Chapter 8
About the Cosserats' Book of 1909

Abstract The Cosserat brothers published in 1909 an original book where they favour a variational formulation of continuum mechanics together with an invariance which they call "Euclidean invariance" of the Lagrangian-Hamiltonian action. This strategy places on an equal footing translations and possible rotational degrees of freedom, yielding in a natural way what is now commonly called the theory of Cosserat or oriented—or polar—continua with asymmetric stresses and the new notion of couple stresses. Here their landmark work is replaced in its epoch making context underlining the influences they benefited from and the influence they have exerted on their direct contemporaries and much later on (in the second-half of the twentieth century). The sociological scientific environment of the early twentieth century and the typical publication strategy of the time are outlined, explaining thus the Cosserats' own strategy. The further reception of their work from 1909 to the Second World War and the revival of interest in it in the nineteen-fifties are examined critically. Finally, the formalization of their work in a new landscape of continuum thermo-mechanics created essentially by Truesdell is evoked together with other influences and further developments.

8.1 Preliminaries

Year 2009 witnessed the commemoration of the centennial of the publication of their (now) celebrated book on the "Theory of deformable bodies" [13]. Celebrations took place in Paris in the form of a EUROMECH Colloquium with many participants from Germany and Russia although attendants came from the world over (Proceedings edited by Maugin and Metrikine [55]), as also at the National "Ponts et Chaussées" school as one of the authors, François Cosserat (1852–1914), was an alumnus from that school. A fac-simile edition of the original book was published on that occasion (2010) with interesting historical comments by M. Brocato and K. Chatzis. This opus by the Cosserat brothers was their longest common contribution to the science of mechanics. The way this was published as

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also the general approach of the brothers concerning this field and their own professional activities require some comments as it appears that neither François nor his young brother Eugène (1866–1931) were professional mathematicians in the field of mechanics. But they were enlightened amateurs with all technical abilities and background of true professionals. Both became members of the Paris Academy of Sciences (François in 1896, and Eugène in 1919). François was even elected President of the French Society of Mathematics (*Société Mathématique de France*) in 1913 one year before his death. Still the way they published is somewhat unusual and also concentrated in time in the period 1896–1914 with the death of François.

François Cosserat was educated at *the Ecole Polytechnique* in Paris with a further specialization in civil engineering at the *Ecole Nationale des Ponts et Chaussées*. This curriculum in the best mathematical and mechanical tradition was typical of many great French "engineers-scientists" of the nineteenth century (among them, Cauchy, Navier, Lamé, Duhamel, Coriolis, Clapeyron, Poncelet, Liouville, Arago and Barré de Saint-Venant). He had a professional career in the fast growing development of railways with the Nord and then the East companies of Railways in France. Eugène, his younger brother by 14 years, was educated in mathematics at the *Ecole Normale Supérieure* in Paris and became a professional (mathematical) astronomer with a career spent almost entirely in Toulouse in the south–west of France. As such he had to teach courses in analysis, astronomy and celestial mechanics.

From 1896 till the death of François in 1914, the Cosserats published together no less than 21 works in the field of theoretical mechanics. Out of these, 14 were short notes—of three or four pages—to the Paris Academy of Sciences. Apart from their long original memoir of 1896 [12] published in Toulouse in a true serial journal,[1] their other publications in the field are scattered in odd places, often as supplements or comments to books by more acknowledged institutional authors: one is a note in the lecture notes of Gabriel Koenigs (published in 1897)—cf. the review by Lovett [43] and citation below—, one is a note of 37 pages in Vol. 1 of Chwolson's Treatise of Physics in its French translation [10, pp. 236–273], another one is a note of seventy two pages in Appell's Treatise of Rational Mechanics [3, Vol. III, pp. 557–629], still another one is an adaptation in French of an article by Aurel Voss (1845–1931) in German on the principles of rational mechanics in the *Encyclopédie des sciences mathématiques pures et appliquées* (the original is the *Encyklopaedie der mathematischen Wissenschaften*), Vol. IV, pp. 1–187—published in 1915 [14] after Francois's death, and finally their now most celebrated opus is a supplement to Chwolson's Treatise of Physics [11, Vol. II, pp. 953–1173], also published with a new pagination (vi+226 pages) as a separate book by Hermann Editeurs in Paris. As noticed by its American reviewer [77], it is

[1] This original paper considers finite strains following G. Green, Kirchhoff and Boussinesq, and already uses the notion of mobile frame. It is a much cited paper by Appell [3] and Truesdell and Toupin [74].

not clear why the Cosserats included their memoir in the translation of Chwolson's treatise, a treatise that lacks any general theoretical treatment of mechanics while the Cosserats' memoir deals with the foundations of analytical mechanics. They may have used this just as a good opportunity. Accordingly, the book version is preferably considered without the rest of Chwolson's nonetheless highly valuable treatise. NASA had an English translation of it made in 1968 as a result of a revival of interest in generalized continuum mechanics in the 1960s.

> The review of Lovett [43] is particularly enlightening concerning the note added by the Cosserats to the Koenigs' lecture notes of 1897. Citing Lovett: *The introduction of this note is peculiarly fortunate for it is high time that kinematics should comprehend the study of deformation and of deformable spaces. The authors have included in their extract certain generalities on curvilinear coordinates, the deformation of a continuous medium in general, infinitely small deformation, use of the mobile trieder* [sic], *and the case where the non-deformed medium is referred to any curvilinear coordinates.*

It seems that the two brothers, together with the husband, E-V. Davaux, of François' daughter, also an alumnus of *Ecole Polytechnique* with a specialization in naval engineering, were very active in translations from the Russian, German and English (including a translation of J. W. Gibbs's "Elementary principles in statistical mechanics" published only in 1932). For Chwolson's treatise, translation may have been done from the Russian and/or the already existing German translation. Note that it was usual in the nineteenth century and the early twentieth century to include comments and possible personal additions to a translation from an original book. The best example of this usage is provided by Barré de Saint-Venant's [5] French translation of A. Clebsch's *Theorie der Elastizität fester Körper* in such a way that the bulk of the book tripled in translation, resulting in a book that was more his than Clebsch's. But in the case of Chwolson's treatise, the Cosserats' supplements do not shed any light on Chwolson's original contents of Vols. 1 [10] and 2 [11]; they seem out of place, as rightly noted by Wilson [76, 77] who nonetheless emphasized their intrinsic importance.

