy, UK, and Spain. When the group first met at CERME 4 in 2005, the idea of networking theories arose: starting from the shared assumption that the existence of different theories is a resource for mathematics education research, we felt that the possibilities of connecting theories (without merging into one big theory) should be further explored. The group developed strategies for networking of theories and decided to investigate strands and issues of these networking practices empirically. From 2005 on, we met regularly at least once a year for commonly conducting empirical research and for reflecting the common practices on the level of theory and methodology. The Networking Theories Group was initiated and coordinated by Angelika Bikner-Ahsbahs, with founding members Michèle Artigue, Ferdinando Arzarello, Marianna Bosch, Tommy Dreyfus, Ivy Kidron, Susanne Prediger, and Kenneth Ruthven. Agnès Lenfant was a member during the first years, while further members joined the group in later years: Stefan Halverscheid, Mariam Haspekian, Cristina Sabena, Ingolf Schäfer, and, as latest member, Alexander Meyer. Meanwhile, Kenneth Ruthven changed his role to a critical friend of the group, Luis Radford also took over the role of critically accompanying this work, and Josep Gascón frequently contributed to our progression from outside in jointly working with Marianna Bosch.

This book is an outcome of these joint efforts in which we document one line of our work (other lines have led to further joint research projects, e.g., Kidron et al. 2008, 2011; Prediger and Ruthven 2007; Artigue et al. 2009, 2011; Bikner-Ahsbahs et al. 2010, 2011).

The book explains and illustrates what it means to network theories, and presents networking as a challenging but nevertheless fruitful research practice between five theoretical approaches: namely the approach of Action, Production, and Communication (APC), the Theory of Didactical Situations (TDS), the Anthropological Theory of the Didactic (ATD), the approach of Abstraction in Context (AiC), and the theory of Interest-Dense Situations (IDS). The book shows how the activity of networking generates questions at the theoretical and practical level and how these questions can be treated.

The structure and content of the book are organized around the most intense experience in these years of common work: starting with one set of video data, we wanted to explore how the analysis of the video differs when conducted with five different theoretical lenses. This raised the issue of the role of data and yielded to the collection of further data that from the theoretical perspectives were needed and led to deepening cooperation and additional research. On the basis of these experiences, the group undertook different case studies of networking while seeking further connections and differences. The methodology of networking of theories evolved while discussing these research practices on a meta-level and is documented in the subsequent chapters.

Although the book is organized systematically and can of course be best read linearly from beginning to end, we also wanted to allow the more spontaneous reader to use it flexibly to follow her or his main interests. Support for nonlinear reading is given by various links between chapters and the index that can help to clarify constructs if the reading includes a case study in which an unfamiliar theory appears. We hope to give the reader an idea not only of the process of networking of theories as a research practice, its strength and weaknesses, but also of the gains and difficulties we have met.

The work of the Networking Theories Group in the years 2006–2013 would not have been possible without financial support for the annual meetings. University Bremen in cooperation with Die Sparkasse Bremen and Nolting-Hauff-Stiftung financed the meetings in 2006, 2008, and 2011 at Bremen University. The meeting of 2007 in Barcelona at IQS – Universitat Ramon Llull was financed by Generalitat de Catalunya (ARCS 2007), and the meetings in Mariaspring in 2010 and 2012 were financed by the Georg-August-University Göttingen and the Ministry of Science and Culture of Lower Saxony, respectively. Finally, TU Dortmund University provided substantial personal resources for the editing process for this volume.

We thank Domingo Paola for sharing with us his interesting video episodes that took place in his classroom. Further, we are grateful to Luis Radford and Kenneth Ruthven for reading the whole book and writing comments from outside advancing the view on the networking of theories. And special thanks goes to Alexander Meyer, Frank Kuhardt and John Evans; without their thorough and constructively critical reading and editing, the book with its complex issues would be much less accessible and coherent.

Bremen, Germany — Angelika Bikner-Ahsbahs
Dortmund, Germany — Susanne Prediger

References

Artigue, M., Cerulli, M., Haspekian, M., & Maracci, M. (2009). Connecting and integrating theoretical frames: The TELMA contribution. *International Journal of Computers for Mathematical Learning*, *14*(3), 217–240.

Artigue, M., Bosch, M., & Gascón, J. (2011). Research praxeologies and networking theories. In M. Pytlak, T. Rowland, & E. Swoboda (Eds.), *Proceedings of the seventh Congress of the European Society for Research in Mathematics Education* (pp. 2381–2390). Rzeszów: University of Rzeszow.

Bikner-Ahsbahs, A., Dreyfus, T., Kidron, I., Arzarello, F., Radford, L., Artigue, M., & Sabena, C. (2010). Networking of theories in mathematics education. In *Proceedings of the 34th conference of the International Group for the Psychology of Mathematics Education – International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 145–175), Belo Horizonte: PME.

Bikner-Ahsbahs, A., Kidron, I., & Dreyfus, T. (2011). *Epistemisch handeln können – aber wie?* [*Knowing to act epistemically – but how?*]. Invited lecture at the annual conference of the German Society of Mathematics Education. Beiträge zum Mathematikunterricht. http://www.mathematik.tu-dortmund.de/ieem/bzmu2011/_BzMU11_1_Einfuehrungen-Hauptvortraege/BzMU11_BIKNER_Angelika_Epistem.pdf. Accessed 30 Oct 2013.

Kidron, I., Lenfant, A., Bikner-Ahsbahs, A., Artigue, M., & Dreyfus, T. (2008). Toward networking three theoretical approaches: the case of social interactions. *Zentralblatt für Didaktik der Mathematik*, *40*(2), 247–264.

Kidron, I., Bikner-Ahsbahs, A., & Dreyfus, T. (2011). How a general epistemic need leads to a need for a new construct – a case of networking two theoretical approaches. In M. Pytlak, T. Rowland, & E. Swoboda (Eds.), *Proceedings of the seventh Congress of the European Society for Research in Mathematics Education* (pp. 2451–2461). Rzeszów: University of Rzeszow.

Prediger, S., & Ruthven, K. (2007). From teaching problems to research problems. Proposing a way of comparing theoretical approaches. In D. Pitta-Pantazi & G. Phillipou (Eds.), *Proceedings of the 5th Congress of the European Society for Research in Mathematics Educatio*n *(CERME 5)* (pp. 1745–1771). Cyprus: ERME.

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Part I
Introduction

Chapter 1
Starting Points for Dealing with the Diversity of Theories

Angelika Bikner-Ahsbahs, Susanne Prediger, Michèle Artigue, Ferdinando Arzarello, Marianna Bosch, Tommy Dreyfus, Josep Gascón, Stefan Halverscheid, Mariam Haspekian, Ivy Kidron, Agnès Corblin-Lenfant, Alexander Meyer, Cristina Sabena, and Ingolf Schäfer

Abstract This chapter presents the main ideas and constructs of the book and uses the triplet (system of principles, methodologies, set of paradigmatic questions) for describing the theories involved. In Part II (Chaps. 3, 4, 5, 6, and 7), the diversity of five theoretical approaches is presented; these approaches are compared and systematically put into a dialogue throughout the book. In Part III (Chaps. 9, 10, 11, and 12), four case studies of networking practices between these approaches show how this dialogue can take place. Chapter 8 and Part IV (Chaps. 13, 14, 15, 16, and 17) provide methodological discussions and reflections on the presented networking practices.

Networking Theories Group: Angelika Bikner-Ahsbahs, Susanne Prediger, Michèle Artigue, Ferdinando Arzarello, Marianna Bosch, Tommy Dreyfus, Josep Gascón, Stefan Halverscheid, Mariam Haspekian, Ivy Kidron, Agnès Lenfant, Alexander Meyer, Cristina Sabena, and Ingolf Schäfer

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For about 15 years, the diversity of theories has been intensively discussed in the mathematics education research community (Ernest 1998; Steen 1999; Lerman 2006; Prediger et al. 2008a; Sriraman and English 2010; and many others). Our Networking Theories Group started in 2005. In this chapter, we make explicit

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the starting points for our way of dealing with this diversity. We will refer to the following questions:

- Why do there exist so many theoretical approaches?
- What exactly do we mean by theories or theoretical approaches, and for what are they needed?
- How can we deal with the diversity of theoretical approaches?

Whereas the most important third question is treated throughout the whole book, this introduction starts with the first two questions.

1.1 Sources for the Diversity of Theoretical Approaches

The first question is easy: one important source for the diversity of theories in mathematics education is that they evolved independently in different regions of the world and different cultural circumstances, including traditions of typical classroom cultures, values, but also varying institutional settings (cf. English and Sriraman 2005, p.452). The (at least equally important) second reason for the existence of different theories and theoretical approaches is the complexity of the topic of research itself. Since mathematics learning and teaching is a multi-faceted phenomenon which cannot be described, understood, or explained by one monolithic theory alone, a variety of theories is necessary to grasp the complexity of the field (Bikner-Ahsbahs and Prediger 2010). A third reason has been outlined by Teppo (1998) in that there are various ways of knowing in the field of mathematics education which are situated in various paradigms and, thus, produce different kinds of theoretical views. Teppo takes the diversity of theories as a sign that the "field of mathematics education is alive and well" (1998, p. 5). We would add that the diversity is not only an *indicator* for the dynamic character of the field, but it is also an *outcome* of the dynamic of theories. This is the fourth source.

The work of the Networking Theories Group, which has grown from the CERME working groups on Theories since CERME 4, started from the claim of diversity as a resource (Artigue et al. 2006). In order to substantiate the claim of diversity as a resource for rich scientific progress, the second question is addressed in the following section (following Bikner-Ahsbahs and Prediger 2010).

1.2 Conceptualizations and Functions of Theoretical Approaches

There is *no shared unique definition* of theory or theoretical approach among mathematics education researchers (see Assude et al. 2008). The large diversity already starts with the heterogeneity of what is called a theoretical approach or a theory by various researchers and different scholarly traditions. Some refer to basic research paradigms (such as the interpretative approach within social constructivism), others to comprehensive general theories (such as the Theory of Didactical Situations), and others to local conceptual tools (such as the modeling cycle) (cf. Prediger 2014). Differences exist in the ways to conceptualize and question mathematical activities and educational processes, in the type of results they can provide, but also in their scopes and backgrounds.

Mason and Waywood distinguish between different characters of theories: *foreground theories* are local theories *in* mathematics education "about what does and can happen within and without educational institutions" (Mason and Waywood 1996, p. 1056). In contrast, a *background theory* is a (mostly) consistent philosophical stance *of* or *about* mathematics education which "plays an important role in discerning and defining what kind of objects are to be studied, indeed, theoretical constructs act to bring these objects into being" (ibid., p. 1058). The background theory can comprise implicit parts that refer to epistemological, ontological, or methodological ideas, for example about the nature and aim of education, the nature of mathematics, and the nature of mathematics education. Taking the notions of foreground and background theory as offering *relative distinctions* rather than an absolute classification, they can help to distinguish different views on theories.

The different understandings of "theory" cannot only be distinguished according to the focus on foreground or background theories, but also according to their general view on the relation between theory and research practices. For analytical reasons, we distinguish a more static and a more dynamic view on theories. A normative *more static view* regards theory as a human construction to present, organize, and systematize a set of results about a piece of the real world, which then becomes a tool to be used. In contrast, a *more dynamic view* regards a theory as a tool in use rooted in some kind of philosophical background which constantly has to be developed in a suitable way in order to answer a specific question about an object. In this sense the notion of theory is embedded in the practical work of researchers. It is not ready for use, but has to be developed in order to answer a given question. In this context, the term "theoretical approach" is sometimes preferred to "theory", and so do we in this volume. Even very well developed theories such as the Theory of Didactical Situations (see Chap. 4) or the Anthropological Theory of the Didactic (Chap. 5) are still in a state of flux and can better be described by a wider and less static view on theories.

Most conceptualizations of theoretical approaches define the *function* of theories as being "to explain a specific set of phenomena as in 'true in fact and theory'" and emphasize "sense-making […as] the subject of theorizing …" (Mason and Waywood 1996, p. 1056). This includes the function of (background) theories as perspectives which help to produce knowledge about *what*, *how*, and *why* things happen in a vague phenomenon of mathematics education. And hence Mason and Waywood conclude: "To understand the role of theory in a research program is to understand what are taken to be the things that can be questioned and what counts as an answer to that questioning" (Mason and Waywood 1996, p. 1056).

Silver and Herbst (2007) also approach the notion of theory in mathematics education in a dynamic way. Comparisons of different theories, with respect to their roles as instruments mediating between problems, practices, and research, show that *theories in mathematics education are mostly developed for certain purposes*. For example:

- theories which mediate practices and research can be understood as "a language of descriptions of an educational practice" or as "a system of best practices" (ibid., p. 56);
- theories which mediate problems and practices can be understood as a "proposed solution to a problem" or a "tool which can help design new practices" (ibid, p. 59);
- theories which mediate research and problems can be understood as "means to transform a commonsensical problem into a researchable problem" or as a "lens to analyze data and produce results of research on a problem" (ibid., p. 50).

Some theories are used to investigate problems or empirical phenomena in mathematics education; others provide the tools for design, and the language to observe, understand, describe, and even explain or predict, (conceptualized) phenomena.

If we approach the notion of theory in this way, from its role in research practices, theories can be understood as guiding research practices and at the same time being influenced by or being the aim of research practices. This dialectic between theory and research (Assude et al. 2008) has to be taken into account in many discourses about the notion of theory. For example, Radford (2008) takes this role into account by describing theories by means of a triplet of three components:

> A "theory can be seen as a way of producing understandings and ways of action based on:
>
> - A system, *P*, of *basic principles*, which includes implicit views and explicit statements that delineate the frontier of what will be the universe of discourse and the adopted research perspective.
> - A *methodology*, *M*, which includes techniques of data collection and data-interpretation as supported by P.
> - A set, *Q*, of paradigmatic *research questions* (templates or schemas that generate specific questions as new interpretations arise or as the principles are deepened, expanded or modified)." (Radford 2008, p. 320)

Radford's conceptualization of theory as "a way of producing understandings and ways of action" again reflects that theories cannot be separated from the research practices in which they are grown and used. Radford considers this triplet as being a dynamic entity which evolves successively through the dialectic relationship of its components. Radford specifically names two ways of supporting the evolution of theories: through producing results, because "the results of a theory influence its components"; and also through the networking of theories (Radford 2012).

In this book, we work with five theoretical approaches, presented in Chaps. 3, 4, 5, 6, and 7. For presenting the theoretical approaches, we decided to follow Radford's triplet of Principles, Methodologies, and Questions. It was interesting to see that two decisions were necessary before this fitted for all five approaches: we had to extend the principles by Key Constructs; and we had to allow different orders among the four components Principles, Key Constructs, Questions and Methodology, since their mutual relationships are conceptualized differently in the five approaches.

1.3 A Journey on Networking Theories

Steen (1999) warned that the diversity of theoretical approaches in mathematics education research is an indicator of missing maturity of the discipline. In contrast, many researchers emphasize that the diversity is not a problem, but a necessity for grasping the complexity of the topic of research (Teppo 1998; Lerman 2006). However, accepting the co-existence of isolated, arbitrary theoretical approaches regularly can cause challenges for communication, for the integration of empirical results (e.g., for practical purposes in classrooms), and for scientific progress (Prediger et al. 2008b, p. 169). That is why we emphasize that the diversity of theoretical approaches can *only* become fruitful *if* connections between them are *actively established*.

During the years of common work in the CERME working groups (Artigue et al. 2006; Arzarello et al. 2008; Prediger et al. 2010; Kidron et al. 2011, 2013), many different strategies and methods for networking of theoretical approaches were developed (see Chap. 8 for an overview).

In this book, we report on the work of the Networking Theories Group (see Preface) on establishing connections among the following five theoretical approaches:

- *Action, Production, and Communication Approach* (introduced in Chap. 3): APC provides a frame for investigating semiotic resources in the classroom. It addresses the use of semiotic resources from a multimodal perspective including the analysis of gestures as a resource for expression and communication.
- *Theory of Didactical Situations* (introduced in Chap. 4): TDS provides a frame for developing and investigating didactical situations in mathematics from an epistemological and systemic perspective that includes a corpus of concepts relevant for addressing teaching and learning processes in mathematics classrooms and beyond.
- *Anthropological Theory of the Didactic* (introduced in Chap. 5): ATD provides a frame for investigating mathematical and didactical praxeologies on the institutional level of mathematics and its teaching and learning conditions. The main idea of the concept of praxeologies is that all human activities comprise and link two parts, a practice and a theory part.
- *Abstraction in Context* (introduced in Chap. 6): AiC provides a frame for investigating learning processes which lead to new concepts and how they are built through phases: the need for a new concept, the process of constructing the new concept, and its consolidation.
- *Theory of Interest-Dense Situations* (introduced in Chap. 7): IDS provides a frame for how interest-dense situations and their epistemic and interest-supporting character are shaped through social interactions in mathematics classes distinguishing three levels: the social interactions and how the participants are involved, the dynamic of the epistemic processes, and the attribution of mathematical value.

For establishing connections among these five approaches, we began by selecting a set of data as an empirical base. The original data provided by the APC team (see Chap. 2 for the presentation of the data) consisted of a video of two students' learning process on exponential functions in grade 10, namely Carlo and Giovanni.

Part II of the book (Chaps. 3, 4, 5, 6, and 7) presents the five theoretical approaches involved in the book. They describe their main principles, methodologies, and paradigmatic questions adding key constructs and – if necessary – additional results and show how these theories are used for analyzing the (for most approaches alien) set of data. Already these first presentations bear testimony of a strong experience recognized in this exercise, namely the need for different data: whereas for the APC team, their video together with the task and the written answers was completely sufficient for conducting an analysis, this data turned out to be insufficient for teams using other theoretical approaches because it does not address their relevant questions and it does not provide the data that is in the center of their methodologies.

That is why the initial set of data had to be appropriated for each approach and extended by background information about the intentions of the teacher, the curriculum of the class, students' previous knowledge, teacher's intentions etc. For making these specific needs for data transparent, the analysis for each theoretical approach in Chaps. 3, 4, 5, 6, and 7 is split into two parts: the first part with only the initial (alien) video and, wherever necessary, the second part with the extended and appropriated set of data.

A second issue was how it would be possible for the different groups to make sense of the given data. Besides the fact that all approaches needed a process of extension and appropriation of the data, they chose different subsets of data to be able methodically to work. This included differences in the focus on the mathematical task. For some theories, the character of the given task is important because they investigate specific questions that can be induced by the design of tasks. For others, the given mathematical learning situation is to be investigated and therefore the situation is taken as it is. Some approaches focus on learning in-depth, others include the teacher behavior or pose further questions to include institutional and societal conditions. These experiences with our home theories investigating alien data pointed to the function of theories as heuristics for research. Since data collection already belongs to the research practice that is specific for a certain approach, this attempt to analyze alien data is a networking endeavor on the theories' methodological level.

Whereas Part II of the book is mainly concerned with making the theoretical approaches understandable (also with respect to their research practices), *Part III* documents different ways of how to deepen the connection of theories. The introductory Chap. 8 presents different networking strategies and profiles on a general level and provides the language and some methodological considerations for networking.

The core of the book is the rest of Part III with four case studies of networking presented in Chaps. 9, 10, 11, and 12, all focused on the set of video data on Carlo, Giovanni, and the exponential function. These case studies not only show the development of new aspects of this research but also how alien and home theories can more deeply be understood by practices of networking:

- *Chapter 9* shows a case study of networking between APC and AiC. In the first case study, the role of gestures for the process of knowledge construction is considered empirically. APC and AiC are linked in a way that gesture studies are included into the frame of AiC through learning from research within the APC-space.

- *Chapter 10* shows a case study of networking between TDS, ATC, and AiC. The case of context, milieu, and the media-milieu dialectic contrasts and compares three complex key constructs and their status within each theory in order to learn how constructs which at a first glance seem to have a similar role in the understanding of teaching and learning can differ in each theory.
- *Chapter 11* shows a case study on networking involving only two theories, APC and IDS. It describes a networking case that starts from a situation of seeming contradiction and leads to a local integration of the new concept of the epistemological gap into both theories.
- *Chapter 12* shows a case study of networking between TDS and IDS. It investigates empirically two phenomena of two different theories and networks the theories by comparing and contrasting these phenomena. This process leads to deepening the understanding of the theories on the one hand and provides insight into the character of the phenomena and their common idea on the other. In addition, the two phenomena are contrasted with a third phenomenon from APC. A reflection from an ATD perspective as an outside view on this case further deepens the comprehension of the phenomena.

The lessons learnt from these different practices of bilateral and trilateral networking were on three levels:

- *On the empirical level*, we could gain deep and complex insights into the empirical and conceptualized phenomena in the videos and the role of data. These insights are reported in Chaps. 9, 10, 11, and 12.
- *On the theoretical level*, the networking gave many impulses for theory development by sharpening theoretical principles or constructs, extending theoretical approaches, building new concepts, posing new questions, or making explicit commonalities but always while keeping the theories' main identities. These developments are documented in Chaps. 9, 10, 11, and 12 and compared and systematized in Chaps. 14 and 15.
- *On a methodological level*, the case studies of networking also offered insights that can be transferred from the concrete cases to networking in principle. These experiences and reflections are made explicit in Part IV of the book, in Chaps. 13, 14, and 15.

Part IV of the book is dedicated to the reflection of networking practices from different perspectives:

- from an internal perspective considering individual and informal experiences (Chap. 13);
- from a bottom-up perspective that tries to systematize the experiences (Chap. 14), their gains and difficulties;
- from a top-down perspective in terms of research praxeologies (Chap. 15);
- and from two external perspectives adopted by our critical friends, Kenneth Ruthven and Luis Radford (Chaps. 16 and 17).

Since the journey of networking of theoretical approaches was very long and intense, this book is only partly able to capture and demonstrate our learning experiences. We started enthusiastically and continued being so, although we met difficulties for which we had to find ways to overcome. One typical difficulty is, for example, the limits arising from our common principle that the theories must not lose their specificity.

The challenge to be theoretically open-minded slowly changed our standpoints. Deep insights and interesting research results helped us carry on and further develop the view on theories, research practices, and their diversity, and to uncover the strengths and weaknesses of our networking enterprise.

In this way, the book intends to offer an opportunity for the readers to partly participate in this networking endeavor and form an opinion and critical standpoint on crucial methodological and meta-theoretical challenges that are as yet far from being completely clarified.

References

Artigue, M., Bartolini-Bussi, M., Dreyfus, T., Gray, E., & Prediger, S. (2006). Different theoretical perspectives and approaches in research in mathematics education. In M. Bosch (Ed.), *Proceedings of the 4th Congress of the European Society for Research in Mathematics Education* (pp. 1239–1244). Barcelona: Fundemi IQS.

Arzarello, F., Bosch, M., Lenfant, A., & Prediger, S. (2008). Different theoretical perspectives in research from teaching problems to research problems. In D. Pitta-Pantazi, G. Phillipou, et al. (Eds.), *Proceedings of the 5th Congress of the European Society for Research in Mathematics Education (CERME 5)* (pp. 1618–1627). Cyprus: ERME.

Assude, T., Boero, P., Herbst, P., Lerman, S., & Radford, L. (2008). *The notions and roles of theory in mathematics education research.* Paper presented by the Survey Team at ICME 11, Mexico.

Bikner-Ahsbahs, A., & Prediger, S. (2010). Networking of theories – An approach for exploiting the diversity of theoretical approaches. In B. Sriraman & L. English (Eds.), *Theories of mathematics education: Seeking new frontiers* (Advances in mathematics education series, pp. 483–506). New York: Springer.

English, L., & Sriraman, B. (2005). Theories of mathematics education: A global survey of theoretical frameworks/trends in mathematics education research. *Zentralblatt für Didaktik der Mathematik, 37*(6), 450–456.

Ernest, P. (1998). A postmodern perspective on research in mathematics education. In A. Sierpinska & J. Kilpatrick (Eds.), *Mathematics education as a research domain: A search for identity* (Vol. 1, pp. 71–85). Dordrecht: Kluwer.

Kidron, I., Bikner-Ahsbahs, A., Monaghan, J., Radford, L., & Sensevy, G. (2011). Different theoretical perspectives and approaches in research in mathematics education. In M. Pytlak, T. Rowland, & E. Swoboda (Eds.), *Proceedings of the 7th conference of the European Society for Research in Mathematics Education* (pp. 2377–2380). Rzeszow: ERME/University of Rzeszow.

Kidron, I., Bosch, M., Monaghan, J., & Radford, L. (2013). Different theoretical perspectives and approaches in research in mathematics education. In B. Ubuz, C. Haser, & M. A Mariotti (Eds.), *Proceedings of the 8th Congress of the European Society for Research in Mathematics Education* (CERME 8), Antalya.

Lerman, S. (2006). Theories of mathematics education: Is plurality a problem? *Zentralblatt für Didaktik der Mathematik, 38*(1), 8–13.

Mason, J., & Waywood, A. (1996). The role of theory in mathematics education and research. In A. J. Bishop et al. (Eds.), *International handbook of mathematics education* (pp. 1055–1089). Dordrecht: Kluwer.

Prediger, S., Arzarello, F., Bosch, M., & Lenfant, A. (Eds.). (2008a). Comparing, combining, coordinating – Networking strategies for connecting theoretical approaches. Thematic Issue of *ZDM – The International Journal on Mathematics Education*, *40*(2), 163–327.

Prediger, S., Bikner-Ahsbahs, A., & Arzarello, F. (2008b). Networking strategies and methods for connecting theoretical approaches: First steps towards a conceptual framework. *ZDM – The International Journal on Mathematics Education, 40*(2), 165–178.

Prediger, S., Bosch, M., Kidron, I., Monaghan, J., & Sensevy, G. (2010). Different theoretical perspectives and approaches in mathematics education research – Strategies and difficulties when connecting theories. In V. Durand-Guerrier, S. Soury-Lavergne, & F. Arzarello (Eds.), *Proceedings of the Sixth Congress of the European Society for Research in Mathematics Education* (pp. 1529–1534). Lyon: Institut National de Recherche Pédagogique.

Prediger, S. (2014, in press). Rolle von Theorien. In R. Bruder, L. Hefendehl-Hebeker, B. Schmidt-Thieme, & H.-G. Weigand (Eds.), *Handbuch Mathematikdidaktik*. Heidelberg: Springer.

Radford, L. (2008). Connecting theories in mathematics education: Challenges and possibilities. *ZDM – The International Journal on Mathematics Education, 40*(2), 317–327.

Radford, L. (2012, May 31–June 1). *On the growth and transformation of mathematics education theories.* Paper presented at the international colloquium: The didactics of mathematics: Approaches and issues. A homage to Michèle Artigue. http://www.laurentian.ca/educ/lradford. Accessed 29 Oct 2013.

Silver, E. A., & Herbst, P. G. (2007). Theory in mathematics education scholarship. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 39–68). Reston: NCTM.

Sriraman, B., & English, L. (Eds.). (2010). *Theories of mathematics education: Seeking new frontiers* (Advances in mathematics education series). New York: Springer.

Steen, L. A. (1999). Review of mathematics education as research domain. *Journal for Research in Mathematics Education, 30*(2), 235–241.

Teppo, A. (1998). Divers ways of knowing. In A. R. Teppo (Ed.), *Qualitative research methods in mathematics education* (pp. 1–16). Reston: NCTM.

Chapter 2
Description of the Data: Introducing the Session of Carlo, Giovanni, and the Exponential Function

Cristina Sabena

Abstract The chapter provides the basic information on the set of data that is used throughout the book. Data from a video recording show two students, Carlos and Giovanni, when investigating the exponential function in a dynamic geometry environment. An interview with the teacher gives background information.

Keyword Data

The common activity in the Networking Group started from considering a single set of data from different perspectives. The basis of the data is a video showing a session from the group-work of two students, Giovanni and Carlo, during a teaching experiment on the exponential function in secondary school. We analyzed the video from different theoretical perspectives.

This *initial set of data* was shared at the beginning of the networking activity. It consists of a video and its verbal transcript (translated into English), the students' written protocols, and some information on the research and didactical contexts. In Sect. 2.1 we present the data, specifying what was actually presented and used in our joint work. The complete transcript can be found in the Appendix A.1.

While this set of data was sufficient within the theoretical framework of the research project in which it was gathered (in an informal project following Paola 2006 and Arzarello et al. 2009), the researchers of other theoretical frameworks needed more data on students' backgrounds, teacher's perspectives and many other aspects. For gathering this *extended set of data*, an interview with the teacher was conducted (see Sect. 2.2.2).

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Further, a second video (here called “extra video on Task 3”) was also considered during the networking process; the video is briefly introduced in Sect. 2.2.3 and its transcript fully presented in the Appendix A.2.

2.1 Initial Set of Data

2.1.1 Research Context

The data come from an Italian long-term teaching project on investigating changing quantities by ICT technologies in secondary schools. The project is supervised by Ferdinando Arzarello and is planned and realized with the active collaboration of the classroom teacher, Domingo Paola (2006).

Students have five hours of mathematics per week. Their teacher (Domingo) has been with them for 5 years. ICT technologies are used extensively in the classroom, in particular dynamic geometry software, spreadsheets and graphic-symbolic calculators. The teaching methods mainly alternate between group-work activities and classroom discussions.

At the time of the experiment (February 2004), students were in grade 10 (second year of secondary school), and already knew about dynamic discrete models of exponential and logistic growth, approached by using different software for graphic-symbolic manipulations. They knew that in a succession defined by recursion that represents an exponential growth, the ratio of two consecutive terms is constant. They had worked with first and second finite differences for functions described by numerical values for $(x, f(x))$ represented in tables. They usually described the features of increasing and decreasing functions using the words “it grows and grows more and more” and “it grows and grows less and less.”

2.1.2 Professional Background of the Teacher

At the time of the project, Domingo was a 50-year-old teacher with long experience in mathematics education, developed through a long-lasting collaboration with many Italian researchers.[1] He was one of the most active Italian “teacher-researchers,” and had published several papers in Italian and international journals and conference proceedings. He was engaged in pre-service and in-service teacher education programs, and took part in innovation projects funded by the Italian Ministry of Education.

[1] Teacher-researchers play a fundamental role in the Italian paradigm of “research for innovation” (see Arzarello and Bartolini Bussi 1998 for a full description). These teachers collaborate closely with researchers, and participate in all phases of classroom-based research, from planning to data analysis.

As a teacher, Domingo believes that the major goal of teaching and learning (in general, and of mathematics in particular) is to foster the formation and development of competences and knowledge essential for an informed, conscious, and critical citizenship. His didactical choices are aimed at this objective.

In his lessons he adopts an informal approach, and exploits different ICT tools (spreadsheets, symbolic-graphic calculators, devoted software for graphs of functions, …) in order to make the students visualize and reason on properties of functions starting from numerical data and a perceptive-descriptive approach. The formalization within the formal mathematical theory follows from the informal approach through technology. As a didactical technique, he poses problems through sheets and files that he prepares himself, with the students working on these in groups of two or three. During the group-work, Domingo supervises the work, resolving possible difficulties with the tools, and providing prompts with regard to the tasks. Classroom discussions follow the group-work sessions: in these lessons, the teacher guides the comparison between the students' productions, and introduces or refines the mathematical notions and methods, by enhancing an argumentative and theoretical approach to mathematics.

2.1.3 Activities and Tools

In the session that is investigated here, the students are involved in exploratory activities that are conducted in pairs using Cabri, a Dynamic Geometry Software (DGS) program. With three tasks presented in written worksheets and DGS files, they explore the graphs of exponential functions $y = a^x$ and of its tangent line[2] (a is a parameter whose value can be changed in a slider).

Carlo and Giovanni work together on a computer with files that the teacher has prepared for the exploration. Figures 2.1, 2.2, and 2.3 show the (translated) text of the worksheets and the configurations in the DGS (that was not on the worksheet but on the computer screen; some screenshots are added for easier reading).

The two students work on three tasks (to which we will sometimes refer as Episode 1, 2, and 3). Task 1 and Task 2 are presented on one worksheet and Task 3 on a second worksheet (since it required opening the new version of the software, Cabri II PLUS). Each task corresponds to a DGS file, which the students have to open and use in their work. The worksheets are translated below.

In Task 1, the students have first to explore the graph of $y = 2.7^x$ in the first DGS file; they can drag a point representing the abscissa x, and for every x, a number representing the ordinate $y = 2.7^x$ appears on the screen (Fig. 2.1). They can use the animation function of DGS to foster the observation of the different velocities of x and y. In Task 2, the students open another file and are asked to explore $y = a^x$ by changing the value of the base of the exponential (Fig. 2.2).

[2] The line is actually a secant line; the secant points are so near that the line appears on the screen as tangent to the graph. This issue had been discussed in the classroom in a previous lesson.

First Worksheet (Task 1)

Task 1

a. Open with Cabri II the file "y = (2.7)^x".

In this file you will see: the point *x* on the *x*-axis and the point *y* = 2.7^*x*, on the *y*-axis.

Move the point *x* on the *x*-axis and check what happens to the point *y* = 2.7^*x* on the *y*-axis; that is, observe how (2.7)^*x* varies as *x* is changing.

In order to make these observations, modify also the measure unit on the *y*-axis of your worksheet. After some trials, use animation. Move the point *x* towards the left until arriving nearly to the end of the field of variation of the negative *x*'s, and then animate with a spring the point x so that it moves from the left towards the right.

Share all the observations that you think interesting on the coordinate movement of the two points, and describe briefly (but clearly) your argument on the sheet that has been given to you.

b. What trend do you think the function *y* = (2.7)^*x* has? Before drawing it on your sheet, agree about what you think are the most important features of the graph of *y* = (2.7)^*x* and then justify the graph that you draw.

Corresponding computer screen (not on Worksheet)

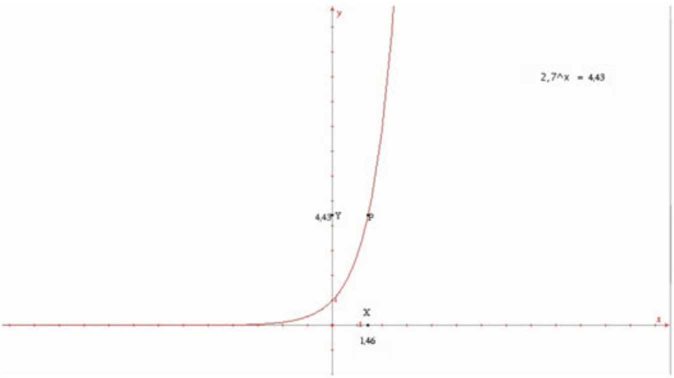


Fig. 2.1 Task 1 and corresponding DGS screen configurations

Task 3 (Fig. 2.3) is more structured than the previous tasks and proposes an exploration in order to highlight both local and global aspects of the exponential variation. It contains:

- the graph of $y = a^x$;
- the points P(x, a^x), H($x + \Delta x$, a^x);
- two sliders, one for Δx and another for a, whose variation allows the students to modify, respectively, the increment Δx and the base of the exponential.

The exploration carried out varying Δx has the didactical goal of highlighting local aspects relative to the value of the slope of the tangent line. The exploration carried

First Worksheet (Task 2)

Task 2

Open the file "a^x" with Cabri II. In it you see a point X on the x-axis, a point a^x, on the y-axis, a point P of coordinate (x, a^x) that, therefore, describes, as x is varying, the function's graphic y = a^x, and finally a ray, on which there is a point A, whose abscissa is the base of the exponential a^x.

That means that, by varying the position of A, you get exponentials with a different base (all bigger than 0: for this there is a precise reason that we will discuss). Then moving the point A changes the base of the exponential. Moving the point P, you run along the graph of an exponential function with a fixed base. Explore, share your impressions (is there something which is not clear and we were not expecting or that is clear and you were expecting). Describe briefly your exploration on the sheet.

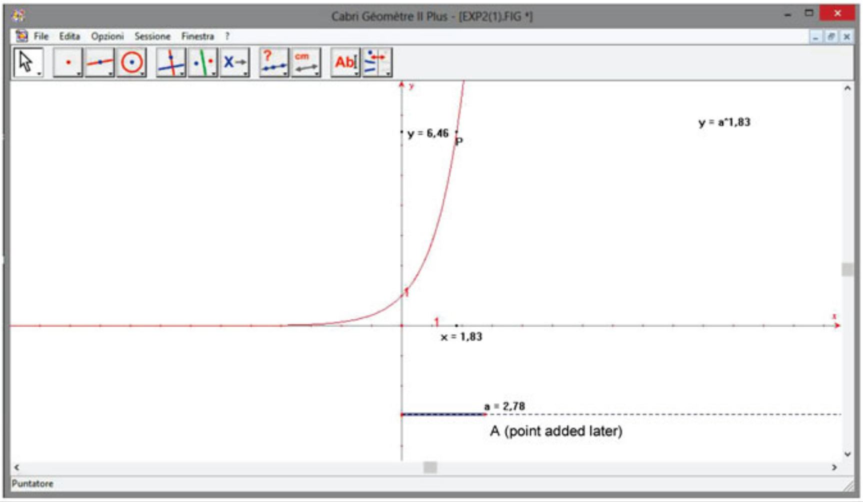


Fig. 2.2 Task 2 and corresponding DGS screen configurations

out dragging P has the goal of shedding light on global aspects of the exponential function, and in particular the variations related to its slope functions (according to the teacher's planning).

2.1.4 The Students Carlo and Giovanni

The video shows two male students, Carlo and Giovanni, who are used to working together during group-work activities in mathematics. We provide brief information about the students (the information has been provided by the teacher).

Carlo reveals good intuition in group-work and is very participative and motivated both in collective activities and in individual work. This attitude has not always been the case. At the beginning of grade 9 (first year of high school), he was

Second Worksheet (Task 3)

Task 3

a. Open the file “exp” with Cabri II PLUS.

Look carefully at the figure: you see that there are some objects that you can move (the points P, *a* and the segment Δx); observe also that the segments PH and Δx have the same length (they have been built so).

Describe briefly the figure, moving first P, then Δx (changing its length), then A; write briefly your observations on the sheet.

b. Look carefully at what happens when Δx tends to 0…
Does such a requirement suggest to you new observations with respect to those you have already discussed and written? Why?

c. Now we want to study how the slope of the linear function that approximates best the function $y = a$^x changes while x is moving. As already said, that means studying how the slope of the line tangent to the function $y = a$^x at the point of abscissa x changes while x moves.

You can use the Cabri worksheet to help you in order to answer the question. Explore and possibly use any other software on the PC you like, or you may also use no software.

Whatever is your decision concerning your (chosen) solving strategies, discuss briefly the features of the graph of the function $m = m(x)$, where m is the slope of the line tangent to the function $y = a$^x at the point of abscissa x.

You are required to say how the slope of the line tangent to the function $y = a$^x at the point of abscissa x changes as x changes.

After discussing your opinions, write briefly the features of the graph of *m=m(x)* on the worksheet and draw also a sketch of the graph itself.

Corresponding Computer screen (not on Worksheet)

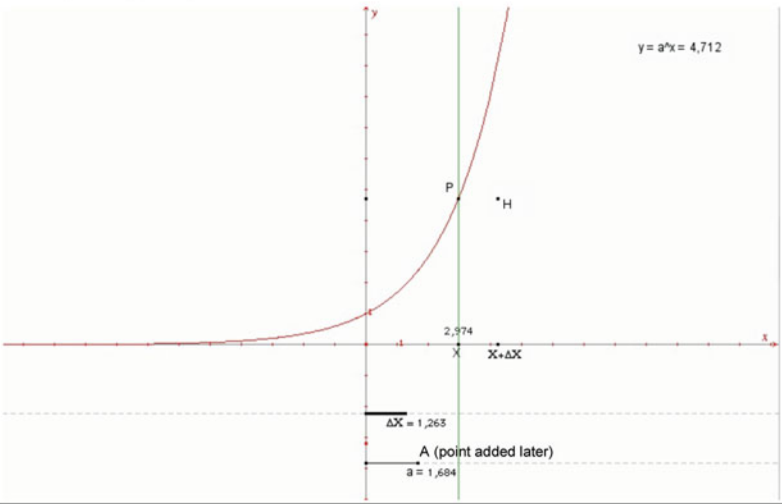


Fig. 2.3 Task 3 and corresponding DGS screen configuration

little involved in school, although his results were sufficient thanks to his capacity of using his possessed knowledge. In grade 10, together with the support of the family, Carlo’s engagement in school has increased, arriving at brilliant results, especially in mathematics. In a short time the student has become one of the most positive students in group-work, both in cognitive and for relational aspects. By the time of the analyzed session, Carlo had obtained excellent results in mathematics, with good expressive capacity.

Giovanni has always participated with commitment and perseverance in the classroom activities and the e-mail exchanges with the teachers and his classmates. From a composition he wrote in the first year of high school on the image of mathematics from primary to secondary school, we can infer that he has some linguistic problems (his mother is German); and his mathematics path is rich from the emotional point of view. In grade 10, he is used to working with Carlo, and they have developed a good rapport.

2.1.5 Overview of the Session

The session in which Carlo and Giovanni worked on Tasks 1–3 has been video-recorded and transcribed. As the English translation of the transcript was the basis of the common analyses, we print it completely in the Appendix A.1. At some critical points, we went back to the Italian version (the original transcript is available from the author of this chapter).

In this paragraph, we give a short overview of Carlo and Giovanni's work, which lasts about one hour. This overview was not given to the researchers at the beginning of the networking activity but helps here to embed the phenomena.

The teacher distributes the worksheets to the students, and they start from Task 1 (Fig. 2.1), opening the corresponding DGS file. Carlo and Giovanni work on one computer. Giovanni sits in front of the screen, uses the mouse and technically carries out the exploration on the screen, whereas Carlo, to his right, is in charge of writing a common solution on the sheets. Despite this "labor division," they work in a very collaborative way and give mutual suggestions for what to explore or write.

Carlo and Giovanni work for roughly twenty-five minutes on Task 1 (lines 0–138 of the transcript), twenty minutes on Task 2 (lines 139–248), and a quarter of an hour on Task 3 (lines 249–379).

In Episode 1, dealing with *Task 1*, the students use Cabri to explore the graphical situation, focusing their attention on the ordinate of the points of the function $y = 2.7^x$. Instead of modifying the measure unit on the y-axis, as suggested on the sheet, they change the base of the exponential, and observe that the function remains of the same kind (lines 0–18). They soon recognize that they are exponential functions (line 18), a kind of function that they had already encountered in previous lessons (even if without focusing on its properties, as they are doing in this lesson). In fact, as can be seen in detail in Sect. 2.2.2 and in particular in the teacher's answers to question 17, about 1 month previously the students had been introduced to continuous linear model and linear discrete dynamic systems, and to discrete exponential models.

Carlo investigates the case with base 1, that is, $y = 1^x$, to check whether the function becomes a line. Then they explore the function for negative abscissas, saying that it goes towards 0, and check the value for 0 (that is, $y^0 = 1$) (lines 19–56).

While activating the Trace function in the DGS, they observe what happens on the screen. They read the worksheet again, wondering whether they had misunderstood it (the changing of the measure unit); they observe the screen, and do not talk very much. They meet some technical problems with the DGS, so they call the teacher and ask about deleting the trace on the screen, and obtaining the coordinate of a point from the DGS (lines 57–88).

After this short teacher intervention, focused on technical instrumental issues, the students continue working on Task 1. Carlo proposes to Giovanni to check whether $(2.7)^2/(2.7)^1=2.7$. They verify their conjecture with the aid of a pocket calculator (lines 89–104). In this way, they get the validation that the graph represents an exponential function (line 105). Carlo starts writing the answer to Task 1, and Giovanni helps him. They also continue to explore the function in DGS, and describe the behavior of the function, first orally, and then in a written form by writing it on their sheets (lines 106–124).

In Episode 2, Carlo and Giovanni begin *Task 2* by reading carefully the corresponding text in the worksheet (lines 139–151). During a short struggle on what to do in the DGS file, Giovanni realizes that they have to move the point corresponding to the base a (the right end of the lowest horizontal segment in Fig. 2.2), so they start together the exploration of the situation (lines 152–159). Carlo proposes to explore the case in which the function has a very big base (by making the parameter a very big), and they make conjectures about the function's behavior (lines 160–171). Then, following Giovanni's suggestion, they explore the case in which a is very small. It is then Carlo who guides the exploration of the case $a=1$ (which corresponds to a line, which as the students show is to be expected), and of the case $a<1$. The students appear satisfied by what they observe on the screen, since the graphs confirm what they expected (lines 172–188). They write their answer, starting again from the case $a>1$ (lines 189–202), and then discussing the case $a>1$ (lines 203–220). The writing of the answer to Task 2 takes some time for the students (from 38:55 to 41:25), since they discuss each claim, sometimes modifying what they had previously said (e.g., now Giovanni says that the function does not reach 0 as value, differently from what he had said before).

The description of the work on Task 3 follows in Sect. 2.2.3.

2.2 Extended Set of Data

In the very initial phase, the researchers in the Networking Group were provided with the video, its transcript (both Italian and English versions), the DGS files, and a summary of the task and its didactic background.

During the process of networking it became evident that there was a need for further information about the background of the teaching experiment. Thus, written protocols, worksheet texts, detailed information on the students' background (as the teacher sees it), and information on the teacher's ideas were added to the already existing data corpus. This information is presented in this section.

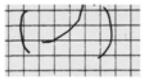
Written protocol relative to Tasks 1 and 2

"The function $y = 2.7^x$ is an exponential function, that is the ratio between $y(n+1)$ and $y(n)$ is constant.

Moving the x point on the left part of the worksheet (on the negative x [plural]) we realize that the y never goes down below the 'zero' and reaches this value for x [plural] equals or less than –5.

Obviously for $x = 0$ the y is e because $2.7^0 = 1$.

Being an exponential function, it increases and increases more and more at the infinite.

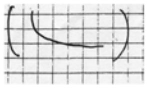


A to the x

Modifying the point A, namely modifying the growth rate, we realize that if it is bigger, the function increases more rapidly, namely for a with the same x interval the y increases more and more and the first differences increase more and more.

For $A = 1$, as we were expecting, we get a line of the type $y = 1$, because by raising 1 to whatever number, you get 1.

For $A < 1$ we see that the graph decreases and decreases less and less but never touching $y = 0$."

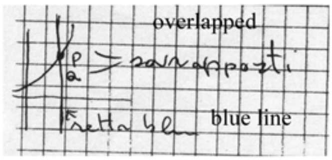


Written protocol relative to Task 3

"Keeping PH constant, that is Δx, as we move little by little towards bigger x [plural] we see that Δy increases more and more.

For Δx very small, that is taking a 'piece' of graph smaller and smaller, we can approximate the blue line and its behavior: for small Δx we can approximate the curve of the graph to many lines that touch each other in a point of always increasing slope.

If Δx is equal to zero, P and Q overlap and the blue line has a function of the type $x = x(P)$ or $x(Q)$.



The more 'a' becomes big, the more the blue line increases its slope, since for a same space Δx, Δy increases more and more.

The slope of the tangent line of an exponential function increases more and more, as x increases, since it is tangent to a curve that increases more and more to infinity.

On the contrary for 'a' < 1 the graph decreases less and less and hence the tangent line will have a negative slope but always smaller."

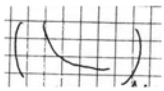


Fig. 2.4 Carlo and Giovanni's written protocols

2.2.1 The Written Protocols

The transcript was accompanied by a written protocol showing all of the students' written notes during the session, presented here in the English translation (Fig. 2.4).[3]

[3] The students sometimes use x in a plural form. As this cannot easily be expressed in English, the hint "[plural]" is sometimes inserted.

2.2.2 Background Information Given by the Teacher

In order to get an extended understanding of the shared data and obtain some background information on the students' learning history, the teacher's learning goals and didactical intentions etc., the participants of the Networking Group collected 18 questions for the teacher which were posed in a written form. The teacher answered 16 of the 18 questions in an interview, and the last two in a written questionnaire. The interview was recorded and translated into English. The questionnaire was answered directly in English.

As different chapters of this book refer to different parts of the interview, we print long parts of the answers for methodological reasons of accountability, the complete answers are given in the Appendix A.3. However, the reader of this chapter might prefer to skip this reading at first and come back to it in the context of the other chapters.

1. *In advance of the lesson, how did you expect the students to work together at the computer? How did you expect them to share roles? What "ground rules" had you tried to establish about joint work at the computer with this class?*

   "[…] With respect to mathematical competences, I hope that students read the text of the posed problems very carefully and that they begin to do some explorations, either mental explorations or with the help of technological tools. These explorations have the aim to create context, to create meaning, to provide experience of problem situations; they encourage the production of conjectures and should motivate students to validate their produced conjectures. I hope that students often ask themselves why they observed some patterns, some regularities. […] I hope that students are able to argue and to support their conjectures and solving strategies in a pertinent and convincing way and with coherence to their mathematical knowledge. […]"

…

3. *What conditions support or hinder learning when students work together like this at the computer?*

   "[…] More in general, in my opinion the use of the PC may hinder learning or, put better, can create obstacles to learning if the tool is used in a uncritical way, for example to obtain answers and not give rise to questions and thought. In my opinion technological tools have to be used to empower the possibility to experience the mathematical environment and mathematical objects. In this way we should use them in the teaching–learning activities. […]"
4. *During a lesson of this type, under what circumstances do you decide to get involved with a pair of students, and what kinds of things do you do?*

   "I enter in a working group if the students call me. Sometimes I enter in a working group if I realize that students are stuck. Other times I enter because I realize that students are working very well and they have very good ideas that need to be treated more deeply. Obviously the type of things that I do vary with the situations, but a constant is that I try to work in a zone of proximal development. The analysis of video and the attention we paid to gestures made me aware of the so-called "semiotic game" that consists in using the same gestures as students but accompanying them with more specific and precise language compared with the language used by students. The semiotic game, if it is used with awareness, may be a very good tool to introduce students to institutional knowledge."

5. *Choose specific examples from the video of your becoming involved with the pair of students. Talk us through them.*

   "In this video my dialogues with the students are few. Anyway, it seems to me that among the more interesting there is the intervention at the minute 53 and 59 seconds. I use a gesture used before by Giovanni. This gesture is towards a little segment that approximates locally the function and I ask: "What is the characteristic of this segment?" My aim is to induce the students to reflect on the fact that it is important to pay attention to the slope of the little segments, because their slope gives information on the growth of the function. Giovanni says "it is twice the previous slope …" I, using his same gesture, say more precisely that "the slope has an exponential growth." At the minute 54 and 24 seconds, I help the students to remember that the characteristic of the exponential successions is that of having the ratio of two consecutive terms constant. Immediately after, I ask the students: "Are you surprised that the graph of the function is so close to zero for small x?" Giovanni, at the minute 55 and 28 seconds says something like "with number smaller and smaller, I have number smaller and smaller." I reword this idea with a more precise language. In the following dialogue, Giovanni and Carlo are able to explain in a comprehensible way the reason why the graph of an exponential function of base greater than 1 is so close to the x-axis for x less than 0 and explodes for high values of x."

6. *What experience did the students previously have of using DGS? Working specifically with function graphs?*

   "The students have known since the beginning of the first school year the software Cabri. Besides, since the beginning of the first school year they have worked on the concept of function, as regards the numerical, graphical, and symbolic aspects. In particular, as regards graphical and numerical aspects, they have also used other software such as spreadsheets, Graphic Calculus, and TI-InterActive. Additionally, they have worked with motion sensors."

7. *Describe the kind of understanding that you expected the students to develop during this lesson.*

   "After having characterized, in previous lessons, exponential growth (I mean exponential successions) as growth for which the ratio of two successive values is constant, I wanted students to understand why a^x with a greater than 1 grows with x more speedily than any polynomial growth. The aim of the DGS file was to make the students understand that an exponential growth is directly proportional to the value of the function itself. This is an important step in understanding why the derivative of an exponential function is still an exponential function of the same base."

8. *What problems/skills/concepts did you expect students to meet in this lesson, and to what extent did you expect them to be able to use these in future lessons?*

   "The main aim of the posed activity was to allow students to develop an understanding of the concept of exponential growth. In previous activities, students faced the study of exponential successions and characterized them as successions for which the ratio between two consecutive terms is constant. With this activity, with the help of Cabri, I wanted the students to understand that exponential functions are functions for which the growth is proportional to the function itself. In other terms, the derivative of an exponential function is proportional to the function itself. This consideration, in my opinion, should allow students to understand why the exponential function a^x with a greater than 1 grows with x faster than any power of x."

9. *How did you plan the lesson and organize the classwork so that this learning would take place?*

   "An idea was that of preparing worksheets in Cabri of increasing difficulty. The first worksheet has only two points, one on the x-axis and the other on the y-axis, tied by the relationship $y = 2.7^x$. […] The second sheet allows students to look at what happens to an exponential function if the base changes, while the third worksheet gives a local and a global approach to the exponential function thanks to the construction of the derivative of an exponential function. […]"

10. *What would it be useful to do after this lesson, to take it beyond the group work shown in the video?*

    "[…] Generally, in the follow-up, I continue the work following two paths. In the first one I pose some problematic situations which, to be solved, ask for exponential models. In the second one I present the properties of exponentials and I introduce the logarithmic function as the inverse function of an exponential. […] Finally I propose some techniques to solve exponential and logarithmic equations and inequations, […]."

11. *What mathematical knowledge do you expect to "institutionalize" – in the sense of giving it some kind of explicit "official" recognition for the future work of the class – following on from this lesson?*

    "[…] I'm strongly convinced that the main function of teaching, not only of math, is to help students to exercise critical thought, to acquire the necessary competences for an informed and aware citizenship. This is the main aim of my lessons and of my work with students. Generally, then, I try to assess in the students the competence to observe and explore situations; to produce and to support conjectures; to understand what they are doing and to reflect on it […] then I try to understand, in my follow-up work, whether students are able to use knowledge constructed with activities like that of the video to face situations which require simple exponential models. […] A typical exercise is like the following: What can you say about the Inequation $1.1^x - 1 > 1000 \cdot x^{100}$? Justify your answer."

12. *Do you expect to find different levels of thinking when you evaluate students' work on this task sequence? If so, what are these levels, and how do you recognize them?*

    "A first level is that of perceiving the different velocity of variation that exists between x and a^x. […] A second level is that of the understanding of how the graph of an exponential function varies when the base varies. A third level, as in the third worksheet of Cabri, is relative to the understanding that the incremental ratio is a function of two variables (the x and the increment h). […] A fourth level is the passage from the local to the global aspects of the derivative. From the gradient to the gradient function. […] Generally, from the third level, the understanding happens only thanks to the direct intervention of the teacher in the small groups and this understanding is consolidated in the mathematical discussions guided by the teacher with the whole class."
13. *Afterwards when the derivative is taught in the formal way, what are the effects of the students having experienced these tasks on their thinking and on your way of explaining?*

    "When I tell about the formal aspects of the derivative I often make some reference to these experiences and activities. It seems to me that also a lot of students are able to make these connections to give meaning to formal aspects. […] For example, the formal calculation of the derivatives can be reduced to the algebra of linear functions if one uses the local linearity of a derivable function. And the local linearity finds its cognitive root in the local straightness of which students have experience thanks to the zooming function of the case they have used."

…

17. *Contextual information about the activity (How does it insert in the didactical path? How is it carried out? In what part of the year?) [written answer in original English]*

    "[…] The worksheet proposed in the videotaped activity is situated in the middle of lesson 7, before the formal approach to the concept of derivative of a polynomial function […] and before the idea of how is it possible to locally approximate a function with a quadratic function.

    The activity intends to clarify the principal features of increasing behaviors and of exponential functions. In particular, it intends to explain the reason why at the increasing of x an exponential of base greater than 1 will increase, definitively, more than any other polynomial function of x, whatever grade of the polynomial. In the project, exponential functions and sequences are used to cope with problem situations coming out from exponential models. […]"

    The most significant and important needs that have brought the creation of the project are:

    - Creating teaching-learning environments that are sensed in the double meaning given by Galileo: linked to senses, perception, but also guided by intellect and theory. […]
    - Engaging students in knowledge building, settlement, reorganizing and communicating, thus providing the teacher tools for obtaining information not only on the products, but also on the cognitive processes, necessary for any serious evaluation escaping the chimera [i.e. wrong idea] of objectivity; […]

18. *Goals, intentions, and methodology as designer of the project (Why is the activity carried out? How? What is its contribution in the global project?) [written answer in original English]*

“[…] The possibilities that new technologies offer to make experiences, to observe, to foster the production of conjectures are a wonderful tool to help students in their approach to theoretical thinking if the didactic contract is clear and includes the justification of the produced conjectures. That means asking, at any school level, questions of the kind: why?

The answer to such kind of question is located, finally, in theories. At the beginning students will tend to explain facts by means of facts. This exercise will lead them, with the guide of the teacher, to seize relationships between facts and thus to feel the need of finding out laws (propositions, axioms, …) that can be chosen to explain the observed facts. When this need is felt, the student is already in the theoretical thinking and the following passages, as the presentation of the organization of the institutional knowledge, i.e. well organized theories, can be done with good hope of success, above all if the necessary didactical attentions are not underestimated. […]

2.2.3 Extra Episode After Task 3 (Extra Video)

The data set was completed by a second video excerpt, containing an episode that occurred immediately after the students completed writing their answers to Task 3. It lasts about a minute and a half, and shows Giovanni and Carlo discussing with the teacher whether the exponential function can be approximated with a straight line, when x is very big. This extra episode provided the fabric for two case studies, which involved APC, AiC, and IDS perspectives (see Chaps. 3, 6, and 7). The transcript can be found in the Appendix A.2.

Before this transcript starts, Giovanni and Carlo worked on Task 3 starts by reading the text on the worksheet, and interpreting it with respect to the DGS file. In particular, they focus on a, PH, and Δx, and discuss whether PH and Δx are the same thing. The students quickly observe that the point P and the base a (which they call “the rate of growth”) can be varied; they also note that Δx can be varied but only after they called the unnamed vertex of the triangle H (see Fig. 2.5; the coordinates of H are (x_Q, y_P)), and identified Δx with PH (lines 249–281).

Varying P, they observe that HQ varies with P, and that as P moves to the left, HQ becomes small and the secant appears to become a tangent. They briefly and vaguely also comment on what happens as PQ gets small (lines 298, 301) and mention the option of varying a, but then return to consider the effect of varying P as PQ is constant. They also explore and comment on what happens for “P near zero,” that is $y_P \to 0$. They identify $PH = \Delta x$ (lines 282–323).

Now the teacher joins them, and participates in the conversation until almost the end of the lesson (line 368). The teacher’s participation is active – he does not only

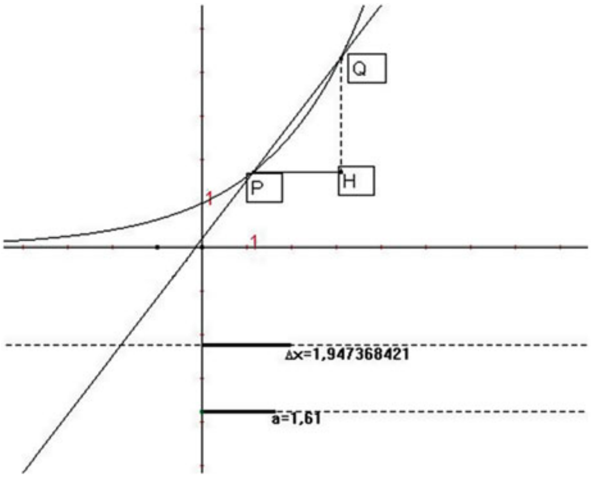


Fig. 2.5 The computer screen configuration (with the triangle PQH highlighted here for clarity)

ask questions but also provides information. In lines 325–343, the first issue discussed with the teacher is what happens as Δx becomes very small. This issue is brought about by the teacher, whereas the students are initially referring to the points P and Q as having the same distance (lines 327–330). While the students focus on the phenomenon that the line becomes (nearly) a tangent, the teacher keeps asking what information this provides for them. Nested within this segment, the students recall that the (secant) line approximates the function better, the closer P is to $y=0$ (lines 331–334).

Under the teacher's questioning and prompting, the students conclude that the exponential function could be approximated by a set of little tangent elements, each steeper than the preceding one (lines 344–353). The teacher then guides a discussion establishing that the "growth percentage" or the ratio between a value and its successor (the teacher's expressions; the students repeat some of them) remains constant and that this is consistent with the growth rate being low: "the function crushes on the x-axis" (according to the teacher) when the values of the function are close to $y=0$ (for small x). The students repeat, in their own words, part of what the teacher says (lines 354–367).

The teacher leaves and the students begin to summarize what they are going to write: that the exponential function can be approximated by little straight line segments of increasing slope; that for small x, these straight-line segments are almost like a (straight) line, that the graph is similar to a line and has a constant rate of growth "at the beginning," that is, $x \to -\infty$ (lines 368–369).

Finally, they turn to the question of what happens when a varies. They seem to keep P and Δx constant and observe that the area of the triangle grows as a grows (lines 370–379).

References

Arzarello, F., & Bartolini Bussi, M. G. (1998). Italian trends in research in mathematics education: A national case study in the international perspective. In J. Kilpatrick & A. Sierpinska (Eds.), *Mathematics education as a research domain: A search for identity* (pp. 197–212). Dordrecht: Kluwer.

Arzarello, F., Paola, D., Robutti, O., & Sabena, C. (2009). Gestures as semiotic resources in the mathematics classroom. *Educational Studies in Mathematics, 70*(2), 97–109.

Paola, D. (2006). Il significato di crescita esponenziale in un ambiente di geometria dinamica. *La matematica e la sua didattica, 1*, 39–58.

Part II
Diversity of Theories

Chapter 3
Introduction to the Approach of Action, Production, and Communication (APC)

Ferdinando Arzarello and Cristina Sabena

Abstract By referring to the data presented in Chap. 2, the chapter introduces the theoretical approach of Action, Production, and Communication (APC) and the related tool of the semiotic bundle. APC provides a frame for investigating semiotic resources in the classroom. It addresses the use of semiotic resources from a multimodal perspective including the analysis of gestures as a resource for thinking and communication.

Keywords Theories • Action/Production/Communication • Semiotic bundle

3.1 APC Approach – An Overview

The APC approach focuses on classroom processes of teaching and learning mathematics, on both cognitive and didactic levels. APC means "Action, Production, and Communication," which are considered to be three fundamental components of mathematical activity in the classroom's social context. These components are to be seen as mutually enriching, and inseparable, and are analyzed with a semiotic lens called a "semiotic bundle."

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In this chapter we introduce the main theoretical elements of this theory, and illustrate it by referring to selected episodes from the video of Carlo, Giovanni, and the exponential function (presented in Chap. 2). For further discussion and examples, the reader may refer to Arzarello (2008), Arzarello et al. (2009, 2011), and Sabena (2007, 2008).

3.1.1 Principles

The APC approach has its foundation mainly in two complementary theoretical assumptions: the multimodal perspective on cognition and communication, and the social-cultural characterization of human activity and thinking. Multimodality has its roots in the psychological theories that emphasize the crucial role of the body in thinking and knowledge development: the most recent is the so-called embodied cognition perspective. The relevance of the social-cultural dimension draws on the work of Vygotsky and Vygotskian scholars.

In the next sections, we show how the integration of these theoretical elements allows us to characterize an interpersonal cognitive space, called *Space of Action, Production, and Communication*, suitable for mathematics learning in a social context. We will elaborate on these notions in the next section.

3.1.1.1 Embodiment and Multimodality

Embodiment is a stream in cognitive science that assigns the body a central role in shaping the mind (for an overview, see Wilson 2002). Even if a certain importance to the body was assigned in other relevant pedagogical theories such as those from Montessori (1934) and Piaget (see Overton 2008), in mathematics education the attention to such a theme was prompted by the provocative book *Where Mathematics Comes From* by Lakoff and Núñez (2000), and then applied by researchers in several studies within the field (e.g., Arzarello and Robutti 2001; Nemirovsky 2003; Edwards 2009).

The new stance emphasized sensory and motor functions, as well as their importance for successful interaction with the environment. Criticizing the platonic idealism and the Cartesian mind–body dualism, Lakoff and Núñez (2000) advocated that mathematical ideas are founded on our bodily experiences and develop through metaphorical mechanisms. A typical example is the notion of set, which is based on the grounding metaphor “sets are containers”: using this cognitive metaphor without effort, we are able to think and say that an element is IN a set, or OUTSIDE a set, and so on, as it would be IN a container or OUTSIDE of it.

The importance of body experiences was not completely new to the field of education: for instance, Piaget (1952) himself stated the sensory-motor experiences as the first steps in concept formation. Against this background, the embodied perspectives brought two interesting novelties: the claim that bodily experiences intervene beyond a first phase of knowing, and permeate all the process of knowledge production; and the metaphors as cognitive mechanisms for abstract concept formation (see the above example of sets conceptualized as containers). However, we agree with Schiralli and Sinclair (2003) and with Radford et al. (2005) in recognizing several limits to the embodied cognition paradigm, in particular concerning the lack of social, historical, and cultural dimensions in the formation of mathematical concepts; for example, there may be cultural means such as speech and symbols which may shape the way in which a metaphor leads to a concept formation.

More recently, embodied stances seem to receive a certain confirmation by neuroscientific results. Specifically, we refer to results on "mirror neurons" and "multimodal neurons," which are neurons firing when the subject performs an action, when he observes something, as well as when he imagines it (Gallese and Lakoff 2005). On the basis of such neuroscientific results, Gallese and Lakoff use the notion of "multimodality" to highlight the role of the brain's sensory-motor system in conceptual knowledge. This model entails that there is not any central "brain engine" responsible for sense-making, controlling the different brain areas devoted to different sensorial modalities (which would occur if the brain behaved in a modular manner). Instead, there are multiple modalities that work together in an integrated way, overlapping with each other, such as vision, touch, and hearing, but also motor control and planning.

On the other hand, in the field of communication design, the term "multimodality" is used to refer to the multiple modes we have to communicate and express meanings to our interlocutors: words, sounds, figures, etc. (Kress 2004). With the overwhelming visual richness of our contemporary technology (web, games, tablets, etc.), and the developing possibilities of interaction with it through our body, a multimodal perspective on both thinking and communicating appears to be of increasing relevance.

3.1.1.2 The Importance of Gestures for Communication and Thinking

The multimodal perspective receives confirmation also from the studies on gestures, which have flourished in the last two decades.

Gestures are part of what is called "nonverbal communication," which includes a wide-ranging array of behaviors such as the distance between people in conversation, eye contact, voice prosody, body posture, and so on. In his seminal work, McNeill (1992) defines gestures as "the movements of the hands and arms that we see when people talk" (McNeill 1992, p. 1). This approach comes from the analysis of conversational settings and has been widely adopted in successive research studies in psychology, in which gestures are viewed as distinct but inherently linked with speech utterances. Nowadays, research in a number of disciplines (such as psychology and all its branches, cognitive linguistics, and anthropology) is increasingly showing the importance of gestures not only in communication, but also in cognition (e.g., see Goldin-Meadow 2003; McNeill 1992). Curiously, Kendon (2000) argues that it has been the interest in cognition prompted by Chomsky's view of linguistics as a kind of purely mental science that has led to the vigorous investigation of gestures by those interested in language:

> If language is a cognitive activity, and if, as is clear, gestural expression is intimately involved in acts of spoken linguistic expression, then it seems reasonable to look closely at gesture for the light it may throw on this cognitive activity. (Kendon 2000, p. 49)

Gestures are usually characterized as follows (McNeill 1992): they begin from a position of rest, move away from this position, and then return to rest. The central part of the movement, generally recognized as expressing the conveyed meaning, is called *stroke* or peak; it is preceded by a preparation phase (hand/arm moving from its resting place, and usually to the front away from the speaker), and symmetrically succeeded by a retraction phase (hand/arm back to the quiescence). Speakers of European languages usually perform gestures in a limited space in the frontal plane of the body, called *gesture space*, which goes roughly from the waist to the eyes, and includes the space between the shoulders. However, differences have been detected according to age (the *gesture space* of children is larger) and different cultural settings.

McNeill (1992) provides also an often-quoted classification of gestures, distinguishing the following categories:

- *iconic* gestures bear a relation of resemblance to the semantic content of discourse (object or event);
- *metaphoric* gestures are similar to iconic gestures, but with the pictorial content presenting an abstract idea that has no physical form;
- *deictic* gestures indicate objects, events, or locations in the concrete world;
- *beats* appear when hands move along with the rhythmical pulsation of speech, lending a temporal or emphatic structure to communication.

More recently, the cohesive function of gesture has been further deepened, and theorized with the notion of *catchment* (McNeill 2005). A catchment is recognizable when some gestures' form features are seen to recur in at least two (not necessarily consecutive) gestures. According to McNeill, a catchment indicates discourse cohesion, and it is due to the recurrence of consistent visuospatial imagery in the speaker's thinking. Catchments may, therefore, be of great importance giving us information about the underlying meanings in a discourse and about their dynamics:

> By discovering the catchments created by a given speaker, we can see what this speaker is combining into larger discourse units – what meanings are being regarded as similar or related and grouped together, and what meanings are being put into different catchments or are being isolated, and thus are seen by the speaker as having distinct or less related meanings. (McNeill et al. 2001, p. 10)

In the classroom context, we believe that a catchment can indicate a student expressing concepts he cannot well express in words. In this sense, catchments are also relevant to analyze concept formation (see the examples regarding Carlo in Sect. 3.2 below). Furthermore, catchments may also give clues about the organization of arguments at a logical level (for a discussion applied to mathematics discourses, see Arzarello and Sabena 2014).

3.1.1.3 The Social-Cultural Dimension and the Role of Signs

As mentioned above, the main limit of embodied cognition is in having neglected the social and cultural dimensions in which mathematical concepts arise and evolve, and the fundamental role of signs therein. With this respect, the APC frame takes a Vygotskian perspective. In particular, according to the *genetic law of cultural development*, namely the general law governing the genesis of higher mental functions, there is a passage from *interpsychic* functions, that are shared on the social level, to *intrapsychic* ones, that relate to the person on the individual level:

> Every function in the child's cultural development appears twice: first, on the social level, and later on the individual level; first, between people (interpsychological), and then inside the child (intrapsychological). This applies equally to voluntary attention, to logical memory, and to the formulation of concepts. All the higher functions originate as actual relations between human individuals. (Vygotsky 1978, p. 57).

Furthermore, Vygotsky claims, cultural development is based on the use of signs. Such a general process, accounting for the formation of human consciousness by progressive individualization of inherently social functions, is called *internalization*.

By virtue of the social meaning, signs serve to the individual to exert a voluntary control on his/her behavior, in a way similar to the road sign signaling to the individual the events to regulate his conduct. Without the meaning, words, mnemotechnic signs, mathematical symbols, and all other symbol systems would be nonsense. *Meaning* has therefore a psychological character, rather than a theoretical/abstract one (Leont'ev and Lurija 1973). Meaning allows the human being to produce those *changes to external things* – that are *signs* – that in a second moment express their transformative action on the interior plane of proper psychological processes, thus allowing him to "control," or "appropriate" the criteria to direct his own behavior. Semiotic mediation accomplishes therefore a fundamental role in the formation of the "plane of consciousness" (Wertsch and Addison Stone 1985).

3.1.2 Key Theoretical Constructs and Methodology

On the bases of the presented principles, the APC approach is based on the idea of Space of Action, Production, and Communication, and its analytical tool: the semiotic bundle. The methodology includes the application of fine-grained analysis, carried out with the aid of video-recording tools. We will present theoretical and methodological tools in the following paragraphs.

3.1.2.1 The Space of Action, Production, and Communication

The notion of Space of Action, Production, and Communication (in short, APC-space) has been introduced by Arzarello (2008). It is a model that intends to frame the processes that develop in the classroom among students and the teacher while working together. The main components of the APC-space are:

- the body;
- the physical world;
- the cultural environment.

These components include students' perceptuo-motor experiences, languages, signs, and resources that they use to act in the environment and to socially interact:

> An APC-space is the unitary system of the three main components listed above, amalgamated in a dynamically evolving unit within a concrete learning situation in the classroom, because of the action and mediation of the teacher, who suitably orchestrates their integration. (Arzarello 2008, p. 162)

The APC-space is a theoretical construct aimed at modeling the didactic setting and the teaching–learning process. Considering the classroom context, the APC-space pinpoints the conditions in which the learning process can be fostered:

> The APC-space is built up in the classroom as a dynamic single system, where the different components are integrated with each other into a whole unit. The integration is a product of the interactions among pupils, the mediation of the teacher and possibly the interactions with artifacts. The three letters A, P, C illustrate its dynamic features, namely the fact that three main components characterize learning mathematics: students' actions and interactions, their productions and communication aspects (ibid., p. 162).

"Space" is to be intended not as a physical entity, but rather in an abstract way, as in mathematics theories. Framed in a socio-cultural perspective, the APC-space is an intersubjective space, involving students and the teacher. It is a typical example of a complex system, in which the global result does not derive linearly from the simple superposition of its components. For an APC-space to be active and to work, it is obviously not sufficient that its components are present in the classroom: bodies, physical world, and cultural environment are certainly always there! The teacher is responsible for the construction of the mathematical knowledge in the classroom, and this responsibility realizes first in the setting of didactic activities, and then in the support of the evolution of the personal senses of the students towards the scientific ones. The teacher is hence an active part of the APC-space. Another important dimension is time: the APC-space, gauged at accounting the teaching–learning process, is a complex dynamic system evolving in time.

3.1.2.2 The Semiotic Bundle

When the students interact (with each other and with the teacher) in the APC-space, the result is not a linear development, but a complex interplay of multimodal actions, productions, and communications. Within the Vygotskian frame outlined above, the semiotic lens can be considered a good tool for observing such an interplay (see also Bartolini Bussi and Mariotti 2008). The semiotic bundle notion is elaborated in the next paragraphs in order to consider, besides linguistic and mathematical semiotic systems, also embodied ones, such as gestures.

The notion is based on Peirce's theorization, according to which a sign is a triad constituted by the sign or *representamen* (that represents), the *object* (that is represented), and the *interpretant* (specifying in which respect the representamen is representing the object). In Peirce's words, anything that can be interpreted by somebody in some respect can be considered as a sign (Peirce 1931–1958, vol. 2, par. 228). The interpretant is the most delicate element, since it constitutes a new sign (conceived in the triadic way), generating a new interpretant, and so on.

Such a characterization of "signs" provides us with two features apt for our needs: the first one regards the generality of the definition of sign, and the second one the dynamicity of the semiotic processes, framed with the idea of the "interpretant."

Basing on this approach to signs, the semiotic bundle notion considers both static and dynamic aspects. It consists in:

> a *system of signs* […] that is produced by one or more interacting subjects and that evolves in time. Typically, a semiotic bundle is made of the signs that are produced by a student or by a group of students while solving a problem and/or discussing a mathematical question. Possibly the teacher too participates to this production and so the semiotic bundle may include also the signs produced by the teacher. (Arzarello et al. 2009, p. 100)

As an example, we can consider the set of gestures and the set of words that are produced during a certain problem-solving activity. The two sets are intertwined, because they are used simultaneously during the activity: so they constitute the elements of the semiotic bundle for that activity.

Differently from other semiotic approaches in mathematics education (e.g., Duval 2006; Ernest 2006), the semiotic bundle includes all the bodily means of expression, such as gestures, gazes, sketches, and so on, as semiotic resources in teaching and learning. Such an approach widens the notion of a semiotic system, so that signs can include gestural and segmented forms of language, which we consider as fundamental components of the multimodal activities in the classroom.

The semiotic enlargement described has also been favored by a refinement of the tools used for the observation of relevant activities in the classroom. Video-recordings play a crucial role, in that they can be examined in detail, in order to carefully analyze the observed processes.

Based on these videos, a transcript including information about gestures is produced, and used for the a posteriori analysis. The analysis, however, is carried out not only by relying on the transcript, but also by looking constantly and repeatedly to the videos. Specifically, the analysis of the semiotic bundle considers dynamics along two dimensions:

- a *diachronic analysis*, focused on the evolution of signs over time, and the transformation of their relationships (in periods with variable length, from a few minutes to years);
- a *synchronic analysis*, focused on the relations among the signs used in a certain moment.

In this way, the focus of the analysis is on the ongoing dynamic contextual teaching and learning processes where the cognitive aspects intertwine with the didactic and communicative ones.

3.1.2.3 The Semiotic Game

Analyzing the teaching and learning activities with the above-mentioned enlarged semiotic lens, general and specific results have been detected (for a general account, see Sabena et al. 2012).

The most important result regards the role of the teacher in the multimodal perspective: the so-called "semiotic game" between teacher and students (Arzarello and Paola 2007; Arzarello et al. 2009). A semiotic game may occur when the teacher is interacting with the students, as in classroom discussions or during group-work. In a semiotic game, the teacher tunes with the students' semiotic resources (e.g., words and gestures), and uses them to make the mathematical knowledge evolve towards scientifically shared meanings. More specifically, the teacher uses one kind of sign (typically, gestures) to tune with the students' discourse, and another one to support the evolution of new meanings (typically, language). For instance, the teacher repeats a gesture that one or more students have just made, and accompanies it with appropriate linguistic expressions and explanations. Such semiotic games can develop if the students produce something meaningful with respect to the problem at hand, using some signs (words, gestures, drawings, etc.). It is apt for the teacher to seize these moments to enact her/his semiotic game. Even a vague gesture of the student can really indicate a certain comprehension level, even when the student has not yet the words to express himself at this level. In a Vygotskian frame, the semiotic game is likely to "work," that is, be useful to the student, if the student is in a zone of proximal development for a certain concept (Vygotsky 1978), so that the teacher may have the chance to intervene in its cognitive development. The intervention is imitative-based, that is, the teacher imitates the students' gestures and accompanies them with certain scientific meanings (expressed in appropriated words), in order that in the following, the students will be able to imitate the teacher's words. At the same time, the teacher encourages the student, signaling that his idea, though not fully or correctly expressed, is on the right way on learning.

3.1.3 Questions

The typical research questions asked within the APC approach are the following:

- What is the role of gestures in the development of mathematical concepts?
- What are the roots of the mathematics representations in students' activities?
- What is the role of the teacher, considering the multimodal perspective?
- How do the different components of the semiotic bundle concur to the conceptualization processes in students?
- What are the different relationships between the components and their evolution in time?

3.2 Illustrating the APC Approach Through Analysis of the Video of Carlo, Giovanni, and the Exponential Function

In the following, we will illustrate the APC approach by selecting excerpts from the video of Carlo, Giovanni, and the exponential function (see Sect. 2.1) and analyzing them in accordance with the notion of a semiotic bundle.

3.2.1 Exploring $\mathbf{y = a^x}$

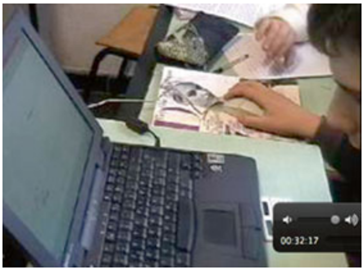
The first episode refers to the students facing Task 2 (see Sect. 2.1.3). After reading the text of the task, Carlo and Giovanni construct $y=a^x$ with the Dynamic Geometry Software. They start exploring the function, according to the task request:

> Then moving the point A changes the base of the exponential. Moving the point P, you run along the graph of an exponential function with a fixed base. Explore, share your impressions (is there something which is not clear and we were not expecting or that is clear and you were expecting). Describe briefly your exploration on the sheet. (See Fig. 2.2 in Sect. 2.1.3)

They decide to consider the case in which the base a is very big. Let us analyze how they start to explore the function in the transcript lines 160–165 (see Appendix for full transcript. In the transcript, underlining designates the part of an utterance during which the speaker gestured.)

160 G we try to move *A*
161 C try to put the *a* very high [*moving his hand upwards, at the top of the screen*]… when we have seen to happen that chaos *[meaning: in a previous lesson]*
162 G no, it always gets… because here it is interrupted… because here it is interrupted
163 C wouldn't it do like this? [*Gesture a*]
wouldn't it do like this? [*Gesture b*]

Gesture in 163 (a): C's quick gesture with right hand



Gesture in 163 (b): like Gesture a with more visible hand, going upwards very steeply

164 G what?

165 C to do <u>like this</u> [*gesture*]

Gesture in 165: C's similar gesture, more evident, with the hand moving very steeply upwards

In this episode, we can observe a semiotic bundle composed of three different kinds of semiotic resources: spoken words, graphical representations on the screen, and gestures. They are strictly interrelated: using words and gestures the students are discussing the behavior of the exponential function and its graph on the screen, when the base is "very big." By using words and gestures, Carlo is making a conjecture of the graph (line 163–165): through words he is indicating the case he is considering ("the *a* very <u>high</u>", line 161) and with gestures he is showing how he is imagining the graph will be (screenshots (a) and (b) in line 163 and in line 165).

While speaking very few words, Carlo performs three gestures, which show similar features: the shape of the hand, and the dynamic movement going upwards (although the concavity changes). This is a case of catchment (McNeill et al. 2001). In the repetition, the gesture becomes bigger and bigger, being performed in a greater space and longer time. Even if we cannot see Carlo's gaze to confirm this, our interpretation is that the student is performing the second and third gesture to show it to Giovanni, who is looking at him (the video shows that Giovanni is turning his head towards Carlo). In the evolving APC-space, Carlo's semiotic resources are used first as thinking tools, in order to produce a conjecture, and then as a communicative means.

As confirmed by the teacher (personal communication), Carlo has some difficulties in expressing his ideas in oral and written language; here we can see that the gesture is co-timed with deictic terms ("like this") that point to the gesture itself: the gesture is indeed part of his thinking and communication means, and in the semiotic bundle, words and gestures complete each other (with reference to the shown screen).

3.2.2 Formulating the Written Answer

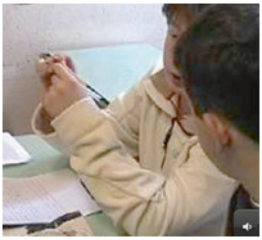
A gesture similar to those discussed in the previous paragraph appears again some minutes later, when the students are about to write the solution:

189 C well… so we write that… let's say: the point A… we put that one thing we had said… [*Gesture a*], we had said that…
I'm still thinking if… [*not understandable*] how I can say… but… also for a <u>same space of the *x*</u> [*Gesture* b], the *y* increases a lot [*Gesture c*]

Gestures in 189:

(a) Carlo's quick gesture in the air

(b) C's fingers close to each other



(c) C's right hand moving upwards

Carlo is offering to write the answer for the teacher. The answer has to be in written language, and still we can observe his difficulty in finding the right words to express what he is proposing: his words have little semantic content ("we put that one thing we had said…"), whereas on the contrary his gesture (screenshot (a) in line 189) offers again the pictorial image he was proposing some minutes before, that is, a graph with a high slope. This is another case of catchment, which can be detected by looking at the semiotic bundle in a diachronic way.

In the second part of line 189, Carlo is connecting the very inclined graph (see the hand moving almost vertically in screenshot b) to the incremental ratio of the function: his fingers are indicating a very small interval on the x-axes, and his words relate this fixed interval of abscissas with increasing increments of the ordinates ("for a <u>same space of the</u> <u>x</u>, the y increases a lot", line 189). Let us notice that the information that x-increments are considered small is expressed only in the gesture (which is therefore non-redundant, in the sense of Kita 2000); however, it is the co-timing with the words that allows the student to connect this information with the variation of the corresponding y-increments: this kind of analysis is typical of the semiotic bundle lens, and witnesses the potential of such an analytical tool.

While Carlo is talking-gesturing, Giovanni is looking at him and following his argument. He immediately agrees, and helps Carlo to find the right words for expressing his ideas in the following lines of the transcript:

190 G yes

191 C eh… how do I say that?

192 G or you can say that with the <u>differences</u> [*in parallel, Carlo gestures*]

Gesture in 192: Carlo is gesturing with two hands parallel to each other; Giovanni is performing a beat gesture

193 C for an [*gesture*] interval [...*inaudible*]
194 G the differences are bigger and bigger
194 C the differences, right?
195 G yes
Carlo finishes writing the answer.

Gesture in 193: C's gesture with two parallel hands is anticipating the word "interval"

The students are now going about producing the written answer. However, this formulation moment is not purely a communicative moment. As a matter of fact, Carlo interrupts himself many times while writing, with seconds of silence, gestures, and words. According to a Vygotskian perspective, the writing act, fostered by the social dimension (the teacher asking for a written answer), has deep influence on the thinking processes.

Giovanni is enriching the semiotic bundle with the word "differences," which Carlo could not find (line 192). While Giovanni is pronouncing it, Carlo is performing a gesture with two parallel open palms (gesture in line 192). Carlo's gesture with the two hands represents the ends of an interval on the x-axes: however, the word "interval" appears only later in his speech (line 193), with respect to which the gesture is anticipatory.

By contrast, there is a perfect interpersonal synchrony between Giovanni's words and Carlo's gesture: such synchrony, which can be detected with a synchronic analysis of the semiotic bundle, is an indication that the students are sharing an active APC-space (Sabena 2007). Another clue in the same direction is provided by the fact that Giovanni is completing Carlo's sentence (lines 193–194), with a perfect timing (there is no time left between the two sentences in lines 193 and 194): due to the close coordination, a careful listening of the video-recording is necessary in order to identify which student is speaking.

Such kinds of semiotic acts are accessible to the researcher only by means of video-recordings and a careful micro-analysis of video and screenshots. They have been observed in students' joint activity also in other contexts (e.g., for algebraic context, Radford et al. 2007), and appear most likely after students have developed a fruitful cooperation in group-work and are deeply engaged in the problem at hand. In the perspective of APC-space, the semiotic bundle analysis provides a suitable lens to seize them, and to study their role in mathematics learning.

