Chapter 2
A Glimpse at the Eighteenth Century: From John Bernoulli to Lagrange

Abstract This essay tries to answer the following question: "What happened to Mechanics between Newton and Lagrange?", hence during what is commonly called the century of the enlightenment or *Age of Reason*. This period where knowledge and learning are the main incentive for intellectuals—of which philosophers and scientists are most representative—, witnessed an evolution that went from the mechanics of point particles (or massive objects seen as such) with Newton to the first elements of continuum mechanics in the hands of Euler and Lagrange. The most famous contributors between these scientists are Jacques (Jacob) Bernoulli, John (Johann) Bernoulli, Daniel Bernoulli, Pierre Varignon, Jean Le Rond d'Alembert, Pierre de Maupertuis, and Leonhard Euler. We peruse the works and the contributions to the formulation and consolidation of various principles by these "mechanicians" on the basis of primary sources—sometimes with original English translations—and accounting for comments by Lagrange and more recent historians–mechanicians such as Jouguet and Truesdell. This period is marked by a great emphasis placed on the notion of living forces, the exploitation of the principle of virtual work, and the expansion of the calculus of variations, which all characterize the "continental" development of the principles of mechanics as opposed to the Newtonian vision. This essay is a prerequisite to the examination in other essays of all what was to be expanded in the thermomechanics of continua during the nineteenth century.

2.1 John Bernoulli and the Principle of Virtual Work

In his discussion on the role of mathematics in science, Truesdell [40, p. 99] asks the following question: "Before 1788 (cf. Lagrange [31]) and after 1687 (cf. Newton) something had happened to mechanics. What was it?" This in fact is the subject matter of the present essay. Lagrange [31, 2nd part of his historical introduction] declares that all really started in this period with the work of John Bernoulli (1667–1748) and his apprehending of the principle of virtual work. John (Jean or

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Johann) Bernoulli is the most remarkable member of the Bernoulli family (or dynasty) [4, 22]. He provides a bridge between the Newtonian period of the ending seventeenth century and the new developments that we are going to discuss here. To be fair, however, we must first acknowledge the contribution of his elder brother James (Jacques or Jacob) Bernoulli (1655–1705) who may have been less creative than John, but nonetheless played an essential role in the dissemination of integral calculus (he coined the word "integral"), in the establishment of the theory of probabilities, and in solving critical problems in mechanics (the isochronous curve, the elastica).

John, with all his bad temper and his unhealthy jealousy, was nonetheless an inspiring mentor. He was instrumental in the adoption of Leibniz's successful differential notation for differential calculus on the continent instead of Newton's fluxions. This, unfortunately, created a kind of dichotomy between British and continental developments in mathematics and applications, that was resolved only in the nineteenth century with the adoption of Leibniz's notation by Cambridgians under the influence of French pedagogues. So we can say that John Bernoulli was a "Leibnizian" as opposed to a "Newtonian". In spite of his strabismus toward Newton, Truesdell had to recognize the mathematical genius and creativity of John (cf. [39, 40]). He had an important epistolary exchange with French mathematicians, e.g., Marquis de l'Hôpital (1661–1704)[1] and Pierre Varignon (1654–1722). He defended in 1694 a doctoral thesis in medicine, but this may have been a première in biomechanics since in it John discussed the movement of muscles.

Main discoveries in mathematics by John Bernoulli are: the exponential calculus, trigonometry treated as a branch of analysis, the study of geodesics, the celebrated solution of the brachistochrone (the catenary), an introduction of the treatment of minima and a foundation for the calculus of variations (to inspire Euler), and more important from the viewpoint of this essay, the enunciation of the principle of virtual work (cf. [5, 6]). This matter is thoroughly discussed by Capecchi [11, pp. 199–209] that we do not need to repeat in detail. In this work John was a Leibnizian, reformulating Leibniz's notion of dead force by introducing the elementary increase $f\,dx$, where dx is an infinitesimal virtual displacement of the point of application of the force f in its direction, so that we would write this as the inner product $\mathbf{f} \cdot d\mathbf{x}$ in modern jargon. It is here an

[1] Both de l'Hôpital and Varignon are known French mathematicians, the first one having left his name attached to the famous "rule of l'Hôpital" and the second having created the "law of the parallelogram of forces" in statics, much before the notion of vectors was invented. John Bernoulli taught the first one on calculus (against a good stipend) by visiting him in France and then continuing by correspondence. De l'Hôpital published a book based on the lessons he had received from Bernoulli. This was the natural cause of a dispute for which Bernoulli had a special talent, but here he was probably right. In the case of Varignon, met in Paris in 1692, the mutual friendship seems to have been sincere, in spite of the somewhat touchy character of Bernoulli who did not hesitate to rebuke his friend on occasions. We must also note that Bernoulli had a fruitful correspondence with the great Leibniz. This was the most major correspondence which Leibniz ever carried out. Decidedly, Bernoulli was a great letter-writer like many scientists and intellectuals of the time.

infinitesimal pulse that defines the new quantity which is an infinitesimal energy increase or *mechanical work* in the future work of Coriolis, more than a hundred years later—here we should not oversee that eighteenth century scientists had no notion of what we call mechanical work. Living forces ("vis viva" $= mv^2$) in the sense of Leibniz are the results of the summation of this work over elementary pulses in time, so that, noting that if $f = p = mv$, then $pdx = mv \cdot vdt = mv^2 dt$ we can write an expression of the type

$$\int p\,dx \propto mv^2. \tag{2.1}$$

Depending on the observer this can be viewed as a mathematical theorem or a principle of conservation. According to Capecchi [11, p. 201] it is practically universally acknowledged that what we now call the principle of virtual work finds its origin in John's considerations. This is what Lagrange was referring to as the *principle of virtual velocities* in 1788 [29–31]. As a matter of fact, Bernoulli refers to a law of virtual work in a letter of 1714 to a naval engineer named Bernard Renau d'Elizaray (Capecchi [11], p. 203), and he defines well his notion of virtual velocities in exchanges of letters with Varignon in the period 1714–1715, and in his discourse "on the laws of the communication of motion" of 1728 (Chap. 3, p. 20) reproduced in his collected works [8, vol 3]: "The virtual velocity is the element of velocity that every body gains or loses, of a velocity already acquired during an infinitesimal interval of time, according to its direction" (English translation from the French by Capecchi).

We shall return to John Bernoulli's works in fluid mechanics in the next two sections mostly devoted to his son, Daniel.

2.2 "Bernoulli's Theorem" by Daniel Bernoulli

Daniel Bernoulli (1700–1782) is the second son of John. He is universally known for his "theorem" although he did many other works in hydrodynamics, mathematics, statistics and physics. He studied in Basel, Heidelberg and Strasbourg. Not so strangely for the time, he obtained a doctoral degree in anatomy and botany (1721). He spent most of his professional career in St. Petersburg (for nine years) and Basel (for almost fifty years). He was a close friend of Euler.

