Chapter 3
What Happened on September 30, 1822, and What Were its Implications for the Future of Continuum Mechanics?

Abstract This contribution offers a discussion about the notion of stress in a general continuum as initially proposed in a magisterial paper by Cauchy in 1822 (but published only in 1828) without using arguments involving molecules. This is here presented in its historical context. Cauchy's view is the currently accepted view among mechanicians and engineers although attempts (including by Navier and Cauchy himself) to start from a molecular description in the manner of Newton and Laplace were constantly offered in both nineteenth and twentieth centuries. The discussion introduces other secondary stress definitions such as those by Piola, Kirchhoff, and more recently Eshelby. The question naturally arises of what happens with the possibility to introduce other internal forces such as hyperstresses (in so-called gradient theories) and couple stresses (e.g., in Cosserat continua), and whether some introduced stresses have associated with them a meaningful boundary condition. Also pondered is the question whether one can identify a stress concept in physical approaches still considering interactions between point particles (lattice dynamics, kinetic theory, nonlocal theory, statistical-mechanics approach). The chapter is concluded by a more in depth discussion of the notion of stress-energy-momentum, culminating in that of pseudo-tensor of energy-momentum in gravitation theory.

3.1 Introduction

Augustin Louis Cauchy (1789–1857) was a brilliant French mathematician with an extremely wide scope of interests including mathematical physics and theoretical mechanics as well as pure questions of algebra (theory of permutations) and analysis (complex analysis and theory of residues). Extremely prolific, he had a production that compares well both in quality and quantity with those of Leonard Euler (1707–1783)—a predecessor in many points- and his contemporaries Carl F. Gauss (1777–1855) and A. Cayley (1821–1895). Born in August 1789 practically one month after the fall of the Bastille (July 14, 1789), he certainly did not become

G. A. Maugin, *Continuum Mechanics Through the Eighteenth and Nineteenth Centuries*, Solid Mechanics and Its Applications 214, DOI: 10.1007/978-3-319-05374-5_3,

a revolutionary. On the contrary, politically, he remained all along his life a convinced legitimist—i.e., in favour of a king in the line of the dynasty (the *Bourbons*) of pre-revolution times. The reader may wonder what is the relationship of this political inclination with his scientific works and career. Indeed, although formed as an "ingénieur-savant"[1] in the best schools created by the French Revolution and Republic and strengthened by Napoleon, he benefited from the fall of the latter and the return of the Bourbon kings in 1815 in being given the positions—which he deserved on a purely technical professional basis—of other scientists who lost their positions for political reasons. Among the positions he was granted we single out that of teaching a course on mechanics at the Faculty of Sciences in Paris in 1821. This, according to Belhoste [1, p. 92], may have "provided the inspiration for further research in mathematical physics", in particular the mechanics of continua, although he had already taught some mechanics at the *Ecole Polytechnique* and at the *Collège de France*. The second fact which may a priori seem irrelevant to a scientific discussion, is the marriage of Cauchy with a certain Miss Aloïse de Bure in 1818. This, as we shall see, had a definite consequence on the manner of publishing his works by Cauchy. Now, the title of the present chapter does not question what happened all over the whole world, but more precisely what happened on that precise day of 1822 at the Academy of Sciences in Paris.

3.2 Preliminary Remarks

In 1822, at age thirty three, Cauchy was already an internationally recognized mathematician when, on September 30, he read a memoir on continuum mechanics to the Paris *Académie*. This was to be the foundational paper in that field of mechanics. This may be referred to as his *first* theory (CAUCHY-1) of general continuum mechanics although he had published before papers on fluid mechanics and he was much interested in the possibility of an elastic medium to transmit waves. That now celebrated date of September 30, 1822, was only the presentation of a theoretical framework that was really published in print only six years later in 1828 in an extended and corrected form. As a matter of fact, Cauchy did not even give a true reading or lecture on the contents of his paper in 1822, nor did he leave a copy with the *Académie*.[2] Only a kind of abstract was given in the bulletin of January 1823 of a learned Paris society, the *Société Philomatique* [this

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[1] For this notion of ingénieur-savant ("engineer-scientist") see [22, 10].

[2] This has been checked in the files of the session of September 30, 1822 with the kind help of the Librarian (Mrs Florence Greffe) of the Paris Academy of Sciences. This date was mentioned by Truesdell [45], but also much before by Duhem (p. 78, Footnote 2, of Duhem [14]). The original record of this session is reproduced at the end of this chapter. It simply says that Cauchy read about his research (probably just the basic ideas) that was to be printed as a long abstract four months later in the Bulletin of the *Société Philomatique*.

is translated into English in the Appendix]. There might have been quarrels of priority with C.L.M.H. Navier (1785–1836)—another great elastician and fluid dynamicist—that delayed the real publication (see [1, pp. 97–98]). All his life Cauchy, an already mentioned prolific author, flooded the *Académie* with notes and memoirs, so much that the *Académie* at a point decided to fix an upper limit to the number of such contributions that any member could submit! This is where the importance of in-laws should not be overlooked. It happens that Cauchy's wife Aloïse was the daughter and niece of the de Bure brothers, Marie-Jacques and Jean-Jacques, renowned Parisian booksellers and publishers. Cauchy frequently used this possibility an as expedient way to publish in print his own lectures at *Polytechnique* and also many of his memoirs. This was the case of his landmark paper of 1822 on continuum mechanics which was published in 1828 (cf. Cauchy [4]) in the second volume of "Exercices de Mathématiques", a kind of privately produced series published between 1826 and 1830. Cauchy was the only author published in this surprising scientific periodical.

3.3 The Main Contents of Cauchy's 1822/1828 Memoir

What we call the *first* theory (CAUCHY-1) of general continuum mechanics and elasticity of Cauchy is a purely phenomenological continuum theory which does not use the notion of constituent "molecules" and at-a-distance interactions between them (contrary to the *second* theory of Cauchy; see below). Cauchy did not build on an uncultivated ground.[3] Euler had already introduced the (restricted) idea that interactions between parts of a fluid were of the *contact* form and materialized in a single scalar, the hydrostatic pressure p. In modern terms, it is said that the applied traction $\mathbf{t}$ at a point of a regular surface is aligned with the local unit normal $\mathbf{n}$ to that surface, i.e.

$$\mathbf{t} = -p\,\mathbf{n}. \tag{3.1}$$

This applies to so-called Eulerian fluids that present no viscosity. The basic idea propounded by Cauchy in 1822 was to generalize (3.1) to all kinds of continua (see the spot-on general title of the abstract published in Cauchy [3]; full memoir Cauchy [4]. His reasoning is that in this state of generalization the relation (3.1) is replaced by a *linear* relationship (a linear vector relation in the language of Gibbs