As to the many Notes to the *Comptes Rendus* of the Paris Academy of Sciences, it was at the time a traditional way to announce a result in brief form so as to provide a priority mark. Cauchy is well known for the flood of such notes that he sent to the Academy. This was also the case of Henri Poincaré and Pierre Duhem among others. Of course, this cannot replace a lengthy well argumented paper with full derivations as many such notes are extremely cryptographic and thus hard to grasp due to their imposed brevity. With all these caveats we can now turn to the real object of this contribution, the "book of 1909".[2]

[2] Orest Danilovich Chwolson (1852–1924)—also written Khvol'son—was a Professor of physics in St Petersburg. He is the author of a five-volume treatise on physics that was translated into German and French in the early twentieth century. The world renowned theoretical physicist Lev D. Landau had a strongly positive appraisal of this treatise.

8.2 The Main Contents of the Cosserats' Book

According to a recent investigation on *Google Scholar* the Cosserats book is cited about 1,500 times. It is a required citation in the introduction of papers dealing with modern oriented or polar continua. But we can safely assume that very few citers have ever seen the book and, of course, even less have read it, reducing the number of the happy few to small integers. There are good reasons for this. First the language, French, would be a common obstacle. But most of the difficulty comes from the state of mind of the authors and their notation since neither tensor nor direct intrinsic notations are used by these authors. We estimate that the book would be reduced to about 80 pages had a direct notation been used in the modern way. But intrinsic vector notation, not to speak of tensor notation, hardly existed at the time.[3] Also, the bias of the authors to work successively for bodies of one, two and three dimensions, if it may have helped the contemporary readers to grasp the basic ideas of their approach, considerably lengthens the progression. The advantage of working at this rhythm is a possible direct comparison with works by famous engineers of the eighteenth and nineteenth centuries who dealt with elastic rods and surfaces. Indeed, in a very professional manner akin to that entertained by previous authors such as Lagrange or Barré de Saint-Venant, the brothers are very generous in accurately citing previous contributors. As proved in many footnotes, the most cited such authors from a relatively old past are Navier, Poisson, Fresnel, Lamé, Helmholtz, Carnot, F. Reeds and Barré de Saint-Venant. Cauchy, although the universally acknowledged founder of general continuum mechanics, is seldom cited perhaps because he does not use variational principles and therefore is more in the Newtonian tradition of the postulate of balance laws. Gabrio Piola (1794–1850) would have been welcomed in the roster of citations because he uses Lagrangian variational principles and is an aficionado of changes of reference configuration (cf. the Piola transformation). But the Cosserats, like most of the French authors of the period, seem to have ignored him.

The most cited contemporary authors certainly are W. Thomson (alias Lord Kelvin) and P. G. Tait (cf. their "Treatise on Natural Philosophy", [71]), Pierre Duhem (1861–1916; cf. his course on hydrodynamics, elasticity and acoustics, [18]), H. Poincaré (1854–1912), Paul Appell (1855–1930), J. Bertrand (1822–1900), G. Darboux (1842–1917), and sometimes W. Voigt (1850–1919). It is less than anecdotic to note that Darboux, Appell and Koenigs were the three members composing the Jury of the doctoral thesis of Eugène. The deepest influences perceived through the unfolding of the book seem to be those of Lagrange and Hamilton for the variational formulation and the notion of action, Green [30] for the notion of potential energy of deformation, and Darboux [16, 17]

[3] Gibbs' [28] book was the first of its type giving an articulated introduction to vector analysis. This may however be a wrong attribution since the book in fact is E. B. Wilson's redaction with an enriched rendering of Gibbs' lectures in vector analysis at Yale; Wilson was only 22 years old when the book was published (see pp. 228–229 in Crowe [15]).

for the theory of surfaces, curvilinear coordinates and the mobile triad. The employed notion of groups, a *première* in continuum mechanics, is not connected with any obvious citation, although we surmise that the views of S. Lie and H. Poincaré may have been influential concerning this very point. Furthermore, the Cosserats are aware of general discussions on the nature and interpretation of the principles of mechanics (works on this subject by Hertz, Poincaré, Mach and Duhem in the period 1890–1909) as shown in many of their footnotes.

In our opinion the best analysis of the book remains the original review written by Wilson[4] [77] from M.I.T, a luminous text that we shall often paraphrase. Wilson had the right state of mind to capture the essential arguments of the Cosserats. First he considers the book as a contribution to the *analytical* mechanics of continua, and this is spot on. In effect, the very object of the book is the deduction of what we now call "field equations" of continua of one, two or three dimensions, from a Lagrangian-Hamiltonian principle of the general form

$$\delta \int_T \int_V W dV dt = 0, \tag{8.1}$$

where T is a time interval, V is a bounded volume element (a filament, a surface or a volume) in the considered physical space, and W is a known function of well-chosen arguments. In standard variational mechanics W is made explicit in terms of an identified kinetic energy and a potential energy so that W is the Lagrangian "volume" (i.e. lineal, surface or true 3D volume) density where the notion of mass (here density) is a basic one. The Cosserats wanted to remain in a sufficiently general framework that may possibly include various types of dynamics (even the special relativistic one with an appropriate definition of the mass).

The importance of the notion of *action* present in (8.1) was emphasized by William R. Hamilton (1805–1865) and Hermann von Helmholtz (1821–1894). But essential to the Cosserats' presentation is their initial remark that the *action* (energy multiplied by time) as introduced by P. L. Moreau de Maupertuis (1698–1759) is invariant under the group of Euclidean displacements. This requirement systematically applied to (8.1) provides the notion of *Euclidean action* in the Cosserats' formalism. From this should be deduced the basic local balance laws of linear momentum, angular momentum and energy, corresponding to the seven parameters (spatial translation and rotation, time translation) of the Euclidean group in E^3 (completed in a ten-parameter group if we include the definition of the centre of mass). What the Cosserats do is to implement this approach in a well tempered manner with the successive examination of one-dimensional bodies (straight line or curved filament), two-dimensional bodies (deformable surfaces such as plates and thin shells (not their vocabulary)) and three-dimensional bodies, with the possible extension to true dynamics (i.e., accounting for inertial effects). Whether this is a good pedagogical way is a

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4 For this see, e.g., Kelvin, reported in Thomson and Tait, Second edition (1879).

disputed matter in modern continuum mechanics where the equations governing slender bodies are rather deduced from the three-dimensional ones by means of some asymptotic procedure associated with the relative smallness of some dimensions.

