Chapter 4
Piola and Kirchhoff: On Changes of Configurations

Abstract The seminal contribution of Gabrio Piola to the foundations of continuum mechanics is critically examined directly on the basis of his publications (1825–1848). This emphasizes the original approach of Piola who favoured a direct projection on the material configuration (where material particles are "labelled"), this yielding the now well known Piola–Kirchhoff stresses in the so-called Piola format of continuum mechanics. Piola is a follower of Lagrange and Poisson, much more than of Cauchy. But he established the connection of his equations with those of the more familiar Euler–Cauchy format (expressed in the actual configuration) of elasticity. Kirchhoff, much more known than Piola because of his renowned works in electricity, spectroscopy and thermo-chemistry, also contributed to the same format as Piola, hence his name attached to that of Piola. The works of Piola acquired a well deserved recognition and an excellent range of applications with the expansion of nonlinear elasticity, the modern theory of material inhomogeneities and the notion of configurational forces.

4.1 Introduction

It is agreed upon [33] that Euler and Lagrange are responsible for the introduction of two kinematical descriptions of the motion of deformable continua, emphasizing the dependence on actual or initial (Lagrangian) coordinates. In their time, this was particularly well exploited in fluid mechanics. However, with the consideration of possible finite deformations, essentially by Cauchy in France [6] and Green in the UK [10], in the framework of elasticity, the relationship between two configurations—the actual one after deformation and perhaps one chosen appropriately to label the "material particles" in a convenient way—became a necessity. It was to be the role of Gabrio Piola in Italy and Gustav Kirchhoff in Germany to clarify this matter, so that the two names are often associated to designate certain entities, e.g., the *Piola–Kirchoff stress tensors*. This possible duality between two kinematic descriptions of course entails the possibility to write the basic equations

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governing the dynamics of continua in two formats, that are now called the Euler–Cauchy and Piola–Kirchhoff formats as they involve the use of different—or "transformed"—tensorial objects (in particular as regards the stress). Of essential importance here is the relationship between tensorial objects expressed in the two different formats. This was practically solved by Piola with the introduction of the (now called) *Piola transformation*, a notion also referred to as pull-back operation (and its inverse the pull-forward [18]). Here we shall critically examine how Piola constructed "his format" of equations by perusing his original works of the period 1825–1848 [24–27] . This format acquires its full importance in the formulation of the theory of material inhomogeneities [20] and the theory of configurational forces [21]. It is in fact our involvement in the expansion of these theories that kindled our interest in the original papers of Piola.

4.2 Piola's Contribution

4.2.1 Some Words of Caution

Gabrio Piola (1794–1850) is an Italian mathematician who was an enthusiastic disciple of Joseph Louis Lagrange (1736–1813), the well known Italian-French mathematician. He is, therefore, an ardent supporter of variational formulations in the Euler–Lagrange tradition. He is the author of generally lengthy memoirs. We must admit that these papers are difficult to read, in reason both of the obsolete mathematical terms and the somewhat antiquated Italian language. We shall focus attention on the memoirs of 1836[1] and 1845 with some comments on that of 1833. In doing so we have decided to translate in modern (intrinsic or indicial) notation many of Piola's mathematical expressions written at a time when neither vector nor tensor notions existed. Thus the motion mapping is given by Piola by the application

$$(a, b, c) \rightarrow (x, y, z)$$

at fixed time, and the summation is indicated by a big S (that we replace by a more familiar Σ). However we shall refer to Piola's equations by an indication such as [26, p. 259, Eq. 137]. Note that Piola's notation for motion and deformation is still used by the Cosserat brothers as late as 1909 (who are not much easier to read [22]). Also, it is remarkable that Piola demonstrates an unconscious capture of a hidden algorithm so that he does not always need to write all components of a vector or tensor equation explicitly, but he gives a hint of this matter to the reader. For the sake of simplification, we do not make any distinction between covariant and contravariant tensors, assuming Cartesian systems of coordinates. But this is not altogether correct.