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[3] The reader will be interested in Truesdell's vision of Cauchy's elaboration of the concept of stress in his Essay "The creation and unfolding of the concept of stress" in pp. 184–238 in Truesdell [45] (this was underlined by J. Casey, private communication). However, in contrast to the present study that emphasizes the story of the concept of stress from and after Cauchy, Truesdell deals with the conceptual stages that led to Cauchy's notion of stress, with works by brilliant predecessors such as Stevins, Galileo Galilei, the Bernoullis, d'Alembert, Euler, Young, and Fresnel. Cauchy himself is parsimonious with citations, and refers to very few scientists with the exception of his contemporary Fresnel.

and Heaviside; cf. Crowe [10]), and not a simple proportionality. That is, in modern intrinsic and Cartesian tensorial notations,

$$\mathbf{t} = \underline{\sigma}.\mathbf{n} \quad and \quad t_i = \sigma_{ij} n_j, \quad i,j = 1,2,3. \tag{3.2}$$

Equation (3.1) corresponds to the special case $\underline{\sigma} = -p\,\mathbf{1}\, or\, \sigma_{ij} = -p\,\delta_{ij}$ where δ_{ij} is Kronecker's delta. This can be viewed as a specific constitutive assumption (i.e., the selection of a specific continuum, the Eulerian fluid). The object $\underline{\sigma}$ is the (Cauchy) *stress tensor*. Of course it was identified with the mathematical notion of tensor (which itself smells of its mechanical origin) only much later by Woldemar Voigt. In his generalization Cauchy considers that the applied traction **t** can be at any angle to the unit normal to the tangent plane of a surface cut in the material body, thus allowing for a contact action of the *shear* type as well as pressure. The true genial point resides in the absolutely general standpoint and its evident conceptual simplicity. To prove (3.2) Cauchy relied on a reasoning involving a special small volume of matter, in his celebrated *tetrahedron* argument, now reproduced in all introductions to continuum mechanics.[4] It was also proved in most cases (no applied couples) that this Cauchy stress is symmetric, having thus only six independent components at most in standard Euclidean physical space. Relying on an argument already introduced by Euler (by looking for the equilibrium of an elementary parallelepiped) the following local dynamical Cauchy equation of motion could be obtained (here in modern notation) [44, 47, 48]

$$\rho\,\mathbf{a} = \rho\mathbf{f} + div\,\underline{\sigma}, \tag{3.3}$$

where vector **a** denotes the acceleration, ρ is the matter density, **f** is an external body force per unit mass, and the symbolism *div* denotes the *divergence* operator applied to the tensor $\underline{\sigma}$. Cauchy could not use this vocabulary as the operation of a divergence was essentially introduced by George Green [21] in electromagnetism in the same year in a practically unknown publication. But if we combine Cauchy's lemma (3.2) and Green's divergence theorem we have the following exploitable result:

$$\int_{\partial B} \mathbf{t}\, ds = \int_{\partial B} \underline{\sigma}.\mathbf{n}\, ds = \int_B div\,\underline{\sigma}\, dv, \tag{3.4}$$

for a simply connected volume B bounded by a regular surface ∂B. Applying this to the following global *balance law* of linear momentum,

$$\frac{d}{dt}\int_B \rho\,\mathbf{v}\, dv = \int_B \rho\,\mathbf{f}\, dv + \int_{\partial B} \mathbf{t}\, ds, \tag{3.5}$$

and localizing this on account of an assumed continuity of fields over B, we are led to the local (Cauchy) balance of linear momentum as Eq. (3.3) with $\mathbf{a} = d\mathbf{v}/dt$, what is the modern way of reaching (3.3).

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[4] See a more technical and rigorous exposition in Noll [39].

Remark 3.1 Very often in the engineering literature the expression *balance laws* and *equations of conservation* are used interchangeably. Like in financial accounting, "balance" carries with it the notion of incoming and outgoing stuff. That is why a quantity like $\underline{\sigma}$ is sometimes referred to as a *flux*.

Remark 3.2 Written in the appropriate coordinate system and in the absence of body source term, an equation such as (3.3) can also be written in the form

$$\frac{\partial}{\partial t}\mathbf{p} - div\,\underline{\sigma} = \mathbf{0}. \tag{3.6}$$

This can be referred to as a *mathematical conservation law*. In particular, with vanishing second term we can say that the quantity $\mathbf{p}$ is strictly conserved in time as

$$\partial\mathbf{p}/\partial t = \mathbf{0}. \tag{3.7}$$

But in statics (no time dependence of fields) or in quasi-statics (possible dependence on time but neglecting acceleration terms), we have the "equilibrium" equation

$$div\,\underline{\sigma} = \mathbf{0}. \tag{3.8}$$

The three possibilities embodied in Eqs. (3.6)–(3.8) can be compared to the Newtonian equation of point-particle motion:

**General Newton equation* (point of constant mass):

$$m\,\mathbf{a} = \sum_{\alpha}\mathbf{F}^{\alpha}, \tag{3.9}$$

where $\mathbf{F}^{\alpha},\ \alpha = 1, 2, \ldots$ is a system of acting forces;

**Statics (Varignon, parallelogram of forces)*:

$$\sum_{\alpha}\mathbf{F}^{\alpha} = \mathbf{0}; \tag{3.10}$$

**Inertial motion (Descartes)*:

$$\frac{d}{dt}\mathbf{p} = \mathbf{0}, \quad \mathbf{p} = m\mathbf{v}, \tag{3.11}$$

**D'Alembert's formulation of* (3.9):

$$\sum_{\alpha}\mathbf{F}^{\alpha} + \mathbf{F}^{a} = \mathbf{0}, \quad \mathbf{F}^{a} = -m\mathbf{a}. \tag{3.12}$$

With these different forms—of which (3.9) and (3.12) are strictly equivalent—we have interpretations at variance depending on the chosen emphasis. With account of the special case (3.10) we have a tendency to consider Newton's equation (3.9) as a definition for the acceleration force. With the special case (3.11) we suffer from another temptation, that of considering (3.9) as a conservation law of linear momentum that is not strictly respected because of the presence of impressed forces. As to (3.12), it is d'Alembert's clever "rewriting" trick to give all "forces" the same status, as understood in his principle of virtual power.

A more or less similar discussion can be envisaged for the continuum Eq. (3.6) through (3.8). What is the primary quantity appearing in these equations? For engineers avoiding a dynamical framework—Eq. (3.8) possibly with an added body force-, the Cauchy stress appears as primary, essentially through the Cauchy Lemma (3.2). But for physicists interested in dynamics, the interpretation of (3.6) as a nonstrict conservation law for linear momentum prevails, the notion of associated flux being only secondary. Finally, with the view of a discriminating physicist and parodying (3.12), we can formally rewrite (3.3) as

$$\mathbf{F}^{ext} + \mathbf{F}^{int} + \mathbf{F}^{a} = \mathbf{0}, \tag{3.13}$$

where

$$\mathbf{F}^{ext} = \rho \mathbf{f},\ \mathbf{F}^{int} = div\underline{\sigma},\ \mathbf{F}^{a} = -\rho \mathbf{a}, \tag{3.14}$$