Some unavoidable words must be said about this famous theorem because it relies on consequences of the *conservation of energy*. It happens that we know a letter from Daniel Bernoulli to Christian Goldbach (1690–1764) dated from Moscow July 17, 1730 (reproduced in pp. 220–221 in Truedsell [39]), where Daniel develops in French the arguments leading to his "theorem". He appeals there to the conservation of energy to show that there exists a relation (his notation; this looks like a differential equation but it is not)

$$v \frac{dv}{dx} = \frac{a - v^2}{2c}, \tag{2.2}$$

where $v_0 = \sqrt{a}$ is a reference velocity, the acceleration of gravity is equal to 1/2 in the chosen system of units, and the quantity $cvdv/dx$ is the accelerating force (i.e., an internal pressure per unit density). This is established by relating the speed of a steady flow of an incompressible fluid (water) in a tube to the pressure this fluid exerts on its walls. The increment dv is the impulsive increment of speed of the water flowing with speed v in traversing the elementary distance dx as "if the tube, supposed horizontal, were suddenly to dissolve into thin air at point x" (Truesdell's words). In modern notation this can be transcribed as the well known equation

$$p + \frac{1}{2}\rho\, v^2 + \rho\, g\, h = const., \tag{2.3}$$

thanks to the mentioned identification with internal pressure (a notion that Daniel Bernoulli did not possess). There is no need to emphasize the importance of Eq. (2.3) in hydrodynamics (think of the Venturi effect: when velocity of the flow increases, then the pressure falls) and aerodynamics where the theorem is used in the proof of the existence of lift on an airfoil. This is beautifully exposed by Anderson [2]. This was not the only result of Daniel who produced a fundamental book on hydrodynamics written in 1734 [9]. This scientific achievement caused a burst of envy and jealousy from the shameless John Bernoulli against his own son, publishing a competitor book with the title *Hydraulica* in 1739. He anti-dated the date of writing this opus to 1732 to pretend to (a false) priority! However, it must also be acknowledged that John's book has many merits. In particular, this book offers the first successful use of the *balance of forces* to determine the motion of a deformable body. This was possible for John because he had recognized that "the fluid on each side of an infinitesimal slice pressed normally upon that slice, with a varying force which was itself a major unknown" [39, p. 121]. With this we are very close to the notion of *internal pressure* and a concrete view of *contiguity* of action in continuum mechanics in a line that both Euler in the period 1749–1752 and Cauchy in the period 1823–1828 will expand. Also, John was the first to practically give the modern form (2.3) to his son's theorem. Bernoulli [10] also discussed the principle of living forces at the time of the death of his father. To be on the safe side, the paranoid John edited himself his collected works in 1742 [8].

2.3 D'Alembert and the Metaphysical Notion of Force

Daniel Bernoulli's work also triggered some envy from a young French philosopher-mathematician, Jean Le Rond d'Alembert (1717–1783). Thus in 1744, this gentleman, well educated in the best college in Paris, but mostly self-taught in mathematics, published his own book on the emerging fluid mechanics [16].

According to Truesdell [39, p. 227], this entry of a newcomer in the field "added nothing to the subject".[2]

But several objective facts must be recorded. In contributions that rapidly followed this opus, d'Alembert (e.g., [17]) achieved correct partial differential equations for axially symmetric and plane flows (of the type now called irrotational flows). This is one of the first considerations of a two-dimensional motion of a continuum. He had already introduced for the first time the notion of *partial differential equations* in a previous work of 1743 on the mechanics of a heavy hanging rope. As noticed by Truesdell [39, p. 228], d'Alembert does not speak of "pressure" but of "forces" that are viewed as "reversed accelerations". This fits well in d'Alembert's vision of reducing hydrodynamics to hydrostatics in accord with the general approach to mechanics that he introduced in his celebrated *Treatise on dynamics* [15] written when he was hardly 26. This treatise is a much

[2] Because Euler was his great hero, Truesdell in all his writings (e.g., [39, 40]) has a tendency to belittle the contributions of d'Alembert on whatever subject, sometimes with innuendos verging on defamation. This kind of *idée fixe* is not supported by many other writers, in spite of the admitted unclear, confusing, and somewhat verbose, and probably too hastily written, works of d'Alembert. But he often pioneered new ideas and wrote these works when he was only 26 or 27 years old, and without much formal education in the field. Furthermore, d'Alembert himself had a tendency to grant a higher value to his contributions to the belles-lettres, among the latter, discourses [19] at the *Académie Française* in his mature age when he has become "perpetual" secretary of this academy. His preliminary discourse (i.e., Introduction) of 1751 to the great Encyclopaedia is a remarkable literary piece. Naturally, the French have a tendency to overestimate d'Alembert. I remember receiving as a prize, when in primary school, a book written at the end of the nineteenth century, the purpose of which was to introduce the youth to the oeuvres of the most representative French scientists (they preferably had to be republican and atheist). This included Lagrange, Lazare Carnot, Chevreul, obviously Pasteur, Marcellin Berthelot, and d'Alembert as a first exemplary contributor. My favourites were Pasteur (he saved young kids from the rabies—I did not know he had also improved the production of beer) and the chemist Chevreul (the latter for three reasons: he died when he was 103; he was born in my native city, and he brought light to the house of many people by inventing the artificial stearic candle). I could not understand what was said of d'Alembert. But many years later I discovered that I was scientifically descending in a straight line from d'Alembert according to Mathematics Genealogy. Moreover, in response to the granting of the name of Diderot to our twin University (Paris VII) I decided to call d'Alembert our Institute of Mechanics, Acoustics and Energetics at the University of Paris VI—that had gained the good name of Pierre and Marie Curie in the mean time. Very little is known about d'Alembert's formation in mathematics. From available documents (in particular, one written by d'Alembert late in his life, his personal notes preserved at the Academy of Sciences in Paris, and studies by specialists of Diderot and d'Alembert such as Pfeiffer [38], it is thought that d'Alembert had only a limited formal education—given in Latin by a rather incompetent teacher - in mathematics while an adolescent at Collège Mazarin (also called "Collège des Quatre Nations"). He was caught by an irrepressible taste for mathematics while he was studying law and then started aborted studies in medicine. It is surmised that he studied by himself L'Hôpital, Varignon, John Bernoulli, Maupertuis, and the Principia of Newton (but this is mostly geometrical while d'Alembert will become an analyst). He thus submitted his first (unpublished) memoir on mathematics to the Academy in 1739 (aged 21), and then published in a row famous lengthy works on the bases of dynamics (his famous treatise at age 26), fluid mechanics, the theory of winds, and the vibrations of strings in an interval of some seven years! Not bad for an autodidact, notwithstanding Truesdell's appraisal.

discussed matter, and obviously demolished by Truesdell as incomprehensible by him (probably himself a too much Newtonian adept to start with). It is of interest to note the full (ambitious) title and sort of summary of the treatise (translation from the French): “Treatise on dynamics, in which the laws of equilibrium and motion of bodies are reduced to the smallest number and are proved in a new way, and where a general principle for finding the motion of several bodies which react mutually in any way” is given. Not a bad abstract!

It is true that, following Leibniz, d’Alembert thinks of the notion of “force” (especially in Newton’s gravitation theory) as an obscure, metaphysical and unnecessary primary notion. Thus all is first to be granted to kinematics. With this he is banishing entirely the Newtonian view of mechanics (even the name of Newton is not mentioned). This agrees well with John Bernoulli’s introduction of the principle of virtual work, where forces are nothing but coefficients of virtual variations. But what d’Alembert added was a re-interpretation of inertial forces as the negative of “forces”, thus giving to dynamics the form of statics on the basis of a principle of virtual “powers”. The problem with this author is that, as correctly remarked by Truesdell, he is extremely difficult to read. We thus admire the thorough analysis and deep interpretation that Jouguet [28] could give of the contents of the Treatise on dynamics. To give a taste to the reader we give in Appendix A attached to this essay an English translation of some parts of d’Alembert’s introduction to his treatise. This text shows the central thinking, methodology, and ambition—not always truly satisfied in the treatise—of the project. Anyway, we should remember d’Alembert’s principle according to which: “If we consider a system of material points connected together so that their masses can acquire different respective velocities whenever they move freely or altogether, the quantities of motion gained or lost in the system are equal”. A flavour of d’Alembert’s statement of his principle is given in Appendix B. It is clear that the modern interpretation of such a text is a challenge for most of us. This is nonetheless what is achieved by Jouguet [28, pp. 197–202]. With this d’Alembert provided the bases on which Lagrange was going to build his grand scheme of mechanics. He had also an interest in the theory of music. Perhaps that he was distracted by, interested in, too many fields to compete creatively with his contemporary Euler in mechanics.