[1] A microfilm copy of the memoirs of 1836 and 1845 was kindly provided to us in 1991 by the Municipal Library of Modena during our stay as a visiting professor of the Italian CNR in Pisa.

4.2.2 The Strategy of Piola

In perusing Piola's works from 1825 to 1848, we can distinguish three main lines that combine together to form a well defined strategy.

The first one is an attempt at avoiding the consideration of infinitesimals—as used by Lagrange—in the conception of the kinematics of moving points, but considering as first principle *the superposition of motions*. This is a line he developed in his competition essay of 1825 [24]. This follows the works of other Italian mathematicians such as Magistrini and Riccardi who criticized Lagrange's use of virtual velocities. This viewpoint studied by Capecchi [3, 4] appears somewhat strange to modern minds, although it does yield the classical form of the equation of motion of a free material point. We shall not dwell further in this matter.

The second line is the a priori consideration of the motion of an ensemble of points in interaction, following Poisson (and also the second theory of continua of Cauchy), and then passing to a limit providing equations for a continuum, with an appropriate definition of what will later on be called the first Piola–Kirchoff stress tensor. This will be examined in greater detail herein below. This is developed at length in Piola [26]. Note that Piola there refers frequently to French mathematicians and mechanicians (Cauchy, Laplace, Poisson, Legendre, Lacroix, and of course Lagrange, hardly a Frenchman to him).

The third line is none other than an application of the Euler–Lagrange variational formulation accounting for possible mathematical constraints such as that due to rigidity. This necessitates the introduction of Lagrange multipliers [26] which are revealed to look like stress tensors. Of course here Piola follows the teaching of his master Lagrange (see the latter's lectures in Lagrange [17]. As shown by Piola [27], in the case of deformable bodies, this formulation in fact leads to the introduction of true stress tensors (Piola stress or Cauchy stress depending on the original definition of rigidity) in a rather formal manner that reminds us of the formulation of the principle of virtual power by Germain [9] or Maugin [19] where linear continuous forms on a set of generalized velocities are introduced a priori with "stresses" as co-factors. Such an approach permits the deduction of the accompanying natural boundary condition involving the stress. As noted by Truesdell and Toupin [33, p. 596, Footnote 3], this was the first deduction of such conditions from a variational principle. Piola's approach will be briefly described in the following paragraphs.

4.2.3 Introduction of the "Piola Format" by Piola

Following Poisson, Piola [26] considers identical point particles of unit mass that we can label (α). Each one is initially at position denoted by (a, b, c) with label (α) and after motion at position (x, y, z) with label (α). In modern notation this would yield the change of position as $x_i^{(\alpha)}$ *or* $\mathbf{x}^{(\alpha)}$ function of $X_K^{(\alpha)}$ *or* $\mathbf{X}^{(\alpha)}$, in Cartesian tensor notation and intrinsic notation, respectively. Thus the kinematic description may be said to be *referential*. With externally applied force $\mathbf{f}^{(\alpha)}$, and a model of interactions between particles (called "molecules") whose exploitation is somewhat obscure, Piola is able to write a variational formulation of the following type (p. 173, Eq. (15))

$$\sum_i \sum_\alpha \left(\frac{d^2 x_i^{(\alpha)}}{dt^2} - f_i^{(\alpha)} \right) \delta x_i^{(\alpha)} + \sum_i \sum_{\alpha,\beta} \phi\left(S_{\alpha,\beta}\right) \delta S_{\alpha,\beta}|_i = 0 \tag{4.1}$$

where the S's—whose details are irrelevant—depend on the relative distances between particles, hence on the x_i. For arbitrary variations of the x_i this formally yields equations of motion of individual particles in the form [26, p. 189, Eq. (41)]

$$f_i^{(\alpha)} - \frac{d^2 x_i^{(\alpha)}}{dt^2} + I_i^{(\alpha)} = 0, \quad \alpha = 1, 2, \ldots \tag{4.2}$$