are volume forces of external, internal, and acceleration origin, respectively. In writing (3.13) we distinguish between *external* forces reserved to *at-a-distance* action (e.g., gravitation, electromagnetism) acting per unit quantity of matter and *internal* forces that account for *contact* action via the second of (3.14) and the notion of Cauchy stress. This "definition" indicates that $\mathbf{F}^{int}$ is defined up to a divergence-free second-order tensor. But the associated natural boundary condition still involves only the initial stress and applied traction. However, this remark makes one ponder the case of an infinite body and the *second* Cauchy's theory (CAUCHY-2) of elasticity proposed by Cauchy in 1828 on the basis of *molecular* considerations [5]. Remember that Cauchy's work of 1822/1828 also proposed a definite theory of linear isotropic elasticity that provides an expression for $\underline{\sigma}$ in terms of infinitesimal strains, with two elasticity coefficients.[5] This was proposed a priori to close the obtained system of differential equations in terms of the elastic displacement gradient. This a priori construct that involves a representation theorem—also attributed to Cauchy—for a second-order tensor is referred to as *Cauchy's elasticity* by Truesdell, Toupin and Noll [47, 48]. Nowadays, this is justified by applying a thermodynamic argument and the consideration of an elasticity potential (in fact following George Green who thereby becomes our second hero).

[5] This was done after correction by Cauchy himself of his initial proposal with only one coefficient; for a general anisotropic body this would yield twenty one independent coefficients at most but its application to specific symmetries requires more group-theoretic reasoning unknown to Cauchy.

3.4 Cauchy's Stress and Hyperstresses

A naturally raised question is what happens to the Cauchy lemma (3.2) when one has to consider a surface that presents irregularities such as an edge where the unit normal may not be uniquely defined. One may also question what happens when in addition to the normal at a regular but curved surface, one tries to account for the geometrical description of the said surface at the second order, thus involving the curvature and the surface variation of the unit normal, hence also the introduction of the tangential derivative. It took some time to ponder these questions and obtain rigorous answers. This was rigorously achieved by Noll and Virga [40] and dell'Isola and Seppecher [11] with an advantage to the latter authors for the brevity of their argument—see also [12] for a more general case. Avoiding the difficult technical points for which we refer the reader to the original authors, we note that these considerations inevitably lead to envisaging the notion of *surface tension* described within the continuum mechanics framework. In addition to the notion of stress à la Cauchy this yields the introduction of the notion of *hyperstress*. This is represented by a third order tensor that is the thermodynamical dual of the second gradient of the displacement vector in elasticity or the second gradient of the velocity in fluids (or the gradient of the density in a "perfect" fluid). This was recognized early in the theory of surface tension by Korteweg [24] and much more recently by Casal [2]. In the case of elasticity, basing on a variational formulation, it was probably Le Roux [28] who first introduced the notion of second gradient theory remarking in the application that the interest for such a formulation appears only for problems with a spatially non-uniform strain. The Cauchy lemma is not used in these energy approaches. As a matter of fact, a formulation such as the principle of virtual power bypasses the Cauchy lemma. Indeed consider that the power of *internal forces* expended in a virtual velocity field $\hat{\mathbf{v}}$ is written as a linear form (for a whole body B)

$$\hat{P}_{int}(B) = \int_B \left(f_i \hat{v}_i + \sigma_{ij}\, \hat{v}_{i,j} + \cdots \right) dB. \tag{3.15}$$

In modern continuum mechanics it is assumed that internal forces for which one needs to construct constitutive equations must be objective (independent of the observer—contrary to externally applied forces and inertial forces that are not subjected to this constraint). So must be the case of their dual partners in (3.15). This rules out the term linear in the velocity itself and the term linear in the skew part of the gradient of the velocity. Being satisfied with a first-order gradient theory, this reduces (3.15) to the expression (here the minus sign is conventional)

$$\hat{P}_{int}(B) = -\int_B \left(\sigma_{ij} \hat{D}_{ji} \right) dB,\ \hat{D}_{ji} := \frac{1}{2}\left(\hat{v}_{j,i} + \hat{v}_{i,j} \right). \tag{3.16}$$

Thus tensor $\underline{\sigma}$ can only be symmetric in this case. In the absence of other internal force (e.g., due to electromagnetic effects), it can therefore be identified to the Cauchy stress when the formulation of the principle of virtual power accounts for the power expended by an applied traction at the regular boundary of B. Pursuing the expansion indicated by the ellipsis in (3.15) will allow one to introduce stresses of higher order, in particular the already mentioned *hyperstress*. We refer to Germain [20] and Maugin [31] for these extensions. This short exercise shows that a weak formulation like the principle of virtual power offers great advantages over the Cauchy type of approach, in particular to obtain a good set of natural boundary conditions at surfaces, corners and apices.

Hyperstresses of another type may be introduced to which a Cauchy type of argument applies. This is the case in media with so-called internal degrees of freedom where each material point, in addition to its translation, is equipped with an internal deformation (called micro-deformation) which in some cases is simply reduced to an internal rigid rotation. This is the case of so-called Cosserat continua. Indeed, the Cosserat brothers were led to consider the possible existence of *internal couples* [6]. They more or less were forced to do that by imposing an invariance (so-called *Euclidean invariance*) on a Lagrangian-Hamiltonian formulation, which invariance treats on an equal footing translations and rotations. This gave rise to the possible existence of a new type of internal force, the *couple stress*, along with that of stress, and as a consequence the possibility to have *non-symmetric* stresses (cf. Le Corre [27]). Such couple stresses also satisfy a lemma of the Cauchy type. But again, the principle of virtual power accounting for the presence of a new velocity field related to the micro-deformation is the most elegant and safe way to deduce all field equations and associated boundary conditions in such a theory.

Gradient theories with hyperstresses and Cosserat media now are part of *generalized continuum mechanics* of which the main characteristic property in effect is to deviate from Cauchy's 1822/1828 pioneering vision (cf. Maugin [33]).

3.5 Stress as a Secondary Notion

In the late 1820s, Cauchy [5], in competition with Navier and Poisson and with a view to envisage anisotropic bodies, decided to construct a linear elasticity theory using arguments involving a molecular picture with kind of interactions *à la* Newton between molecules. This CAUCHY-2 approach bypasses the basic notion of stress tensor. But the general form of the action (repulsion or attraction) of neighbouring molecules on a prototype one must be assumed. The reasoning then consists in making approximations in the infinite series of the involved finite differences to extract no nonsensical continuum equations. This can be achieved only by assuming a specific regular symmetry (a lattice) thus conducing to a generally anisotropic representation of the stress tensor by identification. This recovery of the notion of stress was proposed by Cauchy in his second paper of 1828. However, this does not provide the same number of elasticity coefficients as the first Cauchy's theory because of some constraints brought by the symmetries of molecular interactions (the famous Cauchy-Born relations). Cauchy was to apply his elasticity theory to specific mechanical elements (plates, rods) and to the theory of light propagation in a supposedly elastic medium serving as a support of light vibrations (the ill-fated "ether"). One of his successes was a theory of reflection and refraction at the boundary between two media. This second line fits well in the grand scheme to create a universal molecular physics by Laplace in the manner of Newton in point mechanics and by Ampère in electromagnetism.