Note that the famous *d’Alembert paradox* about the vanishing dragging force on a cylinder placed in a flowing perfect fluid was proved in 1750. This is contrary to common experience. A resolution could be given only with the introduction of discontinuities in the flow field and the notion of wake.

In solid mechanics, d’Alembert also introduced the notion of space-time partial differential equation yielding the wave equation and its paradigmatic solution in 1746. In his later years he published in the form of contributions to some collected works (e.g., [18, 20]). He had much influence on Lagrange whom he mentored in Paris; he obviously discussed with Euler as they often disagreed on many particular points. But, contrary to John Bernoulli, he always remained a gentleman in spite of controversies for which he seems to have developed a special gift for getting involved in. His two main disciples were Pierre-Simon Laplace in analysis and probabilities and Nicolas de Condorcet in economics and statistics. In mathematics he is known for the «theorem of d'Alembert» that says that "any polynomial of degree n with complex coefficients has exactly n (not necessarily distinct) roots in the set of complex numbers" (the theorem was really proved by C.F. Gauss in the nineteenth century) and for his study of the convergence of numerical series. In astronomy, he studied the three-body problem and the precession of equinoxes in 1749. In this he is a precursor of his disciple Laplace.

2.4 The Notion of Internal Pressure and the Fundamental Equations of Hydrodynamics

We certainly agree with Truesdell that Leonhard Euler (1707–1783), the greatest mathematician of the eighteenth century, stands much above d'Alembert in both mathematical creativity and physical intuition. This Euler proved in practice by developing expertly the calculus of variations, solving so many problems, and presenting a theory of fluids that remains intact till our present time, at least for perfect fluids. We cannot peruse the whole work of Euler in the mechanics of continua. This has been achieved by more knowledgeable specialists (among them Truesdell who edited many of the original works in the edition by Orell Füssli Verlag, Basel, and Birkäuser, Basel, in a total of 73 volumes). We are satisfied with a focus on some problems of fluid and solid mechanics.

For us the main two ingredients in fluid mechanics are the notion of *internal pressure*, and the construction of the *field equations* for perfect fluids on the understanding that the notion of *field* has really been introduced. This is indeed the case. For the first ingredient, we may conjecture with Truesdell [39, p. 230] that Euler, while residing at the Berlin Academy since 1741 to become later on its president after the death of Maupertuis, carefully read the prize essays submitted by d'Alembert in 1746 and 1750. This "gave him the final impulse to the creation of the general hydrostatics and hydrodynamics". This was also much influenced by the recent progress in the theory of hydraulics by the Bernoullis, father and son. Thus *pressure* was seen as the action from all sides and from neighbouring elements of fluid on an isolated element of fluid (a "particle"). In modern term, it is *isotropic* and, with Euler, will be viewed as a normal force acting on an element of surface. The notion of *contiguity* is thus definitely reached. Furthermore, it becomes a true *field* that depends on both space and time in the general case of dynamics.

The second argument requires from Euler to think in Newtonian terms to write in 1750 a general *principle of linear momentum* (for whatever body or ensemble of "particles" and not only for a point particle like in Newton), principle that we write here in the condensed form

$$\mathbf{F}(B) = \dot{\mathbf{M}}(B) \tag{2.4}$$

for any portion B of a body, where **F** is the resultant force, **M** is the total linear momentum, and a superimposed dot denotes the time-rate of change. This was expressed by Euler in differential form, a form that suited the mechanics of continua. Now, we cannot do better than reproduce the original text of Euler (Paragraphs XV and XVI of [25], in the old French orthography, but read just like actual French once the writing conventions are known):

> XV. Maintenant je po*ſ*e pour abréger (there were misprints, here corrected, in two of these component equations. GAM):
>
> $$X = \left(\frac{du}{dt}\right) + u\left(\frac{du}{dx}\right) + v\left(\frac{du}{dy}\right) + w\left(\frac{du}{dz}\right); Y = \ldots; Z = \ldots, \tag{a}$$
>
> & l'équation différentielle qui détermine la pre*ſſ*ion p e*ſ*t
>
> $$\frac{dp}{q} = P\,dx + Q\,dy + R\,dz - X\,dx - Y\,dy - Z\,dz\,, \tag{b}$$
>
> dans laquelle le tems r est *ſ*uppo*ſé* con*ſ*tant. Or l'autre équation tirée de la continuité du fluide est:
>
> $$\left(\frac{dq}{dt}\right) + \left(\frac{d.qu}{dx}\right) + \left(\frac{d.qv}{dy}\right) + \left(\frac{d.qw}{dz}\right) = 0, \tag{c}$$
>
> & ce *ſ*ont les deux équations qui contiennent toute la Théorie tant de l'équilibre que du mouvement des fluides, dans la plus grande univer*ſ*alité qu'on pui*ſſ*e imaginer.
> XVI. Lorsqu'il e*ſ*t question de l'équilibre, on n'a qu'à faire évanouïr les trois *viteſſes* u, v & w, & puisque alors les quantités X, Y, & Z, évanouï*ſſ*ent aussi, toute la Théorie de l'équilibre des fluides est contenuë dans ces deux équations:
>
> $$\frac{dp}{q} = P\,dx + Q\,dy + Rdz, \tag{d}$$
>
> le tems t étant con*ſ*tant, &
>
> $$\left(\frac{dq}{dt}\right) = 0. \tag{e}$$

Today's student easily identifies q with the density ρ, (X, Y, Z) as the components of the acceleration and (P, Q, R) as the components of the body force per unit mass, so that Eqs. (a) through (d) are none other than

$$\gamma = \frac{d\mathbf{v}}{dt} := \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}, \tag{a$'$}$$

$$\rho\,\frac{d\mathbf{v}}{dt} = \rho\mathbf{f} - \nabla p, \tag{b$'$}$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0, \tag{c$'$}$$

and

$$\nabla p = \rho \mathbf{f}, \tag{d$'$}$$

in modern intrinsic notation. This says it all. But what about boundary conditions required by this set of equations for complete solution in space? Euler had already pondered this matter before and the only possibility is that pressure corresponds to a *normal* force acting on a surface. We will need the ingenious work of Cauchy in the period 1822–1828 to offer a larger possibility, although the solid case to be discussed also after Euler already hints at a more general situation.

This discussion on fluids may make the reader believe that Euler had no notion of a tangential force. But this is not true because Euler himself dealt with this notion in a problem that had been examined by James (Jacob) Bernoulli a long time before, the elastica (flexible elastic one-dimensional object). On considering the equilibrium of a cut out part of this elastica Euler found that a *shear force* is generally necessary in addition to tension to maintain the balance of this element. He obtained in 1771 the correct governing system of dynamical equations that require considering not only curvature χ (as done by Bernoulli) but simultaneously both normal V and tangential T components of the stress in the form (where s is the arc length)

$$\frac{dT}{ds} - V\chi = -B_t + \rho \ddot{x}_t, \quad \frac{dV}{ds} + T\chi = -B_n + \rho \ddot{x}_n, \quad \frac{dM}{ds} - V = 0. \tag{2.5}$$

Here M is the bending moment and the B's are the components of an applied (body) force. In modern intrinsic notation (2.5) reads (cf. [3])

$$\frac{d\mathbf{S}}{ds} + \mathbf{B} = \rho\, \ddot{\mathbf{x}} \tag{2.5$'$}$$

with a stress "vector" defined by $\mathbf{S} = T\mathbf{t} + V\mathbf{n}$. More remarkably, Euler in 1774 also proposed for this problem a principle of contiguity (contact action), which we can express in this notation as

$$\mathbf{S}_+ = -\mathbf{S}_-, \tag{2.6}$$

where the plus and minus signs refer to the effect of the material on the opposite sides at the point of junction. This obviously is equivalent to the natural boundary condition (here a junction or continuity condition). Equations (2.5$'$) and (2.6) anticipate the equations of Cauchy for both the field and (natural) boundary equations. The engineer Charles Augustin de Coulomb (1736–1806) also recognized the notion of shear stress at about the same time as Euler by examining the effect of "both normal and tangential forces acting on the cross section of a beam subject to transverse terminal load" (cf. [39, p. 236]). Anyhow, the reasoning of Euler yielded what is now known as the Euler-Bernoulli theory of beams with a bending moment given by $M = -\,EId^2w/dx^2$, while plane sections remain plane and there is no shear deformation.