where $I_i^{(\alpha)}$ is the interaction force with other particles that we do not elaborate further. The "tour de force" of Piola rests in the approximation of these interaction terms (pp. 175–200) and passing to some kind of continuum limit that brings the generic local equation of motion to the vectorial form [26, p. 201, Eq. (56)]

$$\mathbf{f} - \frac{\partial^2 \mathbf{x}}{\partial t^2} + div_X \mathbf{T} = \mathbf{0}, \tag{4.3}$$

where $\mathbf{T}$ is an object with nine independent components (for it has no symmetries) and the modern symbol div_X means the divergence operator with respect to the referential coordinates (a, b, c) i.e., X^K. Obtaining (4.3) involves the neglect of supposedly small terms. Equation (4.3) can also be written as

$$f_i - \frac{\partial^2 x_i}{\partial t^2} - \frac{\partial}{\partial X_K} T_{Ki} = 0, \tag{4.4}$$

in Cartesian tensor analysis.

The change of position and its inverse (assuming invertibility in agreement with Lagrange) can be noted [26, p. 202, Eqs. (58)–(59)]

$$(x, y, z) \text{ functions of } (a, b, c) \text{ and time } t$$

and

$$(a, b, c) \text{ functions of } (x, y, z) \text{ and time } t$$

or in modern notation

$$\mathbf{x} = \bar{\mathbf{x}}(\mathbf{X}, t) \text{ and } \mathbf{X} = \bar{\mathbf{X}}(\mathbf{x}, t). \tag{4.5}$$

Let J denote the Jacobian determinant of the first of these transformations (this is denoted H by Piola, p. 204), i.e.,

$$J = \det \mathbf{F},\ \mathbf{F} = \left\{ F_{iK} = \frac{\partial \bar{x}_i}{\partial X_K} \right\}. \tag{4.6}$$

For a continuum, this is as if (4.3) or (4.4) had been written for a body of referential mass density $\rho_0 = 1$. If this is not the case, ρ_0 has to be introduced and (4.3) has to be rewritten as

$$\rho_0 \left(\mathbf{f} - \frac{\partial^2 \mathbf{x}}{\partial t^2} \right) + div_X \mathbf{T} = \mathbf{0}. \tag{4.7}$$

Then Piola would like to compare his equation of motion with the formulation obtained by Cauchy [6] and Poisson [29] in the actual configuration. To do this he needs some work since he must pass to the spatial parametrization of the Eulerian type in terms of the actual position (x, y, z) or $\mathbf{x} = \{x_i;\quad i = 1, 2, 3\}$. He shows that he can introduce a geometrical object $\boldsymbol{\sigma}$ (noted $\boldsymbol{K}$ by Piola) such that [26, p. 204, Eq. (60)]

$$\sigma_{ij} = J^{-1} \frac{\partial \bar{x}_i}{\partial X_K} T_{Kj} \quad or \quad \underline{\sigma} = J^{-1} \mathbf{F}\ \mathbf{T}. \tag{4.8}$$

Reciprocally (Piola 1936, p. 205, Eq. (63))

$$\mathbf{T} = J \mathbf{F}^{-1} \cdot \underline{\sigma} \quad or \quad T_{Ki} = J \frac{\partial \bar{X}_K}{\partial x_j} \sigma_{ji}. \tag{4.9}$$

He establishes identities like [26, p. 205, Eq. (62)]

$$div_X \mathbf{T} = J\ div_x \underline{\sigma}, \tag{4.10}$$

where div_x means the divergence with respect to the (x, y, z) or $\mathbf{x} = \{x_i;\ i = 1, 2, 3\}$ space parametrization. This involves proving the identities

$$\nabla_x \cdot \left(J^{-1} \mathbf{F} \right) = \mathbf{0} \quad \text{and} \quad \nabla_X \cdot \left(J \mathbf{F}^{-1} \right) = \mathbf{0}. \tag{4.11}$$