Nowadays Cauchy's second theory is considered as obsolete while the first theory is the accepted one. But CAUCHY-2 germinally contains the modern theory of lattice dynamics as developed by Max Born and Theodor von Kármán in the early twentieth century and now a tenet of solid-state physics. This **identification** of a stress in a continuum limit of a particle-like theory is not proper to lattice dynamics. It appears where an internal force can be identified as the divergence of a second order tensor [cf. the second of (3.14)], which will then be called "stress". This is the case in the kinetic theory of fluids where after an appropriate expansion in terms of a small characteristic parameter, a continuum equation of linear momentum can be constructed in the series of moments deduced from the Boltzmann equation (see books on kinetic theory). But we must remember that there is no principle requiring the justification of continuum equations from a molecular description, as a logical continuum theory may be entirely autonomous. Nonetheless, the true physicist will be more than happy to be able to establish such a correspondence. The search for this identification is not vain; it progresses constantly and meets some success in a rigorous mathematical framework.[6]

Still another case where the notion of stress can only be secondary is that of materials of which the response exhibits a strong *nonlocal* nature. That is, in principle, the mechanical (or other) response at a material point depends on the values of independent variables (e.g., strains) at points at a far distance form this point, with a natural decrease of influence with increasing distance. This introduces in continuum mechanics a vision à la Newton-Laplace well illustrated by the book of Eringen [17]. Any cut in the material to apply Cauchy's argument would suppress the prevailing action at-a-distance. No wonder, therefore, that such models are usually first constructed in an infinite body and more than often justified by a lattice-dynamic theory with long-distance (with far-neighbour interaction) forces (See e.g. [25]).

From the above we see that there are cases where a computation from a molecular theory allows one to identify a stress tensor at a bulk point via an equation of the type of the second of (3.14), i.e.,

$$\mathbf{F}^{int} \equiv div\underline{\sigma}. \tag{3.17}$$

[6] This is beautifully demonstrated in the recent book of Murdoch [36] after the statistical-mechanics theory of liquids by John G. Kirkwood (1907–1959) where the liquids' properties are calculated in terms of the interactions between molecules.

This is true only modulo a divergence-free tensor. An example of this is the introduction of *Maxwell stresses* in electromagnetism. This can be first illustrated by a simple field theory in which the basic field equation is none other than the Gauss-Poisson equation for the potential ϕ and electric charge density q:

$$\nabla^2 \phi = -q. \tag{3.18}$$

In multiplying both sides of this equation by the vector $\nabla\phi$ and performing elementary manipulations, we are led to the equation of the electrostatic force acting on q as

$$\mathbf{F}^e = q\mathbf{E} \equiv div\underline{\sigma}^e, \quad \underline{\sigma}^e := \mathbf{E} \otimes \mathbf{E} - \frac{1}{2}\mathbf{E}^2\mathbf{1}, \tag{3.19}$$

where $\mathbf{E} = -\nabla\phi$ is a quasi-static electric field and the symmetric tensor $\underline{\sigma}^e$ may be called the Maxwell stress for such fields [35]. For a vanishing q in vacuum this short proof shows that a divergence-type of *conservation law* with vector components can be associated with the scalar Laplace equation [cf. (3.18)]—a fact more than often ignored, but intimately related to Noether's theorem when (3.18) is deduced from a variational principle (see below). In a general magnetized, electrically polarized and conducting continuum a rather long argument starting with the expression of the elementary force acting on electric charges in—relatively slow—motion (the celebrated Lorentz force) allows one to show that the corresponding "internal force" due to electromagnetic fields in a deformable continuum is formally given by an expression of the type

$$\mathbf{F}^{em} = div\,\underline{\sigma}^{em} - \frac{\partial}{\partial t}\mathbf{p}^{em}, \tag{3.20}$$

where the electromagnetic stress tensor $\underline{\sigma}^{em}$ is generally not symmetric and $\mathbf{p}^{em}$ is a linear electromagnetic momentum for dynamical fields. Expressions of these together with the accompanying energy expression can be found in Maugin [32, Chap. 3] after an evaluation made by Maugin and Collet in 1972 and Maugin and Eringen in 1977. In this approach both quantities $\underline{\sigma}^{em}$ and $\mathbf{p}^{em}$ appear as secondary notions. But the representation (3.20) is a patent mark of the ambiguity in interpretation carried by electromagnetic fields that can alternately be considered as giving rise to at-a-distance (*à la* Newton-Laplace) or contact (*à la* Euler-Cauchy) forces.

3.6 Stress as Part of Stress-Energy-Momentum

It is natural to turn next to a space-time formulation propounded by relativistic studies in the twentieth century. First, a naïve consideration puts us on the right track. For instance, [42, 43] introduced an object, now called the Piola-Kirchhoff stress, from the Cauchy stress by the transport/convection (or "pull back") definition (so-called Piola transformation)

$$\mathbf{T} = J_F \mathbf{F}^{-1} \underline{\sigma} \quad or \quad T^K_{.i} = J_F X^K_{,j} \sigma_{ji}, \tag{3.21}$$

where $\mathbf{F}^{-1} := \partial \mathbf{X}/\partial \mathbf{x} = \{\partial X^K/\partial x_j\}$ is the inverse deformation gradient, and $J_F := \det \mathbf{F}$, where $\mathbf{F}$ is the direct deformation gradient between a reference configuration K_R (with material coordinates X^K, $K = 1, 2, 3$). Equation (3.3) with $\mathbf{f} = \mathbf{0}$ is then shown to take the following mathematically strict conservation form:

$$\frac{\partial}{\partial t}(\rho_0 v_i) - \frac{\partial}{\partial X^K} T^K_{.i} = 0, \tag{3.22}$$

where $\rho_0 = \rho J_F$ is the matter density at K_R.

The object **T,** not a traditional tensor since having «feet» in two different spaces, stands for a force in the actual configuration K_t computed per unit area of the reference configuration. With (3.22) one is tempted to introduce a space-time parametrization ($X^\alpha = (X^K, X^4 = t)$) such that (3.22) reads equivalently

$$\frac{\partial}{\partial X^\alpha} T^\alpha_{.i} = 0, \quad T^\alpha_{.i} = \left(T^K_{.i}, T^4_{.i} = -\rho_0 v_i\right). \tag{3.23}$$