As remarked by Wilson [77, p. 242], the Cosserats' book may have proposed "the most general and unifying theory of mechanics" so far (as on 1909). Probably under the influence of Darboux, the Cosserats considered that the "fundamental geometric element in their system is not the point, but the point carrying a system of rectangular axes, that is, the tri-rectangular triedral angle". This is obvious in the case of "an elastic filament that differs from a geometric curve in the way in which a continuous series of rectangular triedral angles differs from the locus of the vertices of the angles". In this case the function W should be "a function of the coordinates of the vertex but also a function of the nine direction cosines of the edges of the angle, and of the first derivatives of these coordinates and direction cosines with respect to time" (in the dynamical case) or the arc length in the case of the elastic filament. All these are Wilson's words.

This function W that is invariant under transformations that belong to the Euclidean group, is said to be an *Euclidean action* density and multiplied by the increment of time dt is the Euclidean action in the time interval dt. For the filament the reasoning can replace dt by an element ds_0 of arc. In this vision the case of straight rods and curves is approached by considering a mobile triad of vectors of which one element is tangent to the line or curve. In the case of two-dimensional bodies the mobile triad has one vector in the plane tangent to the mean surface of the object. In the three-dimensional case the triad has no preferred direction to start with except by convention in a reference configuration. Then the passing from the *motion* of an elastic medium of dimension k to the *equilibrium* of an elastic medium of dimension $k + 1$ is, or should be, "well known to all student of mechanics" [77, footnote in p. 243]: "It is this analogy which enables the authors (the Cosserats) to give a uniform treatment to dynamic and static problems of different nature". That is quite remarkable and seldom considered by most of us as Wilson is rather optimistic concerning this point.

We shall not dwell in detail with the Cosserats' treatment which is somewhat repetitive and not very attractive in modern terms.[5] What is also absolutely important is that this enforcement of the Euclidean group structure leads the Cosserats to consider on an equal footing invariance under spatial translations *and* spatial rotations. That is how they are led to considering *nonsymmetric stress tensors* and the presence of body couples and of a new internal force called *coupled stress tensor* in modern jargon. If some notions may have been readily interpreted for the one- and two-dimensional cases in terms of what was known in the strength of structural elements in the nineteenth century, the three-dimensional case comes up as a new notion, although it is remarked that Kelvin and Voigt may

[5] For this unpleasant aspect to modern eyes, see, for instance, the fantastic and frightening aspect of the individual-component equations in pp. 157–172 of the book of 1909.

have hinted at the presence of body couples. For instance, the formidable equilibrium equations printed in page 137 of the Cosserats' book in the 3D case are now written with an inherent economy of symbols as the equations of equilibrium for stresses $\underline{\sigma}$ and couple stresses $\underline{\mu}$ in the form

$$\nabla \cdot \underline{\sigma} + \mathbf{F} = \mathbf{0}, \tag{8.2}$$

and

$$\nabla \cdot \underline{\mu} + \underline{\sigma}_A + \mathbf{C} = \mathbf{0}, \tag{8.3}$$

where $\underline{\sigma}$ is the nonsymmetric stress tensor, $\underline{\sigma}_A$ is its antisymmetric (or skew) part, $\underline{\mu}$ is the third-order couple stress tensor, and $\mathbf{F}$ and $\mathbf{C}$ are volume densities of externally applied force and couple (the latter in tensor skew symmetric form), respectively. The Cosserats' note [13, p. 137] that Eq. (8.3) with $\underline{\mu} = \mathbf{0}$ was evoked by W. Voigt in a work of 1887 that dealt with the elasticity of crystals involving polarized molecules [75]. This may have prompted Ericksen [22] to envisage a modelling of anisotropic fluids and liquid crystals by means of a field of so-called "director" (one unit vector attached to each material point), clearly a special case of Cosserat continuum. The most obvious case of Eq. (8.3) with $\underline{\mu} = \mathbf{0}$ is that obtained in anisotropic electromagnetic continua as amply documented in our book [48]. The Cosserats are aware of Lord Kelvin's former attempts[6] and the contemporary one of Larmor [39] to build a model of elasticity able to transmit transverse (light) waves but they do not seem to know the work of MacCullagh [45] on the same matter. Passing to the dynamic version of Eqs. (8.2) and (8.3), i.e.,

$$\nabla \cdot \underline{\sigma} + \mathbf{F} = \rho \dot{\mathbf{v}} \tag{8.4}$$

and

$$\nabla \cdot \underline{\mu} + \underline{\sigma}_A + \mathbf{C} = \rho \dot{\mathbf{S}}, \tag{8.5}$$

where $\mathbf{v}$ is the matter velocity and $\mathbf{S}$ is an internal spin (angular momentum), and a superimposed dot denotes the time derivative, is not as trivial a matter as thought by Wilson.[7] As a matter of fact, one had to await a work by Eringen [25], to understand that in parallel with the conservation of mass density, the good construction of (8.5) requires the consideration of a law of conservation of rotational inertia (per unit mass).

From the above-given short analysis we can encapsulate the Cosserats' main contribution in their lengthy memoir of 1909 in two main ingredients. One of these is the deduction of field equations such as (8.2) and (8.3)—which were to yield the fruitful notion of *Cosserat continuum* in the 1950s–1970s. The second ingredient,

[6] The reader may consult Maugin [52] for a historical perspective.

[7] To apply his argument we would have needed to know a four-dimensional static case, whatever that may be.

perhaps more important from the general viewpoint of mathematical physics, is the exemplary use of variational principles and a simultaneous application of a *group theoretical argument*, and this before the proof of her famous theorem by Noether [59]. As emphasized by Wilson in his deeply thought review [77, p. 246], an advantage of the Cosserats' approach is the association it provides "with the transfer of any deductive-intuitional physical science to the corresponding formal-deductive mathematical discipline". This is all the best for mathematically inclined mechanicians of the continuum. Correlatively, it yields a loss in the physical intuition while the latter is also a creative asset: mathematical rigorous form and physical innovation may be antagonistic. From the point of view of the kinematics and deformation theory of the continuum, the Cosserats have learnt their lesson in the finite-strain theory from Green, Kirchhoff, Boussinesq and Duhem. Unfortunately, apart from very general formulas for the W function, they have not provided any more information on possible constitutive equations. Apparently they were not so much interested in problem solutions although their original memoir of 1896 and some of their Notes to the *Comptes Rendus* hinted at progress in the solution of elasticity problems, in two-dimensions in particular. In the memoir of 1909 only the one and two-dimensional models are close to engineering concepts as they allow for a representation of the twisting of rods and shells in addition to their bending as noted by Ericksen and Truesdell [24, p. 297]. Bearing in mind these different characteristic properties, it is salient to examine the contemporary reception of their work and what was more useful in it for further developments, much later in the 1950s–1970s.