2.5 Linear Momentum and Moment of Momentum: Newtonian Versus Variational Formulations

Equation (2.4) reflects the adoption by Euler of Newton's viewpoint concerning the law of linear momentum. But for a rigid continuous body or a system of rigidly linked point particles (with invariant distances between them), one needs to account for a possible mechanical response in rotation. This is a materialization of Newton's third law ("to each action there is always an equal reaction"). It is Euler [24] who formulated this law of *moment of momentum*, which dynamically involves the notion of *angular momentum* along with the notion of inertia about a certain centre of mass. That is, in addition to a law of the form (2.4) we will have to satisfy a law

$$\mathbf{C} = \dot{\mathbf{J}}, \tag{2.7}$$

where $\mathbf{C}$ is the total torque acting upon the body and $\mathbf{J}$ is the total moment of momentum or angular momentum, both being taken with respect to the same fixed point. Both laws (2.4) and (2.7) are valid for discrete systems or continuous bodies. They constitute the *laws of Mechanics of Euler*; he correctly set forth these as applicable to any part of every body in a memoir published in 1776 [26]. It is the evaluation of $\mathbf{J}$ which requires the introduction of the notion of rotary inertia about the mentioned fixed point. According to Truesdell [39, p. 129], Euler's principle of moment of momentum remains even today a "subtle and often misunderstood" (by physicists, not by Truesdell!) law. However it is of universal and everyday use (think of the orientation of satellites seen as rigid bodies in the first approximation, and the application of this law in the mechanics of robots with the introduction of appropriate kinematic descriptors including, beyond Euler's angles, Cayley-Klein parameters, quaternions, orthogonal matrices, and spinors). This Eulerian mechanics of rigid bodies will be perfected to the utmost by scientists such as Lagrange, Poinsot, Poisson and others. We do not deal further with this matter but note that the symmetry of the Cauchy stress in most of modern continuum mechanics is a consequence of the law (2.7) [41].

We cannot close this perusal of Euler's formidable contributions to mechanics without evoking the fact that, albeit a strict Newtonian from many viewpoints [cf. Eq. (2.4) above, and also (2.7) that complements the original Newtonian view], Euler is also one of the true creators of the calculus of variations, which he never hesitated to use in specific problems (e.g., in the buckling problem). Because of this, he is at the root of the variational approach of Lagrange (see next section).

2.6 Calculus of Variations and Analytical Mechanics: Lagrange

Joseph-Louis Lagrange or Giuseppe Ludovico Lagrangia (1736–1813) is neither Newtonian nor Leibnizian or d'Alembertian; he is above all—if we are allowed the joke—Italian (and perhaps Frenchman by adoption), and also a shy and very quiet man who, contrary to some of his colleagues, succeeded to live through the French revolution without any political involvement. He disliked getting involved in controversies and, according to J.-B. Fourier (of series and heat-conduction fame)—a demanding student in the first year of the *Ecole Normale*—, he always answered by a non-committed "je ne sais pas" (I don't know) to all questions asked by students. But he is only second to Euler in the class of mathematicians of the eighteenth century. His creativity blossomed in all fields of mathematics, mechanics of fluids, solids, and celestial mechanics. Apart from C. F. Gauss (1777–1855), only Cauchy may match his inventiveness in mathematics in the successive Revolution, Empire and Restauration periods (1780–1830). Because of limited space, here we can only focus on some of his contributions to mechanics. His canonical equations of motion in arbitrary systems of coordinates are so beautiful and powerful in all of physics[3] that it is often believed that they were God-given like the Holy Scriptures (but with publication via Lagrange's hands of the celebrated book of 1788 [31]). The real story deviates from this ideal vision in the sense that Lagrange was strongly influenced by, among others, John Bernoulli, Maupertuis, d'Alembert, and Euler. In modern terms, Lagrange's works do not create a new paradigm (in the sense of Thomas Kuhn), neither do they provoke an epistemological rupture (according to the expression of Gaston Bachelard). Furthermore, contrary to the principle of virtual velocities by d'Alembert, his famous equations are restricted to the case of non-dissipative processes (cf. the discussion in [34]) at least until the introduction of a dissipation potential by Lord Rayleigh.

The genesis of Lagrange's equations requires some re-construction, which was more or less told by Lagrange himself in the long historical introduction to the two parts of his book. This greatly simplifies our task. In fact, Lagrange dutifully produces in his introduction a rather extended history of the developments of mechanics through the ages, starting with Archimedes, Stevin, Galilei, Descartes, Huyghens, and Roberval, of course on the basis of what was known in his time (the mechanics of the middle ages will be unearthed and thoroughly examined by Pierre Duhem only at the end of the nineteenth century). Because of the

[3] Some physicists place Lagrange equations simultaneously at the top and the base of all of mechanics; This is the case in the first volume of the celebrated course of theoretical physics by Lev D. Landau and Evgeni F. Lifshitz in the former Soviet Union—with many foreign translations—and the standard, much admired, successful and continuously reprinted book on "Classical Mechanics" by Herbert Goldstein in the USA—first edition 1950—, but unjustly criticized by Truesdell [40, pp. 144–147] as not Newtonian enough. James C. Maxwell was a great admirer of Lagrange and did not hesitate to use Lagrange's formalism to study self and mutual inductance in his treatise on electricity ad magnetism.

composition of his book, Lagrange considers separately the cases of statics and dynamics. Concerning the first half of the eighteenth century, he has to pay his tribute to the works of John Bernoulli, Maupertuis, and d'Alembert. We give in Appendix C the English translation of what we think to be the most important statements of this introduction, which provides a magisterial overview and analysis of the principles of mechanics in the eighteenth century.

In the case of statics, Lagrange emphasizes the role played by the "*principle of the lever*" (in some sense, an ancestor of the principle of virtual displacement) and that of the *composition of forces* ("from which one concludes that any two powers (he means "forces") that act simultaneously on a body (Lagrange means a "point"), are equivalent (equipollent) to one force that is represented, in magnitude (Lagrange uses the word "quantity") and direction, by the diagonal of the parallelogram of which the sides represent the magnitude and direction of the two given powers" [31, p. 10]). Anyway, Lagrange clearly concludes that the principle of virtual work as given by John Bernoulli is the most powerful tool. At the end of his study of statics, he develops the hydrostatics of incompressible fluids.