Noting that in his format the mass conservation reads (p. 211, Eq. (72) with Γ—which Piola does not yet call density—standing for ρ)

$$\rho_0 = J \rho, \tag{4.12}$$

where ρ is the actual density at (x, y, z), Piola finally shows that Eq. (4.7) above renders the equation of motion (p. 212, Eq. (74))

$$\rho \left(\mathbf{f} - \frac{d^2 \mathbf{x}}{dt^2} \right) + div_x \underline{\sigma} = \mathbf{0}. \tag{4.13}$$

This he identifies with the equation obtained by Cauchy [6, p. 166] or Poisson [29, VIII, p. 387; X, p. 578]. Accordingly, $\boldsymbol{\sigma}$ is none other than the Cauchy stress tensor for any continuum, whether solid or fluid, while $\mathbf{T}$ deserves to be called the *Piola stress* (*first Piola–Kirchhoff stress* in modern jargon). The word density ("densità") here is used for the quantity J^{-1} since $\rho_0 = 1$ for Piola. Equilibrium is obtained by making the acceleration term vanish in Eq. (13)—[26, p. 215, Eq. (79)].

What is original here with Piola is that he has formulated what we call the "Piola format" of the basic equations of continuity—Eq. (4.12)—and of balance of linear momentum. His "format" involves two configurations with a preference for the referential one for the space parametrization. It is sometimes called the *material* formulation [21] since **X** refers directly to the material "points" that belong to the "material manifold". The only inconvenience is the appearance of geometrical objects such as **F** and **T** that have two "feet" in different configurations and will later on be called *two-point tensor fields*—i.e., tensors depending on two "points"—by Einstein or, here precisely *double vectors*. But it must be understood that all computations are effected by Piola with all explicit scalar components of the introduced objects since he has no notion of a tensor (only introduced in the 1880s by Voigt).

The celebrated *Piola transformation* here is represented by Eq. (4.9) that is even made clearer when applied to a vector field. Let **v** a vector field with components in the actual framework $\mathbf{x} = \{x_i;\ i = 1, 2, 3\}$. The associated vector field **V** in the framework $\mathbf{X} = \{X_K;\ K = 1, 2, 3\}$ is defined by accounting both for the deformation and the volume change; that is:

$$\mathbf{V} = J\mathbf{F}^{-1} \cdot \mathbf{v}. \tag{4.14}$$

This is the Piola transformation—or *pull back* to the reference configuration. The inverse operation is called the *push forward* from reference configuration to actual configuration. Equation (4.9) that defines the first Piola–Kirchhoff stress is thus only a *partial* Piola transformation of the Cauchy stress. This is the rather troubling matter (with many students). But this manipulation allows one to obtain an equation of motion (4.7) with good partial differential derivatives in the space-time parametrization (X_K, t) while this equation still has components in the actual configuration where data in forces are prescribed. For the transformation of boundary conditions on stresses one will have to wait for the formulas obtained by Nanson [23] for the transformation of oriented surface elements. It is not forbidden to construct the full pull-back of the Cauchy stress by completing the transformation (4.9) by defining the fully material stress **S** by {T = transposed; [25, Eq. (45)]; [26, Eq. (132)]}

$$\mathbf{S} = \mathbf{T} \cdot \mathbf{F}^{-T} \ or \ S_{KL} = T_{Ki} F_{iL}^{-1} = J \frac{\partial X_K}{\partial x_i} \sigma_{ij} \frac{\partial X_L}{\partial x_j}. \tag{4.15}$$

This is called the *second Piola–Kirchhoff stress* in modern continuum mechanics. It is a true material tensor; it is *symmetric* by construction if the Cauchy stress is symmetric (which is more than often the case). In contrast, it does no make mathematical sense to speak of the symmetry or non-symmetry of **T**. The thermodynamic importance of **S** will be made clear soon in Green's elasticity derived from a potential.