8.3 Reception and Influence of the Cosserats' Book

Parodying the title of a famous work concerning Leonardo da Vinci by Duhem [20], we could ask "who did the Cosserats read and who read them?" From above made remarks we can safely state that Maupertuis, Lagrange, Hamilton and Kirchhoff must have been primary sources for the bases of the Cosserats' thesis. Much closer to them their contemporaries such as L. Kronecker, G. Koenigs. P. Duhem, H. Poincaré, L. Lecornu (Professor of Mechanics at *Polytechnique*), and G. Darboux have played an essential role in the formation of the authors' background. The same can be said concerning the teachers whom both brothers had in analysis and geometry either at *Polytechnique* or at the *Sorbonne*. Foremost among them is the influence of Darboux with the idea of the mobile triad of vectors. In the case of Duhem, Truesdell had repeatedly pointed out that the idea to attach a triad of rigid vectors (so-called "directors") at each material point in order to describe the orientational changes in some kind of internal rotation goes back to Duhem [19]. But no trace in the Cosserats' opus seems to directly indicate such a borrowing.

The immediate (say in the pre-WWI and early post-WWI period) reception of the Cosserats'works is obvious among mathematically oriented scientists. Of course, Appell who welcomed an addition by the Cosserats in his own treatise of

1909 on rational mechanics easily sided with the Cosserats. Wilson,[8] as a student of Gibbs and a true mathematician, manifests a true enthusiasm for their work as proved by his most favourable review. Cartan [8], the French geometer of Lie-group fame and author of creative developments in modern differential geometry, immediately appreciated the consideration of group arguments in the Cosserats' vision while noting the rich possibility to include the action of distributed couples along with more classical contact forces. This was also true of Ernest Vessiot, another specialist of group theory, who succeeded François Cosserat as president of the French Society of mathematics. On a less "provincial", albeit Parisian level, Heun [34] in his article in the German Encyclopaedia of Mathematics presented a kind of compaction of the Cosserats' arguments for the mechanics of rods. As to Hellinger [33], in a remarkably concise but well informed article to the same encyclopaedia, he correctly captured the new trends in continuum mechanics by accurately citing the most recent works by Boltzmann, Duhem and the Cosserats. But this was published in a tragic period not so favourable to scientific communication. The corresponding volume of this encyclopaedia was never translated into French while all other preceding volumes had been.

One must await a work [68] by Joachim Sudria (1875–1950) to witness an approach truly in the Cosserats' tradition with an unambiguous reference to the notion of Euclidean action. This work was published in Toulouse in a journal in which the Cosserats had published in 1896 and of which Eugène Cosserat was the long time editor (in fact "Secretary") until 1930. It is in this journal that Buhl [7] published an eulogy of Eugène pointing out his role and the influence of Eugene's initial works in geometry in the writing of the papers in common with his older brother François.

Sudria [69] published an up dated version of his memoir as a short monograph. Truesdell told (cf. [4] that it is while perusing works of the 1930s in continuum mechanics that he unburied Sudria's memoir of 1935. Then,—following the (probably unknown to him) advice of Rabbi Rashi of Troyes in Burgundy: "Ask your master his sources" (my citation, GAM)—Truesdell went back in time to uncover the Cosserats' book of 1909. In his usual somewhat grandiloquent style, Truesdell [73] states that "the Cosserats' masterpiece stands as a tower in the field". But he also mentions that "it attracted little attention in its own day and was soon forgotten". This remark may be due to Truesdell's ignorance of citations by French physicists, mathematicians and engineers in the 1920s–1940s [e.g., L.-M.

[8] Edwin Bidwell Wilson (1879–1964) was an American mathematician-physicist who had been a PhD student of J. W. Gibbs at Yale, and became Professor of mathematics first at M.I.T (when he wrote the review of the Cosserats) and then at Harvard. He co-authored a book on vector analysis with Gibbs (first edition, 1901, then several further editions). He was interested in the general principles of physics and mechanics (e.g., relativity), in advanced calculus, and in the differential geometry of surfaces in hyperspaces. Later in his life he contributed much to the developing studies in mathematical economy mentoring Paul A. Samuelson in Keynesian macro-economy. He was well equipped, both intellectually and technically, to apprehend the quintessence of the Cosserats' works.

Roy, J. Delsarte, J. Pacotte, G. Matisse, P. Sergescu, E. Jouguet, and above all R. L'Hermite who places the Cosserats in the top group together with Lamé, Clebsch, Saint-Venant and Duhem while noting the complexity of the Cosserats' development and the lack of possible direct applications save in the one-dimensional case; cf. [6] (Reprint of the Cosserats' book), p. xxxix]. This takes us directly to the second half of the twentieth century with a frantic rebirth of studies on generalized continuum mechanics.[9]

Following the early considerations by Voigt, French crystallographers showed some interest for the case of nonsymmetric stress tensors in the mid 1950s (cf. [40, 41]). But the first manifestations of the use of "directors", the set of unit vectors attached to each material point in the line of Duhem and the Cosserats, are in works by Ericksen and co-workers [23, 24] dealing with structures of one or two spatial dimensions[10] with explicit reference to the Cosserats' book. This would later on be taken over in works by Green and Naghdi [29]. Then a busy period developed in the 1960s–1970s with the introduction of various models of generalized continua, all more or less first basing on a kind of microscopic description. Among these models, some were identified with the so-called Cosserat continua, as essentially governed by Eqs. (8.4) and (8.5), but also christened with other names such as "oriented media" or "micropolar continua" [50]. It has become traditional to refer to Aero and Kuvshinskii [1], Palmov [62] and German authors such as Günther [32], Neuber [58], and Schaefer [65] as pioneers in the field. It became a moral, more than technical, obligation to refer to the Cosserats' book as demonstrated in practically all contributions to the proceedings [38] of a landmark international symposium held in 1967 in Freudenstadt (Black Forest, Germany). These proceedings were rightly dedicated to the Cosserats and Elie Cartan. Of course this feverish citation business was more paying lip service than anything else since most authors had never read—nor even seen—the Cosserats' book. Grioli [31] appears as an exception in not referring to any Cosserats' work. But we do not know if this was by pure honesty or mere ignorance that this author acted.