3.3 Conclusion

In the chapter we have illustrated how the theoretical construct of APC-space can be useful to properly underline the way mathematical concepts are built up by students. The example of Carlo and Giovanni illustrates how three different kinds of semiotic resources intertwine in this complex dynamic process: spoken words, graphical representations on the screen, and gestures. These are typical inhabitants of the APC-space and as such they embody the actions, productions, and communications of students; from its side, the semiotic bundle lens allows pointing out how such components concretely intertwine and evolve in time. We can use a metaphor from physics to point out the differences between the two notions. In dynamics there is the second law of Newton, $\boldsymbol{F}=m\boldsymbol{a}$; to completely understand it, one must operatively define what are force, mass and acceleration. But this is only half of the story – the other half consists in understanding how the three quantities relate each other in expressing a law of physics; the law is a lens that allows the modeling of the motions of classical mechanics. In our case the three components of the APC-space (action, production, communication) are pointed out as basic components of the didactical phenomena in the classroom; the semiotic bundle describes the mutual relationships between them in time. Of course, didactics is not an exact science and the metaphor must be considered *cum grano salis*: the semiotic bundle is not like a physical law but is a construct that qualitatively describes the way the three components of APC relate each other in the classroom, because of the interactions between the students or between the student(s) and the teacher. This phenomenological description possibly points out some didactical phenomena that systematically happen, for example the semiotic game: in the metaphor it corresponds to the use that one can make of the second principle of dynamics to design the trajectory of a rocket. In the same way, a teacher, who is aware of how the components of the APC-space interact in the semiotic bundle, can play a semiotic game to support a student towards a better understanding and formulation of a mathematical concept.

The main result of this approach consists in pointing out not only that more variables than the purely discursive ones are important in the didactical processes, but also defining suitable observation methods in order to give reasons to them. This issue shows the partiality of all those descriptions, which limit to comment only the protocols of the speech or written productions of students. In fact our model aims at better giving account of learning processes as dynamic phenomena, so overcoming the limits pointed out by Freudenthal, when he wrote:

> Indeed, didactics itself is concerned with processes. Most educational research, however, and almost all of it that is based on or related to empirical evidence, focuses on states (or time sequences of states when education is to be viewed as development). States are *products* of previous *processes*. As a matter of fact, *products* of learning are more easily accessible to observation and analysis than are learning *processes* which, on the one hand, explains why researchers prefer to deal with states (or sequences of states), and on the other hand why much of this educational research is didactically pointless. (Freudenthal 1991, p. 87, emphasis in the original)

We have been able to point out a wider range of observables to look at in order to understand the life in the classroom. Of course this does not mean to say that the discursive productions are useless, but only that they must be integrated within the more complex picture given by the semiotic bundle, according to a multimodal perspective. In other chapters of the book (see Chaps. 9 and 11) we will show how this approach can usefully be integrated with other approaches, more based on discursive analysis.

A further by-product of this research consists in indicating a clear position in respect of the complex intertwining between culture and nature in students' performances. The debate about the relationships between the two components has been a must in psychology (McLeod 2007) and has generated considerable discussion also in mathematical education: indeed, the National Association of Mathematics Advisers (http://www.nama.org.uk/index.php) held its 2013 Conference on this issue. For example, the gesture–speech unity (McNeill 1992) of our productions is a typical construct that shows the two aspects to be deeply intertwined: biological and cultural aspects are inextricably bound together in all our performances within the APC-space when we as students (teachers) are learning (teaching) mathematics.

References

Arzarello, F. (2008). Mathematical landscapes and their inhabitants: Perceptions, languages, theories. In E. Emborg & M. Niss (Eds.), *Proceedings of the 10th International Congress of Mathematical Education* (pp. 158–181). Copenhagen: ICMI.

Arzarello, F., & Paola, D. (2007). Semiotic games: The role of the teacher. In J. H. Woo, H. C. Lew, K. S. Park, & D. Y. Seo (Eds.), *Proceedings of the 31st conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 17–24). Seoul: PME.

Arzarello, F., & Robutti, O. (2001). From body motion to algebra through graphing. In H. Chick, K. Stacey, & J. Vincent (Eds.), *Proceedings of the 12th ICMI study conference: The future of the teaching and learning of algebra* (Vol. 1, pp. 33–40). Melbourne: ICMI.

Arzarello, F., & Sabena, C. (2014). Analytic-structural functions of gestures in mathematical argumentation processes. In L. D. Edwards, F. Ferarra, & D. Moore-Russo (Eds.), *Emerging perspectives on gesture and embodiment* (pp. 75–103). Greenwich: Information Age Publishing.

Arzarello, F., Paola, D., Robutti, O., & Sabena, C. (2009). Gestures as semiotic resources in the mathematics classroom. *Educational Studies in Mathematics, 70*(2), 97–109.

Arzarello, F., Ferrara, F., & Robutti, O. (2011). A tool for analysing multimodal behaviours in the mathematics classroom. In *Proceedings of PME 35*. Ankara: Middle East Technical University.

Bartolini Bussi, M. G., & Mariotti, M. A. (2008). Semiotic mediation in the mathematical classroom. Artifacts and signs after a Vygotskian perspective. In L. English (Ed.), *Handbook of international research in mathematics education* (2nd revised ed., pp. 746–783). Mahwah: Lawrence Erlbaum.

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics, 61*, 103–131.

Edwards, L. (2009). Gestures and conceptual integration in mathematical talks. *Educational Studies in Mathematics, 70*(2), 127–141.

Ernest, P. (2006). A semiotic perspective of mathematical activity: The case of number. *Educational Studies in Mathematics, 61*, 67–101.

Freudenthal, H. (1991). *Revisiting mathematics education: China lectures*. Dordrecht: Kluwer.

Gallese, V., & Lakoff, G. (2005). The brain's concepts: The role of the sensory-motor system in conceptual knowledge. *Cognitive Neuropsychology, 21*, 1–25.

Goldin-Meadow, S. (2003). *Hearing gestures: How our hands help us think*. Cambridge, MA: Harvard University Press.

Kendon, A. (2000). Language and gesture: Unity or duality? In D. McNeill (Ed.), *Language and gesture: Window into thought and action* (pp. 47–63). Cambridge: Cambridge University Press.

Kita, S. (2000). How representational gestures help speaking. In D. McNeill (Ed.), *Language and gesture: Window into thought and action* (pp. 162–185). Cambridge: Cambridge University Press.

Kress, G. (2004). Reading images: Multimodality, representation and new media. *Information Design Journal, 12*(2), 110–119.

Lakoff, G., & Nùñez, R. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. New York: Basic Books.

Leont'ev, A. N., & Lurija, A. R. (1973). Le idee psicologiche di Vygotskij. In L. S. Vygotskij (Ed.), *Lo sviluppo psichico del bambino*. Roma: Editori Riuniti.

McLeod, S. A. (2007). *Nature nurture in psychology.* http://www.simplypsychology.org/naturevs-nurture.html. Accessed 21 Oct 2013.

McNeill, D. (1992). *Hand and mind: What gestures reveal about thought*. Chicago: University of Chicago Press.

McNeill, D. (2005). *Gesture and thought*. Chicago: University of Chicago Press.

McNeill, D., Quek, F., McCullough, K.-E., Duncan, S., Furuyama, N., Bryll, R., et al. (2001). Catchments, prosody and discourse. *Gesture, 1*(1), 9–33.

Montessori, M. (1934). *Psicogeometria.* Barcelona: Araluce. (English translation published by Montessori-Pierson Publishing Company.)

Nemirovsky, R. (2003). Three conjectures concerning the relationship between body activity and understanding mathematics. In N. A. Pateman, B. J. Dougherty, & J. T. Zillox (Eds.), *Proceedings of PME 27* (Vol. 1, pp. 105–109). Honolulu: PME.

Overton, W. F. (2008). Embodiment from a relational perspective. In W. F. Overton, U. Muller, & J. L. Newman (Eds.), *Developmental perspectives on embodiment and consciousness* (pp. 1–18). New York: Lawrence Erlbaum.

Peirce, C. S. (1931–1958). *Collected papers, Vol. I–VIII.* (C. Hartshorne, P. Weiss, & A. Burks, Eds.). Cambridge, MA: Harvard University Press.

Piaget, J. (1952). *The origins of intelligence in children*. New York: International Universities Press.

Radford, L., Bardini, C., Sabena, C., Diallo, P., & Simbagoye, A. (2005). On embodiment, artifacts, and signs: A semiotic-cultural perspective on mathematical thinking. In H. L. Chick & J. L. Vincent (Eds.), *Proceedings of the 29th conference of the international group for the psychology of mathematics education* (Vol. 4, pp. 113–120). Melbourne: University of Melbourne, PME.

Radford, L., Bardini, C., & Sabena, C. (2007). Perceiving the general: The semiotic symphony of students' algebraic activities. *Journal for Research in Mathematics Education, 38*(5), 507–530.

Sabena, C. (2007). *Body and signs: A multimodal semiotic approach to teaching-learning processes in early calculus*. Tesi di Dottorato, Università degli Studi di Torino.

Sabena, C. (2008). On the semiotics of gestures. In L. Radford, G. Schumbring, & F. Seeger (Eds.), *Semiotics in mathematics education: Epistemology, history, classroom, and culture* (pp. 19–38). Rotterdam: Sense.

Sabena, C., Robutti, O., Ferrara, F., & Arzarello, F. (2012). The development of a semiotic frame to analyse teaching and learning processes: Examples in pre- and post-algebraic contexts. In L. Coulange, J.-P. Drouhard, J.-L. Dorier, & A. Robert (Eds.), *Recherches en Didactique des Mathématiques, Numéro spécial hors-série, Enseignement de l'algèbre élémentaire: bilan et perspectives* (pp. 231–245). Grenoble: La Pensée Sauvage.

Schiralli, M., & Sinclair, N. (2003). A constructive response to 'Where mathematics comes from'. *Educational Studies in Mathematics, 52*, 79–91.

Vygotsky, L. S. (1978). *Mind in society. The development of higher psychological processes* (M. Cole, V. John-Steiner, S. Scribner, & E. Souberman, Eds.). Cambridge, MA/London: Harvard University Press.

Wertsch, J. V., & Addison Stone, C. (1985). The concept of internalization in Vygotsky's account on the genesis of higher mental functions. In J. V. Wertsch (Ed.), *Culture, communication and cognition: Vygotskian perspectives. The concept of activity in Soviet psychology* (pp. 162–179). Cambridge: Cambridge University Press.

Wilson, M. (2002). Six views of embodied cognition. *Psychonomic Bulletin & Review, 9*(4), 625–636.

Chapter 4
Introduction to the Theory of Didactical Situations (TDS)

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Abstract The chapter briefly introduces the Theory of Didactical Situations (TDS) by referring to the data from Chap. 2. TDS provides a systemic framework for investigating teaching and learning processes in mathematics, and for supporting didactical design. The theory is structured around the notions of a-didactical and didactical situations and includes a corpus of concepts relevant for teaching and learning in mathematics classrooms.

Keywords Theories • Theory of didactical situations

The Theory of Didactical Situations (named TDS in this volume) began to develop in the 1960s in France, initiated by Guy Brousseau who has led its development since that time. A first synthesis was published in 1997 in English (Brousseau 1997) but the theory has since developed considerably in its conceptual notions as much as its research methodologies, as attested to for instance by the special issue of the journal *Educational Studies in Mathematics* (Laborde et al. 2005) or the

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proceedings of the 2009 Summer School devoted to didactical engineering (Margolinas et al. 2011). Important aspects of these developments were motivated by the increasing investment of TDS in the study of ordinary classrooms, and many researchers worldwide have made contributions. This chapter introduces some main elements of TDS structured by principles, questions, methodologies, and key constructs as presented in Chap. 1 of this book, and invites the reader to make sense of these elements through the analysis of the two first episodes of the video of Carlo and Giovanni, introduced in Chap. 2.

4.1 Theory of Didactical Situations: An Overview

TDS is a "home-grown" theory (Sriraman and English 2010); those who have contributed to its development share with the initiator the conviction that the field of mathematics education needs to develop its own theorizations and not just borrow and adapt theories developed in connected fields such as psychology, sociology, or anthropology. In the limited space allocated to this introduction, we focus on three characteristics that create the specific lens through which TDS considers the teaching and learning of mathematics: the systemic nature of teaching and learning; the epistemology of mathematical knowledge; and the vision of learning as a combination of adaptation and acculturation. These characteristics determine the questions that TDS raises and tries to answer, as well as the methodologies it privileges.

4.1.1 Principles

Since its beginnings, TDS has adopted a systemic perspective, conceiving the didactics of mathematics as the study of the conditions for the dissemination and appropriation of mathematical knowledge through educational institutions. This systemic perspective is reflected in the organization of the theory around the idea of situation. A situation is itself a system, "the set of circumstances in which the student finds herself, the relationships that unify her with her milieu, the set of 'givens' that characterize an action or an evolution" (Brousseau 1997, p. 214). TDS is interested in *didactical situations*, that is, those designed and utilized with teaching and learning aims. Brousseau distinguishes two possible perspectives on didactical situations: a vision of these as the student's environment organized and piloted by the teacher; and a broader vision including the teacher and the educational system itself.

A first important characteristic of TDS is the attention it pays to mathematics and its epistemology. In the theory, this sensibility is expressed in different ways, notably through the reference to Bachelard's epistemology and the didactic conversion of his notion of *epistemological obstacle*, and also through the notion of *fundamental situation*. Referring to Bachelard's studies in physics which led to a list of obstacles of epistemological nature, Brousseau (1997, p. 83) extends its application to the field of didactics of mathematics, defining epistemological obstacles as forms of knowledge that have been relevant and successful in particular contexts, including often school contexts, but that at some moment become false or simply inadequate, and whose traces can be found in the historical development of the domain itself (see also Schneider 2013).

A fundamental situation for a given concept is a mathematical situation or, better, a *family of mathematical situations* for which the concept constitutes a priori an optimal solution. This epistemological analysis, connecting mathematical knowledge and situations, constitutes what Brousseau calls today a *theory of mathematical situations*, a first level of modeling and analysis in TDS, the second one being that of *didactical situations*. The well-known situation of puzzle enlargement (Brousseau 1986, 2008) can for instance be seen as a fundamental situation for linearity.

A second important distinction in TDS is linked to the following epistemological characteristic: mathematical knowledge is something that allows us to act on our environment, but the pragmatic power of mathematics is highly dependent on the specific language it creates, and on its forms of validation. This characteristic reflects in TDS through the distinction between three particular types of situations: *situations of action*, *situations of formulation*, and *situations of validation*. The first chapter in Brousseau (1997) illustrates this distinction taking as an example a succession of situations developed around the famous problem "Race to 20", conceived as a fundamental situation for Euclidian division.

The third important characteristic refers to students' cognitive dimension, particularly to the combination of the two processes *adaptation* and *acculturation*. Regarding adaptation, Brousseau's discourse shows an evident proximity with Piagetian epistemology:

> the student learns by adapting herself to a *milieu* which generates contradictions, difficulties and disequilibria, rather as human society does. This knowledge, the result of the students' adaptation, manifests itself by new responses which provide evidence for learning. (Brousseau 1997, p. 30)

But this adaptation is not sufficient; acculturation is necessary to link students' constructions with forms of knowledge that are socially shared, culturally embedded, and institutionally legitimated, being called "savoirs" in French. Such a change in the status of knowledge requires the teacher's didactic intervention and can be achieved in many different ways (Brousseau 1997).

TDS key constructs take these two types of processes into account: independent adaptation through the notions of *a-didactical situation* and *milieu*, acculturation through the notions of *didactical situation* and *didactical contract*, and the relationships between these processes through the dual notions of *devolution* and *institutionalization* (see Sect. 4.1.4).

4.1.2 Questions

The questions that TDS tries to answer are diverse but coherent with these principles. They regard:

- the functioning of didactical systems, leading to the identification of regularities, and their elaboration into didactical phenomena;
- the determination of fundamental situations associated with specific mathematical concepts, and their possible actualizations into didactical situations, taking into account the conditions and constraints of particular educational contexts;
- the dependence between situations and the progression of knowledge in particular domains, paying the necessary attention to both adaptation and acculturation processes.

Even if TDS has the ultimate goal of improving students' mathematics learning, the learner is not at the center of the theory. TDS gives priority to the understanding of how the conditions and constraints of didactical systems enable or hinder learning, and how the functioning of such systems can be improved.

4.1.3 Methodologies

The systemic perspective and TDS ambition of developing didactics as a genuine fundamental and applied field of research have also shaped the methodological development of the theory. Among the diversity of methodologies used in TDS, the systemic view led to being especially valued those methodologies giving access to the complexity of didactic systems, what resulted in an original concept: that of *didactical engineering* (Artigue 1989, 2013). It is a methodology which is structured around a phase of preliminary analysis combining epistemological, cognitive, and didactical perspectives, and aiming at the understanding of the conditions and constraints to which the didactical system considered is submitted, a phase of design and a priori analysis of situations reflecting its optimization ambition; and, after the implementation, a phase of a posteriori analysis and validation. As pointed out by Artigue (2008), the TDS theoretical basis explains:

> the importance given in it to the *a priori* analysis, and the rejection of usual validation processes based on the comparison between the pre and post characteristics of experimental and control groups, at the benefit of an internal comparison between the *a priori* analysis and the *a posteriori* analysis of classroom realizations. (Artigue 2008, p. 11)

The TDS constructs mentioned above are essential tools for researchers to design situations and carry out a priori and a posteriori analyses. In the design of learning situations, for instance, particular attention is paid to the constituents of the milieu organized for the learner, and to the optimization of the possibilities it offers, both in terms of action and feedback, to foster the emergence of the targeted strategies and knowledge. Attention is also paid to the way the devolution process is organized.

An important part of the analyses is devoted to the situation itself: what mathematical sense can emerge from the interactions between the student and the milieu; is the situation sufficiently accessible and efficient to enable the student to have access, by adaptation, to an appropriate meaning of the target concept? This analysis often relies on an epistemological analysis of the concept at stake.

Since the 1990s, the increasing use of TDS for the study of ordinary classrooms has led to the development of new methodologies based on participative and naturalistic observations, but there is no doubt that didactical engineering, which has itself evolved as a methodology (Margolinas et al. 2011), remains the privileged methodology in TDS when a research project includes a design dimension.

4.1.4 Key Constructs

We focus in this part on the key constructs necessary for making sense of the video analysis developed in section "Illustrating the theory through analysis of the video of Carlo, Giovanni, and the exponential function", but do not enter into the recent developments of TDS. Complementary insights will be introduced in other chapters of the book.

4.1.4.1 A-Didactical Situation and Milieu

The notions of *a-didactical situation* and *milieu* are attached to the vision of learning as an adaptation process and to the ambition of optimizing such a process. This means elucidating and creating the conditions for making the target mathematical knowledge emerge from students' interaction with a milieu, as the optimal solution to a mathematical problem. As explained by Brousseau (1997, p. 30), in a-didactical situations the students accept to take the mathematical responsibility of solving a given problem, and the teacher refrains from interfering and suggesting the target mathematical knowledge for making such adaptation processes possible. As Brousseau (1997) stresses:

> The student knows very well that the problem was chosen to help her acquire a new piece of knowledge, but she must also know that this knowledge is entirely justified by the internal logic of the situation and that she can construct it without appealing to didactical reasoning. (p. 30)

Hence comes the name of *a-didactical situation*, the prefix "a" indicating that the situation has been temporally freed from its didactical intentionality. Initially, the development of TDS was tightly linked to the development of a-didactical situations, as evidenced by the impressive work that has been carried out in the experimental center COREM in Bordeaux since 1972.

The milieu is the system with which the students interact in the a-didactical situation and an essential role of the teacher or the researcher is to organize this milieu. It includes material and symbolic resources, possibly calculators, computer devices, or all types of machinery. It conditions the *didactical variables* of the situation, that is to say those which affect the cost and economy of solving strategies. Learning being conceived as an adaptation process, the milieu must be a source of contradictions, imbalances, what is captured through the idea of *antagonist milieu*. The milieu must allow students to experience the limitations of their initial strategies, but its possibilities of action and feedback should also make possible an evolution towards winning strategies, which attest the construction of new knowledge.

Of course, these constraints impose desirable conditions on the problems themselves whose solving is the motive of such a-didactic interaction student–milieu as expressed in Brousseau (2008, p. 249):

- The mathematical knowledge aimed at should be the only good method of solving the problem.
- The assignment (i.e. the given task) should not refer to any of the knowledge that one wishes to have appear. It determines the decisions permitted, the initial state, and the gain or loss represented by the final states.
- Students can start to work with inadequate "basic knowledge".
- They can tell for themselves whether their attempt succeeded or failed.
- Without determining the solution, these verifications are suggestive (they favour some hypotheses, bring in some appropriate information, neither too open nor too closed).
- Students can make a rapid series of "trial and error" attempts, but anticipation should be favoured.
- Amongst the empirically acceptable solutions only one takes care of all objections.
- The solution can be found and tested by some of the students in a reasonable amount of time in an ordinary class, and swiftly shared and verified by the others.
- The situation can be re-used, and will then provide some questions that relaunch the whole process.

Naturally, these conditions describe an ideal and are rarely fulfilled by real scenarios. They constitute a theoretical reference for researchers, helping them to anticipate the a-didactical potential of a given scenario and its limitation, and to better understand the contingency of classroom realizations.

4.1.4.2 Didactical Situation and Contract

Quite soon researchers relying on TDS acknowledged that a theory of mathematical and a-didactical situations is not sufficient for approaching mathematics teaching and learning. The processes of *devolution* and *institutionalization* were introduced for connecting the acculturation and adaptation dimensions of the educational enterprise. Both are under the responsibility of the teacher. Through *devolution*, the teacher makes her students accept the mathematical responsibility of solving the problem without trying to decode her didactical intention, and maintains it, creating thus the conditions for learning through adaptation. Through *institutionalization*, the teacher helps students to connect the contextualized knowledge they have constructed in the a-didactical situation to the target cultural and institutional knowledge and she organizes its decontextualization and transformation into "savoirs".[1] She thus restores the intentionality of the didactic interaction, which was not prominent in the a-didactical situation, and makes acculturation possible.

Devolution is not easy and is in some sense paradoxical. In a didactical situation, the teacher being the voice of the institution has a precise learning aim in mind but, as pointed out by Brousseau (1997, p. 41), "everything that she (the teacher) undertakes in order to make the student produce the behaviours that she expects tends to deprive this student of the necessary conditions for the understanding and the learning of the target notion." If the teacher tells the student what to do, the student cannot learn. In TDS, this paradox of devolution is linked to another essential construct, that of *didactical contract*, which emerged from research work developed by Brousseau with students presenting elective failure in mathematics (Brousseau and Warfield 1999; Brousseau 1980). The concept of didactic contract expresses the fact that teacher–students interactions are subject to rules regarding the mathematical knowledge at stake. These constitute a set of reciprocal obligations and mutual expectations, and are the result of an often implicit negotiation. The rules of the didactical contract remain themselves mostly implicit, in contrast to an ordinary contract, and often only become visible when the contract is broken for one reason or another. The process of devolution is conceived as the negotiation by the teacher of a didactical contract that temporarily allows the transfer of responsibility regarding the knowledge aimed at from the teacher to the student.

This explains why often, in the literature, didactical situations are presented as made of an a-didactical situation and a didactical contract. The didactical contract is source of diverse phenomena and paradoxes. Very early, some of these have been identified: the "Topaze" and the "Jourdain" effects, the metacognitive shift, the improper use of analogy, or the obsolescence of teaching situations. The Topaze effect will be discussed in Chap. 12.

We now invite the reader to gain a deeper insight into the previously introduced concepts through the analysis of the video and its Episodes 1 and 2 with the two students Carlo and Giovanni. However, it must be said that this presentation of TDS is very basic. TDS is much more complex, and it is interesting to notice that many recent developments result from its use for understanding the functioning of ordinary classrooms: scale of didacticity in the didactical contract (Brousseau 1995), refined in different levels of granularity (Hersant and Perrin-Glorian 2005), vertical structuration of the milieu differentiating the students' milieu and the teacher's milieu (Margolinas 1998, 2002). Other developments such as the theory of joint action between students and teachers combine in an original way affordances both of TDS and ATD (Sensevy 2011; Sensevy et al. 2005).

[1] Brousseau distinguishes "knowledge" ("connaissances": individual cognitive constructs) and "knowings" ("savoirs": socially shared cognitive constructs) (Brousseau 1997, p. 72). Thus "savoirs" are depersonalized, decontextualized forms of knowledge. They correspond to the forms in which the scholarly knowledge is expressed.

4.2 Illustrating the Theory Through Analysis of the Video of Carlo, Giovanni, and the Exponential Function

The classroom situation we analyze here with TDS was not conceived within this framework. As mentioned above, analyzing data from classrooms that were not designed within the TDS framework is now a current form of use of the theory.

For analyzing the data, we introduce three situations, associated with the three tasks and the work with the three dynamic geometry files. In this section, we consider only Task 1 and Task 2, in which students are asked to work autonomously and in which the intervention of the teacher is minimal. An a-didactical analysis seems thus appropriate. For carrying it out, we try to characterize the milieu of the situations, anticipate what the interactions with this milieu are likely to produce, compare this analysis with what we know about the mathematical aims of the teacher when proposing these tasks to the students, and question the potential of this situation for making the mathematical knowledge aimed at by the teacher emerge through autonomous adaptation. This a priori analysis is then contrasted with the a posteriori analysis of the video data.

4.2.1 Initial Analysis

In the initial analysis, the use of TDS raised difficulties for at least three reasons. First, the information initially provided was limited to the dynamic geometry files and the tasks, which was insufficient for developing the systemic analysis typical for TDS. The files are essential components of the milieu, but for anticipating the interaction they may produce, we needed information about the mathematical and instrumental knowledge that the students are able to engage in this interaction. Any a priori analysis makes hypotheses at this level considering a *generic student* and her supposed experience; this involves a lot of knowledge about the whole educational system and the particular institution at stake. Of course, these hypotheses are not necessarily fulfilled by the actual individual student, influencing the real dynamics of the situation and its cognitive outcomes, as well as the teacher–students interaction. The varying knowledge of individual students is a normal source of discrepancies with the a priori analysis, which are systematically looked for and questioned in the a posteriori analysis. In this video analysis, filling the gaps of the provided information was all the more difficult for us as the video concerned another educational culture with a different approach to exponential functions (in France, exponential functions are introduced in grade 12 as solutions of differential equations).

The second reason is the form of Task 1, which does not constitute a problem-situation in the sense of Brousseau (1997, p. 214) but is an exploration task:

Open with Cabri II the file "y = (2.7)^x".
In this file you will see: the point *x* on the *x*-axis and the point *y* = 2.7^*x*, on the *y*-axis.

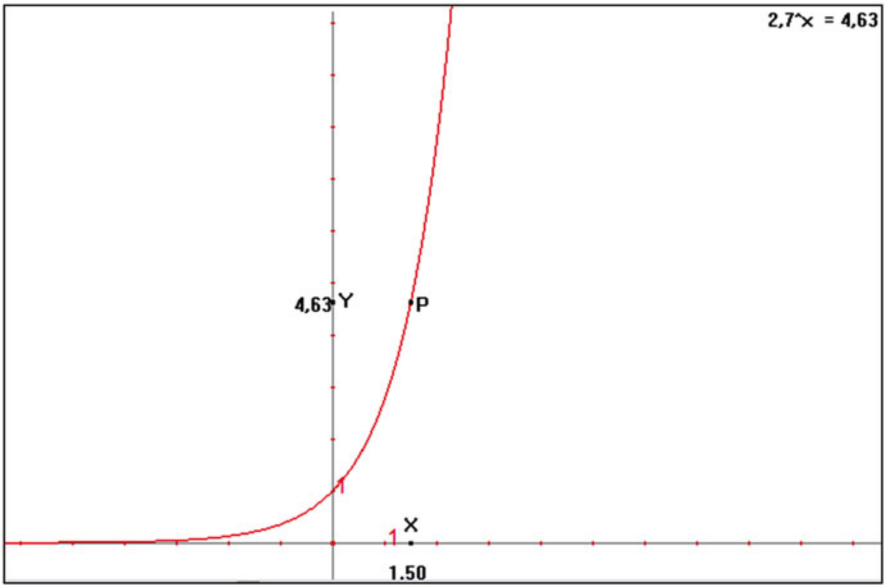

Fig. 4.1 The dynamic geometry file for Task 1

Move the point x on the x-axis and check what happens to the point $y = 2.7$^x on the y-axis; that is, observe how (2.7)^x varies as x is changing.

In order to make these observations, modify also the measure unit on the y-axis of your worksheet. After some trials, use animation. Move the point x towards the left until arriving nearly to the end of the field of variation of the negative x's, and then animate with a spring the point x so that it moves from the left towards the right.

Share all the observations that you think interesting on the coordinate movement of the two points, and describe briefly (but clearly) your argument on the sheet that has been given to you. (Task 1; see Sect. 2.1.3, Fig. 2.1)

In this task, the expectations remain rather fuzzy. What criteria can students have for knowing that they have completed the task? Autonomous work of the students supposes that an appropriate didactical contract has already been negotiated regarding such tasks, in particular helping students appreciate when they can consider that they have solved the task. However, no information on this point was provided in the initial data.

The third reason is the difficulty of categorizing the situation according to the TDS categories: we first categorized it as a *situation of action* as the interaction with the milieu obeys a *dialectics of action*. Nevertheless, in this task, *formulation* has an explicit and important place, and the conjectures produced and made explicit in one task become elements of the milieu for the following ones. Thus the situation is more than a *situation of action*.

Despite these difficulties, partially overcome thanks to the complementary information obtained from the teacher, the use of TDS for the analysis of the video of Carlo and Giovanni was productive, as we show in the following subsections.

4.2.1.1 Initial a Priori Analysis

We present the a priori analysis made with the initial information given before showing how it was refined taking into account the teacher's answers to the questionnaire. Then we contrast it with the a posteriori analysis. Due to the role played by technology, we combine the affordances of TDS with those of the instrumental approach (Guin et al. 2004; Artigue 2002). In this context, the instrumental approach is useful for not underestimating the instrumental knowledge needed for a productive interaction with the milieu and for anticipating how mathematical knowledge and instrumental knowledge, instrumented and paper and pencil techniques, can be combined in the exploration process. One could think that instrumental needs are limited, as files in the Dynamic Geometry Software (DGS) are provided and used as black boxes. However, research shows that, even in that case, actions undertaken and interpretations of the feedback are highly dependent on the students' state of instrumental genesis (Restrepo 2008).

Considering the exploration tasks proposed to the students, we tried to understand what could result from the autonomous interaction of the students with the two DGS files for Task 1 and Task 2. The first file (Fig. 4.1) displays a curve representing the exponential function of base 2.7. This curve has been obtained as the locus of a point P whose coordinates (x, y) are displayed. On the right upper side of the screen, an equality is added: $2.7\hat{}x =$ numerical value, this numerical value being the current value of $2.7\hat{}x$, thus the second coordinate of P.

Students are asked to explore how y varies when x increases, then when x decreases and takes negative values. They are also asked to prepare an animation and it is suggested that they can change the units. There is no doubt that the situation offers a rich potential for a-didactical interaction with the milieu, and that several conjectures can a priori emerge:

- C_1: When x increases, y increases, too, and it increases more and more quickly.
- C_2: When x approaches 0, y approaches 1 (and potentially the inference for $x=0$, $y=1$, even if this cannot be exactly observed).
- C_3: When x approaches 1, y approaches 2.7, the number given in the task and visible on the screen (and potentially the inference for $x=1$, $y=2.7$ even if, once again, this cannot be exactly observed).
- C_4: When x takes negative values which become smaller and smaller, y approaches more and more 0 and from some moment takes the value 0 (note that the final part of this conjecture is not mathematically true but it corresponds to the material evidence provided by exploration with the DGS file).

A good level of instrumental knowledge (Lagrange 2005; Artigue 2002) allows students to infer the above-mentioned conjectures, which go beyond what is observable on the computer screen. It could also lead students to work on the conjecture C_4, which may lead them to question the value 0 taken for negative x, and to change the semiotic register (Duval 1995), moving to the symbolic algebraic register for testing the conjectures and even producing proofs. But such a change requires both a change in semiotic register and a change in attitude: the move from a situation of action to a situation of validation. We hypothesize that this is not likely to appear unless the didactical contract has established and valued such attitudes. Conversely, limited instrumental knowledge concerning discretization processes and their graphical and numerical effects at the interface can lead students to lose time, for instance if they try to obtain the exact values 0 and 1 for x. They can also lose time preparing the required animation which, in our opinion, does not add much to the a-didactical potential of the situation. The fact that the teacher asks for this animation and suggests to change the units, without giving any technical hint, leads us to hypothesize that the students have a good familiarity with the DGS. Nevertheless, if this familiarity has been built in geometry, students may not have developed the instrumental knowledge regarding discretization phenomena which is required when working with functions. We could go on with this a priori analysis, but considering space limitations move to the situation with Task 2.

In this episode (Fig. 4.2), the exploration process uses a second file. In this file, a horizontal half-line with a mobile point A on it has been added; there are thus two mobile points: A and P. The segment joining A to the origin of the half-line (on the y axis) is drawn and an expression "a = numerical value", in which "a" can be interpreted as the measure of the length of the segment or the abscissa of A, is displayed. The curve on which P moves represents the exponential function of base a and the expression "$y = a$^numerical value" is also displayed, the numerical value being the current value of the abscissa of P. It is possible to get the exact value 1 for a, and thus to see the horizontal line which makes the transition between increasing and decreasing exponentials.

Task 2 is fuzzily described: students are simply asked to understand the base a of the exponential function. Nevertheless, the interaction with the milieu can be productive and through the move of point A and its effects on the curve, several new conjectures can emerge:

C_5: If $a > 1$ the exponential function is increasing and the more a increases, the more it becomes vertical (the more it increases quickly).

C_6: If $a < 1$ the exponential function is decreasing and the more a decreases, the more it decreases quickly.

C_7: For $a = 1$ the exponential function is constant (the curve is a horizontal line).

C_8: If x is close to 1, y is close to a (and potentially the inference if $x = 1$ then $y = a$).

C_9:

- If $a > 1$ (respectively $a < 1$): When x takes negative values smaller and smaller (respectively positive values bigger and bigger), y approaches more and more 0 and, from some moment, it takes the value 0 (the final part of these conjectures is not mathematically true but it corresponds to the material evidence provided by exploration).
- And for a in a small interval around the value 1, the moment when y approaches (or "takes" as shown in the file) the value 0 increases and even is not visible on the screen for values of a very close to 1.

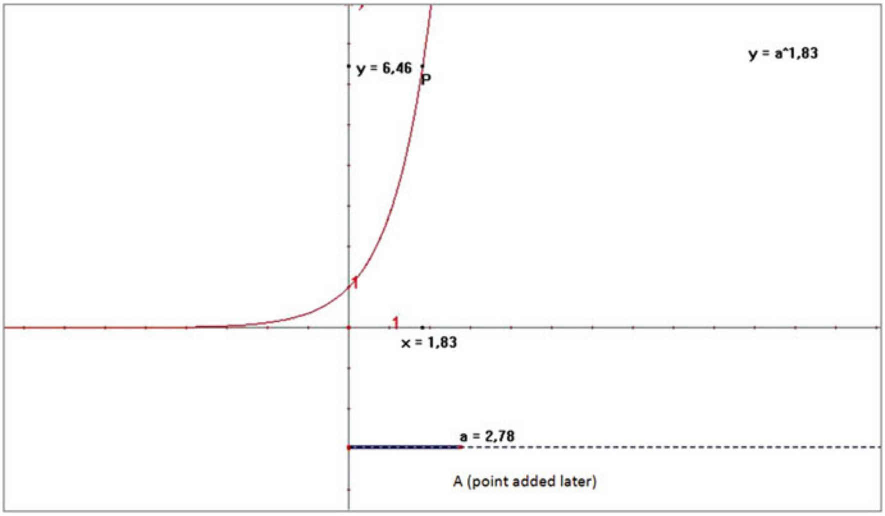

Fig. 4.2 DGS screen configuration for Task 2 (From Fig. 2.2)

According to their degree of instrumentation, the students can move the window or reduce the zoom display in DGS in order to make visible the part of the curve where the x-axis seems to join the curve and thus conjecture that the function has the same shape in any case, except for $a=1$. Otherwise, the limited screen window may lead students to conjecture that the function is of different type for values of a in a small interval around 1, not just for $a=1$.

Of course different formulations of these conjectures are possible. Students can speak of functions or curves, or mix the two notions, and use a more or less mathematical language. Their formulations are a source of information on their comprehension of the notion of function but also on the didactical contract regarding mathematical discourse at this stage of the learning process. These formulations may also be influenced by the use of the instrument. As we did in the first situation, we can investigate what is needed for going beyond conjectures based on graphical evidence. We do not detail this analysis for the second situation, but it shows once again that the emergence of proofs is rather improbable.

4.2.2 Need for Extension of Data and Extended Analysis

For complementing the initial a priori analysis, we especially needed some more information about the didactical contract, the students' mathematical and instrumental background, and the teacher's expectations. The teacher's answers to the questionnaire designed after the initial a priori analysis (Sect. 2.2.2) provided the necessary information.

The didactical contract: The teacher's answers confirm that this kind of exploration task is usual in his classrooms. We can thus suppose the existence of an established didactic contract for exploration tasks in a computer environment, making clear what can be the end of an exploration phase, the kind of writing expected, and the role of the teacher. These answers also show that, beyond its mathematical components, the didactical contract includes some general rules concerning students' interactions: active interaction and collaborative supporting attitude, mutual listening; and concerning teacher–students interactions: the teacher enters in a working group if called by the students, if he realizes that they are stuck, or for provoking deeper reflection on interesting ideas.

The students' mathematical background: This session represents the students' first encounter with (continuous) exponential functions but they have previously met discrete exponential dynamics and associated these to the invariance of the ratio of two consecutive terms. Students could thus mobilize this discrete vision of exponentials and associated algebraic techniques in the exploration. This could help them question graphical evidence, for instance the fact that the exponential function actually reaches the value 0, or conjecture properties not directly accessible at the interface such as $a\text{^}1 = a$.

The students' instrumental background: The teacher's answers confirm the students' familiarity with DGS – they have used it from the beginning of the year. We also learn that they have approached functions numerically, graphically, and symbolically using spreadsheets, Graphic Calculus and TI-Interactive, thus we can suppose that they have already faced discretization phenomena and questions linked to window framing. What is missing is information about the familiarity that students have in working with DGS for studying functions, which requires different instrumental competences from those used for using DGS in geometry. The familiarity gained with other technologies is not necessarily enough for ensuring a possible transfer to DGS. This can have an effect on the conjectures made, as explained in the initial a priori analysis.

The teacher's expectations: The teacher's expectations regarding this situation which precedes the formal introduction of the derivative are high, up to an approach of the differential characteristic property of the exponential function. He writes:

> The activity intends to clarify the principal features of increasing behaviours and of exponential functions. In particular, it intends to explain the reason why at the increasing of x an exponential of base greater than 1 will increase, definitively, more than any other polynomial function of x, whatever grade of the polynomial. In the project, exponential functions and sequences are used to cope with problem situations coming from exponential models. (Answer to question 17, see Sect. 2.2.2)

How the proposed exploration can lead students to the conviction that exponential functions with base greater than 1 dominate any kind of polynomial function is not evident. The a-didactic milieu seems too weak for leading to such a conviction without the teacher's mediation, and even weaker for finding reasons for this phenomenon as expressed above by the teacher.

In the initial a priori analysis, we pointed out the possibility of different levels of language. It is interesting to note that the teacher is sensitive to this question of language, and considers that he has an important role to play in the evolution of the linguistic expressions. He supports this position by a discourse involving the notion of semiotic game, which is extensively discussed in Chap. 3. He also adds that he anticipates such a role for him only from Episode 3. We thus suppose that the didactic contract makes clear for students that they can express the results of the exploration in their own terms. Nevertheless, due to the familiarity already gained in the work with polynomial functions, we expect the use of mathematical terms for expressing and comparing function variations.

In the teacher's answers no allusion is made to proof. We pointed out that the production of proofs for exponential variations did not seem accessible to the students without substantial teacher mediation. Considering the expectations he expresses and the role he anticipates for himself, we consider that his goal is not the production of proof. Rather, he wants the students to make sense of the variations of exponential functions and the role played by the exponential basis through exploration, and access some form of understanding supported by graphical evidence. This seems coherent with the vision he presents of the use of technology: "In my opinion technological tools have to be used to empower the possibility to experience the mathematical environment and mathematical objects" (answer to question 3, Sect. 2.2.2).

4.2.2.1 A Posteriori Analysis

We synthesize now the results of the a posteriori analysis. We pay particular attention to the similarities and discrepancies detected in the comparison with the a priori analysis, and use this comparison for understanding the functioning of the specific didactic system at stake. We start the a posteriori analysis by comparing the students' conjectures with our anticipations.

Students' conjectures: Most of the anticipated conjectures appear, but also some more. In fact, six conjectures are proposed by the students during the first situation and five during the second, which confirms the a-didactical potential of these situations. Discrepancies between a priori and a posteriori analysis are mainly due to Carlo's work in the semiotic register of algebraic expressions to which, in the a priori analysis, we only gave a role of control and validation. Indeed, we linked the production of conjectures to the interaction with the DGS component of the milieu. The first conjecture articulated regards the value of the function for $x=0$ (C_2 in the a priori analysis). It emerges very early (line 3 of the transcript), is directly expressed by Carlo in terms of equality, and proved algebraically. Thus it does not result from the interpretation of the graphical representation. The students then test the effect of a change in the unit of the y-axis and Giovanni notes that the form of the graph does not change (Giovanni, lines 16, 18) while Carlo seems sensitive to the change in the perception of the rate of growth. The second conjecture is articulated by Carlo immediately after line 19 and expresses the fact that if 2.7 is replaced by 1, the graph becomes a straight line (C_7). This conjecture was anticipated but only for Task 2. The first DGS file, indeed, does not allow the students to change the value of the basis of the exponential. The conjecture thus necessarily results here from the connection made by Carlo between the perceptive change in the rate of growth and the algebraic fact that $1^\wedge x = 1$ for every x. The third produced conjecture is C_4: when x goes towards negative values, y gets closer to 0 (Giovanni, line 40), and from some value on, x is always 0 (Giovanni, line 42). The students try to determine the value where the function reaches the value 0 (from line 111); however, for this conjecture we do not notice any attempt of control or validation in the algebraic register. The fourth conjecture states that the ratio $f(2)/f(1)$ equals 2.7 (Carlo, line 90). Once again, it emerges from algebraic work and expresses the connection that students make between this situation and the work developed on discrete exponential models. Connecting 2.7 to particular values of the function, this conjecture plays the role given to C_3 in the a priori analysis, but does it in a way that suggests a deeper connection with the nature of exponential growth. The attempts made by the students for checking this conjecture in the DGS environment are not successful, due to uncontrolled discretization phenomena. The fifth conjecture is the first concerning the variations of the function and refines C_4 (Giovanni, line 109): for negative x, the function decreases towards 0. It apparently results from the manipulation of the software. For finding the exact value for which $2.7^\wedge x$ becomes 0, Giovanni moves the mobile point from the right to the left, which gives a perception of decrease. The sixth conjecture, a variant of C_1, connects variation and limits: $f(x)$ tends towards infinity when x tends towards infinity. Carlo justifies this assertion by invoking the nature of exponential dynamics (line 122) while Giovanni checks it with DGS (line 123).

While dealing with Task 2, the two students quickly understand that by increasing the value of a they obtain exponential curves increasing more and more quickly (C_5). Giovanni expresses this conjecture in ordinary language: “If you change this… that is it becomes more tightened or it increases more or less” (Giovanni, line 152), and Carlo (line 153) goes on using a language more mathematical and involving the idea of increasing rate of change. Later on, when Carlo begins to write the report, another expression emerges: “For the same space, the differences are ever greater” (Carlo, line 192) which shows once again the influence of previous work on discrete dynamics. The second conjecture is C_7 (case $a = 1$) which was already articulated in the exploration of Task 1. It is expressed by Carlo as a confirmation (line 173) but Giovanni does not seem so sure. The end of the episode is not easy to analyze from the transcript but it seems that at least three other conjectures are produced, expressing the decreasing nature of the exponential curve for $a < 1$ (C_6), and the fact that the curve never reaches the x-axis (line 186). Carlo tries to mobilize the algebraic register for justifying this conjecture but using 1 instead of a as the base of the exponential: “If I raise 1 to any number I have not zero” (Carlo, line 199). Later on, another conjecture emerges expressing that when the base a tends towards 0, the curve gets closer to the x-axis but does not touch it (lines 211–215).

Note that, for $a = 2.7$, exploring what happens when x decreases, the students conclude that y reaches 0, whereas in the second exploration they insist on the fact that for $a < 1$, when x increases, y decreases towards 0 but never takes this value. Apparently, they do not make any connection between the two situations. Subtle differences in the information provided at the interface for the two DGS files (in the first file, the y value is displayed and becomes 0 once x is about –5 for instance), in the way the questions are phrased and thus the exploration is carried out (leading to them paying more attention to the coordinates of P in the first situation, and more attention to the global behaviour in the second), may have contributed to this.

The sharing of roles between students: From the beginning, Giovanni takes the mouse and works with the computer while Carlo works mainly with paper and pencil and seems in charge of writing the report. In the first minutes, it seems that he wants to pilot the DGS exploration from this external position, but quickly Giovanni establishes his autonomy. Even if the students collaborate, this sharing of roles creates an evident dissymmetry in their interactions with the milieu, with notable impact as shown above. In fact, we could say that they do not interact with the same milieu; only Giovanni interacts with the milieu of the a priori analysis. This seems the main source of discrepancy between the a priori and a posteriori analysis. We did not anticipate this sharing of roles, and nor did the teacher, although it appears frequently in group-work with technology. When questioned about it, the teacher considers that there is some dysfunction, the interaction between students respecting neither his ethical values nor his vision of cognitive development.

Instrumental issues: Instrumental issues impact the students' relationship with the milieu. Most of the instrumental difficulties met by the students had been anticipated in the a priori analysis, but not all of them, for instance those attached to the distinction between what is fixed or what is not fixed in the first DGS drawing. We supposed that, due to their familiarity with DGS, the students would not have difficulties with this distinction. This was the case for Giovanni, who seemed to understand quickly that the curve was drawn through the use of a formula involving a value 2.7 (lines 30–32) that could be changed into another value. This was not the case for Carlo, who met serious problems, confusing 2.7 with the unit of the y-axis, and thinking thus that by changing the unit he could change the base of the exponential. Note that the difficulties met by the students and anticipated in the a priori analysis were not expected by the teacher. We take this as a symptom of the general underestimation of the complexity of instrumental geneses, and of their mathematical and technical needs. The strong semiotic sensitivity of the teacher did not fully compensate for that.

The role of the teacher: The role of the teacher perfectly corresponds to the anticipation of the a priori analysis. During these two episodes, he only intervenes for solving a few technical problems, and the two students work in full autonomy.

4.3 Conclusion

The video of Carlo and Giovanni comes from a classroom session that has not been designed within the frame of TDS. Nevertheless, these episodes present characteristics which make an analysis through the lens of TDS and the instrumental approach fully pertinent. Students interact with a milieu offering a rich potential of actions and retroactions, and the experience of the students, both mathematical and instrumental, allows them to benefit from this potential, in the autonomous mode which characterizes a-didactical situations. Moreover, the didactical contract makes clear the respective expectations of teacher and students, even if the written exploration tasks do not seem precisely defined.

We developed thus our analysis using the usual techniques of TDS, that is to say, preparing an a priori analysis focusing on the determination of the cognitive potential of an a-didactic interaction with the milieu, for a *generic* and *epistemic* student, that is, a student accepting the a-didactical game and able to invest in it the mathematical and instrumental knowledge supposed by the teacher. We then compared the results of the a priori analysis with the a posteriori analysis of the video. In doing so, we showed that the tools we had used in the a priori analysis allowed us to make realistic anticipations regarding the cognitive a-didactic potential of the two situations, and also to anticipate limitations and difficulties underestimated by the teacher himself. Thanks to this technique, the students' behavior becomes more understandable, and we can separate in the contingency of the actual realization what results from the logic of the situation from what results from other conditions. In particular, we can observe the discrepancies created by the fact that, in the a priori analysis one considers an epistemic and generic student, while in the reality of classrooms teachers work with individuals with different background and motivation, who enter more or less into the game proposed by the teacher and most often do it with different knowledge from that supposed. In this case study, the two students accept the a-didactical game and the devolution process is successful. They behave as epistemic actors of the situation: they try to answer the questions posed by the teacher, using their mathematical knowledge for piloting and making sense of the exploration; they do not try to guess the answers expected by the teacher from some didactical hints (this phenomenon is the object of Chap. 12 on the Topaze effect). This being said, they interact differently with the milieu, in fact not exactly with the same milieu. Comparing the a priori and the a posteriori analysis, we point out new elements not envisaged in the a priori analysis, and identify their effects, both on the cognitive trajectory of each student and on the global trajectory of the group.

We do not pretend that this TDS analysis tells everything that is didactically pertinent about this part of the video. Nevertheless, through its specific lens, it substantially contributes to our understanding of the video.

In this conclusion, we also would like to come back to the issue of "significant unit" for didactical analysis. The vision of what is a significant unit always depends on the adopted theoretical framework. We were able to productively put TDS at the service of the analysis of the episode, but we want to stress again that what this video makes accessible is very limited with regard to the systemic perspective of TDS. What we access is a very small part of the teaching of exponential functions in this classroom and of what makes the teacher able to fulfill the aims he details in his answers to the questionnaire. We have access, for one particular group of students, to a moment of "first meeting" (according to the ATD terminology) with exponential functions. They intervene through technological black-boxes that the students have to explore. Some statements emerge from this exploration whose mathematical status is not clear at this stage. How do these statements situate with respect to the statements produced by other groups? How will the teacher exploit them? How will they be related both in their content and form to the institutional knowledge aimed at? And what use will be made of that knowledge, once institutionalized? What level of technical operationality will be aimed at in the different semiotic registers? We just see a tiny part of a mathematical and didactical organization, something interesting and insightful but very insufficient for someone who would like to understand what can be the teaching and learning of exponential functions in such a context.

References

Artigue, M. (1989). Ingénierie didactique. *Recherches en Didactique des Mathématiques, 9*(3), 281–308.

Artigue, M. (2002). Learning mathematics in a CAS environment: The genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. *International Journal of Computers for Mathematical Learning, 7*(3), 245–274.

Artigue, M. (2008). Didactical design in mathematics education. In C. Winslow (Ed.), *Proceedings from NORMA08 in Copenhagen 2008* (pp. 7–16). Rotterdam: Sense Publishers. https://www.sensepublishers.com/media/699-nordic-research-in-mathematics-education.pdf. Accessed 22 Oct 2013.

Artigue, M. (2013). Didactic engineering in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education*. Berlin/Heidelberg: Springer. (in press).

Brousseau, G. (1980). Les échecs électifs en mathématiques dans l'enseignement élémentaire. *Revue de laryngologie, otologie, rhinologie*, *101*(3–4), 107–131. http://guy-brousseau.com/1388/les-echecs-electifs-en-mathematiques-dans-l%E2%80%99enseignement-elementaire-1980. Accessed 22 Oct 2013.

Brousseau, G. (1986). Fondements et méthodes de la didactique. *Recherches en didactique des mathématiques, 7*(2), 35–115.

Brousseau, G. (1995). L'enseignant dans la théorie des situations didactiques (The teacher in the theory of didactical situations). In R. Noirfalise & M. J. Perrin-Glorian (Eds.), *Actes de la VIIIe Ecole d'Eté de Didactique des Mathématiques* (pp. 3–46). Clermont-Ferrand: IREM de Clermont-Ferrand. http://guy-brousseau.com/2319/l%E2%80%99enseignant-dans-la-theorie-des-situations-didactiques-1995/. Accessed 07 July 2014.

Brousseau, G. (1997). *Theory of didactical situations in mathematics: Didactique des Mathématiques, 1970–1990* (N. Balacheff, M. Cooper, R. Sutherland, & V. Warfield, Trans. & Eds.). Dordrecht: Kluwer.

Brousseau, G. (2008). Research in mathematics education. In M. Niss (Ed.), *Proceedings of ICME-10* (pp. 244–254). Denmark: IMFUFA, Department of Sciences, Systems and Models, Roslkilde University.

Brousseau, G., & Warfield, V. (1999). The case of Gaël. *Journal of Mathematical Behavior, 18*(1), 1–46.

Duval, R. (1995). *Semiosis et Noesis*. Berne: Peter Lang.

Guin, D., Ruthven, K., & Trouche, L. (Eds.). (2004). *The didactic challenge of symbolic calculators.* Dordrecht: Kluwer.

Hersant, M., & Perrin-Glorian, M.-J. (2005). Characterization of an ordinary teaching practice with the help of the theory of didactic situations. *Educational Studies in Mathematics, 59*(1–3), 113–151.

Laborde, C., Perrin-Glorian, M.-J., & Sierpinska, A. (Eds.). (2005). *Beyond the apparent banality of the mathematics classroom*. New York: Springer.

Lagrange, J.-B. (2005). Curriculum, classroom practices and tool design in the learning of functions through technology-aided experimental approaches. *International Journal of Computers for Mathematics Learning, 10*, 143–189.

Margolinas, C. (1998). Le milieu et le contrat, concepts pour la construction et l'analyse de situations d'enseignement. In R. Noirfalise (Ed.), *Analyse des pratiques enseignantes et didactique des mathématiques, Actes de l'Université d'Eté, La Rochelle*. Clermont-Ferrand: IREM. http://halshs.archives-ouvertes.fr/halshs-00421845/fr. Accessed 22 Oct 2013.

Margolinas, C. (2002). Situations, milieux, connaissances: Analyse de l'activité du professeur, cours. In J. L. Dorier et al. (Eds.), *Actes de la 11ème Ecole d'Eté de Didactique des Mathématiques* (pp. 141–157). Grenoble: La Pensée Sauvage. http://halshs.archives-ouvertes.fr/halshs-00421848/fr. Accessed 22 Oct 2013.

Margolinas, C., Abboud-Blanchard, M., Bueno-Ravel, L., Douek, N., Fluckiger, A., Gibel, P., Vandebrouck, F., & Wozniak, F. (Eds.). (2011). *En amont et en aval des ingénieries didactiques. Actes de la 11ème Ecole d'Eté de Didactique des Mathématiques, Clermont-Ferrand 2009*. Grenoble: La Pensée Sauvage.

Restrepo, A. (2008). *Genèse instrumentale du déplacement en géométrie dynamique chez des élèves de 6eme*. Doctoral thesis, Université Joseph Fourier, Grenoble.

Schneider, M. (2013). Epistemological obstacles in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education*. Berlin/Heidelberg: Springer. (in press).

Sensevy, G. (2011). Overcoming fragmentation: Towards a joint action theory in didactics. In B. Hudson & M. Meyer (Eds.), *Beyond fragmentation: Didactics, learning and teaching in Europe* (pp. 60–76). Leverkusen/Farmington Hills: Barbara Budrich Publishers.

Sensevy, G., Schubauer-Leoni, M.-L., Mercier, A., Ligozat, F., & Perrot, G. (2005). An attempt to model the teacher's action in the mathematics class. *Educational Studies in Mathematics, 59*(1–3), 153–181.

Sriraman, B., & English, L. (Eds.). (2010). *Theories of mathematics education: Seeking new frontiers*. Berlin/Heidelberg: Springer Science.

Chapter 5
Introduction to the Anthropological Theory of the Didactic (ATD)

Marianna Bosch and Josep Gascón

Abstract The chapter briefly introduces the Anthropological Theory of the Didactic (ATD) by referring to the data from Chap. 2. ATD provides a frame for investigating mathematical and didactic activities in terms of *praxeologies*, focusing on their components, dynamics, and the conditions that enable their existence and development in a given institutional setting. The main idea of the concept of praxeologies is that all human activities comprise and link two parts, a practice and a theory one.

Keywords Theories • Anthropological theory of the didactic

5.1 Overview

The Anthropological Theory of the Didactic (ATD) is a program of research in mathematics education initiated by Yves Chevallard in the 1980s with the study of *didactic transposition processes* (Bosch and Gascón 2006; see also Chevallard 1985, 1989, 1992a, b) and which has been evolving continuously for the last 30 years. Nowadays, a community of about one hundred researchers, mainly from Europe, Canada, and Latin America, work on the development of this program, focusing on the current problems of spreading knowledge both at school and outside school, concerning mathematics as well as other fields of knowledge.

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A good outline of the approached problems and the obtained results within this framework can be found in the proceedings of the four International ATD Conferences held since 2005 in Spain and France (Bosch et al. 2011; Bronner et al. 2010; Estepa et al. 2006).

The meaning and relevance of ATD has to be understood as a development of the project initiated by the Theory of Didactical Situations (TDS) of a science of *didactic phenomena* called *didactics of mathematics* (cf. Chap. 4 on TDS). In the framework proposed by ATD, the institutional dimension of mathematical and didactic[1] activities becomes much more explicit. Doing, teaching, learning, diffusing, creating, and transposing mathematics, as well as any other kind of knowledge, are considered as human activities taking place in institutional settings. The science of *didactics* is thus concerned with the conditions governing these knowledge activities in society, as well as the restrictions hindering their development among social institutions.

5.1.1 Principles and Key Constructs: Praxeologies

ATD postulates that any activity related to the production, diffusion, or acquisition of knowledge should be interpreted as an ordinary human activity, and thus proposes a general model of human activities built on the key notion of *praxeology.* According to Chevallard (2006):

> *A* praxeology is, in some way, the basic unit into which one can analyse human action at large. […] What exactly is *a* praxeology? We can rely on etymology to guide us here – one can analyse any human doing into two main, interrelated components: *praxis*, i.e. the practical part, on the one hand, and *logos*, on the other hand. *"Logos"* is a Greek word which, from pre-Socratic times, has been used steadily to refer to human thinking and reasoning – particularly about the cosmos. […] [According to] one fundamental principle of the ATD – the anthropological theory of the didactic – no human action can exist without being, at least partially, "explained", made "intelligible", "justified", "accounted for", in whatever style of "reasoning" such an explanation or justification may be cast. *Praxis* thus entails *logos*, which, in turn, backs up *praxis*. For *praxis* needs support just because, in the long run, no human doing goes unquestioned. Of course, a praxeology may be a *bad* one, with its "praxis" part being made of an inefficient technique – "technique" is here the official word for a "way of doing" – and its "logos" component consisting almost entirely of sheer nonsense – at least from the praxeologist's point of view! (Chevallard 2006, p. 23)

Both the practical and theoretical components of a praxeology are in turn broken down into two elements. The *praxis* block is made of "types of tasks" and a set of "techniques" (considering this term in a broad sense of "ways of doing") to carry out some of the tasks of the given type (those in the "scope" of the technique). The *logos* block contains two levels of description and justification of the *praxis*. The first level is called a "technology," using here the etymological sense of "discourse" (*logos*) of the technique (*technè*). The second level is simply called the "theory" and its main function is to provide a basis and support of the technological discourse. In

[1] The adjective "didactic" is used to refer to anything related to the teaching, learning, or study of a given content.

general human activities, the "theory" component is generally more difficult to grasp than the others because it is usually taken for granted, unless in times of difficulties, crises, and questioning of the praxeologies. In return, scientific work provides many examples of how these theoretical assumptions can be made explicit in order to provide more control of the techniques carried out and of their description, justification, and validation.

A praxeology is thus an entity formed by four components, usually called the "four Ts": a type of tasks, a set of techniques, a technological discourse, and a theory. As activities and knowledge can be described considering different delimitations or granularities, a distinction is made between a "point praxeology" (containing a single type of task), a "local praxeology" (containing a set of types of task organized around a common technological discourse) and a "regional praxeology" (which contains all point and local praxeologies sharing a common theory). We will see an example of this distinction in the analysis of the episode of Carlo and Giovanni (see also García et al. 2006; Barbé et al. 2005).

Praxeologies is a useful term when talking about knowledge, mathematics, or any other teaching and learning content, and also about teaching and learning practices, as it provides a unitary vision of these different activities, without considering some of them as more "intellectual," "abstract," "difficult," or theoretically based than the others, and thus without assuming the scale of values usually given to them (mathematics appearing as something related to "thinking" while teaching is more seen as a "practice" than as a "theory").

Praxeologies do not emerge suddenly and never acquire a final shape. They are the result of ongoing activities, with complex dynamics, that in their turn have to be modeled. We will use the term *didactic praxeologies* to refer to any activity related to "setting up praxeologies" (Chevallard 1999). A didactic praxeology is thus a praxeology that aims at making other praxeologies start living in and migrating within human groups. They are an essential part of the functioning and evolution of our societies, indispensable to keeping institutions running, to modifying them, and also to habilitating people to make them work and progress. They are also essential, of course, for the personal development of human beings, to improving their capacity of action and comprehension.

Here appears a sensible point about the relation between institutional and personal praxeologies. In order to answer the question of why people do what they do, what makes it possible for them to do what they do, etc., ATD postulates that what explains the behavior of people are not only their personal idiosyncrasies but also the existence (or availability) of institutional constructions that each person adapts, adopts, and develops either individually or collectively. An ATD analysis therefore starts by approaching *institutional praxeologies* and then referring individual behavior to them, talking in terms of the "praxeological equipment" of a given person. Observable behavior obviously consists of a mixture of personal and institutional ingredients. This dialectic between the personal and the institutional makes it possible to explain both the regularities of our behavior and its personal "footprint." People evolve as they enter different institutions and, at the same time, these individual participations enable institutions to appear, run, and change.

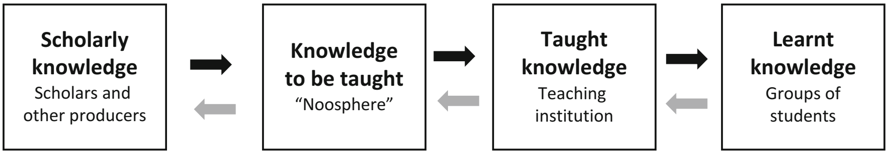

Fig. 5.1 Diagram of the process of didactic transposition

Concerning the dynamics of praxeologies, the ATD assumes an important postulate of the TDS: the fact that any piece of knowledge (i.e., any praxeology) can be considered as an answer provided – explicitly or *de facto* – to a question *Q* (a problem or a difficulty) arising in an institutional setting (or a "situation"). Question *Q* then becomes the "raison d'être" of the praxeology constructed, a rationale evolving as the praxeology develops and integrates into other kinds of activities, for instance to provide answers to other kinds of questions. It often occurs that the raisons d'être at the origin of most praxeologies disappear with time, and people end up doing things out of inertia or habit, without questioning their way of doing nor considering the possibility of changing them. Therefore, an important "research gesture" in didactics is to analyze praxeologies to find out their possible raisons d'être (the historic as well as the contemporary ones) and study the conditions that can make them appear – give them sense – in different institutional settings.

5.1.2 Methodologies and Questions

5.1.2.1 The Praxeological Analysis

One of the first contributions of ATD through the notion of *didactic transposition process* was to make clear that it is not possible to interpret school mathematics properly without taking into account the phenomena related to the way mathematics is introduced and reconstructed at school. What mathematical praxeologies are proposed to be studied at school and why? What are they made of? Where do they come from? Do they live outside school? Where and under what shapes? Didactic transposition processes underline the *institutional relativity of knowledge* and situate didactic problems at an institutional level, beyond individual characteristics of the subjects of the considered institutions (Fig. 5.1).

The process of didactic transposition (Fig. 5.1) refers to the transformations applied to a "content" or a body of knowledge since it is produced and put into use, until it is actually taught and learned in a given educational institution. This notion is not just the description of a phenomenon, but a tool to emancipate the didactic analysis from the dominant vision of educational content. Teaching and learning processes always include some content or piece of knowledge to be taught and

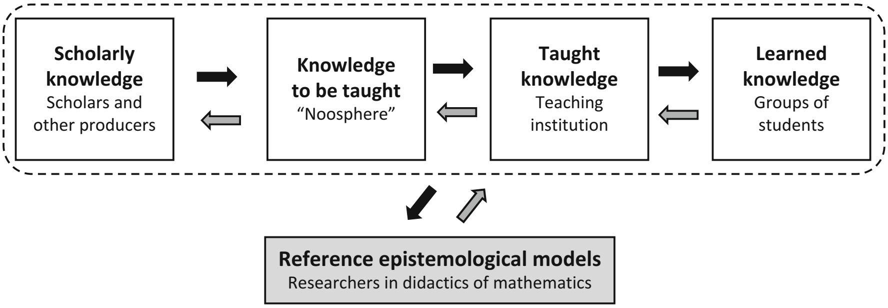

Fig. 5.2 The external position of researchers

learnt. One can take this content as given data or, on the contrary, question its nature and function, considering its *formation* as "knowledge to be taught" through the productions of the *noosphere*-that is, the sphere of those who "think" (*noos*) about teaching-, its relationship to "scholarly knowledge" which usually legitimates its introduction in educational institutions, and the specific form it takes when arriving in the classroom as "taught knowledge," activated by both the teacher and the students. The "knowledge to be taught" can be accessed through official programs, textbooks, recommendations to teachers, didactic materials, etc., which may help in considering also the conditions under which it is constituted and evolves (or remains fixed) in time.

This study should take into account the "scholarly knowledge" produced by mathematicians or other scientists who are recognized as the "experts of the matter" and appears as a source of legitimation of the knowledge to be taught. However, scholarly knowledge should not be considered as the unique reference to which all school mathematical praxeologies are referred to. In order to avoid adopting a particular and "scholarly biased" viewpoint, researchers in didactics need to elaborate their own "reference models" (Fig. 5.2) from which to consider the empirical data of the three corresponding institutions: the mathematical community, the educational system, and the classroom.

5.1.2.2 The Didactic Analysis

A social situation is said to be a *didactic situation* whenever one of its actors (Y) does something to help a person (x) or a group of persons (X) learn something (indicated by a heart ♥). A *didactic system* $S(X; Y; ♥)$ is then formed. The thing that is to be learned is called a *didactic stake* ♥ and is made of questions and/or praxeological components. X is the group of "students of ♥" and Y is the team of "study assistants" (or "study helpers"). The most obvious didactic systems are those formed at school, where Y is ordinarily a "singleton" whose unique member is "the teacher" y. However, there are a multitude of different kinds of didactic systems.

For instance, the authors of this chapter are acting as *Y* to help the reader, *x*, learn something about ATD research.

Given a didactic system $S(X; Y; \heartsuit)$, the praxeological analysis tries to provide answers about the praxeologies the didactic stake ♥ is made of. By contrast, the didactic analysis approaches questions including: What is *X*? What is *Y*? What are the didactic praxeologies put to use by *X* and *Y* and what didactic means have proved necessary to do so? What praxeological equipment can be engendered in *X* as a short-term and as a long-term result of the functioning of $S(X; Y; \heartsuit)$? To answer these and other questions, ATD provides two different general didactic models. The first one, in terms of six dimensions or *didactic moments*, concerns the case where ♥ is a given local praxeology $\mathscr{P}$ and presents a structure of the construction of the different components of $\mathscr{P}$: the *first encounter* with the praxeology, the *exploration* of the type of tasks and the emergence of a technique, the "*work of the technique*" and the study of its scope, the elaboration of a *theoretical environment*, the *institutionalization*, and the *evaluation* of the work done (Chevallard 1999; see also Barbé et al. 2005; García et al. 2006). The second didactic model is more general and aims to include any process of study and research starting from a problematic question *Q*. It is presented and used in the case study on context-milieu-media (Chap. 10).

5.1.2.3 The Ecological Analysis and the Levels of Didactic Codetermination

The study of the *ecology* of mathematical and didactic praxeologies states that, when the teacher and the students meet around an issue at stake ♥, what can happen is mainly determined by conditions that cannot be reduced to those immediately identifiable in the classroom, such as the teacher's and students' praxeological equipment, the teaching material available, the temporal organization of activities, etc. Even if these conditions play an important role, Chevallard (2002) proposed to consider a "scale of levels of didactic codetermination" (see Fig. 5.3).

General educational research usually focuses on restrictions coming from the *generic levels* (above the discipline), while research in specific subject didactics (such as didactics of mathematics, sciences, language, etc.) hardly take them as an object of study, even if they strongly affect the "specific praxeologies" that can exist in the classroom and the way they can evolve. Moreover, even at the *specific levels* (within the discipline), what is commonly considered in didactics research tends to be reduced to phenomena occurring at the *thematic level*, that is, those concerned by the teaching and learning of a specific topic. Consequently, it becomes very difficult for researchers – and even more for the teacher – to question the cultural vision of mathematics and its teaching as proposed by both school and "scholarly" institutions. The way the levels of didactic codetermination are used to analyze the ecology of mathematical and didactic praxeologies is illustrated at the end of the chapter (see also Artigue and Winsløw 2010).

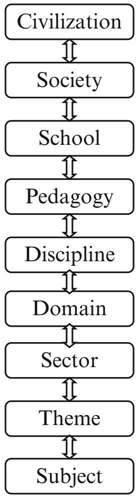

Fig. 5.3 Scale of levels of didactic codetermination

5.2 Illustrating the Theory Through Analysis of the Video of Carlo, Giovanni, and the Exponential Function

5.2.1 Mathematical Praxeologies in the Considered Episode

The description of praxeologies can be carried out at different levels of detail, depending on the kind of problem posed by the researcher. In this case, given the fact that the piece of reality considered does not respond to any specific problem proposed by ATD, we will limit our presentation to an overall illustrative analysis of the mathematical praxeologies involved in the considered episode. We will start by inferring the ingredients of the *praxis* of the mathematical praxeologies and then look for the *logos* used to describe, explain, and justify this praxis.

5.2.1.1 The Technical-Practical Block of Mathematical Praxeologies

In the episode in which Carlo and Giovanni solve Task 1 and 2 (see Sect. 2.1.3 of this book), the mathematical praxeology at stake consists of two related tasks (or point praxeologies), the second one constituting a development of the first. They both integrate into a broader (local) praxeology that we will comment on later, based on the extra material we asked the teacher to provide (see Sect. 2.2.2). Due to lack of space, we will not carry out a detailed analysis of the three point praxeologies that appear in the episode and will only highlight the aspects they have in common. We may consider that Task 1 and 2 (see Figs. 2.1 and 2.2 in Sect. 2.1.3) stem from the same generating question Q, which can be formulated in the following terms: How to describe the variation of exponential functions $y=a^x$, both from a global and a local viewpoint?

In the observed episodes, this question is divided into three sub-questions:

How to *describe* the global variation of $y=2.7^x$ when varying x? (case $a=2.7$)
How to *describe* the global variation of $y=a^x$ for different values of a ($a>0$)?
How to *quantify the local variation* of $y=a^x$ from the study of Δy for different values of Δx and considering the slope of the tangent line of the function at point x?

Obviously, this first task description is being done in terms related to our own mathematical experience and trying to remain close to the considered institution (the Italian secondary school, in this case). Our main empirical material is the worksheet the teacher hands out to the students as a guide to carry out the work. The questions in the worksheet (see Figs. 2.1 and 2.2 in Sect. 2.1.3) are divided into sub-questions that need to be answered. There is no introduction to the tasks proposed, nor are there any references to a more general framework (for instance to study the variation of the exponential function) in which the study takes place. It is possible that the introduction was done before the considered episode, but we do not know.

What are the techniques used to elaborate an answer to the three previous questions? In the case we are working on, we consider the global techniques used, and not only those the students are asked to carry out. In other words, we will consider the techniques useful to provide answers to the previous questions and that appear in the episode as activities carried out both by the teacher and the students, according to a precise distribution of responsibilities into which we will look in Sect. 5.2.2.

In the three considered tasks, the technique contains a specific device: a Cabri Geometry file with interactive graphs elaborated by the teacher, which the students are asked to manipulate and interpret. A certain manipulation of the devices – which the teacher has specified in the tasks – leads them to conjecture some of the "visible" properties of the functions considered from interpreting what is observed on the computer screen (graphic and numeric information). We are faced with a kind of *exploratory techniques* of specific mathematical objects which do not have a standard mathematical denomination (for example "calculating the derivative of a function"). Some of the "gestures" performed when carrying out those techniques are not visible in the video: the part of choosing and providing the experimental device, which the teacher did beforehand. What does appear in the Cabri file (and is observed in the activity the students carry out) is the detail of some of the manipulations of the device, which in some cases figure in relative detail in the task instructions: "*Open… file…*", "*Move the point x on the x-axis*", "*modify also the measure unit on the y-axis*", "*Move the point x towards the left until arriving nearly at the end of the field of variation of the negative x's*", etc. (see Fig. 2.1 in Sect. 2.1.3). The students' participation in carrying out the three tasks consists of performing the indications provided by the teacher and taking charge of the "gestures" that are not indicated: relate the graphic variation of x to the graphic variation of y; interpret it in terms of functional relations; formulate those relations in graphic and functional terms, both verbally and in writing; discuss and reach an agreement about how to draw up the observations; etc. This type of technique may be portrayed as "ostensive" in the sense that it is mainly based on the description of facts (numerical and graphical) which may be observed on a screen, both verbally (orally and written) and graphically (sketches).

An important part of the development of these techniques is the preparation of the computer devices carried out by the teacher. The students intervene at a specific moment of the development of the technique, but only the teacher is in charge of its global use. This situation is different from other mathematical techniques in which the students are fully in charge of generating the device and the gestures (for instance in the case of drawing the graph of a function and interpreting some of its elements, or carrying out a numerical simulation). The students are only asked to prepare a final statement, first orally and then a written version including graphs, so as to provide answers to the questions posed by means of provisional conjectures. They will also need to choose the known elements of the exponential functions in order to partially contrast some of the conjectures formulated (for instance that curve $y=a^x$ is a horizontal line when $a=1$). In the exchange between students, we can observe the functioning of mathematical objects that are essential to the formulation of conjectures and that have previously been integrated in their praxeological equipment: "tangent line", "slope", "effect of the change of units", "to grow more and more", etc.

Given the fact that the episode is situated at an initial stage of the study of the variation of exponential functions, what is observed in the work done by the students is the use of scattered technical elements which, we suppose, will gradually be integrated so as to form more powerful and systematic exploratory techniques. We thus see the emergence of new technical elements such as identifying the secant line with the tangent line (and with the curve itself of the function) when Δx gets close to 0, or the sudden change of behavior of the function when going from the case $0<a<1$ to the case $a>1$. Undoubtedly, more exhaustive technical and theoretical work will be necessary to systematize and institutionalize those elements in further lessons, which are still incipient in the observed episodes.

5.2.1.2 The Technological-Theoretical Block of Mathematical Praxeologies

After having proposed a possible description of the *praxis* of the mathematical activity partially appearing in the episodes, we can turn the attention to the *logos* block, that is, the elements used to "talk about" the work done, to describe and justify it. Some elements make it possible for the practice to be understandable and allow interaction between the students (each one understanding what the other does or says) as well as between the students and the teacher: they are part of the *technology* of the technique. We can mention, for instance, the interpretation of the elements of the Cabri files in functional terms: the correspondence between the graph and the values of the function; the fact that the values of the function are obtained by moving point x on the x-axis; the relationship between the slope of the tangent line and its "growth," etc. Other technological elements, maybe of a less mathematical nature, also contribute to justifying the functioning of the technique of manipulating the graph (correspondence between segment Δx and point P; between segment a and the base of the exponential function; etc.) and to the use of Cabri. Usually the elements of the technological discourse (basically implicit) are built at the same time as the tasks are explored and only rise to the surface in case of difficulty. In fact, the aim of the task partially consists in formulating some of those elements, those of most "mathematical nature" related to the observed variations of the functions.

The second level of justification considered in any praxeology, the level of the theory, corresponds to those suppositions that explain and validate the technological discourse. It contains some aspects of the development and justification of the techniques that are usually taken for granted and, therefore, rarely specified. In this case, two implicit principles seem to "support" the activated praxeologies. The first one – which we could call the *empiricist principle* – consists in assuming that the answers to the questions related to the behavior of an exponential function can be deduced from the simple observation of the images on the screen, using the graphical and numerical information provided. They thus appear as self-justified verifications or, at the most, provisional conjectures that require a subsequent justification. Students say what they say "because it is what they see on the screen" and it seems that "everything that appears on the screen is true." This is the theoretical foundation of ostensive techniques based on the observation of empirical objects.

The second theoretical principle that seems to act (although not always in the same way) is what we could call the *principle of coherence*, which is also essential to the experimental work. We indeed see that some of the affirmations of the students are algebraically validated (for example that $2.7^0 = 1$ or that $1^x = 1$) following the principle of "what is observed has to be compatible with what one already knows." However, this principle does not always function in the same way. For instance, students conjecture that $y = 0$ when the values of x are lower than -5.3, stating what they see on the screen. (Given the fact that numerical values appear to two significant figures and $2.7^{-5.3} \approx 0.0052$ while $2.7^{-5.4} \approx 0.0047$, the Cabri file shows $2.7^x = 0.00$ for all $x < -5.3$.) Here, we see how the two aforementioned principles clash with each other in a certain way, without posing any difficulties to the students, certainly because the teacher remains ultimately responsible for the validity of the activated praxeology.

Finally, we would like to comment that the praxeologies observed "in action" in the video seem to be oriented towards drawing up a global *technological discourse* on how exponential functions vary. In other words, despite having highlighted the practical and theoretical elements of the praxeology involved in the episode, the final result of its setting up basically consists of generating technological elements of a broader praxeology that exceeds the observed work. This special situation makes it difficult to distinguish between the elements of the praxeology at stake (carrying it out consists in producing technological ingredients of a broader praxeology) and the technological and theoretical elements that correspond to those technical elements.

5.2.2 *Didactic Praxeologies*

Besides the description of the mathematical praxeology at stake, the second kind of question that guides the analysis consists of asking: What are the didactic praxeologies put to use by X and Y and what didactic means have proved necessary to do so? In the considered episodes, two types of didactic praxeologies (or two positions in a cooperative didactic praxeology) can be distinguished, depending on whether we consider the teacher or the students to be the main character. We will here focus on describing some of the elements of the didactic praxeology of the teacher (which we may also call the "teaching praxeology") because in general they contribute more to explaining what students do and why they do what they do. It is, however, obvious that, considering that the didactic process is based on cooperation between teachers and students, the praxeologies of both types are always mutually influenced.

In the episode considered, and through the actions of the subjects observed – two students working on a computer in class under the supervision of the teacher – we will try to describe in the first place the (regular) institutional praxeologies that are "activated" by the people observed, or in which they "enter." Given the fact that all praxeologies contain a descriptive and justificatory discourse, their analysis needs to be carried out from an external position in order to grasp this discourse from a critical point of view.

If we respect the chronology of the episode and stick to the point mathematical praxeologies described in the previous section, a first element of the teaching practice is precisely the choice and formulation of the concrete tasks proposed to the students. A second element of this practice is the election of the type of "materials" proposed to provide and validate the answers to the questions posed. And, finally, there is a set of types of didactic task and techniques carried out in order to help students elaborate those answers until turning them into something that may be used again later on.

5.2.2.1 The Practical-Technical Block of Didactic Praxeologies

We assume that the didactic process is centered on the study of a local praxeology about exponential functions and, more precisely, on the variations of exponential functions of the type $y = a^x$. The whole didactic process, which goes from considering the initial question Q until constructing a validated and potentially reusable praxeology, may be described in terms of six *didactic moments*: the *first encounter* with the praxeology and the formulation of the tasks to be carried out, the *exploration* of the tasks and the emergence of a technique to carry out, the *work of the technique*, the elaboration of a *theoretical environment*, the *institutionalization*, and the *validation* of the work done. Even if they can be considered chronologically, the "moments" constitute dimensions of the process of study: they can take place simultaneously and can be repeated at different periods of time. In the case here considered, we may think that the episode corresponds to the moment of the elaboration of the technological-theoretical block of the praxeology.

What is the didactic strategy used by the teacher to make the students experience this moment? To propose two mathematical tasks to be carried out using some Cabri file previously prepared by the teacher and, eventually, other technological means. The way students deal with the tasks proposed shows that this kind of activity is not strange to them. They read the statement and start working without any trouble. We can thus suppose that the didactic technique used by the teacher is common practice in the class. We do not know if it has a specific name or how the authors interpret it (aspects that are part of the *technology* of the didactic praxeology). From our position of external observers, we could classify this didactic technique as the one of "filling gaps": when facing the initial question of describing the properties of the variation of the exponential function, the distribution of responsibilities between the teacher and the students consists of the teacher carrying out an important part of the work (formulating the question, elaborating the Cabri files, giving exact indications of certain gestures to carry out, etc.) and leaving some substantial gaps as gestures for the students to do and questions to answer in writing. The teacher here assumes the why of the questions he formulates, their sequencing and motivation, as well as their functionality (the fact that they will lead somewhere). The students follow the indications of the teacher and have the responsibility of providing a first written formulation, discussing, and drawing up valuable observations, comments, and conjectures on aspects about the functions that are new to the students. The teacher occasionally intervenes during those critical moments to help the students elaborate their answers: gestures concerning the secant lines; the idea of zoom, the fact that with the function graph "[…] *you can approximate it with many small lines*" (53:29); verbal expressions such as "*the growth percentage of the y*'*s*" (54:22); or make the groups of students share some answers as in "*the other group have used a very good example*" (55:32).

As we only see a limited part of this didactic praxeology of the teacher, we are not totally aware of the kind of didactic tasks he feels responsible for, what the destiny of the technical and technological elements activated by the students will be, how these elements are being institutionalized and validated to conform to the final praxeology at stake. Neither do we know the motivation that surrounds this construction, that is, its *raison d'être*.

5.2.2.2 The Technological-Theoretical Block of Didactic Praxeologies

What does the technology and the theory of a didactic praxeology consist of? Just as in any praxeology, it is made up of elements of different natures, well or poorly articulated depending on the case and on the degree of development of the praxeology. In this case, it seems that the didactic praxeology set up by the teacher is not spontaneous, but comes from previous preparation and experimentation supported by elaborated technological-theoretical elements. Some of these elements may be deduced from the details of the episode (the students do not seem astonished by the tasks proposed), others are clarified from the teacher's answers to our questions (see Sect. 2.2.2). However, some aspects will remain blurred. We will infer them as a conjecture from the analysis.

The *technological level* of the didactic praxeology consists in a descriptive and justifying discourse close to the teaching and learning practice. For instance, with respect to the mathematical praxeology at stake, Domingo specifies what kind of answer he wishes to obtain at the end of the study process:

> I wanted the students to understand that exponential functions are functions for which the growth is proportional to the function itself. In other terms, the derivative of an exponential function is proportional to the function itself. This consideration, in my opinion, should allow students to understand why the exponential function a^x with a greater than 1 grows with x faster than any power of x. (Answer to question 8, Sect. 2.2.2)

In fact, he describes this local mathematical praxeology at stake accurately and even proposes an analysis of it in terms of three levels of complexity:

> A first level is that of perceiving the different velocity of variation that exists between x and a^x. […] A second level is that of the understanding of how the graph of an exponential function varies when the base varies. A third level, as in the third worksheet of Cabri, is relative to the understanding that the incremental ratio is a function of two variables (the x and the increment h). […] A fourth level is the passage from the local to the global aspects of the derivative. From the gradient to the gradient function. (Answer to question 12, Sect. 2.2.2)

He even places this local praxeology in a broader one around exponential functions:

> [I follow] two paths. In the first one I pose some problematic situations which, to be solved, ask for exponential models. In the second one I present the properties of exponentials and I introduce the logarithmic function as the inverse function of an exponential. […] Finally I propose some techniques to solve exponential and logarithmic equations and inequations […]. (Answer to question 10, Sect. 2.2.2)

As far as the selected order of the tasks is concerned, he justifies it with the argument of complexity and justifies the necessity of the experimental work with Cabri in terms of the construction of a "cognitive root" for the later "formal" work.

With regard to the criteria to intervene in the independent work of the students, the teacher argues:

> Sometimes I enter in a working group if I realize that students are stuck. Other times I enter because I realize that students are working very well and they have very good ideas that need to be treated more deeply. […] (Answer to question 4, Sect. 2.2.2)

However, in order to justify his interventions in the teamwork, the teacher refers to a broader explanatory framework around the notions of "zone of proximal development" and "semiotic game":

> […] a constant is that I try to work in a zone of proximal development. The analysis of video and the attention we paid to gestures made me aware of the so-called "semiotic game" that consists in using the same gestures as students but accompanying them with more specific and precise language […] (Answer to question 4, Sect. 2.2.2)

Here is where the *didactic theory* shows up. It also includes a certain conception of mathematics, the rationale of teaching it and the mission of schools in society:

> The main aim of the posed activity was to allow students to develop an understanding of the concept of exponential growth […]. This consideration, in my opinion, should allow students to understand why the exponential function […] grows with x faster than any power of x. (Answer to question 8, Sect. 2.2.2)

> I try to assess in the students the competence to observe and explore situations; to produce and to support conjectures; to understand what they are doing and to reflect on it […] (Answer to question 11, Sect. 2.2.2)
>
> the main function of teaching, not only of math, is to help students to exercise critical thought, to acquire the necessary competences for an informed and aware citizenship. (Answer to question 11, Sect. 2.2.2)

We can add another theoretical element the teacher does not explicitly formulate but that seems to support his practice with respect to the mathematical knowledge at stake: the fact that it is not necessary for the teacher to explain to the students why the properties of exponential functions are worthwhile to identify and what is the main purpose of the tasks given to them.

5.3 New Questions Enlarging the Empirical Unit of Analysis

Until now we have just proposed a description in terms of praxeologies of the activities observed (or deduced) from the video and from the extra empirical data gathered. However, the aim of ATD is not just to describe teaching and learning realities, but to *explain* and *question* it from different perspectives, confronting the observed facts with those that could happen and did not, also analyzing the conditions that enable teaching and learning processes to happen in the way they happen, while hindering or impeding other kinds of activities from taking place. As in the case of mathematical praxeologies, when dealing with the description of the didactic praxeologies, the analysis of the observed situation depends on the type of questions we wish to answer as researchers.