Piola still has to be more precise with the notion of *matter density*. He ponders this notion in his Chap. IV (p. 218 on) where, basically, it is mass divided by volume—valid only for a homogeneous volume—as otherwise the correct definition should involve a limit procedure applied to an infinitesimal element of matter at each point. Note that for Newton it was mass that was defined by density multiplied by volume. Starting from Eq. (4.12) and noticing that in Piola's time ρ_0 may at most be a function of (a, b, c) or $\mathbf{X}$, and not a function of time (this is no longer true in the modern theory of material growth [21, Chap. 10]), a laborious computation leads Piola to the *equation of continuity* in the Eulerian form (Piola, p. 235, Eq. 105):

$$\frac{\partial \rho}{\partial t} + \nabla_x \cdot (\rho \mathbf{v}) = 0, \tag{4.16}$$

where $\mathbf{v} = d\mathbf{x}/dt$ is the velocity. Thus (4.16) and (4.13) correspond to (4.12) and (4.7), respectively.

The rest of the impressive Piola's paper of 1836 concerns the introduction of the displacement for a continuum and the notions of dilatation and condensation (in Cauchy's sense), and many more considerations on the molecular description of the material, whose purpose in principle is to deduce explicit expressions for the interactions introduced in Eq. (4.1) above—in particular with the notion of pressure, and a theory of fluids in concurrence with one expanded by Poisson [30, p. 524]. This goes beyond the present focus.

4.2.4 Stresses as Lagrange Multipliers

As a good disciple of Lagrange, Piola exploits the technique of multipliers to account for constraints. Lagrange had done this for the constraint of incompressibility introducing thus a scalar multiplier that is a mechanical pressure. In Piola [25], the author wants to do it for the constraint of *rigidity* of extended bodies, perhaps as a preparation for the case of deformable bodies [27]. The formulation he offers is quite original albeit a little bit involved. We consider again Piola's original notation (a, b, c) and (x, y, z) for the initial and final positions of any point in the body. The a priori motion is written by Piola as

$$x = f + a_1 a + b_1 b + g_1 c,\ y = g + a_2 a + b_2 b + g_2 c,\ z = h + a_3 a + b_3 b + g_3 c, \tag{4.17}$$

which contains twelve scalar parameters. In modern vector and matrix notation this reads

$$x_i = g_i + \sum_{K=1}^{3} a_{iK} X_K,\ i = 1, 2, 3. \tag{4.18}$$

For a true rigid body motion the number of parameters must be reduced to six (three translations and three rotations). Then by astute manipulations (Piola does not possess the notion of matrix) that look terrible to modern eyes, Piola, by eliminating the parameters, succeeds to express the conditions of orthogonality and normality in differential forms for the function $x_i = \bar{x}_i(X_K)$. Two alternate forms are obtained that we rewrite (here in modern notation) as

$$\bar{e}_{ij} = \sum_{K=1}^{3} F_{iK} F_{jK} - \delta_{ij} = 0 \tag{4.19}$$

and

$$\bar{E}_{KL} = \sum_{i=1}^{3} F_{iK} F_{iL} - \delta_{KL} = 0, \tag{4.20}$$

where the deltas are Kronecker symbols and $F_{iK} := \partial \bar{x}_i / \partial X_K$. In setting

$$e_{ij} = \bar{e}_{ij}/2, \quad E_{KL} = \bar{E}_{KL}/2, \tag{4.21}$$

we recognize with the first part of (4.19) and (4.20) the definition of the *left and right Cauchy–Green strain tensors* of non-linear continuum mechanics up to a factor ½ [33], i.e., in intrinsic notation (with the notation of Maugin [20])

$$\mathbf{e} = \frac{1}{2}\left(\mathbf{c}^{-1} - \mathbf{1}\right), \quad \mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{1}_R) \tag{4.22}$$

with

$$\mathbf{c}^{-1} := \mathbf{F} \cdot \mathbf{F}^T, \quad \mathbf{C} := \mathbf{F}^T \cdot \mathbf{F}. \tag{4.23}$$