Explicit reference to—and exploitation of—Euclidean action is much more rare in continuum mechanics. Here we underline the work of Toupin [72] on oriented (Cosserat) continua and our own work [46] on the more general case of so-called *micromorphic* elastic bodies. This variety of continua was introduced in a landmark paper by Eringen and Suhubi [26]. It is equivalent to a Duhem kind of kinematic description with three deformable "directors" and relative-angle changes between these directors in the course of deformation: the microstructure itself is deformable and is in fact subjected to a homogeneous micro-deformation (represented by six additional internal degrees of freedom). The case of rigid micro-rotation and no micro-deformation then corresponds to the Cosserat continuum. A modelling somewhat equivalent to the Eringen-Suhubi one was proposed by Mindlin [56]. In the case of Cosserat continua there appears the problem of the most convenient

[9] See Maugin [52] for a historical perspective.

[10] Cf. Ericksen and Rivlin [23], Ericksen and Truesdell [24].

mathematical representation of the micro-rotation. This is best solved by considering orthogonal transformations and their own representation by an angle and the unit direction of an axis of rotation in the manner of Gibbs [28]—and "our" Wilson—as shown by Kafadar and Eringen [37]. This was duly exploited by Kafadar [36] in an original approach to the classical problem of the "elastica"—and thus back to the spirit of Cosserats' treatment of one-dimensional elastic curves.

The writer was for the first time exposed to a research course involving "directors" in the lectures delivered by a Serbian scientist, Rastko Stojanovic, in Udine (Italy) in July 1970 (cf. Stojanovic lecture notes at the C.I.S.M. referred to as [66]. This was directed at the continuum representation of defective bodies. This gave him the idea to draw an analogy with deformable continua endowed with a continuous distribution of magnetic spins such as in the micromagnetic theory of ferromagnetism, a fashionable subject matter at the time. Then he applied the Euclidean action method of the Cosserats to deduce all relevant coupled field equations, including an equation formally identical to Eq. (8.5) but with all terms bearing a magnetic interpretation (cf. [47], Chap. III; also [53]). This was recently revisited in C.I.S.M. lecture notes [51]. Cherry on top of the cake, a four-dimensional relativistic theory of oriented media was constructed by complementing the triad of spatial "directors" in the Duhem-Cosserat style by the unit-normalized world velocity into a true four-tuple with a view to incorporate spin effects in relativistic continuum mechanics[11] with local Lorentz invariance replacing the Cosserats' invariance requirement [54].

8.4 Concluding Remarks

In recent times most of the Cosserats' work involving a nonsymmetric stress and couple stresses have been formalized in a modern context often under the title of *asymmetric elasticity* (cf. [60]) or *polar* or *microplar media* of the elastic type (cf. [27], Teodorescu [70]) or of the fluid type (cf. [44, 67]) with mathematical results of the same degree of refinement as those dealing with classical continua.[12] The theory of such polar elastic materials has been fully incorporated in the modern framework of configurational forces [49] with the help of Noether' theorem. The formulation of the deformation nonlinear theory of finite-strain Cosserat elasticity has been much clarified by Pietraszkiewicz and Eremeyev [63]—also Eremeyev and Pietraszkiewicz [21]—, and the most recent lecture notes highlight all fundamental geometrical properties and most interesting applications of Cosserat continua (cf [2]). A rapid search on *Google* provides instantaneously more than

[11] For this 4D generalization see Maugin [47, Chap. VI] and Maugin and Eringen [54].

[12] We use this opportunity to mention the seldom cited book of Jaunzemis [35] where elements of generalized continua are nicely introduced. Jaunzemis' career was interrupted by his untimely death at the age of 48 in 1973.

two hundred thousand entries about the Cosserats, although the most recent references concern a local politician from Amiens, the native city (in Picardie, North West of France) of the Cosserat brothers, thus undoubtedly a family connection. But the Amiens textile company specialized in the production of velvet, created in 1794, and of which the parents of the celebrated brothers were the owners, finally closed down in 2012. As to the heritage of the notion of "Euclidean action", it is more diffuse as the Cosserat notion appears somewhat obsolete in a period of full enforcement of Noether's invariance theorem. Concerning this point, we can cite Levy [42] as a general appraisal of this aspect of the Cosserats' works:

> *Cosserat's theoretical research, designed to include everything in theoretical physics that is directly subject to the laws of mechanics, was founded on the notion of Euclidean action [least action] combined with Lagrange's ideas on the principle of extremality and Lie's ideas on invariance in regard to displacement groups. The bearing of this original and coherent conception was diminished in importance because at the time it was proposed, fundamental ideas were already being called into question by both the theory of relativity and progress in physical theory.*

But, nowadays, "Euclidean action" experiences a flourishing vitality in its acception granted in theoretical physics such as in the functional integrals of quantum physics (cf. [57]).

This concludes the present investigation of the subject. But we note that Pommaret [64], a disciple of Vessiot in group theory, and in a sense a "grand-son" of François Cosserat—with whom he shares the same elite education—has expanded never tired efforts to publicize the works of the Cosserats and their relevance to modern group theory in mathematical physics.

PS. Additional biographic information on the Cosserats and their works can be found in Levy [42] , O'Connor and Robertson [61], and Brocato and Chatzkis ([6]; preceding the reprint of the Cosserats' book). Remarks on the Cosserats' work and Duhem's influence can also be found in Casey and Crochet [9].

Appendix A

Partial English Translation of E. and F. Cosserat, "Théorie des Corps Déformables", Hermann, Paris, 1909, by Gérard A. Maugin (Only the First Chapter on General considerations is translated; original footnotes are reported to the end and numbered consecutively. Translator's remarks are placed within square brackets in the main text. This is a verbatim translation without any ambition of literary prowess (Figs. 8.1, 8.2, 8.3).

THÉORIE

DES

CORPS DÉFORMABLES

par MM. E. et F. COSSERAT

I. — CONSIDÉRATIONS GÉNÉRALES

1. Développement de l'idée de milieu continu. — La notion de corps déformable a joué, au siècle dernier, un rôle important dans le développement de la Physique théorique, et FRESNEL (¹) doit être regardé, à l'égal de NAVIER, de POISSON et de CAUCHY (²), comme l'un des précurseurs de la théorie actuelle de l'élasticité. Sous l'influence des idées newtoniennes, on ne considérait encore au temps de ces savants que des systèmes discrets de points. Avec les mémorables recherches de G. GREEN (³), ont apparu les systèmes ponctuels continus. On a essayé depuis d'élargir la conception de GREEN, qui est insuffisante pour donner à la doctrine des ondes lumineuses toute sa portée. LORD KELVIN (⁴), en particulier, s'est attaché à définir des milieux continus en chaque point desquels peut s'exercer un moment. La même tendance s'accuse chez HELMHOLTZ (⁵), dont la controverse avec J. BERTRAND (⁶), à l'égard de la théorie du magnétisme, est très caractéris-

(¹) FRESNEL. — *Œuvres complètes*, Paris, 1866 ; voir l'introduction de É. VERDET.