The first of these has the look of a true (divergence-like) conservation law but it is not really fully space-time in nature since its components still are in three-dimensional physical space. To reach a completely space-time equation one should unite (3.23) with the conservation of energy. As we know now, this was achieved in the first years of the 1900s with Minkoswki's four-dimensional formulation of special relativity. In modern terms this is introduced by noting $x^\alpha = (x^i, x^4 = ct)$ with a hyperbolic space-time metric $g_{\alpha\beta}$ of signature (+, +, +, −) and noting u^α, $\alpha = 1, 2, 3, 4$, the "world" velocity such that $g_{\alpha\beta} u^\alpha u^\beta + c^2 = 0$. Here c is the velocity of light in vacuum taken as a standard of velocity. A definite step forward was taken by Carl Eckart [15] in a paper that is a real pearl, when he proposed that for general continuous matter energy, momentum and stresses could be accommodated in a single notion, the *stress-energy-momentum space-time tensor* $T^{\alpha\beta}$ within a completely covariant format by using systematically the resolution of any space-time object into "proper" components. That is,

$$T^{\alpha\beta} = c^{-2} \omega u^\alpha u^\beta + c^{-2} u^\alpha q^\beta + p^\alpha u^\beta - t^{\beta\alpha} \tag{3.24}$$

where

$$\omega \equiv c^{-2} u_\alpha T^{\alpha\beta} u_\beta, \; q^\beta \equiv -u_\alpha T^{\alpha\gamma} P^\beta_{.\gamma}, \; p^\alpha \equiv -c^{-2} P^{\,\alpha}_{.\gamma} T^{\gamma\beta} u_\beta, \; t^{\beta\alpha} \equiv -P^\beta_{.\gamma} P^\alpha_{.\delta} T^{\delta\gamma}. \tag{3.25}$$

Here $P_{\alpha\beta}$ is the spatial projector such that

$$P_{\alpha\beta} := g_{\alpha\beta} + c^{-2} u_\alpha u_\beta = P_{\beta\alpha}, \quad u_\alpha P^{\,\alpha}_{.\beta} = 0, \quad P^{\,\alpha}_{.\beta} P^\beta_{.\gamma} = P^\alpha_{..\gamma}. \tag{3.26}$$

These are, respectively, a definition, an orthogonality property,[7] and the condition of idempotence. The four elements present in the canonical decomposition (3.25) are but "spatial" covariant forms of the energy density, energy (heat) flux, momentum density, and (Cauchy) stress. The identification of $t^{\beta\alpha}$ as the relativistic generalization of Cauchy's stress is shown by applying the projector $P^{\gamma}_{.\beta}$ to the general balance law (here a strict conservation law)

$$\frac{\partial}{\partial x^{\alpha}} T^{\alpha\beta} = 0, \tag{3.27}$$

in order to obtain its essentially spatial component, i.e., orthogonal to u^{γ} according to the second of (3.26). Now the identification with Cauchy's stress is not so obvious. The matter was pondered by scientists such as Van Dantzig [49] and Costa de Beauregard [7–9]. We refer the reader to these authors. Expression (3.24) does not generally imply that $t^{\beta\alpha}$ is symmetric. But we note that $T^{\alpha\beta}$ is a powerful generalized notion of rich contents compared to the simple stress notion.

Furthermore, in standard general relativity, *minimal coupling* requires to replace the partial derivative in (3.27) by a covariant derivative. That is,

$$\nabla_{\alpha} T^{\alpha\beta} = 0, \tag{3.28}$$

where ∇_{α} is computed in the space-time varying metric $g_{\alpha\beta}$ which is solution of the celebrated Einstein's gravitation equation

$$A_{\alpha\beta} := R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = \frac{8\pi k}{c^4} T_{\alpha\beta}, \tag{3.29}$$

where $R_{\alpha\beta}$ is the Ricci curvature, R is the scalar curvature of space-time, and k is Newton's gravitation constant. The right-hand side of (3.29) provides the source of energy and momentum (of various origins—including mechanical stresses—in particular from electromagnetism in magnetized and electrically polarized bodies; see [18], Vol. 2, Chap. 15); but note that the unknown $g_{\alpha\beta}$ itself is involved in $T_{\alpha\beta}$ so that only a laborious iteration procedure can help obtain, if ever, a solution of (3.29) for the metric. Equation (3.29) requires that $T_{\alpha\beta}$ be symmetric, since this is the case of the Einstein tensor $A_{\alpha\beta}$.[8]

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[7] As a young researcher I used to call "PU" tensorial objects those that are essentially space-like although written in full covariant form. They satisfy typical orthogonality conditions such as the second of (3.26). The hidden play of words was that PU = "Perpendicular to the world velocity **u**" = "Princeton University" for which the author has a definite affection. It is this property that allows for the identification of the space-time tensor $t_{\alpha\beta}$ with Cauchy's stress of classical continuum mechanics [cf. the last of Eq. (3.25)] [See [30], and papers published between 1971 and 1980 in C.R. Acad. Sci. Paris, Journal of Physics (UK), Ann. Inst. Henri Poincaré (Paris), Journal of Mathematical Physics (USA) and J. General Relativity and Gravitation].

[8] The history of the successive missed and successful steps in the production of Eq. (3.29) in the 1910s is a formidable scientific adventure involving, not only Einstein—as we could believe from modern hagiographic treatments—but also Marcel Grossmann, Max Abraham, Gustav Mie, David Hilbert and Emmy Noether, a story that remains to be fully investigated and understood [In particular,

3.7 The *Nec Plus Ultra*: The Eshelby Stress and the Pseudo Tensor of Energy-Momentum

Both Cauchy stress and the first Piola-Kirchhoff stress present the invaluable feature to have associated with them natural boundary conditions on the stress. This follows directly from Cauchy's fundamental lemma. But there are other stress tensors that are more directly related to the concepts of energy and energy-momentum and with which no direct simple boundary conditions are associated. These tensors are often deduced from the original Cauchy and Piola-Kirchhoff stresses via some manipulation. The first of these is the *second Piola-Kirchhoff stress* deduced from the first through the following definition (complete pull-back of the Cauchy stress to the reference configuration):

$$\mathbf{S} := \mathbf{T}.\mathbf{F}^{-T} i.e., \; S^{KL} = T^{K}_{.i} X^{L}_{,i} = J_F X^{K}_{,i} \sigma^{ij} X^{L}_{,j}. \tag{3.30}$$

The interest for this fully material stress tensor is its "energy" contents as, for Green's elasticity deduced from a potential function W per unit reference volume, it is shown that

$$\mathbf{S} = \frac{\partial \overline{W}}{\partial \mathbf{E}}, \quad W = \overline{W}(\mathbf{E}), \quad \mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{1}_R), \quad \mathbf{C} := \mathbf{F}^T\mathbf{F}. \tag{3.31}$$

But there is no direct meaningful boundary condition involving only $\mathbf{S}$ except in small strains where $\mathbf{S}$ readily reduces to the Cauchy stress.