The rigid-body conditions (4.19) and (4.20) read

$$\mathbf{c}^{-1} = \mathbf{1}, \quad \mathbf{C} = \mathbf{1}_R, \tag{4.24}$$

where $\mathbf{1}_R$ is the unit tensor in the reference configuration. $\mathbf{C}^{-1}$, the inverse of $\mathbf{C}$, is called the *Piola (material) finite strain*. The tensor $\mathbf{c}$, inverse of $\mathbf{c}^{-1}$, is called the (spatial) *Finger finite strain*. ($\mathbf{c}^{-1}$ is sometimes noted $\mathbf{B}$ so that $\mathbf{c}$ would be $\mathbf{B}^{-1}$).

Conditions (4.24) are integrated forms while it is usual to express the rigidity condition in time differential form (in terms of the rate of strain) equivalent to Killing's theorem, e.g.,

$$\dot{\mathbf{C}} = 0 \tag{4.25}$$

or in variational form $\delta\mathbf{C} = \mathbf{0}$ or $\delta\mathbf{E} = \mathbf{0}$.

The total virtual work of body forces (per unit mass) for a body of referential volume V_0 is given by

$$\delta W^{ext} = \int_{V_0} \rho_0 \mathbf{f} \cdot \delta\mathbf{x} \, dV_0. \tag{4.26}$$

If this body is to be *rigid*, then either one of the mathematical constraints (4.19) and (4.20) must be taken into account. Following Lagrange [17], this is done by introducing *Lagrange multipliers*, here tensors of components λ_{ij} or Λ_{KL} so that the principle of virtual work for equilibrium is written in any of the following two forms (with summation over repeated indices and $dm = \rho_0 dV_0 = \rho dV$):

$$\int_V \rho \mathbf{f} \cdot \delta \mathbf{x}\, dV - \int_V \lambda_{ij} \delta e_{ij} dV = 0 \tag{4.27}$$

or

$$\int_{V_0} \rho_0 \mathbf{f} \cdot \delta \mathbf{x} dV_0 - \int_{V_0} \Lambda_{KL} \delta E_{KL} dV_0 = 0. \tag{4.28}$$

This is the essence of Piola's argument rewritten in modern formalism. Piola prefers the "material" formulation (4.28) over the Eulerian formulation (4.27). Transformation of (4.28) on account of (4.20, 4.21) and localization yield a local *equilibrium* equation in the form

$$\frac{\partial}{\partial X_K} G_{ki} + \rho_0 f_i = 0, \quad G_{Ki} := \Lambda_{KL} F_{iL} \; or \; \mathbf{G} := \Lambda \cdot \mathbf{F}^T. \tag{4.29}$$

Note that λ_{ij} and Λ_{KL} are related by

$$\Lambda_{KL} = J \frac{\partial X_K}{\partial x_i} \lambda_{ij} \frac{\partial X_L}{\partial x_j}, \tag{4.30}$$

a relation similar to (4.15). The introduced tensorial Lagrange multipliers can be interpreted as "reaction internal forces" needed to maintain the rigidity of the body. These internal forces are undetermined for a rigid body.

Of course the natural question is what happens for a deformable body. This was answered by Piola in his long memoir of 1848 (but presented in 1845). This is more formal in the sense that the reaction internal forces become the true *stresses* in action in the body. They are introduced as *coefficients* of the variation of strains in a linear form. That is (4.28) in principle is replaced by

$$\int_{V_0} \rho_0 \mathbf{f} \cdot \delta \mathbf{x}\, dV_0 - \int_{V_0} S_{KL} \delta E_{KL} dV_0 = 0, \tag{4.31}$$

yielding instead of (4.29)

$$\frac{\partial}{\partial X_K} T_{ki} + \rho_0 f_i = 0, \quad T_{Ki} := S_{KL} F_{iL} \; or \; \mathbf{T} := \mathbf{S}\mathbf{F}^T, \tag{4.32}$$