(²) Voir ISAAC TODHUNTER et KARL PEARSON. — *A History of the Theory of Elasticity and of the Strength of Materials, from* GALILEI *to the present time*, Vol. I, GALILEI to SAINT-VENANT, 1886 ; Vol. II, Part I et II, SAINT-VENANT to LORD KELVIN, 1893. Cet ouvrage remarquable contient une analyse très complète et très précise des travaux des fondateurs de la théorie de l'élasticité.

(³) G. GREEN. — *Math. Papers*, édited by N. M. FERRERS, fac-similé reprint, Paris, A. Hermann, 1903.

(⁴) LORD KELVIN. — *Math. and phys. Papers*, volume I, 1882 ; vol. II, 1884 ; vol. III, 1890 ; *Reprint of Papers on Electrostatics and Magnetism*, 2ᵉ éd. 1884 ; *Baltimore Lectures on Molecular Dynamics and the Wave Theory of Light*, 1904 ; W. THOMSON et P. G. TAIT, *Treatise on Natural Philosophy*, 1ᵉ éd. Oxford 1867 ; 2ᵉ éd. Cambridge 1879-1883.

(⁵) HELMHOLTZ. — *Vorles. über die Dynamik diskreter Massenpunkte*, Berlin 1897 ; *Vorles. über die elektromagnetische Theorie des Lichtes*, Leipzig 1897 ; *Wiss. Abhandl.*

Fig. 8.1 First page of the Cosserats' book of 1909 (Hermann, Paris, 1909) (Note the reference to Green, Kelvin and the Treatise on Natural Philosophy)

Fig. 8.2 François Cosserat (1852–1914) in his uniform of the *Ecole Polytechnique* around 1871 (*Source* http://www-history.mcs.st-andrews.ac.uk/Cosserat-Francois.html)

Fig. 8.3 Sketchy portrait of Eugène Cosserat (1866–1931) in his fifties (*Source* http://www-history.mcs.st-andrews.ac.uk/PictDisplay.html)

Theory of Deformable Bodies

by MM. E. and F. Cosserat

Preface

This volume contains the development of a note on the Theory of the Euclidean action that Appell has thought appropriate to introduce in the second edition [1909] of his *Treatise on Rational Mechanics.* The reproduction of an appendix to the French edition of the *Treatise of Physics* of Chwolson, explains several peculiarities of the editing and the reference that we make to a previous work on the dynamics of the point and of a rigid body, which is here also combined with the work of the Russian scientist. We took advantage of this new print to correct several mistakes in our text.

Presently, we do not seek to deduce all the consequences of the general results that we will obtain; throughout, we make the effort only to rediscover and clarify the classical theories. In order for this kind of checking of the theory of the Euclidean action to appear more complete, in each part of our exposition we will have to establish the form that the equations of deformable bodies take when one is limited to the consideration of infinitely close states; however, this is a point that we have already addressed, with all necessary details, in our first memoir on the *Theory of elasticity* that we wrote in 1896 (*Annales de la Faculté des Sciences de Toulouse*, Vol. X). Moreover, we suppose that the magisterial lessons of G. Darboux on the *general theory of surfaces* are completely familiar to the reader.

Our researches will make sense only when we have shown how one may envision the theories of heat and electricity by following the already followed path. We devoted two notes to this subject in Volumes III and IV of Chwolson's treatise. The *subdivision*, to use a pragmatic language, appears to be a scientific necessity; nevertheless, one must not loose sight of the fact that it answers deep questions. We have tried to provide an idea of these difficulties in our note on the *Theory of slender bodies* published in 1908 in the *Comptes Rendus* of the *Académie des Sciences* and whose contents were also mentioned by Appell in his treatise.

E. & F. COSSERAT

I.- General considerations

1. **Development of the idea of a continuous medium –** The notion of deformable body has played an important role in the development of theoretical physics during the last century [i.e., 19th century], and Fresnel[1] must be considered as one of the precursors of the present theory of elasticity, on an equal stand with Navier, Poisson and Cauchy[2]. Under the influence of Newtonian ideas, only discrete systems of points were still considered at the time of these scientists. Continuous punctual systems appeared with the memorable researches of G. Green[3]. Since then, one has tried to enlarge the conception of Green, which is not sufficient to provide

its full power to the theory of luminous waves. Lord Kelvin[4], in particular, worked hard to define continuous media at each point of which a moment can be exerted. The same trend is emphasized with Helmholtz[5], of whom the controversy with J. Bertand[6] concerning the theory of magnetism is very characteristic. We can go back to the origin of this evolution, on the one hand, with conceptions introduced in the strength of materials by Bernoulli and Euler[7], and on the other hand, to the theory of "couples" due to Poinsot[8]. Thus we are naturally led to gather, under the same geometrical definition, various concepts of deformable bodies that we meet nowadays in natural philosophy [i.e., physics]. A deformable line is a continuous set equipped with one parameter of trihedrons, a deformable surface with a set of two parameters, and a deformable [3D] medium with three parameters ρ_i. In the presence of motion, one must add the time t to these three geometric parametersρ_i. The mathematical continuity that we assume in such a definition, leaves untouched at each point the trace of an invariable [i.e., rigid] solid; therefore, we can foresee that from a mechanical viewpoint moments will appear that are well known and are studied, since Euler and Bernoulli, along elastic lines and on surfaces, and that Lord Kelvin and Helmholtz have tried to embed in a three-dimensional space.

2. **Difficulties presented by the application of the inductive method in mechanics**.