For instance, if we consider the mathematic praxeologies described in the previous section as if they make sense on their own, then we would be assuming the didactic project of the teacher without further analysis and we would only be questioning what the students do, what they learn, and how they learn it. However, if we make a step aside and look at the teacher's whole didactic project, numerous questions arise related, for instance, to the didactic transposition process and the elaboration of the mathematical praxeologies to be taught:

> Where do the proposed tasks come from? What questions could they contribute to answering? What broader praxeology are they supposed to integrate? Why is it important to describe the properties of variation of the exponential function? What is being done with those properties? In which broader praxeology and at what level (practical or theoretical) will the obtained technological elements on the exponential function integrate?

With the extra information gathered in Sect. 2.2, some of those questions can be partially answered. For instance, the broader mathematical praxeology at stake is basically generated by problematic situations modeled by *discrete* exponential functions, a previous work that can motivate the study of the properties of the graphs of continuous exponential functions. At the moment considered in the episode, this work can only be carried out with ostensive techniques in order to conclude that the function depending on the tangent line is proportional to the corresponding exponential function. A first approximation to the notion of derivative and some of the praxeological elements that will be necessary later on for its formal construction are thus obtained. Finally, logarithmic functions are defined as the inverse of exponential functions and, as the teacher indicates, the properties of both are used to propose some techniques of solving exponential and logarithmic equations and inequalities. Given this, the crucial question of the criteria used to choose the structure and dynamics of the mathematical praxeology to be taught should be asked, as well as the conditions needed to make this choice and the restrictions that hinder it. This is part of the analysis of the didactic transposition process that is not being developed here. It requires the elaboration of a *reference epistemological model* about the *theme* of exponential functions and its relationship with the different *sectors* and *domains* of school mathematics to provide researchers with an alternative point of view.

This praxeological analysis about didactic stake ♥ (the characterization of exponential functions through their point and global variation) and the description of the didactic praxeologies used by both the teacher and the students should be completed by an *ecological* analysis about their conditions of possibility. It starts by asking questions such as:

> Where does the didactic praxeology enacted by the teacher and the students come from? How is it built? Is it a common organization in the educational system considered? What institutional conditions, at what level of the scale of didactic codetermination, make it possible to appear? What other alternative organizations exist or could exist?

If we stick at *the level of the discipline*, that is, the teaching of mathematics in grade 10, the teaching strategy followed by the teacher does not seem to correspond to a "standard" content organization, where topics usually have a more classical structure generally imposed by official curricula: the discipline divided into domains or sectors (sometimes called "blocks of content") with a given list of themes or topics in each. Teachers organize, sequence, and program the themes their own way, but they rarely question or, much less, modify the given structure. This curriculum constraint tends to confine the teacher's didactic praxeology at the level of the theme and makes it difficult to draw attention to the *rationale* of the taught mathematical praxeologies because they often appear to be beyond the themes (and even beyond the sectors or domains) where they take place. This is not the case in the teaching process considered here, since the teacher seems to be responsible for the whole organization of the content. It is interesting to ask what kind of institutional as well as personal conditions are necessary to do so. Certainly the teacher's involvement with research in didactics is one of the conditions for this didactic praxeology to exist, since the technological and theoretical discourses underlying it are far from being spontaneous or professionally shared.

The level of the *pedagogy* corresponds to the conditions that are common to the teaching and learning of any discipline in a school institution. In this respect, the considered episode is a good illustration of another phenomenon related to the usual distribution of responsibilities between the teacher and the students in traditional didactic praxeologies. Current curricula tend to refer the main goal of teaching and learning projects to a list of predetermined praxeological elements ("topics," "concepts," "competences," etc.) teachers should teach and students learn. The way these elements are organized, motivated and made available to the students, as well as the reasons for the choices made, are part of the teacher's responsibilities. Students do not participate in this kind of decision, which is even often hidden to them. They are just asked to do things and they usually do them heedfully and obediently. Even if the teaching strategy in the analyzed episode is not a common one and seems modern and innovative, it still contains some remains of the classic "authoritarian" pedagogical gestures: the teacher presents some tasks and an experimental tool and gives instructions to the students without explaining where they come from nor where they lead to; we cannot see any information about the map of the trip students are invited to follow; they do not seem either to be asked to participate in its configuration. The teacher proposes, the students accomplish.

Finally, at the level of the *society*, the episode also illustrates how didactic praxeologies – even the most "elaborated" ones – are always permeable, vulnerable even, to practices with a high cultural value, independently of their didactic "utility" or "productivity." According to the task instructions, students are required to "observe" the properties of the graphs of the functions they see on the screen, "discuss their observations" and then deduce some of the "features" of the graphs. Therefore the teaching strategy seems to be taking advantage of the current fascination for visual representations in our western culture. It thus appears here as a strong condition to facilitate the use of Cabri files as a means for the students' main exploration work. The situation would certainly be more difficult if the experimental work was organized around the observation and manipulation of numerical tables or algebraic formulae, since they tend to appear as meaningless to our common culture. The tasks prepared by the teacher in the sessions following the episode include these kinds of alternative experimental means, but they seem to play a less central role in the whole teaching and learning process.

Because of the loss of its social leadership, school encounters more and more difficulties in giving sense to some didactic practices that are not easily recognized by common culture. In the other sense, school is permeable to some social practices that are easily adopted as didactic ones, while remaining resistant to others. Little is known about the specific ecology of didactic praxeologies at school and how this ecology is related to their existence in other social institutions. This is the reason why researchers are interested in tracking data coming from outside school and in looking into school as *outsiders*, that is, without assuming that anything that happens there is normal or necessary. The theoretical and methodological framework provided by ATD, throughout the delimitation of a unit of analysis that goes far beyond the limits of the classroom activities appears to be a useful tool to emancipate researchers from the "transparency" of didactic facts and from the cultural values about the social and human phenomena they have to approach.

References

Artigue, M., & Winsløw, C. (2010). International comparative studies on mathematics education: A viewpoint from the Anthropological Theory of Didactics. *Recherches en Didactique des Mathématiques, 31*(1), 47–82.

Barbé, Q., Bosch, M., Espinoza, L., & Gascón, J. (2005). Didactic restrictions on the teacher's practice. The case of limits of functions. *Educational Studies in Mathematics, 59*, 235–268.

Bosch, M., & Gascón, J. (2006). Twenty-five years of the didactic transposition. *ICMI Bulletin, 58*, 51–63.

Bosch, M., Gascón, J., Ruiz Olarría, A., Artaud, M., Bronner, A., Chevallard, Y., Cirade, G., Ladage, C., & Larguier, M. (2011). *Un panorama de la TAD. An overview on ATD* (CRM Documents, Vol. 10). Bellaterra (Barcelona): Centre de Recerca Matemàtica.

Bronner, A., Larguier, M., Artaud, M., Bosch, M., Chevallard, Y., Cirade, G., & Ladage, C. (2010). *Diffuser les mathématiques (et les autres savoirs) comme outils de connaissance et d'action*. Montpellier: IUFM.

Chevallard, Y. (1985). *La Transposition Didactique. Du savoir savant au savoir enseigné* (2nd ed., 1991). Grenoble: La Pensée Sauvage.

Chevallard, Y. (1989). On didactic transposition theory: Some introductory notes. In *Proceedings of the international symposium on selected domains of research and development in mathematics education*, Bratislava (pp. 51–62). http://yves.chevallard.free.fr/spip/spip/IMG/pdf/On_Didactic_Transposition_Theory.pdf. Accessed 30 Oct 2013.

Chevallard, Y. (1992a). Fundamental concepts in didactics: Perspectives provided by an anthropological approach. In R. Douady & A. Mercier (Eds.), *Research in Didactique of mathematics, selected papers* (pp. 131–167). Grenoble: La Pensée Sauvage.

Chevallard, Y. (1992b). A theoretical approach to curricula. *Journal für Mathematikdidaktik, 13*(2/3), 215–230. http://yves.chevallard.free.fr/spip/spip/IMG/pdf/A_Theoretical_Approach_to_Curricula.pdf. Accessed 30 Oct 2013.

Chevallard, Y. (1999). L'analyse des pratiques enseignantes en théorie anthropologique du didactique. *Recherches en Didactique des Mathématiques, 19*(2), 221–266.

Chevallard, Y. (2002). Organiser l'étude. 3. Écologie regulation. In J.-L. Dorier, M. Artaud, M. Artigue, R. Berthelot, & R. Floris (Eds.), *Actes de la 11e École d'Été de didactique des mathématiques* (pp. 41–56). Grenoble: La Pensée sauvage.

Chevallard, Y. (2006). Steps towards a new epistemology in mathematics education. In M. Bosch (Ed.), *Proceedings of the IV Congress of the European Society for Research in Mathematics Education* (pp. 21–30). Barcelona: FUNDEMI-IQS.

Estepa, A., García, F. J., & Ruiz-Higueras, L. (2006). *Sociedad, escuela y matemáticas. Aportaciones de la Teoría Antropológica de lo Didáctico (TAD)*. Jaén: Publicaciones de la Universidad de Jaén.

García, F. J., Gascón, J., Ruiz Higueras, L., & Bosch, M. (2006). Mathematical modelling as a tool for the connection of school mathematics. *ZDM – The International Journal on Mathematics Education, 38*(3), 226–246.

Chapter 6
Introduction to Abstraction in Context (AiC)

Tommy Dreyfus and Ivy Kidron

Abstract The chapter briefly introduces the theoretical framework of Abstraction in Context (AiC) by referring to the data from Chap. 2. AiC provides a model of nested epistemic actions for investigating, at a micro-analytic level, learning processes which lead to new (to the learner) constructs (concepts, strategies, …). AiC posits three phases: the need for a new construct, the emergence of the new construct, and its consolidation.

Keywords Theories • Abstraction in context • Epistemic actions

6.1 Abstraction in Context – An Overview

Abstraction in Context (AiC) has been developed over the past 15 years with the purpose of providing a theoretical and methodological approach for researching, at the micro-level, learning processes in which learners construct deep structural mathematical knowledge. Theoretically, AiC attempts to bridge between cognitive and situated theories of abstraction, as well as between constructivist and activity oriented approaches. Methodologically, AiC proposes tools that allow the researcher to infer learners' thought processes. Since we can only give an overview of AiC in the limited space available here, we refer the interested reader to more

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detailed treatments of the theory (Schwarz et al. 2009), the methodology (Dreyfus et al. 2015), and their relationship (Hershkowitz 2009) that have recently been given elsewhere.

AiC is a theoretical framework rather than a full-fledged theory, because its strength lies in suitably choosing and interpolating between elements from cognitive and situated approaches as well as activity theoretical and constructivist elements, and in the development of methodological tools that take these varied aspects into account.

6.1.1 Principles

6.1.1.1 Focus on Abstraction

Understanding the processes by which students construct abstract mathematical knowledge is a central concern of research in mathematics education. In schools, abstraction may occur in a variety of curricular frameworks, classroom environments, and social contexts. The attention to such a variety of contexts requires a hybrid reference to theoretical forefathers that belong to different traditions, Freudenthal and Davydov. Freudenthal (1991) describes what mathematicians have in mind when they think of abstraction. He has brought forward some of the most important insights to mathematics education in general, and to mathematical abstraction in particular. These insights led his collaborators to the idea of "vertical mathematization." Vertical mathematization is a process by which learners reorganize previous mathematical constructs within mathematics and by mathematical means in such a manner that a new abstract construct emerges. In vertical reorganization, previous constructs serve as building blocks in the process of constructing. Often these building blocks are not only reorganized but also integrated and interwoven, thus adding a layer of depth to the learner's knowledge, and giving expression to the composite nature of mathematics.

Davydov was one of the most prominent followers of the historical cultural theory of human development initiated by Vygotsky. For Davydov (1990), scientific knowledge is not a simple expansion or generalization of people's everyday experience. It requires the cultivation of particular ways of thinking, which permit the internal connections of ideas and their essence to emerge; the essence of the ideas and their connections then, in turn, enrich reality. According to Davydov's "method of ascent to the concrete," abstraction starts from an initial, simple, undeveloped and vague first form, which often lacks consistency. The development of abstraction proceeds from analysis, at the initial stage of the abstraction, to synthesis. It ends with a consistent and elaborated form, to which the essence of the ideas and their connections lend concreteness. Hence, it does not proceed from concrete to abstract but from an undeveloped to a developed form.

AiC adopts vertical mathematization and ascent to the concrete as the essential characteristics of processes of abstraction. It investigates how these processes occur in a specific learning environment, a particular social context, and a given curricular context. Giest (2005) points out that Activity Theory is most suitable for this since it proposes an adequate framework for considering processes that are fundamentally cognitive while taking social and other contextual aspects into account. In Activity Theory, individual actions occur in context and make sense only within the activity in which they take place. The kinds of actions that are relevant to abstraction are epistemic actions – actions that pertain to the knowing of the participants and that are observable by participants and researchers. In addition, outcomes of previous activities naturally turn to artifacts in further ones, a feature which is crucial to tracing the genesis and the development of abstraction through a succession of activities that might form part of a curriculum.

As researchers in the tradition of Freudenthal, we are a priori attentive to certain constructs afforded by the activities we observe. In tune with Davydov and a cultural-historical theory of development, we also look at other constructs that may emerge from classroom activities. This is well expressed by Kidron and Monaghan (2009) when dealing with the need that pushes students to engage in abstraction, a need which emerges from a suitable design and from an initial vagueness in which the learner stands:

> … the learners' need for new knowledge is inherent to the task design but this need is an important stage of the process of abstraction and must precede the constructing process, the vertical reorganization of prior existing constructs. This need for a new construct permits the link between the past knowledge and the future construction. Without the Davydovian analysis, this need, which must precede the constructing process, could be viewed naively and mechanically, but with Davydov's dialectic analysis the abstraction proceeds from an initial unrefined first form to a final coherent construct in a two-way relationship between the concrete and the abstract – the learner needs the knowledge to make sense of a situation. At the moment when a learner realizes the need for a new construct, the learner already has an initial vague form of the future construct as a result of prior knowledge. Realizing the need for the new construct, the learner enters a second stage in which s/he is ready to build with her/his prior knowledge in order to develop the initial form to a consistent and elaborate higher form, the new construct, which provides a scientific explanation of the reality. (Kidron and Monaghan 2009, pp. 86–87)

Hence we postulate that the genesis of an abstraction passes through a three-stage process: the need for a new construct, the emergence of the new construct, and the consolidation of that construct.

6.1.1.2 Focus on Context

The C in AiC stands for context. According to AiC, processes of abstraction are inseparable from the context in which they occur. Therefore, it was unavoidable to mention context already in the previous subsection. For AiC, the focus is on the students' processes of construction of knowledge. The "context" integrates any piece of the present and past environment that can influence the individual processes of construction of knowledge. As we show in Chap. 10 on context/milieu there are different approaches towards "context" in different didactic cultures. For AiC, artifacts are conceived as a part of the context. In another theory which privileges the cultural and social dimensions, artifacts are constituents of mathematical activity. For AiC, context has many components. One of them is the social context, often including peers or a teacher; another is the historical context, which refers to the students' prior experiences in learning mathematics; a third is the learning context, which includes, among others, curricular factors, socio-mathematical norms, and technological tools. In any specific activity, tasks given to the students are an essential part of the context.

Chapter 10 of this book deals in depth with the role of context in processes of abstraction. In order to avoid repetitions, we therefore keep this subsection very short and only mention that the context situates processes of abstraction for the learners, while allowing the researcher to focus on the learners' cognitive actions in the given context or situation. Hence, context is the notion that allows AiC to bridge between a cognitive and a situated approach.

6.1.2 Questions

AiC was developed in response to a question that arose in the framework of a research-based curriculum development project (Hershkowitz et al. 2002), namely for what mathematical concepts and strategies students achieved in-depth understanding and retained them in the long term. Hence, the design of task sequences lies at the origin of the questions asked by the originators of AiC, and remains one of their concerns. The research questions AiC attempts to answer include:

- Given a sequence of tasks, what are the intended constructs – the mathematical methods, concepts, and strategies – that the designers intended the students to construct when carrying out the task-based activities? How are these intended constructs structured, how are they related to each other, and how are they based on previous constructs?
- For each of the intended constructs, how did the students go about actually constructing it, and how does each student's construct compare with the intended one? Is it partial, and in what sense?
- Did the students construct alternative or non-intended constructs? Which ones?
- What was the origin for the students' motivation to construct; from where did their need for a new construct originate?
- Which previous constructs were used and consolidated during the constructing process?
- What were the characteristics of the constructing process? Was it sudden or prolonged, continuous or interrupted? Were several constructing processes developing in parallel? If so, how did they interact and influence each other (see, for example, Dreyfus and Kidron 2006)?
- What role did contextual factors play in the process? For example, did groups of students co-construct, and if so, were the group members' constructs compatible in the sense that they can continue co-constructing in the following tasks?

- Did technological tools play a role in the constructing processes, and what role?
- What can we learn from the students' constructing processes about the design of the activities, in particular their micro-design?

As indicated by the last question, one of the aims is to improve the design of sequences of activities, in particular their micro-design. Micro-design includes all local aspects of design from the choice of a particular real-life setting for a task and the potential mathematical limitations imposed by that setting, via the degree of openness of a task, the balance between its qualitative and quantitative aspects, and the degree to which students are encouraged to justify their decision and actions, down to a specific choice of words or a specific formulation of a question.

6.1.3 Key Theoretical Constructs and Methodology

Theory and methodology are closely intertwined in AiC (Hershkowitz 2009). Therefore we cannot describe the key theoretical constructs of AiC, the epistemic actions, without also describing the key methodological aspects of AiC, as the methodology's main purpose is the identification of students' epistemic actions. It is important to point out the dynamic character of the theory: the analyses to identify abstraction processes through the unveiling of its epistemic actions not only helped in the understanding of learners' cognitive processes, the theory as well as the methodology underwent successive refinements (Kidron and Dreyfus 2010a, b; Dreyfus and Kidron 2006). The more technical aspects of the methodology are described elsewhere (Dreyfus et al. 2015).

6.1.3.1 The Dynamically Nested Epistemic Actions Model

The central theoretical construct of AiC is a theoretical-methodological model, according to which the emergence of a new construct is described and analyzed by means of three observable epistemic actions: recognizing (R), building-with (B), and constructing (C). Recognizing refers to the learner seeing the relevance of a specific previous knowledge construct to the problem at hand. Building-with comprises the combination of recognized constructs, in order to achieve a localized goal such as the actualization of a strategy, a justification, or the solution of a problem. The model suggests constructing as the central epistemic action of mathematical abstraction. Constructing consists of assembling and integrating previous constructs by vertical mathematization to produce a new construct. It refers to the first time the new construct is expressed or used by the learner. This definition of constructing does not imply that the learner has acquired the new construct once and forever; the learner may not even be fully aware of the new construct, and the learner's construct is often fragile and context-dependent. Constructing does not refer to the construct becoming freely and flexibly available to the learner: becoming freely and flexibly available pertains to consolidation.

Consolidation is a never-ending process through which a student becomes aware of his or her constructs, the use of the constructs becomes more immediate and self-evident, the student's confidence in using the construct increases, the student demonstrates more and more flexibility in using the construct (Dreyfus and Tsamir 2004), and the student's language when referring to the construct becomes progressively more elaborate. Consolidation of a construct is likely to occur whenever a construct that emerged in one activity is built-with in further activities. These further activities may lead to new constructs. Hence consolidation connects successive constructing processes and is closely related to the design of sequences of activities.

In processes of abstraction, the epistemic actions are nested. C-actions depend on R- and B-actions; the R- and B-actions are the building blocks of the C-action. At the same time, the C-action is more than the collection of all R- and B-actions that contribute to the C-action, in the same sense as the whole is more than the sum of its parts. The C-action draws its power from the mathematical connections, which link these building blocks and make them into a single whole unity. It is in this sense that we say that R- and B-actions are constitutive of and nested in the C-action. Similarly, R-actions are nested within B-actions since building-with a previous construct necessitates recognizing this construct, at least implicitly. Moreover, a lower level C-action may be nested in a more global one, if the former is made for the sake of the latter. Hence, we named the model the dynamically nested epistemic actions model of abstraction in context, more simply the RBC-model, or RBC+C model using the second C in order to point at the important role of consolidation. The RBC-model is the theoretical and micro-analytic lens through which we observe and analyze the dynamics of abstraction in context.

6.1.3.2 A Priori and a Posteriori Analyses

As part of the AiC methodology, an effort is made to foresee trajectories of students' learning: an a priori analysis of the activities (Ron et al. 2010) is carried out before data are collected. Early contacts with the TDS team have reinforced our habit to systematically carry out a priori analyses. Assumptions are first made about the previous knowledge of the students, about constructs they are expected to have acquired during earlier activities (and which may be more or less available to them). Then the question is asked what knowledge constructs are required to deal with each task and to complete it to the designer's or teacher's satisfaction; we also ask what constructs might be helpful but not necessary to deal with the task. We are particularly interested in constructs that have not been relevant in previous activities carried out by the same students. It is our working assumption that the new constructs that emerge for the students when dealing with the task are closely linked to the intended ones. The intended constructs are of course to be distinguished from what students actually construct during the activities, although a close correspondence between intended constructs and learners' actual constructs may be expected if the design and the a priori analysis are adapted to the learner.

The a priori analysis has a considerable influence on the a posteriori RBC-analysis of the data collected, usually by audio and video recordings, from students carrying out the activities. Therefore, we give an operational definition for each intended construct, which fixes under what circumstances the researcher will say that a student is using or expressing a construct that corresponds to the intended one. One aim of the a priori analysis is to focus, at least initially, the researchers' attention on the intended constructs, while keeping an open mind for possible alternative or unintended constructs during the ensuing a posteriori RBC micro-analysis of students' knowledge-constructing processes.

6.2 Illustrating Abstraction in Context in the Case of Carlo, Giovanni, and the Exponential Function

The aim of this section is to illustrate the main notions of AiC as introduced above by means of an excerpt from the work of Carlo and Giovanni. However, for reasons to be explained below, this aim can only be partly realized.

6.2.1 A Priori Analysis

As usual in AiC research, we begin with an a priori analysis. Chapter 4 includes an a priori analysis for Tasks 1 and 2, carried out by the TDS team. They identified nine constructs C_1–C_9, and we assume that had we carried out such an analysis, we would have ended up with a similar list of constructs. We therefore adopt their analysis, and continue here with an a priori analysis of Task 3. Task 3 is very open, and therefore there are not many detailed indications about the constructs that might have been intended by the designer and/or teacher. However, given the quantities that can be varied in the Dynamic Geometry Software (Fig. 2.3, Chap. 2) and the instructions given in the task which relate to this variation, we propose the following constructs as those which the designer/teacher probably intended the students to construct:

C_{10} For any given P, that is, locally, as Δx tends to zero, the slope of the secant tends to the slope of the tangent; the slope of the secants and the tangent are all positive (for $a>1$).

C_{11} As P moves on the graph, the slopes of the corresponding secants (and hence the slope of the tangent) vary. As x grows (P moves to the right), the slope of the tangent grows (for $a>1$). As x decreases (P moves to the left), the slope of the (secants and the) tangent decreases to zero (for $a>1$).

C_{12} As a increases, the slope of the secant (for given x, P) increases (and consequently the slope of the tangent increases as well). As a decreases towards 1, the slope of the secant decreases towards 0. As a becomes smaller than 1, the slope of the secant (and consequently of the tangent) becomes negative; the function is decreasing rather than increasing. The parts of C_{10} that depend on $a > 1$ have to be adapted for $a < 1$.

We stress that these are the constructs that we as AiC researchers found in our a priori analysis. They are not necessarily identical to what the teacher in fact intended, and they may, of course, be different from what the students actually constructed when working on the task.

In the present case, we learned from the answers of the teacher as reported in Sect. 2.2.2 that the intended constructs resulting from our a priori analysis are compatible with the declarations of the teacher, and that according to the teacher they are within reach of the students, given the previous knowledge of the class and the socio-mathematical norms that are characteristic for the class, such as explorations that favor the production of conjectures and should motivate their validation as well as argumentation in support of conjectures (see the answer to question 1 in Sect. 2.2.2).

We further note that these intended constructs give general formulations and properties of the resulting constructs in the specific case of the exponential function. From the teacher's answers (Sect. 2.2.2), we know that this activity was given as preparation before the notion of derivative had been formally introduced: "The worksheet […] is situated […] before the formal approach to the concept of derivative of a polynomial function. […] The activity intends to clarify the principal features of increasing behaviours and of exponential functions. In particular, it intends to explain the reason why at the increasing of x an exponential of base greater than 1 will increase, definitively, more than any other polynomial function of x, whatever grade of the polynomial. In the project, exponential functions and sequences are used to cope with problem situations coming out from exponential models" (the teacher's answer to question 17 in Sect. 2.2.2). So again, our "guess" was confirmed after the event.

We note finally that as researchers we should always expect students to develop other constructs than the ones provided by the a priori analysis. Here especially, because of the open formulation of the task, we may expect constructs different from C_{10}, C_{11}, and C_{12} to emerge for the students. Examples of such "other" constructs in the present case are the following:

C_{11}' As P gets closer to y = 0, the function can be approximated by the secant line.

C* The exponential function can be approximated by many small lines with an increasing slope that join together.

The first of these has been called C_{11}' because it is a complementary construct to (the second part of) C_{11}. On the other hand, C* constitutes a transition from a local to a global view: a construct that seems rather independent of the constructs C_{10}, C_{11}, and C_{12} which were identified a priori; it has therefore been assigned a separate notation. The alternative constructs C_{11}' and C* will play a role in the a posteriori analysis below.

6.2.2 Need for Extension of Data

In AiC, we focus on particular kinds of curricula (see Schwarz et al. 2009) and within these, on tasks with a high potential for supporting the construction of knowledge that is new to the learner. This requires the elaboration of sequences of activities that offer the students opportunities to learn well defined mathematical ideas, for example the notion of integral as an accumulating quantity; or that order is relevant when rolling two dice and therefore getting a 1 and a 4 is twice as likely as getting two 4's, etc. It also requires the elaboration of further activities to apply these ideas as tools in familiar contexts or as tools in contexts that necessitate the elaboration of new ideas. What is common to all these learning aims is that they include adding new connections between students' previous knowledge, hence adding depth to the students' understanding and integrating their knowledge in ways not available to them before. In brief, the design intends to create a didactical sequence aimed at vertical reorganization of students' knowledge.

Most of the tasks that the two students in the analyzed video, Giovanni and Carlo, were asked to work on are not of this kind. These tasks require more phenomenological observation than explanations of the phenomena. For example, Tasks 3a and 3b ask the students to describe the phenomena that occur as Δx tends to zero; these tasks do not require any kind of justification. This suggests that the students had previously experienced the limiting process and were now asked to recall it, and possibly reconstruct it in the case of a new example they may not have dealt with yet; from the point of view of AiC, no new construction was required but the students were offered an opportunity to consolidate some of their previous constructs. In Task 3, the students were asked to "Describe briefly the figure, moving first P, then Δx (changing its length), then A; write briefly your observations on the sheet." This formulation suggests that the students had never explored before what happens when varying the parameters x, Δx, and a, and hence that the teacher intended that, in the course of this exploration, his students would meet situations they had never met before. This would offer the students an opportunity to construct new (to them) knowledge but as long as the requirement is descriptive rather than explanatory or connective, this new knowledge is simply an addition to existing knowledge and no need for vertical reorganization would arise. Even in tasks with more potential, such as studying the effect of changing x on the slope of the linear function that best approximates the function $y=a^x$, the stress in the task formulation is on *how* rather than on *why*. This may have served the teacher's plans: it may have provided a common background for the class to use as basis for a teacher-led whole-class discussion in the next lesson. However, such tasks focusing on phenomenological descriptions are not where we can observe the type of knowledge construction in which AiC researchers are primarily interested. This knowledge construction may then happen during the whole-class discussion. In fact, due to the excellent preparation the students were given, it is likely to happen, but we do not have data about this. Therefore, an AiC analysis of most of the data we have is inappropriate and unlikely to yield results about processes of constructing new knowledge by vertical reorganization.

6.2.3 A Posteriori Analysis

We present here our attempt at analyzing the preceding part of the students' work on Task 3, namely transcript lines 249–301 (see Appendix). Unsurprisingly, we will not be able to identify any constructing actions.

The students start on Task 3 in line 249. Until line 281, they identify parts of the situation on the screen. Only in line 281 do they finally read the task. Until then, a main issue in the discussion focuses on identifying the segment PH with Δx. This identification is not a mathematical relationship but a given of the task. The students need to make this identification in order to get access to the situation, but this is not an epistemic action providing them insight into mathematical connections or relationships. It is a preparatory action and is of interest to us only as such.

In what follows, the students make purely phenomenological observations of what happens as one of the parameters varies, in accord with what they were asked to do in the task. They first seem to vary P; they seem to observe phenomena but do not draw any conclusions; all they say is that QH changes as a consequence of changing P. This could potentially have led to insights such as "the slope changes"; "the slope of the secant changes"; "the slope of the tangent changes"; "the derivative changes" – all depending on the preparation of the students and the requirements of the task. Had the students reached such insights, we would have claimed that they recognized some of their previous constructs as relevant to the present situation, and possibly that they built-with them a dynamic image. But the task does not require such insights and the students' utterances do not indicate such insights. Our interpretation of these utterances is that the students' thinking did not include such insights.

Then, in line 285, Carlo seems to refer to the fact that Giovanni now changes the Δx instead of P. This leads to a mathematically more significant observation that might later become useful, namely that "it approaches slowly … slowly … a tangent" (lines 287, 289, 291, 292). From the point of view of AiC, we might identify this as recognizing a previous construct (tangent) as relevant in the current context. This recognizing action might then act as a seed for a subsequent constructing action, possibly of C_{10}. The role of such seeds for later constructing actions has been discussed elsewhere (Kidron et al. 2010).

Several more observations are made subsequently, namely what happens when Δx increases (lines 294–296) or what is the quality of the tangent approximation when P moves to the left or to the right (lines 298–301). The students correctly observe that as P moves to the left, the approximation is better than when P moves to the right. These observations later become relevant. However, at this stage they are cumulative. They do not require nor provoke any vertical reorganization. They are not used for a purpose like solving a problem or justifying a mathematical relationship, and therefore no building-with actions occur. They do not even qualify as recognizing actions since such an epistemic action, as defined above, implies that the students recognize a specific previously constructed mathematical concept or strategy.

As a consequence of what we wrote in the previous section, the tasks given to Giovanni and Carlo were such that only in very few excerpts of the protocol might an RBC analysis be expected to yield constructing actions; moreover, these excerpts are all concentrated in lines 302–347, and will be analyzed in Chaps. 9 and 10 because they are the same data on which two different networking processes are described in these chapters. Readers who would like to see an RBC analysis that is independent of the data used in this monograph, and demonstrates how such an analysis works in a case where it is appropriate, are referred to the literature, for example to Dreyfus et al. (2015), which focuses on the methodology.

In this chapter, we gave a brief introduction to Abstraction in Context (AiC). We described our view of abstraction as it is grounded in the work of Freudenthal and Davydov, and the notion of context as it is pertinent for AiC. We introduced the idea of epistemic action as it emerges from Activity Theory and the dynamically nested epistemic actions model, which is the key theoretical construct underlying our methodology. In the second part of this chapter, we attempted to demonstrate our methodology using the data of Carlo and Giovanni, and explained why this attempt was only partially successful. We are aware that the present description has its limitations and refer the reader to longer and deeper descriptions available in the literature (Dreyfus et al. 2015, and references therein; Schwarz et al. 2009).

References

Davydov, V. V. (1990). *Types of generalisation in instruction: Logical and psychological problems in the structuring of school curricula* (Soviet studies in mathematics education, Vol. 2, J. Kilpatrick, Ed., J. Teller, Trans.). Reston: National Council of Teachers of Mathematics. (Original work published 1972.)

Dreyfus, T., & Kidron, I. (2006). Interacting parallel constructions: A solitary learner and the bifurcation diagram. *Recherches en didactique des mathématiques, 26*(3), 295–336.

Dreyfus, T., & Tsamir, P. (2004). Ben's consolidation of knowledge structures about infinite sets. *Journal of Mathematical Behavior, 23*, 271–300.

Dreyfus, T., Hershkowitz, R., & Schwarz, B. B. (2015). The nested epistemic actions model for abstraction in context – Theory as a methodological tool and methodological tool as theory. In A. Bikner-Ahsbahs, C. Knipping, & N. Presmeg (Eds.), *Approaches to qualitative research in mathematics education: Examples of methodology and methods* (Advances in mathematics education series). New York: Springer (in press).

Freudenthal, H. (1991). *Revisiting mathematics education. China lectures*. Dordrecht: Kluwer.

Giest, H. (2005). Zum Verhältnis von Konstruktivismus und Tätigkeitsansatz in der Pädagogik. In F. Radis, M.-L. Braunsteiner & K. Klement (Eds.), *Badener VorDrucke* (pp. 43–64). Baden/A.: Kompetenzzentrum für Forschung und Entwicklung (Schriftenreihe zur Bildungsforschung – Band 3).

Hershkowitz, R. (2009). Contour lines between a model as a theoretical framework and the same model as methodological tool. In B. B. Schwarz, T. Dreyfus & R. Hershkowitz (Eds.), *Transformation of knowledge through classroom interaction* (pp. 273–280). London: Routledge.

Hershkowitz, R., Dreyfus, T., Ben-Zvi, D., Friedlander, A., Hadas, N., Resnick, T., Tabach, M., & Schwarz, B. (2002). Mathematics curriculum development for computerised environments: A designer-researcher-teacher-learner activity. In L. D. English (Ed.), *Handbook* *of international research in mathematics education: Directions for the 21st century* (pp. 657–694). Mahwah: Lawrence Erlbaum.

Kidron, I., & Dreyfus, T. (2010a). Justification enlightenment and combining constructions of knowledge. *Educational Studies in Mathematics, 74*(1), 75–93.

Kidron, I., & Dreyfus, T. (2010b). Interacting parallel constructions of knowledge in a CAS context. *International Journal of Computers for Mathematical Learning, 15*(2), 129–149.

Kidron, I., & Monaghan, J. (2009). Commentary on the chapters on the construction of knowledge. In B. B. Schwarz, T. Dreyfus & R. Hershkowitz (Eds.), *Transformation of knowledge through classroom interaction* (pp. 81–90). London: Routledge.

Kidron, I., Bikner-Ahsbahs, A., Cramer, J., Dreyfus, T., & Gilboa, N. (2010). Construction of knowledge: Need and interest. In M. M. F. Pinto & T. F. Kawasaki (Eds.), *Proceedings of the 34th conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 169–176). Belo Horizonte: PME.

Ron, G., Dreyfus, T., & Hershkowitz, R. (2010). Partially correct constructs illuminate students' inconsistent answers. *Educational Studies in Mathematics, 75*, 65–87.

Schwarz, B. B., Dreyfus, T., & Hershkowitz, R. (2009). The nested epistemic actions model for abstraction in context. In B. B. Schwarz, T. Dreyfus & R. Hershkowitz (Eds.), *Transformation of knowledge through classroom interaction* (pp. 11–41). London: Routledge.

Chapter 7
Introduction to the Theory of Interest-Dense Situations (IDS)

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Abstract The chapter briefly introduces the theory of Interest-Dense Situations (IDS) by referring to the data from Chap. 2. IDS provides a frame for how interest-dense situations and their epistemic and interest supporting character are shaped through social interactions in mathematics classes distinguishing three levels: the social interactions and how the participants are involved, the dynamic of the epistemic processes, and the attribution of mathematical value.

Keywords Theories • Theory of interest-dense situations

7.1 Theory of Interest-Dense Situations: An Overview

The development of the Theory of Interest-Dense Situations[1] began around the millennium with the assumption that, in mathematics classrooms, the social situation plays an important role in the question as to whether learning with interest is possible or not. This theory was formulated to determine how to build situations with the potential to support learning mathematics with interest in everyday classrooms. The need for this theory came from the lack of knowledge of how to do so.

Interest research, conducted primarily by educational psychologists, had shown that the impact of interest on learning is especially fruitful (Krapp 1992, 2004; Prenzel

[1] The development of this theory is described in detail in Bikner-Ahsbahs (2005).

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1998; Schiefele and Csikszentmihalyi 1995; Schiefele and Schreyer 1994; Csikszentmihalyi and Schiefele 1993). An intervention study in physics classrooms even demonstrated that the usual decay of individual students' interest (Hoffmann and Häußler 1998) could be halted. But the new settings could not be transferred to mathematics. The only advice for mathematics classrooms by Bauer (1988) concluded that mathematics teachers should employ a wide range of approaches in order to give every child the chance to learn with interest. Research concentrating only on individual interest did not seem helpful for establishing teaching criteria for everyday lessons, and various researchers began to realize the important role of the social contexts in class (Baumert et al. 1998, p. 327; Gardner 1998, p. 41; Renninger 1998, p. 229; Deci 1992, p. 45). However, the social dimension of interest development had been neither conceptualized as a learning theory nor empirically investigated. This lack of knowledge resulted in the need to know more about what situations in everyday classrooms have the potential to facilitate learning with interest, specifically in mathematics. Such knowledge was sought by first carefully considering concepts of individual interest from the point of view of social interactions. Doing so sparked a paradigm shift in looking at interest-based learning as a specific kind of social interaction.

Two conceptualizations of individual interest offered starting points for a paradigm shift: (1) interest seen as a person–object relation (Krapp 1992, 2004; Schiefele et al. 1979) foregrounded the content; and (2) situational interest (Mitchell 1993) is determined by situational conditions. The connection of both concepts to self-determination theory (Deci 1998), which argues that interest arises from the experience of competence, autonomy, and social relatedness, thus provided indicators of how to promote learning with interest in class.

As a person–object relation, interest is observable through *actions which are directed towards the acquisition of new insights, connected with positive emotions, and self-intentional; that is, the reasons for the actions are the objects of interest themselves.* This kind of sustained individual interest can emerge out of situational interest (Hidi and Renninger 2006; Mitchell 1993) which is supported by situational conditions but which could disappear if the conditions change. Situational interest can be maintained if students become deeply involved in an activity and experience its content as meaningful (Mitchell 1993). The paradigm shift occurred through a changing emphasis from interest as an individual concept towards a more collective concept created through social interactions.

7.1.1 Principles

The main ideas of the previously described conceptualization of interest are taken as sensitizing concepts to define the key concept of interest-dense situations (Bikner-Ahsbahs 2005). Interest-dense situations are particularly fruitful epistemic situations which can occur in everyday mathematics courses when the learners work cooperatively and intensely to advance their own and their peers' ideas (*involvement*), construct further and deeper mathematical knowledge (*dynamic of the epistemic process*), and highly value mathematical objects or methods (*attribution* *of mathematical value*). These situations are considered as interest-dense because their underlying epistemic processes encourage students to be more attentive and engaged, thus leading to dense social interactions. Engaged learners indicate situational interest when they become deeply involved in and mark the mathematical constructions as meaningful. In this sense, situational interest can be regarded as a pattern of participation in interest-dense situations.

The approach refers to a specific kind of social constructivism (e.g. Jungwirth 2003; Krummheuer 2000; Steinbring 1998, 1999; Bauersfeld 1993) as its *background theory* (Mason and Waywood 1996). Its basic philosophy reflects Weber's (1921, 1922) view that understanding the social world requires understanding people's actions. The background theory also builds on s*ymbolic interactionism* (Blumer 1969) that has further developed Weber's view. Blumer starts from the fundamental assumption that people act according to their interpretations which are a result of, and can change during, social interactions. Learning mathematics is regarded as a process of constructing mathematical knowledge within social interactions, and individuals may co-construct knowledge by participating in and contributing to these constructions.

7.1.2 Questions

The theory of interest-dense situations is a foreground theory with a middle range scope (Mason and Waywood 1996), situated in the background theoretical framework of interpretative research on teaching and learning. Researchers in this field examine and seek to answer three paradigmatic questions: How are interest-dense situations shaped in various teaching and learning situations? What conditions nurture or hinder the emergence of these situations? How is situational interest supported and maintained? In the development of this theory so far, the teacher has played a central role, and data collection has been limited to a single sixth-grade class (Bikner-Ahsbahs 2005) and to primary school students (Stefan 2012), still narrowing its applicability. However, the theory's scope could be expanded by investigating further situations and contents concerning the three paradigmatic questions, for example processes such as proof and argumentation, interactions at different ages, contribution of signs, and technology. To do so, the theory might need to be broadened by theoretically generating new phenomena and concepts.

7.1.3 Methodology and Key Constructs

IDS-methodology entails the principles and key constructs as tools to investigate interest-dense situations, especially their epistemic processes, but it also has to be open-minded towards the idiosyncratic conditions of mathematics classrooms and how they contribute to build and stabilize IDS and, hence, support situational interest. To avoid subsumption and to adapt to the classroom features, its methodology is based on the principle of reconstruction. Data are gathered according to the concept of theoretical sampling (Strauss 1994; Glaser and Strauss 1967), which calls for cyclical theory-driven data collection and analysis. The principles of symbolic interactions guide reconstructive data analysis to answer a specific question by reinterpreting the interpretations of those involved. Understanding of relevant social interactions then yields understanding of a situation. Since we observe phenomena in everyday classrooms, we take the ethnomethodological view that society is reconstructed in daily life and that actions indicate why people act in certain ways (reflexivity assumption of ethnomethodology; see Garfinkel 2003). On this basis, regularities in classroom interactions are empirically reconstructed on three methodological levels stepwise deepening insight: based on *individual involvement in social interactions* (level 1), *the dynamic of the epistemic process* (level 2) is investigated and gained insights are deepened by analyzing *the attribution of mathematical value* (level 3).

These reconstructions demand enough data for identifying and idealizing key features on the three levels for systematically constructing ideal types (Bikner-Ahsbahs 2003; Gerhard 1986) which, according to Weber (1922, p. 190), yields theoretical insight: ideal types characterize specific features in an idealized way and act as tools to build theories or further theories by re-analyzing existing and new data (Bikner-Ahsbahs 2003, p. 212; 2015).

The first of four steps is the basis of building ideal types (ibid. 2003, p. 215; ibid. 2015). It follows the rules of analysis for interpretative teaching and learning research (Jungwirth 2003). Within this methodologically controlled reinterpretation, we systematically utilize the three levels of notions of utterance (Beck and Maier 1994; Austin 1975). The *locutionary level* is the content level of what is actually said. The *illocutionary level* is that of social relations indicated by how something is expressed and how actions and interactions with others take place. The decision to act at all belongs to this level. The *perlocutionary level* concerns the intended and factual impacts of the individuals' contributions.

We now turn to key constructs describing the emergence of IDS. In line with the theory's principles and methodology they first have been developed empirically based and were then used to construct ideal types.

7.1.3.1 Individual Involvement in Social Interaction Structures

Within social interaction, the individual involvement can be characterized by the participants' orientations with respect to teachers' expectations concerning the mathematical content. In classrooms, a teacher normally has aims and therefore expects the students to produce a specific kind of mathematical meaning. However, he/she may either behave *steered by* his/her own *expectations* leading the students to produce what is expected; or *steered by situations* trying to understand the students' epistemic processes. Within the flow of social interaction, the students also may either behave *dependent* on the teacher's *expectation* trying to produce what

Table 7.1 Merging students' and teachers' behavior

students' behavior / **teacher's behavior**	***expectation-dependent*** (reproduces the teacher's expectations of content)	***expectation-independent*** (reconstructs own meaning)
steered by expectations (expects factual answers from students)	*expectation-dominant*	
steered by situations (tries hard to understand students' constructions of meaning)		*expectation-recessive*

the teacher wants to hear, or they behave *independently* of these *expectations* following their own line of thought.

In mathematics classrooms two ideal types of interaction structures that merge the different kinds of individual involvement may be approximately observed. In Table 7.1, the behavior of students and teachers is described according to how they nurture or hinder the arousal of interest-dense situations. If *expectation-independent student behavior* and *situationally steered teacher behavior* mix, an *expectation-recessive interaction structure* emerges in which both teacher and learners concentrate on and support processes of constructing mathematical meaning independently of the teacher's expectations. *It nurtures the emergence of interest-dense situations*. This interaction structure is fragile because the given mathematical situation and current constructions of mathematical knowledge are the only features that allow orientation for students and teacher. The *expectation-dominant interaction structure* appears if the teacher and students are guided by the teacher's content-specific expectations towards a task. It is more stable but hinders the emergence of interest-dense situation because the teacher guides the students in such a way that they produce exactly what the teacher wants to hear, while the students try to figure out what the teacher wants to hear. If an expectation-dominant interaction structure occurs within an epistemic process the emergence of an interest-dense situation is deeply disturbed. The two remaining fields neither represent interaction structures nor do they address IDS; they even may lead to conflicting situations.

7.1.3.2 The GCSt[2] Model of Epistemic Actions

It is a characteristic of interest-dense situations that they entail fruitful epistemic processes within an expectation-recessive interaction structure. These processes are built through three central collective actions executed within social interactions: gathering and connecting mathematical meanings, and seeing structures. Gathering meanings refers to collecting bits of mathematical meaning that are similar with

[2] The actions of gathering, connecting and structure seeing are collective in the sense that they are built by social interactions.

respect to solving the posed problem. Connecting meanings happens if a limited number of collected bits of meaning are interconnected or linked to other meanings. If there are sufficient gathering and connecting actions then structures can be seen, that is, a system of relationships for which many examples can be found. Structure-seeing is absolutely necessary for a learning situation to even be considered interest-dense. Once the epistemic process is reconstructed by these three actions, it is represented with symbols that give an overview of the whole process (see Fig. 7.2 in Sect. 7.2.2).

Because of the expectation-recessive interaction structure, teacher and students orient themselves towards the epistemic process leading to structure-seeing in various ways. Material may first be gathered; this is the foundation for making connections, and after that, structure-seeing can occur. Such a process can also arrive at structure-seeing if gathering and connecting activities are intertwined. Meanwhile we can explain how a general epistemic need and situational interest mutually further each other (Kidron et al. 2011) and, thus, nurture the epistemic process.

7.1.3.3 Types of Producing Valuable Mathematical Ideas

During the epistemic process, a system of mathematical values is shared among teacher and students directing and supporting the joint epistemic process to produce mathematically substantial ideas. This system is based on an implicit agreement: the students follow this system of values in order to produce mathematically valuable ideas and the teacher assists them. This way different production types are constituted; for example in a competition of ideas mathematical patterns are created or in a quality inspection the validity and significance of a fact is examined thoroughly. Students who participate in producing a mathematically valuable idea that is appreciated by others may identify themselves with that idea, create authorship and agency. The teacher supports this process not only by valuing highly those ideas, but also by accepting fuzzy explanations at the beginning. The students can attach their individual meanings to it and advance the process of interaction by making the expressions in question more precise; the teacher accepts and supports this process of clarification for example by explicitly offering terms to support students' expressing (Bikner-Ahsbahs 2004).

7.1.3.4 Further Methodological Considerations

The theory of interest-dense situations is a social constructivist theory that cannot say much about cognitive processes of individuals and does not provide tools for epistemological analyses. It is a theory for classrooms addressing general and specific features. For example, we assume that, if interest-dense situations occur at all in everyday lessons, every mathematics classroom shapes its own specific types of epistemic processes leading to the emergence of interest-dense situations. As a general tool, the GCSt model helps to investigate them and to represent their process structure (see Figs. 7.1 and 7.2 from Sect. 7.2.2.2). Epistemic structures and production types already gained may provide sensitivity for specific conditions that foster or hinder the emergence of IDS in the single given classroom, but this will not always be possible. For applying the key constructs to another classroom further condition might have to be theorized and included, too. The analysis of the video data in the next section will give an idea of how theory expansion of IDS takes place, how its methodology in the use of methods and techniques are applied even if the data do not meet all criteria of IDS methodology.

7.2 Illustrating the Theory of IDS Through Analysis of the Video of Carlo, Giovanni, and the Exponential Function

In the following sections we will first show how the framework is broadened to make IDS applicable to this episode. We will then pose questions that will be answered by analyses of data concerning the three methodological levels: individual involvement in social interactions, dynamic of the epistemic process, and attribution of mathematical value.

7.2.1 Use of the Theoretical Framework

The given data shaped by three subsequent episodes differs substantially from the classroom material used so far; namely, most of it consists of group-work with computers and without the – normally very relevant – teacher. Thus, in order to address the empirical material in the episodes of Tasks 1–3, it is first necessary to modify IDS in order to broaden its scope and make it applicable to the given data set.

7.2.1.1 Widening the Methodological and Theoretical Background

Social interactions are built around objects, such as computers, with mathematical concepts or visual on-screen representations. The two students in the episode will use the computer to construct knowledge objects, of which there are different kinds. According to Knorr Cetina (1999) there are two types of knowledge objects. Intrinsic knowledge objects, such as the exponential function in the video of Carlo and Giovanni (see Sect. 2.1.3), are imperfect. That is, they lack completeness because they are not fully understood or there is something further to learn about them; this lack of completeness drives the need to know more about them, which leads to involvement in meaningful epistemic processes. Hence, these intrinsic objects, which are shared in the group, have the potential to initiate interest-dense situations and to contribute to the formation of interest. Extrinsic objects, such as tools like the computer mouse, are ready for use. If they are being used in the epistemic process, they normally become only visible if they pose obstacles or disturbances.

The two students, the dynamic geometry files, and the worksheet together shape a social object-related group. The epistemic process within the social interaction refers to the exponential function as a shared intrinsic knowledge object which appears to be incomplete to the students, thus encouraging them to learn more about it. Extrinsic objects can be technical or material objects that are ready for use.

7.2.1.2 Research Questions in the Light of IDS

Even though the emergence of interest-dense situations is the focus of this analysis and this chapter, the research questions should be extended to cover other constituents of the specific situation and confirm whether the extended theoretical framework fits the data:

1. How does this group act on mathematical objects? Do the students collectively construct mathematical knowledge through social interactions? How are the students involved?
2. Which are the intrinsic and extrinsic knowledge objects? Are there epistemic patterns due to the specific constituents in this situation?
3. Can the episodes be regarded as interest-dense situations? Are there conditions that foster or hinder the emergence of an interest-dense situation?

The results of our analysis are presented according to these questions.

7.2.2 Initial Data Analysis

7.2.2.1 Individual Involvement: Analysis of the Social Interaction in the Group

In the video on Tasks 1 and 2, Carlo and Giovanni (see Sect. 2.1) refer to the same objects in their activities. They both construct knowledge about exponential functions through their use of the computer and by referring to the images on the screen. However, the activity is distributed. Generally, Carlo gives Giovanni instructions to do something with the computer (lines 9 and 11), and Giovanni (abbreviated by "G") does what Carlo (abbreviated by "C") wants him to do (line 12) – but this is not always the case (for full transcript, see Appendix):

9 C: yes… modify also the measure unit of the y-axis, that is, you put, instead of 2.7, you put another thing
10 G: the y-axis?
11 C: yes… what have you done?
12 G: oh, I have moved it, I have put it larger like so, as you can see…ok
13 C: but you see that, that is, you must modify 2.7 […]

The computer reacts to the students' input through visible signs – drawings, animations, numbers, algebraic expressions. Both students interpret these signs on the screen in their order of appearance. Giovanni describes what can be seen on the screen most of the time (line 48).

47 C: go towards the negative ones
48 G: when it arrives to minus, at 2.7, it goes, it goes in 0… because then you see when it arrives in 0, you can continue to move, but it remains always on the 0

The roles of the students are quite stable, as shown in the following excerpt, in which Carlo reads the worksheet and writes down the results. Giovanni is in contact with the computer. Even if Giovanni does something on his own, he still regards it as a collective activity:

54 G: to −1 it does not yet go on the 0, wait! **Let us** [*emphasis added by the authors*] go, a little bit more −2, 0, 3, 3… more or less towards the 6

The students interact by building their interpretations on each other's. This is shown through their use of the same words (lines 173–174, 177–178, 182–184), their completion of each other's sentences (lines 185–186), or their references to each other's statements (lines 178–179, 185–186):

173 C: you try to put it a little more low… so… you try with 1… you look: with 1 it's a line
174 G: with 1, it's a line
175 C: we expected this
176 G: uuh
177 C: instead, if it's less than 1, also…
178 G: with a less than 1…
179 C: we expected this so
…

Then, it goes on:

182 G: ehh, this is x, and this P's y
183 C: that is the x
184 G: that is the x
185 C: so you can see
186 G: yes, yes… it never touch the zero, it doesn't touch

The signs on the screen also indicate knowledge objects, on which both students focus. Commenting too much does not seem to be worthwhile. Verbal interaction can be reduced, and long pauses appear. Deictic expressions indicate which aspects the students are examining (lines 183, 184). Carlo even says “you can see” (line 185), while Giovanni describes what he sees. Both negotiate what they are seeing at that moment; their social interaction is built on commonly perceived objects.

In their further work on Task 2, and more intensively in their work on Task 3, the discourse becomes denser; pauses nearly disappear. From line 325 on, the situation changes fundamentally as the teacher joins the group. The computer screen becomes just a tool for representations to which the students and the teacher refer only if necessary. The social interactions between students and teacher become even more intense.

7.2.2.2 Analyses on the Epistemic Level and the Level of Attribution of Mathematical Value

In order to be able to reconstruct typical regularities about how interest-dense situations are fostered or hindered in this class, we would need many more episodes. Since this group acts as a unit to construct mathematical meaning socially, the epistemic action model (GCSt model) can be applied to reconstruct the epistemic process of the abovementioned three episodes. The result of the whole analysis is represented symbolically in Fig. 7.2, its legend in Fig. 7.1.

In the video on Task 1 (see first line of Fig. 7.2), initiations and gathering are the main epistemic actions through which the students explore the dynamic nature of the exponential function by experimenting with the DGS file.

In their work on Task 2, the students learn to change the base of the exponential functions through connecting actions and again familiarize themselves with this

Initiations		Gathering meaning	•	Structure Seeing		Structure seeing & justifying, verifying, proving	
Opposing		Connecting meaning	⊔	Structure seeing & making concrete			
Gesture	G	intrinsic objects	io	Extrinsic object	eo		

Fig. 7.1 Symbols for the compressed process diagram

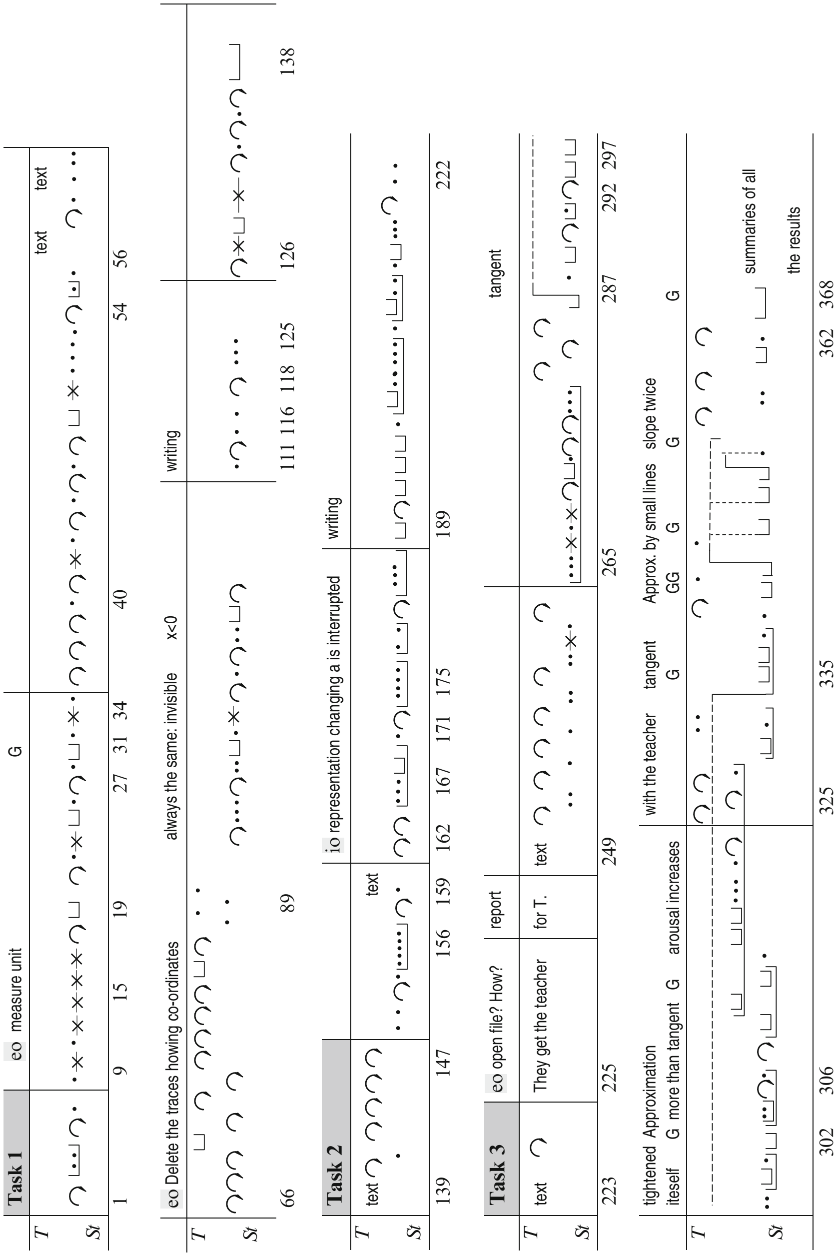

Fig. 7.2 Compressed process diagramme

more complex mathematical situation. Even when gathering takes place here, it becomes part of the connecting actions (Fig. 7.2). As before, the students begin with experimenting and observing. In line 287, the situation changes:

287	C:	look it… slowly… slowly it seems that… I do not know, like, saying tangent
288	G:	eh… yes
289	C:	it seems that it touches it, let's go, let's go, let's go
290	G:	eh, yes… here
291	C:	slowly… slowly
292	C:	it's tangent

Carlo expresses a hypothesis about a certain structure that he sees (line 287), naming it *tangent* and Giovanni agrees (line 288). This is proven through slowly testing the process with the computer. Carlo seems to show increased situational interest (line 287–292) while the computer shows a dynamic situation in which the mathematical idea can be proven. At the end, Carlo sees its structure: "it's tangent" (line 292). The students generate a testing situation through making connections, thus the structure is hypothesized, tested, and labeled.

At the end of the work on Task 3, the teacher joins the group, and the role of the computer in this process changes (327–354). Since the social interactions now take the form of a direct discourse, we will use the three levels of notions of utterances for a discourse analysis.

In line 327, Carlo asks the teacher a question. The teacher does not answer, but instead repeats the principal words as a question: "always the same distance?" (line 328). On the illocutionary level, this gives the students the chance to explore and think again. They then realize that the distance between P and Q[3] is not constant; it may decrease if Q approaches P. In line 331, Giovanni describes the asymptotic behavior of the function: "[…] the nearer P is to y equal to zero, the more this line approximates the function". The reaction of the teacher builds on the previous utterance in trying to understand: "therefore you approach it enough" (line 332); "yes": Giovanni feels accepted and understood (333). In line 334 ("when a function stretches to crush itself on the x-axis"), the teacher adds another idea, but Giovanni on the locutionary level follows the idea of a *nearly tangent* through approximation (line 335–337). The teacher does not disturb this structure-seeing process, but supports it by saying "yes" (336) and then admiring: "uh" (338). Thus, on the illocutionary level, the student's contribution is respected as valuable. In line 340, again, the teacher asks for additional information, directing the students' ideas towards reasoning (on the perlocutionary level): "and so, it gives you some information about what? When the Delta x tends to become very very small, what kind of information do you get?" (line 340). The teacher is successful, since Carlo responds, "if the Delta x becomes small… it means that… the Delta x becomes small when… when between P and Q… that is, the space decreases" (line 341). The teacher strengthens this view: "oh sure, it is almost trivial, isn't it? […]" (line 342) indicating this on the illocutionary level to be important. At the same time he makes clear to look at what is obvious.

[3] Through the points P and Q of the graph of the exponential function a secant is drawn.

The teacher continues, including “tends to” as an additional idea (line 342). Giovanni finishes the teacher’s sentence using the word “tangent” (line 343). This means that the epistemological views of the teacher and the students are similar (see Chap. 11). Giovanni is able to think with the teacher. *Tends to* makes sense in Giovanni’s train of thought; therefore, he is able to integrate this idea. Again the teacher builds on the utterances before this: “and then what kind of information will it give you in this case?” (line 344). On the perlocutionary level, he asks the students to continue deepening to reason within their own view. Giovanni follows this directive metaphorically: “ah, one can say… one can say that the exponential function becomes very little lines…” (line 345). Again the teacher uses the student’s words and changes the direction slightly by adding the idea of approximation: “uh… it could be approximated to some small lines, which however…” (line 346). Giovanni takes this utterance as an invitation (perlocutionary level), and now does nearly the same. He also builds his answer on the utterances of the previous conversation and adds a new perspective of observations about the touch point and the slopes: “that is, that… with increasing slopes, that join together in a, that touch each other in a point” (line 347). The teacher tries to get a better understanding of the students’ aim: “therefore you are imagining to approximate with many small segments” (line 348). On the perlocutionary level, this causes an explanation that shows a deeper understanding, adding again another idea – the idea of zooming in and approximating the graph: “well, if you take it… I don’t know, if you take it with a very large zoom… you can approximate it with many small lines” (line 349). The teacher now appears to be interested in the students’ thoughts, which once again encourages the students to look more deeply and initiates a discussion on the rate of change.

In this situation, the teacher and the students shape an expectation-recessive interaction structure. The teacher’s guidance is done implicitly; mathematical value attribution deepens the students’ epistemic processes and leads to structure-seeing. As additional results, the following patterns may be relevant for nurturing or hindering the emergence of IDS in the classroom:

1. Learning about an intrinsic object is interrupted when an extrinsic object as an obstacle occurs.
2. Intrinsic obstacles do not necessarily induce deep learning processes between peers. From lines 162 to 167, the students observe that representing the graphs is interrupted when the base becomes negative. Since the students agree that this happens because the graph simply gets too high, they feel content. A deepening of learning does not take place, and an interest-dense situation does not emerge.
3. Cyclic patterns appear in which the computer file is used as a tool to experiment: hypothesizing, testing, observing, describing, re-hypothesizing and evaluating, etc.
4. The students get used to the first DGS file by gathering meaning and get familiar with functions by connecting actions through the second DGS file.
5. Shorthand constructions of meanings change into more complex processes of connecting meanings when the students start to write down their results together.
6. When constructing meaning becomes more complex, interactions between the students become more intense. Gestures indicate that they get nervous. Structure-seeing begins in line 287.

7. Before the discourse with the teacher, constructing meaning takes place around what is happening on the computer screen. When the teacher joins the epistemic process, the computer is no longer the source of knowledge. Structure-seeing occurs within a more complex discourse, and the frequency of using gestures to support statements increases (see the case study on gestures in Chap. 9).

By and large, these patterns indicate that the computer is helpful for gathering and connecting meanings, and the teacher is helpful for deepening insight.

7.2.2.3 Emergence of Interest-Dense Situations

There are indications suggesting that an interest-dense situation arises in Task 3 through the three-step task design. The students do not follow the teacher's expectations. They find their own ways of constructing mathematical meaning. In Task 3, students are pushed further to construct mathematical concepts in an epistemic process that prompts structure-seeing. The students themselves do not value the mathematical concepts that they gained explicitly, but the worksheets they complete do. The students are asked to find things that were unexpected, interesting, and so on. This means that the task evaluates as interesting what the students find out beforehand. At the end, the teacher accepts the results and demands that students deepen them by praising the students' comments and using them to steer their conversation and reasoning.

7.2.3 Need for Extended Data and Analysis

The data set has only been analyzed according to how, in these episodes, the emergence of one interest-dense situation is fostered. The second video on Task 3 offers additional data that shows an extra episode in which the emergence of an interest-dense situation is hindered (Chaps. 11 and 12). If additional data of the class were available, the results obtained could be taken as hypotheses to be further investigated towards an overview about how IDS are shaped in this class.

Due to the reflexivity assumption of ethnomethodology, the teacher's interview is not needed, as everything that is relevant is assumed to be activated during and indicated in the lessons.

7.3 Conclusion

The theoretical framework's scope could be widened so that the IDS methodology could be applied to the given features of the data set. Even though further data analysis is needed to reconstruct regularities about how interest-dense situations in this classroom are shaped, we may hypothesize that a three-step design of the task is particularly fruitful. In this, gathering is the main first action, in which students become familiar with the digital worksheet; connecting is the main part of the second step; and structure-seeing is supported in the third by placing the specific concept to be explored into a computer worksheet. This last hypothesis is related to initiating interest formation through task text: "Describe what is interesting, what is expected and unexpected; share your impressions with the other; explore; give arguments and justifications; and write your results down." In this way, the students' results are valued beforehand, initiating involvement in meaningful epistemic processes. In addition, the digital worksheets are designed to test hypotheses, giving the students control over their results so that they can take responsibility for their own learning processes. Extrinsic knowledge objects are quickly removed so that they hardly disturb the flow of ideas. Finally, in the interest-dense situation of Task 3, when the students are ready to learn, the teacher joins by steering the conversation to deepen students' insights implicitly. Our impression is that there is a whole system of features that foster the emergence of interest-dense situations, not just one.

References

Austin, J. L. (1975). *How to* do *things with words: The William James Lectures delivered at Harvard University in 1955* (2nd ed., Eds. J. O. Urmson & Marina Sbisa). Oxford: Clarendon Press.

Bauer, L. A. (1988). *Mathematik und Subjekt*. Wiesbaden: Deutscher Universitätsverlag.

Bauersfeld, H. (1993). Theoretical perspectives on interaction in the mathematical classroom. In R. Biehler, R. W. Scholz, R. Sträßer, & B. Winkelmann (Eds.), *Didactics of mathematics as a scientific discipline* (pp. 133–146). Dordrecht: Kluwer.

Baumert, J., Schnabel, K., & Lehrke, M. (1998). Learning Maths in school: Does interest really matter? In L. Hoffmann, A. Krapp, K. A. Renninger, & J. Baumert (Eds.), *Interest and learning. Proceedings of the Seeon conference on interest and gender* (pp. 327–336). Kiel: IPN 164.

Beck, C., & Maier, H. (1994). Mathematikdidaktik als Textwissenschaft. Zum Status von Texten als Grundlage empirischer mathematikdidaktischer Forschung. *Journal für Mathematik-Didaktik, 15*(1–2), 35–78.

Bikner-Ahsbahs, A. (2003). Empirisch begründete Idealtypenbildung. Ein methodisches Prinzip zur Theoriekonstruktion in der interpretativen mathematikdidaktischen Forschung. *Zentralblatt für Didaktik der Mathematik, 35*(5), 208–223.

Bikner-Ahsbahs, A. (2004). *Interest-dense situations and their mathematical valences. Contribution to the topic study group 24 on students' motivations and attitudes towards mathematics and its study*. 10th international congress for mathematical education, ICME10, Copenhagen. http://www.math.uni-bremen.de/didaktik/ma/bikner/publikationen.html. Accessed 22 Apr 2014.

Bikner-Ahsbahs, A. (2005). *Mathematikinteresse zwischen Subjekt und Situation. Theorie interessendichter Situationen – Baustein für eine mathematikdidaktische Interessentheorie*. Hildesheim/Berlin: Franzbecker.

Bikner-Ahsbahs, A. (2015). Empirically based ideal type construction (translation of Bikner-Ahsbahs 2003). In A. Bikner-Ahsbahs, C. Knipping, & N. Presmeg (Eds.), *Approaches to qualitative research in mathematics education: Examples of methodology and methods* (Advances in mathematics education series). New York: Springer (in press).

Blumer, H. (1969). *Symbolic interactionism: Perspective and method*. Eaglewood Cliffs: Prentice-Hall.

Csikszentmihalyi, M., & Schiefele, U. (1993). Die Qualität des Erlebens und der Prozeß des Lernens. *Zeitschrift f. Pädagogik, 39*(2), 207–221.

Deci, E. (1992). The relation of interest to motivation of behavior: A self-determination theory – Theory perspective. In K. A. Renninger, S. Hidi, & A. Krapp (Eds.), *The role of interest in learning and development* (pp. 43–70). Hillsdale/London: Lawrence Erlbaum.

Deci, E. (1998). The relation of interest to motivation and human needs – The self-determination theory viewpoint. In L. Hoffmann, A. Krapp, K. A. Renninger, & J. Baumert (Eds.), *Interest and learning. Proceedings of the Seeon conference on interest and gender* (pp. 146–164). Kiel: IPN 164.

Gardner, P. L. (1998). The development of males' and females' interest in science and technology. In L. Hoffmann, A. Krapp, K. A. Renninger, & J. Baumert (Eds.), *Interest and learning. Proceedings of the Seeon conference on interest and gender* (pp. 41–57). Kiel: IPN 164.

Garfinkel, H. (2003). *Studies in ethnomethodology*. Englewood Cliffs: Prentice Hall.

Gerhard, U. (1986). Verstehende Strukturanalyse: Die Konstruktion von Idealtypen als Analyseschritt bei der Auswertung qualitativer Forschungsmaterialien. In H.-G. Soeffner (Ed.), *Sozialstruktur und Typik* (pp. 31–83). Frankfurt/New York: Campus Verlag.

Glaser, B., & Strauss, A. (1967). *The discovery of grounded theory: Strategies for qualitative research*. Chicago: Aldine.

Hidi, S., & Renninger, K. (2006). The four-phase model of interest development. *Educational Psychologist, 42*(2), 111–127.

Hoffmann, L., & Häußler, P. (1998). An intervention project promoting girls' and boys' interest in physics. In L. Hoffmann, A. Krapp, K. A. Renninger, & J. Baumert (Eds.), *Interest and learning. Proceedings of the Seeon conference on interest and gender* (pp. 301–316). Kiel: IPN 164.

Jungwirth, H. (2003). Interpretative Forschung in der Mathematikdidaktik – ein Überblick für Irrgäste, Teilzieher und Standvögel. *Zentralblatt für Didaktik der Mathematik, 35*(5), 189–200.

Kidron, I., Bikner-Ahsbahs, A., & Dreyfus, T. (2011). How a general epistemic need leads to a need for a new construct: A case of networking two theoretical approaches. In M. Pytlak, T. Rowland, & E. Swoboda (Eds.), *Proceedings of the 7th Congress of the European Society for Research in Mathematics Education* (pp. 2451–2461). Rzeszów: University of Rzeszów.

Knorr Cetina, K. D. (1999). *Epistemic cultures: How the sciences make knowledge*. Cambridge: Harvard University Press.

Krapp, A. (1992). Das Interessenkonstrukt. In A. Krapp & M. Prenzel (Eds.), *Interesse, Lernen, Leistung* (pp. 297–329). Münster: Aschendorff.

Krapp, A. (2004). An educational-psychological theory of interest and its relation to self-determination theory. In E. L. Deci & R. M. Ryan (Eds.), *The handbook of self-determination research* (pp. 405–427). Rochester: University of Rochester Press.

Krummheuer, G. (2000). Mathematics learning in narrative classroom cultures: Studies of argumentation in primary mathematics education. *For the Learning of Mathematics, 20*(1), 22–32.

Mason, J., & Waywood, A. (1996). The role of theory in mathematics education and research. In A. J. Bishop et al. (Eds.), *International handbook of mathematics education* (pp. 1055–1089). Dordrecht: Kluwer.

Mitchell, M. (1993). Situational interest. Its multifaceted structure in the secondary school mathematics classroom. *Journal of Educational Psychology, 85*(3), 424–436.

Prenzel, M. (1998). The selective persistence of interest. In L. Hoffmann, A. Krapp, K. A. Renninger, & J. Baumert (Eds.), *Interest and learning. Proceedings of the Seeon conference on interest and gender* (pp. 71–92). Kiel: IPN 164.

Renninger, K. A. (1998). The roles of individual interest(s) and gender in learning: An overview of research on preschool and elementary school-aged children/students. In L. Hoffmann, A. Krapp, K. A. Renninger, & J. Baumert (Eds.), *Interest and learning. Proceedings of the Seeon conference on interest and gender* (pp. 165–174). Kiel: IPN 164.

Schiefele, U., & Csikszentmihalyi, M. (1995). Motivation and ability as factors in mathematics experience and achievement. *Journal for Research in Mathematics Education, 26*(2), 163–181.

Schiefele, U., & Schreyer, I. (1994). Intrinsische Lernmotivation und Lernen. Ein Überblick zu Ergebnissen der Forschung. *Zeitschrift für Pädagogische Psychologie, 8*(1), 1–13.

Schiefele, H., Hausser, K., & Schneider, G. (1979). "Interesse" als Weg und Ziel der Erziehung. *Zeitschrift für Pädagogik, 25*(1), 1–20.

Stefan, G. (2012). *Motivation und Interesse im Mathematikunterricht der Grundschule: Genese-Indizierung-Förderung*. Hamburg: Verlag Dr. Kovac.

Steinbring, H. (1998). Epistemological constraints of mathematical knowledge in social learning settings. In A. Sierpinska & J. Kilpatrick (Eds.), *Mathematics education as a research domain: A search for identity* (pp. 513–526). Dordrecht: Kluwer.

Steinbring, H. (1999). Reconstructing the mathematical in social discourse – Aspects of an epistemology-based interaction research. In O. Zaslavsky (Ed.), *Proceedings of the 23rd international conference for the psychology of mathematics education* (Vol. I, pp. 40–55). Haifa: Technion – Israel Institute of Technology.

Strauss, A. L. (1994). *Grundlagen qualitativer Sozialforschung* (UTB für Wissenschaften. Bd. 1776). München: Wilhelm Fink Verlag.

Weber, M. (1921/1984). *Soziologische Grundbegriffe*. Tübingen: J.C.B. Mohr, UTB.

Weber, M. (1922/1985). *Wissenschaftslehre. Gesammelte Aufsätze*. Tübingen: J. C. B. Mohr.

Part III
Case Studies of Networking

Chapter 8
Introduction to Networking: Networking Strategies and Their Background

Susanne Prediger and Angelika Bikner-Ahsbahs

Abstract The chapter provides the language and strategies for networking already published in former *ZDM* issues and books. The concept of networking is clarified and the networking strategies and networking profiles are described. The five theoretical approaches from Chaps. 3, 4, 5, 6, and 7 are compared with respect to the concept of theories as a dynamic way of understanding through the triplet (system of principles, methodologies, set of paradigmatic questions). After that, case studies from Chaps. 9, 10, 11, and 12 are briefly introduced.

Keywords Networking of theories • Methodology • Networking strategies

The comparison of Chaps. 3, 4, 5, 6, and 7 in Part II of this book gave an example of what is meant by the abstract term "diversity of theories". Five theoretical approaches were presented that differ not only in their key constructs, but also in their main questions, principles, methodologies, and the specificity of the results (Radford 2008, 2012). In Part II, the five theoretical approaches and their research practices were presented next to each other. However, the plurality of theoretical approaches can only become fruitful when different approaches and traditions *come into a dialogue*. For this purpose, different networking strategies have been specified (Prediger et al. 2008b) and applied in various projects. Reflection on these projects has offered interesting first contributions to a methodology of networking (Prediger et al. 2008a).

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According to Radford (2008), this networking process takes place in the so-called semiosphere, which – referring to Lotman – he describes as “an uneven multi-cultural space of meaning-making processes and understandings generated by individuals as they come to know and interact with each other” (Radford 2008, p. 318). Core elements of this cultural semiotic space of mathematics education research are theoretical approaches such as those presented in Part II. Cultural exchange within and between theories unfolds the diversity of theories and shapes the semiosphere’s dynamic nature through individuals as they participate in dialogical processes of meaning-making and exchange. Radford characterizes dialogue as the “door for entering the semiosphere” (Radford 2008, p. 318), but a dialogue between theories may also shape and support the development of the semiosphere. The case studies presented in Part III of this book, namely Chaps. 9, 10, 11, and 12 will give examples of such possible dialogues between theories.

This introductory chapter frames the case studies by embedding them into general methodological considerations. For this purpose, we briefly present the landscape of strategies for networking (Sect. 8.1) and discuss how Part II of the book contributes to making theoretical approaches understandable and comparable (Sect. 8.2). Section 8.3 will give an advance organizer for how the networking strategies will be applied in each case study. Section 8.4 presents a first attempt to classify the different aims and benefits of the case studies through the construct of profiles that will be refined later in Chap. 14.

By this structure, we intend to (1) make clear the meta-theoretical and methodological starting points of the case studies, and (2) give advance organizers for the case studies in Chaps. 9, 10, 11, and 12. In Part IV of this book, Chaps. 13, 14, and 15, we will reflect on what we have learnt from the case studies. This includes some refinements of the constructs offered in the present chapter.

8.1 Embedding: Landscape of Networking Strategies

By networking, we mean research practices that aim at creating a dialogue and establishing relationships between parts of theoretical approaches while respecting the identity of the different approaches (cf. Prediger et al. 2008b; Bikner-Ahsbahs and Prediger 2010; Bikner-Ahsbahs 2010).

Given this working definition, there are still many different ways and degrees to bring theoretical approaches into dialogue. For systematizing and reflecting these ways in a conceptual framework, a landscape of networking strategies has been specified that allows distinguishing between different degrees of integration (Prediger et al. 2008b). In this landscape (Fig. 8.1), the strategies *ignoring* other theories and *unifying* theories in a global way serve as the poles on a scale for the degree of integration. Whereas *ignoring* is often guided by a pure relativism concerning theories considered as arbitrary and isolated, the call for a *global unification* is led by the idea of having one unique theory (that Dreyfus 2006 compared to the grand unified theory of which many physicists dream), both being extreme positions.

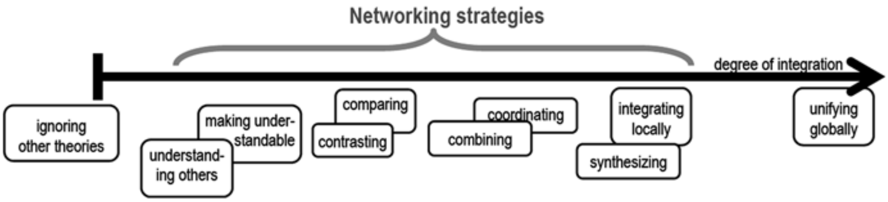

Fig. 8.1 A landscape of strategies for connecting theoretical approaches (Prediger et al. 2008b)

Based on the position that theories are not isolated but can learn from each other, the focus lies on intermediate strategies for finding connections as far as possible (but not further) which can be placed in between the two extremes on the scale in Fig. 8.1. All these intermediate strategies are called networking strategies: "networking strategies are those connecting strategies that respect on the one hand the pluralism and/or modularity of autonomous theoretical approaches but are on the other hand concerned with reducing the unconnected multiplicity of theoretical approaches in the scientific discipline" (Prediger et al. 2008b, p. 170).

In a first approximation, the networking strategies were ordered with respect to the degree of integration of the theories in question. The strategies are structured in pairs: *understanding* and *making understandable*; *comparing* and *contrasting*; *combining* and *coordinating*; and *integrating locally* and *synthesizing*:

- Every attempt to connect theoretical approaches provides the practical experience that it is not trivial to *understand* theories that have been developed in unfamiliar research practices. Hence, all inter-theoretical communication and especially all attempts to connect and apply theories and research results must start with the hard work of *understanding others* and, reciprocally, with *making the own theory understandable.* For understanding a theory, its interplay with the research practices are crucial. Understanding hence refers to all Radford's (2008, 2012) constituents: not only key constructs, but also principles, questions, methodology, and results.
- The most widely used pair of networking strategies is *comparing* and *contrasting* theoretical approaches. Comparing and contrasting only differ in degree, not in substance. Whereas comparing refers to similarities and differences in a more general way of perceiving theoretical components, contrasting is more focused on extracting typical differences. By comparing and contrasting, the specificity of theories and their possible connections and limitations can be made more visible: strong similarities are points for linking and strong differences can make the individual strengths of the theories visible.
- Whereas the strategies of comparing and contrasting are mostly used for a better understanding of typical characteristics of theories and theoretical approaches in view of further developing theories, the strategies of *combining* and *coordinating* are mostly used for a networked understanding of an empirical phenomenon or a piece of data. Following the idea of triangulation, combining and coordinating means looking at the same phenomenon from different theoretical perspectives

as a method for deepening insights into the phenomenon. The distinction between combining and coordinating is drawn according to the degree of integration of theory elements with respect to their compatibility. *Combining* theoretical approaches does not necessitate the complete compatibility of the theoretical approaches under consideration. Even theories with conflicting basic assumptions can be combined in order to get a multi-faceted insight into an empirical phenomenon in view. In contrast, we use the word *coordinating* when a conceptual framework (in the sense of Eisenhart 1991) is built by fitting together elements from different theories for making sense of an empirical phenomenon. A conceptual framework is not a new theoretical approach but a pragmatic bricolage for the purpose of understanding empirical phenomena.
- Whereas the strategies of combining and coordinating mainly aim at deeper insights into an empirical phenomenon, the strategies of *synthesizing* and *integrating locally* are focused on the development of theories by putting together a small number of theoretical approaches into a new framework. We make a gradual distinction between the two related strategies which this time refers to the degree of symmetry of the involved theoretical approaches. The notion *synthesizing* is used when two (or more) equally stable theories are connected in such a way that a new piece of theory emerges. But often, the theories' scope and degree of development is not symmetric, and there are only some constructs or aspects of one theory integrated into an already more elaborate theory or converted and elaborated into another one. This integration should not be mistaken as *unifying totally*, which is why we emphasized the "locally" in the strategy's name *integrating locally*. We call a local integration symmetric if a concept at the border of two theories is worked out and integrated into both theoretical approaches. The latter may be further developed and result in synthesizing.

Of course, the practical work of applying these strategies is more complicated than the model with its strict distinctions made for analytical reasons. Most researchers apply more than one strategy at once (as we do in Part III of this volume, see Sect. 8.4), and an exact topology cannot be given since the degree of integration always depends on the concrete realizations and networking methods. However, the landscape still serves as a useful approximation towards a conceptual framework for discussing and reflecting research practices of networking and their preconditions. It also provides a frame that can describe the development of the networking process (Bikner-Ahsbahs et al. 2010). In the long term, it may help in approaching methodological considerations for connecting theories.

Prediger et al. (2008a) tried to give an overview of many different methods that can be useful for supporting processes of networking, for example:

- cross-experimentation,
- initiate parallel processes of conceptualizing the same problem into different research problems
- convert a problem taken from one approach into a new approach
- interpret the use and role of a notion in two approaches
- parallel analysis,

- compare theories with respect to their articulation in research on the same topic with different focus and data,
- analyze the same empirical phenomena with different approaches.

This book reports on networking practices that started with the last-mentioned method. As we will see, the initial exercise of analyzing the same video led to other methods of networking, and, in this way, the initial exercise allowed a further elaboration of networking methodologies, that is, reflection on the methods, strategies, limits, and benefits (see Chaps. 13, 14, and 15).

8.2 Making Understandable and Comparing Five Theoretical Approaches

Chapters 3, 4, 5, 6, and 7 in Part II of this book can be read as the authors' attempts to make five theoretical approaches understandable. As these chapters have shown, theoretical approaches cannot be explained by their key constructs alone. Understanding a theory means to understand their articulation in research practices which comprise many implicit assumptions. The reference to the same video of Carlo, Giovanni, and the exponential function (presented in Chap. 2) facilitates making explicit some of the implicit aspects of the theoretical approaches.

The applied theoretical approach and the corresponding research practices do not only shape the conceptualization of phenomena, but also influence what counts as relevant questions, analyzable units of data, and adequate methods to answer the questions. However, it was remarkable that although the task was to analyze given (mostly alien) data, three out of the five approaches (TDS, ATD, and AiC) also referred to the *design* of learning situations and tasks, hence included constructive next to the descriptive considerations as a core element in the research and theory formation. The different priorities for designing learning arrangements seem to have shaped also the typical questions posed in the different theoretical approaches and the methodologies for answering them. This observation exemplifies the fact that the design practices are interconnected not only with the research practices but also with the theoretical approach.

The five analyses of the same video now allow a first comparison of the different theoretical approaches:

- *Size:* TDS and ATD are mature theories with a long tradition and large research communities contributing to their development; these theoretical approaches provide many complex key constructs which have evolved over time. In contrast, AiC, IDS, and APC are younger and more local theories, developed for specific purposes and applied in smaller communities.
- *Questions:* Whereas AiC mainly focuses on the learning of the students (in context), APC and IDS mainly focus on the interaction between teacher and students. In contrast, the systemic and epistemological perspective of TDS orients its questions around the functioning of the complex didactical systems and the search

for fundamental situations, and ATD on different institutional settings and their constraints. For AiC, TDS, and ATD, the research questions are deeply connected to different design practices which are typical for their scientific work.

- *Kinds and units of data:* Depending on the different typical research questions, some theories could immediately start an analysis when having only one video, while others needed more information on the curriculum background, teachers' intentions, etc. before having a suitable unit of analysis. These experiences show that "data" does not exist independently from the theoretical approach; rather, every theoretical approach shapes the kind of data constructed for conceptualizing empirical phenomena.
- *Methodical principles*: The AiC, APC, and IDS teams conduct micro-analyses of learning processes of different kinds. The AiC team executes an a priori analysis to capture the expectations of the designer with respect to the intended constructions and a posteriori analyses to learn from the data what additionally has to be taken into account. The IDS team reconstructs social interactions, epistemic processes, and value attribution and aims at aggregating data to build ideal types. The APC team focuses on the semiotic bundle and its synchronic and diachronic analyses in order to disclose multimodal relationships. The TDS team and the ATD team match different methods; for both, design plays an important theory-driven role. The TDS view encompasses epistemologically conducting a priori analyses of the a-didactical potential of the situations and a posteriori analysis with theoretical reflections including characteristics of the didactic contract. ATD considers praxeologies on different institutional levels taking constraints into account. Hence, the methods and methodologies are deeply related to conceptual and procedural tools the theories offer.
- *Objects:* Theories bring specific areas into focus and at the same time leave others aside, namely the focus on abstraction (AiC), on specifically fruitful situations in classrooms with a potential for learning with interest (IDS), on semiotic resources in classrooms (APC), on the epistemological potential of didactical situations (TDS), and on the anthropological nature of human activities in institutions (ATD). Even when using the same data sets, objects and their areas of attention reflect the diversity of theories.