In the case of dynamics, Lagrange thoroughly scrutinizes the various principles proposed since Newton, avoiding none (see Appendix C). In practice he will combine the principle of virtual work with d'Alembert's astute proposal to view inertial forces as negative applied forces, i.e., a kind of reformulation of Newton's law appropriately multiplied by virtual velocities and summed over all bodies composing the system. The step that will glorify this work among physicists is the consideration of a kinematical description by means of generalized systems of coordinates (see Fourth Section, p. 282 on, in which we witness for the first time the appearance of the functional derivative—in page 285). There follows from this the introduction of the quantity $T - V$, where T is the kinetic energy and V is the potential of interacting forces (this will later be called a Lagrangian L) and, using an argument on the homogeneity of functions (due to Euler), he proves the conservation of the integral $T + V$, which contains the principle of living forces. By invoking the calculus of variations he further shows the validity of Maupertuis' principle of least action. In a sense, with this work Lagrange has unified in a construct typical of an analyst all what concerns the mechanics of systems of points in the absence of dissipative processes. He also provides interesting applications to the oscillations of a linear system of bodies (pp. 320–380).

Lagrange died in 1813, but he had already prepared an extension of his *Analytical mechanics* in a second volume. This was completed (from Lagrange's papers) and edited by J. Binet (1786–1856), G. Prony (1755–1839) and S. F. Lacroix (1765–1843). This is reproduced in Lagrange [33] as reprinted from the third and fourth editions with comments and additions by Joseph Bertrand (1822–1900) and Gaston Darboux (1842–1917)—with more than sixty pages of notes by V. Puiseux (1820–1883), J.-A. Serret (1819–1885), O. Bonnet (1819–1892), J. Bertrand, A. Bravais (1811–1863), and Lagrange himself. What is of highest interest for us here is, apart from many solutions and applications to rotational motions and celestial mechanics, the development of Lagrange's view of the fluid mechanics of incompressible and compressible fluids (pp. 250–312). Readers will not be surprised that

Lagrange adopts here what will become known as the "Lagrangian" kinematical description (p. 253 on). That is, he introduces the initial coordinates (a, b, c) of a fluid "particle" to be later in time at placement (x, y, z). Thus,

$$x = \bar{x}(a, b, c; t), \text{etc.}$$

He thus write the continuity equation as (in modern notation)

$$dm = \rho \, dx \, dy \, dz = const. \quad \text{or} \quad \rho_0 = \rho J.$$

He introduces a scalar "Lagrange" multiplier to account for incompressibility. He obtains thus the three equations of balance of linear momentum equations in the "Lagrangian" format. But he also shows how to revert to an Eulerian description (Equation F in p. 264). He is quite honest in admitting that it may be easier to deal with Euler's format (p. 265). He does the same for compressible fluids where pressure will now be a constitutive quantity.[4] He concludes this second volume with simple wave problems in one dimension (e.g., in a flute or an organ pipe).

In this book and previous works (e.g., in Mémoires sur le calcul des variations, Torino, 1760), Lagrange greatly contributes to the definite form of what may be called the δ-calculus, that is, the calculus of variations. As already mentioned, this was initiated by Euler in the period 1755–1760 in his study of maxima and minima. This author even introduced what is now called the *Euler-Lagrange equations*, of which the above recalled Lagrange equations are a special case. Dahan-Dalmenico [14] has critically examined this contribution of Lagrange to one of the most useful and efficient tools in theoretical physics.

It is hard not to express admiration in view of Lagrange's book. This is a true monument that is beautifully organized and practically happily concludes the development of the principles of mechanics through the eighteenth century. Lagrange has a style of his own, being fully analytical, somewhat formal (even to the taste of Truedsell [39, p. 132]) and using no argument of geometry. He is even proud of the fact that no figures illustrate his exposition (although a few illustrations may have been welcomed). Lagrange in fact introduced a privileged tool for the "algebraïzation" of Mechanics. The second important point is that Lagrange, after some previous works by Clairaut [13] and Maupertuis [37] but above all with his introduction of generalized coordinates, really inaugurates an era where the recognition of *invariance* in mathematical physics has become fundamental. No wonder that perhaps with some exaggeration W. R. Hamilton called the *Mécanique Analytique* a "kind of scientific poem".

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[4] In the period 1825–1848, Gabrio Piola (1794–1850), an ardent supporter of his "compatriot" Lagrange, will follow the same line of approaching first the equations of motion for finitely deformable bodies (elasticity) in a reference configuration and then transforming them to the actual configuration for the sake of comparison with Cauchy's format of 1828. To do that Piola had to use what is now called an inverse "Piola transformation" of the stress (see [35]). From this we infer that Lagrange had already used a similar transformation (in a rather unidentified manner) for perfect fluids.

2.7 The Age of Reason: Conclusion and Things to Come

The period we spanned in this essay practically corresponds to what is called in history the *Age of Enlightenment* or the *Age of reason* (in French, the "*Siècle des lumières*", in German, the "*Aufklärung*"). What is usually meant by this denomination is a period in which one thinks of reforming society by using reason and to advance knowledge through the scientific method. Scientific thought is promoted together with a challenge of ideas that are too much grounded in tradition and faith. One of our heroes in this essay, d'Alembert, epitomizes the enlightened scientist who simultaneously wants to improve society and teach it through a pharaonic enterprise such as the production of the great "Encyclopédie" (ou *Dictionnaire raisonné des sciences, des arts et des métiers*) directed by him and Denis Diderot (1713–1784), and sold (by subscription) all over Europe in about 20,000 copies in spite of the bulk of thirty-five volumes. About a hundred "philosophers" (today we would say "intellectuals" of all kinds) contributed to this formidable enterprise, including Voltaire (1694–1778), J.-J. Rousseau (1712–1778), and Montesquieu (1689–1755). Many of the contributions on scientific subjects are due to d'Alembert himself.

The enlightenment influenced both American and French revolutions and inspired among others the American Declaration of Independence and the French Declaration of the Rights of Man and of the Citizen. Baruch Spinoza (1632–1677; a philosopher much appreciated by scientists such as Albert Einstein) and John Locke (1632–1704) were inspirations for this movement that spread all over Europe and European colonies in the Americas. In Germany, Immanuel Kant in 1784 tried to answer the question: "Was ist Aufklärung?" A partial answer is "Sapere aude" (dare to know). It has mostly to do with the advance of knowledge in all forms. From the scientific viewpoint, Newton may have sparked the original steps of the movement in the early 1700s. It is symptomatic that we technically concluded the present essay with the publication of Lagrange's "Analytic Mechanics" in 1788, just one year before the French revolution (perhaps not always the best realization of the Enlightenment). We can now summarize the achievements in mechanics in this remarkably active period with the following list:

- the formulation of integral calculus (introduction of the term "integral"; Jacob Bernoulli)
- the principle of virtual work (John Bernoulli, Lagrange [32])
- the parallelogram of forces (Varignon)
- Bernoulli's theorem (Daniel Bernoulli in 1730 [9])
- the general equations of hydraulics [7]
- the concept of shear stress (John Bernoulli, Euler)
- the principle of least action [36, 37]
- two-dimensional motion/partial differential equations [16]
- the wave equation (d'Alembert)
- d'Alembert's principle of virtual velocities [15, 20, 32]
- the notion of internal pressure as a field (D'Alembert; Euler [25])

- the fundamental equations of hydrodynamics [25]
- the principle of linear momentum [24]
- the equations of motion of rigid bodies (Euler)
- the principle of moment of momentum [24]
- the calculus of variations (John Bernoulli [6], Euler [23], Lagrange [29, 30])
- analytic mechanics [31].