The primary form of mechanics is inductive; this is what one clearly perceives in the theory of deformable bodies. This theory has first borrowed from the mechanics of invariable [rigid] bodies the propositions relative to the notion of static force, that were applied with the principle of solidification ["rigidification" due to Cauchy]; then the relation between the effort and the deformation was hypothetically first established (generalized Hooke's law), and then only one looked a posteriori under what conditions this was conserved (Green). Carnot[9] already mentioned, one century ago, the defect of this method, where it is constantly called for a priori notions, and where the followed path is not always safe. The static force in fact does not have the effect of a constructive definition, in our classical form of mechanics, and the influence of the reform that Reech[10] proposed regarding that matter in 1852 remained practically unknown until our present time. Perhaps that this is due to the long uncertainty in which elasticians remained concerning the rational foundation that can be attributed to Hooke's law. Analogous hesitations have indeed been manifested, almost in the same form, in other domains of physics[11].

In order to escape from these difficulties, Helmholtz tried to construct what is called an *energetics*, that relies on the principle of least action and on the very idea of energy, the force, whatever its nature, becoming then a secondary notion of deductive origin. But the principle of a minimum in natural phenomena[12] and the concept of energy[13] itself bring us to confront the defects of the inductive method. Why a minimum and what definition to be granted to energy to avoid having simply a physical theory, but a truly mechanical theory? Helmholtz does not seem to have left an answer to these questions. However, he contributed to establish more completely than done before the distinction between the two notions, energy and action, that apparently are identified in classical dynamics. We believe that one must start from the latter [the action] to make perfectly precise the views of Helmholtz and to give to mechanics, or more generally, theoretical physics, a perfectly deductive form.

3. **Theory of Euclidean action**

When we are concerned with the motion of a point, the essential element that enters the definition of action is the Euclidean distance between two infinitesimally close positions of the mobile point. We have shown previously[14] that one can deduce from this single notion all fundamental definitions of classical dynamics, those of quantity of motion [linear momentum], of force and of energy.

Here we propose to establish that we can follow an identical path in the study of the static deformation or the dynamics of discrete systems of points and continuous bodies, and that we arrive thus to the construction of a general theory of action in both extension and motion, that embraces all that, in theoretical physics, is directly governed by the laws of mechanics.

Here also, the action will be the integral of a function of two infinitesimally close elements in time and in the space of the considered medium. Introducing the condition of invariance under the group of Euclidean displacements and defining the medium, as indicated in the first paragraph above, *the density of action at a point will have the same remarkable form that the one already met in the dynamics of the point and of invariable bodies*. Let, with the notations of the *Leçons* of M. Darboux, (ξ_i, η_i, ξ_i), (p_i, q_i, r_i) be the geometric velocities of translation and rotation of the elementary trihedron, and (ξ, η, ξ), $(p, q, r)$the corresponding velocities relative to the motion of this trihedron; The action will be the integral

$$\int_{t_1}^{t_2} \int \cdots \int W(\rho_i, t; \xi_i, \eta_i, \xi_i, p_i, q_i, r_i; \xi, \eta, \xi, p, q, r) d\rho_1, \ldots, d\rho_i \ldots dt.$$

It will suffice to consider the variation of this action to be led to the definition of the quantity of motion, those of efforts and moment of deformation, of the external force and moment, and finally those of the energy of deformation and of motion, via the intermediary of the notion of work.

In this theory, statics will become entirely autonomous, in agreement with the views of Carnot and Reech; we will simply have to take for this purpose a density of action W independent of the velocities (ξ, η, ξ), (p, q, r), that is, to consider a body devoid of inertia, or else a body with inertia on the condition to regard deformation as a *reversible transformation* in the sense of M. Duhem. On the other hand, having recourse to the notion of *hidden arguments* [i.e., arguments that do not appear explicitly in W], we will recover all the concepts of mechanical origin that are employed in physics, for instance, those of flexible and inextensible lines [strings], flexible and inextensible surfaces, invariable [i.e., rigid] bodies, as also less particular definitions as proposed for a deformable line since D. Bernoulli and

Euler till Thomson and Tait, for the deformable surface since Sophie Germain and Lagrange till Lord Rayleigh, and for the deformable media since Navier and Green till Lord Kelvin and W. Voigt.

Envisaging both deformation and motion, we shall arrive in a purely deductive manner at the idea that is contained in the principle of d'Alembert, which relates only to the case where *an action of deformation separates fully from the kinetic action*. Finally, if we suppose that the deformable body is not submitted to any action from the external world and if we introduce, as a consequence, the fundamental notion of *isolated system*, of which M. Duhem[15], and the M. Le Roy[16] have shown the necessity for a rational construction of theoretical physics, we shall naturally be led to the idea of a minimum that Helmholtz had already considered as a starting point, while simultaneously there will appear the principle of conservation of energy, which is at the basis of our present scientific system.

4. **Critique of the principles of mechanics.-** As we just sketched it, the theory of Euclidean action brings a first contribution to the critique of the principles of mechanics.

Its generality allows one to foresee that there are singular phenomena, both in the action on the motion and in the deformation of extension, for example the aspect of solids in a plastic state or near fracture, and that of fluids submitted to large forces. In ordinary circumstances, this generality can be reduced by the consideration of a state that is infinitesimally close to the natural state; this is a point that we already mentioned in our preceding note.

But we can still assume that one or two dimensions of the deformable body become infinitesimally small and then envisage what is called a *slender body*[18]. This notion was developed in 1828 by Poisson, also a short time afterwards, by Cauchy; their aim, like that of all elasticians preoccupied later by this arduous question, was to build a passage between the distinct theories of bodies with one, two and three dimensions. We know that an important part of the works of Barré de Saint-Venant and Kirchhoff is related to a discussion of the researches by Poisson and Cauchy. However, these scientists, and then their followers, did not exhibit the true difficulty of the matter; this difficulty resides, *generally, in the fact that the zero value of the introduced parameter is not an ordinary point, as admitted by Poisson and Cauchy, not even a pole, but an essential singular point*. This important fact justifies the separate studies of lines, surfaces and [3D] media that the reader will find in the present work[19].

[Here the Cosserats touch upon a fundamental problem that concerns *singular perturbations*; this matter will be solved only in the 1960s-1980s for the limit reduction to slender bodies with the correct asymptotic methods (in particular "asymptotic integration" and the "zoom technique") developed by Gol'denveizer in Russia, Ambartsumian in Armenia, Berdichevsky in Moscow, and Ph. Ciarlet and Destuynder in France].