These first aspects of comparison show that it is the concrete analysis of one set of videos that facilitates the access to the design (of teaching and learning arrangements) and research practices connected to the theories.

For networking these different approaches and their research practices towards a higher degree of integration, further networking strategies have been applied in four case studies. By these case studies, we intend to contribute to the overall methodological question of how to deal with the diversity of theories (Question 3 in Chap. 1), here refined to Question 3′:

> How can we network different theoretical approaches, that is, what methods, strategies, and meta-theoretical constructs are needed for creating a dialogue and establishing relationships between parts of theoretical approaches while respecting the identity of the different approaches? What can we learn from networking practices empirically, theoretically, and methodologically and where are the limits?

The Networking Group decided not to conduct only abstract discussions on these questions but treat them as empirical (second order) research questions. So we involved ourselves in four case studies of concrete research practices which were supposed to give local answers to these big questions. Among all the different attempts of networking the Networking Theories Group has experimented with (see Chap. 15), this was the most fruitful one for the group's methodological long-term aim: understanding and reflection on strands and issues of networking practices.

8.3 Outlook on the Four Case Studies for Networking

The four case studies in the following Chaps. 9, 10, 11, and 12 each use different networking strategies, each with respect to selected aspects of two or three out of the presented five theoretical approaches. We describe them briefly here, on the one hand as first concretization of the strategies presented in Sect. 8.1, on the other hand as an advance organizer for the core chapters of the book:

- In *Chap.* 9 (Case study of the epistemic role of gestures – networking between APC and AiC), the two analyses of a video scene are coordinated with respect to the epistemic role that gestures play in the epistemic process. Gestures played a prominent role in the theoretical construct of the Semiotic Bundle in APC. AiC has learned from APC how to systematically engage in gesture analyses, and has hence locally integrated one aspect of the methodology. In this way, the concept of epistemic gestures emerged. This new concept is an example where the asymmetric local integration on the methodological level of networking led to enrichments of both theories, namely by raising new questions and developing a new concept without touching the principles.
- In *Chap.* 10 (Case study of context/milieu – networking between AiC, TDS, and ATD), the networking process started from the common vision that learning and teaching processes depend on the context in which they develop. The idea of context is conceptualized differently by the three theories. A broader notion of the idea of context could be elaborated by comparing the three complex theoretical key constructs of context, milieu, and the media-milieu dialectic. This comparison of related but not equal constructs revealed a deeper theoretical understanding of the key architecture of the three theoretical approaches and the use of data served for illustration and as a base for theoretical reflection. The networking strategy of contrasting allowed the insightful showing of limits of the theoretical approaches and the nature of concepts within their theories.
- *Chapter* 11 (Case study of the epistemological gap – networking between APC and IDS) starts by comparing the analyses of the same scene in the video with seemingly contradictory results. By trying to *coordinate* the analyses and to harmonize these contradictory results, the new concept of epistemological gap emerged and was included into both theories. This chapter thus provides an example of the networking process of local integration.

- *Chapter* 12 (Case study of the Topaze effect – networking between IDS and TDS): In the first spontaneous data analysis, each group reconstructs different phenomena in the video. The case study *compares and contrasts* two of them within a cyclic networking process of analyzing separately, sharing the results, reflecting on the process, re-analyzing the data, etc. The attempts to *combine* the analysis and the results led to a deeper understanding of the episode and the theoretical constructs involved on the one hand and on the other hand to providing deepened insights into the character of the two conceptualized phenomena and the common empirical idea that the two phenomena try to capture.

8.4 Networking with Different Profiles

When the discussion on networking practices started in the CERME working groups, there immediately arose a need not only to distinguish between different networking strategies but also to distinguish the networking practices by complementary starting points and aims with respect to theoretical and empirical considerations (Bikner-Ahsbahs et al. 2010, p. 164).

The first attempt to draw this distinction resulted in specifying two dichotomic profiles: prototypically, networking practices with a bottom-up profile start from empirical data or phenomena and aim at a deeper understanding of these data or phenomena. In contrast, a top-down profile mostly starts from theoretical considerations and aims at theoretical progress (Arzarello et al. 2008). Although we briefly classify the four case studies here with respect to these prototypical profiles, the reflection on them in Chap. 14 will show that, in reality, both profiles appear in each case, only with different priorities.

The case study on gestures (Chap. 9) consequently follows empirical aims, namely understanding the role of gestures in the video of Carlo, Giovanni, and the exponential function. Two different approaches are coordinated in order to gain insights into the empirical situation.

The case study on the Topaze effect (Chap. 12) also started with the aim of deepening understanding of an empirical phenomenon. In these cases, networking also fulfills the classical purpose of triangulation of data analysis by two theoretical approaches. But in the case of the Topaze effect, the networking went beyond the empirical phenomenon and contributed to a further development of the theories: seeing through different theoretical lenses obliged the researchers to rework the concept of the Topaze effect that had been taken for granted within TDS and to rethink the boundaries of the Funnel pattern as described by Bauersfeld (1978). This development left the theories' principles unchanged. Realizing limitations in the theoretical approaches motivated a sharpening of theoretical constructs *within* these theories, not *between*.

The case study on the epistemological gap (Chap. 11) took the seemingly contradictory results of analyses of the same data as a starting point and through elaborating their understanding revealed a new concept that could be integrated into both theories. This case study shows the relevance of replication studies and the coordination of results.

In contrast to these bottom-up profiles (starting from the data), the case study on context/milieu (Chap. 10) provides an example for a top-down profile. It starts from three strong theories and compares and contrasts one of the most complex constructs of each. This comparison contributed to making the theories more explicit, especially for AiC, but it also revealed a common aspect all the three theories share.

The chapters are ordered according to their mutual dependency. However, the order with respect to complexity would have definitely placed Chap. 10 as the last one.

References

Arzarello, F., Bosch, M., Lenfant, A., & Prediger, S. (2008). Different theoretical perspectives in research from teaching problems to research problems. In D. Pitta-Pantazi, G. Phillipou, et al. (Eds.), *Proceedings of the 5th Congress of the European Society for Research in Mathematics Education (CERME 5)* (pp. 1618–1627). Cyprus: ERME.

Bauersfeld, H. (1978). Kommunikationsmuster im Mathematikunterricht. Eine Analyse am Beispiel der Handlungsverengung durch Antworterwartung. In H. Bauersfeld (Ed.), *Fallstudien und Analysen zum Mathematikunterricht* (pp. 158–170). Hannover: Schroedel.

Bikner-Ahsbahs, A. (2010). Networking of theories: Why and how? In V. Durand-Guerrier, S. Soury-Lavergne, & F. Arzarello (Eds.), *Proceedings of the sixth Congress of the European Society for Research in Mathematics Education* (pp. 6–15), Lyon, 2009. Lyon: Institut National de Recherche Pédagogique.

Bikner-Ahsbahs, A., & Prediger, S. (2010). Networking of theories – An approach for exploiting the diversity of theoretical approaches. In B. Sriraman & L. English (Eds.), *Theories of mathematics education: Seeking new frontiers* (Advances in mathematics education series, pp. 483–506). New York: Springer.

Bikner-Ahsbahs, A., Dreyfus, T., Kidron, I., Arzarello, F., Radford, L., Artigue, M., & Sabena, C. (2010). Networking of theories in mathematics education. In M. M. F. Pinto & T. F. Kawasaki (Eds.), *Proceedings of the 34th conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 145–175). Belo Horizonte: PME.

Dreyfus, T. (2006). Linking theories in mathematics education. In A. Simpson (Ed.), *Retirement as process and as concept* (pp. 77–82). Festschrift for Eddie Gray and David Tall, presented at Charles University, Prague.

Eisenhart, M. (1991). Conceptual frameworks for research. Ideas from a cultural anthropologist; implications for mathematics education researchers. In R. Underhill (Ed.), *Proceedings of the thirteenth annual meeting of psychology of mathematics education – North America*. Blacksburg: PME.

Prediger, S., Arzarello, F., Bosch, M., & Lenfant, A. (Eds.). (2008a). Comparing, combining, coordinating – Networking strategies for connecting theoretical approaches. Thematic Issue of *ZDM – The International Journal on Mathematics Education*, *40*(2), 163–327.

Prediger, S., Bikner-Ahsbahs, A., & Arzarello, F. (2008b). Networking strategies and methods for connecting theoretical approaches: First steps towards a conceptual framework. *ZDM – The International Journal on Mathematics Education, 40*(2), 165–178.

Radford, L. (2008). Connecting theories in mathematics education: Challenges and possibilities. *ZDM – The International Journal on Mathematics Education, 40*(2), 317–327.

Radford, L. (2012). *On the growth and transformation of mathematics education theories.* Paper presented at the international colloquium: The didactics of mathematics: Approaches and issues. A homage to Michèle Artigue, May 31 to June 1, 2012 http://www.laurentian.ca/educ/lradford. Accessed 7 Nov 2013.

Chapter 9
The Epistemic Role of Gestures: A Case Study on Networking of APC and AiC

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Abstract In this case study, the epistemic role of gestures is considered empirically. The analysis of gestures is included into the AiC analysis of a small excerpt of the data from Chap. 2 by means of the notion of semiotic bundle, which forms a crucial component of the APC-space. For this purpose, APC and AiC are coordinated and then locally integrated in an asymmetric way.

Keywords Networking of theories • Gestures • Epistemic

9.1 Introduction

Chapter 6 deals with Abstraction in Context (AiC) as a theoretical framework for analyzing processes of constructing abstract mathematical knowledge by the so-called RBC analysis (recognizing, building-with, constructing). In that chapter, we explained why the AiC team, when presented with the tasks and the transcript

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illustrated in Chap. 2, immediately focused on Task 3. The main reason for this was that Task 3 offered the students an opportunity to construct new (to them) knowledge about notions they had never met before. These notions were specified in the a priori analysis carried out in Chap. 6. We remind the reader that the aim of the a priori analysis is to identify and formulate the constructs which (according to the researchers' judgment) the teacher intended the students to construct. We also note that in this chapter the AiC team relates to an RBC analysis of the knowledge-constructing processes with limited attention to different parts of the context (the use of the computer, the role of the teacher), which will be attended to in Chap. 10.

The most productive situation for an RBC analysis of individual students' constructing of knowledge tends to be students working in pairs because their discussions often provide the researcher with information on their thought processes. For Carlo and Giovanni, this was not the case. When attempting to carry out an RBC analysis of the students' work on Task 3, the AiC team found that their utterances were not many and often vague, and hence the data were too sparse to analyze and difficult to interpret.

This is where the contacts between the AiC team and the APC team became important. The APC team's analyses are multimodal. This multimodality includes, in particular, a focus on the learners' gestures, in addition to their verbal utterances. The role of gestures in APC is central. The AiC team learned from the APC team and their semiotic bundle (SB) analysis (Chap. 3) how to pay attention to gestures. The question arose, whether gestures could provide some of the data the AiC team lacked in order to carry out an effective RBC analysis, and how the RBC analysis might change as a consequence of taking gestures into account. In particular, this raises the issue of the epistemic function of gestures, and more specifically, whether and in what sense gestures can contribute to the construction of knowledge.

In order to examine these issues, the AiC team used the methodological experience of the APC team in interpreting gestures and adopted some of it. This was facilitated by the fact that, in some sense, the two teams start from rather close positions: both are fundamentally interested in student cognition (and additional aspects), and both employ a micro-analytical approach to data analysis. On first sight, one might therefore ask whether networking was even an issue. Were the two teams attempting integration or were they only trying to smooth out minimal differences? Were the differences indeed minimal? In fact, while both approaches have a strong socio-cognitive tenet, and while their micro-analytic methods of data analysis may be similar in grain size, the foci of the two teams are rather different: focus and interpretation depends on the researchers' interest and theoretical frame. The APC team focuses mainly on the semiotic resources observable in the classroom while students solve problems or discuss a mathematical task; hence the focus is on what they do, produce, and on their interactions (among themselves, or between them and a teacher). The observable semiotic resources include utterances, gestures, and inscriptions (utterances, graphs, sketches, formulas). In such a sense, they scrutinize also the role gestures may play in the formation of mathematical knowledge. Hence the communicative function of gestures is extremely important to APC researchers in their SB analysis, as are phases of student–teacher interactions, as well as thinking tools (see Chap. 3).

AiC researchers, on the other hand, are primarily interested in the construction of knowledge. Hence, their focus is mainly on the learners, the teacher forming part of the context, and researchers consider gestures as relevant only insofar as they have an epistemic function in the construction of knowledge.

These differences lead to different research questions and different choices of data for analysis. What makes data interesting for AiC researchers and what makes data interesting for APC researchers? What can we say about the nature of data required, or at least desired, for analyzing gestures with SB (see Chap. 3) or with RBC (see Chap. 6)? First of all, and trivially, when the aim is to analyze gestures, only excerpts with gestures are relevant, and this already considerably restricts the choice of data. Secondly, the APC team favors excerpts in which gestures have a communicative function in the learning process; for the AiC team, on the other hand, the main criterion for the choice of excerpts with gestures is the potential for the emergence of new constructs. They are interested in the role gestures might have in the process of constructing knowledge; as pointed out in more detail below, the main function of such gestures is epistemic, and they may well be isolated in time, and made by a learner to and for him- or herself without social interaction. As a consequence, the situations of Tasks 1 and 2, which were intended as a preparatory phases for constructing knowledge, were of less interest to the AiC team, and the team focused on the situation of Task 3, that is, lines 249–379 in the transcript (see Appendix).

As a consequence of these different data requirements, and of the different foci of the teams, and in spite of the closeness, in some sense, of the two approaches, we were left with a very small intersection of data that could have formed a basis for parallel and then comparative or common analysis, as is often done in research that networks two theoretical frameworks (see, for example, Chaps. 11 and 12). Therefore, this chapter will be somewhat different from the subsequent case studies. It will relate to the two theoretical frameworks in an asymmetric way for two reasons: the dearth of data for parallel analysis as well as the asymmetrical aim of using one theoretical framework in order to enrich the analysis of the other one. In the following section, we present an attempt to integrate gesture analysis into the RBC analysis of knowledge construction in the situation of Task 3 (Sect. 9.2); this attempt to integrate gestures has been based on the SB methodology and carried out by the two teams together. We will then discuss methodological aspects of the RBC analysis that are related to this integration of gestures (Sect. 9.3), as well as theoretical consequences for AiC, and feedback to APC (Sect. 9.4); and we will end with a reflection on the process (Sect. 9.5).

The central aim of the chapter in the framework of this monograph is to show that one team (AiC) can exploit the experience of another team (APC) in order to explore how to improve its methodology, in this case how to incorporate gesture analysis according to SB into the RBC analysis, and how this team's theoretical framework can grow methodologically and theoretically in the process.

9.2 The Data and Their RBC Analysis

Our aim in this section is to demonstrate how data based on the students' gestures complement the data based on their utterances and how this combined data set allows carrying out an RBC analysis of the students' constructing actions while working on Task 3. We focus in particular on the second part of the students' work on this task (lines 302–349 in the transcript), because this is the excerpt that allows us to best demonstrate our analysis.

As described in Chap. 2 (Fig. 2.3), Task 3 had been designed by the teacher in order to give the students an opportunity to explore the exponential variation at both the local and the global level. Besides the graph of $y = a^x$, it contains the points $P(x;a^x)$, and $H(x+\Delta x;a^x)$; it was also supposed to contain the point $Q(x+\Delta x;a^{x+\Delta x})$, and the students related to this point as if it were there (see Fig. 2.4). It also contained, two sliders, whose variation allowed the students to modify, respectively, the increment Δx and the base a of the exponential. The computer screen configuration is shown in Fig. 9.1.

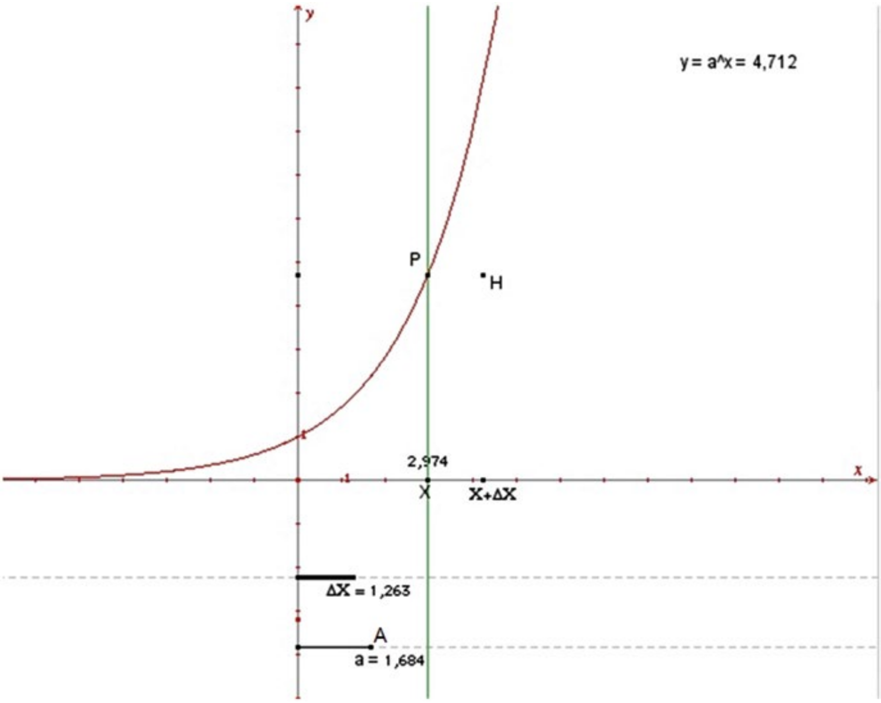

Fig. 9.1 The computer screen configuration of Task 3 (see Fig. 2.3)

9.2.1 Narrative Summary of the Students' Work on Task 3

As mentioned, we focus on the part of the activity in lines 249–379. We first give a narrative summary of what happened in this excerpt. The aim of the narrative summary is to serve as a frame of reference for the reader during the following micro-level analysis (Sects. 9.2.2 to 9.2.7). The narrative summary does not pretend to be an analysis or to be objective; we are aware that any view, even if it is descriptive, is influenced by the viewer's selections.

Episode 3a (lines 249–281). The students quickly observe that the point P and the base a (which they call "the rate of growth") can be varied; they also note that Δx can be varied, and identify Δx with PH.

Episode 3b (lines 282–324). Varying P, the students observe that HQ varies with P, and that as P moves to the left, HQ becomes small and the secant appears to become a tangent. They briefly and vaguely also comment on what happens as PQ gets small (298, 301) and mention the option of varying a, but then return to consider the effect of varying P. They also explore and comment on what happens for what they call "P near zero," by which they mean $y_P \rightarrow 0$. Now the teacher joins them, and participates in the conversation until almost the end of the lesson (until line 368). The teacher's participation is active – he not only asks questions but provides information.

Episode 3c (lines 325–343). The first issue discussed with the teacher is what happens as Δx becomes very small; while the students focus on the phenomenon that the line becomes (nearly) a tangent, the teacher keeps asking what information this provides them. Nested within this episode, the students recall, in a different formulation, that the (secant) line approximates the function better, the nearer "P is to zero" (lines 331–334).

Episode 3d (lines 344–353). Under the teacher's continued questioning and later his suggestion of the term "approximation," the students conclude that the exponential function can be approximated by a set of little tangent elements, each steeper than the preceding one.

Episode 3e (lines 354–367). The teacher guides a discussion establishing that the "growth percentage" or the ratio between a value and its successor (these are the teacher's expressions; the students repeat some of them) remains constant and that this is consistent with the growth rate being low. "The function crushes on the x-axis" (according to the teacher) when the values of the function are close to $y=0$ (for small x). The students repeat, in their own words, part of what the teacher says. The teacher leaves and the students begin to summarize what they are going to write: that the exponential function can be approximated by little straight line segments of increasing slope; that for small x, these straight-line segments are almost like a single (straight) line, and hence that "at the beginning," that is for $x \rightarrow -\infty$, the graph is similar to a line and has a constant rate of growth (366–369).

Episode 3f (lines 370–379). Finally, they turn to the question of what happens when a varies. They seem to keep P and Δx constant and observe that the area of the triangle PQH grows as a grows.

9.2.2 Lines 300–302: Behavior of the Tangent Line for Small x

In Episode 3b, the students have chosen and fixed a rather small value of Δx, and explore the behavior of the tangent line, focusing in particular on the case in which P is to the left of the origin, which they express as “P is small.” (In the transcript, underlining designates the part of an utterance during which the speaker gestured.)

300	G	we can say that if P it’s small, that is more like a tangent, it seems, if you take it much small	
301	C	a single point	
302	G	eh, it can be approximated to one line, with P very small, then instead as long as it increases…	*Gesture in 302*

A significant gesture occurs in line 302: Giovanni is positioning and slowly moving rightwards his left hand on the desk, as shown in the picture. He refers to “if P it’s small” (line 300) or “with P very small” (line 302); this can be interpreted in two almost equivalent ways as x_P being small (close to $-\infty$) or as y_P being small (close to zero). It is a moot question which one he means. It is much more important what he would like to say about P being small: “that is, it seems rather like a tangent” (line 300), and “it can be approximated to a line” (line 302). He seems to find it difficult to express what he means in words; his left hand, positioned horizontally in front of the computer on the table, is an additional means of expression. We interpret the combination of his utterances and his gesture as expressing his image of what happens when P is small. Possibly, expressing his image also helps him construct a more definite image of what happens when P is small, namely that the graph of the exponential function is similar to a straight line and therefore well approximated by its tangent. Hence the gesture is non-redundant with respect to the student’s words (in the sense of Kita 2000; see Chap. 3 for a discussion about the characterization of gestures).

During collaboration with the APC team, the AiC team learned to consider gestures such as the one made by Giovanni in line 302 in a manner similar to how they were used to look at verbal utterances: as expressing information. Here, the information expressed is that when P is small, the tangent line has an almost horizontal position. The AiC team also learned to distinguish between redundant and non-redundant gestures. The gesture in line 302 is non-redundant in the sense that it adds information beyond the one in the utterance. This additional point of view allowed us to interpret line 302 as a building-with action: Giovanni recognizes the notion of tangent as relevant for the situation he is currently dealing with and builds-with previous constructs including "tangent" and a certain possibly rather vague notion of "approximation."

9.2.3 *Lines 308–313: Behavior of the Function for Large* x

Still in Episode 3b, Giovanni summarizes his insight from a different point of view:

308	G	yes, if we move P we can see that the point, eh, sorry the HQ segment becomes smaller, it decreases…and this, the point QH, can you see?		
309	C	because P and Q have always the same distance		
310	G	yes		
311	C	ok, so ok, ok, so ok, because if it means that they increase, the more you move them over there, it increases very very much	*Gesture in 311: C quickly moves the hand upwards to the right*	
312	G	yes		
313	C	because it's an exponential function		

In line 309, Carlo joins the action. The link between line 308 and 309 is not obvious even though Giovanni (in line 310) expresses agreement with Carlo (line 309). If the students had gestured here, the researchers might have had better access to the exchange. But in this instance, we were not so fortunate. Noticing that "P and Q have always the same distance" (line 309), Carlo considers the opposite end of the x-axis (line 311); he also gestures, waving his right hand with the pen in the air in a repeated upward movement to the right. While the gesture, because of its wavy nature, is not more definite than the verbal expression, gesture and speech mutually support each other: the words indicate that there is an increase; the gesture shows that it is on the right side of the screen and becoming larger. The AiC team learned from the APC team to consider such instances as a semiotic bundle of gesture and speech. Moreover, the gesture by Carlo is anticipatory with respect to his words. This fits the "information packaging hypothesis" proposed by Kita (2000) and the "growth point" model by McNeill (2005) (see Chap. 3 for a theoretical elaboration on gestures).

Together, words and gestures give a rather clear expression to the middle part of the intended construct C_{11} (see the a priori analysis in Chap. 6 for details): "As x grows (P moves to the right), the slope of the tangent grows (for $a > 1$)." The third part of C_{11} ["As x decreases (P moves to the left), the slope of the (secants and the) tangent decreases to zero (for $a > 1$)"] is implicit already in line 308 (and earlier in line 302); hence we interpret this excerpt as evidence that the students are in the process of constructing C_{11}.

9.2.4 Line 316: Constructing the Dependence on x (C_{11})

The next significant gesture occurs briefly afterwards, in line 316:

316 G eh, ok, when the P it's very close to the 0, the line that passes for Q and H <u>represents</u> [*begins gesturing on the desk by screenshot (a)*] <u>more and more</u> [*gesture in screenshot (b)*] the function… <u>the smaller it is</u> [gesture (*c*)]

Gestures in 316 (a) *(b)* *(c)*

When Giovanni gets to "represents more and more the function" (line 316), he puts his right hand on the table, next to the computer, with the thumb and index finger touching the table and approaching each other while the hand moves to the right (see screenshots in transcript line 316); he then repeats the same movement again. Giovanni looks at Carlo's face; Carlo looks at the screen; nobody looks at the gestures. We note that the gestures, though explicit, seem to be almost automatic, expressing, together with the words, Giovanni's thinking. Our interpretation follows.

APC interpretation. Giovanni's gesture repeats many times the small back-and-forth movement of index and thumb, while softly moving his hand towards the right. Through this catchment (according to McNeill et al. 2001, a catchment is recognizable when some gesture form features are seen to recur in at least two, not necessarily consecutive, gestures), he metaphorically expresses the limiting process of QH tending to zero. In this way, he is able to pictorially add more information to his words (P is very close to zero), showing that this gesture is non-redundant as well.

AiC interpretation, as enriched by the APC interpretation. "P is very close to zero" together with the movement of the hand to the right (although it "should" move to the left, physically it is much easier to move the right hand to the right than to the left) refers to the *y*-coordinate of P only, as its *x*-coordinate moves in the direction of $-\infty$. The movement of P, expressing that x approaches $-\infty$, is expressed only by the gesture, not in words – once again, the gesture is non-redundant. Simultaneously, the thumb represents the point Q and the index finger the point H, and as the hand moves to the right the index finger and the thumb metaphorically express the limiting process that QH becomes smaller and tends toward zero. In addition, the words, though not the gesture, express that Q is almost on the *x*-axis and hence the line through H and Q represents the graph of the function ever better.

Giovanni thus gave expression to his constructing the third part of C_{11}, namely the variation of the slope as P moves on the graph to the left (see Chap. 6 for details). Taking all of this together, we claim that the students as a pair have now constructed C_{11}, at least in some vague form that relates to fixed Δx and increasing or decreasing segment HQ rather than to the slope of the tangent (which is the ratio of these two quantities). In fact, this represents a variant $C_{11'}$ of C_{11}: "As P gets closer to $y = 0$, the exponential function can be approximated by the secant line." This vague form lacks some of the aspects of the intended construct, for example the fact that the (slope of the) tangent is obtained as a limit of a sequence of (slopes of) secants – the first part of C_{10} ["For any given P, that is, locally, as Δx tends to zero, the slope of the secant tends to the slope of the tangent; the slope of the secants and the tangent are all positive (for $a > 1$)"]. According to Davydov (1972/1990), it is typical and expected that constructs start from a vague form and then progressively become more elaborate and precise. Here, while C_{11} is already quite elaborate, it is still partial since C_{10} is absent, and the full form of C_{11}, as identified in the a priori analysis, relies on C_{10}.

9.2.5 Line 331: Completing the Co-construction of C_{11}

The teacher has joined the group, and Giovanni repeats his conjecture about what happens as "P becomes small":

331 G yes, look… [*pointing at the screen*] and then we have discovered also that the nearer P is to [*Carlo's gesture (a)*] y equal to zero, the more this line approximates [*gesture (b1) on the desk*] the [*gesture (b2) in the air*] function

Gestures in 331:

(a) Carlo's gesture accompanying Giovanni's statement in 331 | *(b1) Giovanni's gesture repeating the one from 316* | *(b2) Giovanni's gesture representing decrease (to the left)*

The fact that the teacher joins them (line 324) gives the students a chance to repeat their finding about the behavior as "P becomes small," this time (line 331) in clearer words than before (line 316). Their explanation is supported by short gestures accompanying the utterance in line 331 by both students: Carlo gesturing a flat movement with his right hand (screenshot 331a), Giovanni repeating a shorter version of the gesture he had made already in line 316 (here screenshot 331b1), and then a gesture representing the function graph decreasing to the left (screenshot

331b2). The AiC team interprets the progressively more elaborate language as a sign of consolidation of the construct, in this case C_{11} and/or $C_{11'}$.

This interpretation by the AiC team, while acceptable, is incomplete. The APC team points to signs of a close collaboration between the two students. This is witnessed by the fact that the verbal statement in line 331 made by Giovanni is illustrated by Carlo's gesture in screenshot 331a: right hand suspended horizontally in front of him. This gesture is synchronous to Giovanni's words "the nearer P is to." Such synchrony within the semiotic bundle has been called an "interpersonal synchrony" (Sabena 2007). It may be interpreted as a sign of the fact that an APC-space has been built by the students through their common work in the problem-solving activity. This interpersonal synchrony shows that the constructing action has been a co-construction. The consideration of the gestures, in addition to and together with the verbal utterances, thus eliminates the vagueness of the AiC interpretation with respect to who has constructed C_{11}.

Giovanni's first gesture is a recurrent gesture, a catchment in McNeill's terms (McNeill et al. 2001). Following McNeill, the APC team interprets the catchment as a signal that this idea has been internalized by Giovanni and is recalled here in his reasoning. The AiC team interprets the recurrence as additional evidence for consolidation, supplementing the "progressively more elaborate language" criterion (Chap. 6).

9.2.6 Lines 335–343: The Limit as Δx Tends to 0

The teacher now takes the lead, giving the students little opportunity to independently construct their knowledge, and the AiC team little reason to perform an RBC analysis. The thoughts move quickly under the lead of the teacher. The teacher focuses the discussion on the transition as Δx tends to 0. The students had twice identified Δx with PH and briefly varied the slider determining this quantity in order to confirm the identification, but they had never commented on the effect of the varying Δx. Nevertheless:

335 G and moreover another thing, if the Delta x is very small…

337 G [*pointing at the screen*] the line becomes nearly a tan.., a <u>tangent</u> [*gesture*]

Gesture in 337: G holds the fingertips of the flat vertical left hand against the interior of the flat vertical right hand, while moving the right hand upward

339 G to the, to the function

340 T and so, it gives you some information about what? When the Delta x tends to become very very small, what kind of information do you get?

341	C	if the Delta x becomes small… it means that…[*looking at the screen, where Giovanni is moving something using the mouse*] the Delta x becomes small [*gesture*]	*Gesture in 341 (a): C is pointing with index and thumb (the "Delta gesture")*	
		when… when between P and Q… that is [*gesture*] the space decreases	*Gesture in 341 (b): C is moving his open hand vertically from the bottom upwards*	
342	T	oh sure, it is almost trivial, isn't it? Therefore he was saying that this line tends to become…		
343	G	[*gesture*] tangent. [*C nods*]	*Gesture in 343: C's anticipatory gesture: puts his hand in a horizontal position*	

The APC team provides the following analysis. Giovanni's gesture and words are mutually supporting each other. The gesture comes toward the end of line 337, when he says "nearly a tangent" and holds the fingertips of the flat vertical left hand against the interior of the flat vertical right hand, while moving the right hand upward. This gesture can be interpreted as showing in an iconic way how the secant becomes a tangent, and can therefore be interpreted as at least a partial construction of C_{10} by Giovanni.

During the gesture, but not before, Giovanni turns his head toward the teacher. Carlo, meanwhile, yawns (line 337) and seems uninterested. A little earlier he had asked whether he had to consider always the same distance between P and Q (line 327), and now, asked about what happens "when the Delta x tends to become very very small" (line 340), he answers in a circular way: "the Delta *x* becomes small when… when between P and Q… that is, the space decreases" (line 341). As can be observed along the whole transcript (see Appendix), Carlo has many difficulties in expressing himself thoroughly with verbal utterances. However, he often accompanies his vague words with gestures, which shed some light on his stream of thought. In line 341, he performs two gestures. The first one (screenshot a) is a sort of pinching gesture, performed with pointed index and thumb, and it indicates that the Delta x considered is small. The APC team has observed this gesture many times during the teaching experiment; it is performed both by the students and by the teacher. It has been called "Delta gesture" (Arzarello et al. 2009; Sabena 2007), since it usually appears co-timed with utterances referring to increments of the *x* or the *y* variables. It shows iconic features with respect to a segment in the Cartesian plane and it is rooted in the students' activities with the finite differences of functions in the previous year. In this episode, the two fingers appear very close to each other, since the attention, as the teacher prompts (line 340), is directed to consider the Delta x as becoming small.

The second gesture performed by Carlo in the same fragmented sentence is constituted by his open right hand moving vertically from the bottom upwards (screenshot 341b). It refers iconically to the tangent line in the right part of the screen (as the outcome of making the Delta x tending to zero).

The teacher pushes towards the idea that the secant is becoming a tangent (line 342). The two students react to the teacher's prompt in two different ways: Giovanni (in line 343) using words, and Carlo using an iconic gesture (screenshot 343), which anticipates Giovanni's words (and which is not seen by Giovanni, who is looking at the screen).

9.2.7 Lines 344–349: The Limit as Δx Tends to 0

In Episode 3d starting with line 344, the teacher asks an open question: "and then what kind of information will it give you in this case?", which has an immediate effect on Giovanni – instead of explaining to the teacher what he already knows, he is now expanding his knowledge using verbal and gestural ways of expressing his thinking (lines 345, 347, and 349):

345 G ah, one can say [*gesture (a)*]… one can say that [*so far G has kept the gesture, while looking at it silently]*.
[*gesture (b)*]
the exponential function <u>becomes</u> [*gesture (c)*]
very <u>little</u> [*gesture (d)*] lines…

Gesture in 345 (a)

Further Gestures in 345:

(b) G joins his fingers on the desk and traces a trait rightwards

(c)

(d) G moves his right hand little by little upwards

Giovanni's gestures sequence rightwards is repeated twice.

346	T	uh… it could be approximated to some small lines, which however…		
347	G	that is [*gesture (a)*], that…with increasing slopes [*gesture (b)*], that join together [*gesture (c)*] in a, that touch each other in a point [*gesture (d)*]	*Gesture in 347 (a): G's two-hands configuration*	

Further Gestures in 347:

(b) G's right hand moving upwards	*(c) G's left hand touching the right palm*	*(d) G's left index touching the right palm*
		

348	D	therefore you are imagining to approximate with many small segments		
349	G	well [*gesture (a)*],	*Gesture in 349 (a): initial phase of Giovanni's "Delta gesture"*	
		if you take it… I don't know, if you take it with a very large zoom… you can approximate it with many small lines [*gesture (b)*]	*Gesture in 349 (b): G final phase of the Delta gesture. The gesture has been kept during the whole sentence, a little larger and moved rightwards and upwards with higher slope (as before the right hand).*	

This episode has a greater degree of complexity than the previous ones because construction of new knowledge occurs during interaction between the students and the teacher, and because of the degree of complexity of the knowledge under consideration. Moreover, part of this episode (specifically the utterances in lines 344–347) was identified by both teams as central for their analysis of the learning process. Therefore, the collaboration of the two teams on this episode was more one of parallel or even common analysis than of APC ideas supporting the AiC analysis. For these reasons, our presentation intertwines the two analyses, while pointing to the origin of some of the interpretations in AiC or in APC.

Giovanni intends his gestures in line 345–349 to be seen by the teacher; this can be concluded from the orientation of his body. The gestures are part of Giovanni's discourse to the teacher; they are communicative (as opposed, for example, to his gestures in line 316).

We have already commented on Giovanni's catchments in lines 316 and 331 and have discussed their role in the consolidation of C_{11}/C_{11}'. Similarly, we have commented on the two catchments – the repetition by Giovanni of the gesture with the palm (screenshots 345a, c, d and 347a–d) and his repetition of the gesture with two pointed fingers (screenshots 345b and 349a, b) – saying that they support the consolidation process. In fact, McNeill describes catchments as a "thread of recurring gestural imagery" (McNeill 2005, p. 19): as such, they show how language and imagery can contribute to making sense of the mathematical concepts through their dialectic. Through the blending of imagistic and discursive aspects, catchments can contribute to making apparent the new concepts; namely, they have an epistemic function, because they contribute to constructing knowledge (this aspect is grasped and underlined by the teacher, in his comment to Giovanni's productions in line 348).

In Giovanni's gestures in line 345, the APC team recognizes two catchments, expressing two approximation processes: (i) on the left, namely that the function is approximated by the line $y = 0$ when x tends to $-\infty$; (ii) on the right, namely that small slices of the tangent approximate the function also in this case. They have an iconic feature, insofar as both represent some aspects of the relationship between the graph of the function and that of a line (resp. the line $y = 0$, and the tangent to the function's graph). But they have also a metaphoric aspect, which is expressed through the repetition of the gesture: the two catchments capture the limit process through the dynamic character conveyed by the repetition of the same gesture. In this sense catchments may indicate the epistemic character of gestures.

Specifically, Giovanni's second gesture in screenshot 345b repeats the gesture previously made in lines 316 and 331. The third gesture in screenshot 345c repeats the gesture previously made in line 337. This latter gesture is then elaborated in line 347, when the hands move vertically, representing the movement of the line along the exponential function (which can be observed in the DGS file).

The AiC team observes that the second gesture in screenshot 345b appears to be very similar to the earlier gestures but it occurs in a different context. This gesture seems to show a decreasing interval. Earlier (in line 316 and in line 331), the decreasing interval was QH. Here (in line 345) the decreasing interval is the interval on which the exponential function is taken to be approximately straight. In other words, the gesture is now associated with the more general meaning "tends to zero." Hence the catchment expresses much more than repetition and more than consolidation – it expresses a generalization of the context in which Giovanni sees and applies the notion of convergence to zero.

Giovanni continues gesturing: he shows the tangent line, repeating his gesture from line 337. Both in words and gestures, there is first one tangent, and then many tangent bits. Hence, there is more than catchment here: in the second repetition, the meaning of many tangent bits is added to that of tangent. Giovanni refers to "the exponential function becomes very little lines" (line 345) and at the same time holds his right hand up, moving it in a way that is clearly not smooth and conveys quite well different secant or tangent bits at different places. This is a considerable mental jump that has been made by means of the hand – from a single tangent to a sequence of tangent segments that join together to approximate the exponential function. The gestures clearly express that there is a construction of knowledge. At the same time, the language is evolving and becoming more elaborate. It is difficult to tell whether his own hand movements or the teacher's "it could be approximated to some small lines" (line 346) allowed Giovanni to express himself more clearly in line 347, adding that with increasing slope the bits join together, and in line 349 that they approximate the function. Most probably, it was a combination of both. In any case, this is another instance where Giovanni at first lacked the words to express what he saw in his mind and hence another case of significant gestures supporting his construction of knowledge. Here Giovanni constructed C* ["The exponential function can be approximated by many small lines with an increasing slope that join together"], making the transition from the previous local construct – the geometric representation of the derivative is a tangent – to a global view of a continuous, piecewise linear approximation to the exponential function by joining together many small tangent line segments whose slope increases monotonically. This concludes the AiC analysis.

Meanwhile, the APC analysis continues: an analysis of this last segment of Episode 3d (lines 345–347), in view of theoretical notions proposed by McNeill, reveals another aspect of the way gestures can contribute to the production of abstraction in context. This segment is composed of three successive components:

(a) first a gesture from the desk to the air with the right hand (screenshot 345a);
(b) then the repetition of the gesture (pinching gesture on the table) of a previous set of catchments (which referred to the graph where $x<0$, but now the attention is in the part where $x>0$) (screenshot 345b);
(c) finally, the two hands are raised in the air keeping them with extended fingers touching each other: the left hand is moved to touch the right at a certain angle (screenshot 345c); the teacher echoes Giovanni's words and prompts for further elaboration (line 346); Giovanni repeats gestures (screenshots 347a–d).

Only towards the end of the episode is Giovanni able to express in words what he has intuited and represented in the second catchment. In the language of McNeill, the gesture in screenshot 345a can be interpreted as an index of a "growth point."

According to McNeill (2005), a growth point (GP) marks the starting point for the emergence of newsworthy information prior to its full articulation. A growth point combines both imagery and linguistic components in a dialectical way: “A GP contains opposite semiotic modes of meaning capture – instantaneous, global, non hierarchical imagery with temporally sequential, segmented, and hierarchical language” (McNeill 2005, p. 18). In a growth point, the two modes are simultaneously active in the mental experience of the speaker, creating a dialectic, and, therefore, a sort of instability. The process ends when the growth point “is unpacked into an increasingly well-formed, hence increasingly stable, structure on the static dimension” (ibid., p. 18). The unpacking of the growth point provides a resolution of the dialectic; this resolution is shown by a linguistic form, often accompanied by a gesture: “Images vary materially from no apparent gesture at all to elaborate multidimensional displays; but, hypothetically, imagery is ever present. What varies is the amount of materialization” (ibid., p. 18).

An unpacking of the gesture in screenshot 345a is given by the words that accompany the gestures in the screenshots 345c and 347d where Giovanni expresses the idea that small slices of the tangents approximate the graph of the function. The index finger is touching and almost pushing on the hand (screenshot 347d): the gesture expresses in a global way both “touch” and “point” (this second meaning is anticipated in the gesture, with respect to words). The growth point marks the starting point of this refinement process: Giovanni first recalls his previous idea on what happens on the left part of the graph with the pinching gesture (screenshot 347a), then changes his focus to the right part of the graph and uses his peripersonal space (i.e., the space being immediately around the body) in the air to represent his refined ideas about the tangent. Possibly the echoing and prompting words of the teacher encourage him to finally express his intuition in words. This concludes the APC analysis.

9.3 Comments on the Analysis

In the specific case at hand, our analysis of the constructing actions focused on the role of gestures in the construction of knowledge. From the semiotic bundle methodology, the AiC team learned that modes of expression tend to be strictly linked with each other, and that the interpretation of one of them is linked to the interpretation of the others. Specifically, we have paid particular attention to utterances that are accompanied by significant gestures and to gestures that invited utterances and helped learners to formulate their thoughts. According to the AiC perspective, gestures are significant if they do more than underline the importance of the speaker’s words or point to a specific object that is intended by speech (e.g., “this”). In some cases (e.g., line 337) this function is mainly communicative; in other cases (e.g., lines 302 or 316) it includes the students’ attempts to clarify thoughts to themselves, and hence to contribute to a constructing action: this may give the gesture a crucial role in the construction of knowledge. In still other cases (e.g., line 347), gestures are communicative while contributing to a constructing action, and hence significant for the social construction of knowledge.

The role of gestures in the constructing process is a double role: on the one hand, gestures with an epistemic function support and possibly influence the constructing process by allowing the learner to realize the shape or movement or other spatial aspects of the object "under construction." Showing an aspect of the construct kinesthetically by means of a static or dynamic gesture may support the learner in mentally creating that construct. On the other hand, gestures may also raise the learner's, the teacher's, a peer's, or a researcher's awareness of the constructing process, thus obtaining a communicative function in addition to the epistemic one. Since gestures may be anticipatory while words appear only later in the constructing process, gestures may draw the attention of a teacher or peer to the learner's thinking. Similarly, gestures may help the researcher to identify and interpret the initiation, the end, and other features of the constructing process.

As a side remark, we note that our case study illustrates the methodological nature of data. The data we use here are based on recordings that were made during the relevant class period. These recordings include videotapes and students' written productions. However, the videotape is not the data, nor is its transcript; in fact, the original transcript included the words as spoken by the students but no mention of their gestures. In view of the expected analysis, we revised the transcript so as to include gestures (and we may have disregarded information that could be of interest to other researchers with other research aims). Hence, the videotape became data for us once we transcribed it with focus on verbalizations and gestures.

In order to detail the possible roles of gestures, we revisited some of the excerpts discussed above: while Giovanni gestures in line 302, Carlo seems to pay no attention to the gesture; and while Carlo gestures in line 311, he does not look at Giovanni, nor does Giovanni seem to even notice Carlo's gesture (he looks at the screen). We infer that, at this stage, the function of their gestures is to illustrate or clarify the mathematical objects and their behavior to themselves rather than to communicate to the other. This strengthens our argument that the gestures play a role in the construction of knowledge of the learner as he reflects without necessarily communicating with another person. Similarly, in line 316, we noted above that the gestures, though explicit, seem to be almost automatic, expressing, together with the words, Giovanni's thinking. Carlo did not look at the gestures and the gestures did not have a communicative function.

In line 316, some aspects of the mathematical situation are expressed by means of words and gestures, others by words only, and still others by gestures only. Neither the words alone, nor the gestures alone, would have been easy to interpret. It is in the multimodal combination that they lend us confidence that our interpretation is accurate. Hence, this is a case where gestures helped the researcher interpret the constructing process.

Repeating gestures have several functions and cases. From the AiC point of view, repeating a gesture such as Giovanni's repetition of his 316 gesture in line 331 and again in line 345 seems to indicate consolidation, which in this specific case now occurs with a communicative function that it did not have before. We surmise that gestures may also (and actually do in the present case) support the learner to more firmly establish knowledge that is still fragile. However, we are much more fascinated by another repetition, namely Giovanni's repetition in line 337 of his 316 gesture. It is important to note that the 337 context is quite different from the 316 one. In line 316, Giovanni considered the vertical segment QH as x tends to $-\infty$; in other words, his gesture appears to relate to the process by which a (positive) difference of functional values becomes ever smaller. In line 337, on the other hand, he appears to relate to positive values of x and considers the interval in which the exponential function may be approximated by a straight line segment; the underlying idea is that the smaller this interval, the better the approximation, and his gesture relates to this interval, also a positive quantity, becoming ever smaller. While the constructs identified in the a priori analysis are quite different (and some do not explicitly appear in the a priori analysis because they seemed of minor importance to the researchers), Giovanni's gesture indicates the commonality of the two contexts, in both of which a positive quantity becomes ever smaller and potentially tends to zero. This is an unexpected (by the researchers) construct that demonstrates Giovanni's process of abstraction in a neat manner. Such "generalizing catchment" suggests that this gesture attached with the idea of a very small and decreasing interval might become a "standard gesture" for Giovanni in a diversity of contexts where the consideration of small intervals whose length decreases to 0 is relevant.

9.4 Insights from Networking

In the introduction (Sect. 9.1), we explained that only a very small intersection of the available data could have formed a basis for parallel and then comparative or common analysis by the APC and the AiC teams; this small intersection was the transcript lines 344–347. This in itself may be considered as a theoretical insight: in spite of the closeness of two theoretical frameworks, in this case APC and AiC, both of which are socio-cognitive and employ a micro-analytical approach to data analysis, the differences in focus may be such that the two teams tend to concentrate their analyses on different parts of the learning process. Moreover, even in episodes where gestures have an epistemic function, and therefore both approaches have something to say, it is not clear whether the two analyses can be integrated into a coherent picture of the episode. In order to make this issue more concrete, we present a comparative analysis of the transcript lines 344–347 under consideration, based on the interpretations presented in Sect. 9.2.7.

9.4.1 Comparative Analysis of Lines 344–347

The researchers of both teams agree in their analyses that Giovanni expresses, by his words and gestures, significant new understandings in lines 345 and 347. The APC team stresses that Giovanni expresses how he sees, at that specific moment, the mathematical objects approximating the exponential function. In fact, the sequence of gestures by Giovanni tells us that he is imagining the exponential function as composed of (or “approximated by,” as the teacher specifies) many little line segments. This sequence occurs after the teacher, supervising the group-work, asks what happens when Δx becomes very small. The APC analysis takes into account the teacher’s intervention and didactic choices. In particular, we observe that what we can call the “didactic memory of the students” (in analogy to that of the teacher, studied by Brousseau and Centeno 1991) can play a role in the building of new knowledge and in linking it to the “didactical past.” More precisely, some signs emerge from the past history of the students and help them in picturing and acting on the new situation. In fact, in the previous year they had used and shared the “Delta gesture” to indicate a difference of values of a function for different values of the x’s that increased at a constant step (they were studying functions through the so-called finite difference method). This recollection that emerges from their didactic memory is used as a sign to represent the mathematical object and as a tool to enter the new situation: for example, from this sign and through its modifications Giovanni starts his reasoning about the properties of the function a^x. By considering the teacher’s didactic choices – the approach to functions made via finite increments, analysis of the function behavior through the behavior of its tangent line (whose slope is easy to compute) – we may explain Giovanni’s view of the function, and why the students thus “read” and “see” a function graph as composed of or as approximated by many little consecutive segments: they see the graphs/functions through their increments. Keeping the x-increment constant (as usually done in previous activities), it is the y-increment that expresses the increment of the function, and gives information on the slope. The almost omnipresence of the gesture with two fingers extended (the “Delta gesture”), which has been shared in the classroom since grade 9 in activities in which functions were studied using the finite differences technique, is thus related to this modality of seeing the functions.

The AiC team with its focus on the learners and their cognitive processes stresses the newness (to Giovanni) of viewing the exponential function as a sequence of little tangent lines. The two teams perceive these utterances with gestures as inserted differently in the flow of the students’ activity. The AiC team connected it to the preceding focus in line 337 on the appearance of a tangent as Δx tends to 0, and the consequent view of the little lines as bits of tangents, whereas the APC team related it to the teacher’s choices, mainly the choice of approximating a function by finite increments, in which the x-increment is first being kept constant, and later made tending to 0. For AiC, the teacher and the students’ previous experiences are considered as an important part of the context that may influence the learner’s process of construction of knowledge. Thus, while this choice imposed by the teacher (and ignored by the AiC analysis in this chapter but not in Chap. 10) can explain what the AiC team perceived as “a considerable mental jump from a single tangent to a sequence of tangent segments,” the APC team uses the connection to the previous occupation with tangents to explain the nature of the line segments as seen by the student.

Similarly, different functions of the gestures are considered by the two teams: while the AiC team focused on the epistemic function of Giovanni's gestures in their potential contribution to the elaboration of his understanding, and the support they give to his increasing power to express himself better in words, the APC team focused on the communicative function of the gestures, Giovanni using them to transmit his understandings to the teacher. Neither the difference in the way the two teams – AiC and APC – see the excerpt as inserted in the flow of the activity, nor the difference in the way the two teams see the function of the gestures, leads to contradictions. Indeed, quite the contrary: they complement each other and point to failures in each team's analysis to grasp and describe the complexity in a more comprehensive way. In this sense, the analyses have been coordinated successfully. Together, the two analyses provide far deeper insight than each one separately. We point out that the additional understandings contributed by each analysis (and hence by each theoretical framework) to the other one happened in a case where the two approaches are similar in that they both consider evolving cognitive and social aspects of the situation by means of micro-analysis of an enhanced transcript (enhanced by a description of the gestures). It is thus not surprising that the analyses are compatible, but it is surprising that they are nevertheless so different and complement each other so extensively. Hence, we were led to the question of when a gesture is meaningful for APC and when it is meaningful for AiC; an outcome of this question is the notion of epistemic gesture to be discussed in the next subsection.

9.4.2 Epistemic Gestures

In our networking process, we considered a gesture to be significant if its epistemic function is to contribute to the construction of knowledge. We call such gestures *epistemic gestures*. Examples of epistemic gestures are those in lines 302, 311, and 316. In fact, we noticed in some of these instances that while Giovanni gestures (line 302), Carlo seems to pay no attention to the gesture; and while Carlo gestures (line 311), he does not look at Giovanni, nor does Giovanni seem to even notice Carlo's gesture (he looks at the screen). We infer that, at this stage, the function of the students' gestures is to illustrate or clarify to themselves the mathematical objects and their properties rather than to communicate to one another. This strengthens our argument that these gestures play a role in the construction of knowledge. Hence, these are epistemic gestures par excellence. The gestures analyzed later (those in lines 337 and 345) occurred during a conversation with the teacher and serve, at least in part, a communicative function, as has also been observed by the APC team; in such cases, it is more difficult to decide on the epistemic function of the gesture, but we venture the claim that at least the gestures in screenshot 345d, where Giovanni says "the exponential function becomes very little lines" while moving his right hand little by little upwards, serves both a communicative and an epistemic function in that it allows Giovanni to create, in his mind's eye, the image of the approximating sequence of tangent line elements.

We conclude that epistemic gestures may, but need not necessarily, serve a communicative function; using notions from AiC, one of their characteristics is that they form, often together with verbal expressions, an inseparable part of the students' reorganization of their knowledge into a new construct. This justifies the term "epistemic gestures."

To the APC team, the meaningfulness of a gesture emerges from two sources: (i) the relationships with the other signs in the semiotic bundle (for instance, a gesture may be genetic with respect to a written sign; or it can add meaning to co-occurring words); and (ii) with respect to the evolution of mathematical meanings in the activity. An example is the Delta gesture, which in this episode is associated with the local approximation of the function by means of little tangent lines. Thus, pointing gestures may also be important for the semiotic bundle, as well as repeated gestures (catchment), which may provide hints of the learners' line of thinking.

While these criteria are formulated in different terms and stress important aspects that have been neglected by the RBC analysis, they are fundamentally consonant with the AiC characterization: source (i) has appeared in a natural way also in the AiC description above since it is usually only in combination with verbal mode that a gesture can be identified as being epistemic. Source (ii) refers, just as AiC does, to the meaning students associate with the mathematical objects they deal with, but adds depth by stressing the evolutionary aspect of these meanings more explicitly than AiC. In summary, while there is a great deal of resonance between the ways AiC and APC consider the epistemic function of gestures, the two approaches mutually enrich each other and hence the interaction between the teams allowed us to deepen the analysis. The notion of epistemic gestures is an excellent example of this.

9.5 Reflection

Our networking process was driven by a common but somewhat vague research question (Q in Radford's 2008 triplet P, M, Q), what gestures can contribute to the process of constructing knowledge. This question obtained more definite forms as the work progressed: Can gestures have an epistemic function that supports the construction of knowledge? Can gestures that have an epistemic function be characterized? Can the specific epistemic function gestures play in a particular constructing process be identified?

The two teams strove to provide some elements to answer these questions, by analyzing chosen episodes, at first each team according to their own methodology, and then by combining the analyses (in terms of the networking landscape, see Chap. 8). Since this was not successful because of the differences in research questions and in choices of data for analysis, a need for coordinating arose. For this purpose, we were looking for excerpts from the transcripts that were of high interest to both teams.

The approaches of both teams are fundamentally interested in student cognition (and additional aspects), and both employ a micro-analytical approach to data analysis. Hence it was surprising that we faced difficulties in choosing excerpts for common analysis that could then be coordinated. These difficulties can be explained by differences in the underlying principles (P) of the two approaches:

- APC is most interested in the evolution of signs in the social interaction, which includes both the teacher as an intervening subject and the didactic choices of the teacher in the classroom (i.e., the social and cultural dimension, according to a Vygotskian perspective). In other words, the teacher and the didactic trajectory are considered as part of the analyzed elements.
- AiC is most interested in the evolution of meanings of single students or small groups of students, within a socio-constructivist perspective; the didactic choices and the teacher's interventions are considered as part of the context.

As a consequence, the result of the networking process consists mainly in an exchange at a methodological level (M) that led to a local integration of the semiotic bundle tool into the AiC methodology. The networking process did not progress beyond the methodological level, and we suggest that the differences between the principles (P) account for this as well.

As a result of the networking process, we found that there are gestures with an epistemic function and that some gestures that are relevant for analyzing the construction of knowledge belong to this category of epistemic gestures. In some cases, it was necessary to include epistemic gestures into the RBC analysis as potential epistemic actions; in other cases, including epistemic gestures as potential epistemic actions enriched the RBC analysis. This led to a broadening of AiC, some aspects of which will now be discussed. We do not distinguish between methodological and theoretical aspects because there is no clear borderline between methodology and theory in AiC (Hershkowitz 2009).

The very notion of the epistemic function of gestures obtained its importance through the networking process described in this chapter. This epistemic function is perceived somewhat differently by APC and by AiC. AiC tends to consider the epistemic function of a single gesture within the process of "thinking" or constructing or even of formulating. APC tends to consider the epistemic function of a sequence of gestures in the overall flow of ideas within the social interaction. Both approaches have gained from this an added point of view. The stress of the SB analysis on the evolutionary aspect of meanings led to an important benefit for AiC: looking at the meaningfulness of a gesture with respect to the evolution of mathematical meanings in the activity stresses the evolutionary aspect which is crucial for AiC.

As compared with earlier RBC analyses, the evidence we admitted and paid attention to in the present analysis was broader since gestures were considered as potential indicators of epistemic actions. There was also a change in the questions we were asking, such as: How and why did a student use gestures instead of or in addition to words? How did this help the student to form ideas? Were gestures repeated or modified along the constructing process? What thought processes may be expressed by repeating gestures? Were gestures, like words, becoming more and more elaborate and clear? These were additional questions the AiC team asked, not questions that replaced previously asked questions. The analysis presented in Sect. 9.2 shows how the interpretations and answers to these additional questions enabled a deep analysis of the knowledge-constructing process with respect to both abstraction and consolidation.

In the analysis, gestures played an eminent role in deciding whether a constructing action occurred; repeated gestures, and in particular gestures repeated in a different context, had an especially important role expressing generalization. Gestures may have a distinct advantage over words in this respect since they may be repeated as is, whereas somewhat different words are likely to be chosen in a different context.

Repeated gestures have been interpreted as a sign for consolidation. We have discussed this in detail in the case of line 331 where the repeated gesture for a decreasing interval is accompanied by more precise language, a prime criterion for consolidation in previous research. While the language is becoming more precise, the gesture in its repetition becomes more evident. McNeill gives descriptions about catchment that are reminiscent of consolidation: the repeated gesture becomes more elaborate, more abstract with repetition (see the discussion above about catchments).

In some cases, AiC researchers encounter methodological problems in analyzing groups of students due to a dearth of information on particular students. Those examples of collaboration between the students in which the verbal statement made by one student is illustrated by the gesture of another student help the AiC researchers to better understand the interplay between the social and the cognitive dimensions. An analysis that takes into account the gestures highlights how social interaction, by means of coordination between the gestures of one student and the words of another student, enables the flow of ideas and the development of the constructing process. This comment might be especially useful in those cases in which the AiC researchers analyze the construction of knowledge of a group of learners and decide to consider the group as an entity.

We already mentioned that, according to Davydov, it is typical and expected that constructs start from a vague form and then progressively become more elaborate and precise. Indeed, the view of abstraction underlying AiC is based on Davydov's (1972/1990) ideas, according to which the process of abstraction starts from an undifferentiated and possibly vague initial notion, which is not necessarily internally and externally consistent. The development of abstraction proceeds by establishing an internal structure by means of links and results in a differentiated, structured, consistent entity. Reinforcing previously accepted fragile knowledge by means of gestures, especially when the learner lacks the words to express what he sees in his mind, is therefore consistent with the description of the genesis of abstraction as expressed by Davydov. By means of repeating the gestures the learner is able to further elaborate his previous fragile knowledge.

In spite of the asymmetry of the networking process described in this chapter, the interaction between the teams was not unidirectional. At the beginning of the networking process, the APC team had the strong conviction, based on gesture studies (e.g. by McNeill), that the gestures help the learner to think, and not only to communicate. After the interaction with the AiC team and the networking process, the APC team could refine their claim by means of the more precise definition of “epistemic gesture.” Possible links between McNeill’s theory (2005) and AiC theory can be the following: the growth point can constitute the beginning of a building process, and the catchment can be a signal of the consolidation process.

While the analysis presented in this chapter is of a single case study, it raises several questions for further research and might well lead to theoretical developments about the role of gestures for processes of abstraction in the future. Questions that arise include: Are gestures important tools of constructing knowledge in other mathematical content areas, which ones, and to what extent? Do iconic and metaphoric gestures play different roles in the constructing process? Do gestures play an especially important role in processes of abstraction that are related to generalization such as happened in this analysis?

So what case of networking does this case study present? We noticed that the data described in Chap. 2 was insufficient for the AiC team to carry out an RBC analysis. The reason for this is that the data was collected within a different theoretical framework and for a different purpose than carrying out an analysis of knowledge construction. This led the AiC team to learn about APC and to expand their view profoundly. The influence of APC on AiC led to additional methodological tools, insights, questions, and results. In terms of Radford’s (P, M, Q) triplet, the AiC team may not have established new principles, but we did ask new questions, and use new methods that led to results and insights. In terms of the networking landscape, we have been coordinating two analyses from different perspectives and then locally integrating them in an asymmetric way, leading from APC to AiC. This greatly enriched AiC but also provided insights to APC. In fact the APC group has been encouraged to study the epistemic function of gestures, which is a new idea for the group. Based on this, the group is now considering fresh aspects of gestures in mathematics (e.g. catchments and growth points, see McNeill 2005), which are considered in the literature of gestures in everyday conversation but are new for entering into the analysis of gestures in order to reveal interesting aspects of mathematical thinking.

This type of asymmetric networking may well be more easily and more broadly applicable by the wider research community than the deeper networking experiences to be presented in the following chapters, because cases where a research team is in need of additional theoretical ideas or methodological tools in order to understand phenomena are frequent. Our case study elucidates what might happen in such a case.

References

Arzarello, F., Paola, D., Robutti, O., & Sabena, C. (2009). Gestures as semiotic resources in the mathematics classroom. *Educational Studies in Mathematics, 70*(2), 97–109.

Brousseau, G., & Centeno, J. (1991). Rôle de la mémoire didactique de l’enseignant. *Recherches en didactiques des mathématiques, 11*, 167–210.

Davydov, V. V. (1990). *Types of generalisation in instruction: Logical and psychological problems in the structuring of school curricula* (Soviet studies in mathematics education, Vol. 2; J. Kilpatrick, Ed., J. Teller, Trans.). Reston: National Council of Teachers of Mathematics. (Original work published 1972.)

Hershkowitz, R. (2009). Contour lines between a model as a theoretical framework and the same model as methodological tool. In B. B. Schwarz, T. Dreyfus & R. Hershkowitz (Eds.), *Transformation of knowledge through classroom interaction* (pp. 273–280). London: Routledge.

Kita, S. (2000). How representational gestures help speaking. In D. McNeill (Ed.), *Language and gesture: Window into thought and action* (pp. 162–185). Cambridge: Cambridge University Press.

McNeill, D. (2005). *Gesture and thought*. Chicago: University of Chicago Press.

McNeill, D., Quek, F., McCullough, K.-E., Duncan, S., Furuyama, N., Bryll, R., Ma, X.-F., & Ansari, R. (2001). Catchments, prosody and discourse. *Gesture, 1*(1), 9–33.

Radford, L. (2008). Connecting theories in mathematics education: Challenges and possibilities. *ZDM – The International Journal on Mathematics Education, 40*(2), 317–327.

Sabena, C. (2007). *Body and signs: A multimodal semiotic approach to teaching-learning processes in early calculus*. Dissertation, Università degli Studi di Torino.

Chapter 10
Context, Milieu, and Media-Milieus Dialectic: A Case Study on Networking of AiC, TDS, and ATD

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Abstract The case study of context, milieu, and media-milieu dialectic is a networking case that links TDS, AiC, and ATD. It compares and contrasts three concepts and their status within each theory, in order to learn how concepts which at a first glance seem to have a similar role in the understanding of teaching and learning in each theory differ.

Keywords Networking of theories • Context • Milieu • Media-milieu

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10.1 Introduction

In this chapter we analyze a case study of networking between TDS, ATD, and AiC. As observed in the previous chapters, the foci of the three theoretical approaches are different. In particular, AiC focuses on the learner and his or her cognitive development, while TDS and ATD focus on didactical systems. The three theoretical approaches are sensitive to issues of context but, due to these differences in focus, context is not theorized and treated in the same way. In the next sections, we explain how context is theorized in each of the three theoretical approaches and show some consequences for the analysis of the video episode. We might expect some complexity in the effort of creating a dialogue between the three theories in relation to constructs such as context, milieu, and media-milieus dialectic. However, this case study has its own characteristics. We will observe how the dialogue between the three theories appears as a progressive enlargement of the focus, showing the complementarity of the approaches and the reciprocal enrichment, without losing what is specific for each one.

In Sect. 10.2, we explain the notions of context, milieu, and media-milieus dialectic. Section 10.3 offers a first classification of similarities and differences between the three theories. Separate analyses are presented in Sect. 10.4. A dialogue between AiC, TDS, and ATD with regard to "context," "milieu," and "media-milieus dialectic" is described in Sect. 10.5 and concluding remarks are discussed in the last section of the chapter.

10.2 The Notions of Context, Milieu, and Media-Milieus Dialectic

In this section we explain the meanings of the terms context (for AiC), milieu (for TDS) and media-milieus dialectic (for ATD), each of them being a cornerstone for the theory.

10.2.1 What Is "Context" for AiC?

As explained by Dreyfus and Kidron in Chap. 6, the nested epistemic actions model for abstraction in context is apt for describing processes of abstraction in their specific context. The contextual factors that may influence a process of abstraction include the physical setting, the tasks on which learners work, and the tools (such as paper and pencil or computers and software) that are available to them. They also include students' personal histories and previous constructions of knowledge.

Furthermore, any process of abstraction takes place in a particular social setting and thus the context also includes social relationships among students and between students and teachers. As a consequence, context becomes an inseparable component of the process of constructing knowledge because students act in a manner that seems appropriate and relevant to them in the given context. The role of context is crucial in learning processes and the complexity of learning processes goes back, at least in part, to the contextual influences on the learner's construction of knowledge. Hence, we believe that a better understanding of the role of context is likely to lead to a better understanding of learning processes. Some parts of the context have a *dynamic nature*: the learner interacts with the context. This may be the case for social interactions or interaction with the computer. The influence of contextual factors on the process of construction of knowledge is an object of analysis with the AiC lenses, especially the influence of context on the epistemic actions (see Chap. 6). For example, the relations between the learner and the computer as a dynamic partner were analyzed in Kidron and Dreyfus (2010). The study describes how the integration of knowledge structures was facilitated by the potential offered by the computer and the learner's ability to make sense of the resources offered by the computer.

The influence of social interactions on processes of abstraction has already been analyzed by Dreyfus et al. (2001). The authors have considered processes of abstraction in pairs of collaborating peers and investigated the distribution of the process of abstraction in the context of peer interaction. This was done by carrying out two parallel analyses of the protocols of the work of the student pairs, an analysis of the epistemic actions of abstraction, as well as an analysis of the peer interaction. The parallel analyses led to the identification of types of social interaction that support processes of abstraction.

10.2.2 What Is "Milieu" for TDS? How Is It Related to A-Didactical and Didactical Situations?

As explained in Chap. 4, the notion of *milieu* is attached to the vision of learning as an adaptation process and to the ambition of optimizing such a process. The milieu is defined as the system separate from any didactical intentionality with which the students interact in the *a-didactical situation.* In line with the idea of learning through adaptation, it should be a source of contradictions, or at least disequilibria, what is captured through the idea of *antagonist milieu*. However, the possibilities of action and feedback it offers should also make possible an evolution towards winning strategies, which lead to the construction of new knowledge. The milieu includes material and symbolic artifacts, and possibly other learners, depending on the social organization of the situation. Note that for interacting with the milieu, learners always need to mobilize some of their already constructed knowledge. Some but not all authors include this knowledge into the milieu.

One essential role of the teacher is to organize this milieu, but in TDS she is not considered herself as a component of the milieu. Organizing the milieu can mean: selecting the problems the learners will have to solve and fixing the values of their *didactical variables*, the way these problems are introduced and managed, the tools and means at students' disposal, and the social organization of the classroom.

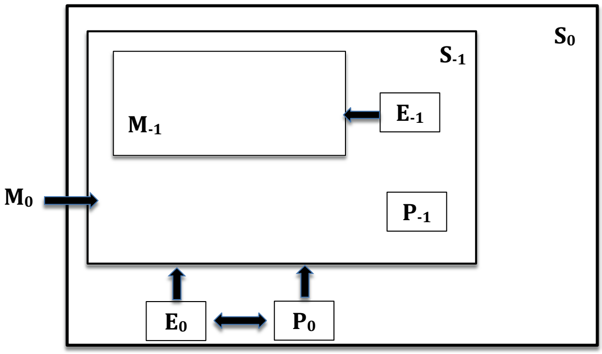

Fig. 10.1 Simplified schema of nested milieus

Milieu is a dynamic object. As long as students' interaction with it develops, new constructions emerge, new representations are built, and the milieu progressively enriches. Quite often, in classrooms, the teacher too contributes to this evolution of the milieu (called semiogenesis) by providing additional information and tools. The reason for this may be that the students do not mobilize previous knowledge as they are supposed to do, that they need some help in interpreting and benefiting from the interactions with the milieu, or that classroom constraints create the necessity of accelerating the dynamics of the situation.

The construct of milieu has been regularly discussed and reworked since its introduction in the 1980s (see for instance Dorier et al. 2002; Perrin-Glorian 1999; Brousseau 1990, 1997; Margolinas 1995), and one essential development has been its vertical and nested structuring. The networking process obliged us to enter into this vertical structure, in fact its negative levels, and more particularly into the levels S_0 and S_{-1} corresponding respectively to the didactical and a-didactical situations (see Fig. 10.1).

We briefly introduce this part of the structure. The simplified definition that we gave of the milieu above corresponds in fact to the level S_{-1}, and to the a-didactical milieu. In the a-didactical situation S_{-1}, students (E_{-1}) are modelled as learners interacting with the a-didactical milieu (M_{-1}), and the teacher (P_{-1}) is outside the system, in the position of an observer. In the nested structure, this a-didactical situation is itself the milieu of the didactical situation S_0. In S_0, the didactical intentionality re-emerges, the teacher acts as a teacher (P_0) and the student as a student (E_0), their interaction being regulated by the rules, mainly implicit, of the didactical contract. S_0 is the level of the structure where the knowledge developed through a-didactical interaction with the milieu is made explicit, partially decontextualized, and connected to the institutional forms of knowledge aimed at. As proved by many studies, this structure is very helpful for understanding the complex relationships between a-didactical and didactical processes in teaching and learning, resulting from the fact that learning is both an adaptation and an acculturation process as pointed out in Chap. 4. In fact, the nested structure of the milieu includes more levels, both positive and negative, the situation at level n being systematically the milieu for the situation at level $n+1$. For more details, see Perrin-Glorian (1999) and Margolinas (1995).

10.2.3 What Are "Media," "Milieu," and "Media-Milieus Dialectic" for ATD?

In the Anthropological Theory of the Didactic, teaching and learning processes or, more generally, processes of study and inquiry, are described using a very broad frame, the Herbartian formula (Chevallard 2008, 2012), named after Johann Friedrich Herbart (1776–1841), a German philosopher and the founder of pedagogy as an academic discipline. The starting point of the process requires a question Q (not be mixed up with the research questions of a theoretical approach as used in Chaps. 4, 5, 6, and 7), a group of persons X with the project to study question Q, and a team of "study aides" Y which can be eventually empty. This induces the formation of a didactic system around question Q: $S(X; Y; Q)$ the functioning of which must lead to the production of an answer $A^♥$ (where the heart ♥ indicates that it is the answer given by X and Y; see Chap. 5) to question Q, a process represented as:

$$S(X; Y; Q) \hookrightarrow A^♥$$

This is the *reduced* Herbartian schema. To produce $A^♥$, however, $S(X; Y; Q)$ needs "materials"; these materials make up the *didactic milieu* M established by $S(X; Y; Q)$ and represented as follows in the *semi-developed* Herbartian schema:

$$[S\,(X; Y; Q) \rightarrow M] \hookrightarrow A^♥$$

In the didactic milieu M, it is customary to distinguish two main categories: on the one hand, M accommodates already existing and "labelled" *answers* $A^◊$ drawn from available "resources" (including members of $X \cup Y$); and on the other hand, it may contain *other works* O which, among other things, can be theories, experiments, questions, brought into the milieu M by members of $X \cup Y$. The didactic milieu is represented generically as $\left[M = \left\{A_1^◊, A_2^◊, \ldots, A_k^◊, O_{k+1}, \ldots, O_m\right\}\right]$; hence the *developed* Herbartian schema:

$$[S(X; Y; Q) \rightarrow \{A_1^◊, A_2^◊, \ldots, A_k^◊, O_{k+1}, \ldots, O_m\}] \hookrightarrow A^♥.$$

Sometimes, the didactic system does not seem to be formed around a question Q but around a given praxeology P (usually designated as a "content" or a "piece of knowledge") that students X have to "learn" or "appropriate": it is what was called a *didactic stake* in Chap. 5. However, even if praxeologies can be considered on their own and in a decontextualized way, they always originally appear as the result of the inquiry of some questions arising in institutional settings, which give praxeologies their rationale or raisons d'être. We can thus consider that the didactic system is formed around these questions, even if they are initially unseen by X (and even by Y), or around the questions: What is P? What is it for? How to use it? Etc.

The *didactic milieu* of the Herbartian formula can include an *a-didactic milieu* in the sense given by TDS, that is, a system of objects acting as a fragment of "nature" for Q, able to produce objective feedback about its possible answers without any didactic intention towards X. According to Brousseau (1997), there is no construction of knowledge without such an a-didactic milieu; there can only be *imitation*, that is, the reproduction of somebody else's answer. As in TDS, and even if the a-didactic milieu is usually given or produced by the teacher Y, this is not necessarily the case: the production and organization of an appropriate a-didactic milieu for Q is an essential aspect of the study process carried out by both X and Y. For instance, in scientific work (where X is a team of researchers and Y the leader(s) or supervisor(s) of the investigations), finding or creating an appropriate experimental milieu for the study of a given phenomenon can be one of the most challenging issues to tackle.

Related to the notion of milieu introduced by TDS, the main development brought by ATD is the following. Usually, in the construction of answer $A^{\heartsuit}$, using a set of objects without any didactic intention $\{O_j\}$ is not enough; it is also necessary to use other answers $A^{\diamond}$(also called "cultural works") produced outside the didactic system as the results of other study processes carried out in different institutions to propose answers to different questions Q' more or less related to Q. In these cases, X and Y need to access these already-produced answers and they will do that through some *media*. In a larger sense, media refers to any means addressed to a certain type of audience presenting information about the world or a part of it: any media production in the usual sense (journal, paper, video, etc.); an essay or treatise; a lecture; an informal report or just a system of rumors; etc. Use of media can be considered as carrying out a *didactic intention* towards a given issue or question. To answer a question Q, one of the first things that can be done is to look around, in the media available, for possible existing answers to Q. The aim of the media is to present knowledge or information to others.

The *media-milieus dialectic* appears when considering the different kinds of general *didactic gestures* performed by X and Y in the interaction with M to produce $A^{\heartsuit}$. Both elements $A_j^{\diamond}$ and O_k intervene in the media-milieus dialectic, which can be initially presented as follows. When starting the study of a question Q, a basic action is to look for already available answers $A_j^{\diamond}$ either to Q or to a question Q' that seems related to Q. These answers are produced in different institutions which identified them by a "label" (this is the reason for the upper index $\diamondsuit$) and which are the last respondents of their validity. The answers are made available through different kinds of media: books, treatises, articles in journals or encyclopedias, videos, online resources, etc. However, $A_j^{\diamond}$ are still not the definitive answer $A^{\heartsuit}$ to Q. For the moment, they can only play the role of "conjectures" or "elements of answer" to Q, needing to be patched together and validated in relation to Q. This is the role of the other objects O_k. The media-milieus dialectic corresponds to this continuous interaction between available (partial) answers given by the media and their testing through the interaction with an a-didactic milieu. Of course, already available answers $A_1^\lozenge \ldots A_{j-1}^\lozenge$ are also part of this milieu once they have been tested and can be used as a fragment of nature for X and Y. It may be said that the media-milieus dialectic consists in contrasting previously available answers from the media to convert them into an experimental milieu (as "sure knowledge") and to work with the objects of the milieu so as to get new information from them (new knowledge to be tested), that is, to convert them into media. As we can see, the notions of "media" and "milieu" do not refer to a property of objects but to their use in the process of study of a given question Q. For example, a computer program can be used as a medium to get information about a given issue, or as a milieu to test a conjecture obtained by other means. When a person P asks another person Z a question, P can be using Z as a medium to obtain new information about the issue asked, or as a milieu, for instance to check that Z already knows the answer.

When looking at traditional teaching systems, the dialectic of the media and the milieus is very weakly balanced. There exist some activities where students overuse a small number of media (the teacher and the textbook, for instance) without feeling the need to test the validity of the information they get from them; while other activities (like a session of practical exercises, for instance) seem not to allow access to any extra media (other students' or other people's answers). This is a situation very different from specialist work, when any partial answer given is acutely searched and also continuously checked by as many means as possible.

10.3 Similarities and Differences: A First Classification

Definitively, "context," "media-milieus," and "milieu" do not mean the same thing in the approaches considered. A look at the AiC components of context shows that a computer program, a teacher, the web, as well as peers can all act as either media or milieus. Furthermore, there is an important difference in the way the questions concerning the contextual influences are formulated in the different theoretical approaches. The different ways in which the three theoretical approaches take the interactions with the context/milieu into consideration are mirrored by the different questions asked by the researchers. For example, in relation to the role of the teacher, TDS researchers might ask what milieu the teacher is making available to the students and how she is managing its evolution in order to establish a meaningful connection with the mathematical knowledge aimed at. AiC researchers might ask how the teacher's intervention influences the students' construction process as described by means of the RBC epistemic actions. ATD researchers in their turn might ask what responsibilities the teacher and the students are assuming in the media-milieus dialectics and what conditions enable them to manage it.

The three theories share some similarities in the central role assigned to the construction of mathematical knowledge in the analysis. For example, TDS and ATD look for conditions for a study process not to be reduced to a simple copy of previously elaborated answers. Also in AiC, abstraction is defined as an activity of vertically reorganizing previous mathematical constructs within mathematics and by mathematical means so as to lead to a construct that is new to the learner (see Chap. 6). Nevertheless, the research questions posed in the three theories are different: AiC researchers are interested in finding out how a given path of thinking works, while TDS and ATD researchers will rather ask: what produces the “path” (the answer given by the students to the question Q), what makes it possible to happen, is it something from the milieu, allowing the students to work autonomously (in an a-didactic situation) or, on the contrary, is it something coming from the teacher or from some other media proposed by her? The three theories will also consider the possible paths that, even if virtually possible, do not really happen, and ask about the reasons for that. Nevertheless, there is a crucial difference: for AiC the emphasis is on the way students develop their answer, strategy, or “thinking”; for TDS and ATD it is on the conditions making this development possible and the restrictions hindering other possible answers or strategies.

10.4 Separate Analyses

As a first step, the networking efforts start by analyzing the episodes in line with the way each theory views the role of context: for AiC, the focus is on the influence of the context on the learners’ process of constructing knowledge; for ATD the focus is on the media-milieus dialectic; while TDS researchers are interested on the potential and limitation of the milieu.

10.4.1 AiC: The Contextual Influences on the Construction of Knowledge

For the purpose of illustrating the contextual influences on the construction of knowledge, the AiC team decided to analyze the episode in which Carlo and Giovanni treat Task 3, investigating how the slope of the line tangent to the graph of the function $x \to a^x$ at the point of abscissa x changes with x. They focus on lines 249–379 of the transcript (see Appendix). The reason for this is discussed in the chapter on gestures (Chap. 9), namely, that this is the most complex situation and that one may expect constructing actions here as opposed to situations on Task 1 and 2, where mostly descriptive expressions of how the quantities under discussion behave might be expected. The worksheet which includes the task given to the students is described in Chap. 2 and a narrative summary of the students’ work on Task 3 is described in Chap. 9 on gestures.

The first part of the AiC analysis consists of an a priori analysis of the learning situation. The aim of the a priori analysis is to identify the constructs that the students are expected to construct, while considering the task and the context. This a priori analysis will be followed by an a posteriori analysis of the learning event, using the RBC-model.

10.4.1.1 A Priori Analysis

The TDS researchers have proposed in Chap. 4 an a priori analysis in terms of constructs for the episodes on Task 1 and 2. We would probably largely accept their analysis into nine constructs C_1–C_9; however, they have not done a similar analysis for the episode on Task 3. As already mentioned in Chap. 6 on Abstraction in Context, the AiC team proposed the following intended constructs as being those intended by the designer/teacher to be constructed. We repeat them here for the convenience of the readers:

C_{10} For any given P, that is, locally, as Δx tends to zero, the slope of the secant tends to the slope of the tangent; the slope of the secants and the tangent are all positive (for $a > 1$).

C_{11} As P moves on the graph, the slopes of the corresponding secants (and hence the slope of the tangent) vary. As x grows (P moves to the right), the slope of the tangent grows (for $a > 1$). As x decreases (P moves to the left), the slope of the (secants and the) tangent decreases to zero (for $a > 1$).

C_{12} As a increases, the slope of the secant (for given x, P) increases (and consequently the slope of the tangent increases as well). As a decreases towards 1, the slope of the secant decreases towards 0. As a becomes smaller than 1, the slope of the secant (and consequently of the tangent) becomes negative; the function is decreasing rather than increasing. The parts of C_{10} that depend on $a > 1$ have to be adapted for $a < 1$.

While these intended constructs have been formulated on the basis of the tasks given to the students, they are compatible with the declarations of the teacher and they are also within reach of the students, given the previous knowledge of the class and the socio-mathematical norms that are characteristic of the class.

10.4.1.2 A Posteriori Analysis

In the a posteriori analysis the AiC team gives an account of constructions by the students, Carlo and Giovanni. As mentioned in Chap. 6 on AiC and in Chap. 9 on gestures, in addition to the (partial) constructing of C_{10}, C_{11}, and C_{12}, we expected the students to develop other constructs. We observed the following, which might be considered as an enlargement of the intended constructs observed in the a priori analysis:

C_{11}' As P gets closer to $y = 0$, the function can be approximated by the secant line.

C^* The exponential function can be approximated by many small lines with an increasing slope that join together.

C^* is a transition from a local to a global view.

The AiC analysis in the previous chapters on AiC and gestures did not explicitly focus on the different components of the context and their role in the process of constructing knowledge. This is what we propose to do in the following. We discuss the students' constructing actions, focusing on *the role of different contextual* *factors in the construction of knowledge*: (1) the task, (2) the learners' personal history, (3) the computer software, (4) the teacher, and (5) the teachers' learning goals.

A first contextual element is *the task* itself and a first question is: why can the task make sense for the students? The AiC researchers were missing for their analysis a more specific task design. In Task 3, the task design does not require the students to answer specific questions but encourages them to explore, and to report on the exploration. On the other hand, especially in Task 3 the students appear to act with mathematical purpose; they vary quantities in a way that allows them to learn and construct new knowledge.

Secondly, the task also interacts with the *learners' personal history* and this part of the context is related to socio-mathematical norms. Giovanni and Carlo are used to exploring mathematical objects and situations, even when they are not given any specific tasks to carry out. In our experience, most students find it difficult to explore and even students who are used to exploring, and whose teachers have stressed exploration activities, might be lost in the situation as given by the teacher because there are only very general instructions; there are no tasks with clear goals, no questions. Trying to answer the question of how the task makes sense for the students, our first thinking was that this kind of situation in which students are asked to "explore and describe what is happening" could be "normal" when working with the Dynamic Geometry Software, at least in this classroom. Indeed, Giovanni and Carlo knew the software. Besides, they have earlier worked on the concept of function, as regards the numerical, graphical, and symbolic aspects. In other words, the reason why the task can make sense for the students is that appropriate conditions have been previously instituted.

One of these appropriate conditions concerns *the computer software* as the third contextual element. Interacting with the computer, the students vary quantities in a way that allows them to construct new knowledge. It enables them to use multiple graphical representations: specifically, this enabled them to carry out the transition from a static graphical view to a more dynamic graphical view. This transition is expressed in the following utterances:

287	C	look it… slowly… slowly it seems that… I do not know, like, saying, tangent
288	G	eh… yes
289	C	it seems that it touches it, let's go, let's go, let's go
290	G	eh yes… here
291	C	slowly… slowly
292	G	it's tangent

The students vary Δx and as a consequence of their interaction with the computer, a dynamic view of the secants turned out to be the emergent tangent. This is expressed in line 287. At this first stage, the students do not refer to the notion of slope, but rather to the geometrical objects that they can see, that is, the secants which become tangent. However, their view might be connected to the following part of C_{10}:

C_{10} For any given P, that is, locally, as Δx tends to zero, the slope of the secant tends to the slope of the tangent […]

We might define this construction as "the geometric representation of the derivative as a tangent." We see it as a local view and we will refer later to the transition from this local view to a more global view. In line 293, we observe the potential offered by the computer allowing the learners to check a new idea:

293 C if instead you make the contrary, increasing, increasing the differences

In line 316 the interaction with the computer facilitates the construction of

C_{11}' As P gets closer to $y=0$, the function can be approximated by the secant line.

316 G Eh, ok, when the P it's very close to the 0, the line that passes through Q and H represents more and more the function… the smaller it is [moves the right hand on the table, first showing something like a decreasing interval and then circling]

Construct C_{11}' is still fragile but continuing the work and interacting with the computer enables the students to consolidate fragile knowledge by means of checking.

By interacting with the computer, a previous fragile construct is consolidated. This was the case concerning the process of construction of C_{11}' as quoted by one of the students, Giovanni: "…if the point P is very near to zero, this line approximates very much the exponential function" (in line 368). The following utterances demonstrate how the language becomes progressively more and more precise and this fact shows the consolidation of C_{11}':

331 G the nearer P is to y equal to zero, the more this line approximates the function
368.2 G Then, if the point P is very near to zero, this line approximates very much the exponential function. Also here even if numbers are very small, it increases not so much, hence like a line…

It might be of interest to note that the consolidation of C_{11}' has been discussed in the gestures chapter (Chap. 9) in terms of catchment. On the other hand, an analysis is never complete; specifically, the AiC analysis in Chap. 9 ignored, to a large extent, the influence of context. In the present discussion, we refer to the influence of the computer context on the consolidation process.