In the transition period[5] of the French revolution, Lazare Carnot (1753–1823), both a successful politician (the “organizer” of the victory in 1792 as a kind of Minister of War and scientific adviser to the Convention) and an engineer-scientist by formation at the Military Engineering school of Mézières (the ancestor of the Ecole Polytechnique) pondered the principles of mechanics with a specific interest in their applications to mechanical “machines” (cf. [12]). Carnot was essentially a disciple of d’Alembert since in his book of 1803 (originally published 1783, p. 47) he wrote about “a metaphysical and obscure notion, that of force”. He simply states the following alternative [12, Introduction]: “There are two ways to envisage Mechanics, in its principles. The first one is to consider it as the *theory of forces*, i.e., the causes that impress the motions. The second one is to consider it as the *theory of movements* in themselves. Carnot prefers the second avenue. A thorough examination of the principles as enunciated by Carnot is given by Jouguet [28, pp. 72–77, 203–210]. Since Carnot’s essay was originally published in 1783, we may consider that its contents somewhat anticipated Lagrange, but certainly not with the same acuity and success. Jouguet [28, pp. 203–210] also discussed the presentation of the principles of mechanics by Fourier [27] in his “Mémoire sur la statique…”. Other scientists who pondered the principle of virtual velocities in the early nineteenth century are André-Marie Ampère (1775–1836) and Louis Poinsot (1777–1859) both in 1806 (cf. [1]).

The things to come in the nineteenth century were:

- the general notion of stress (Cauchy in the period 1822–1828) to be perfected by Piola, Kirchhoff and Boussinesq
- nonlinear deformations (Cauchy, Green, Piola, Kirchhoff, Boussinesq)
- the notion of mechanical work (Coriolis)
- the notion of thermo-mechanical couplings (Duhamel, 1837; F. Neumann)
- Heat conduction (Fourier)
- the creation of thermo-statics and thermo-dynamics (Sadi-Carnot, Kelvin, Clausius, Mayer, Helmholtz)
- the mathematics of elasticity (Lamé, Clebsch, Saint-Venant, Boussinesq, Love, the Cosserats)
- the equations for viscous fluids (Navier, Saint-Venant, Stokes)
- criteria of plasticity (Tresca, Lévy, Saint-Venant, 1870s)

[5] This is a transition period because it witnessed the creation of a new way to teach engineering sciences in the class room in combination with solid mathematical bases in the “*Grandes Ecoles*” of which the *Ecole Polytechnique* is the paragon (creation 1794).

- the anisotropy of deformable solids (Duhamel, F. Neumann, Voigt)
- visco-elasticity (Kelvin, Maxwell, Voigt, Boltzmann)
- the science of energetics (Rankine, Duhem, Mach)
- the notion of internal degrees of freedom (Duhem, the Cosserats)

All these are the objects of study of the remaining essays in this book.

Appendix A

Partial English Translation of J. Le Rond d'Alembert, (1743), Traité de dynamique, 1st Edition, David, Paris (1743) [after the reprint of the second augmented edition of 1758 by Gauthier-Villars Publishers, in two volumes, Paris, 1921] by Gérard Maugin. Translator's remarks are placed within square brackets in the main text. This is a verbatim translation without any ambition of literary prowess.

From the preliminary discourse

PP XVIII – XIX [on motion and extension]

....

The motion and its general properties are the first and principal object of mechanics; this science supposes the existence of motion, and we shall assume it as well advocated and recognized by all physicists. Concerning the nature of motion, philosophers, on the contrary, are much more divided on its definition. I admit that nothing seems more natural than to conceive of the motion as the successive application of the mobile to the different parts of infinite space, that we imagine as the locus of bodies [Here d'Alembert thinks essentially of point particles]; but this idea supposes a space of which parts are impenetrable and immobile, but everybody knows that the Cartesians (a sect which, indeed, does not exist any more), do not dissociate space from bodies and that they regard extension and matter as a unique thing. We must admit that starting from such a principle, motion would be the most difficult thing to conceive, and that a Cartesian would soon better come to negate its existence than to try to define its nature. Finally, how much absurd this opinion of philosophers may look, and with so little clarity and precision there are in the metaphysical principle on which they made the effort to lean, we will not try to refute it here; we shall be satisfied with remarking that, to have a clear idea of motion, we cannot avoid distinguishing, at least in thought, two kinds of extension: one, that must be regarded as impenetrable and that properly constitute the bodies; the other which, simply considered as extension, without examining on whether it is penetrable or not, provides the distance from one body to another one, and the parts of which can be considered as fixed and immobile, can serve to judge of the rest or motion of bodies. It is therefore always allowed to us to conceive of an infinite space as the locus of bodies, either real or hypothesized, and to regard motion as the transport of a mobile from one place to another one in space.

…………

PP XXVI – XXVII [on principles and the primary role of motion]

…………

If the principles of the force of inertia, of the compounded [or "composed"] motion and of equilibrium differ essentially from one another, as we cannot avoid to agree, and from another side these three principles suffice to mechanics, then this is to have reduced this science to the smallest possible number of principles that to have built on these principles all the laws of motion of bodies in any circumstances, as I tried to do in this treatise.

Concerning the proofs of these principles in themselves, the scheme I have followed to grant them all the clarity and simplicity that they likely deserve, has always been to deduce them from the consideration of the motion alone [this indicates the marked preference of d'Alembert for kinematics over the notion of forces], viewed in the simplest and clearest manner. All that we perceive distinctly well in the motion of a body is that it traverses a certain space and it takes a certain time to achieve this. This, therefore, is the only idea from which we must extract all principles of mechanics, when we want to prove them in a neat and precise manner; thus we shall not be surprised that, as a consequence of this way of thinking, I have, so to say, distracted the view from the *motive causes* [author's emphasis] to uniquely envisage the motion that produced them; that I have proscribed the forces inherent in the body, obscure and metaphysical beings, that are capable only to dispense darkness on a science so clear by itself.

It is for this reason that I thought appropriate not to enter the examination of the famous question of living forces.

……………

PP XXIX- XXX [on equilibrium, virtual velocities, and *vis viva*]

……………

But everybody agrees on the fact there is equilibrium between two bodies when the product of their masses by their virtual velocities, that is, the velocities with which they have a tendency to move, are equal from both sides. Thus, at equilibrium the product of mass by velocity, or what is equivalent, the quantity of motion, can represent the force. All agree also on the fact that in a retarded motion the number of overcome obstacles is like the square of the velocity, so that a body that closed a spring with a certain velocity will, for example, close four springs like the first one with a doubled velocity….From this the partisans of the living forces [*vis viva*] conclude that the force of bodies that actually move is generally the product of mass by the square of the velocity.

…………….

PP XXXVII – XXXIX [on the contents of the treatise]

…………….

Having given to the reader a general idea of the object of this work, it remains to say a word about the form that I thought appropriate to give to it. I have tried in the first part, as much as possible, to express the principles of mechanics in a form accessible to the people of the trade; but I could not avoid the use of differential calculus in the theory of varying motions [i.e., non-uniform motions]; I am forced to do that by the nature of the subject. Moreover, I have done in such a way as to encapsulate [obviously not a term in use at the time of d'Alembert] in this first part a sufficiently large number of things in a rather little space, and if I did dot enter any detail in the relevant matter, it is only because I remained focused on the exposition and development of the essential principles of mechanics, and having for purpose to reduce this work to what new ingredients it can contain, I did not thought appropriate to enlarge it with an infinity of particular propositions that can easily be found elsewhere.

The second part, in which I propose to treat of the laws of motion between bodies, is the largest part of the work. That is what led me to give to this book the title of "Traité de dynamique". This name, that properly signifies the science of powers or motive causes, could at first seem inadequate since I envisage Mechanics rather like the science of effects than that of causes [this is the most important point for d'Alembert; exit the notion of force to start with]; nonetheless as the word "*Dynamics*" is very much used among scientists to designate the science of the motion of bodies, which act among themselves in whatever way, I thought to keep the term, in order to announce to geometers by the very title of this treatise that I envisage principally to aim at perfecting and enlarging this part of Mechanics. As it is no less curious than difficult, and relevant problems compose a very large class, the most famous geometers have particularly dealt with it for the last few years; but they succeeded until now to solve only a small number of problems of this class, and only in particular cases. Most of the solutions they provided to us rest on principles that nobody has ever proved in a general manner, such as, for example, that of the living forces. I therefore thought to spend some time on the subject, and show how to solve the questions of dynamics by a unique very simple and direct method which consists only in the already above mentioned combination of the principle of equilibrium and of the compounded motion. I show the exploitation of this in a small number of selected problems, some already known, and others entirely new, and finally others even badly solved by the best mathematicians [notice that d'Alembert is rather avaricious with citations; he does not give names].