The second stress in this class is the *material Eshelby stress* $\mathbf{b}$ [19] which can be introduced thus. Apply $F^{i}_{.L}$ to Eq. (3.22) and account for the fact that for an elastic body $W = \tilde{W}(\mathbf{F})$ and $\mathbf{T} = \partial \tilde{W}/\partial \mathbf{F}$. This manipulation provides a mathematically strict conservation law (for homogeneous bodies) in the form

$$\frac{\partial}{\partial t}\mathbf{P} - div_R\, \mathbf{b} = \mathbf{0}, \tag{3.32}$$

wherein we have set

$$\mathbf{P} = -\rho_0 \mathbf{v}.\mathbf{F}, \quad \mathbf{b} := -(L\mathbf{1}_R + \mathbf{T}.\mathbf{F})$$

with

$$L = \frac{1}{2}\rho_0 \mathbf{v}^2 - W. \tag{3.33}$$

(Footnote 8 continued)
were Einstein's equations first written down by Hilbert with the help of Noether since only Eq. (3.29)—with all terms present—could be in agreement with Noether's invariance theorem that associates a conservation laws with a "good" field equation in a variational treatment (see Sect. 3.7 about the Eshelby stress)? Indeed, while the general covariance of the basic Eqs. (3.29) and (3.28) is a tenet (see the discussion in Norton [41]), the Noetherian relationship between these two—field and conservation (in that order)—equations is an acknowledged requirement.

The last quantity is akin to a Lagrangian density. This hints at a possible derivation of **b** via a Lagrangian-Hamiltonian variational principle, in which, in effect, application of the Noether's [38] theorem for material space translations provides (3.32) automatically via "Noether's identity". The main interest in the conservation Eq. (3.32) is its role in capturing field singularities—for instance in the theory of fracture—since quantities such as the so-called material momentum (or pseudo-momentum) **P** and the Eshelby stress **b** are at least second order in the motion and the associated deformation. Tensor **b** can be rewritten in two alternate forms as

$$\mathbf{b} = -(L\mathbf{1}_R + \mathbf{C}.\mathbf{S}) = -L\mathbf{1}_R + \mathbf{F}^T.\frac{\partial L}{\partial \mathbf{F}}. \tag{3.34}$$

The first of these shows the relationship of **b** with the Mandel stress $\mathbf{M} := \mathbf{F}.\mathbf{T} = \mathbf{C}.\mathbf{S}$ which plays a definite role as driving force in many material structural rearrangements (e.g., in finite-strain plasticity, growth; see [29, 34]). The second of (3.34) exhibits the field-theoretic origin of the notion of Eshelby stress in pure elasticity. This was recognized early by J. D. Eshelby who called **b** the *energy-momentum tensor* of elasticity or *Maxwell stress* of elasticity. The last denomination holds good by analogy with a tensor such as in $(3.19)_2$. The first coinage is not entirely correct since **b** remains three-dimensional (essentially "spatial"—i.e., 3D—albeit "material") while the notion of energy-momentum (see above) requires a four-dimensional treatment. In spite of its usefulness demonstrated at length in our book [34], the nonsymmetric material tensor **b** is not associated with a physically obvious boundary condition.[9]

But the remark concerning the second of (3.34)—a canonical formula in analytical mechanics—naturally takes us back to a relativistic treatment such as in general relativity. It is not difficult to show that $(3.34)_2$ indeed is *minus* the purely space-like part of a four-dimensional energy-momentum tensor as deduced in a four-dimensional formulation.[10] Pondering now the case of Einstein's general relativistic theory of gravitation, we must realize, as emphasized by Landau and Lifshitz in their remarkable "Theory of fields" [26, Section 100] that the covariant form (3.28) does not in general express a conservation law of any truly meaningful physical quantity. The reason for this is that, on account of the expression of the covariant divergence in terms of the usual divergence (3.28) reads [here $g = \det\,(g_{\alpha\beta})$]

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[9] Some authors (e.g., [23]) have proposed to consider Eq. (3.32) as autonomous being posited—for any material behaviour—as a general balance law—a "new" equation of physics—by some kind of trick involving a boundary flux of energy together with stresses. The artificiality of this type of reasoning as well as the erroneous concept of the novelty of (3.32) in physics is shown in the Appendix A5.2 of our book [34]. Furthermore, we have also shown that an equation such as (3.32) with a possibly nonvanishing right-hand side could be established without a variational formulation at hand and no application of any Noether theorem (Chap. 5 in Maugin [34])—but with a mimicking of Noether's identity. This fortifies the view of the secondary nature of stresses such as **M** or **b** compared to the Cauchy stress.

[10] For this see Eq. (4.26) in Maugin [34] and select the ϕ^α there as the three components of the direct motion $\mathbf{x} = \bar{\mathbf{x}}(\mathbf{X}, t)$ between the reference (material) configuration and the actual (physical, i.e., Eulerian) one.

$$\nabla_\alpha T^{\alpha}_{.\beta} = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\alpha}\left(\sqrt{-g}T^{\alpha}_{.\beta}\right) - \frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\beta}T^{\mu\nu} = 0. \tag{3.35}$$

The quantity which must be conserved is the 4D (four-dimensional) momentum of matter *plus* gravitational field. But the latter is not included in $T^{\alpha}_{.\beta}$ by its very definition. In other words, this 4D momentum must be the *canonical four-momentum* associated with the *whole* physical system. This means that the corresponding conservation law must present a flux ("stress-energy-momentum tensor") that includes a term that accounts for the gravitational effect. A possible solution is given by Landau and Lifshitz [26] where the additional contribution to the energy-momentum tensor is called the *pseudo-energy momentum tensor of gravitation*. We denote by $G^{\alpha}_{.\beta}$ this new object so that the looked for local conservation law should read

$$\frac{\partial}{\partial x^\alpha}\left[\sqrt{-g}\left(T^{\alpha}_{.\beta} + G^{\alpha}_{.\beta}\right)\right] = 0 \tag{3.36}$$

Accordingly, the four-momentum defined by

$$P^\alpha = \frac{1}{c}\int \sqrt{-g}\left(T^{\alpha\beta}_{.} + G^{\alpha\beta}_{.}\right)dA_\beta \tag{3.37}$$

will be conserved, where dA_β is a space-time surface element and the integration is taken over any infinite space-time hypersurface that contains the whole of three-dimensional space. Here $G^{\alpha\beta}$ is symmetric although not a true tensor (hence the denomination of *pseudo*-tensor). The Landau-Lifshitz definition of $G^{\alpha\beta}$ may be given as

$$G^{\alpha\beta} = \frac{1}{2(-g)}\frac{c^4}{8\pi k}\frac{\partial^2}{\partial x^\mu \partial x^\nu}\left((-g)\left(g^{\alpha\beta}g^{\mu\nu} - g^{\alpha\mu}g^{\beta\nu}\right)\right). \tag{3.38}$$

As usual with his characteristic economy of words and formulas and his dedication to the beauty of the said equations, ([13], Eq. (32.5))[11] Paul Dirac establishes a formula for $G^{\alpha}_{.\beta}$ by applying Noether's theorem to the Lagrangian density L_g of the gravitational field with

$$G^{\alpha}_{.\beta}\sqrt{-g} = \frac{\partial L_g}{\partial g_{\gamma\delta,\alpha}}g_{\gamma\delta,\beta} - L_g g^{\alpha}_{.\beta}. \tag{3.39}$$

One obtains thus

$$2\left(8\pi k/c^4\right)G^{\alpha}_{.\beta}\sqrt{-g} = \left(\Gamma^{\alpha}_{\mu\nu} - g^{\alpha}_{\nu}\Gamma^{\sigma}_{\mu\sigma}\right)\left(g^{\mu\nu}\sqrt{-g}\right)_{,\beta} - L_g g^{\alpha}_{\beta}, \tag{3.40}$$

where the Γ'*s* are Christoffel's symbols. This is Einstein's [16] proposal that may have been inspired by contemporary works by David Hilbert and Emmy Noether.