In line 349, we observe the construction of C* – a transition from the previous local view observed in lines 287–289 ("the geometric representation of the derivative as a tangent") to a global view: the exponential function can be approximated by many small lines which have an increasing slope. This is expressed in the following utterances:

345 G ah, one can say… one can say that the exponential function becomes very little lines…
346 T uh… it could be approximated to some small lines, which however…
347 G that is with increasing slopes that join together in a…, that touch each other in a point
348 T therefore you are imagining to approximate with many small segments
349 G well, if you take it… I don't know, if you take it with a very large zoom… you can approximate it with many small lines

Since the teacher was involved in this part of the transcript, we will analyze *the role of the teacher* as the fourth contextual element influencing the students' constructing process. AiC analyzes the role of the teacher as a part of the context in order to identify how the RBC actions might have been influenced by the role of the teacher. AiC researchers would see semiotic games as an integral part of the teacher's actions and the teacher's actions as an integral part of the context. AiC researchers ask how the teacher's intervention influences the students' construction and consolidation process as described by means of the RBC epistemic actions. RBC lenses reveal the influence of the teacher on the students' construction of knowledge.

The interview with the teacher supports this view. In the interview, the teacher was asked how he decides to get involved with a pair of students:

> I enter in a working group if the students call me. Sometimes I enter in a working group if I realize that students are stuck. Other times I enter because I realize that students are working very well and they have very good ideas that need to be treated more deeply. Obviously the type of things that I do vary with the situations, but a constant is that I try to work in a zone of proximal development. The analysis of video and the attention we paid to gestures made me aware of the so-called "semiotic game" that consists in using the same gestures as students but accompanying them with more specific and precise language compared with the language used by students. The semiotic game, if it is used with awareness, may be a very good tool to introduce students to institutional knowledge. (Answer to question 4, Sect. 2.2.2)

In order to better analyze the role of the teacher as a contextual influence, we will take into account, as the fifth contextual element, *the teacher's learning goals* which we consider as a part of the context as well.

Knowing the intended constructs makes a difference in the analysis. The interview with the teacher shed new light on the contextual influence on the different modes of thinking. It offered a kind of a priori analysis exposing the expectations of the teacher at the different stages of the teaching learning experience. The teacher (Chap. 2) explained the "raison d'être" of the entire project:

> Engaging students in knowledge building, settlement, reorganizing and communicating, thus providing the teacher tools for obtaining information not only on the products, but also on the cognitive processes, necessary for any serious evaluation escaping the chimera [i.e. wrong idea] of objectivity. (Answer to question 17, Sect. 2.2.2)

This view is appropriate to the RBC+C lenses which focus on process aspects of construction of the knowledge constructs rather than on outcomes. The teacher focuses on the cognitive processes of the learner and this is appropriate to RBC analysis which focuses on the learner. Nevertheless, we will point to some differences between the teacher's expectation of the use of the semiotic game and the RBC analysis of the cognitive process of construction of knowledge.

We mentioned earlier that, in line 349, we observe the construction of C*: a transition from the previous local view observed in lines 287–289 ("the geometric representation of the derivative as a tangent") to a global view: the exponential function can be approximated by many small lines which have an increasing slope. This transition is mentioned in the teacher's answer to question 9 where he was asked how he planned the lesson:

> …the third worksheet gives a local and a global approach to the exponential function thanks to the construction of the derivative of an exponential function. (Answer to question 9, Sect. 2.2.2)

The teacher's expectation of the use of semiotic games is directed towards a future opportunity for the students to better understand the formal view:

> When I tell about the formal aspects of the derivative I often make some reference to these experiences and activities. It seems to me that also a lot of students are able to make these connections to give meaning to formal aspects. (Answer to question 13, Sect. 2.2.2)

The previous local view observed in lines 287–289 ("the geometric representation of the derivative as a tangent") corresponds to what the teacher calls LOCAL – a little segment that approximates the function locally.

Let us consider the GLOBAL view which is expressed in C* – a transition from a local view to a global view: the exponential function can be approximated by many small lines which have an increasing slope. This view as expressed in the RBC analysis is different from the teacher's view. For AiC, GLOBAL was the global approach to the exponential function – the envelope – but for the teacher LOCAL–GLOBAL is the transition from the local to the global aspects of the derivative. More precisely, for the teacher, the global is the function derivative compared with the derivative at a point as a local approximation. The transition from the local to the global aspects of the derivative will be reached by recognizing the characteristics of the derivative function (itself an exponential). The teacher expressed his aim:

> …to pay attention to the slope of the little segments, because their slope gives information on the growth of the function. (Answer to question 5, Sect. 2.2.2)

The AiC researchers observed the students' construction of the view: an exponential function can be approximated by a sequence of tangent line elements… "well, if you take it… with a very large zoom… you can approximate it with many small lines…" as expressed by Giovanni in line 349. The AiC researchers observed a global view of an envelope in addition to an idea of how the slope increases. There is no expression by the students of the expected construction by the teacher that the slope has an exponential growth. In the following the teacher expressed this expected construction:

> My aim is to induce the students to reflect on the fact that it is important to pay attention to the slope of the little segments, because their slope gives information on the growth of the function. Giovanni says "it is twice the previous slope…" I, using his same gesture, say more precisely that "the slope has an exponential growth." (Answer to question 5, Sect. 2.2.2)

This view was expressed by the teacher in his intervention, as observed in the transcript, BUT not by the two students. The AiC researchers did not find any indication of such a construction by the students.

10.4.2 TDS Analysis of the Potential and Limitation of the Milieu

10.4.2.1 A Priori Analysis

The a priori analysis of the potential of the a-didactical milieu for this episode on Task 3 is coherent with the a priori analysis piloted by AiC: the interaction with the software file guided by the text defining the task can reasonably lead to the conjectures mentioned in this a priori analysis. We complement it below by some elements which explain why, in our a posteriori analysis of this situation, we will focus on the part during which the teacher directly interacts with Carlo and Giovanni.

It is indeed interesting to note that, compared with the Tasks 1 and 2 guiding the work with the two first files, the text of Task 3 is longer and provides substantial information. For instance, it explicitly mentions the notion of tangent (recalling its status of best linear approximation around a given point) and explicitly associates it with the decreasing of Δx towards 0. The transition from a local to a global perspective on the derivative is carefully detailed and supported by the introduction of the specific functional notation $m = m(x)$. The task description also mentions the possibility of using other software and it is reasonable to think that the teacher has in mind the software Graphic Calculus which has already been used in this class for exploring polynomial functions, their tangents, and their derivatives in a similar way. We interpret this as a sign that the teacher thinks this help is necessary in order for the students to engage in a productive interaction with the material milieu (here the file), and especially use it for moving from a local vision to a global vision on the derivative.

In fact, in his answer to Question 12, the teacher shows that he is perfectly aware of the difficulty of such a move, and of the limitation of the a-didactical interaction with the milieu for achieving it. In the interview, he structures his expectations into four different levels. The transition from a local to a global perspective on the derivative corresponds to the fourth and last level and he points out that:

> Generally, from the third level, the understanding happens only thanks to the direct intervention of the teacher in the small groups and this understanding is consolidated in the mathematical discussions guided by the teacher with the whole class. (Answer to question 12, Sect. 2.2.2)

The teacher also expresses his expectations that the interaction with the file will lead students to conjecturing that the derivative of an exponential function is also an exponential function:

> The aim of the DGS file was to make the students understand that an exponential growth is directly proportional to the value of the function itself. This is an important step in understanding why the derivative of an exponential function is still an exponential function of the same base. (Answer to question 7, Sect. 2.2.2)
>
> With this activity, with the help of Cabri, I wanted the students to understand that exponential functions are functions for which the growth is proportional to the function itself. In other terms, the derivative of an exponential function is proportional to the function itself. This consideration, in my opinion, should allow students to understand why the exponential function a^x with a greater than 1 grows with x faster than any power of x. (Answer to question 8, Sect. 2.2.2)

This is of course the fundamental characteristic of exponential functions, but examining the file and the precise questions posed to the students in the task, we consider that such conjectures, contrary to those mentioned above, are quite unlikely to result from mere interaction with the a-didactical milieu in this episode.

Our a priori analysis leads us thus to conjecture the risk of a gap between the teacher's expectations and the potential of the a-didactical interaction with the milieu, and thus to pay specific attention in the a posteriori analysis on the strategies that the teacher uses for going beyond these limits, if this conjecture turns out to be true.

10.4.2.2 A Posteriori Analysis

The exchanges before the intervention of the teacher show the students undertaking exploration, following the guiding text, observing increasing lengths of HQ as P moves to the right and that the line (PQ) better approximates the function when P moves to the left. They try to make sense of their observations, but the expression and argumentation is rather fuzzy, and utterances are not so easy to interpret. Carlo also seems to conjecture that PQ is constant, which of course is not mathematically the case but can appear so when HQ is small with respect to HP, that is to say when P is close to the x-axis, and this is the first question he asks the teacher when he joins the group. But Giovanni contradicts him, moves to the approximation result mentioned above, and then to what happens when Δx decreases to 0, articulating that the line becomes a tangent. Carlo adds that PQ is decreasing, showing that his attention is still on PQ, but the teacher tries to orient the discussion in a more productive direction:

342	T	oh sure, it is almost trivial, isn't it? Therefore he was saying that this line tends to become…
343	G	tangent
344	T	and then what kind of information will it give you in this case?

Thanks to the answer provided by Giovanni in line 345 ("one can say that the exponential function becomes very little lines") and the gesture accompanying it, an episode can start in which the teacher–students interaction allows the students to increase the cognitive benefit of their a-didactical interaction with the milieu. The teacher rephrases Giovanni's utterance in line 346 ("it could be approximated to some small lines, which however…"), and Giovanni follows in line 347: "that is, that… with increasing slopes, that join together in a, that touch each other in a point." After consolidating this first achievement, the teacher asks in line 350: "and such lines which features have they?" Once again, Giovanni's answer offers an opportunity for going further as it introduces the idea of constant ratio between the slopes of successive segments which, appropriately worked out, could lead to the property that the derivative is also an exponential function as aimed at by the teacher in line 351: "they have… well, they may have a function, a slope are, possibly always twice than before." But, this time, the teacher does not jump in it with something like: "well, I don't know if the slope is twice, but… in any case… their slope increases, does it?" (line 352), but rather comes back to the growth ratio of the exponential function itself and to the compatibility of this property with the observation that the function crushes on the *x*-axis. His interaction with other groups mentioned in line 365 may have contributed to this orientation of the interaction.

It is interesting to point out that in his answer to Question 5, the teacher mentions this episode as especially interesting, explaining:

> I use a gesture used before by Giovanni. This gesture is towards a little segment that approximates locally the function and I ask: "What is the characteristic of this segment?" My aim is to induce the students to reflect on the fact that it is important to pay attention to the slope of the little segments, because their slope gives information on the growth of the function. Giovanni says "it is twice the previous slope …" I, using his same gesture, say more precisely that "the slope has an exponential growth." At the minute 54 and 24 seconds, I help the students to remember that the characteristic of the exponential successions is that of having the ratio of two consecutive terms constant. Immediately after, I ask the students: "Are you surprised that the graph of the function is so close to zero for small *x*?" Giovanni, at the minute 55 and 28 seconds says something like "with number smaller and smaller, I have number smaller and smaller." I reword this idea with a more precise language. In the following dialogue, Giovanni and Carlo are able to explain in a comprehensible way the reason why the graph of an exponential function of base greater than 1 is so close to the *x*-axis for *x* less than 0 and explodes for high values of *x*. (Answer to question 5, Sect. 2.2.2)

Thanks to this answer, we access the didactical technique (semiotic game) he consciously uses in this episode. From the perspective of TDS, this episode is quite interesting. The a posteriori analysis confirms the limitation of the a-didactical milieu anticipated in the a priori analysis. However, it also shows a specific technique used by the teacher for compensating this limitation. In the didactical situation S_0, through the technique of semiotic game, the teacher succeeds in extending the outcomes of the students' a-didactical interaction with the milieu in S_{-1}.

In actual classroom situations, even when tasks are carefully designed for fostering learning through adaptation, limitations such as those observed here are frequent. Research shows that in such cases, teachers' actions are not necessarily as productive as is the case here. On the contrary, they often degenerate into didactical phenomena such as Topaze effects which just maintain the fiction that students have learnt what they were supposed to learn (see Chap. 12). Semiotic games thus appear as a didactical technique which can be used for linking in a productive way the a-didactical and didactical levels of classroom situations, and extend in a didactical phase the potential of a-didactical interaction.

10.4.3 ATD Analysis of the Media-Milieus Dialectic

Let us analyze the episode under consideration using the Herbartian formula and the media-milieus dialectic provided by ATD. We can consider that the question *Q* in the Herbartian formula – according to the instructions in Task 3 – is the study of the "features of the graph of the function $y = m(x)$, where *m* is the slope of the line tangent to the function $y = a^x$ at the point of abscissa x" or, in other words, "how the slope of the line tangent to the function $y = a^x$ at the point of abscissa x changes as x changes" (quoting Task 3 from Fig. 2.3). In this episode we can observe the functioning of a didactic system formed around this question Q and where X is a couple of students (Giovanni and Carlo) and Y is the teacher. The Herbartian formula now leads to the question: what are the elements $A_j^\diamond$ and O_k intervening in the study process and how is the media-milieus dialectics managed by both the students and the teacher?

First of all, let us notice that the answer $A^\heartsuit$ the students are asked to provide – an explanation about the behavior of $y = m(x)$, that is, a piece of a "technology" in the ATD sense – is rather difficult to validate experimentally. Thus, what is the status we can give to the file prepared by the teacher in a way that, according to him, it may help the students with their explorations? The students are asked to interact with the files and extract some information about the "features" of the observed graphs. In a sense, they have to *read* or *interpret* what they see on the screen through some manipulations tightly specified by the teacher. We can thus consider that the files act as media, presenting – even if it is done in a quite hidden way – some previously elaborated answers $A_j^\diamond$ and which deliver partial information about the question Q at stake. Conjectures C_{10}, C_{11}, and C_{12} indicated above are part of the answers that could be extracted, even if, as shown by TDS analysis, the media provided seem to have some limitations if the students are left alone with it. However, these limitations do not constitute any constraint to the didactic process since this is carried out by the students *and the teacher*. The interventions of the teacher and his use of the semiotic games can thus be considered as part of the didactic gestures necessary to deal with the media provided by him.

Of course, even if it occasionally requires some help from the teacher, students need to know how to "read" the files in order to obtain the information requested. It is during this "reading" that the media-milieus dialectic starts running, as the students contrast the information on the screen with some other previous knowledge they have about function graphs, exponential functions, and growth variation. These are the objects O_k of the *a-didactic milieu*, the objects that are already available and the existence of which is stable enough to act as a fragment of nature. The following sequences show how new observed properties are contrasted with some previously known features, that are thus acting as an a-didactic milieu:

255	C	well… P moves on the graph
256	G	yes, and also a
257	C	a is the rate of growth
258	G	perfect
…		
303	C	but like… try to put it $a = 1$, it must result
304	G	a line…
305	C	$a = 1$ we know it already… than you must do less than 1
…		
311	C	ok, so ok, ok, so ok, because if it means that they increase, the more you move them over there, it increases very very much
312	G	yes
313	C	because it's an exponential function

When the media-milieus dialectic stops being productive, the students search another medium. The classic didactic gesture in this case is to ask the teacher, to use him as one of the media. Then a new media-milieus dialectic starts running as the teacher does not limit himself to providing new information; he also intervenes to validate or question the answers found by the students, refusing to act as a medium and returning the question to the students, as in the following interactions:

327 C we wished, practically, is there always the same distance between P and Q?
328 T always the same distance?
329 G no no, it decreases
…
351 G they have… well, they may have a function, a slope are, possibly always twice than before
352 T well, I don't know if the slope is twice, but… in any case… their slope increases, does it? In this case, when this function increases

Another form of the media-milieus dialectic takes place when the teacher mentions the result obtained by another group:

365 T yes, the other group have used a very good example: if we take 10 % of 5 cents it is 0,5… it doesn't exist, isn't it? It is as it did not exist; if we take 10 % of 5 million euro on the contrary thing start changing, isn't it? It is a considerable amount of money… here the hypothesis are the same… and it is ok; now you go on in this way. Where have you arrived?

Here the teacher is using the answer provided by another group as a way to validate Giovanni and Carlo's proposal: when there is no experimental milieu, as is the case here with the conjecture provided by the students (the *explanation* of how $y = m(x)$ grows), there is always the possibility of contrasting the conjecture with different media and seeing if the different answers provided are coherent with each other. This is what we can call the "contrast between media" strategy, a very usual form of the media-milieus dialectic, currently used by scientists, journalists, and also students, to check their results.

Finally, we can find at the end of the protocol some of the materials that will compose the final answer $A^{\heartsuit}$ provided by Giovanni and Carlo, a partial result of the didactic system *S* (*Carlo & Giovanni*; *Teacher*; *Q*) that will supposedly be later on incorporated into the answer of the larger didactic system *S*(*Whole class*; *Teacher*; *Q*):

368.1 G that if the *x* increases again, the line passes through P and Q and is almost constant, it becomes almost a tangent… this because if we take a very big zoom we can approximate the exponential function with many lines, which have an increasing slope…
368.2 G Then, if the point P is very near to zero, this line approximates very much the exponential function. Also here even if numbers are very small, it increases not so much, hence like a line… and then we can write that we were waiting for it even if the ratios are constant at the beginning… it was almost a line… [not understandable]

369 C hence we write that it is a graph with a constant rate of growth, of a… of a if x is always the same… [not understandable] but the y's…

…

379 C it is ok! Otherwise it had no sense… that maintaining PH constant and therefore also the Δx's constant we notice that.. [not understandable] while P increases, P increases more and more, that is the Δy's increase; they increase more and more

It is interesting to notice that, in the episode observed, the class works in a complete autarchy regarding outside media: the teacher brings the information into the class, through the files and his own knowledge; the students are expected to obtain all the desired information only with the means they are given and their previous knowledge. We do not know if, after the work done during the observed session, some other external media are being consulted (such as mathematical books, encyclopedias, internet files). The traditional functioning of our current mathematical teaching systems shows a tendency to avoid these types of media and limit the work to the information provided by the teacher or "extracted" by the students. It implies an important loss in terms of the elaboration of strategies to validate the answers presented by these media and, as a consequence, a tendency to take the traditional media (teacher, lesson notes, and textbooks) as previously granted and without any need to be contrasted with a milieu.

10.5 Dialogue Between AiC, TDS, and ATD with Regard to "Context," "Milieu," and "Media-Milieus Dialectic"

10.5.1 Different Analyses, Different Priorities

The three analyses provided in the previous sections illustrate the differences between the three theories as observed in our first classification in Sect. 10.2 as well as the shared epistemological sensitivity. The three analyses demonstrate how these similarities and differences are practically expressed in the analysis of the episode and therefore allow the dialogue between the three theories.

In order to better understand the dialogue between AiC, TDS, and ATD with regard to "context", "milieu," and "media-milieus dialectic" we come back to the different priorities of the theories with regard to the focus of analysis. For example, AiC researchers focus on the learner. This is not the case for TDS researchers, as expressed in Chap. 4:

> Even if TDS has the ultimate goal of improving students' mathematics learning, the learner is not at the center of the theory. TDS gives priority to the understanding of how the conditions and constraints of didactical systems enable or hinder learning, and how the functioning of such systems can be improved. (Sect. 4.1.2)

And a similar position is adopted by ATD researchers who consider a larger environment of conditions and constraints for the evolution of didactic systems. Since in AiC the focus of analysis is on the learner, all other factors such as the task, the computer, the teacher, and the learning goals are considered as contextual factors. Therefore, the notion of context for AiC is especially wide since it includes the external world of the learner and part of his internal world. This might include notions which are not necessarily considered as part of the milieu for TDS or ATD. This reflection is well illustrated in the differences in the a priori analyses of TDS and AiC. The AiC a priori analysis is concerned with the learner's intended constructs. The TDS a priori analysis included already a conjecture on the role of a gap between the teacher's expectation and the potential of the a-didactical interaction with the milieu. It seems that for TDS researchers the "context" is already taken into account and is structured already in the a priori analysis.

As a consequence, the AiC researchers learnt the importance of the TDS a priori analysis but also the fact that some excerpts might add direct knowledge to the analysis of the cognitive processes which might be missed if one focuses first on the cognitive processes and only then analyses the influence of other parts of the context.

In addition, some insight is offered while reflecting on the question: what can another theory (semiotic games) offer to the three existing theories (AiC, ATD, and TDS) in terms of insight regarding relationships between the existing theories? In the previous section we observed that the AiC analysis of the role of the teacher demonstrates that the students' construction of knowledge was not as expected by the teacher. The AiC analysis points to the limitations of the semiotic games. This is in accord with TDS and ATD discussion on the limitation of the milieu.

The three theories agree therefore on the limitation of the semiotic games and the limitation of the milieu. Nevertheless, the insights offered by each theory are different and we may say that they complement each other. We explain this complementarity as follows.

AiC offers a fine-grained analysis of the students' epistemic processes and makes subtle evolutions visible in the process of construction of knowledge. TDS and ATD offer to AiC the benefits of a more systematic engagement in a priori analysis for anticipating the possible effect of contextual characteristics on epistemic actions. TDS and ATD observed the entire situation from the beginning. Both support the AiC a posteriori analysis. This is done by means of the analysis of the role of the teacher, first as an observer in the a-didactical situation and then as an active actor exploiting the milieu provided by the a-didactical situation. In other words:

- TDS complements the AiC analysis in analyzing how the teacher extends the outcomes of the a-didactical interaction. The TDS analysis seems to start where the AiC analysis stops.
- This link between the a-didactical and didactical levels is offered by the ATD analysis as well. For ATD, the limitation of the milieu does not constitute any constraint to the didactic process since this process is carried out by both the students and the teacher. The ATD media-milieus dialectic permits taking into account the different ways one can use the context. This capacity is not offered by AiC nor TDS.
- AiC offers the possibility to discern which element of context leads to the conceptualization and to the construction of new meaning on the part of the students.

10.5.2 The Subtle Interaction Between Contexts and Theoretical Approaches

The complexity being addressed by the notion of context is well known. A first problem is that what is considered as a part of the context in one theory is not necessarily considered as that in another theory. There are different approaches towards context in different didactic cultures.

We can illustrate this comment by taking some different parts of context analyzed by AiC researchers and see how it fits for the TDS or ATD analysis. For example, taking the learner's personal history as part of the context according to AiC, one could ask how TDS and ATD take it into account. Concerning personal history, a distinction may be between the history of a specific individual student (AiC), a student with a typical history for a specific situation (TDS), and taking into account the institutional background of the whole process (ATD). The focus of each of the teams corresponds closely to the elements of the theory within which this team works. For example, in their a priori analysis neither the TDS researchers nor the ATD ones would take into account personal data concerning the learners' personal history. Even in the a posteriori analysis there is no description of each learner's individual trajectory of thinking. By the ATD approach, students are considered as "normal subjects" of the class, that is, of the didactic system $S(X; Teacher; Q)$. The focus is on the functioning of this system and its ecology: the conditions that make the functioning possible and the restrictions that hinder other possible evolutions. This study would need, however, some extra information about the teaching process in which the episode takes place, to know what type of tasks corresponds to the questions proposed, what kind of production is expected from the students, and what praxeological elements are made available to make this production possible.

In a similar way, some other elements of context could be considered, for example the teacher. Contrary to AiC, the teacher for TDS is neither an element of the context nor a component of the milieu: he is an actor. TDS is interested in relations between systems and the teacher is an element of the system. TDS does not theorize the context in itself; but through the different levels of the notion of milieu, characteristics of the context are progressively taken into account in the analysis, from those which are controlled by the teacher through the organization of the material milieu and selection of appropriate didactical variables and up to the conditions which influence design decisions outside the classroom. This can be linked to the distinction evoked in Chap. 4 between two different perspectives on didactical situations: a restricted vision of these elements as the student's environment organized and piloted by the teacher and a broader vision including further elements such as the teacher and the educational system itself.

Taking context into account supposes an enlargement on the unit of analysis considered at an early stage of development of research in mathematics education, where the focus was essentially put on the students and their knowledge development. From its beginnings, the TDS has gone beyond the simple consideration of the student and the mathematical activity. It started to consider *situations*, that is, problems arising in institutional settings about something that is already there, the *milieu*. The construction of mathematical knowledge cannot thus be carried out *in a vacuum*, it needs to suppose the existence of something "external": a milieu in a situation. A given piece of reality can then correspond to different possible situations depending on what is considered as the related milieu. The notion of situation thus allows consideration of different types of activities concurring in the teaching and learning process (students doing mathematics, teacher helping students do mathematics, etc.), and different pieces of reality at different levels of generalization.

Also, for ATD researchers, when facing a given "piece of reality" such as an activity carried out in the classroom, the focus is on the institutional conditions making this activity possible, using for instance the different levels of codetermination (see Chap. 5). scale of levels of didactic codetermination emphasizes that the concrete actions of a teacher and a group of students in a classroom may depend on the mathematical activity and on the domain or sector where this activity takes place (a reasoning about functions and their variations does not need the same elements as a reasoning about random variation of a statistical variable, for instance). But it also depends on determinants related to how the activity is organized, for example in a school context, with a specific pedagogy (in the case considered, the students are used to interacting and discussing with the teacher, to using computers, and understanding that they are supposed to give an answer to the questions posed, etc.). And, as was stated in Chap. 5 on ATD, the existence of a task of the kind "look at the computer and tell something about what you see" seems related to a current social practice where new information seems to come from the direct observation of phenomena (and not, for instance, from the study of old books…). The hierarchy of levels of codetermination does not exist in other frames such as AiC or TDS. It is the approach of research problems in terms of the *ecology* of mathematical and didactic praxeologies that drives researchers to look for conditions and restrictions that go beyond the narrow space of the classroom (and even of the school). Compared with other approaches, ATD proposes a huge expansion of the "external" world that should be taken into account to explain why things happen the way they happen and, furthermore, why many other things that could happen never really happen.

10.6 Concluding Remarks About Networking Strategies

10.6.1 Proximity

A dialogue between different approaches can only start when a point of contact is found. In this case, we may talk about a common "epistemological sensibility" of AiC, TDS, and ATD, which can be noticed in the a priori analyses provided by each frame. These analyses are the starting point of the dialogue between the approaches and, in a sense, they seem to answer each other. For instance, AiC researchers refer to an analysis provided by TDS researchers about the episodes on Task 1 and 2 and they use and complete it in the episode on Task 3 in terms of an analysis of intended constructs. The ATD researchers take a further step when they refer to a (hypothetical) teaching project used as a reference and enlarge the description of students' activities using the media-milieus dialectics. This initial proximity seems essential for a dialogue to start and become productive, showing the complementarity of the approaches and the reciprocal enrichment, without losing what is specific to each one.

10.6.2 TDS as "Mediator"

Another contextual condition which enhances the dialogue is that TDS is in an intermediate position, like a theoretical frame which permits to establish a bridge, a communication between AiC and ATD with their different foci which are very far from each other. The dialogue has then appeared as a progressive enlargement of the focus which, in a sense, is central in the notion of context. For AiC, with its focus on the learner, the context integrates any piece of the present and past environment that can influence the individual epistemic processes. For TDS, with its focus on situations, there is a kind of split between an explicitly theorized part (in terms of milieu and didactical contract) and a part that is not explicitly taken into account by the theory and that could also play the role of "context" as in AiC. With ATD, again the vision is enlarged since the whole teaching project is taken into account and the focus is on the conditions for a given didactic process to exist and evolve in a given direction.

In this theoretical dialogue on questions about "context," we should differentiate between the notion of media-milieus dialectic and the hierarchy of levels of codetermination. ATD's hierarchy of levels of codetermination permits enrichment of the theorization of TDS. Nevertheless, for what concerns the media-milieus dialectic, the situation is different for ATD and TDS even if both theories start their analysis with the same observation of the limitation of the milieu. In TDS analysis, we observe the view of a situation with an a-didactical potential. TDS analysis demonstrates the limitation of this potential and the need for an action from the teacher. The analysis shows how the teacher extends the outcomes of the a-didactical interaction. ATD analysis starts with the same observation of the limitation of the milieu but demonstrates an absence of a media-milieus dialectic which could have been expected and questions the reasons for this absence.

10.6.3 Different "Units of Analysis"

An interesting, and also revealing, point is the fact that, in the analysis, AiC researchers focus on the autonomous work of the students, while TDS researchers pay more attention to the episode where the students interact with the teacher, and ATD approaches the teacher's overall strategy. The AiC analysis shows the richness of the knowledge constructed by the students during their interaction with the context and the role of the different elements of the context in this construction. As a counterpart, the TDS analysis pays less attention to the students' constructions. The TDS analysis is guided by the conjecture (coming from the a priori analysis) of the gap between the power of the milieu and the teacher's expectations, which is supposed to lead to an impasse where the milieu should be enriched in order to avoid a Topaze effect (see Chap. 12). We should also notice another reason why the TDS analysis pays less attention to the students' constructions: the difficulty for the TDS researchers to achieve a fine grained analysis of the first exchanges between the students and their constructing processes. AiC researchers with their analytical tools have no problem in carrying out this analysis. Nevertheless, their tools, in contrast to TDS tools, are less operational when the teacher interacts with the students. ATD researchers take a further step when they take the whole episode into account and focus on the aim of the didactic process and the strategies used by both the teacher and the students to make the needed praxeological ingredients available. Without this, ATD researchers are not able to give sense to the students' interaction, nor to their interaction with the teacher.

As we see, the theories have different foci and the potential of the analytical tools is different for each theory. These differences can explain the fact that each group of researchers can learn and evolve from the others' analyses.

References

Artigue, M., & Winslow, C. (2010). International comparative studies on mathematics education: A viewpoint from the anthropological theory of didactics. *Recherche en Didactique des Mathématiques, 30*(1), 47–82.

Artigue, M., Kynigos, C., & Mariotti, M. A. (2010). Contextual issues seen through the ReMath filter: Some illustrative examples. In M. M. F. Pinto & T. F. Kawasaki (Eds.), *Proceedings of the 34th conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 295–304). Belo Horizonte: PME.

Brousseau, G. (1990). Le contrat didactique: le milieu. *Recherches en Didactique des Mathématiques, 9*(3), 309–336.

Brousseau, G. (1997). *Theory of didactical situations in mathematics*. Dordrecht: Kluwer.

Chevallard, Y. (2008, November). Afterthoughts on a seeming didactical paradox. In *International conference "Efficacité & Équité en Éducation"*, Rennes, France. http://yves.chevallard.free.fr/spip/spip/IMG/pdf/Yves_Chevallard_-_On_Inquiry-Based_Instruction.pdf. Accessed 28 Nov 2013.

Chevallard, Y. (2012). Teaching mathematics in tomorrow's society: A case for an oncoming counterparadigm. In *Preconference proceedings of 12th international Congress on Mathematical Education 2012*, Seoul, Korea. http://www.icme12.org/upload/submission/1985_F.pdf. Accessed 28 Nov 2013.

Dorier, J. L., Artaud, M., Artigue, M., Berthelot, R., & Floris, R. (2002). *Actes de la 11e Ecole d'Eté de Didactique des Mathématiques*. Grenoble: La Pensée Sauvage.

Dreyfus, T., Hershkowitz, R., & Schwarz, B. (2001). Abstraction in context II: The case of peer interaction. *Cognitive Science Quarterly, 1*, 307–368.

Kidron, I., & Dreyfus, T. (2010). Interacting parallel constructions of knowledge in a CAS context. *International Journal of Computers for Mathematical Learning, 15*(2), 129–149.

Margolinas, C. (1995). La structuration du milieu et ses apports dans l'analyse a posteriori des situations. In C. Margolinas (Ed.), *Les débats de didactique des mathématiques, Annales 1993–1994* (pp. 89–102). Grenoble: La Pensée Sauvage.
Perrin-Glorian, M. J. (1999). Problèmes d'articulation de cadres théoriques: l'exemple du concept de milieu. *Recherches en Didactique des Mathématiques, 19*(3), 279–322.

Chapter 11
The Epistemological Gap: A Case Study on Networking of APC and IDS

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Abstract The case study of the epistemological gap involves two theoretical approaches, APC and IDS. It describes a networking case that starts from a situation of seemingly contradictory analyses, develops a common methodology, and leads finally to conceptualizing and locally integrating the new concept of the epistemological gap into both theories.

Keywords Networking of theories • Epistemological analysis

11.1 Introduction

In this chapter we present the networking process developed by two teams,[1] namely the APC team using the Space of Action, Production, and Communication theory with its Semiotic Bundle construct (Chap. 3) and the IDS team using the Theory of Interest-Dense Situations (Chap. 7) in which a partial integration of a new theoretical construct, the epistemological gap, took place for both approaches after trying to coordinate seemingly contradictory analysis.

The networking in this case study has its origin and empirical base in a short video excerpt, referred to in Chap. 2 as "extra video" on Task 3 (see Sect. 2.2.3).

[1] A first partial account of this networking process has been presented in CERME6 (Working Group 9: Different theoretical perspectives and approaches in research); see Arzarello et al. (2010).

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It lasts about a minute and a half, and shows Giovanni and Carlo discussing with the teacher what happens to the exponential function for very large x.

The need for a second video was raised by the IDS team. In fact, in order to progress with the networking, the IDS-analysis did not need so much the interview data with the teacher that other teams needed (see Sect. 2.2.2), but rather needed additional data about typical social interactions in the class. To be able to reconstruct these typical social interactions, videos from about 20 lessons would be needed in order to shape an appropriate empirical base. However, only one additional video was available from the same classroom in the same school year, and this was the extra video on Task 3 (see Sect. 11.2 below).

The first networking step was to analyze the video from the two perspectives separately. Each team carried out an initial analysis of the episode (as reported in Sect. 11.3). Each of the initial analyses in its own way described a teacher–students interaction that did not lead to a successful outcome, but neither of the two analyses could provide an explanatory account of the empirical phenomenon. On the contrary, the analyses appeared almost contradictory. This surprising result triggered the necessity to carry out a joint analysis that started a coordination process between the two teams. The result was a local integration of the methodologies of the two theories. In Sect. 11.4.1, we describe the process as well as the result of our coordinating strategy.

In a spiral process this coordinated analysis brought about the necessity of further theoretical reflection, especially considering the epistemological dimension. In an interplay between the theoretical reflection and the data analysis, we developed a new concept, which we called the *epistemological gap*, and which could provide a satisfactory explanation (for us) of the empirical phenomenon previously identified (Sect. 11.4.2).

A local integration based on the epistemological dimension was thus realized for both theories. The new tool for analysis, produced in the networking activity, deepened our understanding of the data and opened routes for reflection that were new for each perspective. In the final section (Sect. 11.5) we report our reflections on our networking enterprise.

11.2 The Empirical Base

In the extra video, Giovanni and Carlo discuss with their teacher (T) what happens to the exponential function for very large x. This episode occurred immediately after the students had finished Task 3 (see Sect. 2.1.3). The result of the exploration was

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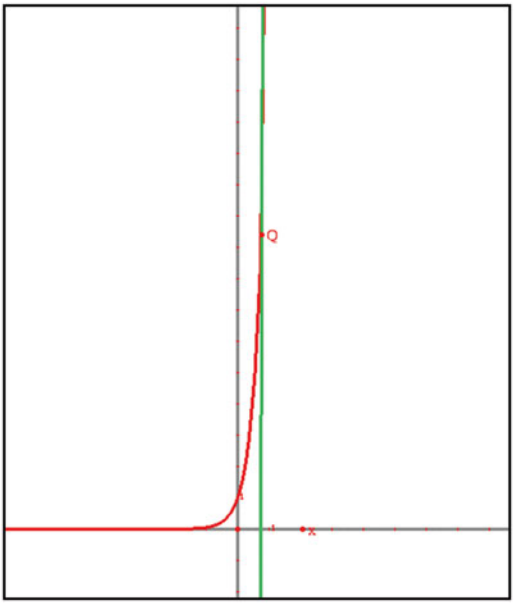

Fig. 11.1 The graph shown on the computer screen

still visible on the computer screen (Fig. 11.1). It shows a secant built by two points very close to each other leading to a quasi-tangent line, which the students and the teacher refer to as the "tangent line."[2]

Due to the specific methodologies of APC and IDS, we report in this chapter the transcript integrated with information about the connotation of the speech, and the occurrence of accompanying gestures. Some screenshots are added from the video, in order to better account for the gestures: they are reported immediately after the corresponding speech line.

In the transcript, underlined words indicate that they are simultaneous with the gestures.

1 G but always for *a* very big <u>this straight line</u>, [Gesture] when they meet each other, there it is again…that is it approximates the, the function very well, because …

Gesture in 1: G is pointing at the line in the screen

2 T what straight line, sorry?

[2]The configuration on the screen is not reported by the video-camera, nor captured in any other way, rather it is reconstructed by the APC team as reported in Fig. 11.1. Detailed information on the line (e.g., exactly which value of *a* is chosen) is therefore not available.

3 G this here [*pointing at the screen*], for *x* very, very [*Gesture*] big

Gesture in 3: G's hand goes upwards

4 T [*Gesture a*] will they meet each other [*Gesture b*]? [*challenging connotation*]

Gestures in 4:

(a) T pointing two forefingers

(b) T crossing the two pointed forefingers

5 G that is [cioè[3]], yes, yes they meet each other [*gesture*]

Gesture in 5: G's two forefingers touching each other

6 T but after their meeting, what happens? [*continuing to keep the hands in the same configuration as in line 5*]

7 G eh…eh, eh no…, it makes so

8 T ah, ok, this then continues [*gesture a*], this, the vertical straight line [*gesture b*], has a well fixed *x*, hasn't it? The exponential function later goes on increasing the *x*, doesn't it [*gesture c*]? Do you agree? Or not?

Gesture in 7: G crosses the left hand over the right one; T is keeping the previous gesture

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[3]The expression "cioè" in Italian means literally "that is." Over-used by teenagers, it introduces a reformulation of what was just said. As is likely in this case, it can have the connotation of "I am sorry but."

Gestures in line 8:

(a) T moving rightwards his left hand

(b) T's right hand vertically raised

(c) T moving rightwards his right hand

9 G yes […]

10 T [*addressing C*]: He [*G*] was saying that this vertical straight line [*pointing at the line in the screen*] approximates very well [*gesture*] the exponential functions

Gesture in 10: T raises both hands

11 G that is, but for x that are very…very big

Gesture in 11: G moves his left hand upwards

12 T and for how big x? 100 billions?
$x = 100$ billions?

Gesture in 12: T raises his hand at his right and keeps it fixed

13 G because at a certain point…, that is, if the function [*gesture 13a*] increases more and more, more and more [*gesture 13b*], then it also becomes almost a vertical straight line [*gesture 13b*]

Gestures in line 13:

(a) G raises his left hand

(b) G moves his hand upwards

(c) final position of G's hand after moving upwards

14 T eh, this is what it seems to you by looking at; but imagine that if you have $x = 100$ billions [*gesture*], there is this barrier…is it overcome sooner or later, or not? [*connotation: suggesting the answer yes*]

Gesture in 14: T keeps his right hand in the vertical position

15 G yes

16 T and so when it is overcome [*gesture 16a*], this x 100 billions [*gesture 16b*], how many x do you still have at disposal, after 100 billions? [*gesture 16c*]

Gestures in 16:

(a) T crosses left forefinger over right hand

(b) T raises his right hand

(c) T moves right hand rightwards, repeatedly

17 G infinite

18 T infinite… and how much can you go ahead after 100 billion [*repeating the gesture 16c*]?

19 G infinite points

20 T then the exponential function goes ahead on its own, doesn't it?

11.3 Initial Analysis from IDS and APC Perspectives

Initially, the two teams carried out the analysis of the episode separately according to the two perspectives. The result is summarized in the following sections.

11.3.1 Initial Analysis According to the Theory of Interest-Dense Situations

To work out the IDS-analysis (cf. Chap. 7), we have to consider the central question: How is the emergence of an interest-dense situation supported or hindered? (cf. Bikner-Ahsbahs 2005). To answer it, an analysis of speech acts at three levels is conducted. The locutionary level considers the direct meaning of what is said, while the non-locutionary level considers, on the one hand, the information given by saying something and the way it is said (illocutionary level) and, on the other hand, the intended and actual effect on the listener (perlocutionary level) (Austin 1975; cf. Beck and Maier 1994).

In line 1, Giovanni begins to construct mathematical meanings about the growth of the exponential function in broken language as described above. The teacher interrupts him: by apologizing he indicates illocutionarily that he normally would not interrupt the student, but in this case an interruption seems necessary to him. By saying "sorry," he also might want Giovanni to feel accepted. Asking "what straight line" (line 2) indicates either that there is something problematic with the straight line or that the teacher wants to clarify which straight line exactly is meant. Pointing at the line on the screen, Giovanni refers to an answer on the locutionary level, but also adds the condition for his explanation given in the task in line 1: "for very big *x*." The teacher's question "will they meet each other?" locutionarily requires information whether the graph and the line meet, but illocutionarily questions the truth of the condition of Giovanni's beginning explanation in line 1 "when they meet each other…." Therefore, and through connotation, the teacher's question is challenging to Giovanni. It is not clear whether the teacher suggests a negative or positive answer, but the teacher's finger crossing gesture (screenshot 4b) might support the latter, as does the intonation. Giovanni follows the teacher's crossing gesture and answers that "they meet" (line 5), indicating through intonation and by doubling the word "yes" (illocutionarily) that he has no doubts about the fact that they meet. Since Giovanni has perceived the line to be constructed as a secant his certainty is based on his previous experience. On the locutionary level, we would see only the questions and the answers. On the non-locutionary levels there is negotiation underneath.

Looking only at lines 1–5, an interest-dense situation is about to emerge because Giovanni is deeply involved in the mathematical problem and he starts to construct further-reaching mathematical meanings. Based on the theory of interest-dense situations we can predict how the teacher could support or hinder the emergence of interest density. By focusing on the student's ideas and supporting their further construction, the teacher would support emergence of interest density; but by acting according to his own thinking process or his expected answers, he would interrupt its emergence. Going further in our analysis, we will show that there is a negotiation which proceeds underneath the locutionary level.

In line 6, the teacher builds on the student's utterance by asking what happens after the two lines meet. Imitating the teacher's finger crossing (that he gives as a hint, perlocutionarily), by his hands Giovanni shows how the two graphs cross each other. However, the teacher does not seem to be content because he explains his view of the situation in line 8 and gives a more elaborate answer to his own question. Giovanni does not fulfill the teacher's expectations. In this way the teacher establishes an argumentation as a proof by contradiction, following his own train of thought and not that of the student. In line 8, he constitutes the basis of his argument. In order to include Giovanni into the process, his rhetorical questions "Do you agree? Or not?" demand Giovanni's agreement (line 9). Summarizing Giovanni's statement addressed to Carlo (line 10), the teacher puts forward the statement that he wants to prove false. However, Giovanni modifies and restricts the range of the statement's validity by "but for x that are very…very big." This utterance (locutionarily) adds a condition, but illocutionarily Giovanni corrects the teacher. Hence, he only partially agrees, because his description was based on "very…very big x" (line 11). Again, Giovanni indicates that his train of thought is a bit different. Perlocutionarily Giovanni succeeds at this moment because the teacher changes his focus, by locutionarily taking up the student's idea in the question: "for how big x?" (line 12). Giovanni seems to feel encouraged to explain: "because at a certain point…, that is, if the function increases more and more, more and more, then it also becomes almost a vertical straight line" (line 13). Because of Giovanni's deep involvement and the dynamic of the epistemic process, the situation has the potential to lead to an interest-dense situation. It is the attribution of mathematical value, which is not yet expressed. On the illocutionary level, the teacher indicates understanding of Giovanni's point of view (line 14) by saying "this is what it seems to you by looking at," but he also implies that the student's way of arguing is not the correct way. In this way, he divests Giovanni of his argumentation base, that is, the diagram on the screen. Through the word "imagine" he refers to another argumentation base but asks whether $x = 100$ billion will be overcome, suggesting that the answer is positive. Giovanni agrees. The teacher now finishes proving the statement (line 10) to be wrong by a proof of contradiction that he orchestrates through social interaction. The teacher works his argument out. "there is this barrier… is it overcome sooner or later, or not?" he closes (line 14), demanding agreement. Lacking an argumentation base, Giovanni now gives up following his own train of thought (lines 15–20). The emergence of interest-density dries up.

11.3.2 Initial Analysis According to the APC Model

The APC team analyzed the episode by focusing attention on the semiotic resources shown in the teacher–students interaction, that is, on the *semiotic bundle* that is the combination of words, gestures, and representations in the Cabri file. As discussed in

Chap. 3, the semiotic bundle construct is the main analysis tool of micro-processes within the APC approach.

The basic point of discussion between students and teacher concerns the behavior of the exponential function for a large base *a* and large values of *x*. In his first utterances (lines 1–3) Giovanni claims that in such conditions the straight line that appears on the screen, that is, the quasi-tangent line, can be a good approximation of the exponential function. Such a conjecture is fostered by the image from the Dynamic Geometry Software the students are using (see Fig. 11.1): the quasi-tangent line appears almost vertical, and the exponential function comes to be perceptually confounded in it.

However, the teacher misinterprets Giovanni's words: whereas Giovanni is referring to the tangent line (he points also to it on the screen, lines 1 and 3), the teacher appears to interpret the student's words as referring to a vertical asymptote (lines 4–6). There are two hints for this misinterpretation:

- Giovanni says "when[4] they meet each other" (line 1): he seems to refer to the fact that the tangent approximates the function well near the tangent point.
- The teacher starts speaking (line 4) without giving Giovanni time to complete the sentence.

Hence there is a conflict between Giovanni's gesture, pointing to the tangent on the screen, and the teacher's gesture, which shows the vertical asymptote. A possible origin of this misinterpretation can be traced to the teacher's professional knowledge regarding the exponential functions and teaching–learning processes about it – what in literature has been called "specialized content knowledge" (Ball et al. 2008) of the teacher.

Asking about a hypothetic meeting of the function with the straight line, the teacher is representing the graphs by means of an iconic gesture (screenshots 4a, b and subsequent pictures): his right forefinger stands for a straight line, and his left forefinger moves in an upwards inclined way to represent the exponential function graph. In his subsequent interventions, Giovanni (lines 5–7, and corresponding gestures) is tuning with the teacher's semiotic resources, both speech and gesture. With his hands, he represents the graph of the exponential crossing the straight line (gesture in line 7): he is answering the teacher's question by showing the behavior of the function through the gesture. The teacher (line 8) accepts such an answer and endeavors to make explicit the idea that the domain of the exponential function is not limited, and therefore its graph intersects any vertical line. To do so, he uses both speech and gestures (see lines 8–20, and the related pictures).

We can now focus more closely on the dynamics using the semiotic bundle lens. In order to include Carlo in the discussion, the teacher reports Giovanni's observation (line 10). By repeating and rephrasing Giovanni's words (line 10) he is tuning with the student's speech. But through gestures (screenshots 10, 16b, c), he is trying to make apparent a specific feature of the graph of the exponential function, that is, the fact that it *crosses* any vertical line. The teacher is demonstrating what we call a *semiotic game* (see Chap. 3), in that he is tuning with one semiotic resource, and is

[4]In English "when" may have the meanings of "if" and of "where." In this case, the sense is "where."

using another resource used by a student to make meanings evolve in order to align them with the culturally established mathematical ones.

The gesture appears as a powerful resource in the teacher's hands, in order to prompt the students' imagination. In fact, the gesture allows the teacher to refer to what cannot be seen in the representation on the screen and may be difficult for the students to understand from a purely verbal description (the graph for very large x). In particular, gestures seem to be a suitable means to refer to very large values and to evoke their infinite quantity (screenshot 16c).

As to Giovanni, we see that he does not appear to have profited from the teacher's semiotic game. In fact, considering lines 11–13 (and the related pictures), we can see that in his words he is expressing the idea that the function will become "almost a vertical straight line"; and his gestures appear very different from the teacher's. Whereas the teacher's gestures place large values of x in the correct location with respect to his gesture space[5] (hand moving rightwards: screenshots 8c, 12, and 16c), the student's gestures place large values of x to a high location in space (hand moving upwards, screenshots 11, 13a, b, c): he is probably referring to the values taken by the exponential functions, rather than to the abscissas. As a result of the analysis, we can conclude that the episode shows an example of a non-successful semiotic game.

11.4 Networking of the Approaches

The two analyses were exchanged between the teams. Through this exchange, the researchers made a strong effort *to make themselves understandable and to understand the other's perspective*, which constituted an important networking strategy (see Chap. 8).

This initial step led to *contrasting* the two analyses: we acknowledged their complementarity, but also felt that the two results had feeble explanatory power. Thus, we were led to a new common question: what is the deeper reason why the epistemic process (socially and semiotically) is not successful in this episode? Guided by this common question, the two teams jointly carried out a common analysis, through the *coordination* of the two theories. The resulting coordinated analysis is presented in the next Section.

11.4.1 Coordinating the Two Analyses

Based on the theoretical account and the two initial analyses, the IDS and APC teams considered the two perspectives as providing *complementary analytical tools and, thus, complementary interpretations* of the data. In fact, each view shed light on different aspects of the teacher–students interaction. The strength of the

[5]The gesture space (McNeill 1992) is the area in front of the speaker's body, in which he performs the majority of his gestures.

IDS-perspective is in the possibility to predict the emergence of interest-dense situations according to the type of social interactions that hinder or foster it. It includes the analysis of the locutionary and non-locutionary levels of speech, reconstructs epistemic processes within social interactions, and shows negotiations underneath the content level. According to this perspective, the student and the teacher are not able to merge their argumentations in the episode although there is a lot of negotiation about whose train of thought will be followed. Neither the teacher nor the student is able to engage with the other's perspective. The analysis shows a gap that cannot be overcome. However, the IDS-approach is, from an epistemic point of view, unable to provide tools to find out why this is the case.

On the other hand, by looking at a wide range of signs (in Peirce's sense), the APC-analysis identifies phenomena that could go unnoticed under standard linguistic-based analysis, such as the semiotic game between teacher and student. The analysis with the semiotic bundle provides the means to observe and to properly describe this game. From our classroom observations at several school levels, we can say that students did succeed to learn by means of semiotic games in other cases (see, e.g., Arzarello et al. 2009). This episode was one of the first in the APC team's research in which things appeared to go wrong. It was therefore a good occasion to investigate the *scope* and the *limits* of the semiotic game construct. Using the APC-frame one could observe that in the above episode the semiotic game shows the gesture–speech resources in the opposite direction with respect to semiotic games previously analyzed as "successful" (called "standard semiotic games" for the moment). In standard semiotic games, in fact, the teacher tunes with the students' gestures and uses speech to foster meaning development; in the above episode it was the other way round: tuning with speech (line 10) and fostering meaning through gestures. Using the semiotic bundle lens one can identify this difference, but within the theory it is not possible to say why this semiotic game is not working.[6]

The discussion so far led us to argue that the simple juxtaposition of the two perspectives was not enough to deeply understand what went wrong in the interaction. Since the student and the teacher referred to different resources in their argumentation, we conjectured that the reason for this gap might be located in the epistemological viewpoints of the student and the teacher, that is, in their views on relevant knowledge and ways of knowing. The idea of epistemological viewpoints was elaborated during the networking process.

The coordinated analysis of this episode was accomplished by re-analyzing the entire transcript line by line. The analysis in line with the theory of IDS was complemented with attention to the gestures. The speech–gesture in line with APC was complemented with attention to the non-locutionary levels. Figure 11.2 depicts the resulting lens of the analysis, from the coordination of the two previous ones which connects two levels of meaning-making with speech and gestures.

[6] Realizing that the semiotic game had not worked properly led us to suspect that the episode showed a case of a "Topaze effect," as described in the TDS theory. This was the prompt for further elaborating the networking process, and the outcome is reported in Chap. 12. The reader will see that, finally, the episode is considered to be neither a case of a genuine semiotic game, nor of a genuine Topaze effect.

	Speech	Gestures
Locutionary level		
Non-locutionary level		

Fig. 11.2 Two-level analysis of semiotic resources, deriving from a coordination of the two perspectives

There is no space here to report on the complete analysis. However, we provide the main parts to illustrate how our networking strategy was implemented.

At the beginning of the episode, Giovanni's words and gestures convey information at the locutionary level. Giovanni is trying to express his ideas on the behavior of the function for large *x*: his gestures and his words complement each other. The pointing gesture (line 1) specifies the reference of his words, making clear that he is referring to the straight line represented in the Cabri file. His gesture (line 3) complements the information given in words, by showing the behavior of the function when *x* is very large.

However, the gesture–speech analysis at the locutionary level has already been carried out by the APC team. The substantial novelty brought about by the coordinated analysis is constituted by considering the non-locutionary aspects. In line 4, the teacher is asking a question at the locutionary level, that is, he is asking whether the line and the function will meet. But at a non-locutionary level:

– his words have a challenging connotation,
– his gesture illocutionarily suggests that he is thinking about a vertical line (right forefinger in screenshot 4a).

line 4 Speech Gestures

Locutionary level 4 T: <u>will they meet each other</u>?

(a)

(b)

Non-locutionary level *The sentence is spoken with challenging connotation (illocutionary).*

The vertical right forefinger suggests that the teacher is thinking about a vertical line, and referring to it in his question illocutionarily.

line 5	Speech	Gestures
Locutionary level	G: that is, yes, yes <u>they meet each other</u>	
Non-locutionary level	*The sentence is uttered with a connotation like: "Of course, they will meet" (illocutionary).* *The teacher's gesture (showing how the crossing point will look, directing Giovanni's answer) more or less forces Giovanni to approve the teacher's gesture (perlocutionary level).*	

If we consider the prosody in Giovanni's answer (line 5), we can hear that he feels very sure of his words (illocutionary level). At the same time, his gesture (two forefingers touching each other, screenshot 5) is completing (locutionarily) the verbal answer by expressing how the line and the function will meet: they will have a tangent point.

In line 5, while Giovanni is answering, the teacher keeps his gesture (screenshot 5, introduced in screenshot 4b). The gesture shows a configuration in which the function is crossing the line, thus suggesting an answer to the question he has just asked (perlocutionary level, Chap. 7).

The teacher continues keeping his gesture (line 6), until the student indicates agreement with him. Giovanni changes his gesture from touching the forefingers to crossing hands (gesture in line 7), deictically saying "it makes so." Locutionarily he shows how the graph of the function and the line meet. At the non-locutionary level his speech and gesture show that the student is trying to agree with or to follow the teacher's perspective.

This is the only case in the entire episode in which Giovanni shows a gesture similar to the teacher's. In all the other cases, Giovanni's gestures have very different configurations.

In the following lines (8–12), the teacher's gestures illustrate the graphical situation that he is speaking about, thus complementing at the locutionary level his verbal utterances. However, the constant presence of the right finger or hand kept vertical constitutes a catchment (in the sense of McNeill 2005) and at the illocutionary level it tells us that the vertical "barrier" is crucial in his argumentation all the time. Note that the barrier is mentioned locutionarily in an explicit way in the speech only later (line 14).

Finally, the teacher's last gesture consists of his right hand moving repeatedly rightwards (screenshot 16c). This movement is not only depicting a graphical situation in an iconic way, but also at a non-locutionary level it is suggesting the answer ("infinite") to the student, which Giovanni takes up (line 17).

11.4.2 A Local Integration Based on the Epistemological Dimension

The joint analysis process produced an *integration at a methodological level* between the two theories, as shown in the tables above.

Furthermore, during this process, a new idea arose, consisting in hypothesizing the existence of an *epistemological gap* between teacher and student in the episode. The idea at this stage was just a sensitizing idea for the different epistemological viewpoints, and at the beginning was not clearly defined (rather it was quite fuzzy!), yet we felt that it helped to deepen our understanding. In order to clarify whether it could provide a suitable means for understanding the episode, we started on the one hand to apply it to the data, and on the other hand to frame it theoretically. Indeed, by applying it to the data und theorizing about it, we elaborated the epistemological gap concept and began to see it as being valuable. Data analysis and theoretical reflection mutually enriched each other in a nonlinear process, until a satisfactory understanding of the episode was reached and the epistemological dimension was theorized through two new constructs: the *epistemological view* and the *epistemological gap*. Their integration into both theoretical approaches provided a new, symmetrical case of local integration of common new theoretical constructs (see Chap. 8).

As a starting point, we elaborated a working definition for the epistemological gap – which was new to both of the theories – and through a spiral process we checked it against the data and theorized about that. Since space is insufficient here to present the entire process, we now present our final definition, and apply it to the data analysis. The notion of epistemological gap is based on two domain-specific concepts: the “personal epistemology” and the “epistemological view” of mathematics.

Personal epistemology has been described in the literature as a theory-like background view that an individual holds about the nature of knowledge and the nature of knowing (Feucht and Bendixen 2010, p. 10 ff.; Lising and Elby 2004). The *nature of knowledge* encompasses aspects of certainty (stable–fragile) and simplicity (simple–complex); and the *nature of knowing* specifies the kind of justification and sources that are taken as legitimate in the specific domain (Hofer and Pintrich 1997). Someone’s personal epistemology can be different in different domains (Lising and Elby 2004) and can be regarded as part of the belief system of an individual that influences learning processes. Hofer and Pintrich (1997) have shown that personal epistemology is not stable over one’s lifetime, and that it develops in a domain-specific way.

Considering that we are analyzing a teaching–learning situation with social interactions constituting an epistemic process towards solution of a mathematical problem, we can adapt the given definition in order to define a student’s or teacher’s *personal epistemology of mathematics* as a theory-like background view that the student or teacher holds about the nature of mathematical knowledge, and the nature of knowing it. In the mathematics classroom, the personal epistemologies of students and teacher are influential. In addition, the teacher's *personal epistemology towards mathematics teaching and learning* plays a crucial role. The importance of recognizing the professional knowledge of mathematics teachers has been highlighted for example by Ball and colleagues (Ball et al. 2008); our focus is more specifically on the epistemological dimension of professional knowledge as part of the teacher's personal epistemology.

Boaler and Greeno (2000) have shown that personal epistemologies of learners depend on the kind of epistemic climate they experience in the class. Because of this influence, we may assume that personal epistemology becomes partly visible in processes of knowledge construction. What is taken as legitimate knowledge and knowing in a specific task is determined by one's personal epistemology and at the same time by the affordances and aims of the task, the social and instructional environment, the tools available, and the development of the current learning process. Thus, when faced with a mathematical task, students base their actions on their personal epistemologies towards mathematics, and through the process of working with the task they build and develop their *epistemological view*. In other words, we call the collection of aspects of the nature of what is taken as mathematical knowledge and as legitimate knowing in mathematics the *epistemological view* in a specific mathematical task. These epistemological views develop over time in the learning process and have an impact back on the personal epistemologies, which change more slowly.

Hence, the epistemological view of a *student* is individual, locally dependent on the current mathematical task situation, and is shown through *epistemic and semiotic actions* and within social interactions. It is not static; on the contrary, it can be enriched and widened within the process of working with the task.

Due to his professional knowledge about teaching and learning mathematics, the *teacher* may build several possible epistemological views on the same situation within the task. These views may anticipate the students' views and they are dependent on the teacher's personal epistemology as well as on many didactical variables, such as the students' age, knowledge, ability, the curriculum, the tools available, the processes the students are familiar with, and so on.

After this necessary theoretical digression, we now come back to the episode, apply the introduced notions, and define the *epistemological gap*.

From the point of view of a researcher, the student's and teacher's epistemological views are only accessible through their semiotic productions and epistemic actions (see, e.g., Chap. 12). They are revealed by including the non-locutionary dimension of the semiotic resources, which we developed in our coordinated analysis (see Sect. 11.4.1). Through this analysis of the video data, we identified that there was a gap between the teacher's and the students' epistemological views. We will call it an *epistemological gap* and now explain this notion in greater detail.

In the first part of the excerpt, Giovanni is trying to express his interpretation of the exponential function in the case of large x (line 1). The teacher interrupts the student, prompts him with questions, and does not allow him to properly complete his argument (lines 2–7). Then the teacher performs a semiotic game articulated in a tuning in words and a dissonance in gestures (lines 6–8): the teacher is using gestures to focus on the possibility of vertical asymptotes to the exponential function. This semiotic game is different from "standard" ones. In particular, there are two main differences:

1. The teacher tunes with the student's words and uses gestures to express further meanings (whereas, usually, it is the other way round: the tuning is with gestures, and words are used to better articulate meanings).
2. The teacher does not repeat the words exactly, but rephrases them, by inserting the word "vertical" (whereas, usually, one semiotic resource is repeated as it is expressed by the student).

This refined analysis suggests that, in the teacher's interpretation, Giovanni is referring to a vertical line that the exponential function is crossing. However, from the student's semiotic resources we get no hints that Giovanni is thinking of a vertical line as an asymptote. In fact, he is deictically referring (with both speech and gesture) to the screen images, which show the exponential function and its secant line (line 3). Using the notions discussed above, we can say that the student and the teacher are showing two different epistemological views of the same situation, and therefore an epistemological gap is apparent:

- Giovanni is concerned with what happens for "very big x" (line 11), but he is relying on the *visual perception* of the exponential function graph, which is provided by the software.
- The teacher is focusing on *mathematical properties* of the exponential function, in particular to its lack of a vertical asymptote.

Giovanni is building his epistemological view about what happens to exponential functions for large x on the experience of exploring them with the digital learning environment. The nature of Giovanni's *knowledge* is therefore strongly experiential and empirical, since the computer shows empirical facts and images. This background knowledge is enriched by experiential knowledge developed through exploring the graph of exponential functions with the computer, extrapolating what he observes, and using what he knows about the construction of the objects on the screen. In the specific environment, the tangent was constructed by approaching a specific point on the graph using secants, and results can be seen on the computer screen. They are gained by extrapolating what is observed for very large x. The task is interpreted as getting a description about what can happen for very large x based on what can be seen. Also the source of *knowing* is based on experiences with the slope of the exponential function: the justification refers to what is explored and is visible on the screen. Terms such as "approximating" are used by the student intuitively and informally.

On the other hand, the teacher refers to more theoretical knowledge from logic and limits as analytical concepts of calculus. He is basing his argument on his mathematical knowledge of exponential functions, and more generally of functions and limits. Therefore when he speaks of "approaching," he is using it in a way coherent with the theory about formal limits. Since "almost" does not exist in this theory, he overlooks it. Also the teacher's nature of *knowing* differs from that of the student, since the teacher is referring to a proof by contradiction, which is part of the official mathematical epistemology.

This analysis has shown a local integration of the new concept of epistemological gap which is the product of a long period of networking efforts. However, there are two further steps of the networking methodology still to be carried out: elaborating the status of the new concept within the two theories and reflecting on the networking process itself.

11.4.3 Including the Epistemological Dimension into APC and IDS

By introducing the new epistemological dimensions, we could explain the phenomena that we identified within the APC-approach and the IDS-theory: the failure of the semiotic game, and the drying up of the interest-dense situation.

How can the epistemological dimension be related to the semiotic game? Let us consider the final part of the episode. The teacher and the student are performing different gestures: for example, in screenshot 11 the student's hand is moving upwards, to indicate large values of the function, whereas in the gesture in screenshot 12, the teacher's hand is moving rightwards. We interpret the dissonance – we could say this semiotic gap – in gesture as a signal that the teacher and the student have different epistemological views: the teacher's hand goes *to the right*, based on the fact that, being defined for every x, the exponential function cannot stop (screenshot 16c). Giovanni moves his hand *upwards* (gestures in screenshots 11, 13a, b, c): these gestures (and in particular their location) suggest that he is considering the points on the graph, without stressing the distinction between x-values and y-values, as the teacher does.

The teacher's reference to a vertical line, which is a key part of his argument, is firstly introduced through his extended forefinger (screenshot 4a), and then made more explicit through words ("vertical straight line," line 8) and a whole hand gesture showing a "barrier" (screenshots 10–12). The word "barrier" is finally uttered in line 14.

Line 14 was crucial for us in identifying the epistemological gap: in this line the teacher is starting his argumentation as a proof by contradiction, and at the same time he is telling the student (both at a locutionary and an illocutionary level) that he should not trust completely the images on the screen, and rather should follow his argumentation by imagining.

Basing on his *personal epistemology towards mathematics teaching and learning*, the teacher has to contradict the student's epistemological view. In fact, we know from the teacher's interview (Sect. 2.2.2) that as a teacher he tries to work within a zone of proximal development for the students (Vygotsky 1978). To do so, he uses different kinds of semiotic resources, including speech and gesture. Sometimes he tries to tune with those of the students in order to support them (as in the case of semiotic games); other times he introduces new words or gestures to offer the students means to enter into his epistemological view, as he is doing in line 14 with the word "imagine" and the configuration he represents with gestures. However, to be successful these didactic actions require that the teacher's and the student's epistemological views are close. If there is an *epistemological gap* between teacher and student, we can hypothesize that this gap prevents the semiotic game and more generally the semiotic interaction from working successfully. And, this way, the teacher is not addressing the zone of proximal development of the students.

Taking the concept of epistemological view into the IDS-analysis, we are able to explain why an interest-dense situation does not emerge. In the episodes before this extra video, the students have built some knowledge through gathering and connecting mathematical meanings and structure-seeing based on their interpretations of what they have experienced with the computer. Since the tasks previous to this one have been implemented by the teacher, Giovanni takes the experiences and visual representations on the computer screen as a legitimate source for argumentation, as has been accepted before. This interpretation can be supported by Giovanni's behavior in sticking to his own train of thought and referring to the image on the screen (lines 3, 5, 7, and 11). Thus, Giovanni shows an experiential epistemological view based on visual representations. The teacher seems to be aware of this view because he explicitly rejects the students' epistemological view as a legitimate source for argumentation (line 14).

Already in line 10, the teacher has started an argumentation process that is not based on visual experiences but on a proposition that the teacher imputes to Giovanni (line 10). The teacher takes this proposition as a hypothesis that he disproves within social interactions by the use of theoretical knowledge about exponential functions. The term "approximation" is interpreted differently by the two. In Giovanni's view, approximation means coming near (line 11), but the teacher takes this term as a theoretical part of the proposition that he starts to disprove by a proof by contradiction. As the teacher deprives Giovanni of his visual argumentation base, he says "imagine that if you have $x = 100$ billions…" (line 14); in this way he offers Giovanni imagination as a source for a legitimate argumentation that is different from the visualizations on the screen. However, for Giovanni, imagination separated from visual perception does not provide suitable arguments. We observe an epistemic situation with an epistemological gap that the social interactions do not bridge. The student adheres to his epistemological view and refuses to follow the teacher. The teacher does not accept the student's epistemological view as valid. Since the two epistemological views are not compatible and the student does not yet have access to the teacher's view, only the teacher could interact based on an epistemological view much closer to the student's one. Since this does not happen, in order to fulfil the teacher's expectations the student can only either stop interacting at all or reduce his participation.

In fact, Giovanni only partly fulfils the teacher's expectations. He does not fully give up his epistemological view because in line 19 he paraphrases the expected answer "infinite" into "infinite points". This is an indicator that Giovanni thinks of the points on the graph that grow towards infinity but not of the x-values as the teacher refers to them. Since the teacher's personal epistemology does not allow acceptance of the student's epistemological view as legitimate in the task at hand, and the student does not have enough means to enter into the teacher's epistemological view, he cannot get involved in the social interaction about the task deeply enough. Therefore the interest-dense situation cannot develop suitably.

Theoretically, we can say that epistemic actions emerging in social interactions are based on the epistemological view of the interlocutors. When an epistemological gap between students and teacher occurs, the epistemic process can only proceed and the emergence of interest density can only be supported if this gap is bridged, either by the teacher or by the students.

11.5 Reflection and Conclusions

Going from a coordination of two complementary analyses to a local integration of a new theoretical construct into both approaches was possible for this networking process since IDS and APC share many *common features*:

- The view on data: in both cases, the data concerns processes of teaching and learning mathematics in regular classrooms. Even if we investigate them in teaching experiments we assume that the students act according their everyday practices (methodology).
- The unit of analysis: both theories use micro-perspectives taking into account every single utterance or semiotic action (methodology).
- Both approaches are "transformation oriented" in the following sense. Ulich (1976) distinguishes between two paradigms: *stability-oriented* and *transformation-oriented*. In a stability-oriented paradigm, the objects of investigation are of a stable nature and can therefore be investigated separately from their constitution. In a transformation-oriented paradigm, objects are regarded as dependent on their constitution, and they can only be investigated looking at their processes of creation. Results in a transformation-oriented paradigm are, for example, patterns of constitution. Our paradigms are transformation-oriented, since we look at changes and are interested in the patterns of change. The epistemological gap is a pattern that is constituted within the current situation through teacher–student interaction.
- Both approaches focus on the students' actions and interactions with each other and the teacher, with respect to the evolution of their mathematical ideas.

Reflecting on the networking activity carried out in this case study, we can refer to Radford's quadruplet (Radford 2008, 2012) [(P, M, Q), R]. The separate analyses done by means of the IDS- and APC-theories (P1, M1, Q1) and (P2, M2, Q2) brought similar results to the fore: learning was not successful since the emergence of interest density was interrupted (R1: result 1) and the semiotic game was different from those in the successful cases (R2: result 2). By contrasting the two theories in the analyses, the idea of an epistemological gap appeared. The analyses seemed to complement each other. Therefore, we worked out a common coordinated analysis, locally integrating tools of the two theoretical approaches on the methodological level (M), that is, *micro-analysis of the video encompassing both verbal and non-verbal dimensions, and locutionary and non-locutionary ones*. A more consistent understanding

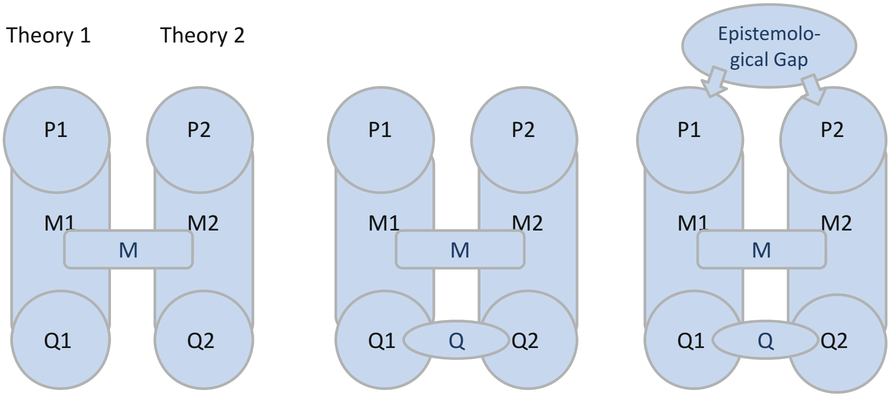

Fig. 11.3 Using Radford's (2008) categories to describe the process from coordination to local integration [P = Principles; M = Methodology; Q = Questions]

of the teacher–student interaction came out of this, indicating that the explanatory power that was lacking in our separate analyses could be provided in the common analysis by answering the common question Q, why the construction of knowledge was not successful. This made us include an epistemological dimension of the activity with the conjecture that there seemed to be an epistemological gap between the teacher's and the students' behavior in their interaction (Fig. 11.3).

Starting from a working definition about what is meant by the epistemological gap, a common re-analysis of the data was worked out that answered our question. The explanatory power of the new concept on the one hand became a common result R (Radford 2012) of the networking process and on the other hand initiated a spiral process of mutually improving the theoretical understanding of the concept of the epistemological gap and the empirical understanding of the extra video.

Finally, it was theoretically investigated how this new concept fits into the two sets of principles. In fact both theories carefully analyze relationships between different aspects of students' and teacher's actions and productions in the classroom: IDS considers mainly the different levels of discourse (locutionary, illocutionary and perlocutionary), while APC studies the relationships between the different semiotic productions in the classroom through the semiotic bundle lens. Both theories consider the dynamical and reciprocal evolution in time of such components and point out their possible convergence in dramatic moments, when they deeply interact possibly producing new knowledge: for example, when the situation becomes highly interest-dense or when a semiotic game is successful.

The joint analysis through both theories underlines that the dynamics of the construction of new knowledge can be successful provided the different discursive and semiotic components synchronize and converge. We can use a metric metaphor to describe this process: it is as though the mutual "distances" between the different semiotic and discursive components diminished more and more. This can happen if the students are in a zone of proximal development with respect to the piece of knowledge to be built. In such cases the actions and productions of teacher and students converge towards a shared knowledge, which is built up through a progressively shared epistemological basis. But sometimes this convergence process does not happen, for example in cases when a common epistemological basis is missing. Then there is an epistemological gap between the actors, and the process of building new knowledge is broken. As already pointed out, such a gap can be properly grasped only through the coordinated analysis of the two approaches, namely through extending the discursive analysis of IDS towards the APC model.

References

Arzarello, F., Paola, D., Robutti, O., & Sabena, C. (2009). Gestures as semiotic resources in the mathematics classroom. *Educational Studies in Mathematics, 70*(2), 97–109.

Arzarello, F., Bikner-Ahsbahs, A., & Sabena, C. (2010). Complementary networking: Enriching understanding. In *Proceedings of the 6th Congress of the European Society for Research in Mathematics Education* (pp. 1545–1554), January 28th–February 1st 2009, Lyon France. © INRP 2010. www.inrp.fr/editions/cerme6. Accessed 28 Nov 2013.

Austin, J. L. (1975). *How to do things with words: The William James Lectures delivered at Harvard University in 1955* (2nd ed., J. O. Urmson & M. Sbisa, Eds.). Oxford: Clarendon Press.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*(5), 389–407.

Beck, C., & Maier, H. (1994). Mathematikdidaktik als Textwissenschaft. Zum Status von Texten als Grundlage empirischer mathematikdidaktischer Forschung. *Journal für Mathematik-Didaktik, 15*(1–2), 35–78.

Bikner-Ahsbahs, A. (2005). *Mathematikinteresse zwischen Subjekt und Situation. Theorie interessendichter Situationen – Baustein für eine mathematikdidaktische Interessentheorie*. Hildesheim/Berlin: Franzbecker.

Boaler, J., & Greeno, J. G. (2000). Identity, agency, and knowing in mathematics worlds. In J. Boaler (Ed.), *Multiple perspectives in mathematics teaching and learning* (pp. 171–198). Westport: Ablix.

Feucht, F., & Bendixen, L. D. (2010). Personal epistemology in the classroom: A welcome and guide for the reader. In L. D. Bendixen & F. C. Feucht (Eds.), *Personal epistemology in the classroom. Theory, research and implications for practice* (pp. 3–28). Cambridge: Cambridge University Press.

Hofer, B., & Pintrich, P. R. (1997). The development of epistemological theories: Beliefs about knowledge and knowing and their relation to learning. *Review of Educational Research, 67*, 88–140.

Lising, L., & Elby, A. (2004). The impact of epistemology on learning: A case study from introductory physics. *American Journal of Physics, 73*(4), 372–382.

McNeill, D. (1992). *Hand and mind: What gestures reveal about thought*. Chicago: University of Chicago Press.

McNeill, D. (2005). *Gesture and thought*. Chicago: University of Chicago Press.

Radford, L. (2008). Connecting theories in mathematics education: Challenges and possibilities. *Zentralblatt für Didaktik der Mathematik, 40*(2), 317–327.

Radford, L. (2012, May 31–June 1). *On the growth and transformation of mathematics education theories.* Paper presented at the International colloquium: The didactics of mathematics: Approaches and issues. A hommage to Michèle Artigue. http://www.laurentian.ca/educ/lradford. Accessed 28 Nov 2013.

Ulich, D. (1976). *Pädagogische Interaktion. Theorien erzieherischen Handelns und sozialen Lernens*. Weinheim/Basel: Beltz.

Vygotsky, L. S. (1978). *Mind in society. The development of higher psychological processes* (M. Cole, V. John-Steiner, S. Scribner, & E. Souberman, Eds.). Cambridge, MA/London: Harvard University Press.

Chapter 12
Topaze Effect: A Case Study on Networking of IDS and TDS

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Abstract The case study of Topaze effect shows a networking practice of connecting two theoretical approaches, TSD and IDS. It investigates empirically two phenomena, Topaze effect and funnel pattern, of the two theories and networks the theories by comparing and contrasting these phenomena including also the semiotic game phenomenon. This process leads to deepening the understanding of the strengths and blind spots of the two theories on the one hand and provides enriched insight into the character of the phenomena and their common idea on the other.

Keywords Networking of theories • Social interaction • Topaze effect

For the extra video on Task 3 (extensively discussed in Chap. 11), the European Networking Group also experienced another surprising effect: looking at the video, three research teams had initially identified diverging conceptualized phenomena, each well known in their respective perspectives: the TDS team identified a Topaze effect, the IDS team a funnel communication pattern, and the APC team a semiotic game. Each of these identifications valued the episode differently, positively for one

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of these perspectives and negatively for the two others. This contrast motivated the participating research teams to understand the situation more deeply. From that starting point, our inquiry led us to revisit the nature of these specific phenomena and the way research conceptualizes them. In this way, we were able to reveal unexpected complementarities between the three different perspectives. This research practice is here presented as a fourth case study on networking, mainly between IDS and TDS.[1]

12.1 Analyses from the TDS Perspective

After the initial identification of the empirical phenomenon as a Topaze effect, the TDS team considered carefully whether the video really showed such an instance. For presenting the answer, here we first explain the effect by describing its emergence in the TDS, common instances, and criteria of identification (Sect. 12.1.1). Then, we illustrate how we used this concept for interpreting the video. This work led us to question the concept of Topaze effect itself and its use in research. We finally add some reflections, and broaden the discussion by describing similar contexts and considering additional theoretical constructs that will be related to the Topaze effect.

12.1.1 Topaze Effect as a Construct in TDS

12.1.1.1 Emergence of the Topaze Effect

The Topaze effect emerged in the TDS (Brousseau 1997, p. 25) as one of the paradoxes attached to the central concept of didactical contract (see Chap. 4). A didactical contract, which is inherent in any didactical interaction, denotes the – only partly explicit – expectations of teachers and students regarding the mathematical knowledge at stake in their interactions. Such a didactical contract automatically generates paradoxical injunctions (Warfield 2006).

On the teacher's side:

> her [the teacher's] aim is to lead the student to learn and understand some concept, and her indication that that concept has been learned will be some set of behaviors on the part of the student, but anything that she does that aims directly at producing that set of behaviors deprives the student of the opportunity to learn the concept itself. (Warfield 2006, p. 35)

On the student's side:

> if he accepts that the contract requires the teacher to teach him everything, he doesn't establish anything for himself, so he doesn't learn any mathematics. On the other hand, if he refuses to accept any information from the teacher, the didactical relationship is broken and he can make no progress at all. In order to learn, he must accept the didactical relationship but consider it temporary and do his best to reject it. (ibid., p. 35)

[1] A summary of the chapter has been published in Haspekian et al. (2013).

Some effects of these paradoxical injunctions on the teachers' practices are identified by Brousseau (1997, pp. 25–27): the Topaze effect, but also the Jourdain effect, the improper use of analogy, or the meta-cognitive shift. Here we consider the Topaze effect.

12.1.1.2 What Is a Topaze Effect?

The Topaze effect (Brousseau 1997, p. 25) takes its name from the famous scene of Marcel Pagnol's play named *Topaze*. In this scene, a master called Topaze is dictating a text to a weak student. One of the sentences is: "Des moutons étaient dans un pré." In the French pronunciation, the orthographical signs for plural (the "s" of "moutons" and the "en" of "étaient") are silent. The student writes: "Des mouton était dans un pré," missing these signs. In his successive attempts at making the student correct his mistakes, Topaze, who cannot simply give the correct spelling of the words, uses different artifices to transform radically the way he pronounces these words: "*des moutonsses étai-hunt*…." Finally, when the student adds "s" and "en," it is reasonable to hypothesize that he eventually decoded the teacher's intention and hints, but did not really understand why these corrections were needed. However, Topaze and the student can maintain the fiction that the student actually understood.

In coherence with this description, the Topaze effect denotes the following: the teacher poses a question, expecting a particular answer that she does not obtain; the teacher tries to get it by posing easier and easier questions. If along the process, the target knowledge disappears completely or nearly completely (as this knowledge is no longer necessary to answer the teacher's questions), we have a Topaze effect. One of its characteristics is that the didactical contract is not apparently broken: the expected answer has been produced and the teacher did not give the answer. However, the knowledge necessary to produce this answer is not identical to the knowledge which the teacher aimed at originally.

12.1.1.3 Common Instances of Topaze Effects

The paradoxes of the didactical contract and their effects become especially visible when teachers move from pedagogies based on knowledge transmission towards those inspired by constructivism. Within the constraints imposed, particularly time constraints, designing tasks whose solving allows the students to produce the target knowledge through a-didactic interaction with a "milieu" (cf. Chap. 4) is difficult. When these conditions are not fulfilled, which happens frequently, maintaining the didactical contract and the fiction of a pure constructive pedagogy often leads to Topaze effects. Short-cuts between action and institutionalization, even in situations with evident a-didactic potential, can also lead to Topaze effects as shown for instance in Artigue (2009).

12.1.1.4 Criteria for Identifying Topaze Effects in Teacher–Students Interactions

Taking into account the definition above and the research literature, we articulate four criteria which could be used for identifying an instance of the Topaze effect:

(a) The teacher has a precise expectation in terms of students' answers.
(b) There is a substantial distance between the students' initial productions and utterances and these expectations.
(c) One can observe a succession of questions or dialogue piloted by the teacher for obtaining the expected answer drastically reducing the mathematical meaning of it.
(d) When the expected answer is produced, the teacher tries to maintain the fiction that the answer is really significant and that the didactical contract has not been broken.

Considering the extra video on Task 3 with these criteria, a first glance led us to conjecture the existence of a Topaze effect.

12.1.2 Analysis of the Video: Questioning the Topaze Effect Interpretation

The systematic analysis of the video within the perspective described in Sect. 12.1.1 leads to questioning the affordance of the a-didactic milieu for answering the teacher's question at the beginning of this episode: "What happens to the exponential function for very large x?"

The affordances of the milieu are: the DGS files, the social interactions, the experience already gained by the students through their explorations with this digital media, and its results. These experiences are expressed in the students' written report:

"Being an exponential function, it increases and increases more and more at the infinite.

- For Δx very small, that is taking a "piece" of graph smaller and smaller, we can approximate the blue line and its behavior: for small Δx we can approximate the curve of the graph to many lines that touch each other in a point of always increasing slope.
- The more "a" becomes big, the more the blue line increases its slope, since for a same space Δx, Δy increases more and more.
- The slope of the tangent line of an exponential function increases more and more, as x increases, since it is tangent to a curve that increases more and more to infinity.
- On the contrary for "a" < 1 the graph decreases less and less and hence the tangent line will have a negative slope but always smaller." (cf. Chap. 2, Fig. 2.4, Task 3)

Considering these elements, and the analysis already developed of the first video on Tasks 1 and 2, we hypothesize from a priori analysis that interaction with the teacher can reasonably lead to insight into the following asymptotic behavior of the exponential function: this is an increasing function which "tends towards infinity" when x increases towards infinity, and whose increasing slope also "tends towards infinity." We also hypothesize that the connection between this type of behavior and the phenomenon that the graph is becoming more and more vertical for increasing values of x will require the teacher's help for becoming accessible to the students.

What happens in the actual interaction? Giovanni's first utterance is a bit distant from these expectations: "but always for *a* very big this straight line, when they meet each other, there it is again… that is it approximates the, the function very well, because…" (see Appendix, transcript of the extra video, line 1). What attracts his attention seems indeed to be the quality of the approximation provided by the "secant" (meaning a locally nearly tangent) for large values of x. Perceptively, the secant and the curve seem to go on together after the contact points (which is not the case for smaller values of x where the secant and the curve progressively separate). In fact, the exponential function being convex, after the second point of intersection of abscissa $x+\Delta x$, the curve must stay above the secant. Thus if the secant looks nearly vertical, the curve and the secant necessarily look glued together. From that, the students' erroneous impression results in that the affine approximation has a global character and not just a local one. Changing the scale or reasoning on the respective rates of change of affine and exponential functions may lead the students to see that this interpretation is invalid.

Considering the teacher's habits, he rather strangely chooses another way: he interrupts Giovanni who was at the point of developing an explanation ("because…") and asks him: "what straight line, sorry?" He then expresses doubts about the possibility of intersection (line 4), uses the expression "vertical straight line" not used by Giovanni himself (line 8), insists on the fixed abscissa of the line (line 8), and then addresses Carlo (line 10): "He [G] was saying that this vertical straight line approximates very well the exponential function." This succession of verbal exchanges and accompanying gestures lead us to think that, for him, the line shown by Giovanni is no longer a secant but a vertical line, and that he interprets Giovanni's utterances as the claim that a vertical straight line can approximate very well an exponential function, a claim that he cannot accept and thus tries to invalidate.

Why such a reaction? Our interpretation is that his expertise in the educational use of ICT makes the teacher especially sensitive to Giovanni's first utterance, relating it to common mistakes made by students in such environments due to the fact that the graphical representation of a function with a sufficiently big slope cannot be perceptively distinguished from a vertical line. Teacher experts in ICT use know that students do not necessarily see the contradiction between this perceptual evidence and the fact that a function defined for every real number cannot have a vertical asymptote or no part of its graphical representation can be a vertical line.

Many characteristics of the observed interaction indicate that, being used to facing such mistakes, the teacher has developed specific techniques to manage them, and that these techniques rely on the potential of semiotic games (as shown in Chap. 3). He does not make particular efforts to elucidate what this idea of good approximation means for Giovanni. Regardless of what Giovanni's perception of the curve is, he wants to convince the students that the exponential curve cannot have a vertical approximation because it must cross any vertical line.[2] When Giovanni, in defending his vision regarding the approximation, says (line 13): "that is, if the function increases more and more, more and more, then it also becomes **almost**[3] a vertical straight line," the teacher replies (line 14): "eh, this is what seems to you by looking at; but imagine that if you have $x = 100$ billions, there is this barrier…is it overcome sooner or later, or not?", introducing the word "barrier" and an extraordinarily large value "100 billions." After getting Giovanni's agreement, he insists (line 16): "when it is overcome, this x 100 billions, how many x do you still have at disposal, after 100 billions?", until reaching the conclusion that the exponential function will go on for infinite values of x beyond this barrier.