The elegance of a solution to a problem consisting above all in exploiting only a few direct principles, one should not be surprised that the uniformity that prevails in all my solutions, and that I have principally in view, renders them a little longer than if I had deduced them from less direct principles. The proof that I would have been obliged to give of these principles would indeed have distracted me from the brevity that I had searched by using them, and the largest part of the present book would have reduced to a shapeless accumulation of problems that did not deserve to see light, in spite of the variety that I tried to expand and the difficulties that accompany each of them.

Also, as this second part is mostly aimed at those who, already learned in differential and integral calculus, have become familiar with the principles established in the first part, or are already familiar with solutions of problems known and ordinary in Mechanics, I must tell that, in order to avoid circumlocutions, I have often used the obscure term of *force* [that the author strongly dislikes], and other terms that are commonly used when treating the motion of bodies; but I never pretended to attach to these terms more ideas than those which result from the principles that I proved, either in the [introductory] Discourse or in the first part of this treatise.

Finally, from the same principle that leads me to the solution of all problems of Dynamics I deduce also several properties of the centre of gravity, some of them entirely new, the others having only been proved in a vague and obscure manner, and I conclude this book with a proof of the principle that is commonly called the *conservation of living forces* [the theorem of the kinetic energy].

Appendix B

English Translation of the "principle of d'Alembert" as given by him in his Traité de dynamique, 1st Edition, David, Paris (1743) [after the reprint of the second augmented edition of 1758 by Gauthier-Villars Publishers, in two volumes, Paris, 1921, pp. 82-83] by Gérard Maugin. Translator's remarks are placed within square brackets in the main text. This is a verbatim translation without any ambition of literary prowess.

Definition. — In what follows I shall call motion of a body [here he means a point] the velocity of this same body considered on account of its direction, and call quantity of motion, in the ordinary sense, the product of mass by velocity.

General problem. — Let a system of bodies be disposed in any manner with respect to one another, and let us suppose that we impose to each of these bodies a particular motion that it cannot follow because of the action exerted by other bodies: find out the motion that each body will take.

Solution. — Let A, B, C, &c. the bodies that compose the system, and suppose that we imposed motions [here d'Alembert means velocities] a, b, c, &c that are forced, because of the mutual action, to change in the motions **a**, **b**, **c**, &c. It is clear that we can regard the motion a impressed on body A as composed of the motion **a** that it took and another motion α; that we can equally regard the motions b, c, &c. as composed of motions, **b**, β, **c**, γ, &c, from which there follows that the motion of the bodies A, B, C, &c., among them would have been the same if, instead of giving impulses a, b, c, &c., one would have given simultaneously the double impulsions **a**, α; **b**, β; **c**, γ; &c. But, by superimposition, the bodies A, B, C,

&c. have by themselves taken the motions **a**, **b**, **c**, &c. Therefore, the motions α, β, γ, … must be such that they do not disturb the motions **a**, **b**, **c**,…, i.e., if the bodies have received only the motions α, β, γ, &c. these motions should have destroyed themselves mutually and the system would have remained at rest.

From this there results the following principle to find out the motion of several bodies that interact among themselves: *Decompose the motions a, b, c,* &c *impressed on each body, each in two other motions* **a**, α; **b**, β; **c**, γ; &c *that are such that that if we had imposed to the bodies only motions* **a**, **b**, **c**, &c, *they could have conserved their motion without reciprocal hindrance and that we had impressed only motions* α, β, γ, &c. *the system would have remained at rest; it is clear that* **a**, **b**, **c**, &*c are the motions that the bodies will take by virtue of their action*. Q. E. D.

Appendix C

Partial English translation of J.-L. Lagrange (1788), Méchanique analitique, 1st Edition, Veuve Desaint, Paris (1788) [after the reprint of the second revised edition of 1811/1815 with new title “Mécanique analytique” in two volumes, Veuve Courcier, Paris; reprinted by Gabay, Paris, 1997, from the fourth edition edited by J. Bertand and G. Darboux] by Gérard Maugin. Translator’s remarks are placed within square brackets in the main text. This is a verbatim translation without any ambition of literary prowess.

P 10 [on the principle of composition of forces]

………

The second fundamental principle of statics is that of the composition of forces. It is based on the supposition: that if two forces act simultaneously on a body [Lagrange means a “point”], along different directions, then these forces are equivalent to a unique force capable of impressing to the body the same motion as the two forces would give him separately. But a body that we make move uniformly along two different directions simultaneously necessarily takes a path that is along the diagonal of the parallelogram of which it would have followed separately the two sides by virtue of each of the motions. From this one concludes that any two powers [he means “forces”] that act simultaneously on a body [Lagrange means a “point”], are equivalent to one force that is represented, in magnitude [Lagrange uses the word “quantity”] and direction, by the diagonal of the parallelogram of which the sides represent the magnitude and direction of the two given powers”. This is in what the principle called the *composition of forces* consists.

This principle suffices by itself to determine the laws of equilibrium in all cases; because, by composing all forces two by two, we must arrive at a unique force that is equivalent to all forces, and this, consequently, must vanish in the case of equilibrium, if there is no fixed point in the system. But if there is such a point, the direction of this unique force must go through this fixed point. This is what we can see in all books on statics, and particularly in the "Nouvelle Mécanique" by Varignon, where the theory of machines [e.g., pulleys] is deduced uniquely from the principle we just spoke about.

....

PP17-18 [on the principle of virtual velocities]

....

Now I come to the third principle, that of virtual velocities. By "*virtual velocity*" one must understand the velocity that a body in equilibrium is ready to receive, whenever equilibrium is broken, that is, the velocity that the body would really take in the first instant of its motion, and the considered principle consists in that the powers [he means "forces"] are in equilibrium when they are as the inverse ratio of their virtual velocities, estimated along the direction of these powers.

If we examine the conditions of equilibrium in the lever and other machines, it is easy to recognize this law that the weight and the power are always in the inverse ratio of the space that each of these can travel in the same time: however, it seems that the ancients were not aware of this.

....

PP 20-21 [on the principle of virtual velocities cont'd]

The principle of virtual velocities can be rendered quite general in the following manner:

If any system of bodies or points as we want, is acted upon by any system of powers, is in equilibrium, and we give to this system any small motion, then by virtue of the fact that each point travels an infinitesimally small space that expresses its virtual velocity, the sum over powers each multiplied by the space that the point where it is applied travels along the direction of the same power, will always be equal to zero, regarding as positive the small distances followed in the direction of the powers and as negative those travelled in the opposite direction [Emphasis by Lagrange].

In so far as we know, John Bernoulli was the first to have perceived this great generality of the principle of virtual velocities, and its usefulness to solve problems of statics. This is what we witness in his letters to Varignon, dated 1717, and that the latter placed at the head of New Mechanics, a section entirely devoted to showing the truth and uses of this principle by the different applications.

The same principle then inspired the principle that Maupertuis proposed in the *Mémoires de l'Académie des Sciences de Paris* for the year 1740, under the name of the *law of rest* [Cf. Maupertuis, 1740], and that Euler developed further and rendered more general in the *Mémoires de l'Académie de Berlin* for the year 1751. Finally, it is the same principle that provides a basis for the principle that Courtivron [probably Marquis de Courtivron (1717–1785) specialist of optics] gave in *Mémoires de l'Académie des Sciences de Paris* for the years 1748 and 1749.