[11] See also the problem proposed in Landau and Lifshitz [26] at the end of their Section 100.

Contrary to the Landau-Lifshitz definition, the Einstein-Dirac pseudo-tensor $G^{\alpha\beta}$ is not symmetric. Of course (3.39) reminds us of the second of formulas (3.34) since these are canonical definitions in field theory.

In practice, the spatial part of (3.37) is obtained by considering the hypersurface $x^4 = const.$ so that we have the 3D space integral

$$P^i = \frac{1}{c}\int \sqrt{-g}\left(T^{i4} + G^{i4}\right)dV. \tag{3.41}$$

In the absence of gravitational field, this establishes a correspondence with the balance of a canonical momentum obtained in Eq. (3.32). This closes our disquisition about the notion of stress started with Cauchy's 1822/1828 pioneering work.

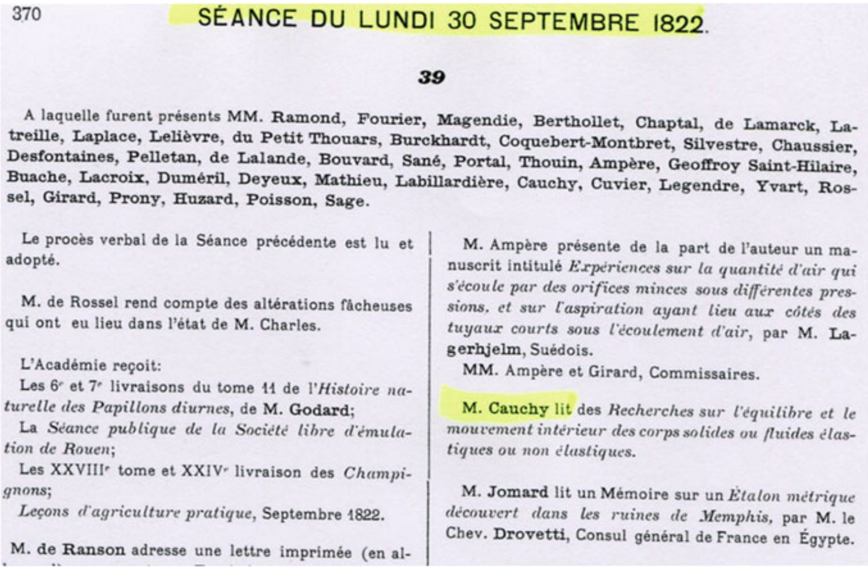
3.8 Conclusion

As witnessed by the above given exposition there is a long way between Cauchy's inception of the stress concept and Einstein-Dirac's pseudo tensor of stress-energy-momentum. We have explored this evolution in a rather pedestrian manner. What fundamentally remains from this is the essential role played by stress tensors or energy-momentum tensors that appear as true fluxes so that a corresponding physically meaningful conserved quantity (momentum) can be constructed. This is satisfied by Cauchy's initial construct of the stress because it provides at once the associated natural mechanical boundary conditions. This also holds for the first Piola-Kirchhoff stress, but not for derived definitions such as those of the second Piola-Kirchhoff stress, Mandel stress and Eshelby's stress in classical continuum mechanics. It is this kind of physical-theoretical argument which materialized in the introduction of the pseudo-tensor of energy-momentum in Einstein's gravitation theory as shown in the foregoing section. Having recurrently emphasized the role of Noether's theorem [37, 38], we also note that Cauchy's original proposal of 1822 and Green's divergence theorem (possibly generalized to space-time) accordingly remain the two basic tenets of continuum theory in spite of all progress achieved since that innocuous—but memorable for our community—day of September 30, 1822. We can say that 1828, with the inception of Green's divergence theorem and Cauchy's detailed presentation of his lemma, was a true *annus mirabilis* for continuum mechanics, providing thus the *fons et origo* of this science.

Acknowledgments Heartful thanks go to Dr Martine Rousseau in Paris and Professor James Casey in Berkeley for their critical careful reading of this contribution that led to much improvement and readability. Mme Florence Greffe ("Conservateur du Patrimoine") from the Archives Library of the Paris Academy of Science is to be thanked for her definite help in providing the "birth certificate" of continuum mechanics.

Appendix A

Augustin L. Cauchy (1789–1857)



370 SÉANCE DU LUNDI 30 SEPTEMBRE 1822.

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A laquelle furent présents MM. Ramond, Fourier, Magendie, Berthollet, Chaptal, de Lamarck, Latreille, Laplace, Lelièvre, du Petit Thouars, Burckhardt, Coquebert-Montbret, Silvestre, Chaussier, Desfontaines, Pelletan, de Lalande, Bouvard, Sané, Portal, Thouin, Ampère, Geoffroy Saint-Hilaire, Buache, Lacroix, Duméril, Deyeux, Mathieu, Labillardière, Cauchy, Cuvier, Legendre, Yvart, Rossel, Girard, Prony, Huzard, Poisson, Sage.

Le procès verbal de la Séance précédente est lu et adopté.

M. de Rossel rend compte des altérations fâcheuses qui ont eu lieu dans l'état de M. Charles.

L'Académie reçoit:
Les 6^e^ et 7^e^ livraisons du tome 11 de l'*Histoire naturelle des Papillons diurnes*, de M. Godard;
La *Séance publique de la Société libre d'émulation de Rouen*;
Les XXVIII^e^ tome et XXIV^e^ livraison des *Champignons*;
Leçons d'agriculture pratique, Septembre 1822.

M. de Ranson adresse une lettre imprimée (en al-

M. Ampère présente de la part de l'auteur un manuscrit intitulé *Expériences sur la quantité d'air qui s'écoule par des orifices minces sous différentes pressions, et sur l'aspiration ayant lieu aux côtés des tuyaux courts sous l'écoulement d'air*, par M. Lagerhjelm, Suédois.
MM. Ampère et Girard, Commissaires.

M. Cauchy lit des *Recherches sur l'équilibre et le mouvement intérieur des corps solides ou fluides élastiques ou non élastiques*.

M. Jomard lit un Mémoire sur un *Étalon métrique découvert dans les ruines de Memphis*, par M. le Chev. Drovetti, Consul général de France en Égypte.

The "birth certificate of modern continuum mechanics": Cauchy's reading of his ideas at the September 30, 1822 session of the *Académie des Sciences* in Paris (Procès verbal de l'Académie des Sciences, Tome VII, Décembre 1822, Imprimerie d'Abbadia, Hendaye, 1916; kindly provided by Mrs Florence Greffe, Acad. Sc. Paris; May 2013) The remarkable roster of scientists among the above list of attending academicians is stupendous: e.g., Fourier, Magendie, Berthollet, Chaptal, Lamark, Laplace, Lacroix, Cauchy, Cuvier, Legendre, Prony, Poisson.