All along the exchange, we see strong connections with gestures, which is typical for semiotic games: the teacher passes from pointing to crossing the two forefingers when he expresses doubts about the meeting of the curve and the line; a gesture insists on the fixed value of the abscissa for the vertical line when his left hand moves vertically; his right hand moves rightwards when saying that the exponential function later goes on, simulating a barrier with his right hand…. Nevertheless, the coherence between students' and teacher's gestures which is characteristic of such games (cf. Chap. 11) is lacking here. The teacher's gestures seem to be more useful for supporting a pre-determined argumentation than to help Giovanni elaborate his own views and gestures. Giovanni continues to defend his vision, for instance when he says (line 5): "yes, yes they meet each other" or resists[4] giving up his vision, which can be seen in his answer in line 13 already quoted above. Eventually, though, Giovanni seems to give up.

Can we interpret this episode as a Topaze effect? Let us come back to the criteria articulated above. We do not exactly know the answer(s) the teacher expected when raising the initial question, but this is not the most important point. The Topaze effect, if there is one, is not linked to this answer but to Giovanni's first utterance interpreted by the teacher in terms of a vertical asymptote. Undoubtedly, the teacher has a precise expectation regarding this point: he wants the two students to reject this claim. If we consider the second criterion, there is substantial distance between Giovanni's initial utterance and the teacher's expectation, but the exact meaning of this utterance is not clear. Regarding the third criterion, the succession of questions or dialogue is piloted by the teacher. Is it a clear Topaze pattern, progressively reducing drastically the mathematical substance of the answers? Undoubtedly, the teacher retains the full mathematics responsability and strongly orientates the students' answers. From the beginning of the episode, he imposes his interpretation

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[2] This connection is not obvious as explained above, all the more so as Giovanni did not say that the line or the curve were vertical, but it is not discussed at all in this episode.

[3] Emphasis added by us.

[4] We can speak about resistance, but if we take for granted that the teacher, through connotation, challenges the students, there is no resistance: Giovanni takes over the challenge and gives an answer.

of Giovanni's first utterance and develops an argumentation which the students have to adhere to rather than contribute to. His reaction when Giovanni tries to explain his position again (line 13) shows this very clearly: "eh, this is what it seems to you by looking at; but…" (line 14) and he introduces the 100 billion value for x. At the end, Giovanni's contribution is reduced to few words: "yes" (line 15), "infinite" (line 17), "infinite points" (line 19), directly induced by the teacher's discourse: "Is [this barrier] overcome sooner or later?" (line 14), "how many x do you still have at disposal, after 100 billions?" (line 16), "how much can you go ahead after 100 billions?" (line 18). The episode ends by the following sentence of the teacher: "then the exponential function goes ahead on its own, doesn't it?" (line 20), a formulation that goes beyond what has been already said, expressing in an anthropomorphic way the fact that the line and the curve must separate, and once again looking for the agreement of the students. This seems more important than sustaining the fiction that the answer is significant and that the didactical contract has not been broken. However, obtaining the agreement of the students is also a way of keeping the didactical contract alive.

For all these reasons, this episode has many more facets which cannot be easily assigned to a Topaze effect. Of course some characteristics of the Topaze effect are present:

- The teacher takes the mathematics responsibility, and he sets the mathematical goal that he wants students to reach.
- Students' utterances are not really considered for themselves but only up to the point to which they contribute to the teacher's goal.
- The last utterances (after line 14) are rather typical of the linguistic format of the Topaze effect, with one student eventually saying the word expected by the teacher, directly prompted by his question.

However, in this episode, the teacher does not hide either his expectations or his arguments from the students. In the last utterances, Giovanni's contribution drastically reduces, but not necessarily because he cannot contribute more to the mathematical exchange. Instead, he seems more to give up and play the minimal role he may play for not breaking the didactical relationship. Everything happens as if Giovanni was *"playing" the role he has to play in a typical Topaze scene*, without being personally convinced. From the beginning, he expresses his vision confidently. When he gives up we cannot be sure that he has been convinced that he is wrong.

For all these reasons, even if some criteria of a Topaze effect are fulfilled, it would not be appropriate to interpret this episode as a Topaze effect. It is interesting to mention a similar conclusion given in another case (Artigue 2009). That article analyzed an Italian classroom episode and raised the issue of the dependence of effects such as the Topaze effect upon more global characteristics of the classroom culture and associated patterns of interaction. But we also point out that this episode cannot be considered as an illustration of the potential of semiotic games as described in Chap.11.

12.1.2.1 Comment by the APC Team

In fact, a semiotic game can be described as follows, considering the same structure of criteria used for the Topaze effect. This comparison will make evident the major differences between the two didactical phenomena, as well as the difference between a genuine semiotic game and what happened in the extra video on Task 3:

(a) The teacher has a precise expectation regarding the mathematical reference in the students' answers.
(b) The teacher is observing that the students' initial semiotic productions are close to the expected mathematical productions: this proximity is an indicator for the teacher that the student is entering a zone of proximal development for the concept at stake.
(c) There is a student–teacher interaction where typically the teacher "lends" the student the right words and/or signs to express verbally and/or symbolically what he judges to be on the way to the right answer considering the students' semiotic productions. To stress the correctness of students' answers, he echoes students' non-verbal productions: hence he takes the responsibility of a multimodal production, which supports the students to formulate what they were grasping in fuzzy and imprecise language in more proper mathematical language. In doing so, the mathematical meaning of the students' answer increases. The students are put in the condition of being able to express it in the shared scientific language.
(d) When the expected answer is produced, the teacher underlines that the answer is really significant and that the didactical contract has been fulfilled.

Comparing this characterization of the semiotic game with that of the Topaze effect, it is possible to see how both start from the same assumption about the teacher (a), but then follow a parallel but very different path (b, c, d).

The previous discussion of the video points out that there are some characteristics which suggest a semiotic game. Namely, the teacher takes the mathematical responsibility and makes a multimodal production. However, this production is not based on the production of the student: Giovanni's gestures are not echoed by the teacher. Instead, he produces a different gesture guided by his idea about Giovanni's possible mistake. The last exchanges are completely different from what happens in the final part of a semiotic game. Giovanni says the expected answer to which the teacher guided him, according to the plot of the Topaze effect, and this is not a verbal description of what he had previously pictured through his gestures and utterances.

Similarly to what happens for the pseudo-Topaze effect illustrated above, we have here a sort of denaturated semiotic game.

There could be many reasons for the students and the teacher playing a pseudo-Topaze effect game and a pseudo-semiotic game (desire of avoiding a conflict, student/teacher personalities, classroom culture, etc.) so that one can hypothesize that this happens more frequently in the mathematics classroom than one might think. It would be very interesting to look for such phenomena in a more general way: possibly there is a sort of "comedy of errors" that may happen in the classroom behind such games, which are only partially described by the Topaze effect or the semiotic game.

12.1.3 Discussion: Back to the Construct of Topaze Effect

This analysis led us to deepen the reflection about the Topaze effect and, for that purpose, to look more systematically at how the literature handles it. Strangely, this did not turn out to be easy. Most often, when the existence of a Topaze effect was mentioned, it was without providing detailed elements of the episode, as if its identification was not problematic and not deserving detailed justification. We explain this situation, at least partly, by the fact that the idea of Topaze effect obtained a strong adhesion within the research community immediately after its introduction by Brousseau. Everyone recognized a familiar phenomenon and from that moment could label it. The Topaze effect naturalized too quickly in the sense that it was so directly adopted and integrated in didactic analyses that the notion remained somehow not sufficiently worked out.

Another point is that, in the few cases providing a detailed description, we hardly found the qualification of Topaze effect fully convincing. Without entering into details, let us illustrate this point. In the example given by Hersant (2004), the aim of the teacher is that students express the application of a percentage of increase (5 %) as a multiplication by a decimal number (0.05). Instead, she just gets as an answer the operation "÷ 20." She tries to get help from a pupil but quickly reduces her explanation to elucidating the way 0.05 can be obtained from the two numbers 100 and 5 without referring to the sense of the problem (deposit and interest). In this way, the exercise becomes purely formal. In this episode, the teacher has a clear expectation in terms of answer; the way the answer is produced and then shared in the classroom does not guarantee that the pupils have understood why the multiplication by 0.05 links the deposit and interest. The fiction of shared understanding which closes the episode creates thus a proximity to a Topaze effect. Nevertheless, we do not observe the progressive disappearance of knowledge under the pressure of the didactic contract characteristic of the Topaze effect.

This is also the case in most episodes mentioned by Novotna and Hospesova (2007, 2008) where teachers reduce the students' mathematical work to solving simple and isolated tasks. This certainly affects the meaning of mathematical knowledge for students, but the episodes do not show teachers facing the paradoxical injunctions of the didactic contract through an absence of answer or an unexpected answer. They are not progressively negotiating down the mathematical demand of the task for maintaining the fiction that the didactical game functions as it should do.

Our position is that such a broad extension of the applicability of the Topaze effect is not helpful. It could make us forget that the functioning of classrooms is based on joint action between students and teachers. As stressed by Sensevy (2012), if one essential characteristic of the didactic joint action is that it imposes some level of "didactic reticence"[5] on the teacher, effective joint action requires the teacher to regularly relax this reticence for making the interaction of the students with the a-didactic milieu productive. There is a permanent tension for the teacher between directly helping students in their difficulties with a situation (thus relaxing the didactic reticence) and not saying too much. In the language of the theory of joint action, Topaze effects correspond to situations when, for maintaining the didactical contract, the progressive relaxation of the didactic reticence destroys completely the epistemic game. In some sense, the distance of a particular episode involving such a relaxation to a Topaze effect measures up to what point this relaxation alters the epistemic game.

12.2 Analyses from an Interactionist Perspective

Whereas the TDS team at first identified a Topaze effect and then started to restrict this spontaneous conceptualization, the interactionist perspective adopted in the IDS approach had suggested the identification of a phenomenon that emerges through social interactions, namely a stepwise execution of a funnel communication pattern. This pattern produces a solution of a task while the teacher stepwise narrows his answer expectations to make the student express the expected one.

The more thorough analysis of the video in the TDS-approach (in Sect. 12.1) has shown that this episode does not fulfill all the criteria of the Topaze effect. In this section, we report on how the IDS team analyzed the video in order to find out if this implies that we do not observe a funnel pattern. Before presenting the answer to this question (in Sect. 12.2.2), we explain the theoretical approach in which the funnel pattern is embedded.

12.2.1 Patterns of Social Interactions

Based on empirical data of mathematics classrooms, Bauersfeld was the first to reconstruct a funnel communication pattern (Bauersfeld 1978, p. 162). The funnel communication pattern is a specific interaction pattern, and might explain how the Topaze effect emerges, becoming solidified step by step during a process of social interaction in a mathematics classroom. Bauersfeld calls the funnel pattern *narrowing*

[5] "didactic reticence" is the term used by Sensevy (2012) to characterize the teacher's tension that Brousseau (1997) describes in the didactical relation. The teacher cannot directly mention to the students the things he wants them learn (because of the hypothesis that the learning happens through an interaction with a milieu, putting the teacher in a tension of not saying directly what he could be tempted to say). This constraint for the teacher to remain silent on things he knows is called the "réticence didactique."

of actions by expected answer [our translation of "Handlungsverengung durch Antworterwartung"] (ibid., p. 162) and states that at any moment in the process, a change in this pattern would be possible but with reducing grades of freedom. The funnel pattern begins with an open question or task and is constituted through five steps (ibid., p. 162, our translation):

(a) The student does not recognize the mathematical operation or is not able to draw an adequate conclusion.
(b) The teacher asks an additional question but gets a false answer or does not get any answer.
(c) The teacher continues his effort to get at least part of the expected answer. Understanding is no longer approached.
(d) Missing the expected answer, the teacher tends to narrow his efforts, aiming at just what is expected being said, no matter who says it. Self-determined behavior of the students decreases and at the same time the situation becomes more and more emotionalized.
(e) The process is finished as soon as the answer occurs, no matter whether the student or the teacher has produced it.

According to Bauersfeld (1978), the teacher experiences the impact of interaction patterns only pre-consciously. In his prototypic example, the teacher wanted to adapt her assistance to the level of students' competencies. However, she overlooked that the main goal of learning mathematics insightfully could not be reached this way. In order to gain deeper insight, students would have to change their views in the direction of seeing a new mathematical structure. According to Bauersfeld, this insight could be initiated by a change of the didactical model or a change of means of description that provides access to building necessary knowledge for getting insights (1978, p. 166ff.). However, he leaves open how exactly such a change could be achieved.

The perspective of looking at social interactions in mathematics classrooms builds on the same background theory of social constructivism (cf. Jungwirth 2003) as the theory of interest-dense situations (cf. Chap. 7) but with the focus on how interaction patterns are constituted. The five steps that shape a funnel pattern cannot be understood as a result of cause and effect, they rather are constituted by zugzwangs during the interaction that are the results of mutually normative obligations (cf. Voigt 1983, p. 198) such as the teacher's obligation to make the student learn and the student's obligation to show having reached the goal. Voigt distinguishes *experience patterns* and *interaction patterns*. (Voigt 1983). For example, a teacher may observe a specific mistake such as *the sum of two fractions is built by adding the numerator and the denominator separately* that occurs repeatedly in different mathematics classrooms in similar ways. As a result, the teacher builds an *experience pattern* about this mistake that may direct his behavior whenever adding fractions is a topic in the class.

While experience patterns are built by individuals, an *interaction pattern* emerges in the social interaction of a group based on the individuals' experience patterns. An interaction pattern consists of successive actions that are guided and coordinated by zugzwangs and routine actions (Voigt 1983, p. 198). For example, a teacher may react to the beforementioned mistake of adding fractions by asking the whole class "does someone want to react?" Given that the class has experienced this teacher reaction whenever there has been a mistake, they take it as an indicator of being mistaken and are forced to express it even without knowing the right answer. If the students do not explain why, it is the teacher's turn to ask why there is a mistake. The individual experience patterns of the teacher and the students have guided their behavior. The teacher's first question forces the students to react; a zugzwang occurs. Shaped by two routine utterances (*does someone want to react* and *this is wrong*) and mutual obligations to act in a specific way, this social interaction reaches a point where the students are forced to reason on the mistake. An interaction pattern of handling a mistake is constituted. *Routine actions* have facilitated class interaction (ibid., p. 207).

A funnel pattern is a specific interaction pattern that is carried out step by step in the course of everyday action routines concerning interlocutors' perceptions, expectations, and interpretations. Methodically, interaction patterns can be reconstructed with the help of an interpretative approach, and, specifically, a sequential turn-by-turn analysis (Jungwirth 2003).

In an expectation recessive interaction structure (cf. Chap. 7) the teacher focuses on the students' line of thought, hence a funnel pattern can never occur within such an interaction structure and therefore cannot appear within interest-dense situations. As described in Chap. 11, the constitution of an interest-dense situation is not successful in the analyzed video. Is there a funnel pattern that may hinder its constitution?

12.2.2 Is the Video an Example of a Funnel Pattern?

The episode begins with the students' attempt to answer the question of what happens to the function for very large x. Giovanni seems to work out an answer in broken language, hence, while talking and gesturing the train of thought is worked out (line 1). This can be interpreted as a kind of exploration in which the language cannot be used precisely. However, Giovanni cannot finish his contribution because the teacher interrupts him (line 2). Since the teacher normally offers a lot of space for exploring (cf. Chap. 2), there must be a good reason for his behavior. He seems to have observed a potential mistake that he might have found in other classrooms: the mistake that students believe that the exponential function has a vertical asymptote. Therefore, he might have wanted to prevent the two students from going in the wrong direction. However, Giovanni resists by explaining that his train of thought follows the task "this here (pointing at the screen), for x very, very big" (line 3). Again the teacher focuses on what he thinks was the wrong point: graph and line meet and the line is observed as vertical. For him it seems that the student lacks insight into his mistake. In a challenging way the teacher asks, "will they meet each other" (line 4). The student tries to imagine the graph for very large x, connecting the idea of a large x with what he observes on the screen: there he perceives the approximation between the graph and the tangent line constructed as a secant between two narrow points. Giovanni describes this straight line as almost vertical (line 13). Following his own train of thought, he says, “that is, yes, yes they meet each other” (line 5), just as the inscription on the screen had been constructed. The teacher now crosses his fingers as a hint while saying, “but after their meeting, what happens?” (line 6). Giovanni does not seem to agree: “eh…eh, eh no…, it makes so” (line 7), imitating the finger crossing, but differently from the teacher: his hands touch while crossing fingers, referring to the idea of approximation as he has seen it on the screen (screenshot 7).

Thus, again, he follows his own line of thought referring to the visualization on the computer screen and the construction of the lines. The student interprets the situation according to what was visible on the screen whereas the teacher has a mistake in mind. The teacher starts with “ok,” meaning that he accepts the answer, and explains: “ah, ok, this then continues, this (showing the graph after the finger crossing with his hands), the vertical straight line, has a well fixed x, hasn’t it? The exponential function later goes on increasing the x, doesn’t it? Do you agree? Or not?” (line 8, that a vertical line has a fixed x whereas for the exponential function x can vary). The question, “Do you agree? Or not?” (line 8) constitutes a zugzwang for the student since both statements are true. If the student did not agree, he would show that he did not understand. Thus, Giovanni is obliged to say “yes” (line 9). Stating something that cannot be doubted is a routine action to get agreement on which further argumentation can be built. But “yes” does not mean that Giovanni agrees with the whole situation. Agreement is expected just for the different roles of the variable x and this cannot be withdrawn.

In line 10 the teacher expresses his interpretation of the student’s first utterance. At this point the experience patterns of the student and the teacher become clearer:

- *The teacher’s view including possible experience patterns*: The teacher normally would support the students’ thinking, but in this case he reacts differently. In his view he observes a wrong idea about the exponential function. *Straight line* is interpreted as vertical probably because the student points to the screen showing an almost vertical line. Since the teacher drops into the students’ exploration, he does not exactly know how the students have got the inscription on the screen. He might want to prevent the students from a wrong understanding about the slope of the exponential function and therefore insists on clarifying it. In line 10 the teacher formulates a proposition transforming Giovanni’s utterance into a statement, directing it as an explanation to Carlo.
- *The student’s view including possible experience patterns*: The computer shows a diagram in which the straight line is nearly a tangent to the graph of the exponential function; for very large x the slope increases. The student envisions the approximation at the tangent point and knows that this line is not vertical (line 13). The work with the computer serves as an argumentation base on which the teacher usually builds supporting the students’ thinking.

If we compare this with the situation that Bauersfeld describes, we observe a deep difference. The starting point is an open question, but the student here is deeply involved in getting insight. There is nothing wrong when the teacher interferes. The teacher seems to observe a mistake and therefore interrupts Giovanni. He is so focused on the supposed mistake that he is not able to listen exactly, but instead hears what is in his mind. This mistake might belong to an experience pattern of the teacher. The student does not try to decode the teacher's expectation. Since the teacher is acting steered by expectations and the student tries to follow his own way of thinking, we observe a conflicting situation (cf. Chap. 7, Table 7.1; cf. Bikner-Ahsbahs 2005, p. 192ff.). Therefore the beginning of the interaction is not a starting point for a funnel pattern. It is not even a starting point for an interaction pattern. It begins with a potential for an interest-dense situation – and maybe it would have become one if the teacher had built on the student's perception.

In line 11 Giovanni again legitimates his answer "for x that are very … very big," but does not refuse the term vertical which the teacher used in line 10. Giovanni still is involved in his own thinking but, with the following utterance of the teacher, Giovanni is somehow trapped: "for how big x? 100 billions? $x = 100$ billions?" (line 12). Giovanni's utterance becomes the starting point of staging an interaction as a proof by contradiction, that is, orchestrating the social interaction by attributing roles the teacher and the students have to play. The short question "how big" takes over the student's utterance. But this confuses Giovanni. The student again describes his view of the function and how it develops for large x: "because at a certain point…, that is, if the function increases more and more, more and more, then it also becomes almost a vertical straight line" (line 13). The teacher asks for a fixed point x but Giovanni's image about the graph of the exponential function together with a never-ending slope is more dynamic. Giovanni grasps something that is not yet part of his knowledge and thus cannot be expressed by words. The teacher interprets "almost vertical" as "approximating a vertical asymptote," which is not correct. Claiming that the screen offers a wrong impression, the teacher refuses the student's base of argumentation: "eh, this is what it seems to you by looking at" (line 14), and goes back taking a large, but fixed, value "$x = 100$ billions" as a starting point for building a proof by contradiction. He continues: "is it [this barrier] overcome sooner or later, or not?" (line 14). Giovanni is committed to agree because the teacher's statement is true, thus he confirms with a one-word sentence: "yes" (line 15). In the teacher's view the value of x ($x = 100$ billions) is only one example for a specific barrier. To clarify this crossing argument, he says, "when it is overcome, this x 100 billions, how many x do you still have at disposal, after 100 billions?" (line 16), and indicates with his hands that he expects "infinite" as an answer (line 17). That is just the answer of Giovanni but the connection with the original problem meanwhile is lost. The teacher confirms this answer by repeating "infinite" and asks how far the exponential function can reach according to the x-values: "and how much can you go ahead after 100 billion?" (line 18). However, the answer is "infinite points," which can be interpreted as points of the graph of the exponential function, illustrating it by an upward gesture. The teacher, however, interprets this answer as x-values that the exponential function may reach. Finally, the teacher indicates the separation of the graph from the original straight line.

Like for the Topaze effect, conceptualizing the phenomenon as a funnel pattern does not completely reflect the complexity of the situation. Since the original task has changed implicitly, Giovanni's resistance can be understood as keeping involved in the original task but the teacher seems to interpret his behavior as resistance to getting insight into the false view the teacher has identified. Therefore criteria (a) and (b) of the funnel pattern are not fulfilled. From line 14, two levels of actions are conducted. On the perlocutionary level (cf. Chaps. 7 and 11), the teacher's intention is to convince the student by a proof of contradiction which he does not expect the students to produce by themselves. Understanding is not basically approached (part of criterion c). On the illocutionary level, criterion (d) becomes fulfilled, but partly for different reasons: without an argumentation base, the self-determined behavior of the student is given up, as is shown by him producing the expected one-word agreement that the teacher tries to get by his narrow questions. In addition, the situation becomes emotionalized. From the teacher's view, criterion (e) is fulfilled as he himself finishes the proof and the student agrees, but the student's behavior indicates that he has not fully given up his own view.

We can also conclude that the teacher's experience pattern of a specific mistake attracts the teacher's attention by key words and makes the teacher blind to the students' views. Giovanni's experience pattern (saying that the teacher supports the students' way of arguing) empowers his effort to explain his own view. Since these two experience patterns do not fit, a conflicting situation occurs. The teacher seems to approach this conflict through previously proven routines, such as demanding confirmation of incontestable facts and disempowering the student by depriving him of his argumentation base. Therefore, this episode might be typical for correcting a mistake shaped by four partly overlapping phases: the teacher's identification of a mistake, parrying the student's approaches of justification, extracting the argumentation base from the student, and stage-managing a proof of contradiction to convince the student. Together, these partly result in a funnel pattern: the teacher offers statements or narrow questions with obviously clear answers that the student feels obliged to confirm by either saying yes or answering with one- or two-word sentences. Thus the emergence of an interest-dense situation is not disturbed by a funnel pattern – rather, the partly resulting funnel pattern is a consequence of the disruption of the emergence of an interest-dense situation.

12.3 Networking of the Approaches

12.3.1 TDS and IDS

The Topaze effect and the funnel pattern seem to describe a similar empirical phenomenon in which the difference between the teacher's specific content expectations and the students' possibilities to act mathematically cannot be bridged and therefore they interactively establish a process of keeping the didactical contract alive and fulfilling the teacher's expectation without producing mathematical insight. Both conceptualized phenomena describe a fiction about learning: on the surface the answer is produced, but this does not necessarily go hand in hand with students' insight into mathematics.

In this case, the two analyses converge in a refutation: the video neither shows a Topaze effect nor a funnel pattern. Instead, the two analyses clarify the complementary nature of the concepts: one perspective is able to describe aspects that the other perspective has left aside. The TDS-analysis classically highlights the epistemological view of the situation by referring to the potential of interaction with the milieu and the difficulties met by the teacher for maintaining the didactical contract when this potential is insufficient. In this case, however, the teacher suddenly no longer accepts the students' arguments because he considers them to be in conflict with the expected mathematical knowledge. Therefore, the teacher moves the didactical contract and tries to make the students enter a new game, that of a proof by contradiction. As Giovanni resists this change, the interaction does not constitute a Topaze effect. We observe a "split in two" of the initial situation (Comiti and Grenier 1995): the teacher and the student interact but they do not play the same game. This "split" apparently vanishes when Giovanni gives up, thus the alignment of Giovanni with the teacher gives the impression of a Topaze effect.

The interactionist perspective underlying the IDS approach shows how the specificity of the situation emerges step by step through social interactions constituting mathematical meanings that are not necessarily based on insight and how emotional experience is intensified in parallel. But this analysis also shows that the underlying crisis is not solved. The fiction that the teacher and the student share their views is covered, since the student does not really give up his view although he gives up his resistance. Here, the final steps of a funnel pattern are part of *socially staging a proof by contradiction* that itself is part of an interaction pattern of *socially correcting a mistake*.

At a more general level, the TDS view that the didactic contract must be kept even if it is just fictional offers a deeper reason why interaction patterns may occur: if the contract had to be kept no matter how, teacher and student could become unconfident and just try to keep their roles as teacher who is teaching and as student who is learning. A possible consequence might be that they carefully orient themselves according to the supposed expectation of the other, step by step producing what the teacher wants to hear. We observe two argumentation bases: on the one hand, the inscription on the computer screen and the work behind it; and, on the other hand, the teacher's background knowledge to which the student does not have access. When the student's argumentation base is withdrawn, the social contract that the teacher supports the student is broken. In order to keep the didactic relationship, the student gives up his resistance officially. Two levels of thinking occur: the surface or stage level on which the teacher and the student socially produce a "proof by contradiction"; and the backstage level on which the student keeps his view alive. Through acting on these two levels, the fiction of agreement can be kept.

The second task, "proving that the exponential function cannot be approximated by any vertical straight line," emerges within the situation. The milieu with which the students interact is not at all a support for producing the proof expected. Therefore the teacher has to create a fictive milieu through his actions, discourse, and gestures in order to reach a solution. From the interactionist view, this is done by an interaction pattern. In order to make learning more likely, the milieu would have to be changed. In saying that the teacher should change the didactical model, Bauersfeld (1978, p. 70) goes in the same direction, but his theoretical approach does not offer a frame for constructing such a milieu. However, this is just what TDS is able to offer, for instance by zooming in another screen that would provide evidence that the tangent line never can be vertical.

12.3.2 Comment from the ATD Approach

To go further in the networking of the approaches, the analysis proposed by ATD (cf. Chap. 5) can also be brought into the scene. The split of the situation pointed out in the previous TDS and IDS interpretations can be brought back to an initial characteristic (or even contradiction) of the teacher's didactic strategy, labeled as an "empiricist approach": by looking at the graph and interacting with it, students are supposed to be able to "read" some characteristics of exponential functions and formulate them as general properties of these functions. However, in the video, the asymptotic behavior of the function is not "visible" on the graph; it should be inferred by what is observed but cannot be "seen." While the students go on with the description of their observations following the didactic contract previously established in the class, the teacher moves to another milieu: the logical reasoning that is supposed to work against the very empirical evidence perceived on the graph. The media-milieu dialectic put into the scene to convey all the work (cf. Chap. 10) becomes problematic: how can the students know when the milieu provided by the dynamic geometry file is adequate and when it is not? How can the teacher convince them that what has been supporting the production of knowledge about exponential functions in previous learning situations (the description of what is "seen" on the DGS files) is no longer valid? The "empiricist principle" that seems to partially support the teacher's didactic strategy as was pointed out in the first ATD analysis (see Sect. 5.2) is here showing its limitations.

12.4 Reflection

What is the nature of the networking undertaken here? It seems close to the strategy "comparing and contrasting", but certainly more than "contrasting" took place. Comparing and contrasting the two analyses (from the TDS view and the social constructivist view) led to an awareness of the resonance between the two conceptualized phenomena on the one hand and their specificities on the other. It deepened the understanding of the strength of the epistemic position in TDS and showed that the analysis of interactions led by this epistemic view was less sensitive to other characteristics of the social interaction. On the other hand, the idea of the didactic contract that produces mutual obligations for social interaction, and the insight that an insufficient milieu forces the teacher to change the milieu within the situation, have both deepened the understanding of the episode from the interactionist view underlying the IDS approach. Contrasting both analyses showed how they could complement each other, one consolidating the other and contributing to a joint improved understanding of the episode, in this way providing a coherent picture of the situation.

In this sense, we experienced scientific progress, namely a step of developing theoretical understanding towards increasing explicitness of the theories' principles (Radford 2008) and improving connectivity (Bikner-Ahsbahs and Prediger 2010, p. 506). We now reflect on what we mean by "progressing," describing it based on the networking in this case.

12.4.1 Progressing by Working Out a More Mature and Dense Understanding of Our Own Home Theory

Often episodes in situations are labelled as a Topaze effect when forms of interaction between teacher and students progressively reduce the mathematical responsibility of students. However, such an interpretation can be an oversimplification. While initially interpreting the video in terms of the Topaze effect, we fell into such a trap. The necessity of overcoming the apparent contradiction between the proposed interpretations of the different research teams obliged us to go back to the foundational text by Brousseau, looking for further texts elaborating this notion and for examples of Topaze effects in the literature. In doing so, we discovered that this notion had "naturalized" very quickly, without any substantial work of elaboration. Moreover, when looking for examples in the literature, we observed a tendency to label very diverse forms of relaxation of the didactic reticence as Topaze effects. Networking thus helped the TDS team to become conscious of this premature "naturalization," which may constitute a barrier for researchers of other cultures but also for the advancement of our own research. Thus, we have been able to create a vision in terms of degree of proximity with the theoretical object by deconstructing and then reconstructing the Topaze effect as a theoretical construct and, in this way, increasing the Topaze effect's epistemological density through the process of networking. Contrasting it with the concept of a-didactic situations even clarified both roles in TDS as limit concepts, corresponding to extreme types. This idea of proximity also connects to the role ideal types have in the IDS methodology and also can be applied to using the funnel pattern as an analysis tool.

Concerning the interactionist view underlying the IDS approach, networking through comparing the analyses and the data showed the strong influence of previous experience on the behavior of the student and the teacher. Therefore, looking only at the current situation of social interaction as is commonly done in the pure interactionist approach may be superficial. Our analyses also yielded the reconstruction of an interaction pattern as a socially constructed proof by contradiction, which we have not been aware of before. If experienced repeatedly, this interaction pattern may become part of students' experience pattern supporting the development of competencies in proving. Therefore, interaction patterns are not always misleading, and might even be fruitful. Finally, through networking, different levels of acting became more explicit. Routine actions are actions on the stage level whereas additional views of insight supported by individual interest may stay on the backstage level not shown in social interactions.

12.4.2 Progressing by Focusing on Limitations of Our Home Theory

Through networking, limitations of our own theoretical background can become more apparent than in other research approaches. The work with alien data creates an antagonist milieu (in the sense of TDS, cf. Chap. 4) for the networking enterprise that gives rise to some resistance for the analyses. This work requires convincing and negotiating, which in turn demands more in-depth argumentation and theoretical foundation. For instance, the resistance of Giovanni, and what it reveals about the classroom culture and the place given to the students in it, can only be partially captured by the TDS constructs such as the a-didactic situation and the milieu, and even the didactic contract. The pure interactionist's lens does not focus on epistemology. Furthermore, this lens has no concept for the mathematical arrangement similar to the concept of milieu in TDS. Hence, in contrast to TDS, there is no tool that may suggest how to change the mathematical arrangement. Some of these limitations, however, have been partly overcome (Fetzer 2013; Steinbring 2005).

12.4.3 Progressing by Taking into Account Complementary Views on Data

The TDS team, while analyzing the video, could not totally explain why the situation does not degenerate into a complete Topaze effect. Working and discussing with the IDS team helped them to tackle the data from another perspective, linked here to the culture of the classroom and its social interactions, and made them pay more attention to emotive aspects. These aspects might explain why finally Giovanni gave up. TDS does not directly take them into account; however, such aspects are taken into account in the IDS perspective. The methodology of the interactionist approach helps in understanding the situational conditions, for instance by helping us to address the role of the experience pattern that partly predetermines how Giovanni is acting. In addition, the IDS approach is concerned with interest development. This helps us to understand that Giovanni is committed to understanding the mathematical situation, but that an interest-dense situation has not been established. But interest indicated at the beginning cannot vanish so quickly; therefore Giovanni is empowered to keep his view alive even if on the surface level he seems to give up.

12.4.4 Progressing by Comparing and Contrasting Notions That Are "Close" to Each Other

The networking process led to a comparison of the Topaze effect with the funnel pattern by looking more deeply at their characterizations. This made their complementarity apparent, especially in regard to the fact that both focus on the common idea of keeping the fiction of having taught and learnt mathematics. Through this comparison, some fundamental differences (in the principles as described by Radford 2008) between the two theories were identified. In the interactionist perspective, reasoning partially escapes the consciousness, as it is intertwined with routines and zugzwangs in patterns of interaction. Mathematical meaning is perceived as emergent through social interaction. Epistemological considerations are left aside. In the TDS, the starting point and its strength is epistemological even if interactions and their optimization are a central issue, too.

12.5 Conclusions

In the process of networking, gaining insight developed in different directions for the different researchers. However, the trajectories also point to a similarity: superficially, TDS researchers initially interpreted the situation as a Topaze effect; after a deep analysis, they came to the conclusion that there is a not complete Topaze effect, but rather a "split" in the situation. On the other side, IDS researchers initially thought of a funnel pattern, and after the analysis finally concluded with a restrictive funnel pattern. In these processes, *something* similar happened to both groups of theorists.

What is the nature of this similarity? Both theories tried to grasp their conceptualized phenomenon. Both terms describe limit concepts, corresponding to extreme types. The origin of the similarity is necessarily intrinsically linked to the data. Reflecting upon that, both groups of theorists arrived at the central idea of *fiction*. They tried to catch what seems important regarding their paradigms and principles as underlines Radford (2008). Through combining by cross-analyses, a common idea considered in the two theories could be identified: the fiction in the student–teacher exchange concerning the mathematics learnt. Both theories describe this fiction differently, one with the Topaze effect, the other one with the funnel pattern. We can learn from such work that there might be similar ideas in different theory cultures that are described differently because of the preferred focus each theory has. This leads to the question whether such a fiction belongs to teaching and learning as an intrinsic feature. In our opinion, such a fiction can exist to some degree each time the teacher feels obliged by the didactic contract to give substantial responsibility to the students in the production of new knowledge, because the milieu and forms of joint action do not necessarily allow this objective to be fulfilled.

References

Artigue, M. (coord.) (2009). *Integrative theoretical framework – Version C*. Deliverable 18, ReMath Project. http://remath.cti.gr. Accessed 29 November.

Bauersfeld, H. (1978). Kommunikationsmuster im Mathematikunterricht. Eine Analyse am Beispiel der Handlungsverengung durch Antworterwartung. In H. Bauersfeld (Ed.), *Fallstudien und Analysen zum Mathematikunterricht* (pp. 158–170). Hannover: Schroedel.

Bikner-Ahsbahs, A. (2005). *Mathematikinteresse zwischen Subjekt und Situation. Theorie interessendichter Situationen – Baustein für eine mathematikdidaktische Interessentheorie*. Hildesheim/Berlin: Franzbecker.

Bikner-Ahsbahs, A., & Prediger, S. (2010). Networking of theories – An approach for exploiting the diversity of theoretical approaches. In B. Sriraman & L. English (Eds.), *Theories of mathematics education: Seeking new frontiers* (Advances in mathematics education series, pp. 483–506). New York: Springer.

Brousseau, G. (1997). *Theory of didactical situations in mathematics* (Ed. & Trans., N. Balacheff, M. Cooper, R. Sutherland, & V. Warfield.) Dordrecht: Kluwer.

Comiti, C., & Grenier, D. (1995). Two examples of "split situation" in the mathematics classroom. *For the Learning of Mathematics, 15*(2), 17–22.

Fetzer, M. (2013). *Counting on objects in mathematical learning processes. Network theory and networking theories.* Contribution to CERME 8 in Antalya, Turkey.

Hersant, M. (2004). Caractérisation d'une pratique d'enseignement des mathématiques: le cours dialogué. *Revue Canadienne de l'enseignement des sciences, des mathématiques et des technologies, 4*(2), 243–260.

Haspekian, M., Bikner-Ahsbahs, A., & Artigue, M. (2013). When the fiction of learning is kept: A case of networking two theoretical views. In A. Heinze (Ed.), *Proceedings of the 37th conference of the international group for the psychology of mathematics education* (Vol. 3, pp. 9–16). Kiel: IPN.

Jungwirth, H. (2003). Interpretative Forschung in der Mathematikdidaktik – ein Überblick für Irrgäste, Teilzieher und Strandvögel. In G. Kaiser (Ed.), Qualitative empirical methods in mathematics education – Discussions and reflections. *Zentralblatt für Didaktik der Mathematik, 35*(5), 189–200.

Novotna, J., & Hospesova, A. (2007). What is the price of Topaze? In J.-H. Woo, H.-C. Lew, K.-S. Park, & D.-Y. Seo (Eds.), *Proceedings of PME 31* (Vol. 4, pp. 25–32). Seoul: PME.

Novotna, J., & Hospesova, A. (2008). Types of linking in teaching linear equations. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano, & A. Sepulveda (Eds.), *Proceedings of PME 32 and PME NA 30* (Vol. 4, pp. 49–56). Morelia: PME.

Radford, L. (2008). Connecting theories in mathematics education: Challenges and possibilities. *ZDM – The International Journal on Mathematics Education, 40*(2), 317–327.

Sensevy, G. (2012). About the joint action theory in didactics. *Zeitschrift für Erziehungswissenschaft, 15*(3), 503–516.

Steinbring, H. (2005). *The construction of new mathematical knowledge in classroom interaction – An epistemological perspective* (Mathematics education library, Vol. 38). Berlin/New York: Springer.

Voigt, J. (1983). Die heimliche Organisation von Aufgabenlösungsprozessen im Mathematikunterricht einer achten Klasse. In H. Bauersfeld, H. Bussmann, G. Krummheuer, J.-H. Lorenz, & J. Voigt (Eds.), *Lernen und Lehren von Mathematik. Untersuchungen zum Mathematikunterricht, IDM Band 6* (pp. 172–222). Köln: Aulis-Verlag.

Warfield, V. (2006). *Invitation to didactique.* www.math.washington.edu/~warfield/Inv%20to%20Did66%207-22-06.pdf. Accessed 29 Nov 2013.

Part IV
Reflections

Chapter 13
Beyond the Official Academic Stage. Dialogic Intermezzo

Stefan Halverscheid

Abstract The chapter goes beyond the official academic stage and presents the researchers' personal experiences during their networking practices and their reflections on these processes. These experiences and reflections were collected by individually arranged interviews and then assembled.

Keyword Networking of theories

13.1 The Story of This Networking Project

In this book, the products of the Networking Theories Group are presented in ways in which readers expect researchers to write: from the perspective drawn from established theories (Part II), from the networking point of view with several theories on a common research interest (Part III), and from a research methodological point of view that reflects upon the research processes in the networking activities (Part IV, following this chapter).

Throughout the work in the Networking Theories Group, the researchers went through individual stages of experiences with the networking of theories. This chapter presents an attempt to tell their stories about this networking experience. It should be viewed as the result of a somewhat journalistic approach: the chapter is organized via critical questions written in italics below, which serve as guides. These questions were asked to five researchers from the five teams and theories involved in the book. Given that it was difficult to string together material from approximately 8 h of interviews in excerpts of direct speech, the responses are summarized and the questions slightly altered.

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In the interviews, the expression “home theory” turned out to be useful when referring to the theoretical approach with which a researcher usually works. As the relationship between the home theory and other theoretical approaches is central in the networking process, it is used here occasionally.

13.2 The Initiation of This Networking Project

Mathematics education is still a nascent scientific discipline. Why did you start the networking project at this stage? Wasn’t there enough work left to do on your home theory?

The Networking Theories Group started to work after the Congress of European Research on Mathematics Education (CERME) conference in 2005. In a panel and in a working group, intensive discussions on theoretical perspectives and paradigms had taken place. All interviewees had noticed limitations of their home theories or the need to integrate approaches and results of other theories within their own research. Because the scholars were internationally cross-linked, some of them felt the need to get to know other theories in a depth that is hard to achieve to a satisfactory degree simply by reading other scholars’ papers and listening to some talks. The group included representatives of both established and relatively new theoretical approaches.

All participants were convinced that connecting the theoretical approaches was important to further develop mathematics education, even though most of them said retrospectively that they had no experience in doing so before the project, and that the notion of “networking” emerged slowly. However, the Theory of Didactical Situations and the Anthropological Theory of the Didactic are different in their history, as the second emerged as a development of the first, and a mutual dialogue between them has occurred from scratch.

As empirical researchers at heart, the members of the Networking Theories Group agreed from the beginning that common work on empirical data was important. Whereas different theoretical approaches can be worked out from an abstract point of view, words on this “meta” level can be misleading. To make sure that the scholars were talking about the same thing while networking of theories, the group decided to work on data. Besides, the histories of the theories involved in this project have in common that they were shaped by and developed for the examination of empirical data. It appeared, therefore, natural to initiate networking processes in the context of data analyses.

Everybody reads others’ papers or attends talks. Isn’t this already a sort of networking experience?

No, this is actually only a small part of it. The group started first by introducing the theories to one another (understanding others and making theirs understandable). Looking back, this encounter was considered to be necessary but not sufficient by all interviewees. Some scholars described these first meetings as the beginning of their networking. They presented their own views with those of the others, such as by comparing and contrasting views on a theoretical level. Others considered the readings of the video analyses from different theoretical perspectives (Part II in this book) as the beginning of their networking. From the point of view of several interviewees, the networking of theories started at the moment when joint research questions were worked out or even with the common process of writing on research papers. All agreed that the potential scope of networking activities is very limited unless it leads to joint research.

13.3 The Work in This Networking Project

What was the main motivation for examining the episodes of Carlo, Giovanni, and the exponential function? Was it just an exercise for networking theories, or did it deserve interest in its own right?

The researchers recalled different primary motivations for working on the videos. The AiC team (working on the theory of epistemic actions of abstraction in context) noticed well before participating in this project that the comprehension of gestures in their research might be useful. At the same time, the APC team (working with the semiotic bundle construct) was interested in studying the context of epistemic actions. The video of Carlo and Giovanni provided a good opportunity for both groups to work together.

The Topaze effect case study (cf. Chap. 12) was also initiated by the analyses of different home theories. Understanding the three competing interpretations of the same episode was the challenge for this group. It is a remarkable feature of networking dynamics that the question as to whether there is a Topaze effect was brought up by the APC team. The Topaze effect and the funnel pattern, which were both early considerations for the episode, were hypotheses that served as driving forces.

These patterns were identified in the first analyses and discussions. It was striking that the same episode could be interpreted as a semiotic game, on the one hand, and as a Topaze effect or a funnel pattern, on the other hand. It soon became clear that a deeper analysis involving the corresponding home theories would be needed. This, finally, led to certain theoretical clarifications for these notions within the respective home theories.

Every theory has developed its own way to work with data over the years. Isn't this a major obstacle to the networking of theories?

The networking activity started from the assumption that even if the data collection was shaped by the inscription in APC, there was space for analysis of this data using other theoretical approaches. Looking back, the problems were rather underestimated at the beginning.

The data in a research context is made for research purposes and this is shaped by the theoretical approaches. Theories also require a certain type of data. For instance, in terms of extent versus detail, different theories require different extents and different details. The intense connection between data and theoretical frames is highlighted by the experiences of the need for different data and for the knowledge of phenomena that are typical for certain theoretical constructs. For instance, talking about a Topaze effect only makes sense using the Theory of Didactical Situations. Similarly, the funnel pattern is shaped within the interactionist perspective of Bauersfeld (1993), underlying the IDS. At the beginning of the work of the networking teams, some researchers reported that they worked with notions from other theories, such as the Topaze effect, on a somewhat metaphorical level. When the discussion became deeper, the specialists' points of view were important, and a thorough theoretical penetration of these notions was deemed necessary (cf. Chap. 12).

Some scholars asked themselves, to what extent did the theoretical background of the involved teacher have an impact on the video, and how should this be dealt with in the analysis. Domingo is a teacher-researcher who works with the semiotic bundle. In particular, he is trained in employing gestures in classroom situations. One of the researchers of the other teams described the beginning of her analysis in the following manner: "First, we found it strange how Domingo acts." The design of the instructional tasks in the video is less appropriate for certain teams than for others. For instance, the tasks posed by Domingo were rather explorative and, as such, not specific enough for AiC in the sense that the knowledge that was intended to be constructed could not easily be identified. For the involvement of the APC group in the project, on the other hand, it was helpful to have a teacher whose approach stimulated the use of gestures and whose video ensured that the gestures were captured.

For some networking teams, it was difficult to identify sufficient manageable data on an episode (cf. the documentation in Chaps. 2, 3, 4, 5, 6, and 7 with initial and extended data). More than usual, it was necessary to agree to certain compromises concerning the data. For some theories, for instance, it is everyday business to consider how a teacher acts in a certain situation; if this question becomes central, it is difficult for theories such as the ATD that do not consider actions at all. Furthermore, the AiC theory does not describe the role of a teacher in a prominent way: it considers the teacher as a part of the context, which is a very flexible construct to handle.

Being already experienced in the home theory at the data-recording stage normally allows the researchers to identify potential problems and fix some of them right away. However, this will remain a challenge in networking theories where scholars need to satisfy the needs of the involved theoretical approaches both in the design process and while recording data. Additionally, even if this were achieved, the data needs would still change when being utilized in other approaches (cf. Chap. 14 for a detailed methodological discussion on the role of data).

13.4 Looking Back on This Networking Project

It is not a new thing in (mathematics) education that different theoretical approaches are employed to consider the same situation. Schoenfeld (2002) describes how to deal with results from different theoretical backgrounds in the triangulation method. Before the start of this networking project, Hannula et al. (2004) considered, for instance, four different frameworks for affect to "evaluate these frameworks from different perspectives." Is networking theories much ado about nothing new?

All interviewees believe that this networking project went beyond what is described as triangulation. The researchers reported on a couple of experiences that they had had before these networking practices and that went beyond the method of triangulation. The main point seemed to be that networking theories influenced the view on the involved theories or even influenced the theories themselves (cf. Sect. 14.2.1 for a discussion on the differences from triangulation).

Scholars who use networking practices question theoretical approaches and the values that come with them but strengthen them in the end. For the APC team, it was difficult when their result that the semiotic game was successful for the extra video starting after Task 3 was confronted by other analyses that pointed out epistemic differences in the discussion between the teacher and the students. Apart from the obvious research question regarding whether these results would be reconcilable, the other teams' first analyses were a blow to the positive values attributed to a "successful" semiotic game. In the end, the theoretical clarifications proved to be necessary, but they confirmed the initial analyses. In the case of APC, this process put the underlying values into the perspective of the other theories by the introduction of the "epistemological gap": the positive attribution to the statement that even if the semiotic game works this does not yet imply that the underlying epistemic actions reflect this in the intended manner (cf. Chap. 11).

Are there hands-on effects of networking practices?

Networking of theories helps to make theoretical notions more precise; in fact, this even applies to highly developed theories, such as TDS. Its legacy includes a variety of phenomena in the teaching and learning of mathematics that are shaped by the theory and examined empirically within this approach, such as the Topaze effect. Confronted with the data examined here, it still became necessary to go back to the theoretical roots of this effect and to make the definition of this phenomenon more precise.

Networking of theoretical approaches pushes the involved approaches to their limits. In this project, this was almost a permanent experience for the Anthropological Theory of Didactics (ATD). The networking project pushed the researchers to conduct analyses that were motivated by an unusual point of view. For instance, the ATD team normally does not initially focus on the teacher and her or his actions unless the teaching and learning process is described. For the AiC, the teacher only appears as part of the context, which can be given more or less importance in a flexible manner. Other theoretical approaches frequently work on understanding the teacher's role. For the ATD team, the unusual work of the material caused a constant feeling of not being at home, but their members still regard this experience as an interesting and rewarding one.

There is no doubt that networking of theories enriches research practices or, in terms of ATD, research praxeologies (see Chap. 15 and Artigue et al. 2011). The construct of research praxeologies offered an approach to better understand and manage the interaction between researchers, which was underestimated before the networking theories project began. In a similar way, the repertoire of the Theory of Didactical Situations was especially rewarding for the networkers, as it provided some elements to describe the common research strategies used. The a priori analysis was often mentioned as an example of what will influence research in the future. The notion of milieu was a complex, albeit very inspiring, tool, which helps the researcher to understand the necessity for networking strategies to have a shared empirical set of objects available for the analysis (cf. Chap. 10). Additionally, it provides a rich collection of studied situations and phenomena that several groups are interested in. The anthropological approach to localizing institutional constraints of teaching and learning and to describing their praxeologies is also named by all interviewees as inspiring, even if the theoretical link – for example, to the Theory of Interest-Dense Situations – is not clear.

Networking of theories creates a networking spirit in everyone involved. By knowing the limits of the home theory better, the networkers unanimously report experiencing a networking attitude in their normal research. "I could do networking alone now," one of the researchers said, adding, "but it would be very risky."

What was important in order to get the networking of theories to work?

All of the networkers emphasized the role that the people in the group played in getting the networking going. There was no overall strategic master plan; instead, the commitment of several researchers led at certain stages to strategies that were worked out in Prediger et al. (2008) (cf. Chap. 8). The approaches varied in the first stage; then, later, the strategies used for networking were still diverse, but the scholars had more of a feeling that they had been able to build tasks and methods for the networking work. In this way, some interviewees recall the work as similar to their usual research at that particular stage.

Everyone's motivation was needed to maintain interest in reaching this point, but the basic motivation differed in the subjective views of the researchers. Some scholars were interested in the comparison of theories, in general. For them, it was like an exercise in networking theories. Some scholars were interested in the particular elements of other theories. Others were motivated by the differences in the explanations in the preliminary analyses, as in the analyses of APC and IDS, for example.

All of the researchers underlined the importance of the personal and social dimensions of networking. Especially in times with no progress, people who push for continuing the project are necessary. The interviewees stressed the importance of a working cooperation- and confidence-based atmosphere that allows for considering problems several times from different points of view.

Are any two theories on the teaching and learning of mathematics suitable for networking?

Certainly, some obstacles were identified while the research teams defined their networking case studies. Early in this discussion, the metaphor of a lens was utilized to describe how the various theories could be used to analyze different "grain sizes" in the data. The APC and ATD approaches are at the extreme ends of the resolution of the lens, even if they both explicitly attribute an important role to the semiotic dimension of mathematical and didactic activities, as shown in Arzarello et al. (2008).

Corresponding grain sizes are neither necessary nor sufficient to do networking together, however. In the networking of theories, most researchers worked on different data sources than those that they usually treat. The usual grain sizes differ in this area, i.e. the different approaches have different units of analysis. The APC construct semiotic bundle typically addresses distinct and short pieces of the data: because a one-second gesture may help to explain the result of a longer working process, this also can be of interest to other theories, even though they would not go down to such short grain sizes. Slightly bigger grain sizes are used to determine whether a classroom episode can be qualified as interest-dense in the IDS approach. Even though interest-dense situations typically can last for a couple of minutes or even up to a whole lesson, the methods used in this theoretical approach address different grain sizes, including very short ones, such as a gesture or a sign.

One of the researchers hypothesized that a useful feature for making networking successful is a common interest in phenomena of comparable timescales. This is meant differently from the aspect of grain sizes. For example, it is traditionally not considered primarily relevant in ATD to determine whether a learning process of a couple of minutes is interest-dense or of another quality, unless the process is not questioned and located within a broader teaching and learning project. However, because questions on this timescale are the main motivation of IDS, it is difficult to work out a research question of relevance to both theories (hence, questions are considered to be an important component of theories, cf. Chap. 1). The representatives of both theories have thought independently on the reasons for this difficulty and have come to similar explanations. They believe that if the overlap of both research interests and possible grain sizes is small, it is difficult to participate in a joint networking effort though, for example, the Theory of Interest-Dense Situations can use insights gained from ATD. Similarly, even though the need for different data remains, ATD could more easily find common research questions with semiotic or distinct epistemic approaches if they address what is learnt before considering the question of how this is achieved.

13.5 Looking into the Future: What's Next?

Has the investigation of the video been an exercise that is now completed, or would some of the results be worth considering in more detail?

Some results of the project show that the analyses were not merely exercises for the members of the Networking Theories Group and that the underlying research questions on the video episode warrant interest in their own right. Nevertheless, it would be very challenging to produce data that make it possible to involve all theories of the group together. Because no teacher would produce a Topaze effect deliberately, it is difficult to investigate how it is constituted by social interactions in the classroom. Besides, producing an example of a veritable Topaze effect would not fundamentally add to the theoretical knowledge because this phenomenon is situated as an idealized limit concept.

A learning situation with gestures seems more likely to be producible. To obtain this, the idea was brought up to start from Domingo's course design, picking out certain key features and asking teachers to adapt these in their design of a course. This could enable the researchers to study the semiotic game under the conditions in which an epistemic gap occurs on a broader empirical basis.

Can research designs be thought of to understand more about networking?

The challenge for a follow-up design would be – again – to find a learning situation that fits the aims of several research groups. Surely, it would remain difficult to find empirical material that is both interesting and usable for every theory involved.

Although several ideas for a networking design appeared during the networking process, it is not clear which theories could be utilized to fully exploit the knowledge that the researchers gained in terms of networking praxeologies and on which one can, thus, rely for developing a vision of possible research dynamics in the area. On the other hand, the networking praxeologies should prove fruitful in new contexts and research questions, and previous research projects might be better understood with the help of these. This should also provide new insights into research processes of networking theories.

References

Artigue, M., Bosch, M., & Gascón, J. (2011). Research praxeologies and networking theories. In M. Pytlak, T. Rowland, & E. Swoboda (Eds.), *Proceedings of the seventh Congress of the European Society for Research in Mathematics Education CERME 7* (pp. 2381–2390). Rzeszów: University of Rzeszów.

Arzarello, F., Bosch, M., Gascón, J., & Sabena, C. (2008). The ostensive dimension through the lenses of two didactic approaches. *ZDM – The International Journal on Mathematics Education, 40*(2), 179–188.

Bauersfeld, H. (1993). Theoretical perspectives on interaction in the mathematical classroom. In R. Biehler, R. W. Scholz, R. Sträßer, & B. Winkelmann (Eds.), *Didactics of mathematics as a scientific discipline* (pp. 133–146). Dordrecht: Kluwer.

Hannula, M., Evans, J., Philippou, G., & Zan, R. (2004). *Affect in mathematics education – Exploring theoretical frameworks.* Research forum. International Group for the Psychology of Mathematics Education. www.emis.de/proceedings/PME28/RF/RF001.pdf. Accessed 2 Dec 2013.

Prediger, S., Bikner-Ahsbahs, A., & Arzarello, F. (2008). Networking strategies and methods for connecting theoretical approaches – First steps towards a conceptual framework. *ZDM – The International Journal on Mathematics Education, 40*(2), 165–178.

Schoenfeld, A. H. (2002). Research methods in (mathematics) education. In L. English (Ed.), *Handbook of international research in mathematics education* (pp. 435–488). Mahwah: Erlbaum.

Chapter 14
Networking as Research Practices: Methodological Lessons Learnt from the Case Studies

Angelika Bikner-Ahsbahs and Susanne Prediger

Abstract The methodological reflection on the case studies from Chaps. 9, 10, 11, and 12 starts with elaborating the data–phenomena distinction by which the role of data and phenomena in empirical networking practices can be grasped deeply. The gradual distinction between more empirical and more conceptualized phenomena clarifies the status of empirical situations, data, and theory and how these are linked to phenomena. Looking at the case studies, data–phenomena distinction is referred to the networking strategies and the monitoring role of research questions. The chapter finishes with summarizing potential empirical, theoretical, and methodological benefits of networking practices.

Keywords Networking of theories • Methodology • Role of data • Role of phenomena

The authors of Chaps. 9, 10, 11, and 12 have already reflected on the difficulties and gains of their networking practices within each of the chapters. In this chapter, we present the methodological lessons learnt from a more general perspective. We start in Sect. 14.1 with the role of data and phenomena and discuss in Sect. 14.2 the empirical and theoretical benefits of networking practices between theoretical approaches.

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14.1 Lessons Learnt on the Role of Data and Phenomena

14.1.1 Looking Back

The specific idea of starting the networking in 2006 was to challenge the researchers with the demand to analyze alien data (i.e., data coming from a study in another research frame). The choice of data contributed to producing an "antagonist" milieu for the research groups they had to deal with. In fact, since data were given and were not fully suitable for analysis, this caused different reactions:

- using experiences of previous studies with digital technologies for developing a hypothetical a priori analysis of the situation to enrich the understanding of the data at hand (by the TDS-team in Chap. 4);
- including the teaching material of the whole course into the analysis (the strategy of the ATD-team in Chap. 5);
- using just a part that is suitable enough (the strategy of the AiC team in Chap. 6);
- enlarging the theoretical frame in order to apply the theory to the given data (as decided by the IDS-team in Chap. 7);
- including additional foci into the given data to take into account additional aspects into the transcript (by the APC and AiC teams in the case of epistemic gestures in Chap. 9).

The important role of data became even more visible in two specific needs that the different teams expressed: the need for further data that fitted better to the usual analysis; and the need for reflecting about the role of data as a link between theory and empirical issues, here the video episode. The first need resulted in collecting additional data and conducting the case studies of networking as presented in Part III of the book. The second need was already present in the presentation of the theories in Part II and was focused on in more detail in some case studies, for example in the case of epistemic gestures (Chap. 9).

The important role of data also appeared in other networking activities conducted by the Networking Theories Group which are documented outside this book. In one networking activity, researchers from different theoretical approaches translated a common problem in classrooms to different research questions and sketched possible research designs (Prediger 2008). While comparing the different research questions and designs, the different kinds of desire for data became apparent. In another activity, one research question posed by the TDS team was translated into research questions of the other teams and led to distinguishing between problems and phenomena (see for example Artigue et al. 2011). In many networking practices, the issue of insufficient data is apparent. In the case study on the epistemic role of gestures (Chap. 9), the authors even wondered why the two teams – that methodologically have a lot of features in common – experienced difficulties in selecting a common piece of data.

The problem of inadequate data in networking practices can be understood more deeply by distinguishing the notions of data and phenomenon, as discussed in the following section.

14.1.2 The Data–Phenomenon Distinction

According to Knipping and Müller-Hill (2013), qualitative research in mathematics education should follow the principle of clearly distinguishing data and phenomena where data are the means to identify and investigate phenomena in mathematics education:

> As a practical and methodological consequence of a clear conceptual distinction between data and phenomena, a large amount of research effort has to be spent to face the resulting problem of detecting a genuine phenomenon rather than some artefact of the experimental setting. (Knipping and Müller-Hill 2013, p. 3)

In their paper, Knipping and Müller-Hill refer to the work of Bogen and Woodward in philosophy of science who describe data as being "idiosyncratic to particular experimental contexts" (1988, p. 317), whereas "phenomena, by contrast, are not idiosyncratic to specific contexts. We [Bogen and Woodwad] expect phenomena have stable repeatable characteristics which will be detectable by means of a variety of different procedures which may yield quite different kinds of data" (ibid., p. 317).

If we accept Bogen and Woodward's definition of phenomena for mathematics education, then phenomena are constructed by human beings who realize these stable and [repeated or] repeatable characteristics in mathematics education as an instance of a more general pattern. Given that mathematics education is still a young discipline, its phenomena often are not very clear or even not well known. Therefore, research in this field is not only conducted to investigate phenomena but also to identify, disclose, and describe phenomena. Some phenomena can easily be detected and shared based on common-sense knowledge of mathematics education without any strong theoretical foundation; others need more profound conceptualizations and sophisticated methodological and methodical arrangements to make them accessible for the human perceptual system. For example, one member of the Networking Theoreis Group described the following phenomenon: "in one situation a child may be able solve a specific task but later the same child is not able to solve it anymore" (problem and succeeding networking activities documented in Prediger 2008). The research teams separately translated the description of this phenomenon into research questions and developed a research design for its investigation. Through this translation, the phenomenon was conceptualized in different nuances by the different theoretical frameworks. Hence, the phenomenon changed its status. In the first case we talk about an *empirical phenomenon*, and in the latter about a *conceptualized phenomenon*, although of course no phenomenon can be perceived completely independently of the theoretical approach or even from simple pre-assumptions. That is why we understand the distinction between more empirical and more conceptual phenomena as a gradual one according to the degree in which the theory guides the conceptualization of the phenomenon. Figure 14.1 roughly sketches these connections between data, more empirical and more conceptualized phenomena in the interplay between theory and reality (here, concretely, episodes of teaching and learning) which can, on the one hand, be perceived as an interplay between particular and general, but also (as a second dimension) between the more vague to the more theoretically focused and structured perspective that allows us to see connections.

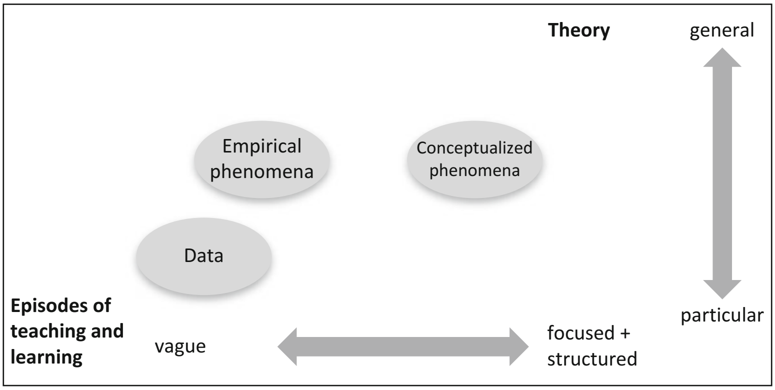

Fig. 14.1 Rough localization of data, empirical phenomena, and conceptualized phenomena for each theoretical approach

In contrast to data, phenomena are not directly perceivable, as they are constructions. However, data themselves are not relevant; for research they are only interesting as means to "constitute evidence for the existence of phenomena" (Woodward 1989, p. 394). The reason for underlying problems with data in the networking of theories is not the data themselves but the kinds of phenomena that the research teams normally are used to identifying and investigating through data. The given video of Carlo and Giovanni struggling with the exponential function and its transcription were taken from a larger study on the introduction to variation and calculus in the first years of secondary school (see Chap. 2). The role of different semiotic resources, including gestures and embodied ones, was of great importance for the APC team in this study. The semiotic bundle notion and the semiotic game phenomenon were built up along with the video analysis of this and other episodes in the project. Although these kinds of phenomena were not relevant in the other approaches, the other teams attempted to analyze the alien data by taking them as a constitutive means for identifying and analyzing one or more home phenomena that were not intended by the APC team. The teams strongly experienced that the given data were only partly appropriate for this endeavor.

In contrast to our terminology of more empirical and more conceptualized phenomena, for Artigue et al. (2011) phenomena only exist in research contexts as a result of theory-driven investigations; hence, all phenomena are regarded as already being conceptualized:

> In a first approach, we can characterise didactic phenomena as empirical facts, regularities that arise through the study of research problems. Some of these phenomena enrich the initial theoretical framework to produce new interpretations and techniques or research methodologies, while others remain at the level of "results obtained" and are reinvested to formulate new problems or to propose new diagnostic and practice-development tools. (Artigue et al. 2011, p. 2383)

In the quotation, Artigue et al. (2011) also address the degree of conceptualizing phenomena that may be a result of specific research and lead to new problems, products, etc., but also to theoretical concepts whose status is determined by the relationship to the other concepts and principles of the theory. This is in line with the gradual distinction between empirical and conceptual phenomena.

In the case of the Topaze effect (Chap. 12), two conceptualized phenomena which had been the results of previous research within two different theoretical frames were networked. Through this networking process the two conceptualized phenomena and their theoretical status were strengthened and further conceptualized. This revealed a "too early naturalization of a phenomenon" as a new phenomenon in the culture of mathematics education and led to deepening insight into the theories' blind spots but also to uncovering the empirical phenomenon of the fiction "that teaching [in the two phenomena] has led to learning" as the underlying idea which both teams agreed upon.

Taking the view of Bogen and Woodward, data are means to identify more general phenomena and investigate claims about phenomena within theories. For example, the AiC team (in Chap. 9) broadened their notion of what constitutes data in that they admitted gestures to be data. In the second stage the AiC team identified that gestures may shape part of the constructing process, hence conceptualized this phenomenon resulting in the term "epistemic gesture." The AiC team stated, "As compared with earlier RBC analyses, the evidence we admitted and paid attention to in the present analysis was broader since gestures were considered as potential indicators of epistemic actions" (Sect. 9.5). In our view, it is not the data themselves that provide evidence for a phenomenon but the way data are used to provide evidence for the constitution of phenomena, the way they are freed from their complexity and "their highly irregular coincidences" (Bogen and Woodward 1988, p. 326), the way they are analyzed and interpreted, and this is determined by the theory's methodology and principles. Bogen and Woodward also emphasize, "Often the characteristics which data must have to be useful as evidence can only be purchased at the cost of tolerating a great deal of complexity and idiosyncrasy in the causal processes which produce data" (ibid., p. 319). This by-product of producing data also holds in research practices in mathematics education and it explains why alien theorists might be able to find some evidence of their home phenomena in given data, as has been shown in Chaps. 3, 4, 5, 6, and 7 of Part II and the case studies in Chaps. 9, 10, 11, and 12 of Part III.

14.1.3 Relating Research Questions to the Data–Phenomena Distinction

In the case study on epistemic gestures between APC and AiC (Chap. 9), the AiC team started to integrate gestures into their methodology of studying epistemic processes of constructing knowledge and asked "whether and in what sense gestures can contribute to the construction of knowledge" (Sect. 9.1). In spite of the integration of a common type of data into the case study, research questions were different in the two approaches. An outcome of the two data analyses and their comparison was the discovery of an interesting phenomenon, namely that some gestures supporting the constructing process were used without the producer and his peer looking at them. The discovery of this phenomenon raised dialectic questions and pushed the networking process further. The deeper analysis on the epistemic function of these gestures and the succeeding common, combined research process about the epistemic role of gestures in the knowledge construction processes resulted in the notion of an epistemic gesture. This combined research process demanded choosing *common data for analysis.* The difficulties of finding such data required intense work about the idea of epistemic gesture and the role of gesture for AiC and was analytically focused on a very small piece of the given video for which the transcript had to be refined and enriched. Only after this step did the AiC team state, "the videotape became data for us once we transcribed it with focus on verbalizations and gestures" (Sect. 9.3). In contrast to this way of approaching data, the APC team normally starts from observing the video and not from interpreting the transcript.

This step of modifying the data with respect to the phenomenon under question shows how data are made in different research practices from the same video, addressing the epistemic role of gestures in the AiC approach and the communicative function of gestures in the APC approach. The dialectic between the teams' questions and data usage helped to detect the new phenomenon, and reflecting the different views helped to clarify the process of conceptualization: the AiC team sees the epistemic function of gestures in single gestures which contribute to constructing knowledge; the APC team identifies the epistemic function in sequences of gestures as part of the semiotic bundle. The epistemic function of gestures here seems to be a concept at the boundary of the two approaches which allows establishing a locally integrated methodology but no local integration on the level of principles. The principles are not close enough.

In contrast to the case study on gestures, in the case of context, milieu, and media-milieu dialectic (Chap. 10) the researchers were able to choose a common piece of data which was much larger and not explicitly reported on. Data were not chosen for common analyses but *to refer to for separate analyses* that served as the basis for comparing and contrasting the role of the three concepts in their home theoretical background. In this case study of networking, data served as a common reference pool, but not as a resource for research itself. The questions reflect this role of data in the networking process, since:

> TDS researchers might ask what milieu the teacher is making available to the students and how she is managing its evolution in order to establish a meaningful connection with the mathematical knowledge aimed at. AiC researchers might ask how the teacher's intervention influences the students' construction process as described by means of the RBC epistemic actions. ATD researchers in their turn might ask what responsibilities the teacher and the students are assuming in the media-milieu dialectics and what conditions enable them to manage it. (Sect. 10.3)

Questions for analyses in AiC stress the epistemic process itself, whereas researchers in TDS and ATD ask how this process is made possible. Already these questions indicate an interesting point in networking: researchers were able to build on ideas and results of the other analyses in a complementary way. This made them identify *epistemological sensitivity* as an underlying proximity in their respective approaches. That means, the three theories share the aim to understand the epistemological nature of the episode, while, at the same time, each of the three theories accesses data in its own ways. The teams pose different questions concerning contextual influence. Through comparing and contrasting these questions, researchers tried to elucidate the ways in which the three theoretical approaches address the issue of contextual dependence of teaching and learning processes through their concepts. Context in AiC is everything that does not belong to the epistemic process itself, but does influence the construction of mathematical knowledge. The milieu is, for TDS, the main concept describing the environment with which the learner interacts in order to produce a mathematical piece of knowledge. The dialectic of media-milieu clarifies the dynamic nature of the milieu being changed by media. The three concepts are accessed by different data or different foci on data in a complementary way sharing *epistemological sensitivity*, which facilitated establishing connections and reflecting on them. The researchers claim that such proximity seems to be crucial for undertaking the networking practice between their theories. However, we see it as an open question how the networking could function without such proximity.

In the case of epistemological gap (Chap. 11), the data were not a problem at the beginning. Both teams were able to use the same video, namely the extra video starting after Task 3 (see Appendix for a complete transcript), as a common piece of data for separate analyses answering home questions. But the ways in which the data reflected the core questions were different. The IDS team focused more on the discourse whereas the APC team focused on the gesture–speech interplay. Questions in the networking process were very interesting because they directed the attention of the two teams. The first question was: Which of the two results are more suitable for understanding the episode? This made the two teams reconsider the raw data to refine the utterances, include the students' protocols, and produce a written transcript in which gesture pictures and speech intonation were included. In this case, the data was reworked for a more common analysis. This first step led to a refinement of the concept of semiotic game and raised another question: What is the deeper reason why Giovanni reduces to be engaged in such a short situation? This question brought the idea of the epistemological gap as a vague idea to the fore. During the following months, both teams took this episode as a prototype represented by the refined data set that evidenced the phenomenon of an epistemological gap. Its mechanism was still only vaguely understood. Since the two theories did not offer an appropriate theoretical frame to conceptualize this empirical phenomenon, a literature review was conducted. Concepts and results from research on personal epistemology could be included into the two approaches, leading to a process of conceptualizing the phenomenon of epistemological gap and clarifying the mechanism of it. In this way, a local integration of a new construct at the boundary of both theories has emerged and connected the two approaches within the semiosphere (Radford 2008;

cf. Sect. 8.1 in this book). This case of epistemological gap does not only demonstrate the data–phenomena distinction but also the gradual difference between a more empirical phenomenon and a more conceptualized phenomenon. The latter is part of a theory while the former is a construction which may still appear more pre-theoretical and less elaborate.

The previous reflections on research questions and their relations to the data–phenomena distinction show that questions, explicitly or implicitly posed, may guide researchers' attention in research practices and their mediating between data and phenomena. In networking processes, often dialectic research questions from different approaches mediate the comparing and contrasting of theories. The resulting synthesized common questions seem to support processes of coordinating which may lead to a local integration.

14.1.4 Relating Networking Strategies to the Data–Phenomena Distinction

The four case studies show that processes of networking may lead to uncovering an underlying proximity or even an empirical phenomenon underlying the theories' concepts. They also show that the researchers are often unaware of these proximities at the beginning of a networking process but they can be achieved as a result. Since common proximities or empirical phenomena allow for complementary views, they may be a starting point for the networking strategy of coordinating. If such a common empirical phenomenon is first uncovered, it may be vague at the beginning, like the epistemological gap, but further investigated in a process of coordinating showing how far networking processes can reach. By a process of conceptualizing, the empirical phenomenon changes its character and status, and may be worked out and finally conceptualized. In this way, the empirical phenomenon turns into a conceptual phenomenon that then belongs to the theoretical approach and may finally result in a local integration, as in the case of epistemological gap (Chap. 11).

The case study of networking on the Topaze effect (Chap. 12) started with two such conceptualized phenomena to which different questions directed the separate analyses. Through networking, the underlying common empirical phenomenon was able to be uncovered and at the same time the nature of both conceptualized phenomena as being limit concepts was clarified. This case and the case of context-media-milieu (Chap. 10) showed the fruitfulness of the networking strategy of comparing and contrasting, even without further degrees of integration. This was different in the cases of the epistemological gap and of the epistemic gesture: the networking strategy of coordinating encompassed conceptualizing a phenomenon and even led to local integration of new constructs in both cases.

Empirical phenomena – even if different theories share them – may be elaborated differently in different theories, bringing to the fore complementary views. The other way round, a shared empirical phenomenon may be hidden in concepts of different theories but can be uncovered through networking processes. In both cases, the networking strategies of comparing and contrasting are especially fruitful for revealing the complementary nature of differently conceptualized phenomena.

By these methodological reflections, the role of data for the networking practices is also clarified. The networking practice also depends on the kind of data used. As long as data are used separately and modified with respect to each theoretical approach, the networking practice may reach the stage of combining because researchers stay within their home theoretical approach. As soon as common questions are investigated, the choice of common data may become difficult within this strategy because the home theories look at different empirical phenomena and possibly slightly different data. At this stage, the strategy of coordinating may help to overcome difficulties. The intermediate strategy of coordinating seems to be that of transforming separate views towards a more integrating view on the empirical phenomena. At this stage the phenomena may also change their nature, from a more empirical towards a conceptual status, leading finally to local integration.

14.2 Lessons Learnt on the Empirical and Theoretical Benefits of Networking Between Theoretical Approaches

What can we generally gain from networking of theoretical approaches? We discuss our methodological considerations on different benefits in two steps: in Sect. 14.2.1, we summarize possible *empirical* benefits and *theoretical* benefits; and in Sect. 14.2.2 we show the strong interdependence between both.

14.2.1 Empirical and Theoretical Benefits from Networking Practices

Considering the same empirical material from different theoretical lenses is not a new research practice; it has often been applied by many researchers in terms of theory or *perspective triangulation* (e.g., Schoenfeld 2002). The notion *perspective triangulation* was introduced by Denzin (1970), and was presented by him (together with method triangulation, data triangulation, or investigator triangulation) as research practices for increasing validity of an empirical analysis. During the last 40 years, though, it became evident that a systematic triangulation of theoretical lenses often does not offer increased validity: if different theoretical lenses capture different aspects of research objects or conceptualize the research objects in different ways, their results are not comparable. However, additional theoretical views mostly focus on additional and complementary aspects which altogether deepen and broaden the understanding of an empirical situation and thus shape “triangulation as a research strategy” for increasing research quality (cf., e.g., Flick 2007, p. 20ff.).

In this sense, the networking practices as presented in Chaps. 9, 10, 11, and 12 might be perceived as practices of classical perspective triangulation, seeing the substantial empirical benefits received by complementary insights into complex empirical phenomena. However, we put emphasis on the fact that the presented cases of networking go beyond perspective triangulation in three aspects:

1. *Empirical benefits:* Sometimes, perspective triangulation is naïvely discussed as a practice of "different theoretical lenses for the *same* data." Different theoretical approaches rarely deal with the same data since data is constructed within a theoretical frame; this point was extensively discussed in Sect. 14.1. Instead of a simple perspective triangulation on the same empirical material, our networking practices enhanced the empirical benefits by enlarging and reshaping data while connecting the approaches.
2. *Theoretical benefits:* As was argued by the data–phenomenon distinction in Sect. 14.1, networking activities do not only aim at a deeper understanding of empirical phenomena, as will be discussed below.
3. *Methodological benefits:* The methodological reflection of possibilities, benefits, and limits constantly accompanies the dialogue between theoretical approaches. In this sense, networking practices also aim at increased methodological awareness.

Coming back to the benefits for the theoretical approaches themselves, networking of theories can facilitate the development of theories in four directions (Bikner-Ahsbahs and Prediger 2010):

(a) *Explicitness:* Starting from the claim that a theory should make its background theories and its underlying philosophical base (especially its epistemological and methodological foundations) as explicit as possible, the maturity of a theory can be measured by the degree of its explicitness: the more implicit assumptions are explicitly stated and the more parts of the philosophical base shape explicit parts of the background theory, the more we would consider the theory to be *mature*. A step towards such a development took place in the case of the Topaze effect through uncovering blind spots and some limitations of the theories (Chap. 12).

(b) *Empirical scope:* Formal theories have a large empirical scope. They characterize empirical phenomena in a global way and often cannot exactly be concretized through empirical examples (Lamnek 1995, p. 123). On the other hand, contextualized and local theories have a limited scope but their statements can more easily be made concrete by the empirical content (see Krummheuer and Brandt 2001, p. 199). This proximity to empirical phenomena makes contextualized theories a suitable background to guide practice in schools. However, developing local theories in order to *enlarge their empirical scope* can be an important direction for theory development. This happened for example to AiC in Chap. 9.

(c) *Stability:* Stability is a long-term aim for theory development on a longer time scale. A new theory might be a bit fragile because its concepts and the relationships among its key concepts are still in progress, for example in IDS. Through networking with other theories, IDS concepts proved to be fruitful (Chaps. 11 and 12), its principles could be strengthened (Chaps. 7 and 12), and the disclosure of empirical phenomena (common to other approaches) unfolded its complementary view on specific empirical phenomena (Chaps. 11 and 12).

(d) *Connectivity:* Science is characterized by argumentation and interconnectedness, as Fischer (e.g., 1993) emphasizes. This can, for example, be realized by establishing relationships through linking theories, by declaring commonalities and differences. Hence, establishing *argumentative connectivity* is another important direction for the development of theories. This direction has been touched on in all case studies since argumentative connectivity is an intrinsic feature of networking practices in general.

14.2.2 Interdependences Between Empirical and Theoretical Progress

Although the discourse on different networking profiles (see Chap. 8, following Arzarello et al. 2008) might suggest that networking practices either aim at theoretical or empirical benefits, our case studies show that both can often be connected since the development of empirical analysis, conceptualized phenomena, and theoretical constructs often interdepend.

These interdependences are also highly connected to the role of results. Radford (2012) added research results to his triplet (questions, methodology, principles) for describing theory as a fourth component: research results as the source for the dynamic development of theories. New results may enlarge the amount of phenomena that can be investigated and the number of key constructs. However, they also may have an impact at least on enlarging and understanding more deeply the home methodologies, paradigmatic questions, and also principles. In networking practices, results play an important role in understanding more deeply what networking approaches, their principles, methodologies, and questions mean, too. The four case studies gave examples that networking may:

1. uncover underlying empirical phenomena that later can be investigated and yield new constructs within the theories or at the border of them (Chaps. 9 and 11);
2. yield new constructs at the border of theoretical cultures. According to Lotman (1990, p. 134), the new dynamic of cultural development comes from the periphery, therefore concepts at the boundary of theories may lead to new research directions,

integrating theoretical views or providing complementary or supplementary considerations (Chaps. 9 and 11);

3. lead to clarifying methodological aspects such as the role of data and phenomena in the networking research (Chap. 9);
4. build new networking methodologies such as cross-methodologies including cross-data collection, cross-task design, cross-experimentation, and cross-analyses which all have a cyclic pattern of interconnected research actions followed by an exchange that leads to a refinement of the research actions etc. (for example Chaps. 10 and 11);
5. strengthen the understanding of theories by clarifying their foci, what also is taken as relevant, what is left aside, and finally identifying blind spots and thus making assumptions more explicit (Chaps. 10 and 12);
6. produce results about implicit practices in research cultures such as the naturalization of phenomena within a research culture (Chap. 12).

References

Artigue, M., Bosch, M., & Gascón, J. (2011). Research praxeologies and networking theories. In M. Pytlak, T. Rowland, & E. Swoboda (Eds.), *Proceedings of the seventh Congress of the European Society for Research in Mathematics Education* (pp. 2381–2390). Rzeszów: University of Rzeszów.

Arzarello, F., Bosch, M., Lenfant, A., & Prediger, S. (2008). Different theoretical perspectives in research from teaching problems to research problems. In D. Pitta-Pantazi, G. Phillipou, et al. (Eds.), *Proceedings of the 5th Congress of the European Society for Research in Mathematics Education (CERME 5)* (pp. 1618–1627). Cyprus: ERME.

Bikner-Ahsbahs, A., & Prediger, S. (2010). Networking of theories – An approach for exploiting the diversity of theoretical approaches. In B. Sriraman & L. English (Eds.), *Theories of mathematics education: Seeking new frontiers* (Advances in mathematics education series, pp. 483–506). New York: Springer.

Bogen, J., & Woodward, J. (1988). Saving the phenomena. *The Philosophical Review, CXVII*(3), 304–352.

Denzin, N. K. (1970). *The research act in sociology*. Chicago: Aldine.

Fischer, R. (1993). Wissenschaft, Argumentation und Widerspruch. In R. Fischer, M. Costazza, & A. Pellert (Eds.), *Argument und Entscheidung: zur Idee und Organisation von Wissenschaft* (pp. 29–44). Wien: Profil.

Flick, U. (2007). *Triangulation – eine Einführung* [Triangulation – An introduction]. Wiesbaden: VS-Verlag für Sozialwissenschaften.

Knipping, Ch., & Müller-Hill, E. (2013). *The problem of detecting phenomena amid a sea of noisy data*. Paper presented at the 8th Congress of the European Society for Research in Mathematics Education (CERME 8), Antalya, Turkey.

Krummheuer, G., & Brandt, B. (2001). *Paraphrase und Traduktion. Partizipatorische Elemente einer Interaktionstheorie des Mathematikunterrichts in der Grundschule*. Weinheim: Beltz.

Lamnek, S. (1995). *Qualitative Sozialforschung Band 1: Methodologie*. Weinheim: Beltz.

Lotman, Y. M. (1990). *Universe of the mind. A semiotic theory of culture*. Bloomington: Indiana University Press.

Prediger, S. (2008). How are theoretical approaches expressed in research practices? A report on an experience in comparing theoretical approaches with respect to the construction of research problems. *ZDM – The International Journal on Mathematics Education, 40*(2), 277–286.

Radford, L. (2008). Connecting theories in mathematics education: Challenges and possibilities. *ZDM – The International Journal on Mathematics Education, 40*(2), 317–327.

Radford, L. (2012). *On the growth and transformation of mathematics education theories.* Paper presented at the international colloquium: The didactics of mathematics: Approaches and issues. A homage to Michèle Artigue. May 31 to June 1, 2012. http://www.laurentian.ca/educ/lradford. Accessed 28 Nov 2013.

Schoenfeld, A. (2002). Research methods in (mathematics) education. In L. D. English (Ed.), *Handbook of international research in mathematics education* (pp. 435–487). Mahwah: Lawrence Erlbaum.

Woodward, J. (1989). Data and phenomena. *Synthese, 79*, 393–472.

Chapter 15
Reflection on Networking Through the Praxeological Lens

Michèle Artigue and Marianna Bosch

Abstract In this chapter, the Anthropological Theory of the Didactic (ATD) is given a different status, its lenses and constructs being used for reflecting on the networking enterprise itself. For this purpose, the notion of praxeology first introduced for modeling mathematical and didactic activities is extended to research practices. This extension leads us to consider that the proper level for addressing networking issues is in fact the level of research praxeologies, and to reflect on the collaborative work carried out by the different teams and its outcomes in the light of this perspective. Along the way, we also rely on other constructs, and especially on the ideas of milieu and media-milieu dialectics.

Keywords Networking of theories • Research praxeology • Methodology

In the previous chapters, the Anthropological Theory of the Didactic (ATD) introduced in Chap. 5 has been just one of the theories involved in the networking process of the Networking Theories Group. In this chapter, we give it a different status, using its lenses and constructs for reflecting on the networking enterprise itself, following ideas initially presented in Artigue et al. (2011a). For this purpose, the notion of praxeology first introduced for modeling mathematical and didactic activities is extended to research theories and practices. This extension leads us to consider that the proper level for addressing networking issues is in fact the level

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of research praxeologies, and to reflect on the collaborative work carried out by the different teams and its outcomes in the light of this perspective. Along the way, we also rely on other constructs, and especially on the ideas of milieu and media-milieu dialectics (see Chaps. 4, 5, and 10).

15.1 Introduction

As explained above, in the previous chapters the Anthropological Theory of the Didactic (ATD) has just been one of the theories involved in the networking process engaged around the video provided to the group by the Italian team. It was not given a particular status, and the Networking Theories Group (networking group or simply group in the following) used constructs in some sense neutral with respect to the different theories for organizing the presentation of the different approaches, and for situating its networking efforts. It used for instance the categorization proposed by Radford in terms of Principles, Questions, and Methodologies for introducing the different theoretical approaches, and systematically referred to the scale of networking processes proposed in Prediger et al. (2008) for situating achievements in the four case studies. However, for the authors of the present chapter, the idea progressively emerged that this theory could provide useful tools for approaching the idea of networking itself, and for analyzing the networking efforts of the group and their outcomes.