And, in general, I believe that I can venture to say that all general principles that could eventually be discovered in the science of equilibrium, will be none other than the same as the principle of virtual velocities, envisaged in a different manner, and of which they will differ only in their expression.

Moreover, this principle is not only very simple and very general by itself, but it also has the precious and unique advantage to be translated in a general formula which comprises all problems that we can propose concerning the equilibrium of bodies. We shall expose this formula in its whole extent: we shall even try to present it in an even more general manner than done until now, and give of it new applications.

As to the nature of the principle of virtual velocities, we must agree with the opinion that it is not sufficiently obvious by itself to be erected as a primary principle, but we can view it as the general expression of equilibrium, deduced from the previously mentioned two principles.

…………

PP 223-224 [on d'Alembert's dynamics and other principles of dynamics]

…………

The *Treatise on dynamics* by d'Alembert, which was published in 1743, put an end to kinds of challenges, by offering a direct and general method to solve, or at least to put in equations, all problems of dynamics that we can imagine. This method reduces all laws of motion of bodies to those of the equilibrium, and thus gives to dynamics the form of statics. We have already remarked that the principle used by Jacques [Jacob] Bernoulli in the search for a centre of oscillations, had the advantage to make this search depend on the condition of equilibrium of a lever; but it was reserved to d'Alembert to envisage this principle in a general manner, and to give to it all the simplicity and fruitfulness that it deserved.

If we impress to several bodies motions that are forced to change as a result of their mutual action, it is clear that we can regard these motions as composed of those they will really take and of motions that are destroyed; from what it follows that the latter must be such that the animated bodies be in equilibrium under their own motions.

This is the principle that d'Alembert gave in *his Treatise on dynamics* and of which he made a fruitful usage in several problems, and above all in that of the precession of equinoxes. This principle does not readily provide the equations needed for the solution of problems of dynamics, but it teaches how to deduce them from conditions of equilibrium. Therefore, by combining this principle with ordinary principles of the equilibrium of the lever, or of the composition of forces, we can always find the equations for each problem; but the difficulty to determine the forces that must be destroyed, as well as the laws of equilibrium among these forces, often renders cumbersome and laborious the application of the principle; and the solutions that follows are most often more complicated than those that would be deduced from less simple and less direct principles, as we can be convinced in the second part of the same *Treatise on Dynamics*.

…………..

In the first part of this work, we have reduced the whole of statics to a unique general formula that provides the laws of equilibrium for any system of bodies acted upon by as many forces as we like. Thus, we shall also reduce to a unique general formula the whole of dynamics; because, to apply to the motion of a system of bodies the formula of its equilibrium, it will suffice to introduce forces that arise from the variations of the motions of each body, and that must be destroyed. The development of this formula, on account of conditions that depend on the nature of the problem, will give all the equations needed for the determination of the motion of each body; and it will remain to integrate these equations, what is a matter of analysis.

One of the advantages of the relevant formula is to offer immediately the general equations which contain the principles or theorems known under the names of conservation of living forces, conservation of the centre of gravity, conservation of the moment of rotation or principle of areas, and the principle of the least quantity of action. These principles must be considered like general results of the laws of dynamics, rather than primary principles in this science; but as they are often employed as such in the solution of problems, we are due to speak about them here, by indicating in what they consist, and to whom scientists, in order to leave nothing untouched in this preliminary exposition of the principles of dynamics.

The first of these four principles, the one concerning the conservation of living forces, was found by Huyghens, but in a form slightly different from the one that it receives now. ....

Until now this principle was viewed as a simple theorem of mechanics; but when John Bernoulli adopted the distinction made by Leibniz between dead forces or pressures which act without actual motion, and living forces which accompany this motion, as well as the measure of the latter by the product of mass and the square of the velocity, he saw in the principle in question only a consequence of the theory of living forces, and a general law of nature according to which the sum of the living forces of several bodies is conserved while these bodies act mutually on each other by simple pressures, and is constantly equal to the simple living force that results from the action of actual forces that move the bodies. He gave to this principle the name of *conservation of living forces*, and he used it with success to solve some problems which had not been solved before, and of which the solution looked difficult by direct methods.

....

The great advantage of this principle is to provide immediately a definite equation between the velocities of bodies and the variables that determine their position in space; do that when by the nature of the problem, all these variables reduce to a unique one, this equation suffices to solve it completely, and this is the case of the problem of the centre of oscillations. In general, the conservation of living forces always provides a first integral of the different differential equations for each problem, what is of utmost usefulness on several occasions.

The second principle is due to Newton who, at the beginning of his *Principes mathématiques* [Principia mathematica], proves that the state of rest or motion of the centre of gravity of several bodies, is not altered by the reciprocal action of these bodies, whatever; so that the centre of gravity of the bodies that act on each other in any manner either by means of threads or levers, or by the law of attraction, etc, without any action nor any external obstacles, is always at rest or moves uniformly along a straight line.

Since then, d'Alembert has given to this principle a larger extent by making it known that if each body is acted upon by an accelerating force that acts along parallel lines or are directed toward a fixed point, and acts proportionally to the distance, the centre of gravity must follow the same line as if the bodies were free; to what we can add that the motion of this centre generally is the same as if all forces of bodies, whatever, were applied along its own direction.

It is visible that this principle is used to determine the motion of the centre of gravity, independently of the respective motions of the bodies, and so that it can always provide three definite equations relating the coordinates of bodies and the time, which equations will be the integrals of the differential equations of the problem.

The third principle is much less old than the preceding two ones, and seems to have been discovered simultaneously by Euler, Daniel Bernoulli [son of John] and d'Arcy [Patrice, Comte; 1725–1779], but in different forms. According to the first two of these authors, this principle consists in the fact that in the motion of several bodies about a fixed point, the sum of the product of the mass of each body by it velocity of circulation about this centre and by its distance from this centre, is always independent of the mutual action that the bodies exert on one another, and is conserved in as much there are neither action, nor external obstacles. Daniel Bernoulli gave this principle in the first volume of the *Memoirs of the Berlin Academy* published in 1746, and Euler gave it during the same year in the first volume of his *Opuscules*. ……The principle of d'Arcy [21], as given to the Académie des Sciences in the memoirs of 1647 (published in 1752) ……is a generalization of a beautiful theorem of Newton on the areas described by any centripedal forces. ….. He made of it a kind of metaphysical principle which he called the *conservation of action* to oppose it, or to substitute it to, the *principle of least action*…..

Now I come to the fourth principle that I call the *principle of least action* by analogy with the principle that Maupertuis [37] gave under this denomination….. This principle, only envisaged analytically, consists in the fact that in the motion of bodies interacting with one another, the sum of the masses by the velocities and by the travelled spaces is a *minimum*. This author deduced from it the laws of reflection and refraction of light as well as those of the shock of bodies in two memoirs in 1744 [36] and 1746 [37].

But its applications are much too specific to serve to establish its truth as a general principle. They indeed have something vague and arbitrary that can only render uncertain the consequence that we could deduce from the exactness of the said principle. That is, it seems to me that we should be wrong in granting to this principle the same status as to the other principles that we just exposed….. This principle, combined with that of the living forces and expanded along the rules of the calculus of variations directly provides all equations that are necessary for the solution of each problem; and from this is born an equally simple and general method to treat the questions that concern the motion of bodies; but this method is itself only a corollary of the method that is the object of the second part [on dynamics] of the present book, and has simultaneously the advantage to be extracted from the first principles of Mechanics.

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