Appendix B

A. L. Cauchy—1823: Researches on the equilibrium and internal motion of solid bodies or fluids, whether elastic or non-elastic.

Bulletin of the Société Philomatique, pp. 9–13, 1823, Paris.

(Translation from the French by G.A. Maugin)

The present researches were undertaken on the occasion of the publication of a memoir by M. Navier on August 14, 1820. Its author, with a view to establishing the equilibrium equation of an elastic plane, had considered two kinds of forces, some produced by dilatation or contraction, and the other by the flexion of this plane. Moreover, he had supposed, in his computations, that both these forces are perpendicular to lines or faces on which they are exerted. It seemed to me that these two kinds of forces could be reduced to one kind only, which should be always called tension or pressure, and is of the same nature as the hydrostatic pressure exerted by a fluid at rest on the surface of a solid body. However, the new "pressure" will not always be perpendicular to the faces on which it act, and is not the same in all directions at a given point. Expanding this idea, I arrived soon at the following conclusions.

If in a solid body, whether elastic or not elastic, we succeed to render rigid—in thought. [GAM]—and invariable a small volume element bounded by any surfaces, this small element will be subjected on its different faces and in any point of each of these, to a determined pressure or tension. This pressure or tension will be of the same type as the pressure that a fluid exerts on an element of the boundary of a solid body, save for the difference that the pressure exerted by a fluid at rest on the surface of a solid body is directed normal to this surface, from the outside to the inside, and is independent at each point of the orientation of the surface with respect to the coordinate planes, while—in our case [GAM]—the pressure or tension exerted at a given point can be oriented perpendicularly or obliquely to this surface, sometimes from the outside to the inside if there is condensation [i.e., contraction, GAM] and sometimes from the inside to the outside if there is dilatation, and it can depend on the angle made by the surface with the relevant planes. Furthermore, the pressure or tension exerted on any plane can easily be deduced, in both amplitude and direction, from the pressures or tensions exerted on three given orthogonal planes. I had reached this point when M. Fresnel, who came to me to talk about his works devoted to the study of light and which he had presented only in part to the Institute, told me that, on his own, he had obtained laws in which elasticity varies according to the various directions issued from a unique point, a theorem similar to mine. However, the theorem in question was far from being sufficient for my projected object of study, at that period, that was to formulate the general equations of equilibrium and internal motion of a body; and it is only in recent times that I succeeded to establish the proper new principles that yielded this result, and that, now, I will make known.

From the above mentioned theorem, it follows that the pressure or tension at each point is equivalent to the inverse of the vector radius of an ellipsoid. Three pressures or tensions that we call *principal* correspond to the three axes of this ellipsoid, and we can show [This remark here is in agreement with the last researches of M. Fresnel (See the Bulletin of May 1822)] that each of these is perpendicular to the plane on which its acts. Among these principal pressures or tensions there are a maximum pressure or tension and a minimum one. The other pressures or tensions are distributed symmetrically about these three axes. Moreover, the pressure or tension normal to each plane, i.e., the component, perpendicular to a plane, of the pressure or tension exerted on this plane, is proportional to the inverse of the squared vector radius of a second ellipsoid. Sometimes, this second ellipsoid is replaced by two hyperboloids, one with one sheet, the other with two sheets, which have the same centre, the same axes, and are asymptotic at infinity with a common second- degree surface, of which the edges point in the direction for which pressure or normal tension reduces to zero.

This being said, if we consider a solid body of varying shape and subjected to arbitrary accelerating forces, in order to establish the equilibrium equations of this solid body it will be sufficient to write that there is equilibrium between the motive forces that act on an infinitesimal element along three axes of coordinates, and the orthogonal components of external pressure or tension that act on the faces of this element. We will thus obtain three equations of equilibrium that include, as a particular case, the corresponding equations for the equilibrium of fluids. But, in a general case, these equations contain six unknown functions of the coordinates x, y, z. It remains to determine the value of these six unknown quantities. But the solution of this last problem varies with the nature of the body and its more or less perfect elasticity. Now we shall explain how one can solve this problem for elastic bodies.

When an elastic body is in equilibrium by virtue of arbitrary accelerating forces, one must assume that each molecule has been displaced from the position it occupied when the body was in its natural state. As a consequence of these displacements, there are around each point different condensations or dilatations in different directions. But it is clear that each dilatation produces a tension, and each condensation produces a pressure. Furthermore, I prove that the various condensation or dilatation about this point, decreased by or augmented of the unit, become equal, up to the sign, to the vector radii of an ellipsoid. I call *principal condensations or dilatations* those that occur along the axes of this ellipsoid, about which the others are distributed symmetrically. This being set, it is clear that in an elastic body, tensions or pressures depending only on the condensations or dilatations, are directed in the same directions as the principal condensations or dilatations. In addition, it is natural to assume, at least when the displacements of molecules are small, that the principal tensions or pressures are proportional to the principal condensations and dilatations, respectively. Admitting this principle, we arrive immediately at the equilibrium equations of an elastic body. In the case of very small displacements, the component, perpendicular to a plane, of the pressure or tension exerted on that plane, always is in the same ratio with the condensation or dilatation that occurs in the same direction, and the formulas for equilibrium reduce to four partial differential equations of which each one determine separately the condensation or dilatation in volume, while each of the others serves to fix the displacement parallel to one of the coordinate axes.

The equations of equilibrium of an elastic body being set, it is now easy to deduce by ordinary means the equations of motion. The latter still are four in number, and each of them is a linear partial differential equation with an added variable term. These equations are integrated by use of methods that I exposed in a previous memoir. One of these equations contains only the unknown that represents the condensation or dilatation in volume. In the particular case where the acceleration force becomes constant and keeps everywhere the same direction, this equation reduces to the propagation of sound in air, with the only difference is that the constant it contains, instead of depending on the height of a supposedly homogeneous atmosphere, depends on the linear dilatation or condensation of a body in a given pressure. One must conclude from this that the speed of sound in an elastic body is constant, like in air, but it varies from one body to another one depending on the matter of which it is made. This constancy is all the more remarkable that the displacements of molecules considered successively in fluids and elastic solids obey different laws.

My memoir is concluded by the formation of the equations of the internal motion of solid bodies completely devoid of elasticity. To arrive at this it is sufficient to suppose that in these bodies the pressures or tensions about a point in motion do not depend any more on the total condensations or dilatations that correspond to the absolute displacement measured from the initial positions of the molecules, but only, after any lapse of time, on the very small condensations or dilatations that correspond to the respective displacement of the different points during a short interval of time. One therefore finds that the volume condensation is determined by an equation similar to that governing heat, what establishes a remarkable analogy between the propagation of the caloric [the supposed “fluid” carrying heat. GAM] and the vibrations of a body entirely devoid of elasticity.

In a forthcoming memoir, I shall give the application of the obtained formulas to the theory of elastic plates and strings.

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