Why this idea? In ATD, as explained in Chap. 5, mathematical and didactic practices are modeled in terms of praxeologies. A basic assumption in the theory is that this notion of praxeology can be productively used for modeling any forms of human practice, not just those attached to the production or dissemination of mathematical knowledge. If we take this assumption seriously, it should also be possible and productive to model our research practices in such a way, and especially those developed for achieving networking goals. When adopting such a position, immediately many questions emerge: How to express research practices through the (task, technique, technology, and theory) filter imposed by the model of praxeologies? What changes in perspectives does it induce? There is no doubt, for instance, that the fact that in ATD theories are embedded in praxeologies and not treated as autonomous entities leads to questioning of the nature of the networking enterprise itself. What does it mean exactly to network "theoretical frameworks"? Can this idea make sense without considering the whole research praxeologies of which these theoretical frameworks are part? What exactly have the teams involved in the networking group networked? Can such a perspective help in understanding the potential and limitations of the work undertaken, identifying and organizing its outcomes, designing more effective networking practices? What challenges does it raise?

These questions have paved the way for the reflection we have developed and that we invite the reader to share with us in this chapter. In the next section, we will extend the notion of praxeology to research praxeologies, insisting on the dynamic character of these objects and the crucial role that didactic phenomena play in these dynamics. Then we will use this extension to reflect on the collaborative work carried out by the different teams and its outcomes.

15.2 From Theoretical Approaches to Research Praxeologies

Theories are often presented in a static way as a structured network of concepts (see for instance Niss 2007). In this book, we have adopted a dynamic and operational vision by referring to Radford's elaboration in terms of principles, methodologies, and paradigmatic questions (Radford 2008). Considering theories as elements of research praxeologies is also adopting a pragmatic and dynamic vision of theories, trying to make clear how they inform and shape the practical research work, and conversely how they progressively emerge from it and integrate its results. In this section, we first introduce how research practices can be interpreted in terms of praxeologies, then discuss the connection between their practical and theoretical blocks, emphasizing the bridging role played by didactical phenomena, and illustrating our discourse by some examples taken from previous chapters.

15.2.1 What Is a Research Praxeology?

As any other praxeology, research praxeologies are composed of an amalgam of pieces that can be described by a set of four elements [T/τ/θ/Θ]. The pair [T/τ] corresponds to the practice (or know-how) of research, with the *types of tasks* T that are approached and the *techniques* τ used to carry them out. We can consider that, at its core, the types of tasks are mainly composed of the research questions and problems approached. Formulating a problem, looking for appropriate *milieus*, organizing the experimental work, putting it into practice, gathering data, analyzing it, relating the observations to other investigations and previous results, discussing and evaluating the results obtained, etc. are examples of different types of tasks carried out in a research project. However, a research practice contains much more other action: presenting a result obtained at a local seminar, giving a talk at an international conference, preparing a funding application to the national government, reviewing a paper for a journal, supervising a PhD project, etc. These should thus be considered as part of research praxeologies and it must be clear that, in their own way, they also contribute to the development of the different theoretical approaches. The *techniques* correspond to the different possible "ways of doing" that can be used to carry out a task of a given type, with usually many slight variations and sometimes strong differences between them. When some of these techniques acquire a rather stable, systematic and justified form, we usually talk about "research methods" or "methodologies", as each team of the networking group has tried to present in Part II of the book.

The block [θ/Θ] of research praxeologies forms the *technological-theoretical* discourse used to describe, justify, and interpret both the research practice and the results obtained. The first component, the *technology* θ is the first level of description, explanation, and justification of the practice. It includes the methodological discourse used for explaining and justifying the choices made in terms of research methods.[1] It also provides a preliminary description of the results obtained, before they integrate the theory, once their stability has been proved and they can be considered as basic assumptions. The *theory* Θ is a second level of justification of the practice. It is made up of the main principles, notions, and properties that are considered as unquestionable. It is interesting to see, in the chapters of Part II, how this basic discourse can vary from one framework to another as they are based on different primary terms: "interest-dense situations" in IDS, "semiotic bundle" in APC, "epistemic actions" or "context" in AiC, "praxeologies" in ATD, "didactic and a-didactic situations" in TDS. There are many other elements of the theory Θ that remain implicit in each framework, for instance the priority given by AiC and APC to the students' constructions of knowledge, while TDS and ATD initially focus on the institutional construction of knowledge; the focus of AiC, TDS, and ATD on the epistemic dimension of teaching and learning activities; the reasons for choosing a given type of empirical data; etc.

These first chapters of the book presenting the main theoretical frameworks also show to what extent the practical and theoretical blocks of praxeologies are mutually dependent. For instance, the presentation of TDS makes clear that this theory orients research questions towards the study of didactic systems, not towards the cognitive functioning of individual learners. In contrast, AiC orients research questions towards the understanding of such a cognitive functioning, and in it the didactic systems to which the individual learners belong are taken as elements of the context. Each type of question generates its own research tasks. Quite often, researchers rely on familiar techniques for solving these tasks, but research also leads to the creation of specific methodologies (techniques and associated technological didactic discourses). As mentioned in Chap. 4, for instance, the methodology of didactical engineering emerged in TDS and since the 1980s it has played a crucial role in TDS research praxeologies.

The results obtained and their theoretical exploitation, thus the theoretical block of praxeologies, are in turn shaped by the research tasks articulated and the techniques used for carrying them out. For instance, the solving of research tasks oriented by ATD will not lead to the identification of "epistemic actions" in the sense of AiC; and, reciprocally, the solving of research tasks oriented by AiC will not lead to the identification of the constraints conditioning the ecology of mathematical knowledge in a given institutional context in the sense of ATD.

Such interdependence of the different elements of research praxeologies leads us to conjecture that the networking of theories should be approached at the level of *research praxeologies*, and that, for being productive, the methodologies developed for such networking should allow researchers to consider both the practical and theoretical block of research praxeologies and their

[1] The term "methodology" usually denotes both the research methods or 'techniques' and the discourse developed around these methods.

interactions. The language used for expressing and supporting these networking practices is not neutral from this perspective. It must allow researchers to share the know-how of research praxeologies. If focused only on theories, it may reinforce the risk of underestimating the crucial role played by the practical block of research praxeologies. Up to what point did the networking group limit this risk, and how?

The presentation of ATD (see Chap. 5) also makes clear that the progression of knowledge goes along with the progressive structuration of praxeologies: point-wise praxeologies, characterized by a precise type of task and technique, organized into local structures sharing a same technological discourse, and at a next level into regional structures sharing some theory. Within this perspective, theoretical networking should oblige researchers to situate themselves at a regional level, considering that each piece of theory shelters a diversity of point and local research praxeologies. This is not an easy condition to satisfy, considering the constraints to which research projects are submitted. Up to what point have the different networking strategies allowed the networking group to address this difficulty, with what consequences? Another point is that all theoretical frameworks involved do not have the same size, in other words the same level of regionality. For instance, IDS and AiC are much more local than ATD and TDS. How has this affected the networking enterprise and its results?

15.2.2 The Dynamic Dimension of Praxeologies

Research praxeologies, as any other praxeological form, are living entities that evolve and change, which affects at the same time their four components and their interactions. The evolution of the practical block [T/τ] produces new theoretical needs that make the theoretical block [Θ/θ] progress and, reciprocally, the evolution of concepts, interpretations, or ways of thinking and the emergence of new results lead to the construction of new techniques and the formulation of new problems. In this dynamic, the two-level structure of the theoretical block of praxeologies has an important functionality. As said before, the *technological* discourse (θ) produces a first description, explanation, and justification of the research tasks approached (the *questions*, in the model provided by Radford (2008), the techniques used to approach them, and the first *results* obtained by this work). The *theoretical* discourse (Θ), as a second level of justification, contains the basic notions, conceptualizations, and principles used in the technological discourse and in the practical block. In most praxeologies, this second level is mainly implicit: it is made of the "folk knowledge" everybody uses without being conscious of it. In research praxeologies, it is important to make it explicit in order to control the assumptions made and to make them evolve if necessary. It is, however, a very stable hard core (in the sense of Lakatos 1978) of regional research praxeologies. The *technological level* of justification thus plays the "transactional" role of including the first results obtained in the practical block as preliminary descriptions of regular facts and phenomena, then transferring the most robust of these results to the theoretical block in the form of new principles to adopt and new germs of methodologies and problems.

The notion of didactic transposition in ATD can be a good example of this transactional role of the "technology" between the practical elements of research praxeologies (types of tasks and techniques used to approach them) and the theory. At the beginning, the process of didactic transposition was obtained as a result of the analysis of different mathematical school contents, to show that the mathematical knowledge that is taught at school can be questioned and compared to the scholar knowledge where it comes from and that legitimates its introduction at school. It thus appeared as a *result* of the investigations carried out, the description and explanation of a regularity observed, an element of the *technological* discourse. It was the explanation of a (hypothetical) phenomenon. Then new types of problems started to be raised (new *types of tasks*) using this process: how the didactic transposition of some given school content is carried out, how it affects the conditions of its teaching, what happens when the didactic transposition is interrupted, etc. After some research about the transposition of different contents, it became an assumption made in ATD (and also TDS) that any content involved in any teaching and learning process comes from a didactic transposition process, an assumption giving rise to a new *theoretical* ingredient. This result is no longer questioned; on the contrary, it leads to new research *techniques*, those of analyzing the taught contents, looking for the way they are described as "knowledge to be taught", and tracing their evolution from the scholar institutions to the school ones.

As in any other scientific discipline, and depending on the maturity of the field, research praxeologies can appear as different kinds of amalgams, more or less organized. It is the historical development of the field that helps structure these praxeological amalgams, making them more coherent and easier to diffuse, according to different didactic and institutional transposition processes that we are starting to know better. It seems reasonable to conjecture that while each of the didactic perspectives studied in this book can be considered as mature, this is not the case for the research praxeologies that the networking enterprise caused to emerge on top of these. Networking tasks have been articulated and germs of techniques developed, but, at this stage, these are certainly more craft techniques than well described and acknowledged research methodologies; the theoretical block of these praxeologies is still in an emergent state. This makes a dynamic vision of research praxeologies all the more important here.

15.2.3 The Role of Phenomena in the Dynamics of Research Praxeologies

In Artigue et al. (2011a, b), we argue that, for understanding the dynamics of research praxeologies, specific attention should be paid to the notion of didactic phenomenon, due to its emergence at the interface between the practical and theoretical blocks of research praxeologies: "In a first approach, we can characterise didactic phenomena as empirical facts or regularities that are raised through the study of research problems. Some of these phenomena can enrich the initial theoretical frame to produce new interpretations and new techniques or research methodologies, while others remain at the level of the "results obtained" and are reinvested to formulate new problems or to propose new diagnosis and practice-development tools, thus enriching the *technology*" (ibid., p. 2383).

In Chap. 12, for instance, three didactic phenomena are considered in the analysis of video-2: the Topaze effect, the funnel pattern, and the semiotic game. These three phenomena are incorporated in theoretical frameworks, respectively in TDS, Bauersfeld's Interactionism, and APC, and detached from the particular research praxeologies where they emerged. The Topaze effect is part of the theory of didactical contract in TDS, and identified as one of the didactic effects of the paradoxical nature of the didactic contract. The idea of semiotic game has been incorporated into APC and, beyond its theoretical status, it has become a didactical technique helping teachers align students' utterances with institutionalized forms of knowledge. Through the associated processes, these phenomena have been objectified and decontextualized, which explains why we could so easily invoke them for interpreting the video-2 episode. We can say that in both cases the *technological level* of the TDS and APC research praxeologies have evolved, even if the main principles and conceptualizations (the *theory*) remain stable.

15.3 Analyzing Networking Through the Praxeological Lens

For analyzing the networking enterprise through the praxeological lens, we first consider the tasks and techniques which have been developed along the project. We then come to the knowledge produced in terms of networking by solving these tasks. In terms of praxeologies, we thus study the emergence and dynamics of networking praxeologies, from their practical block to their theoretical block. Within such a praxeological perspective, all components are equally important and the lessons from this project involve all of them.

15.3.1 The Practical Block of Networking Praxeologies

15.3.1.1 Starting from a Technical Artifact: The Videotape of Two Students at Work

From the beginning of the project, the idea that its realization would require the sharing of a common object of study was clear to the different researchers involved in the networking group. The Italian team proposed to use a video associated with one of its projects, and the proposal was accepted. It seemed to the involved researchers that a video, while certainly influenced by the particular project at stake, its theoretical background, and the questions addressed, was an object open enough for starting a productive networking enterprise. However, at that time, the group did not discuss in depth the reasons that could made this video a "good transactional object." Its choice was partly one of convenience: taking an object already there made it possible to start the project immediately, exploring the networking potential of this object. Its limitations would certainly help select or develop more appropriate objects if needed. In fact, the video was a technical artifact inscribed in an APC research praxeology, and much more shaped by APC than the group initially imagined:

- the session was designed by a teacher-researcher working in close collaboration with the Italian colleagues;
- the Italian team was especially interested in the role of components of the semiotic bundle, and this had strongly influenced the way the students' activity and exchanges, as well as the interaction between the students and the teacher, were captured;
- the complementary information the Italian team thought necessary to give us was influenced by what they looked for in the data, and the information they needed for securing their interpretations.

In addition, the session had taken place in an educational system and culture that most of the members of the networking group were not very familiar with. However, as evidenced by the previous chapters, through the tasks designed around this artifact and the techniques developed, the group succeeded in transforming it into a transactional object and part of a productive milieu for its networking activities and emerging praxeologies. Analyzing the whole process through the praxeological lens thus led to investigating how tasks and techniques were progressively created, and what can be learnt from this activity in terms of networking praxeologies.

A first task spontaneously emerged: the different teams should analyze the video, each one with its specific theoretical lens. However, the networking project required anticipating and organizing the communication between the different analyses. This was achieved through a system of common questions, and through different techniques, progressively built. Two especially productive elements resulted. First, the difficulties the teams all had in using their technological and theoretical tools for developing the analysis from the video and the contextual information provided by the Italian team. This observation led to a first productive common question: each team was asked to identify exactly what it missed for carrying out the analysis of the video. It was also asked to make clear why it felt this limitation so problematic, and to connect the invoked reasons to the principles, questions, and methodologies specific to its approach. The answers to this question and their comparison played a key role in situating the different theoretical approaches with respect to each other, and understanding the respective lenses they used for approaching the "real world" and the influence of these lenses on their research practices. From that phase also resulted a questionnaire for the teacher-researcher. His answers, accompanied by a second short episode (video-2), complemented the material milieu the networking group was interacting with.

15.3.1.2 The Evolution of Milieus and Tasks

The second productive element came from these additional data: the description by the teacher of his didactic use of semiotic games in the answers to the questionnaire (cf. Chap. 2). For a diversity of reasons, all teams noticed this element. Once again, a specific task and a new process of study were built around this element, which transformed it into a transactional object. The technique used was the following. First, the TDS team was asked to associate a question with this element. The question, articulated in the TDS theoretical discourse, was about the possible relationship between semiotic games and a phenomenon of limitation of the a-didactic milieu. Each team was then asked to re-formulate this question within its own theoretical discourse before trying to answer it. Re-formulations, the work carried out in answering the resulting questions, and the answers eventually provided were then exchanged and discussed; new questions emerged, leading to work at the level of theoretical constructs and phenomena, and to progress in the networking enterprise. For instance, the use of video-2 for making sense of the teacher's discourse around semiotic games led to the case study reported in Chap. 12, in which the possible connections between the ideas of Topaze effect and funnel pattern were systematically investigated. More globally, each case study involving a few teams around the study of specific questions is the result of such a process.

A retrospective look at the whole enterprise shows this regular move from the contact with the initial then complemented milieu, to research questions and tasks collaboratively negotiated to exploit this milieu. These tasks organize the work of each team and pave the way towards productive exchanges around this work. In a second phase, these tasks and the work carried out for working them out become a new shared milieu with which the teams interact for answering questions and tasks situated at a more meta-didactic level. One of the first examples of such a move, not reported in this book, was the moment when, from the observation that all analyses of video-1 paid specific attention to the social dimension of the learning process, it was decided to clarify the ways this attention was expressed and theoretically instrumented in the different discourses, and compare them (Kidron et al. 2008). Chapter 10 on context, milieu, and media-milieu dialectic in fact obeyed a similar logic. It is worth noticing that, in these two cases, the move to a meta-didactic level had also as a consequence that the teams involved were obliged to take into account their respective theories at a regional level.

15.3.1.3 Some Less Successful Attempts

If we take seriously the needs of the networking enterprise in terms of contact with the range of research praxeologies associated with a given technological-theoretical block, there is no doubt that the initial milieu and its extensions mentioned above have evident limitations. It only allows approaching the research praxeologies of the different teams very partially. Retrospectively, we interpret a task proposed by Ken Ruthven at one of our first meetings as an attempt to overcome these limitations. The task had no link with the videos. It proposed to question our respective theoretical approaches through the way we would transform a teacher question into a research question. The starting point was thus an object external to the different research praxeologies involved, but it came from an empirical system shared by all of us: the profession of mathematics teacher in a European country. The example selected was the following:

> How is it that some students can learn to tackle a particular type of mathematical problem successfully (as shown by their performance in the class), but be unable to do so two weeks or months later? What strategies can the teacher use to reduce the likelihood of this occurring?
>
> Answer this question along the following guidelines and write 2 to 4 pages:
>
> (a) How do you – a priori – answer this question and what are your basic assumptions?
> (b) How do you transform the raised problem into a research question starting from the question above?
> (c) What is your research design?
> (d) What type of results would you expect?

All teams answered these questions, which were also proposed to the researchers involved in the Theory Working Group at the 5th Conference of the European Society for Research in Mathematics Education, and the eight responses received were presented and discussed at the conference (Prediger and Ruthven 2007). However, within the networking group, the task was no further exploited. Retrospectively, we see two reasons for this. On the one hand, the task started from an observation shared by all of us in our respective educational contexts, but it was too disconnected from the work we were engaged in for not being perceived as an artificial exercise; on the other hand, the initial milieu for this task did not offer sufficient potential of retro-action for dealing with the heterogeneity of the answers provided. Enriching the initial milieu would have been thus necessary for developing a productive networking activity. However, at that time, our understanding of the conditions to be satisfied for initiating productive networking praxeologies was not developed enough. This track was abandoned.

This was also the case for an initial attempt made at connecting directly our respective principles and key concepts through a system of conceptual maps. We worked on this task at one of our first meetings but did not find the results very convincing and gave up. Retrospectively, this attempt that was not further developed confirms our vision that connecting theories and concepts cannot be achieved without involving strategies that allow researchers to situate these within research praxeologies, and create appropriate milieus for that. At this starting stage of the networking, working at the level of the theory was only useful to point out the differences between the approaches, without helping in the mutual understanding of each other's visions and the searching for commonalities to promote collaborative analyses.

15.3.1.4 General Comments

We will not enter further into these attempts, but they must not be omitted from this retrospective reflection. They show that, in this new area of research, praxeologies are in a state of emergence. Tasks and techniques for solving them, that is to say appropriate methodologies, cannot be simply borrowed from the practical blocks of the research praxeologies familiar to us. In particular, the constitution of milieus and the organization of appropriate media-milieu dialectics likely to produce knowledge regarding networking are not obvious. Drawing the lessons from this particular networking enterprise imposes thus to precisely look at the tasks successively created along the process and the milieus arranged for these, not only at the results obtained. We conjecture that an important reason for the success of this project is that the networking tasks designed made it possible to overcome the limitation of an approach focused on the theories themselves. The anchoring of tasks in the analysis of two videos helped the teams engage some practical blocks of their respective research praxeologies and consider them as objects of study. The succession of tasks taking into account the questions progressively emerging from this study, and the associated evolution of the milieus with which the researchers interacted, played also a crucial role for addressing the different components of research praxeologies and their dialectic interactions. Another essential point is the way the different researchers contributed themselves to the milieu. Compared with networking efforts carried out by a single researcher, this networking enterprise engaged researchers with different backgrounds and theoretical expertise. This expertise contributed to the antagonist dimension of the milieus at stake. In most of the tasks collectively designed, researchers acting as elements of the milieu offered resistance to the interpretations or claims that other teams could propose; they obliged them to make visible implicit assumptions and arguments, naturalized in their research praxeologies. This antagonist role was reinforced by the fact that many researchers were not really familiar with the other theoretical approaches involved.

Beyond the level of tasks and milieus, the techniques used in the networking process were a combination of familiar research techniques and specific techniques used for carrying out the collaborative work planned. For instance, as made clear in the different chapters of the book, each team used its own techniques for analyzing the videos and the complementary material. Reading these analyses, one can grasp the technical diversity at stake, despite the limitation of the material involved, the essential pieces of it being a 1-hour video showing two students working essentially in an autonomous way, and a very short video complementing it. The specific techniques used for collaborative work included those usual in collaborative scientific work: presentations and discussions, group work on specific issues and collective reports, co-writing of texts, both in regular face-to-face meetings and at a distance. However, the evolution of tasks went along with an evolution in the organization of all these ingredients, the collaborative work taking a cyclic nature: formulation of a question, team work on this question, exchange and comparison of the work developed and its outcomes, reflection on its networking potential, new questions, etc. And, at the end, a systematic reflective stance with the interpretation of the whole process in terms of the ordered structure of networking processes. As shown by the case studies, the generating questions were of a different nature: from questions directly emerging from the analysis of the data as in Chap. 12 already mentioned, to more general questions such as in Chap. 10 in which the aim of the case study is to understand how three of the theoretical approaches involved, AiC, TDS, and ATD, take in charge the idea of context. However, one characteristic of the technical work developed in the case studies is its anchoring in the data shared by the networking group, and especially the two videos.

15.3.2 The Theoretical Block of Networking Praxeologies

A retrospective analysis of the networking enterprise through the praxeological lens must go beyond the practical dimension of networking praxeologies and consider their theoretical block. The current emerging state of these networking praxeologies does not make this an easy task: the technological and theoretical discourses are not fully articulated. However, as pointed out in the introductory chapter of the book, there is no doubt that this networking enterprise relies on theoretical principles. For instance, it considers theoretical diversity as a normal state of the field of mathematics education, not a sign of some scientific immaturity. It adopts a dynamic and functional vision of theories. These principles are expressed using a language and references familiar to the community of mathematics education. Along the development of the project, some aspects of a theoretical discourse progressively consolidated and became more specific. One example is provided by the differentiation between different forms of networking and their ordering along a networking axis. The networking group has systematically used this structure for situating its networking efforts and their outcomes, as attested by the different case studies, and this technological tool resulted in being useful. Another example is the more recent idea of networking profile introduced in Chap. 8.

Creating categories and hierarchies is often a first step in the development of a theoretical discourse. These constructions confirm thus that networking praxeologies are emerging. For approaching their theoretical block, it is certainly appropriate to consider the interface between the theoretical and practical block, the place where results emerge which can contribute to the development of a technological discourse and contribute to the praxeological dynamics. A first point to be mentioned is that the results of the networking work go beyond networking. As evidenced by several case studies, the tasks designed and the way they were carried out questioned the different theoretical approaches involved, not just their possible connections. A typical example is provided by Chap. 12, in which the interpretation of the same episode by three different phenomena led to a process of deconstruction–reconstruction of these phenomena, the reconstruction being influenced by the contact established among them. Even when there is no such process of deconstruction–reconstruction, each case study has as a result a deepening of the understanding of each theoretical approach by the researchers already experts in it. This could have been anticipated. In this long-term process, each theoretical approach, except APC, has been questioned on its capacity to make sense of data shaped by another educational and didactic culture; the interpretations each team provided have been systematically confronted with alternative views strongly defended by their authors; theoretical constructions have been challenged by researchers who did not understand them but wanted to make sense of them and of their potential.

However, whatever is the interest of such progression in the understanding of our own or other theories, what was expected were results in terms of networking. As shown by the different chapters, the project has produced such results, and they cover the different levels of the landscape for networking strategies mentioned above. This is not the place for listing them here. In line with the praxeological lens we adopt in this chapter, we prefer to focus on the way these results may support the emergence of a proper technological discourse.

Let us give an example, considering once again Chap. 12. In this chapter a connection is established between the Topaze effect and the funnel pattern. This is achieved through the following process. First, each phenomenon is situated within its theoretical environment and precisely described. Then the functional proximity between the two phenomena is made clear: the two of them are identified as ways of maintaining the fiction that learning has occurred when the conditions for such learning do not exist; this makes it possible to subsume the two phenomena under a common umbrella. However, the analysis makes clear how the characteristics of each theoretical approach shape the way this fiction is expressed, giving complementary insights on it. By doing so, the analysis makes visible the strength and limitation of each approach. The whole process results thus in an original technological discourse having clear networking characteristics.

The work carried out shows other possible formats for the emergence of a technological discourse attached to networking praxeologies. Without having the ambition of identifying all of them, we would like to give another example, considering Chap. 11. In this case, the two research praxeologies at stake are APC and IDS. When considering a given episode from a short video excerpt, they raise the common question (or research task) of how to explain a hypothetical failure of the teacher–students interaction. Then the technological elements provided by each approach as possible explanations appear to be contradictory. A common work starts which remains at the technological level: there is no contest of the basic principles of each frame nor of the type of methodologies used (both at the theory level). The final result is an enrichment of both technologies by a new emerging concept, that of "epistemological gap". We can forecast that, if the concept remains productive and robust in its use for approaching new tasks and in instrumenting new techniques, it could become a basic notion of the praxeology and enter its theory. What is sure is that the development also affects the practical block as the new analysis provided would lead to the raising of new problematic questions and the development of both analytical techniques. As the authors pointed out, this special case of networking praxeologies is certainly made possible by the proximity of their theoretical principles: view on data, unit of analysis, orientation, etc.

As a counterexample, a case of success and failure of networking can be mentioned referring to Chap. 9 on the epistemic role of gestures. A quite similar theoretical proximity between AiC and APC (at least at the level of the unit of analysis) enables both approaches to be enriched by the other – inclusion of the gesture analyses in AiC and of the epistemic dimension in APC. However, an attempt to include the ATD team in the networking initially failed due to the difficulties of the ATD researchers in combining their analysis with those of the AiC and APC teams. In the ATD theory, gestures are part of the praxeologies and, thus, of the knowledge that is to be taught and learnt and of the didactic strategies used to do so. This distance from the AiC and APC assumptions about the mediator role of gestures acted as a barrier for the integration of the ATD analysis in the common work.

15.4 Conclusion

Analyzing this networking enterprise through the praxeological lens makes clear that, within this project, specific networking praxeologies have been developed. Considering the questions raised in the introduction, and the risk of underestimating the crucial role played by the practical block of research praxeologies, there is no doubt that this risk has been avoided. If, during the first meetings, some attempts were made at connecting directly the different theoretical approaches through descriptions and maps trying to link the main concepts of each theory, quickly the strategies evolved to tasks allowing the researchers to mobilize both the practical and the theoretical block of their research praxeologies and make the whole praxeologies the object of joint study. This is certainly one reason for the success of the enterprise that the praxeological lens helps identify. What also contributed to the success of the enterprise was the fact that these research praxeologies were not considered as isolated objects, but were engaged in the solving of common questions around a shared set of data. One can observe here an evident proximity with the strategy developed in the European project ReMath, whose networking ambition was also clear regarding the semiotic potential of digital technologies. In ReMath, indeed a system of cross-experimentations was developed, common questions articulated about these cross-experimentations, and case studies carried out (Artigue et al. 2009). Common questions addressed and case studies are thus common ingredients of the two projects. In ReMath, however, cross-experimentations played a crucial role in the networking praxeologies developed. Each team was asked to experiment with two digital tools: one familiar, because produced by the team itself; and the other alien, because produced by another team from another country with a different theoretical background. The case studies focused thus on the comparison of the two pairs of experimentations of the same digital tool. Networking praxeologies were thus different, but the two projects shared the same vision of theories as dynamic and functional objects. ReMath also had the vision that networking could only be achieved through the production of specific tasks allowing making visible how theoretical concerns impacted the design of digital tools and their didactic use. The cross-experimentation process was one of the techniques used for making visible the tacit part of design and research practices. The techniques used in our project are certainly less demanding from an experimental perspective, but, in some sense, the limitation of the experimental constraints has allowed the focusing of the work on the progressive definition of tasks and constitution of milieus making us able to maximize the profit that could be taken from the limited corpus of data used. And the long term of this project with no external limit in time made this progression possible.

Such characteristics contrast with many earlier efforts made at networking theoretical frameworks, even if the word networking was not used. For instance, the Special Issue of *Educational Studies in Mathematics* (Zan et al. 2006) results from a Research Forum at the 28th PME conference, held in 2004 in Bergen, and considers the diversity of theoretical frameworks used in research on affect in mathematics education. As expressed by the editors, a special feature of the Special Issue is “to show how different frameworks can help in interpreting and intervening in students’ learning processes, through the analysis of an empirical account of a particular student’s solving of a mathematical problem in the classroom” (Zan et al. 2006, p. 118). However, in the six articles constituting the issue, the place attached to the analysis of this empirical account is very limited, and the different analyses are just juxtaposed.

In spite of the lessons provided by this experience, it however remains a very limited experience. Only a tiny part of the praxeological complexity of the research frames has been involved in the networking process. The networking tasks presented in this book, built around the collaborative study of a particular set of data, cannot engage the entire set of questions where each of the research praxeologies can show its potential, as well as its limitations. This is true for the five theories involved but is especially obvious for “big theories” such as ATD and TDS.

In the introduction of this chapter, we also raised the issue of the different sizes of the theory involved. In fact, this networking experience shows that differences in size are not necessarily an obstacle to networking processes, when adequate points of contact between theories are identified. For instance, Chap. 10 involves three theories of very different size: AiC, TDS, and ATD. As shown in Chap. 10, the networking process was associated in that case with a progressive extension of the perspective from the cognitive and individual perspective underlying AiC to the institutional perspective underlying ATD. TDS acted as an intermediate level, which on the one hand could be connected to AiC through its cognitive roots and vision of learning as an adaptation process, and on the other hand was connected to ATD through its systemic perspective and vision of learning also as an acculturation process. Moreover, the possibility of connection between AiC and TDS-ATD, already connected for decades within the French didactics community, was reinforced by a shared concern with the epistemology of the discipline. This shared concern was for instance made clear by the convergence between the a priori analysis made by AiC and TDS researchers.

Another fundamental element of the networking technology, its description and justification, is what we can call its “didactic component”. The main condition for networking to develop is the diffusion of research praxeologies among the community of researchers – a diffusion that is not just an acknowledgement of what is done in the different frames, their specificities, differences, and commonalities, but a high degree of comprehension at all levels of the research praxeologies. What we have called the “dialogue” between research praxeologies (Artigue et al. 2011a, b; Trigueros et al. 2011), the condition for researchers from different approaches to work together, needs special teaching, learning, and study conditions of the problems raised by the others, the methodologies used, the notions used to interpret the work done, and the kind of results obtained. It clearly appears at this point that the very reading of the others’ productions (papers, communications, informal analysis, teaching productions, etc.) is far from being enough to enable fruitful dialogues to develop. The craftsmanship dimension of research needs people meeting face to face, seeing the other carry out the research analyses, questioning and explaining the research gestures observed, trying to imitate the practice of others before fully understanding it. The results obtained in terms of research production are maybe not necessarily relevant; they are, however, absolutely crucial for the personal share of these research implicit skills and competences. The workshop activities that are not shown in this book, the walks, meals, informal discussions, the share of failure experiences, as well as some successes, are also part of networking praxeologies and should not be underestimated. The humility, modesty, patience, generosity of the participants – especially those with a deeper research background – are part of the conditions that should integrate a networking praxeology to make it effective. In fact, such practices are not new. They are normal ingredients of researchers' activity each time their work involves different communities, all the more different disciplines. What is new, however, is to take them as objects of study, to investigate their particular characteristics and ecology, to understand their dynamics and try to make them more effective, to identify their outcomes, and to share the resulting knowledge with the research community at large. For that purpose, ATD can be a useful lens.

References

Artigue, M. (coord.). (2009). *Integrative theoretical framework – Version C*. Deliverable 18, ReMath Project. http://remath.cti.gr. Accessed 29 Nov 2013.

Artigue, M., Bosch, M., & Gascón, J. (2011a). Research praxeologies and networking theories. In M. Pytlak, T. Rowland, & E. Swoboda (Eds.), *Proceedings of the seventh Congress of the European Society for Research in Mathematics Education* (pp. 2381–2390). Rzeszów: ERME/ University of Rzeszów.

Artigue, M., Bosch, M., & Gascón, J. (2011b). La TAD face au problem de l'interaction entre cadres théoriques en didactique des mathématiques. In M. Bosch et al. (Eds.), *Un panorama de la TAD. An overview of ATD* (CRM documents, Vol. 10, pp. 33–56). Barcelona: Centre de Recerca Matemàtica.

Kidron, I., Lenfant, A., Bikner-Ahsbahs, A., Artigue, M., & Dreyfus, T. (2008). Toward networking three theoretical approaches: The case of social interactions. *ZDM – The International Journal on Mathematics Education, 40*(2), 247–264.

Lakatos, I. (1978). *The methodology of scientific research programmes: Philosophical papers* (Vol. 1). Cambridge: Cambridge University Press.

Niss, M. (2007). Reflections on the state of and trends in research on mathematics teaching and learning. From here to utopia. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 1293–1312). Greenwich: Information Age.

Prediger, S., & Ruthven, K. (2007). From teaching problems to research problems. Proposing a way of comparing theoretical approaches. In D. Pitta-Pantazi & G. Phillipou (Eds.), *Proceedings of the 5th Congress of the European Society for Research in Mathematics Education (CERME 5)* (pp. 1745–1771). Cyprus: ERME.

Prediger, S., Bikner-Ahsbahs, A., & Arzarello, F. (2008). Networking strategies and methods for connecting theoretical approaches: First steps towards a conceptual framework. *ZDM – The International Journal on Mathematics Education, 40*(2), 165–178.

Radford, L. (2008). Connecting theories in mathematics education: Challenges and possibilities. *ZDM – The International Journal on Mathematics Education, 40*(2), 317–327.

Trigueros, M., Bosch, M., & Gascón, J. (2011). Tres modalidades de diálogo entre APOS y TAD. In M. Bosch et al. (Eds.), *Un panorama de la TAD. An overview of ATD* (CRM documents, Vol. 10, pp. 77–116). Bellaterra: Centre de Recerca Matemàtica.

Zan, R., Brown, L., Evans, J., & Hannula, M. S. (2006). Affect in mathematics education: An introduction. *Educational Studies in Mathematics, 63*, 113–121.

Chapter 16
From Networked Theories to Modular Tools?

Kenneth Ruthven

Abstract This chapter offers a critical appreciation of the networking project. It notes the origins of the participating theories in prior networking and draws out commonalities and contrasts in their pedagogical preoccupations. It highlights the opportunities that coordinated analysis and theory breakdown provide for elaboration of the participating theories and appropriation between them. Finally, it suggests that rather than conceiving synthesis in terms of an integration of theories, an alternative is to adopt a modular viewpoint which acknowledges the decomposability of theories into component analytic tools and the composability of tools from different theories.

Keyword Networking of theories

16.1 Introduction

This book arises from a sustained collective enterprise to explore the "networking of theories" within mathematics education. It reports, in particular, on collaborative work aimed at developing a sharper conceptualisation of the motivating idea of networking theories as well as productive methods aimed at operationalising that idea. The opening of Part I sketches the history of the enterprise, explains its motive and broad method, announces the five theories to be networked, and previews the structure of the book. What I did not find here was some reflection on the process by which the project came to focus on these five specific theories and perhaps on the exemplarity of this particular collection.

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Guided by Radford (2008), the project embraces a perspective in which a "theoretical approach" (in preference to "theory") is taken to be dynamic (rather than static) and is considered to be "embedded in the practical work of researchers" (Chap. 1, p. 6). Thus, a theoretical approach is seen as taking the form of an evolving "tool in use" which requires adaptation to each new situation (Chap. 1, p. 6) (rather than an already finished "tool to be used"). To reframe this in different terms, the book takes the fundamental objects of its interest as theorising communities of research practice rather than simply the theoretical reifications associated with such communities. To push this point further, the book focuses on communities of research practice which identify with a particular theoretical approach rather than those which embrace a more pragmatic theoretical bricolage. Nevertheless, as the next section will demonstrate, all five of the theoretical approaches examined in this book originated in the appropriation of reifications from various communities of research practice to create a hybrid practice drawing on multiple theoretical approaches.

In the light of the view that a theoretical approach is an evolving tool in use which requires adaptation to each new situation, it is also interesting to note that the participants in this enterprise found it necessary to adapt their organising theory of theories (based on Radford's triplet of Principles, Methodologies and Questions) in order to adequately survey the five theoretical approaches and present each of them:

> We had to extend the principles by Key Constructs, and we had to allow different orders among the four components Principles, Key Constructs, Questions and Methodology, since their mutual relationships are conceptualized differently in the five approaches. (Chap. 1, p. 7)

Much of the networking activity reported and reflected upon within the book involved parallel, then joint, analyses of a dataset originating from a previous teaching experiment that had been framed in terms of one of the participating theories. The final chapter of Part I of the book introduces this dataset, filling out the context in which it was gathered. Predictably, the original dataset had to be extended in order to become usable by the other theoretical approaches, highlighting the way in which any theoretical lens frames and filters experience and evidence:

> During the process of networking it became evident, that there was a need for further information about the background of the teaching experiment. Thus, written protocols, worksheet texts, detailed information on the students' background (as the teacher sees it) and information on the teacher's ideas were added to the already existing data corpus. (Chap. 2, p. 20)

16.2 The Origins of the Participating Theories in Prior Networking

The chapters in Part II of the book provide an overview of the participating theoretical approaches. Each of them arose through the appropriation and coordination of components from several prior theoretical approaches not specific to mathematics education.

APC (Action, Production and Communication) is characterised as having "its foundation mainly in two complementary theoretical assumptions: the multimodal perspective on cognition and communication, and the social-cultural characterization of human activity and thinking" (Chap. 3, p. 32). A particularly important resource drawn from the first body of theory is McNeill's theorised typology of gesture (Chap. 3, pp. 34–35), and from the second body of theory, a Vygotskian model of the socially and semiotically mediated growth of mental functions (Chap. 3, p. 35).

Likewise, AiC (Abstraction in Context) is described as "suitably choosing and interpolating between elements from cognitive and situated approaches as well as activity theoretical and constructivist elements" (Chap. 6, p. 86), and as making "hybrid reference to theoretical forefathers that belong to different traditions, Freudenthal and Davydov" (Chap. 6, p. 86). Each furnishes notions of abstraction through the ideas of "vertical mathematization" from the former and of "ascent to the concrete" from the latter.

Again, IDS (Interest Dense Situations) is reported as drawing first on "conceptualizations of individual interest… as a person-object relation", of "situational interest determined by situational conditions", and of "the connection of both concepts to self-determination theory" (Chap. 7, p. 98); second, on "a specific kind of social constructivism…[in which] learning mathematics is regarded as a process of constructing mathematical knowledge within social interactions, and individuals may co-construct knowledge by participating in and contributing to these constructions" (Chap. 7, p. 99); and more specifically on "interpretative teaching and learning research" (Chap. 7, p. 99), itself drawing on a philosophical theory of language concerned with "levels of… utterance" (Chap. 7, p. 100).

The expositions of these first three theories make quite extensive reference to their prior theoretical resources. Perhaps this is because these approaches have been developed more recently so that their sources not only remain salient in the collective memory of the research team associated with each theory but also continue to make explicit contributions to their research practice. Certainly, the longer history and wider community of the other two theories in play in this book has resulted in their developing a more autonomous form, going well beyond the foundational resources which they drew from prior theoretical approaches.

However, for TDS (Theory of Didactical Situations), important foundational sources appear to have been Bachelard's historical epistemology, particularly through "the didactic conversion of his notion of *epistemological obstacle*" (Chap. 4, pp. 48–49); and Piaget's genetic epistemology, particularly the idea that "the student learns by adapting herself to a *milieu* which generates contradictions, difficulties and disequilibria" (Chap. 4, p. 49). Likewise, for ATD (Anthropological Theory of the Didactic), important foundational sources appear to have been the sociological idea of "didactic transposition" proposed by Verret (1975) as the basis of a theory of school knowledge, and the "anthropological" notion of "praxeological knowledge" proposed by Bourdieu (1973) as the basis of a "science of practices".

Clearly, then, all five of the theoretical approaches on which this book focuses have combined key ideas from various prior theories and adapted them to focus specifically on some issue of mathematics education. At the same time, a process of dissociation has taken place, so that the functioning of these borrowed components is no longer disciplined – at least not directly – by the theory of origin as whole. Rather, the emphasis within each of these new theoretical approaches has been on coordinating these disparate borrowed elements to form a system, and on elaborating that system to create a conceptual framework through which the new theoretical approach can acquire its own identity and integrity.

There is a further important relationship between TDS and ATD, reflecting their shared origins in the French community of *didactique des mathématiques*. Not only have these theories been subject to similar influences and drawn common ideas from this wider community, but one is consciously a development of the other:

> The meaning and relevance of ATD has to be understood as a development of the project initiated by… TDS of a science of *didactic phenomena.* (Chap. 5, p. 68).

This means that ATD draws some of its central ideas from TDS. For example:

> ATD assumes an important postulate of… TDS: the fact that any piece of knowledge (i.e. any praxeology) can be considered as an answer provided —explicitly or *de facto*— to a question… (a problem or a difficulty) arising in an institutional setting (or a "situation"). (Chap. 5, p. 70)

The distinctive new contribution of ATD is to provide a much broader theorisation of this institutional dimension. Although recently, "developments such as the theory of joint action between students and teachers combine in an original way affordances both of TDS and ATD" (Chap. 4, p. 53), such developments do not feature in this book where TDS and ATD maintain separate identities.

Ultimately, the problematic of ATD is rather different from those of the other four participating theories, and its concerns somewhat distant from theirs. Consequently, ATD proved less well adapted to analysing the classroom episodes which provided the focus of much of the joint networking activity. For that reason, in this chapter I will focus on the other four theories.

16.3 The Pedagogical Preoccupations of the Participating Theories

The development of each of these four theoretical approaches has been motivated by the goal of formulating, in more systematic terms, a particular pedagogical approach to mathematics. There is an important commonality between the four theoretical approaches in question – AiC, APC, IDS and TDS – in that they all appear to privilege a pedagogy which gives a central place to relatively extended investigative or problem-solving activity by students within carefully engineered task environments designed to support the development of target mathematical concepts. Beyond this, however, there are important differences between these pedagogical approaches: for example, in the design logic underlying task sequences, and in the mediating role of the teacher in task-based activity.

Differences of both these types can be seen very clearly through comparison of the AiC and APC approaches. The development of AiC as a framework for analysing the didactical design of task sequences of a very particular character is made clear:

> In AiC we focus on particular kinds of curricula… and within these, on tasks with a high potential for supporting the construction of knowledge that is new to the learner… In brief, the design intends to create a didactical sequence aimed at vertical reorganization of students' knowledge. (Chap. 6, p. 93)

Seeking to use the AiC framework to analyse student activity on an APC-inspired task, it seems that one crucial difference between the two approaches is the emphasis of the former on situations calling for mathematical explanation and justification:

> Most of the [APC-inspired] tasks that the two students in the analyzed video… were asked to work on are not of this [AiC-specified] kind. These tasks require more phenomenological observation than explanations of the phenomena. (Chap. 6, p. 93)

Likewise, a distinctive form of pedagogical interaction, in which the teacher plays a very specific role, is highlighted within the APC framework: the "semiotic game".

> In a semiotic game, the teacher tunes with the students' semiotic resources (e.g. words and gestures), and uses them to make the mathematical knowledge evolve towards scientifically shared meanings. More specifically, the teacher uses one kind of sign (typically, gestures) to tune with the students' discourse, and another one to support the evolution of new meanings (typically, language). (Chap. 3, p. 38)

Not only is this deliberate scheme of interaction between teacher and students distinctive to APC; within AiC, no privileged interactional role is accorded to the teacher, who is regarded simply as one potential element within a broadly framed notion of context:

> For AiC, context has many components. One of them is the social context, often including peers or a teacher; another is the historical context, which refers to the students' prior experiences in learning mathematics; a third is the learning context, which includes, among others, curricular factors, socio-mathematical norms, and technological tools. (Chap. 6, p. 88)

Nevertheless, on the evidence of the video-recorded episodes analysed in the book, the pedagogy associated with APC also envisages substantial phases of largely autonomous student exploration of task situations analogous to those characteristic of the pedagogy associated with AiC.

These differences lead to correspondingly different motors of, and indicators of, "learning" being highlighted by the two approaches. In AiC, the key marker is evidence of a particular type of coordination by the student of prior concepts to create a new one:

> The model suggests constructing as the central epistemic action of mathematical abstraction. Constructing consists of assembling and integrating previous constructs by vertical mathematization to produce a new construct. It refers to the first time the new construct is expressed or used by the learner. This definition of constructing does not imply that the learner has acquired the new construct once and forever; the learner may not even be fully aware of the new construct, and the learner's construct is often fragile and context dependent. (Chap. 6, p. 89).

Whereas AiC focuses on student productions as marking the emergence of a new concept from a constructive learning process, APC focuses on the potential that such productions provide for subsequent interaction with the teacher through the semiotic game in a learning process conceived as much as imitative as constructive:

> Even a vague gesture of the student can really indicate a certain comprehension level, even when the student has not yet the words to express himself at this level…. [T]he semiotic game is likely to… be useful to the student… in a zone of proximal development for a certain concept… so that the teacher may have the chance to intervene in its cognitive development. The intervention is imitative-based, that is, the teacher imitates the students' gestures and accompanies them with certain scientific meanings (expressed in appropriated words), in order that in the following, the students will be able to imitate the teacher's words. (Chap. 3, p. 38)

This comparison can be extended to the remaining theoretical approaches. At the heart of TDS is a distinctive pedagogical approach grounded in the careful design of task specification and environment to create an "adidactical situation" capable of "making the target mathematical knowledge emerge from students' interaction with a milieu, as the optimal solution to a mathematical problem" (Chap. 4, p. 51). Again, seeking to use this TDS framework to analyse student activity on an APC-inspired task reveals an important difference in their approaches to task formulation:

> [The task tackled by students in the video] does not constitute a problem-situation… but is an exploration task… In this task, the expectations remain rather fuzzy. What criteria can students have for knowing that they have completed the task? (Chap. 4, p. 55)

While TDS like APC and AiC envisages substantial phases of student work on task situations involving little or no interaction with the teacher, the design logic of TDS firmly sequences such student-moderated activity as falling between two particular forms of teacher moderation:

> The processes of *devolution* and *institutionalization* were introduced for connecting the acculturation and adaptation dimensions of the educational enterprise. Both are under the responsibility of the teacher. Through *devolution*, the teacher makes her students accept the mathematical responsibility of solving the problem without trying to decode her didactical intention, and maintains it, creating thus the conditions for learning through adaptation. Through *institutionalization*, the teacher helps students to connect the contextualized knowledge they have constructed in the a-didactical situation to the target cultural and institutional knowledge and she organizes its decontextualization and transformation into "savoirs". (Chap. 4, pp. 52–53)

While the "semiotic game" within APC pedagogy shows concern with acculturation, it is not clear that this extends as far as the decontextualising and transformational characteristics of "institutionalization" within TDS pedagogy. Indeed, the two techniques appear quite different both in their interactional characteristics and in their conception of the learning process. Moreover, the active, if differing, roles explicitly accorded to the teacher within TDS and APC stand in contrast to AiC:

> Contrary to AiC, the teacher for TDS is neither an element of the context nor a component of the milieu: he is an actor. (Chap. 10, p. 173)

Finally, let me extend this comparison to IDS. First, development of IDS was motivated by pedagogical concerns which were as much affective as cognitive:

> The development of the theory of interest-dense situations began… with the assumption that in mathematics classrooms, the social situation plays an important role for the question as to whether learning with interest is possible or not. This theory was formulated to determine how to build situations with the potential to support learning mathematics with interest in everyday classrooms. (Chap. 7, p. 97)

Indeed, IDS sees the affective and cognitive aspects of situations as interacting through their link to the culture of the classroom and the qualities of task-based interaction:

> Interest-dense situations are particularly fruitful epistemic situations which can occur in everyday mathematics courses when the learners work co-operatively and intensely to advance their own and their peers' ideas (*involvement*), construct further and deeper mathematical knowledge (*dynamic of the epistemic process*) and highly value mathematical objects or methods (*attribution of mathematical value*). (Chap. 7, pp. 98–99)

The two ideal types of interaction structure that IDS identifies appear rather similar to contrasting forms of didactical contract identified within TDS. In particular, the "expectation-recessive interaction structure" which IDS takes as fostering interest-dense situations closely resembles the type of didactical contract which TDS regards as necessary to create a genuinely "adidactical situation" within TDS:

> If *expectation-independent student behaviour* and *situationally steered teacher behaviour* mix, an *expectation-recessive interaction structure* emerges in which both, teacher and learners, concentrate on and support processes of constructing mathematical meaning independently of the teacher's expectations. *It nurtures the emergence of interest-dense situations*. (Chap. 7, p. 101)

Moreover, the argument of IDS that "these situations are considered as interest-dense because their underlying epistemic processes encourage students to be more attentive and engaged, thus leading to dense social interactions" (Chap. 7, p. 99) appears conducive to the process of "devolution" within TDS through which "the teacher makes her students accept the mathematical responsibility of solving the problem without trying to decode her didactical intention, and maintains it, creating thus the conditions for learning through adaptation" (Chap. 4, p. 52).

However, there are key differences between the scope of IDS and that of the other three theories under discussion:

> The theory of interest-dense situations is a social constructivist theory that cannot say much about cognitive processes of individuals and does not provide tools for epistemological analyses. (Chap. 7, p. 102)

Nevertheless, central to IDS is a model of "epistemic actions":

> [F]ruitful epistemic processes within an expectation-recessive interaction structure… are built through three central collective actions executed within social interactions: gathering and connecting mathematical meanings, and seeing structures. Gathering meanings refers to collecting bits of mathematical meaning that are similar with respect to solving the posed problem. Connecting meanings happens if a limited number of collected bits of meaning

> are interconnected or linked to other meanings. If there are sufficient collecting and connecting actions structures can be seen, that is a system of relationships for which many examples can be found. (Chap. 7, pp. 101–102)

Indeed, there are parallels in the terminology of "epistemic actions" that IDS shares with AiC, and in the trios of action types proposed by these two theories, although AiC seems to characterise the logic of such action types more precisely:

> [T]he emergence of a new construct is described and analysed by means of three observable epistemic actions: recognizing, building-with and constructing. Recognizing refers to the learner seeing the relevance of a specific previous knowledge construct to the problem at hand. Building-with comprises the combination of recognized constructs, in order to achieve a localized goal such as the actualization of a strategy, a justification or the solution of a problem…. Constructing consists of assembling and integrating previous constructs by vertical mathematization to produce a new construct. (Chap. 6, p. 89)

16.4 Coordinated Analysis as an Opportunity for Appropriation

The core of Part III, and of the book as a whole, consists of four chapters reporting directly on case studies of analytic networking between theories. The first example, presented in Chap. 9, is a relatively straightforward one. The microanalysis of learning and teaching activity which characterises both AiC and APC often calls for relatively high inference (and therefore less confident conclusion) from the evidence available. Influenced by APC's attention to the whole "semiotic bundle", the networking activity leads AiC to embrace consideration of gesture as well as utterance, broadening the spectrum of evidential information available, and so potentially reducing the level of inference required and rendering conclusions more confident. This is particularly so if, as the chapter suggests, "modes of expression tend to be strictly linked with each other, and… the interpretation of one of them is linked to the interpretation of the others" (Chap. 9, p. 142). The preoccupations of AiC encourage particular attention to gestures which (may) contribute to the constructing of concepts, and this leads to a distinction being proposed between an epistemic function of such gestures (for example, through representing an idea kinaesthetically) and a communicative function (for example, through raising others' awareness of the constructing process).

The AiC team, in particular, then, considers that this networking activity has given rise to productive development of that theoretical approach. It provides an example in which networking leads one theoretical approach (AiC) to appropriate from another (APC), on a more permanent basis, an analytic tool (gesture analysis) which extends its scope for analysis (from a semiotic frame focusing on utterances alone to one incorporating both utterances and gestures). While, in principle, a similar synthesis might have arisen through networking between AiC and the theory of gesture which originally influenced APC, the greater congruence between the concerns and methods of these two theories was conducive to their forming a networking partnership, providing a more indirect link between AiC and the gestural source theory of APC.

Equally, the reflection that Chap. 9 offers on the networking experience concludes that “together, the two analyses provide far deeper insight than each one separately” (Chap. 9, p. 146). More specifically, perhaps because of important underlying congruences between the concerns and methods of the two theories, “neither the difference in the way the two teams [read] the flow of the activity, nor the difference in the way [they] see the function of the gestures leads to contradictions”; rather the two analyses “complement each other and point to failures in each team’s analysis to grasp and describe the complexity in a more comprehensive way”. Thus this networking activity also provides an example of how developing a coordinated use of two compatible and complementary theoretical frameworks can enrich an analysis.

In Chap. 10, the focus of networking is on the comparison of apparently cognate concepts from different theories, notably the notions of *context* in AiC, *milieu* in TDS (and *milieu / media* in ATD). As noted earlier, within AiC, *context* serves as a comprehensive placeholder for a range of types of contextual shaping of the process of conceptual construction by students, whereas, within TDS, *milieu* is a central component of the architecture of an adidactical situation. Although the *milieu* can take different forms, it has a precise and distinctive function within the theoretical system: it is the task environment with which students interact and which provides them, in particular, with feedback on their constructions. Thus, whereas *milieu* is tightly bound into the conceptual system at the core of TDS and closely defined, *context* gives AiC much broader licence to incorporate whichever of “such a variety of contexts” (Chap. 6, p. 86) appear relevant into the conceptual framework for a particular analysis. This treatment of context as placeholder provides a convenient point of entry for some external theorisation of contextual features into an AiC analysis. In this respect, then, AiC appears more open to stronger forms of networking than TDS.

Chapters 9 and 10 both involve networking between AiC and other theories. There is greater congruence between the concerns and methods of AiC and APC in Chap. 9, than between AiC, TDS, and ATD in Chap. 10. Indeed, the concluding reflections to Chap. 10 note how important it was that some point of contact could be found as a base for networking between the theoretical approaches: in this case, “a common epistemological sensibility” (Chap. 10, p. 174) dislayed in the substantial commonalities of *a priori* analysis within each approach. But, in the *a posteriori* analysis, AiC surveys the role of different contextual elements in students’ construction of knowledge, ranging over the task, the learners’ personal history, the computer software, the teacher, and the teachers’ learning goals (Chap. 10, pp. 161–162). By contrast, TDS homes in on “the limitation of the a-didactical milieu” in play and identifies “a specific technique used by the teacher for compensating this limitation” (Chap. 10, p. 168). The two analyses eventually converge when AiC “points to the limitations of the semiotic games” while TDS highlights “the limitation of the milieu” (Chap. 10, p. 172). The complementarity of the two approaches is that while AiC’s fine-grained analysis of epistemic processes reveals subtle evolutions at the student level in the course of knowledge construction, TDS offers an overview of the didactical system which provides a more systematic anticipation of factors likely to affect the unfolding of knowledge construction (Chap. 10, p. 172). Here again, then, the coordination of compatible and complementary theoretical approaches enhances the richness of the analysis possible. However, unlike the case reported in Chap. 9, there appears to be no residual appropriation of tools by one theory from another as a result of the networking activity.

16.5 Theory Breakdown as an Opportunity for Elaboration

The empirical sections of Chaps. 11 and 12 both focus on the same classroom episode, lasting about a minute and a half. This episode involves, on first impression, two students and their teacher interacting about a task; and, on closer examination, the unfolding of some form of breakdown in the pedagogy of semiotic games. In these two chapters, arrays of theoretical resources are brought forward to analyse this episode, providing opportunities to assess and compare them in application.

The results of the initial analyses presented in Chap. 11, using the IDS and APC approaches "appeared almost contradictory" (Chap. 11, p. 180) to the two teams involved. The initial IDS analysis of the episode detects a pattern of interaction in which the teacher rejects the unfolding line of thinking being expressed by one student, and introduces a different line of thinking which is then developed through a series of forced rhetorical questions. Because neither teacher nor student grasps the other's perspective, there is no basis for successful negotiation between them. In line with the concern of IDS with generating interest-dense situations, the way in which this pattern undermines the basis for the student's engagement in the task is highlighted.

The initial APC analysis of this same episode detects a crucial misinterpretation by the teacher of the student's line of thinking as misconceived in a particular way. In line with APC's semiotic focus, what is highlighted is the way in which the teacher's ensuing repetition and rephrasing of the student's words is accompanied by gestures intended to draw attention to a particular feature of the graph which could serve to challenge what the teacher imagines to be the student's misconceived idea. An extension of this line of analysis then identifies how, in this episode, there is an inversion of the typical structuring of semiotic registers in the teacher contribution to interaction. Rather than the normal pattern of the teacher overlaying a sanctioned verbal narrative on gestures taken from, or referring to, (the germ of) an approved line of thinking preferred by a student, here the teacher incorporates fragments from the student's verbal narrative to a disapproved line of thinking into his own spoken contributions, accompanying them with gestures aimed at highlighting the basis for a counterargument to the student's line of thinking.

Nevertheless, the coordinated analysis that was then undertaken suggests that the two theoretical approaches are compatible and that they provide complementary insights. This analysis leads to the introduction of the notion of there being an "epistemological gap" between teacher and student in the episode. The suggestion is that, at root, the interactional pattern that unfolds originates in the gap between a student reasoning from visual perception as against the teacher reasoning from mathematical properties (Chap. 11, pp. 192–193). Here, this "gap" phenomenon is linked to a further available theory of "personal epistemologies" and "epistemological views". Indeed, this example illustrates the way in which the approach to analysis characteristic of this book extends, not infrequently, beyond strict adherence to the disciplined use of well defined constructs from the theory that is explicitly in play to incorporate more casual use of ideas drawn from the analyst's own wider repertoire of commonsense and scientific thinking. In this case, perhaps a more powerful extended theorisation for the purposes of mathematics teaching would combine a genetic epistemology of mathematical thinking (to locate the positions defining the gap within a developmental model) with a theory of mathematical knowledge for teaching (to relate the breakdown to a blind spot on the part of the teacher in terms of such knowledge).

In Chap. 12, the interactional pattern in this same episode is analysed in terms of its fit to ideal types associated with three of the theories: the Semiotic Game of APC (which is the theory associated with the intended pedagogy), the Topaze Effect of TDS, and the Funnel Pattern from the interactionist tradition on which IDS draws. Here, I was surprised that the IDS analysis had recourse to the Funnel Pattern, characterised as "narrowing of actions by expected answer" (Chap. 12, pp. 210–211), rather than to the core notions of its own theory. Specifically, in this episode we might well be witnessing the transition from an "expectation-recessive" interaction structure (as described earlier) to an "expectation-dominant" one:

> The *expectation-dominant interaction structure* appears if the teacher and students are guided by the teacher's content-specific expectations towards a task. It is more stable and hinders the emergence of interest-dense situation because the teacher guides the students in such a way that they produce exactly what the teacher wants to hear, while the students try to figure out what the teacher wants to hear. If an expectation dominant interaction structure occurs within an epistemic process the emergence of an interest-dense situation is deeply disturbed. (Chap. 7, p. 101)

From the individual and coordinated analyses undertaken, the video-recorded exchange between teacher and student proved not to correspond to any of the proposed ideal types, suggesting a need for further elaboration of this aspect of all three theories. However, the participating researchers report that the process of networking their theories through comparing these ideal types and examining their degree of fit to the observed interaction led to their "experience[ing] a scientific progress, namely a step of developing theoretical understanding towards increasing explicitness of the theories' principles… and improving connectivity" (Chap. 12, p. 218).

16.6 Conclusion

The four case studies in Part III provide strong evidence, of the benefits of the networking which took place, first in deepening the analyses produced of the episodes under study, and second in developing the participating theories through stimulating clarification and refinement of existing constructs and (more rarely) the appropriation and development of new ones. Certain features of the “empirical” networking activity were clearly conducive to producing a decentring on the part of each of the research teams which encouraged reflection and development: working to translate and augment “data” produced under a different theoretical approach to meet the needs of their own theory; comparing and contrasting independent analyses of the same episode produced by teams applying different theories; developing a more coordinated analysis of an episode which incorporated and harmonised insights from two or more theories. In the course of such networking activity, research teams found themselves obliged to think more deeply about aspects of their theory in order to engage with the differing perspectives of other teams. These features of the networking activity are highlighted in the second half of Chap. 14.

While it is not always easy to typify particular examples of networking, it is clear that the activity across these case studies illustrates most of the processes envisaged in Chap. 8:

> understanding and making understandable, comparing and contrasting, combining and coordinating, and integrating locally and synthesizing. (Chap. 8, p. 119)

The exception seems to be the last of these processes – synthesizing. The final qualifying phrase of the question motivating the enterprise (shown by my italics) suggests some discomfort with this from the outset:

> How can we network different theoretical approaches, i.e. what methods, strategies and meta-theoretical constructs are needed for creating a dialogue and establishing relationships between parts of theoretical approaches *while respecting the identity of the different approaches?* (Chap. 8, p. 122)

This is hardly surprising, of course, in a project which involved several research teams each already with a strong commitment to a particular theory. Nevertheless, as shown by the way in which, at their inception, all these theoretical approaches borrowed and combined components from disparate sources, theoretical identities too are fluid and transient, as indeed the project’s guiding idea of evolving theoretical approaches also implies.

Equally, the way in which these tightly focused but broadly commensurable theories were used in a coordinated way in some of the case study analyses points to another way of thinking about synthesis. Rather than conceiving synthesis in terms of achieving an integration of theories, an alternative is to view it in terms of increasing the integrability of their components, in the sense of developing the scope for coordinated adaptation of tools drawn from several theories, or even common to several (as already exemplified by TDS and ATD), each chosen for use in a particular study because of the distinctive functionality that it contributes to the proposed analysis. This involves adopting a modular viewpoint, both with respect to the decomposability of theories into component analytic tools and with regard to the composability of tools from different theories; through the possibility either of one theory borrowing tools from another or of new theoretical frameworks being improvised which combine tools from several source theories to address a new type of question or an old type of question in a new way. The manner in which the component of context seems to function as a placeholder in AiC, open to imported theorisation of any of the many aspects of context, hints at this possibility. In this vision, one of the functions of networking would be to develop, first the modularity of individual theories by identifying their component tools (as well as the overarching way in which that particular theory organises them), and second the commensurability of theories by establishing some kind of "shared sensibility" which would facilitate the borrowing and combination of such tools. This is an idea which has already received some discussion around specific examples from mathematics and science education, including TDS (Ruthven et al. 2009). But, of course, this vision reflects my membership of a research community with a strong pragmatic motivation that embraces theoretical bricolage.

References

Bourdieu, P. (1973). The three forms of theoretical knowledge. *Social Science Information, 12*(1), 53–80.

Radford, L. (2008). Connecting theories in mathematics education: Challenges and possibilities. *ZDM – The International Journal on Mathematics Education, 40*(2), 317–327.

Ruthven, K., Laborde, C., Leach, J., & Tiberghien, A. (2009). Design tools in didactical research: Instrumenting the epistemological and cognitive aspects of the design of teaching sequences. *Educational Researcher, 38*(5), 329–342.

Verret, M. (1975). *Le temps des études*. Paris: Honoré Champion.

Chapter 17
Theories and Their Networking: A Heideggerian Commentary

Luis Radford

Abstract The chapter briefly discusses the construct of theory and the contribution of networking theories to mathematics education research. It starts by a reflection on the meaning of theories in general and in mathematics education in particular. Dwelling upon Heidegger's etymological analysis of theory, it stresses the ineluctably tension between the phenomena a theory tries to account for and the manner in which the account is carried out. The comment concludes by suggesting that networking mathematics education theories offers a unique possibility to grasp a thematized and systematic array of sides of educational problems.

Keywords Networking theories • Heidegger • Semiosphere • Methodologies

17.1 Theory

The concept of theory is an elusive one that often escapes the realm of definitions, regardless of how hard we try to pin it there. Buried under numerous layers of meaning, theory seems to appear differently depending on the discipline that evokes it. Some of us grew up thinking of theory as a kind of lens through which we perceive, interpret, and interact with our surroundings. This is the meaning of theory that we inherited from the ancient Greeks, who, as we well know, cast knowledge in a metaphor of vision. Martin Heidegger reminds us, indeed, that the word *theory* derives from the Greek verb *thēorein*, a verb that comes in turn from two root words: *thea*

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and *horaō*. *Thea*, from where the word theater derives, "is the outward look, the aspect, in which something shows itself, the outward appearance in which it offers itself" (Heidegger 1977, p. 163). *Eidos* is the name Plato uses to refer to which shows itself in the phenomenological realm, that is, a *presence that makes itself present* (e.g., an idea, a thing), more specifically the "aspect in which what presences shows what it is" (p. 163). To know, *eidenai*, is to have seen this aspect. *Horaō* means "to look at something attentively, to look it over, to view it closely" (p. 163). For the ancient Greeks, then, theory consisted in looking "attentively on the outward appearance wherein what presences becomes visible and, through such sight—seeing—to linger with it" (p. 163).

The modern term observation, which comes from the Latin word *contemplatio*, refers to the Greek lingering vision metaphor and moves the term theory into new territory. Although it stresses the visual metaphor through which theory is conceived, as in *vita contemplativa*, it adds a new array of efforts that have to be made in order to render visible the thing to be seen. With da Vinci and Galileo the meaning of theory changes: the border line separating *bios theōrētikos* (a theoretical form of life) and *bios praktikos* (a practical and productive form of life) somehow vanishes and theory appears as an endeavor where one strives to manipulate something, to work over it, to pursue it, "to entrap it in order to secure it" (Heidegger, p. 167). And it is *objectness*, that is, this feature of entrapping something as an object to be secured, that, according to Heidegger, characterizes the modern concept of science—a concept that "would have been as strange to medieval man [*sic*] as it would have been dismaying to Greek thought" (p. 168).

Thus, when Euclid proves the Pythagorean Theorem, he resorts to the original *bios theōrētikos*: Euclid's proof consists indeed in attentively looking at the outward appearance of the right triangle and the squares built on the sides; when areas are compared, he is looking at the relations over and over, closely, lingering, waiting so to speak for the relational presences to become visible through sight. When Galileo is busy measuring time using a large pail filled with water descending along a channel carved on an inclined plane, he resorts to a conception of theory or theoretical approach where the original senses of *bios theōrētikos* and the *bios praktikos* have merged. Galileo's deeds illustrate very well how manipulation and planning become important in the modern concept of theory. From the Renaissance on, mathematics moves too from its central place within a *bios theōrētikos* to a synthesis that starts featuring the dimensions of a *bios praktikos* where reckoning comes to the fore: mathematics becomes not merely the science of reckoning:

> in the sense of performing operations with numbers for the purpose of establishing quantitative results …. Mathematics [becomes] the reckoning that, everywhere by means of equations, has set up as the goal of its expectation the harmonizing of all relations of order, and that therefore "reckons" in advance with one fundamental equation for all merely possible ordering. (Heidegger 1977, p. 170)

There are several aspects that come to the fore in the concept of theory. A theory is always a theory of *something*—an object-area. A theory is always about the mattering and happening of specific kinds of entities that Heidegger calls *presences*. It is in this sense that theories work as filters that discriminate between presences and their importance. But when we do so, we highlight presences and link them in ways that make them appear as congruent wholes. These links that we create between presences comprise *meaning*.

To bestow meaning on what otherwise would remain an overwhelming flux of sensorial data, we codify our experience of the world in more or less explicit ways: we create patterns of understanding and action. Although culturally codified shared experience comprises a vast territory, some parcels of it are highlighted and expressed through language; they acquire the status of *principles*. Acting hence as filters, these principles (P) allow us to refer to *presences*—problems, questions, tasks, situations. As a result, questions, problems, and tasks (Q, in short) are already imbued with a theoretical layer. It is this theoretical layer that allows us to recognize for instance two tasks as *similar* or even a task *as such*. But this does not mean that the theoretical principles of a theory predate genetically the problems or the tasks. There is a fundamental dialectical relationship between them. P and Q are formed simultaneously; they co-emerge.

This picture, however, is incomplete in an essential way. For the systematic actions that we undertake to cope with a task—i.e., the methodology, M—is consubstantial with the principles P and questions Q that we use to recognize or formulate a task as such. This is why a theory—or a theoretical approach—can, analytically speaking, be thought of as a triple (P, M, Q) (Radford 2008) only if we do not forget that there is a profound entanglement between these three “components” of a theory and that none of them can be reduced to the others or serve as the constitutive basis for the others. Because of their mutual genetic constitution, we should talk about these components as being in *trialectical* existence.

Now, to talk about a theory as a trialectical entity means to conceive of it as something dynamic, an entity in movement with layered descriptions of reality that emphasizes at certain times P, Q, or M, or two or all three of them. What is characteristic of a theory is that, in its movement, it produces results. Results may refer to new interpretations of presences (i.e., the *objects* of the theory), the identification of new presences or relationships between presences, etc. The results of a theory may require some adjustments and the transformation of its components, P, Q, and/or M.

The dynamic dimension of a theory, however, cannot be limited to the manner in which it is affected by its own results. Theories develop not only through the internal trialectical relationship of its own components. Theories are produced within cultural formations and live and interact with other theories.

17.2 The Semiosphere

Following semiotician Yuri Lotman’s (1990) ideas, I have suggested (Radford 2008) that we can think of theories in general, and theories in mathematics education in particular, as evolving in a *semiosphere,* that is, a multi-cultural, heterogeneous, and dynamically changing space of conflicting views and meaning-making processes generated by theories and their different research cultures. It is in a semiosphere that theories live, move, and evolve. It is in a semiosphere that theories come into a relationship.

What characterizes what has been termed the networking of theories is the explicit goal of bringing theories together. That is, to put them in explicit relationship so that theories get connected or networked within a same research project.

There are different possible forms of connectivity. In their seminal paper, Prediger et al. (2008) identify some of them, including "comparing" and "contrasting," "coordinating" and "combining," "integrating locally" and "synthesizing."

As suggested previously (Radford 2008), the possible forms of connectivity are constrained and afforded by the nature of the theories, but also by the research goal of the connectivity research project. In general terms, a network N of theories T_1, T_2, T_3, … can be seen as a set of connections c_1, c_2, c_3, …, where c_k involves at least two theories T_i, T_j (in what follows, to simplify, I will assume that only two theories are networked).

Using the semiosphere's spatial metaphor, theories T_i and T_j can be visualized as being "closer" or "further" depending on their own (P_i, M_i, Q_i) and (P_j, M_j, Q_j) structures. The connection c_k of T_i and T_j requires the identification of research questions Q_{ij} (tasks, problems, etc.) that guide the enterprise as well as the building of a new methodology M_{ij} to answer the research questions under consideration.

One of the key research questions that have been investigated within the networking theories research community is the manner in which the analysis of classroom events differs when conducted through different theories or theoretical lenses. At the level of methodologies a typical example (used in this book) has been the analysis of a common videotaped lesson or segment of it under different theories. Another example of methodology consists in the creation of educational tools within a theory that are then used in the classroom and analyzed through the lenses of that and other theories (Radford 2014). This endeavor has led the corresponding research teams to learn from each other, to improve and refine their own theories, to understand them better, and to become more sensitive to other ways of theorizing.

As the chapters of this book show, the networking task is not easy, but it is rewarding. The task is not easy, among other reasons, because theories may use the same theoretical names with different meanings. They may resort to different theoretical principles and conceptualize differently the basic phenomena under scrutiny; they may also resort to different methodologies or to have a different set of concerns leading to different research questions. A networking task hence requires an open mind from the outset. It requires the capability of opening oneself to others and moving across theoretical approaches in a cautious and reflective way. By being confronted by, or immersed into, new theories, new predispositions towards new emerging shared interpretative situational contexts become available. The structuring background of shared reality shifts and new forms of action and understanding become possible. Researchers become endowed with new possibilities to look at their home theories and to see the familiar through new stances that make the familiar look unfamiliar and hence open to scrutiny, critique, and change.

In a networking task there is always a tension that results from putting together different theories. The tension is not, however, something to be seen in negative terms. It is this tension that pushes the networking task further and moves the theories to each other. The tension does not need to end up in a harmonious synthesized point in the semiosphere. In his thoughtful critique Kenneth Ruthven (Chap. 16 in this book) notes that the synthesis of theories is the kind of connectivity that does not seem to appear in the examples shown in the book. Certainly, a synthesis is the most difficult kind of connection to achieve. But there may be an *unresolved synthesis*, that is, a synthesis where theories do not disappear to create a new entity, yet the theories are radically shaken and transformed. The synthesis appears not in a new single entity, but in the imprint that the other theories leave in the transformed theory.

17.3 The Question of Learning

In this section, to illustrate the tensions that ineluctably arise in the networking of theories, I would like to comment on learning.

Learning mathematics is indeed one of the central concerns of most theories or theoretical approaches featured in this book—other important concerns revolve around knowledge and how it appears or is practiced in a manner conforming with the institutional dimension in which it operates, as in the *Anthropological Theory of the Didactic* (ATD). Learning can be conceptualized in different ways and operationalized in distinctive manners. Even the questions that are asked about it vary from one theoretical approach to another. Following Heidegger's ideas, let me suggest that learning mathematics can be considered as a *presence* whose presencing is differently framed by educational theories in accordance with the lenses they provide and the manipulative (i.e., methodological) endeavors that they make to reveal its presence in the classroom—to make it come to stand and lie in unconcealment (Wrathall 2011), that is to become object of thought and consciousness. Let me also suggest that, at its most general level, learning is a process of tuning with life and that its being is interwoven in threads of objective, subjective, conceptual, aesthetical, ethical, and political matters. What theories provide us with are not really truths, but *moments* of learning-as-being. For instance, when researchers resort to the *Abstraction in Context* theoretical approach (AiC), they posit the problem of learning as distinctive kinds of students' deeds and focus on actions that can be identified as "building-with," "recognizing," and "constructing." Learning appears—or is expected to appear—through these lenses that unavoidably transform it into an object of specific form. The expected object, regardless of the theory, is always partial, as it has undergone a process of filtration or a process of entrapping that seeks to secure its recognition in the advent of its presencing. Hence, we recognize some aspects of learning, but not the whole of it.

Referring to the sciences in general, Heidegger notes their impotence in grasping their topical presences in their totality and suggests that "this impotence of the sciences is not grounded in the fact that their entrapping securing never comes to an end" (Heidegger 1977, p. 176). And he goes on to dispel the idea that the problem would be merely methodological. The problem, in fact, is *ontological*. He continues:

> [the impotence of the sciences] is grounded rather in the fact that in principle the objectness in which at any given time nature, man [*sic*] history, language, exhibit themselves always itself remains only *one* kind of presencing, in which indeed that which presences can appear, but never absolutely must appear. (Heidegger 1977, p. 176; emphasis in the original)

Regardless of their theoretical sophistication, concepts remain, and will remain, precarious vis-à-vis the presences they strive to reveal.

But again, the transcendence of the presences they strive to reveal does not stem from the insufficiency of our methodologies or concepts, but from the presences' ontological constitution. The reason is not to be found in the idea that presences such as learning are Kantian "things in themselves" or immutable Platonic beings. Their transcendence has rather to do with their own fluid nature: they are moving pointers refracting the complexities of life—not objects to be grasped, like apples with our hands, but mobile pointers that invite us to historical journeys through which to explore our place and possibilities as humans in the historical, cultural world of practice.

Within this line of thought, if learning is a process of tuning to life, learning changes with life, nature, and the individuals that come to inhabit and transform nature and the cultural world.

It is this unique possibility of offering us a thematized and systematic array of sides of learning and other crucial problems that I find of vital importance to mathematics education in the networking theories research field.

References

Heidegger, M. (1977). *The question concerning technology and other essays*. New York: Harper Torchbooks.

Lotman, Y. (1990). *Universe of the mind. A semiotic theory of culture*. London: I. B. Taurus.

Prediger, S., Bikner-Ahsbahs, A., & Arzarello, F. (2008). Networking strategies and methods for connecting theoretical approaches: First steps towards a conceptual framework. *ZDM – The International Journal on Mathematics Education, 40*(2), 165–178.

Radford, L. (2008). Connecting theories in mathematics education: Challenges and possibilities. *ZDM – The International Journal on Mathematics Education, 40*(2), 317–327.

Radford, L. (2014). On the role of representations and artefacts in knowing and learning. *Educational Studies in Mathematics, 85*(3), 405–422. doi:10.1007/s10649-013-9527-x.

Wrathall, M. (2011). *Heidegger and unconcealment. Truth, language, and history*. New York: Cambridge University Press.

Appendix

Data on the Episodes of Carlo, Giovanni, and the Exponential Function – Transcript and Teachers' Interview

Cristina Sabena and Alexander Meyer

In the following, we print the complete transcript of the two videos presented in Chap. 2 and discussed throughout the book. The tasks to which these transcripts refer are printed in Sect. 2.2. All translation was carefully done, but only those excerpts of the transcripts which were used in Chaps. 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12 have been controlled and edited several times. The original Italian transcript is available from the authors and contains many more screenshots; here we only print those that were used in any of the chapters.

In the transcript, underlined words indicate that they are simultaneous with the gestures.

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A.1. Complete Transcript for the Episodes on Task 1–3

Carlo and Giovanni Working on Task 1 (See Fig. 2.1)

1	C	x… try to put it on…on x/2 is
2	G	on 0, it's 0… yes, it must go on 0
3	C	it's 1 on 0
4	G	yes
5	C	when (2.7)^0, a number to the 0 gives 1
6	G	ah… yes
7	C	put it on the 5
8	G	which is it? This, 2.7
9	C	yes… modify also the measure unit of the y-axis, that is you put, instead of 2.7, you put another thing
10	G	the y-axis?
11	C	yes… what have you done?
12	G	oh, I have moved it, I have put it larger like so, as you can see…ok
13	C	but you see that, that is, you must modify 2.7, you do not have to modify…
14	G	are you sure?
15	C	yes… if not, which is the utility of jumping in 20, 10 instead that of 5, [not understandable] it's useless… if you put let us say 3,5 on the x-axis… what are you doing?
16	G	it does not change anything
17	C	how does it not change anything? It grows faster
18	G	yes, but it's always an exponential function
19	C	try to put it… put 1… it should be…put 1… let us see… eh… not now, it should not remain always, always a straight line, sorry, if you put 1
20	G	wait, if we had put it here on 1
21	C	sorry
22	G	ah not, because it's always… [not understandable] that is this point…
23	C	I've understood, but if, if I move x, while x in changing, even if x is 100, however 1^100 is 1
24	G	eh no, because here it's as if you made… [not understandable]
25	C	yes?
26	G	eh, yes
27	C	if you put 1 here…
28	G	eh no, because we need a formula, to do…to do it 2.7
29	C	ehhh?
30	G	that is, they have a formula for…
31	C	sorry, if here it says to change it, it means that, that you must change the 2.7 and you must change it in 1… 1 is a straight-line, in any way you move it, it remains on 1, also, that is 1 to the…
32	G	eh no, here the level must be always the fixed one
33	C	it's not the true, if here it says: you can change the measure unit on the y-axis

34	G	ah, the measure unit is this; 1, 2, it's not that, the measure unit is this
35	C	then we must, what must we do?… move them… make…
36	G	just a moment
37	C	no no… it is better if to you make it on the line
38	G	ah… towards left?
39	C	yes
40	G	this is approaching 0… [not understandable]
41	C	go, go
42	G	eh, it doesn't work… this one is always 0
43	C	move it a little bit, also in this case… go on 0
44	G	on the 0 it gives 1
45	C	put, put 5 of measure unit
46	G	5
47	C	go towards the negative ones
48	G	when it arrives to minus, at 2.7 it goes, it goes in 0… because then you see when it arrives in 0, you can continue to move, but it remains always on the 0
49	C	yes… but before, when, when it's just 0, it's 1, there
50	G	yes, when it's 0
51	C	it's on 1
52	G	eh, it's 1
53	C	then you go to 1, it's 1
54	G	to −1 it does not yet go on the 0, wait! Let us go, a little bit more −2,0, 1, 3, 3… more or less towards the 6
55	C	an then at the 0 then it goes
56	G	yes, then it goes
57	C	then… you animate with a spring the point X, so that it moves from left to right… what are you doing?
58	G	eh, it disappeared from the scream… ah…. Here it is
59	C	[reading the worksheet] Share all the observations that you think interesting on the coordinate movement of the two points and write a sketch of your argument on the protocol that has been given to you.
60	G	practically… now I verify with the trace… then, trace… uh? [not understandable] enlarge in the sense of the measure unit?
61	C	yes, what, to try these points…
62	G	it could be this one
63	C	yes
64	G	however this one disappears that is
65	C	then…
66	G	how can we delete the trace?
67	C	is there a way to do it, to see the coordinates here on the y-axis?
[The teacher arrives]		
68	G	one thing, how can we delete the trace?
69	T	so to cancel, to cancel the trace it's enough clicking here, you move a little bit and you cancel down
70	G	and to find the coordinates of this point?
71	T	what do you think? Let us see, the coordinates of this point, that is the one on the y-axis?
72	G	no, ah no, no, no, of the point, wait a moment… of this… it could be…

73	T	but in your opinion which is the x? the x of that point?
74	C	the x is this one
75	G	the x is this one
76	T	eh and the y?
77	G	the y is this one
78	T	ok, do you agree?
79	C	yes
80	T	do you wish to see just concretely, these coordinates represented by a number?
81	G	yes
82	T	well, it's enough that you put, I think, coordinates and equations. Is this there from somewhere, isn't it? To the right, move, coordinates and equations and you click on the point of which you want, if you want that and read than
83	G	ah
84	T	now what are you going to put as x, which is?
85	G	as x it is 0.31
86	T	ok
87	G	and as y it is 2.3
88	T	perfect
89	G	so.. here we can know that…
90	C	look if the… the ratio, if dividing, if the ratio between 2… x on 2 and on 1 gives 2.7
91	G	the x…
92	C	on 1 how much is it?… 2.85…2.68
93	G	68
94	C	go up
95	G	eh, because it's 1.9
96	C	on 2
97	G	…. This… ah, 7
98	C	yes, but if you put those you don't change some, some things… if you put 5 before and then you do it for 10…
99	G	eh no, it's always the same
100	C	no
101	G	eh, yes, look here, even if you put here with the index… [not understandable] it's always like so
102	C	go up, go up
103	G	eh, it doesn't change anything
104	C	up no, put it on, on 2… wait…
105	G	so this is exponential, then we can also increase, I think… like…
106	C	when…
107	G	at a certain point…
108	C	for the negative, the negative x's
109	G	for the negative x's the function decreases up the point… up to this point here where it's 0
110	C	which is it?
111	G	this one, 5.34 and the… the function… and the point never goes under 0, under 0
112	C	which is the point? 5.34?
113	G	yes… we have not written of the x-axis…
114	C	for a smaller x
115	G	for x smaller than…

116 C for x smaller or equal
117 G smaller than 5.34… the x of the point goes to 0… then…
118 C put x equal to 0
119 G with the x equal 0
120 C it should be 1
121 G it's 1… ok
122 C if it's an exponential function, it goes to the infinity
123 G yes it increases more and more
124 C have you a pen?
125 G then
126 C try to change the unit measure… eh.. we cannot change, change just the 2.7
127 G ah.. it's why I don't know how to do
128 C I think that we must do like so, using, if you change the measure until, if, that stuff there, if you put 500, to each smaller value it gives to you… try to change 2.7
129 G eh, but I don't' know how to do it
130 C let's go there, click and change it
131 G no, you change only… this
132 C try
133 G that is only a label
134 C try it; Luca (another boy) can we change 2.7?… in the second? … What?
135 Luca: are you still at the first exercise? When you have finished do the second and the third one
136 C and there can we do that?
137 Luca: it's written
138 G here there is nothing to say… any longer…

Carlo and Giovanni Working on Task 2 (See Fig. 2.2)

139 C [reading the text of Task 2] what do you think about the graph of the function y = (2.7)^x? … We have said it
140 G what?
141 C here it asks you which is the graph?
142 G yes, it's an exponential function
143 C then… [not understandable] we have already answered… open the Cabri file a^x with Cabri 2 and do a^x
144 G wait… open a^x
145 C in it you will see a point X on the x-axis, a point a^x on the y axis
146 G yes… and then?
147 C a point P of coordinate (x, a^x) that, therefore, it describes, at the varying of the x, function's graphic y = a^x and finally a symmetry… a symmetry on these it's put a point A, the whose abscissa is the base of the exponential a^x … the base of the exponential… you try to understand what it's means

148 G ah yes, it's what that I'm understanding… I trying to understand… [not understandable]

149 C base of exponential a^x

150 G I've not understood what we can modify

151 C the base

152 G ah, but you see, if you change this… that is it become more tightened or it increases more or less

153 C it increases the rate of growth

154 G yes!

155 C of the function… so the rate of growth change

156 G here, it jumps

157 C try to put it as 2.7

158 G 2.7, we put it on the point 1… over the point 1… 2.73, ok… on the point 2

159 C look, it means… changing the position of *a*, we can have exponential with different bases, all greater of zero: for this choice there is a very precise reason on which we will discuss in class.

159.1 C [reading the worksheet] So, moving the point A you change the exponential base, moving the point P you cover the diagram of one exponential function with fixed base. Do some exploration, you exchange eventual impressions: there is something that is not clear, than you did not expect or that instead there is clear and you expected? Brought back on the protocol synthetic trace of your exploration.

160 G *[Underlining designates the part of an utterance during which the speaker gestured.]*

160 G we try to move *A*

161 C try to put the *a* very high [*moving his hand upwards, at the top of the screen*]… when we have seen to happen that chaos *[meaning: in a previous lesson]*

162 G no, it always gets… because here it is interrupted… because here it is interrupted

163 C wouldn't it do like this? [*Gesture a*]
wouldn't it do like this? [*Gesture b*]

Gesture 163 (a): C's quick gesture with right hand

Gesture 163 (b): like Gesture a with more visible hand, going upwards very steep

164 G what?

165 C to do like this [*gesture*]

Gesture in 165: C's similar gesture, more evident, with the hand moving very steep upwards

166 G no, that only if…

167 C I know, but because it too high

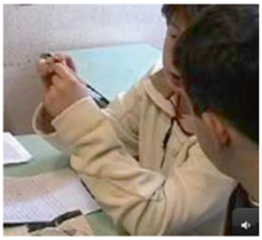
168 G that is?