where $\mathbf{T}$ indeed is the first Piola–Kirchhoff stress. But this is not entirely correct because there can be a traction $\mathbf{t}$ acting on the boundary of the body so that, introducing also acceleration forces to obtain the dynamical case (4.31) should be re-written as

$$\int_{V_0} \rho_0 \ddot{\mathbf{x}} \cdot \delta \mathbf{x} dV_0 = \int_{V_0} \rho_0 \mathbf{f} \cdot \delta \mathbf{x} \, dV_0 - \int_{V_0} S_{KL} \delta E_{KL} dV_0 + \int_{\partial V_0} \mathbf{t} \cdot \delta \mathbf{x} \, dS_0. \quad (4.33)$$

This is quite similar to the formulation of the *principle of virtual power* used in modern times by Germain [9] and Maugin [19] —see also Truesdell and Toupin [33, p. 596, Eq. (232.4)]—that considers the right-hand side of Eq. (4.33) as a linear continuous form on virtual velocities including that of the gradient of the motion. Equation (4.33) allows one to obtain the local balance of linear momentum—here in the Piola format (4.7)—as also the accompanying natural boundary condition for stresses. It seems that Piola was really the first to deduce the stress boundary condition from a variational principle [27, Part 2, p. 52]. The original variational principle by Piola goes back to 1833 (Part 3) and 1848 (Part 2, pp. 34–38, 46–50). Hellinger [11, Chap. 4, Paragraph 3d] also dealt with the same variational principle. In addition, Piola formulated analogous variational principles for one-dimensional and two-dimensional systems [27, Chap. 7]. This can be compared with variational formulations by the Cosserat brothers [7]. Pierre Duhem [8] formulated a principle of virtual work that looks very much like the one of Piola for equilibrium [4, p. 390].

To conclude this point, we recall that George Green (1793–1841) introduced in his celebrated memoir of 1839 the same finite-strain tensors as Cauchy, hence the association of the two scientists in the denomination of these tensors. Furthermore, he simultaneously introduced the notion of strain energy W per unit of referential volume, $W = W(\mathbf{E})$, such that

$$\delta \int_{V_0} W(\mathbf{E}) dV_0 = \int_{V_0} \delta W(\mathbf{E}) dV_0 = \int_{V_0} \mathbf{S} : \delta \mathbf{E} dV_0 \quad (4.34)$$

and

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{E}} \; or \; S_{KL} = \frac{\partial W}{\partial E_{KL}}. \quad (4.35)$$

That is why the second Piola–Kirchhoff stress is also referred to as the *energetic stress* while $\mathbf{T}$ may also be called the *nominal stress* because it is evaluated per unit area in the reference configuration although it still behaves as a vector in the actual configuration.

Among the Italian disciples of Piola we must count Eugenio Beltrami (1836–1900) and to a lesser degree Enrico Betti (1823–1892). Like Lamé, Beltrami was a great amateur of curvilinear coordinate systems and his differential methods favoured the early development of tensor calculus in Italy. More to our point, in the application of the principle of virtual work (e.g., in Beltrami [1]), he considered internal forces (stresses) and deformations as dual variables, which is our modern view point with the notion of separating duality between two vector spaces (e.g., in Maugin [19]). As to Betti, although having started as a "Newtonian", he later on based his continuum mechanics on potential energy, strains and the principle of virtual work [2]. This is discussed by Capecchi [4, pp. 392–393].