In concluding these preliminary observations, we shall remark that the theory of Euclidean action relies on the notion of *differential invariant* taken in its simplest form. If we enlarge this notion in such a way as to understand the idea of *differential parameter*, modern theoretical physics appears as an immediate extension, from the *Eulerian viewpoint*, of mechanics per se, and we are naturally led to the principles of the theory of heat and to the actual electric doctrines. This new field of research, in which here we start to enter by deducing from the consideration of deformable bodies the idea of radiation of energy, will be explored more completely in a further work [This ambitious programme was never really formulated; F. Cosserat died in 1914]. We shall thus introduce a new precision in the views of H. Lorentz[20] and H. Poincaré[21] in what is called the *principle of reaction* in mechanics.

(Original) Notes

1. Fresnel, *Oeuvres complètes*, Paris, 1866; see the introduction by E. Verdet.
2. See Isaak Todhunter and Karl Pearson, *A history of the theory of elasticity and the strength of materials, from Galilei to the present time*, Vol. I, Galilei to Saint-Venant, 1886; Vol. II, Part I and II, Saint-Venant to Lord Kelvin, 1893. This remarkable work contains a very complete and precise analysis of the works by the founders of the theory of elasticity.
3. G. Green, *Math. Papers*, edited by N.M. Ferrers, facsimile reprint, Paris, A. Hermann, 1903.
4. Lord Kelvin, *Math. and Phys. Papers*, Vol. I, 1882; Vol. II, 1884, Vol. III, 1890; *Reprint of papers in electrostatics and Magnetism*, 2nd edition; *Baltimore Lectures on Molecular Dynamics and the Wave Theory of Light*, 1904; W. Thomson and P.G. Tait, *Treatise on Natural Philosophy*, First edition, Oxford 1867; 2nd edition, Cambridge, 1879-1883.
5. Helmholtz, *Vorles. über die Dynamik diskreter Massenpunkte*, Berlin , 1897; *Vorles. über die elektromagnetische Theorie des Lichtes*, Leipzig, 1897; *Wiss. Abhandl.* Three volumes, Leipzig, 1892-1895.
6. J. Bertrand, *C.R.* 73, p.865; 75, p. 860; 77, p. 1049; See also H. Poincaré, *Electricité et optique, II, Les théories de Helmholtz et les expériences de Hertz*, 1891, p. 51 ; 2nd edition, 1901, p.275.
7. See Todhunter and Pearson – op. cit.
8. Auguste Comte, *Cours de philosophie positive*, 6th edition, Paris, 1907; Vol. I, p. 338: « Whatever, in truth, the fundamental qualities of the conception of M. Poinsot, with respect to statics, one must nonetheless recognize, it seems to me, that it is above all for perfecting dynamics that it is essentially destined; and I can assure you, considering this point, that this conception has not exerted its most important influence so far".
9. Carnot, in his essay of 1783 on "Machines in general" that became in 1793, *Les Principes fondamentaux de l'équilibre et du mouvement* , has searched to reduce mechanics to principles and precise definitions devoid of any metaphysical character and of any vague terms about which philosophers quarrel without reaching any understanding. This reaction led Carnot a little too far, since he went to the point of contesting the legitimacy of the

expression of force, for him an obscure notion, and to which he wanted to substitute exclusively the idea of motion. For the same reason, he could not admit as rigorous any of the known derivations of the rule of the parallelogram of forces, "the very existence of the word force, in its expression, rendering this derivation impossible, by the nature of things itself" (Ch. Combes, Ed. Phillips and Ed. Collignon, *Exposé de la situation de la mécanique appliquée*, Paris, 1867).

10. F. Reech, Cours de mécanique, d'après la nature généralement flexible et élastique des corps, Paris, 1852. This work was written by the illustrious Marine engineer in view of the reform of the teaching of mechanics at Ecole Polytechnique. Since then, his ideas were exposed by J. Andrade, *Leçons de mécanique physique*, Paris, 1898, and by Marbec, Chief engineer in the Navy, in his elementary teaching of mechanics at the school of "Maistrance" [forming non-commissioned officers as mechanical specialists in the French Navy; GAM] in Toulon (1906). See also J. Perrin, *Traité de Chimie physique, les principes*, Paris, 1903.
11. The remark by Lord Kelvin, in his Baltimore Lectures, p. 131, on the work of Blanchet, is particularly of interest in this regard; he mentions that Poisson, Coriolis and Sturm (*C.R.* **7**, p. 1143), as well as Cauchy, Liouville and Duhamel (1841) have accepted without objection the 36 coefficients that Blanchet had admitted in the generalized Hooke law. Lord Kelvin also has opposed from the same viewpoint the law of at-a-distance force of Weber, in the first edition of the *Natural Philosophy*. More recently, the application of the static adiabatic law to the study of waves of finite amplitude has been criticized for the same reason by Lord Rayleigh, and we know that Hugoniot has proposed a dynamic adiabatic law.
12. Maupertuis himself felt the danger of the principle that he introduced when he wrote in 1744: "We do not know enough what is the purpose of nature, and we can misinterpret the quantity of motion that we must regard as its expense, in the production of its effects"; Lagrange first has intended to make of the principle of least action the basis of his analytical mechanics, but later on he recognized the superiority of the method which consists in considering virtual works.
13. Hertz, *Die Prinzipien der Mechanik,* etc., 1894; See especially the introduction.
14. *Note sur la dynamique du point et du corps invariable*, Tome I, page 236.
15. P. Duhem, *Commentaire aux principes de la thermodynamique*, 1892; *La théorie physique, son objet et sa structure*, 1906.
16. E. Le Roy, *La science positive et les philosophies de la liberté*, Congrès int. de philosophie, T.I., 1900.
17. E. and F. Cosserat, *Sur la mécanique générale*, *C.R.*, 145, p. 1139, 1907.
18. E. and F. Cosserat, *Sur la théorie des corps minces*, *C.R.*, 146, p. 169, 1908.
19. It must be that the interest and the importance of the theories of lines and deformable surfaces are today so badly appreciated that the

Encyclopaedia of pure and applied mathematics [*Enz. math. Wiss.*], presently published in Germany, grants them no room. W. Thomson and Tait have avoided to omit them in their *Natural Philosophy*, and they presented them before the theory of three-dimensional elastic bodies. Similarly, P. Duhem, *Hydrodynamique, Elasticité, Acoustique*, Paris, 1891.

20. H. Lorentz – *Versuch einer Theorie der electrischen und optischen Erscheinungen in bewegten Körpern*, Leiden, 1895; reprinted in Leipzig in 1906, *Abhandl. über theoretische Physik*, 1907; *Encycl. der Math. Wissenschaften*, V2, Elektronen Theorie, 1903.
21. H. Poincaré, *Electricité et optique*, 2nd edition, 1901, p. 448.

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