169 C we had said that happened
170 G eh.. no… only on the conclusion
171 C yes. This is changed, it grows fastest, just before of… of x as a… now we try to do the opposite, low, low
172 G well
173 C you try to put it a little more low… so… you try with 1… you look: with 1 it's a line
174 G with 1, it's a line
175 C we expected this
176 G uuh
177 C instead, if it's less than 1, also…
178 G with a less than 1…
179 C we expected this so
180 G yes
181 C why… but you try to move this to see this… no, no, you leave it, at least you don't change the rate… Are the same the co-ordinates of the point P?
182 G ehh, this is x, and this P's y
183 C that is the x
184 G that is the x
185 C so you can see
186 G yes, yes… it never touch the zero, it doesn't touch
187 C you have gone out
188 G yes, yes… it never touches the zero
189 C well… so we write that… let's say: the point A… we put that one thing we had said… [*Gesture a*], we had said that…
I'm still thinking if… [*not understandable*] how I can say… but… also for a same space of the x [*Gesture* b], the y increases a lot [*Gesture c*]

Gestures in 189:

(a) Carlo's quick gesture in the air

(b) C's fingers close to each other



(c) C's right hand movingupwards

190 G yes
191 C eh… how do I say?
192 G you can also say that using the difference. For the same space the difference are ever greater
193 C the difference? Yes
194 G yes
195 C well, if x change, the y is never zero
196 G yes
197 C and if…
198 G because if…

199 C because if I raise 1 to any number I have not zero
200 G yes yes
201 C any number
202 G well, if a is less than 1…
203 C less than 1… the function decreases, it increases, increases less and less
204 G yes
205 C it increases, increases, decreases, decreases less and less
206 G no
207 C yes
208 G it increases less and less
209 C that is… turn it out, just a moment, it decreases less and less
210 G yes
211 C yes, the function decreases less and less… less and less… decreases less and less… but it doesn't touch the axis $y = 0$… it doesn't touch
212 G wait… write "with respect to the x", because it does touch the y-axis
213 C ov the y-axis
214 G no, it doesn't touch the axis of y? It touches it
215 C the y is never zero
216 G ah yes…
217 C let's put on $y = 0$
218 G and… and then we have to take into consideration a bigger then 1 and a bigger then 0
219 C it is this
220 G ok, that one, and then…

[Here they have completed the first worksheet (Tasks 1 and 2).]

Carlo and Giovanni Working on Task 3 on the Second Worksheet (Fig. 2.3)

221 C the second worksheet
222 G can you wait a moment? We look if there is some other thinks to do… no
223 C well, you have to open in file exp with Cabri2 plus
224 G so we can close this
225 C you have to open Cabri2 plus… the file Cabri2 plus exp
226 G no
227 C where is it? We have to search it? On the other way we ask to prof… prof… [They call the teacher to get help with the file]
228 T I come soon as possible
229 C you can say as where if the file of second worksheet?

230	T	yes
231	C	we can’t find it
232	T	what? The file of second worksheet?
233	C	yes
234	T	because…
235	C	to find the file of first worksheet we took more than 10 minutes
236	T	you are right, what is exp?
237	G	yes
238	T	it is this, it is this with Cabri
239	C	2 plus
240	T	ok, well… we open it directly… it is in this folder
241	G	ok
242	T	if you need, you can call me again…
243	C	prof
244	T	yes
245	C	at the first and second questions we have replied together, because we have seen the answer of second question in the first
246	T	it’s ok… you have problems at this point?
247	C	no, no… we wasted a lot of time at the beginning with this, we wasted a lot of time finding file… at least Luca helped us because we could’nt find the folder
248	T	it’s ok
249	C	[Reading the text of Task 3] you have to look with attention at the figure, you see that there is some point that you can move: P, Dx, and a … wait
250	G	P
251	C	Dx
252	G	Dx
253	C	it is this. It’s right?
254	G	ok
255	C	well… P moves on the graph
256	G	yes, and also a
257	C	a is the rate of growth
258	G	perfect
259	C	ok…you notice that the segments PH…PH
260	G	PH… yes
261	C	and Dx have the same length, it is Dx
262	G	yes, yes
263	C	no?
264	G	ah, yes, yes, yes
265	C	well, yes, it is Dx, that… they are make in order to have the same length… eh, it have got to, I hope the they are the same thing
266	G	what?
267	C	PH and Dx
268	G	ah
269	C	PH and Dx are the same thing
270	G	no… if they are the same… sorry, if PH…
271	C	PH is equal to Dx

272	G	yes, PH, but not HQ
273	C	no, not HQ
274	G	ah
275	C	if you make the usual point, this would have to increase
276	G	yes, sure
277	C	you try, you try to increase this
278	G	look
279	C	you see, PH increases
280	G	yes
281	C	then, well, it is the same… you have to describe shortly the figure moving first P, then Dx, then…
282	G	with P, the graph changes with P. We look at what happen
283	C	P changes on the graph, it changes as x and y vary
284	G	hold on, also the segment QH changes, look at, if you move it… more…, this increases here
285	C	yes, wait a moment, do… to put it instead down
286	G	ah
287	C	look it… slowly… slowly it seems that… I do not know, like, saying tangent
288	G	eh… yes
289	C	it seems that it touches it, let's go, let's go, let's go
290	G	eh yes… here
291	C	slowly… slowly
292	G	it's tangent
293	C	if instead you make the contrary, increasing, increasing the differences
294	G	the differences
295	C	yes… it's also increasing the differences
296	G	uh
297	C	this one here, slowly it's like if it tightened itself
298	G	problem: if I move the segment PQ also like so, and I put Dx very small, in this case, also if, it seems getting worse
299	C	eh, well, because one moves just
300	G	we can say that if P it's small, that is more like a tangent, it seems, if you take it much small
301	C	a single point
302	G	eh, it can be approximated to one line, with P very small, then instead as long as it increases…

Gesture in 302

303	C	but like… try to put it a = 1, it must result…
304	G	a line…
305	C	a = 1 we know it already… than you must do less than 1
306	G	look at it but… that is… look, excuse me, look at here… the line
307	C	comes back with a positive

308 G yes, if we move P we can see that the point, eh, sorry the HQ segment becomes smaller, it decreases…and this, the point QH, can you see?

309 C because P and Q have always the same distance

310 G yes

311 C ok, so ok, ok, so ok, because if it means that they increase, <u>the more you move them over there</u> [*gesture*], it increases very very much

Gesture in 311: C quickly moves the hand upwards to the right

312 G yes

313 C because it's an exponential function

314 G and the P…

315 C it would be yet….[not understandable]

316 G eh, ok, when the P it's very close to the 0, the line that passes for Q and H <u>represents</u> [*begins gesturing on the desk by screenshot (a)*] <u>more and more</u> [*gesture in screenshot (b)*] the function… the smaller it is [gesture (*c*)]

Gestures in 316 (a)

(b)

(c)

317 C eh yes, because Q, HQ decreases always more

318 G this becomes always more… [not understandable] it can be become simpler until. This.. that is

319 C PH and _x are the same, because the same of the… of the function

320 G yes

321 C then since that, being, being always the same distance from P to Q, Q it seems me..

322 G yes, yes, yes

323 C yes, yes, yes

324 G wait a moment…distance from the point…

[The teacher arrives]

325 T ehh, if you wish to drag, you must use this

326 G ah

327 C we wished, practically, is there always the same distance between P and Q?

328 T always the same distance?

329 G no no, it decreases

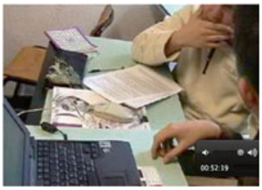
330 C does it decrease?

331 G yes, look… [*pointing at the screen*] and then we have discovered also that the nearer P is to [*Carlo's gesture (a)*] y equal to zero, the more this line approximates [*gesture (b1) on the desk*] the [*gesture (b2) in the air*] function

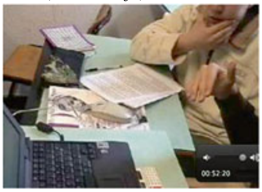
Gestures in 331:

(a) Carlo's gesture accompanying Giovanni's statement in 331

(b1) Giovanni's gesture repeating the one from 316



(b2) Giovanni's gesture representing decrease (to the left)



332 T therefore you approach it enough [not understandable]

333 G yes

334 T when a function stretches to crush itself on the x-axis

335 G and moreover another thing, if the Delta x is very small…

336 T yes

337 G [*pointing at the screen*] the line becomes nearly a tan.., a tangent [*gesture*]

338 T uh

Gesture in 337: G holds the fingertips of the flat vertical left hand against the interior of the flat vertical right hand, while moving the right hand upward

339 G to the, to the function

340 T and so, it gives you some information about what? When the Delta x tends to become very very small, what kind of information do you get?

341 C if the Delta x becomes small… it means that… [*looking at the screen, where Giovanni is moving something using the mouse*] the Delta x becomes small [*gesture*]

Gesture in 341 (a): C is pointing with index and thumb (the "Delta gesture")

when… when between P and Q… that is [*gesture*] the space decreases

342 T oh sure, it is almost trivial, isn't it? Therefore he was saying that this line tends to become…

Gesture in 341 (b) C is moving his open hand vertically from the bottom upwards

343 G [*gesture*] tangent. [*C nods*]

Gesture in 343: Carlo's anticipatory gesture: puts his hand in a horizontal position

344 T and then what kind of information will it give you in this case?

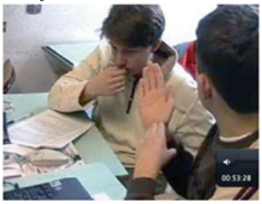
345 G ah, one can say [*gesture (a)*]… one can say that [*so far G has kept the gesture, while looking at it silently*].
[*gesture (b)*]
the exponential function becomes
[*gesture (c)*]
very little [*gesture (d)*] lines…

Gesture in 345 (a)

Further Gestures in 345:

(b) G joins his fingers on the desk and traces a trait rightwards

(c) G's open hands positioned one after the other



(d) G moves his right hand little by little upwards

Giovanni's gestures sequence rightwards is repeated twice.

346 T uh… it could be approximated to some small lines, which however…

347 G that is [*gesture (a)*], that…with increasing slopes [*gesture (b)*], that join together [*gesture (c)*] in a, that touch each other in a point [*gesture (d)*]

Gesture in 347 (a): G's two-hands configuration

Further Gestures in 347:

(b) G's right hand moving upwards

(c) G's left hand touching the right palm

(d) G's left index touching the right palm

348	D	therefore you are imagining to approximate with many small segments
349	G	well [*gesture (a)*], if you take it… I don't know, if you take it with a very large zoom… you can approximate it with many small lines [*gesture (b)*] *Gesture in 349 (a): initial phase of Giovanni's "Delta gesture"*  *Gesture in 349 (b): G final phase of the Delta gesture. The gesture has been kept during the whole sentence, a little larger and moved rightwards and upwards with higher slope (as before the right hand).* 
350	T	and such lines which features have they?
351	G	they have… well, they may have a function, a slope are, possibly always twice than before
352	T	well, I don't know if the slope is twice, but… in any case… their slope increases, does it? In this case, when this function increases
353	G	yes, when it climbs on
354	T	have you observed that… now I ask a directed question to you… when you have seen how the exponential function grows, let us say the growth percentage of the y's; does it remain constant or not? … Does the ration between a value and its successive remain constant?
355	C	sure
356	G	yes, it's remains constant
357	C	we have already written it here
358	T	well
359	C	exponential, that is the between the y of the point and its successive is constant, always…
360	T	does it surprise you the fact that the function crushes on the x-axis? Here it seems that the function increases not much and here the function increases very much. Does it surprise you such a type of increasing with a constant ratio or is it natural?
361	C	… well, yes…
362	G	sure, because before the numbers are small and with small numbers the ratio is always between nearer points
363	T	eh
364	G	if the numbers are big, the ratio…
365	T	yes, the other group have used a very good example: if we take 10 % of 5 cents it is 0,5… it doesn't exist, isn't it? It is as it did not exist; if we take 10 % of 5 million euro on the contrary thing start changing, isn't it? It is a considerable amount of money… here the hypothesis are the same… and it is ok; now you go on in this way. Where have you arrived?

366 G here

367 T it is ok

368 G let's write this… so we can write that [exploring the screen with the mouse]…that if the x increases again, the line passes through P and Q and is almost constant, it becomes almost a tangent… this because if we take a very big zoom we can approximate the exponential function with many lines, which have an increasing slope… Then, if the point P is very near to zero, this line approximates very much the exponential function. Also here even if numbers are very small, it increases not so much, hence like a line… and than we can write that we were waiting for it even if the ratios are constant at the beginning… it was almost a line… [not understandable]

369 C hence we write that it is a graph with a constant rate of growth, of a… of a if x is always the same… [not understandable] but the y's…

370 G well, we try to do… [not understandable] wait, with a great a the triangle's area increases

371 C what?

372 G look, the area of this triangle PHQ

373 C are they a triangle?

374 G yes

375 C why is it a triangle?

376 G it is a triangle: look at it, do you see how small it is. You can see that it is a triangle

377 C PHQ… the one with the line?

378 G yes, yes

379 C it is ok! Otherwise it had no sense… that maintaining PH constant and therefore also the Δx's constant we notice that.. [not understandable] while P increases, P increases more and more, that is the Δy's increase; they increase more and more

A.2. Transcript of the Extra Video

An Extra Episode About the Exponential Function

At the end of the exploration described in the first video, Carlo and Giovanni observed the computer screen in Fig. A.1 and the teacher asked them the following question: What happens to the exponential function for very big x? We present a short excerpt from the interaction between the teacher and the two students about this question.

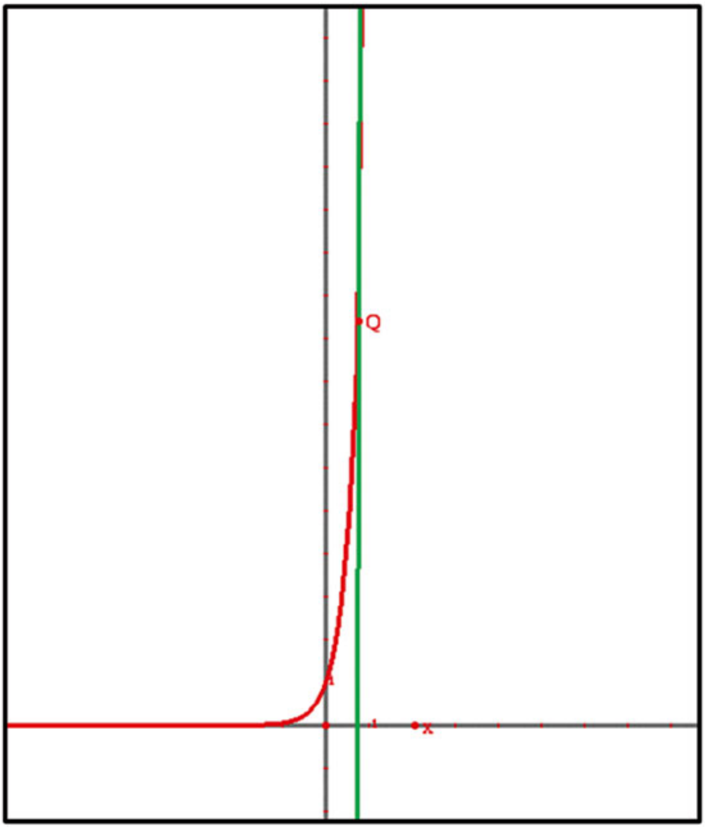
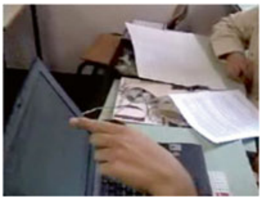


Fig. A.1 Computer screen configuration in the extra episode

1 G but always for *a* very big this straight line, [Gesture] when they meet each other, there it is again…that is it approximates the, the function very well, because …

Gesture in 1: G is pointing at the line in the screen



2 T what straight line, sorry?

3 G this here [*pointing at the screen*], for *x* very, very [*Gesture*] big

Gesture in 3: G's hand goes upwards

4 T [*Gesture a*] will they meet each other [*Gesture b*]? [*challenging connotation*]

Gestures in 4:

(a) T pointing two forefingers

(b) T crossing the two pointed forefingers

5 G that is [cioè[1]], yes, yes they meet each other [*gesture*]

6 T but after their meeting, what happens? [*continuing to keep the hands in the same configuration as in line 5*]

Gesture in 5: G's two forefingers touching each other

7 G eh…eh, eh no…, it makes so

8 T ah, ok, this then continues [*gesture a*], this, the vertical straight line [*gesture b*], has a well fixed *x*, hasn't it? The exponential function later goes on increasing the x, doesn't it [*gesture c*]? Do you agree? Or not?

Gesture in 7: G crosses the left hand over the right one; T is keeping the previous gesture

[1]The expression "cioè" in Italian means literally "that is". Over-used by teenagers, it introduces a reformulation of what was just said. As it is likely in this case, it can have the connotation of "I am sorry but".

Gestures in line 8:

(a) T moving rightwards his left hand

(b) T's right hand vertically raised

(c) T moving rightwards his right hand

9 G yes […]

10 T [*addressing C*]: He [*G*] was saying that this vertical straight line [*pointing at the line in the screen*] approximates very well [*gesture*] the exponential function

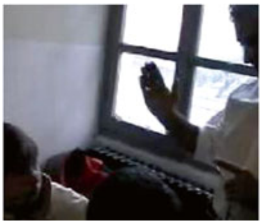
Gesture in 10: T raises both hands

11 G that is, but for x that are very… very big

Gesture in 11: G moves his left hand high wards

12 T and for how big x? 100 billions? $x = 100$ billions?

Gesture in 12: T raises his hand at his right and keeps it fixed



13 G because at a certain point…, that is, if the function [*gesture 13a*] increases more and more, more and more [*gesture 13b*], then it also becomes almost a vertical straight line [*gesture 13b*]

Gestures in line 13:

(a) G raises his left hand

(b) G moves his hand upwards

(c) final position of G's hand after moving upwards

14 T eh, this is what it seems to you by looking at; but imagine that if you have x = 100 billions [*gesture*], there is this barrier…is it overcome sooner or later, or not? [*connotation: suggesting the answer yes*]

Gesture in 14: T keeps his right hand in the vertical position

15 G yes

16 T and so when it is overcome [*gesture 16a*], this x 100 billions [*gesture 16b*], how many x do you still have at disposal, after 100 billions? [*gesture 16c*]

Gestures in 16:

(a) T crosses left forefinger over right hand

(b) T raises his right hand

(c) T moves right hand rightwards, repeatedly

17 G infinite

18 T infinite… and how much can you go ahead after 100 billion [*repeating the gesture 16c*]?

19 G infinite points

20 T then the exponential function goes ahead on its own, doesn't it?

A.3. Teacher's Interview Transcript

In order to get an extended understanding of the shared data and obtain some background information on the students' learning history, the teacher's learning goals and didactical intentions etc., the participants of the Networking Group collected 18 questions for the teacher which were posed in a written form. The teacher answered 16 of the 18 questions in an interview, and the last two in a written questionnaire. The interview was recorded and translated into English. The questionnaire was answered directly in English.

1. *In advance of the lesson, how did you expect the students to work together at the computer? How did you expect them to share roles? What "ground rules" had you tried to establish about joint work at the computer with this class?*
   "My expectations are relative both to mathematical topics and competences, and to relational and emotional aspects. The expectations relative to mathematical knowledge that come into play in the activities vary strongly with the proposed activities. It is in fact easier to find expectations relative to competences and to relational and emotional aspects, which characterize almost every activity that I propose to the students.

   With respect to mathematical competences, I hope that students read the text of the posed problems very carefully and that they begin to do some explorations, either mental explorations or with the help of technological tools. These explorations have the aim to create context, to create meaning, to provide experience of problem situations; they encourage the production of conjectures and should motivate students to validate their produced conjectures. I hope that students often ask themselves why they observed some patterns, some regularities. As regards relational aspects, I hope students interact actively; that they listen to the words of other students. I hope that students are able to argue and to support their conjectures and solving strategies in a pertinent and convincing way and with coherence to their mathematical knowledge. I hope that students are able to help fellow students who are in some difficulties. I hope that students understand that in small-group work it is important to collaborate seriously in order to have a good product.

   With respect to the emotional aspects, I hope that students work serenely, but seriously; I hope that they overcome the anxiety tied to the awareness that they are object of observations. I hope that they are not afraid to give the teacher a lot of information about their mathematical competences and solving processes. My expectations are that the students will gradually succeed in looking at the teacher as someone who is there to help them to acquire critical awareness and not only as a person who judges their performance.

   I expect all these things, but I'm not so naïve and blind to not understand and see that, in order to realize all this things, it is necessary to devote a lot of time and patience."

2. *Choose some episodes from the video of the lesson that broadly meet your expectations about how students will work together at the computer and some others that do not. Talk us through them.*

   "Some episodes from the video that meet my expectations:

   At the beginning the students have difficulty in working. From the minute 3 and 25 seconds to the minute 8 and 50 seconds they reach few results; they frequently stop to speak. It seems that students don't understand well the problem and the Cabri worksheet. Notwithstanding this, the interaction is balanced: there isn't just one student who speaks and another who only listens. The discussion is poor, but there is some research through interaction. Students collaborate very well, for example, from the minute 29 and 52 seconds to the minute 30 and 35 seconds when they realize the effect that the variation of the base has on the exponential graph. They collaborate very well also from the minute 32 and 30 seconds to the minute 33 and 50 seconds where it is possible to see also some interesting gestures, which reveal understanding and communication. They use the software to validate conjectures and Carlo says, "We expected this" (minute 33 and 25 seconds).

   From minute 36 and 40 seconds to 36 and 55 seconds there is a good exchange of ideas on exponential growth. From minute 45 and 50 seconds to 46 and 13 seconds there is a good collaboration to interpret the figure of the Cabri worksheet and the text of the activity.

   Some episodes from the video that don't meet my expectations:

   From the minute 11 and 0 seconds to 21 and 15 seconds, Carlo stays with paper and pencil and seems not to be interested in what Giovanni is doing with the pc. It seems that Carlo is waiting for some results from Giovanni. At the minute 18 and 15 seconds it seems that Giovanni tells Carlo what to write. I don't see interaction, discussion, but only a passive attitude from Carlo.

   In general, I don't like that Carlo uses only paper and pencil and Giovanni uses only the pc. This subdivision of the role may be useful in order to reach the final result more quickly, but may be an obstacle to the process of construction of meaning."

3. *What conditions support or hinder learning when students work together like this at the computer?*

   "I observed that the possibility of using the mouse, the keyboard, and the availability of good visibility on the pc-screen helps students to collaborate actively and then to construct knowledge and to learn. According to this consideration it should be better that students work individually at the pc. On the other hand, if students work individually, they do not have the possibility to exchange and to share ideas. According to this consideration, it should be better that students work in groups that favor the sharing of ideas and solving strategies. In my opinion a good compromise for working with the pc are dyads (pairs of students) or, if necessary, three, but no more than three. More in general, in my opinion the use of the PC may hinder learning or, put better, can create obstacles to learning if the tool is used in a uncritical way, for example to obtain answers and not give rise to questions and thought. In my opinion technological tools have to be used to empower the possibility to experience the mathematical environment and mathematical objects. In this way we should use them in the teaching–learning activities. It is a way of use which is very different from the way in which technological tools are often used in daily life."

4. *During a lesson of this type, under what circumstances do you decide to get involved with a pair of students, and what kinds of things do you do?*

   "I enter in a working group if the students call me. Sometimes I enter in a working group if I realize that students are stuck. Other times I enter because I realize that students are working very well and they have very good ideas that need to be treated more deeply. Obviously the type of things that I do vary with the situations, but a constant is that I try to work in a zone of proximal development. The analysis of video and the attention we paid to gestures made me aware of the so-called "semiotic game" that consists in using the same gestures as students but accompanying them with more specific and precise language compared with the language used by students. The semiotic game, if it is used with awareness, may be a very good tool to introduce students to institutional knowledge."

5. *Choose specific examples from the video of your becoming involved with the pair of students. Talk us through them.*

   "In this video my dialogues with the students are few. Anyway, it seems to me that among the more interesting there is the intervention at the minute 53 and 59 seconds. I use a gesture used before by Giovanni. This gesture is towards a little segment that approximates locally the function and I ask: "What is the characteristic of this segment?" My aim is to induce the students to reflect on the fact that it is important to pay attention to the slope of the little segments, because their slope gives information on the growth of the function. Giovanni says "it is twice the previous slope …" I, using his same gesture, say more precisely that "the slope has an exponential growth." At the minute 54 and 24 seconds, I help the students to remember that the characteristic of the exponential successions is that of having the ratio of two consecutive terms constant. Immediately after, I ask the students: "Are you surprised that the graph of the function is so close to zero for small x?" Giovanni, at the minute 55 and 28 seconds says something like "with number smaller and smaller, I have number smaller and smaller." I reword this idea with a more precise language. In the following dialogue, Giovanni and Carlo are able to explain in a comprehensible way the reason why the graph of an exponential function of base greater than 1 is so close to the x-axis for x less than 0 and explodes for high values of x."

6. *What experience did the students previously have of using Cabri? Working specifically with function graphs?*

   "The students have known since the beginning of the first school year the software Cabri. Besides, since the beginning of the first school year they have worked on the concept of function, as regards the numerical, graphical, and symbolic aspects. In particular, as regards graphical and numerical aspects, they have also used other software such as spreadsheets, Graphic Calculus, and TI-InterActive. Additionally, they have worked with motion sensors."

7. *Describe the kind of understanding that you expected the students to develop during this lesson.*
"After having characterized, in previous lessons, exponential growth (I mean exponential successions) as growth for which the ratio of two successive values is constant, I wanted students to understand why a^x with a greater than 1 grows with x more speedily than any polynomial growth. The aim of the DGS file was to make the students understand that an exponential growth is directly proportional to the value of the function itself. This is an important step in understanding why the derivative of an exponential function is still an exponential function of the same base."
8. *What problems/skills/concepts did you expect students to meet in this lesson, and to what extent did you expect them to be able to use these in future lessons?*
"The main aim of the posed activity was to allow students to develop an understanding of the concept of exponential growth. In previous activities, students faced the study of exponential successions and characterized them as successions for which the ratio between two consecutive terms is constant. With this activity, with the help of Cabri, I wanted the students to understand that exponential functions are functions for which the growth is proportional to the function itself. In other terms, the derivative of an exponential function is proportional to the function itself. This consideration, in my opinion, should allow students to understand why the exponential function a^x with a greater than 1 grows with x faster than any power of x."
9. *How did you plan the lesson and organize the classwork so that this learning would take place?*
"An idea was that of preparing worksheets in Cabri of increasing difficulty. The first worksheet has only two points, one on the x-axis and the other on the y-axis, tied by the relationship $y=2.7^x$. Students moving the point on the x-axis should what is meant by y increasing very fast with x. In fact as one moves the point on the x-axis, one sees that the point on the y-axis seems not to be moving between $x=0$ and $x=1$, and from $x=1$ the growth of y is very fast. The second sheet allows students to look at what happens to an exponential function if the base changes, while the third worksheet gives a local and a global approach to the exponential function thanks to the construction of the derivative of an exponential function. Another idea has been that of asking the students to work in small groups or in couples so as to encourage discussion and the exchange of ideas."
10. *What would it be useful to do after this lesson, to take it beyond the group work shown in the video?*
"The aim of the group work was to consolidate the meaning of the concept of exponential growth and to specify some characteristics of exponential functions, particularly the fact that their derivative is still an exponential function. Generally, in the follow-up, I continue the work following two paths. In the first one I pose some problematic situations which, to be solved,

ask for exponential models. In the second one I present the properties of exponentials and I introduce the logarithmic function as the inverse function of an exponential. In particular, I insist on the formula for the change of the base, showing that it is proved by the fact that an exponential function of base *a* can be written as the product of a constant by an exponential function of base *b*. Finally I propose some techniques to solve exponential and logarithmic equations and inequations, underlining the fact that these techniques can be applied only to particular types of equations and inequations and that, in general, approaches that use numerical and graphical approaches are necessary. All this should not be rushed; the time required can vary from three to six months of work. Obviously during this period other topics and didactic activities are also done."

11. *What mathematical knowledge do you expect to "institutionalise" –in the sense of giving it some kind of explicit "official" recognition for the future work of the class– following on from this lesson?*
"I think that the teaching and learning of math today must be deeply different from what it has always been in the past, at least in Italian schools. Mathematics has often been partly responsible for increasing social differences. It has often been used as a tool of selection. Often the teaching–learning has been reduced to a teaching–training in techniques of symbolic manipulation learned by imitation, in a mechanical way. A didactic of this type takes the students away from critical thinking and understanding. I'm strongly convinced that the main function of teaching, not only of math, is to help students to exercise critical thought, to acquire the necessary competences for an informed and aware citizenship. This is the main aim of my lessons and of my work with students. Generally, then, I try to assess in the students the competence to observe and explore situations; to produce and to support conjectures; to understand what they are doing and to reflect on it. On the other hand exponential growth and exponential functions are relevant as mathematical topics and then I try to understand, in my follow-up work, whether students are able to use knowledge constructed with activities like that of the video to face situations which require simple exponential models. Obviously there are also some specific exercises that help me to understand if students have understood the differences between polynomial and exponential growths. A typical exercise is like the following: What can you say about the Inequation $1.1^x - 1 > 1000 \cdot x^{100}$? Justify your answer."
12. *Do you expect to find different levels of thinking when you evaluate students' work on this task sequence? If so, what are these levels, and how do you recognize them?*
"A first level is that of perceiving the different velocity of variation that exists between x and a^x. Generally students don't meet any difficulty in observing and describing this. Generally they succeed in associating the points that are moving on the x-axis and the y-axis with the graph of the exponential function $y = a^x$. A second level is that of the understanding of how the graph of an exponential function varies when the base varies. A third level, as in the third worksheet of

Cabri, is relative to the understanding that the incremental ratio is a function of two variables (the x and the increment h). At this level students are faced with the local aspect of the concept of derivative: they are faced with the concept of gradient. A fourth level is the passage from the local to the global aspects of the derivative. From the gradient to the gradient function. The recognition of students' understanding of these different levels happens through the observation of their discussions in the small working groups or through the questions that they pose to the teacher. Generally, from the third level, the understanding happens only thanks the direct intervention of the teacher in the small groups and this understanding is consolidating in the mathematical discussions guided by the teacher with the whole class."

13. *Afterwards when the derivative is taught in the formal way, what are the effects of the students having experienced these tasks on their thinking and on your way of explaining?*
"When I tell about the formal aspects of the derivative I often make some reference to these experiences and activities. It seems to me that also a lot of students are able to make these connections to give meaning to formal aspects. In particular there are some points for which these experiences allow to simplification and clarification of the theory. For example, the formal calculation of the derivatives can be reduced to the algebra of linear functions if one uses the local linearity of a derivable function. And the local linearity finds its cognitive root in the local straightness of which students have experience thanks to the zooming function of the case they have used."
14. *Are the kinds of tasks which students are working on in this lesson, typical of your approach to teaching? Are they typical of the approach to teaching which would generally be found amongst teachers working with similar classes in your school? And amongst teachers working with similar classes in other schools in your educational system?*
"These kinds of tasks are typical of my approach to teaching. On the web, at *http://www.matematica.it/paola/Corso%20di%20matematica.htm*, there are many teaching–learning activities that are of the same type and that I propose to students. These kinds of tasks are, however, not typical either among teachers working with similar classes in my school, nor among teachers working with similar classes in other schools in my educational system. Obviously, though, even if it is not so usual in Italian schools, I'm not alone in this sort of didactical approach!"
15. *Did any difficulties arise the first time you used this sequence of tasks, or indeed on later occasions? What kinds of difficulties? And how did you deal with them? During the lesson itself? And in replanning the lesson for future use?*
"The difficulties that arise when one follows a didactical approach of this type consist mainly in a low level of involvement of some students. This didactical approach requires the ability to work, to think, to became responsible for one's own path of formation. Passive behavior is an obstacle to serious and efficient work. There are some students who would prefer just to repeat what the teacher or the textbook says; these students would prefer to be engaged in mechanical and repetitive tasks; these students would prefer

simply to have training. This behavior of some students poses the greatest difficulty for the didactical approach I propose. The difficulty extends over a long period, and it is important also to involve the parents of the students in the formation of this approach."

16. What aspects of this lesson do you think it would be possible to export to a normal school and class in your system and what aspects would it be more difficult to undertake there? Why?

    "For me, not only it is possible, but I think it is necessary to choose this didactical approach in today's schools. For certain, it is necessary to have the courage to break away from the didactical tradition which has been used in the past. But it needs much more courage to continue to choose a didactical approach which is not adequate for the role and the function of the school today. Everything depends on the teacher: the willingness that she or he has in order to look for solving the problems of students' motivation; above all, the willingness that she or he has to create a teaching–learning environment which favors critical and aware thinking."

17. *Contextual information about the activity (How does it insert in the didactical path? How is it carried out? In what part of the year?) [written answer in original English]*

    "The analyzed activity is usually proposed to my student in the second part of the first year. It is part of the 7th lessons (http://www.matematica.it/paola/Corso%20di%20matematica.htm). Generally, the schema I follow for the first three years is the following:

    First year

    1. Numbers, combinatorial, first algorithms, introduction to computation with letters ("Lezione4").
    2. Functions. Linear functions. The statistic linear correlation: ("Lezione1")
    3. Continuous linear models and linear discrete dynamic systems. Discrete exponential models ("Lezione2").
    4. Local approximation of a function with a linear function ("Lezione3").

    Second year

    5. Quadratic functions ("Lezione 6").
    6. Local approximation of a function with a quadratic function ("Lezione 7").
    7. Continuity in a point and the Fundamental theorem of Calculus ("Lezione 8").
    8. Examples of non linear continuous models and non-linear discrete dynamic systems ("Lezione 6").

    Third year

    9. Points, lines, planes. Vectors, directions, distances, angles. Circumference and sphere (to be prepared).
    10. Linear uniform motion and the line in the Cartesian Plane (first lesson of the third year).

11. Parabolic motion and the parabola in the Cartesian Plane (second lesson of the third year).
12. Circular uniform motion, the circumference, the ellipse and the hyperbola in the Cartesian Plane (third lesson of the third year).
13. Harmonic motion, harmonic functions and trigonometry (fourth lesson of the third year).
14. Exponential functions. The logarithm (to be prepared, even if an introduction to such functions, in particular to the exponential, has already been carried out in the first 2 years).

By now the lessons are built with TI-InterActive!, a software by Texas Instrument that allows to build interactive worksheets. At the time of the video, lessons were Word documents, since the school had not yet bought TI-InterActive!.

The worksheet proposed in the videotaped activity is situated in the middle of lesson 7, before the formal approach to the concept of derivative of a polynomial function (computation of the slope of the secant by means of the incremental ratio; simplification of numerator and denominator by the increment h; computation of the slope of the tangent line in x for h tending to 0) and before the idea of how is it possible to locally approximate a function with a quadratic function.

The activity intends to clarify the principal features of increasing behaviors and of exponential functions. In particular, it intends to explain the reason why at the increasing of x an exponential of base greater than 1 will increase, definitively, more than any other polynomial function of x, whatever grade of the polynomial. In the project, exponential functions and sequences are used to cope with problem situations coming out from exponential models.

In the following there is a description of the project, coming out by the website. It intends to clarify needs, aims, specific goals, structural, methodological, and content choices of the project."

The most significant and important needs that have brought the creation of the project are:

- Creating teaching-learning environments that are sensed in the double meaning given by Galileo: linked to senses, perception, but also guided by intellect and theory. Furthermore, they were meant to be also reasonable from a didactic or ecological point of view;
- Offering to all the students opportunities of creating meanings for objects and competences needed for a critical thinking allowing them to participate in a conscious way, as citizens, to their choices in the public life;
- Engaging students in knowledge building, settlement, reorganizing and communicating, thus providing the teacher tools for obtaining information not only on the products, but also on the cognitive processes, necessary for any serious evaluation escaping the chimera of objectivity;

- Making the students autonomous in the use of a good manual (schoolbook) and in the reading of scientific papers proper for their level;
- Providing teachers with structured material, both from a content and didactic points of view.[2]

18. *Goals, intentions, and methodology as designer of the project (Why is the activity carried out? How? What is its contribute in the global project?) [written answer in original English]*
"They are already expressed in the "Needs" above, but perhaps the more effective formulation of the aims comes from a deep reflection of our function as teachers.

Why teaching mathematics?

We think that the principal and primary purpose of school is that of helping students to acquire also in autonomous way the knowledge and competences essential to take consciously and critically part to the choices of public life, and therefore we think that mathematics must concur, as the other subject matters, to this purpose. In this sense it appears fundamental to gradually introduce students to the theoretical knowledge. Mathematics is a suitable subject matter in this sense, as long as meanings do not evaporate in recipe and lists of empty formulas, becoming for the students words of an unknown language to quickly repeat before forgetting their sound.

The possibilities that new technologies offer to make experiences, to observe, to foster the production of conjectures are a wonderful tool to help students in their approach to theoretical thinking if the didactic contract is clear and includes the justification of the produced conjectures. That means asking, at any school level, questions of the kind: why?

The answer to such kind of question is located, finally, in theories. At the beginning students will tend to explain facts by means of facts. This exercise will lead them, with the guide of the teacher, to seize relationships between facts and thus to feel the need of finding out laws (propositions, axioms, …) that can be chosen to explain the observed facts. When this need is felt, the student is already in the theoretical thinking and the following passages, as the presentation of the organization of the institutional knowledge, i.e. well organized theories, can be done with good hope of success, above all if the necessary didactical attentions are not underestimated. The request of "explaining why", that of working in small groups and not only individually, trying to reach a sharing of strategies, are aimed at helping students to pass from a tacit knowledge (knowing how to do a think, but being unable to explain to others haw I do it, or why I do in that way) to a conscious, explicit knowledge that can be communicated.

Together with this very general goal, there are the following ones, more specific and apparently linked to contents:

[2]There is plenty of material at teachers' disposal, edited or on the web. For instance, the materials produced by the Italian Association of Mathematics. Those materials are rich in proposals of sensed didactical activities. However, they are not structured in a parcours (sequence). The project intends to insert them in a structured didactical parcours.

- The approach to statistics and probability thinking (tools for managing situations of uncertainty, and so necessary for social life);
- The approach to multidimensional thinking (also as characteristic of organizing and representing information);
- The approach to the study of changing quantities, with particular reference to mathematics modelling to describe situations and foresee their evolution.

These goals are in our view to be reached by all students, and thus they are pertinent to the first 2 years of secondary school.[3]

Such goals give the rationale for the organization of contents and the proposal of the specific learning goals. We have accordingly set up a corridor, with the activities aimed at the essential competences and contents, and several rooms that complement of deepen the issues according to the teacher choices. In the corridor we have avoided to ask students strong competences on techniques (that are not among the goals in the project), with the proposal of fostering the use of computing tools

A very important role in our parcours is given by geometric intuition as instrument to give meaning to formal aspects: synthetic geometry as an intuitive base for the construction of meaning,

More than as corpus of organized knowledge in an hypothetical-deductive system (which can be deepened in the final years of secondary school).

Finally, we thing it is important to stress that in the proposed activities the numerical graphic and symbolic aspects are always used with the same dignity and importance for the construction of meaning of mathematical objects.

Methodological approaches are flexible and different: work in couplet, individual work, small group work, class discussions lead by the teacher (with the goal of drawing a balance), more standard frontal lessons to settle the knowledge or to recover certain contents of competences.

We think that the teacher is essential: his role is very delicate and changes according to the needs, as it happened in the shops-studios of the Renaissance. In any case, the teacher always has the responsibility of sharing the knowledge in the classroom and of directing the meanings, and he must be conscious of that. In fact, he is the only one in the classroom to know what is the arrival point and has the duty of leading there all the students, in a manner so as everyone is conscious of what happened along the road and could then reach autonomously the place and others in the nearby, even if unknown.

The didactic approach is constructivist, especially for what concerns the activities proposed in the "corridor". The fundamental topics and the essential competences should be acquired through organized activities, structured around sheets (schede) in which questions are posed, and problems proposed to individual, couple or group work. The phase of consolidation of knowledge constructed during the activities, is post-posed to the end of a relatively long parcours; the construction of meanings inevitably asks long times in the didactics…

[3] School in Italy is compulsory until grade 10. Grades 9 and 10 are the first 2 years of secondary school (5 years of primary school +3 years of middle school, then 5 years of secondary school).

Bibliography

Artigue, M. (1989). Ingénierie didactique. *Recherches en Didactique des Mathématiques, 9*(3), 281–308.

Artigue, M. (2002). Learning mathematics in a CAS environment: The genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. *International Journal of Computers for Mathematical Learning, 7*(3), 245–274.

Artigue, M. (2008). Didactical design in mathematics education. In C. Winslow (Ed.), *Proceedings from NORMA08 in Copenhagen 2008* (pp. 7–16). Rotterdam: Sense Publishers. https://www.sensepublishers.com/media/699-nordic-research-in-mathematics-education.pdf. Accessed 22 Oct 2013.

Artigue, M. (coord.) (2009). *Integrative theoretical framework – Version C*. Deliverable 18, ReMath Project. http://remath.cti.gr. Accessed 29 Nov 2013.

Artigue, M. (2013). Didactic engineering in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education*. Heidelberg: Springer (in press).

Artigue, M., & Winslow, C. (2010). International comparative studies on mathematics education: A viewpoint from the anthropological theory of didactics. *Recherche en Didactique des Mathématiques, 30*(1), 47–82.

Artigue, M., Bartolini-Bussi, M., Dreyfus, T., Gray, E., & Prediger, S. (2006). Different theoretical perspectives and approaches in research in mathematics education. In M. Bosch (Ed.), *Proceedings of the 4th Congress of the European Society for Research in Mathematics Education* (pp. 1239–1244). Barcelona: Fundemi IQS.

Artigue M., et al. (2009a). *Integrative theoretical framework – Version C*. Deliverable 18, ReMath Project. http://remath.cti.gr. Accessed 11 Dec 2013.

Artigue, M., Cerulli, M., Haspekian, M., & Maracci, M. (2009b). Connecting and integrating theoretical frames: The TELMA contribution. *International Journal of Computers for Mathematical Learning, 14*(3), 217–240.

Artigue, M., Kynigos, C., & Mariotti, M. A. (2010). Contextual issues seen through the ReMath filter: Some illustrative examples. In M. M. F. Pinto & T. F. Kawasaki (Eds.), *Proceedings of the 34th conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 295–304). Belo Horizonte: PME.

Artigue, M., Bosch, M., & Gascón, J. (2011a). Research praxeologies and networking theories. In M. Pytlak, T. Rowland, & E. Swoboda (Eds.), *Proceedings of the seventh Congress of the European Society for Research in Mathematics Education* (pp. 2381–2390). Rzeszów: ERME/University of Rzeszów.

Artigue, M., Bosch, M., & Gascón, J. (2011b). Research praxeologies and networking theories. In M. Pytlak, T. Rowland, & E. Swoboda (Eds.), *Proceedings of the seventh Congress of the European Society for Research in Mathematics Education CERME 7* (pp. 2381–2390). Rzeszów: University of Rzeszów.

Arzarello, F. (2008). Mathematical landscapes and their inhabitants: Perceptions, languages, theories. In E. Emborg & M. Niss (Eds.), *Proceedings of the 10th International Congress of Mathematical Education* (pp. 158–181). Copenhagen: ICMI.

Arzarello, F., & Bartolini Bussi, M. G. (1998). Italian trends in research in mathematics education: A national case study in the international perspective. In J. Kilpatrick & A. Sierpinska (Eds.), *Mathematics education as a research domain: A search for identity* (pp. 197–212). Dordrecht: Kluwer.

Arzarello, F., & Paola, D. (2007). Semiotic games: The role of the teacher. In J. H. Woo, H. C. Lew, K. S. Park, & D. Y. Seo (Eds.), *Proceedings of the 31st conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 17–24). Seoul: PME.

Arzarello, F., & Robutti, O. (2001). From body motion to algebra through graphing. In H. Chick, K. Stacey, & J. Vincent (Eds.), *Proceedings of the 12th ICMI study conference: The future of the teaching and learning of algebra* (Vol. 1, pp. 33–40). Melbourne: ICMI.

Arzarello, F., & Sabena, C. (2014). Analytic-structural functions of gestures in mathematical argumentation processes. In L. D. Edwards, F. Ferarra, & D. Moore-Russo (Eds.), *Emerging perspectives on gesture and embodiment* (pp. 75–103). Greenwich: Information Age Publishing.

Arzarello, F., Bosch, M., Gascón, J., & Sabena, C. (2008a). The ostensive dimension through the lenses of two didactic approaches. *ZDM – The International Journal on Mathematics Education, 40*(2), 179–188.

Arzarello, F., Bosch, M., Lenfant, A., & Prediger, S. (2008b). Different theoretical perspectives in research from teaching problems to research problems. In D. Pitta-Pantazi, G. Phillipou, et al. (Eds.), *Proceedings of the 5th Congress of the European Society for Research in Mathematics Education (CERME 5)* (pp. 1618–1627). Cyprus: ERME.

Arzarello, F., Paola, D., Robutti, O., & Sabena, C. (2009). Gestures as semiotic resources in the mathematics classroom. *Educational Studies in Mathematics, 70*(2), 97–109.

Arzarello, F., Bikner-Ahsbahs, A., & Sabena, C. (2010, January 28–February 1). Complementary networking: Enriching understanding. In *Proceedings of the 6th Congress of the European Society for Research in Mathematics Education* (pp. 1545–1554), Lyon, France. © INRP 2010. www.inrp.fr/editions/cerme6. Accessed 28 Nov 2013.

Arzarello, F., Ferrara, F., & Robutti, O. (2011). A tool for analysing multimodal behaviours in the mathematics classroom. In *Proceedings of PME 35*. Ankara: Middle East Technical University.

Assude, T., Boero, P., Herbst, P., Lerman, S., & Radford, L. (2008). *The notions and roles of theory in mathematics education research.* Paper presented by the Survey Team at ICME 11, Mexico.

Austin, J. L. (1975) *How to do things with words: The William James Lectures delivered at Harvard University in 1955* (2nd ed., J. O. Urmson & M. Sbisa, Eds.). Oxford: Clarendon Press.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*(5), 389–407.

Barbé, Q., Bosch, M., Espinoza, L., & Gascón, J. (2005). Didactic restrictions on the teacher's practice. The case of limits of functions. *Educational Studies in Mathematics, 59*, 235–268.

Bartolini Bussi, M. G., & Mariotti, M. A. (2008). Semiotic mediation in the mathematical classroom. Artifacts and signs after a Vygotskian perspective. In L. English (Ed.), *Handbook of international research in mathematics education* (2nd revised ed., pp. 746–783). Mahwah: Lawrence Erlbaum.

Bauer, L. A. (1988). *Mathematik und Subjekt*. Wiesbaden: Deutscher Universitätsverlag.

Bauersfeld, H. (1978). Kommunikationsmuster im Mathematikunterricht. Eine Analyse am Beispiel der Handlungsverengung durch Antworterwartung. In H. Bauersfeld (Ed.), *Fallstudien und Analysen zum Mathematikunterricht* (pp. 158–170). Hannover: Schroedel.

Bauersfeld, H. (1993). Theoretical perspectives on interaction in the mathematical classroom. In R. Biehler, R. W. Scholz, R. Sträßer, & B. Winkelmann (Eds.), *Didactics of mathematics as a scientific discipline* (pp. 133–146). Dordrecht: Kluwer.

Baumert, J., Schnabel, K., & Lehrke, M. (1998). Learning Maths in school: Does interest really matter? In L. Hoffmann, A. Krapp, K. A. Renninger, & J. Baumert (Eds.), *Interest and learning. Proceedings of the Seeon conference on interest and gender* (pp. 327–336). Kiel: IPN 164.

Beck, C., & Maier, H. (1994). Mathematikdidaktik als Textwissenschaft. Zum Status von Texten als Grundlage empirischer mathematikdidaktischer Forschung. *Journal für Mathematik-Didaktik, 15*(1–2), 35–78.

Bikner-Ahsbahs, A. (2003). Empirisch begründete Idealtypenbildung. Ein methodisches Prinzip zur Theoriekonstruktion in der interpretativen mathematikdidaktischen Forschung. *Zentralblatt für Didaktik der Mathematik, 35*(5), 208–223.

Bikner-Ahsbahs, A. (2005). *Mathematikinteresse zwischen Subjekt und Situation. Theorie interessendichter Situationen – Baustein für eine mathematikdidaktische Interessentheorie*. Hildesheim/Berlin: Franzbecker.

Bikner-Ahsbahs, A. (2010). Networking of theories: Why and how? In V. Durand-Guerrier, S. Soury-Lavergne, & F. Arzarello (Eds.), *Proceedings of the sixth Congress of the European Society for Research in Mathematics Education* (pp. 6–15), Lyon, 2009. Lyon: Institut National de Recherche Pédagogique.

Bikner-Ahsbahs, A. (2015). Empirically based ideal type construction (Translation of Bikner-Ahsbahs 2003). In A. Bikner-Ahsbahs, C. Knipping, & N. Presmeg (Eds.), *Approaches to qualitative research in mathematics education: Examples of methodology and methods* (Advances in mathematics education series). New York: Springer (in press).

Bikner-Ahsbahs, A., & Prediger, S. (2010). Networking of theories – An approach for exploiting the diversity of theoretical approaches. In B. Sriraman & L. English (Eds.), *Theories of mathematics education: Seeking new frontiers* (Advances in mathematics education series, pp. 483–506). New York: Springer.

Bikner-Ahsbahs, A., Dreyfus, T., Kidron, I., Arzarello, F., Radford, L., Artigue, M., & Sabena, C. (2010). Networking of theories in mathematics education. In M. M. F. Pinto & T. F. Kawasaki (Eds.), *Proceedings of the 34th conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 145–175). Belo Horizonte: PME.

Bikner-Ahsbahs, A., Kidron, I., & Dreyfus, T. (2011). *Epistemisch handeln können – aber wie?* [Knowing to act epistemically – But how?]. Invited lecture at the annual conference of the German Society of Mathematics Education. Beiträge zum Mathematikunterricht. http://www.mathematik.tu-dortmund.de/ieem/bzmu2011/_BzMU11_1_Einfuehrungen-Hauptvortraege/BzMU11_BIKNER_Angelika_Epistem.pdf. Accessed 30 Oct 2013.

Blumer, H. (1969). *Symbolic interactionism: Perspective and method*. Eaglewood Cliffs: Prentice-Hall.

Boaler, J., & Greeno, J. G. (2000). Identity, agency, and knowing in mathematics worlds. In J. Boaler (Ed.), *Multiple perspectives in mathematics teaching and learning* (pp. 171–198). Westport: Ablix.

Bogen, J., & Woodward, J. (1988). Saving the phenomena. *The Philosophical Review, CXVII*(3), 304–352.

Bosch, M., & Gascón, J. (2006). Twenty-five years of the didactic transposition. *ICMI Bulletin, 58*, 51–63.

Bosch, M., Gascón, J., Ruiz Olarría, A., Artaud, M., Bronner, A., Chevallard, Y., Cirade, G., Ladage, C., & Larguier, M. (2011). *Un panorama de la TAD. An overview on ATD* (CRM documents, Vol. 10). Bellaterra (Barcelona): Centre de Recerca Matemàtica.

Bourdieu, P. (1973). The three forms of theoretical knowledge. *Social Science Information, 12*(1), 53–80.

Bronner, A., Larguier, M., Artaud, M., Bosch, M., Chevallard, Y., Cirade, G., & Ladage, C. (2010). *Diffuser les mathématiques (et les autres savoirs) comme outils de connaissance et d'action*. Montpellier: IUFM.

Brousseau, G. (1980). Les échecs électifs en mathématiques dans l'enseignement élémentaire. *Revue de laryngologie, otologie, rhinologie*, *101*(3–4), 107–131. http://guy-brousseau.com/1388/les-echecs-electifs-en-mathematiques-dans-l%E2%80%99enseignement-elementaire-1980. Accessed 22 Oct 2013.

Brousseau, G. (1986). Fondements et méthodes de la didactique. *Recherches en didactique des mathématiques, 7*(2), 35–115.

Brousseau, G. (1990). Le contrat didactique: le milieu. *Recherches en Didactique des Mathématiques, 9*(3), 309–336.

Brousseau, G. (1995). L'enseignant dans la théorie des situations didactiques (The teacher in the theory of didactical situations). In R. Noirfalise & M. J. Perrin-Glorian (Eds.), *Actes de la VIIIe Ecole d'Eté de Didactique des Mathématiques* (pp. 3–46). Clermont-Ferrand: IREM de Clermont-Ferrand. http://guy-brousseau.com/2319/l%E2%80%99enseignant-dans-la-theorie-des-situations-didactiques-1995/. Accessed 07 July 2014.

Brousseau, G. (1997). *Theory of didactical situations in mathematics*. Dordrecht: Kluwer.

Brousseau, G. (2008). Research in mathematics education. In M. Niss (Ed.), *Proceedings of ICME-10* (pp. 244–254). Denmark: IMFUFA, Department of Sciences, Systems and Models, Roslkilde University.

Brousseau, G., & Centeno, J. (1991). Rôle de la mémoire didactique de l'enseignant. *Recherches en didactiques des mathématiques, 11*, 167–210.

Brousseau, G., & Warfield, V. (1999). The case of Gaël. *Journal of Mathematical Behavior, 18*(1), 1–46.

Chevallard, Y. (1985). *La Transposition Didactique. Du savoir savant au savoir enseigné* (2nd ed., 1991). Grenoble: La Pensée Sauvage.

Chevallard, Y. (1989). On didactic transposition theory: Some introductory notes. In *Proceedings of the international symposium on selected domains of research and development in mathematics education,* Bratislava (pp. 51–62). http://yves.chevallard.free.fr/spip/spip/IMG/pdf/On_Didactic_Transposition_Theory.pdf. Accessed 30 Oct 2013.

Chevallard, Y. (1992a). Fundamental concepts in didactics: Perspectives provided by an anthropological approach. In R. Douady & A. Mercier (Eds.), *Research in Didactique of mathematics, Selected papers* (pp. 131–167). Grenoble: La Pensée Sauvage.

Chevallard, Y. (1992b). A theoretical approach to curricula. *Journal für Mathematikdidaktik*, *13*(2/3), 215–230. http://yves.chevallard.free.fr/spip/spip/IMG/pdf/A_Theoretical_Approach_to_Curricula.pdf. Accessed 30 Oct 2013.

Chevallard, Y. (1999). L'analyse des pratiques enseignantes en théorie anthropologique du didactique. *Recherches en Didactique des Mathématiques, 19*(2), 221–266.

Chevallard, Y. (2002). Organiser l'étude. 3. Écologie regulation. In J.-L. Dorier, M. Artaud, M. Artigue, R. Berthelot, & R. Floris (Eds.), *Actes de la 11e École d'Été de didactique des mathématiques* (pp. 41–56). Grenoble: La Pensée sauvage.

Chevallard, Y. (2006). Steps towards a new epistemology in mathematics education. In M. Bosch (Ed.), *Proceedings of the IV Congress of the European Society for Research in Mathematics Education* (pp. 21–30). Barcelona: FUNDEMI-IQS.

Chevallard, Y. (2008, November). Afterthoughts on a seeming didactical paradox. In *International conference "Efficacité & Équité en Éducation"*, Rennes, France. http://yves.chevallard.free.fr/spip/spip/IMG/pdf/Yves_Chevallard_-_On_Inquiry-Based_Instruction.pdf. Accessed 28 Nov 2013.

Chevallard, Y. (2012). Teaching mathematics in tomorrow's society: A case for an oncoming counterparadigm. In *Preconference proceedings of 12th International Congress on Mathematical Education 2012*, Seoul, Korea. http://www.icme12.org/upload/submission/1985_F.pdf. Accessed 28 Nov 2013.

Comiti, C., & Grenier, D. (1995). Two examples of "split situation" in the mathematics classroom. *For the Learning of Mathematics, 15*(2), 17–22.

Csikszentmihalyi, M., & Schiefele, U. (1993). Die Qualität des Erlebens und der Prozeß des Lernens. *Zeitschrift f. Pädagogik, 39*(2), 207–221.

Davydov, V. V. (1990). *Types of generalisation in instruction: Logical and psychological problems in the structuring of school curricula* (Soviet studies in mathematics education, Vol. 2, J. Kilpatrick, Ed., J. Teller, Trans.). Reston: National Council of Teachers of Mathematics. (Original work published 1972.)

Deci, E. (1992). The relation of interest to motivation of behavior: A self-determination theory – Theory perspective. In K. A. Renninger, S. Hidi, & A. Krapp (Eds.), *The role of interest in learning and development* (pp. 43–70). Hillsdale/London: Lawrence Erlbaum.

Deci, E. (1998). The relation of interest to motivation and human needs – The self-determination theory viewpoint. In L. Hoffmann, A. Krapp, K. A. Renninger, & J. Baumert (Eds.), *Interest and learning. Proceedings of the Seeon conference on interest and gender* (pp. 146–164). Kiel: IPN 164.

Denzin, N. K. (1970). *The research act in sociology*. Chicago: Aldine.

Dorier, J. L., Artaud, M., Artigue, M., Berthelot, R., & Floris, R. (2002). *Actes de la 11e Ecole d'Eté de Didactique des Mathématiques*. Grenoble: La Pensée Sauvage.

Dreyfus, T. (2006). Linking theories in mathematics education. In A. Simpson (Ed.), *Retirement as process and as concept* (pp. 77–82). Festschrift for Eddie Gray and David Tall, presented at Charles University, Prague.

Dreyfus, T., & Kidron, I. (2006). Interacting parallel constructions: A solitary learner and the bifurcation diagram. *Recherches en Didactique des Mathématiques, 26*(3), 295–336.

Dreyfus, T., & Tsamir, P. (2004). Ben's consolidation of knowledge structures about infinite sets. *Journal of Mathematical Behavior, 23*, 271–300.

Dreyfus, T., Hershkowitz, R., & Schwarz, B. (2001). Abstraction in context II: The case of peer interaction. *Cognitive Science Quarterly, 1*, 307–368.

Dreyfus, T., Hershkowitz, R., & Schwarz, B. B. (2015). The nested epistemic actions model for abstraction in context – Theory as a methodological tool and methodological tool as theory. In A. Bikner-Ahsbahs, C. Knipping, & N. Presmeg (Eds.), *Approaches to qualitative research in mathematics education: Examples of methodology and methods* (Advances in mathematics education series). New York: Springer (in press).

Duval, R. (1995). *Semiosis et Noesis*. Berne: Peter Lang.

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics, 61*, 103–131.

Edwards, L. (2009). Gestures and conceptual integration in mathematical talks. *Educational Studies in Mathematics, 70*(2), 127–141.

Eisenhart, M. (1991). Conceptual frameworks for research. Ideas from a cultural anthropologist; Implications for mathematics education researchers. In R. Underhill (Ed.), *Proceedings of the thirteenth annual meeting of Psychology of Mathematics Education – North America*. Blacksburg: PME.

English, L., & Sriraman, B. (2005). Theories of mathematics education: A global survey of theoretical frameworks/trends in mathematics education research. *Zentralblatt für Didaktik der Mathematik, 37*(6), 450–456.

Ernest, P. (1998). A postmodern perspective on research in mathematics education. In A. Sierpinska & J. Kilpatrick (Eds.), *Mathematics education as a research domain: A search for identity* (Vol. 1, pp. 71–85). Dordrecht: Kluwer.

Ernest, P. (2006). A semiotic perspective of mathematical activity: The case of number. *Educational Studies in Mathematics, 61*, 67–101.

Estepa, A., García, F. J., & Ruiz-Higueras, L. (2006). *Sociedad, escuela y matemáticas. Aportaciones de la Teoría Antropológica de lo Didáctico (TAD)*. Jaén: Publicaciones de la Universidad de Jaén.

Fetzer, M. (2013). *Counting on objects in mathematical learning processes. Network theory and networking theories.* Contribution to CERME 8 in Antalya, Turkey.

Feucht, F., & Bendixen, L. D. (2010). Personal epistemology in the classroom: A welcome and guide for the reader. In L. D. Bendixen & F. C. Feucht (Eds.), *Personal epistemology in the classroom. Theory, research and implications for practice* (pp. 3–28). Cambridge: Cambridge University Press.

Fischer, R. (1993). Wissenschaft, Argumentation und Widerspruch. In R. Fischer, M. Costazza, & A. Pellert (Eds.), *Argument und Entscheidung: zur Idee und Organisation von Wissenschaft* (pp. 29–44). Wien: Profil.

Flick, U. (2007). *Triangulation – eine Einführung* [Triangulation – An introduction]. Wiesbaden: VS-Verlag für Sozialwissenschaften.

Freudenthal, H. (1991). *Revisiting mathematics education. China lectures*. Dordrecht: Kluwer.

Gallese, V., & Lakoff, G. (2005). The brain's concepts: The role of the sensory-motor system in conceptual knowledge. *Cognitive Neuropsychology, 21*, 1–25.

García, F. J., Gascón, J., Ruiz Higueras, L., & Bosch, M. (2006). Mathematical modelling as a tool for the connection of school mathematics. *ZDM – The International Journal on Mathematics Education, 38*(3), 226–246.

Gardner, P. L. (1998). The development of males' and females' interest in science and technology. In L. Hoffmann, A. Krapp, K. A. Renninger, & J. Baumert (Eds.), *Interest and learning. Proceedings of the Seeon conference on interest and gender* (pp. 41–57). Kiel: IPN 164.

Garfinkel, H. (2003). *Studies in ethnomethodology*. Englewood Cliffs: Prentice Hall. [First published 1967.]

Gerhard, U. (1986). Verstehende Strukturanalyse: Die Konstruktion von Idealtypen als Analyseschritt bei der Auswertung qualitativer Forschungsmaterialien. In H.-G. Soeffner (Ed.), *Sozialstruktur und Typik* (pp. 31–83). Frankfurt/New York: Campus Verlag.

Giest, H. (2005). Zum Verhältnis von Konstruktivismus und Tätigkeitsansatz in der Pädagogik. In F. Radis, M.-L. Braunsteiner, & K. Klement (Eds.), *Badener VorDrucke* (pp. 43–64). Baden/A.: Kompetenzzentrum für Forschung und Entwicklung (Schriftenreihe zur Bildungsforschung – Band 3).

Glaser, B., & Strauss, A. (1967). *The discovery of grounded theory: Strategies for qualitative research*. Chicago: Aldine.
Goldin-Meadow, S. (2003). *Hearing gestures: How our hands help us think*. Cambridge, MA: Harvard University Press.
Guin, D., Ruthven, K., & Trouche, L. (Eds.). (2004). *The didactic challenge of symbolic calculators*. Dordrecht: Kluwer.
Hannula, M., Evans, J., Philippou, G., & Zan, R. (2004). *Affect in mathematics education – Exploring theoretical frameworks.* Research forum. International Group for the Psychology of Mathematics Education. www.emis.de/proceedings/PME28/RF/RF001.pdf. Accessed 2 Dec 2013.
Heidegger, M. (1977). *The question concerning technology and other essays*. New York: Harper Torchbooks.
Hersant, M. (2004). Caractérisation d'une pratique d'enseignement des mathématiques: le cours dialogué. *Revue Canadienne de l'enseignement des sciences, des mathématiques et des technologies, 4*(2), 243–260.
Hersant, M., & Perrin-Glorian, M.-J. (2005). Characterization of an ordinary teaching practice with the help of the theory of didactic situations. *Educational Studies in Mathematics, 59*(1–3), 113–151.
Hershkowitz, R. (2009). Contour lines between a model as a theoretical framework and the same model as methodological tool. In B. B. Schwarz, T. Dreyfus, & R. Hershkowitz (Eds.), *Transformation of knowledge through classroom interaction* (pp. 273–280). London: Routledge.
Hershkowitz, R., Dreyfus, T., Ben-Zvi, D., Friedlander, A., Hadas, N., Resnick, T., Tabach, M., & Schwarz, B. (2002). Mathematics curriculum development for computerised environments: A designer-researcher-teacher-learner activity. In L. D. English (Ed.), *Handbook of international research in mathematics education: Directions for the 21st century* (pp. 657–694). Mahwah: Lawrence Erlbaum.
Hidi, S., & Renninger, K. (2006). The four-phase model of interest development. *Educational Psychologist, 42*(2), 111–127.
Hofer, B., & Pintrich, P. R. (1997). The development of epistemological theories: Beliefs about knowledge and knowing and their relation to learning. *Review of Educational Research, 67*, 88–140.
Hoffmann, L., & Häußler, P. (1998). An intervention project promoting girls' and boys' interest in physics. In L. Hoffmann, A. Krapp, K. A. Renninger, & J. Baumert (Eds.), *Interest and learning. Proceedings of the Seeon conference on interest and gender* (pp. 301–316). Kiel: IPN 164.
Jungwirth, H. (2003). Interpretative Forschung in der Mathematikdidaktik – ein Überblick für Irrgäste, Teilzieher und Strandvögel. In G. Kaiser (Ed.), Qualitative empirical methods in mathematics education – Discussions and reflections. *Zentralblatt für Didaktik der Mathematik, 35*(5), 189–200
Kendon, A. (2000). Language and gesture: Unity or duality? In D. McNeill (Ed.), *Language and gesture: Window into thought and action* (pp. 47–63). Cambridge: Cambridge University Press.
Kidron, I., & Dreyfus, T. (2010a). Justification enlightenment and combining constructions of knowledge. *Educational Studies in Mathematics, 74*(1), 75–93.
Kidron, I., & Dreyfus, T. (2010b). Interacting parallel constructions of knowledge in a CAS context. *International Journal of Computers for Mathematical Learning, 15*(2), 129–149.
Kidron, I., & Monaghan, J. (2009). Commentary on the chapters on the construction of knowledge. In B. B. Schwarz, T. Dreyfus, & R. Hershkowitz (Eds.), *Transformation of knowledge through classroom interaction* (pp. 81–90). London: Routledge.
Kidron, I., Lenfant, A., Bikner-Ahsbahs, A., Artigue, M., & Dreyfus, T. (2008). Toward networking three theoretical approaches: The case of social interactions. *ZDM – The International Journal on Mathematics Education, 40*(2), 247–264.
Kidron, I., Bikner-Ahsbahs, A., Cramer, J., Dreyfus, T., & Gilboa, N. (2010). Construction of knowledge: Need and interest. In M. M. F. Pinto & T. F. Kawasaki (Eds.), *Proceedings of the 34th conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 169–176). Belo Horizonte: PME.

Kidron, I., Bikner-Ahsbahs, A., & Dreyfus, T. (2011a). How a general epistemic need leads to a need for a new construct: A case of networking two theoretical approaches. In M. Pytlak, T. Rowland, & E. Swoboda (Eds.), *Proceedings of the 7th Congress of the European Society for Research in Mathematics Education* (pp. 2451–2461). Rzeszów: University of Rzeszów.

Kidron, I., Bikner-Ahsbahs, A., Monaghan, J., Radford, L., & Sensevy, G. (2011b). Different theoretical perspectives and approaches in research in mathematics education. In M. Pytlak, T. Rowland, & E. Swoboda (Eds.), *Proceedings of the 7th conference of the European Society for Research in Mathematics Education* (pp. 2377–2380). Rzeszow: ERME/University of Rzeszow.

Kidron, I., Bosch, M., Monaghan, J., & Radford, L. (2013). Different theoretical perspectives and approaches in research in mathematics education. In B. Ubuz, C. Haser, M. A. Mariotti (Ed.), *Proceedings of the 8th Congress of the European Society for Research in Mathematics Education* (CERME 8), Antalya.

Kita, S. (2000). How representational gestures help speaking. In D. McNeill (Ed.), *Language and gesture: Window into thought and action* (pp. 162–185). Cambridge: Cambridge University Press.

Knipping, Ch., & Müller-Hill, E. (2013). *The problem of detecting phenomena amid a sea of noisy data*. Paper presented at the 8th Congress of the European Society for Research in Mathematics Education (CERME 8), Antalya, Turkey.

Knorr Cetina, K. D. (1999). *Epistemic cultures: How the sciences make knowledge*. Cambridge: Harvard University Press.

Krapp, A. (1992). Das Interessenkonstrukt. In A. Krapp & M. Prenzel (Eds.), *Interesse, Lernen, Leistung* (pp. 297–329). Münster: Aschendorff.

Krapp, A. (2004). An educational-psychological theory of interest and its relation to self-determination theory. In E. L. Deci & R. M. Ryan (Eds.), *The handbook of self-determination research* (pp. 405–427). Rochester: University of Rochester Press.

Kress, G. (2004). Reading images: Multimodality, representation and new media. *Information Design Journal, 12*(2), 110–119.

Krummheuer, G. (2000). Mathematics learning in narrative classroom cultures: Studies of argumentation in primary mathematics education. *For the Learning of Mathematics, 20*(1), 22–32.

Krummheuer, G., & Brandt, B. (2001). *Paraphrase und Traduktion. Partizipatorische Elemente einer Interaktionstheorie des Mathematikunterrichts in der Grundschule*. Weinheim/Basel: Beltz.

Laborde, C., Perrin-Glorian, M.-J., & Sierpinska, A. (Eds.). (2005). *Beyond the apparent banality of the mathematics classroom*. New York: Springer.

Lagrange, J.-B. (2005). Curriculum, classroom practices and tool design in the learning of functions through technology-aided experimental approaches. *International Journal of Computers in Mathematics Learning, 10*, 143–189.

Lakoff, G., & Nùñez, R. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. New York: Basic Books.

Lamnek, S. (1995). *Qualitative Sozialforschung Band 1: Methodologie*. Weinheim: Beltz.

Leont'ev, A. N., & Lurija, A. R. (1973). Le idee psicologiche di Vygotskij. In L. S. Vygotskij (Ed.), *Lo sviluppo psichico del bambino*. Roma: Editori Riuniti.

Lerman, S. (2006). Theories of mathematics education: Is plurality a problem? *Zentralblatt für Didaktik der Mathematik, 38*(1), 8–13.

Lising, L., & Elby, A. (2004). The impact of epistemology on learning: A case study from introductory physics. *American Journal of Physics, 73*(4), 372–382.

Lotman, Y. M. (1990). *Universe of the mind. A semiotic theory of culture*. Bloomington: Indiana University Press.

Margolinas, C. (1995). La structuration du milieu et ses apports dans l'analyse a posteriori des situations. In C. Margolinas (Ed.), *Les débats de didactique des mathématiques, Annales 1993–1994* (pp. 89–102). Grenoble: La Pensée Sauvage.

Margolinas, C. (1998). Le milieu et le contrat, concepts pour la construction et l'analyse de situations d'enseignement. In R. Noirfalise (Ed.), *Analyse des pratiques enseignantes et didactique des mathématiques, Actes de l'Université d'Eté, La Rochelle*. Clermont-Ferrand: IREM. http://halshs.archives-ouvertes.fr/halshs-00421845/fr. Accessed 22 Oct 2013.

Margolinas, C. (2002). Situations, milieux, connaissances: Analyse de l'activité du professeur, cours. In J. L. Dorier et al. (Eds.), *Actes de la 11ème Ecole d'Eté de Didactique des Mathématiques* (pp. 141–157). Grenoble: La Pensée Sauvage. http://halshs.archives-ouvertes.fr/halshs-00421848/fr. Accessed 22 Oct 2013.

Margolinas, C., Abboud-Blanchard, M., Bueno-Ravel, L., Douek, N., Fluckiger, A., Gibel, P., Vandebrouck, F., & Wozniak, F. (Eds.). (2011). *En amont et en aval des ingénieries didactiques. Actes de la 11ème Ecole d'Eté de Didactique des Mathématiques, Clermont-Ferrand 2009*. Grenoble: La Pensée Sauvage.

Mason, J., & Waywood, A. (1996). The role of theory in mathematics education and research. In A. J. Bishop et al. (Eds.), *International handbook of mathematics education* (pp. 1055–1089). Dordrecht: Kluwer.

McLeod, S. A. (2007). *Nature nurture in psychology.* http://www.simplypsychology.org/naturevs-nurture.html. Accessed 21 Oct 2013.

McNeill, D. (1992). *Hand and mind: What gestures reveal about thought*. Chicago: University of Chicago Press.

McNeill, D. (2005). *Gesture and thought*. Chicago: University of Chicago Press.

McNeill, D., Quek, F., McCullough, K.-E., Duncan, S., Furuyama, N., Bryll, R., Ma, X.-F., & Ansari, R. (2001). Catchments, prosody and discourse. *Gesture, 1*(1), 9–33.

Mitchell, M. (1993). Situational interest. Its multifaceted structure in the secondary school mathematics classroom. *Journal of Educational Psychology, 85*(3), 424–436.

Montessori, M. (1934). *Psicogeometria.* Barcelona: Araluce. (English translation published by Montessori-Pierson Publishing Company.)

Nemirovsky, R. (2003). Three conjectures concerning the relationship between body activity and understanding mathematics. In N. A. Pateman, B. J. Dougherty, & J. T. Zillox (Eds.), *Proceedings of PME 27* (Vol. 1, pp. 105–109). Honolulu: PME.

Novotna, J., & Hospesova, A. (2007). What is the price of Topaze? In J.-H. Woo, H.-C. Lew, K.-S. Park, & D.-Y. Seo (Eds.), *Proceedings of PME 31* (Vol. 4, pp. 25–32). Seoul: PME.

Novotna, J., & Hospesova, A. (2008). Types of linking in teaching linear equations. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano, & A. Sepulveda (Eds.), *Proceedings of PME 32 and PME NA 30* (Vol. 4, pp. 49–56). Morelia: PME.

Overton, W. F. (2008). Embodiment from a relational perspective. In W. F. Overton, U. Muller, & J. L. Newman (Eds.), *Developmental perspectives on embodiment and consciousness* (pp. 1–18). New York: Lawrence Erlbaum.

Paola, D. (2006). Il significato di crescita esponenziale in un ambiente di geometria dinamica. *La matematica e la sua didattica, 1*, 39–58.

Peirce, C. S. (1931–1958). *Collected papers, Vol. I–VIII* (C. Hartshorne, P. Weiss, & A. Burks, Eds.). Cambridge, MA: Harvard University Press.

Perrin-Glorian, M. J. (1999). Problèmes d'articulation de cadres théoriques: l'exemple du concept de milieu. *Recherches en Didactique des Mathématiques, 19*(3), 279–322.

Piaget, J. (1952). *The origins of intelligence in children*. New York: International Universities Press.

Prediger, S. (2008). How are theoretical approaches expressed in research practices? A report on an experience in comparing theoretical approaches with respect to the construction of research problems. *ZDM – The International Journal on Mathematics Education, 40*(2), 277–286.

Prediger, S. (2014). Rolle von Theorien. In R. Bruder, L. Hefendehl-Hebeker, B. Schmidt-Thieme, & H.-G. Weigand (Eds.), *Handbuch Mathematikdidaktik*. Heidelberg: Springer (in press).

Prediger, S., & Ruthven, K. (2007). From teaching problems to research problems. Proposing a way of comparing theoretical approaches. In D. Pitta-Pantazi & G. Phillipou (Eds.), *Proceedings of the 5th Congress of the European Society for Research in Mathematics Education (CERME 5)* (pp. 1745–1771). Cyprus: ERME.

Prediger, S., Arzarello, F., Bosch, M., & Lenfant, A. (Eds.). (2008a). Comparing, combining, coordinating – Networking strategies for connecting theoretical approaches. Thematic Issue of *ZDM – The International Journal on Mathematics Education*, *40*(2), 163–327.

Prediger, S., Bikner-Ahsbahs, A., & Arzarello, F. (2008b). Networking strategies and methods for connecting theoretical approaches: First steps towards a conceptual framework. *ZDM – The International Journal on Mathematics Education, 40*(2), 165–178.

Prediger, S., Bosch, M., Kidron, I., Monaghan, J., & Sensevy, G. (2010). Different theoretical perspectives and approaches in mathematics education research – Strategies and difficulties when connecting theories. In V. Durand-Guerrier, S. Soury-Lavergne, & F. Arzarello (Eds.), *Proceedings of the sixth Congress of the European Society for Research in Mathematics Education* (pp. 1529–1534). Lyon: Institut National de Recherche Pédagogique.

Prenzel, M. (1998). The selective persistence of interest. In L. Hoffmann, A. Krapp, K. A. Renninger, & J. Baumert (Eds.), *Interest and learning. Proceedings of the Seeon conference on interest and gender* (pp. 71–92). Kiel: IPN 164.

Radford, L. (2008). Connecting theories in mathematics education: Challenges and possibilities. *ZDM – The International Journal on Mathematics Education, 40*(2), 317–327.

Radford, L. (2012, May 31–June 1). *On the growth and transformation of mathematics education theories.* Paper presented at the International Colloquium: The didactics of mathematics: Approaches and issues. A homage to Michèle Artigue. http://www.laurentian.ca/educ/lradford. Accessed 28 Nov 2013.

Radford, L. (2013). On the role of representations and artefacts in knowing and learning. *Educational Studies in Mathematics*, *85*, 405–422. doi:10.1007/s10649-013-9527-x.

Radford, L., Bardini, C., Sabena, C., Diallo, P., & Simbagoye, A. (2005). On embodiment, artifacts, and signs: A semiotic-cultural perspective on mathematical thinking. In H. L. Chick & J. L. Vincent (Eds.), *Proceedings of the 29th conference of the international group for the psychology of mathematics education* (Vol. 4, pp. 113–120). Melbourne: University of Melbourne, PME.

Radford, L., Bardini, C., & Sabena, C. (2007). Perceiving the general: The semiotic symphony of students' algebraic activities. *Journal for Research in Mathematics Education, 38*(5), 507–530.

Renninger, K. A. (1998). The roles of individual interest(s) and gender in learning: An overview of research on preschool and elementary school-aged children/students. In L. Hoffmann, A. Krapp, K. A. Renninger, & J. Baumert (Eds.), *Interest and learning. Proceedings of the Seeon conference on interest and gender* (pp. 165–174). Kiel: IPN 164.

Restrepo, A. (2008). *Genèse instrumentale du déplacement en géométrie dynamique chez des élèves de 6eme*. Doctoral thesis, Université Joseph Fourier, Grenoble.

Ron, G., Dreyfus, T., & Hershkowitz, R. (2010). Partially correct constructs illuminate students' inconsistent answers. *Educational Studies in Mathematics, 75*, 65–87.

Ruthven, K., Laborde, C., Leach, J., & Tiberghien, A. (2009). Design tools in didactical research: Instrumenting the epistemological and cognitive aspects of the design of teaching sequences. *Educational Researcher, 38*(5), 329–342.

Sabena, C. (2007). *Body and signs: A multimodal semiotic approach to teaching-learning processes in early calculus*. Tesi di Dottorato, Università degli Studi di Torino.

Sabena, C. (2008). On the semiotics of gestures. In L. Radford, G. Schumbring, & F. Seeger (Eds.), *Semiotics in mathematics education: Epistemology, history, classroom, and culture* (pp. 19–38). Rotterdam: Sense.

Sabena, C., Robutti, O., Ferrara, F., & Arzarello, F. (2012). The development of a semiotic frame to analyse teaching and learning processes: Examples in pre- and post-algebraic contexts. In L. Coulange, J.-P. Drouhard, J.-L. Dorier, & A. Robert (Eds.), *Recherches en Didactique des Mathématiques, Numéro spécial hors-série, Enseignement de l'algèbre élémentaire: bilan et perspectives* (pp. 231–245). Grenoble: La Pensée Sauvage.

Schiefele, U., & Csikszentmihalyi, M. (1995). Motivation and ability as factors in mathematics experience and achievement. *Journal for Research in Mathematics Education, 26*(2), 163–181.

Schiefele, U., & Schreyer, I. (1994). Intrinsische Lernmotivation und Lernen. Ein Überblick zu Ergebnissen der Forschung. *Zeitschrift für Pädagogische Psychologie, 8*(1), 1–13.

Schiefele, H., Hausser, K., & Schneider, G. (1979). "Interesse" als Weg und Ziel der Erziehung. *Zeitschrift für Pädagogik, 25*(1), 1–20.

Schiralli, M., & Sinclair, N. (2003). A constructive response to 'Where mathematics comes from'. *Educational Studies in Mathematics, 52*, 79–91.

Schneider, M. (2013). Epistemological obstacles in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education*. Heidelberg: Springer (in press).

Schoenfeld, A. H. (2002). Research methods in (mathematics) education. In L. English (Ed.), *Handbook of international research in mathematics education* (pp. 435–488). Mahwah: Erlbaum.

Schwarz, B. B., Dreyfus, T., & Hershkowitz, R. (2009). The nested epistemic actions model for abstraction in context. In B. B. Schwarz, T. Dreyfus, & R. Hershkowitz (Eds.), *Transformation of knowledge through classroom interaction* (pp. 11–41). London: Routledge.

Sensevy, G. (2011). Overcoming fragmentation: Towards a joint action theory in didactics. In B. Hudson & M. Meyer (Eds.), *Beyond fragmentation: Didactics, learning and teaching in Europe* (pp. 60–76). Leverkusen/Farmington Hills: Barbara Budrich Publishers.

Sensevy, G. (2012). About the joint action theory in didactics. *Zeitschrift für Erziehungswissenschaft, 15*(3), 503–516.

Sensevy, G., Schubauer-Leoni, M.-L., Mercier, A., Ligozat, F., & Perrot, G. (2005). An attempt to model the teacher's action in the mathematics class. *Educational Studies in Mathematics, 59*(1–3), 153–181.

Silver, E. A., & Herbst, P. G. (2007). Theory in mathematics education scholarship. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 39–68). Reston: NCTM.

Sriraman, B., & English, L. (Eds.). (2010). *Theories of mathematics education: Seeking new frontiers*. Berlin/Heidelberg: Springer Science.

Steen, L. A. (1999). Review of mathematics education as research domain. *Journal for Research in Mathematics Education, 30*(2), 235–241.

Stefan, G. (2012). *Motivation und Interesse im Mathematikunterricht der Grundschule: Genese-Indizierung-Förderung*. Hamburg: Verlag Dr. Kovac.

Steinbring, H. (1998). Epistemological constraints of mathematical knowledge in social learning settings. In A. Sierpinska & J. Kilpatrick (Eds.), *Mathematics education as a research domain: A search for identity* (pp. 513–526). Dordrecht: Kluwer.

Steinbring, H. (1999). Reconstructing the mathematical in social discourse – Aspects of an epistemology-based interaction research. In O. Zaslavsky (Ed.), *Proceedings of the 23rd international conference for the psychology of mathematics education* (Vol. I, pp. 40–55). Haifa: Technion – Israel Institute of Technology.

Steinbring, H. (2005). *The construction of new mathematical knowledge in classroom interaction – An epistemological perspective* (Mathematics education library, Vol. 38). Berlin/New York: Springer.

Strauss, A. L. (1994). *Grundlagen qualitativer Sozialforschung* (UTB für Wissenschaften. Bd. 1776). München: Wilhelm Fink Verlag.

Teppo, A. (1998). Divers ways of knowing. In A. R. Teppo (Ed.), *Qualitative research methods in mathematics education* (pp. 1–16). Reston: NCTM.

Ulich, D. (1976). *Pädagogische Interaktion. Theorien erzieherischen Handelns und sozialen Lernens*. Weinheim/Basel: Beltz.

Verret, M. (1975). *Le temps des études*. Paris: Honoré Champion.

Voigt, J. (1983). Die heimliche Organisation von Aufgabenlösungsprozessen im Mathematikunterricht einer achten Klasse. In H. Bauersfeld, H. Bussmann, G. Krummheuer, J.-H. Lorenz, & J. Voigt (Eds.), *Lernen und Lehren von Mathematik. Untersuchungen zum Mathematikunterricht, IDM Band 6* (pp. 172–222). Köln: Aulis-Verlag.

Vygotsky, L. S. (1978). *Mind in society. The development of higher psychological processes* (M. Cole, V. John-Steiner, S. Scribner, & E. Souberman, Eds.). Cambridge, MA/London: Harvard University Press.

Vygotsky, L. S. (1997). *Collected works*, *Vol. 4* (R. Rieber, Ed.). New York: Plenum.

Warfield, V. (2006). *Invitation to didactique.* www.math.washington.edu/~warfield/Inv%20to%20Did66%207-22-06.pdf. Accessed 29 Nov 2013.

Weber, M. (1921/1984). *Soziologische Grundbegriffe*. Tübingen: J.C.B. Mohr, UTB.

Weber, M. (1922/1985). *Wissenschaftslehre. Gesammelte Aufsätze*. Tübingen: J. C. B. Mohr.

Wertsch, J. V., & Addison Stone, C. (1985). The concept of internalization in Vygotsky's account on the genesis of higher mental functions. In J. V. Wertsch (Ed.), *Culture, communication and cognition: Vygotskian perspectives. The concept of activity in Soviet psychology* (pp. 162–179). Cambridge: Cambridge University Press.

Wilson, M. (2002). Six views of embodied cognition. *Psychonomic Bulletin & Review, 9*(4), 625–636.

Woodward, J. (1989). Data and phenomena. *Synthese, 79*, 393–472.

Wrathall, M. (2011). *Heidegger and unconcealment. Truth, language, and history*. New York: Cambridge University Press.

Zan, R., Brown, L., Evans, J., & Hannula, M. S. (2006). Affect in mathematics education: An introduction. *Educational Studies in Mathematics, 63*, 113–121.

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