4.3 The Role of Kirchhoff

Kirchhoff (1824–1887) is one of the German giants in continuum mechanics for the 19th century, although his reputation in electricity, spectroscopy, black-body radiation, and thermo-chemistry is at the same if not higher prestigious level. It is in Königsberg that Kirchhoff took lectures with Neumann, a specialist of the strength of materials. He later became a professor of physics in Breslau, Heidelberg and finally Berlin. Kirchhoff made many important contributions to continuum mechanics and the mechanics of structures [31]. For instance, he proposed a correct model for the bending of plates by means of a variational principle. The two-dimensional equation was deduced from a variational principle (principle of virtual work) in which a reduced potential energy accounts for a set of basic kinematic hypotheses concerning the section of the plate normal to the middle surface and the neglect of any stretching of the elements of the middle plane for small deflections. This much improved the tentative theory proposed earlier by Sophie Germain (1776–1831). This is now referred to as *Kirchhoff–Love theory of plates* after Love (1863–1940) who extended Kirchhoff's approach to the case of thin shells. Kirchhoff also studied theoretically and experimentally the vibrations of plates on the basis of his model. He also subsequently extended his theory of plates to include the case of not too small deflections.

If Kirchhoff is cited here it is because he also considered finite deformations, especially in Kirchhoff [14] —apparently independently of Piola—which is also reported in his lectures on mechanics in Kirchhoff [16]. He was thus led [14, pp. 763–764 and p. 767] to introducing stress tensors similar to those of Piola, hence the two names jointly attached to these geometric objects, even though the role of Kirchhoff in this very subject seems rather minor compared to that of Piola. Both Kirchhoff [14], and later on Poincaré [28; Paragraph 40], explained the "non-symmetry" of **T**, but the present notation is clear enough. We can also note that if the Cauchy stress is symmetric, then we can also say that **T** *is symmetric with respect to* **F**, because the local equation of moment of momentum (in the absence of internal spin, applied couple, and microstructure [7] $\underline{\sigma} = \underline{\sigma}^T$ also reads

$$\mathbf{F} \cdot \mathbf{T} = (\mathbf{F} \cdot \mathbf{T})^T. \tag{4.36}$$

It must be emphasized that Kirchhoff's works in elasticity were among his first scientific works in the late 1840s and early 1850s, and then on and off during the rest of his career. Thus in 1850, he wrote down a variational principle which looks somewhat like Piola's one recalled in Eq. (4.31) above. However the second term was expressed in terms of the principal dilatations (in Cauchy's words)—Kirchhoff [13]. This was further expanded in Kirchhoff [15], the most important paper by Kirchhoff in elasticity according to Todhunter [32, p. 63]. Jungnickel and McCormmach [12, p. 295] rightly remark that his work in elasticity provided useful analogies for his works in electricity.

4.4 Conclusion

The interest for the developments recalled in the foregoing two sections mainly rests on two ingredients: one is the importance given to objects defined partially or entirely in the reference configuration such as the Piola–Kirchhoff stress tensors, clearly of utmost interest to Piola. The other is Piola's and Kirchhoff's interest in variational formulations of the type of the principle of virtual work. These formulations remained fashionable and efficient for a large part of the nineteenth century as evidenced by the works and lecture notes of Clebsch [5], Duhem [8], Poincaré [28], the Cosserat brothers [7], and Hellinger [11]. Concerning the first ingredient, it is with the development of *non-linear elasticity* in the 1930s–1950s in Italy, the UK and the USA—as richly illustrated in the encyclopaedia synthesis of Truesdell and Toupin [33]—that the necessity of clearly distinguishing between the actual configuration and a reference one was made clear to all students in continuum mechanics. This has become common practice. More recently, this fruitful approach to the deformation theory in general was enhanced by an inclusive definition of *material inhomogeneity*: this is the possible continuous or discontinuous dependency of material properties (e.g., elasticity) on the material point **X** itself, an "element" of the material manifold (this view was forcefully emphasized in Maugin [20]). A corollary of this was the required consideration of the full projection of all field equations on this material manifold leading to a true "material" mechanics of continua in the Piola–Kirchhoff format and the systematic introduction of the (completely material) *Eshelby stress tensor* as the most relevant internal-force concept and its application in the form of *configurational forces* to the theory of defects. This is amply documented in our treatise [21] which owes much in spirit to Eshelby (1916–1981), but also retrospectively much to the original thinking of Piola that we tried to capture in the present contribution.

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