Chapter 13
Epilogue

Abstract This epilogue briefly summarizes the foregoing essays while emphasizing the ways in which this critical perusal was approached, and what appears to be the large themes that received a special magnification and, perhaps, a biased presentation. It underlines the main breakthroughs as well as the secondary ones. It highlights the role of scientists who left essential prints in this history of scientific ideas. It finally outlines the observed timid beginnings of future theories of coupled fields in thermo-mechanics.

13.1 On the Method

Nowadays, two sometimes irreconcilable approaches are considered in political history. One, under the influence of structuralism, favours a global approach epitomizing great movements of ideas and philosophical tendencies (with an emphasis on general themes, sociological and economical background), and the other still basing on chronology, dramatic events, national heroes, great names and even myths for more ancient times, what provides a tempo that is useful to the youth in forming a consolidated view of history.[1] In the present book dealing with

[1] If we compare the examined period with the one considered in our previous book [13] which included the two World Wars of the twentieth century, we find that this was a relatively quiet one. Of course there were wars. An important one, very much similar to World War One in extent and casualties was the Seven-Years war (known as the French and Indian War in the USA) that included England, the Netherlands, and Prussia on one side and France, Austria, Russia, and Sweden on the other with battle fields on three continents (Europe, India, North and Central America) and various seas. It lasted from 1756 to the Treatise of Paris in 1763. This date marks the true birth of a powerful British Empire and the disappearance of French possessions in India and North America (Canada, East of the Mississippi river; the rest of Louisiana, west of the Mississippi river, from New Orleans to the Canadian boarder was sold to the USA by Napoleon in 1803), but with a status quo in Europe. Other conflicts were the Napoleonic wars, the wars of independence in Italy and Greece, the British-Russian war, and the French-Prussian war of 1870, and of course the war of independence in the USA and the unfortunate American Civil War in

G. A. Maugin, *Continuum Mechanics Through the Eighteenth and Nineteenth Centuries*, Solid Mechanics and Its Applications 214, DOI: 10.1007/978-3-319-05374-5_13,

one aspect of the history of scientific ideas—as witnessed by the different essays—, we have preferred a mixed attitude, sometimes emphasizing general themes such as the evolution of the principles of mechanics—or of their wording—, Newtonian versus "continental" viewpoints, combination of continuum mechanics and thermodynamics in a true "thermo-mechanics", and also paying more than justified attention to some remarkable individuals who left their name attached to a theorem or a principle, although these persons are not truly sanctified. Here chronology plays an important role. It is obviously cogent to know that Daniel Bernoulli's celebrated theorem came before Cauchy's postulate on stresses, and the true understanding of the conservation of energy, or that the Navier-Saint-Venant-Stokes equations, although involving dissipation, were written down independently of the second law of thermodynamics that was not yet clearly expressed. In the last case this has left a print since there still are many workers in viscous fluids, and even more in non-Newtonian fluids, who practically do not refer to thermodynamics.[2]

The main result of our mixed attitude has been a series of essays that generally follow the arrow of time while underlining the role played by scientists who brought seminal ideas and contributed the most remarkable breakthroughs. These scientists are not unknown to the majority of students and practicing scientists, because their names are classically attached to familiar theorems or commonly used mathematical objects. Among the names that recur in the above-given essays, we find those of: John and Daniel Bernoulli, Varignon, d'Alembert, Euler, Lagrange, Cauchy, Lamé, Piola, Kirchhoff, Green, Duhamel, Neumann, Carnot, Kelvin, Helmholtz, Clausius, Stokes, Maxwell, Saint-Venant, Boussinesq, the Cosserat brothers, Duhem, Poincaré, and Caratheodory. Parodying what American magazines do for film stars, these individuals form, in some sense, the "hall of fame" of our discipline for the eighteenth and nineteenth centuries. These are all first roles, although some secondary roles may have played a crucial part at some specific time. An unavoidable filtering process took place with our biased

(Footnote 1 continued)
1861–1865. But all these did not alter much the scientific world where international exchanges (e.g., between England and France or France and Germany) continued uninterrupted except in case of physical impossibility (e.g., during a blockade). This is in sharp contrast with what happened in the twentieth century. The great influential event in fact was the French Revolution started in 1789, not because "it did not need savants", [Supposedly, this is what was said by some philistine revolutionaries when the chemist Lavoisier lost his head (but he was not executed because he was a scientist. Remember that Laplace, Lagrange, Monge, Coulomb, Lazare Carnot and others went through this period without physical damage; d'Alembert had died of natural causes in 1783).] but because it instituted a new type of framework for scientific studies with the creation of engineering (polytechnic) schools, a model that was to spread all over the world during the nineteenth century. This was particularly beneficial to the advances in continuum mechanics and its application to mechanical and civil engineering along with implementation of good mathematics (sometimes created for this very purpose) as illustrated by Cauchy, Navier, Fourier, Ampère, Fresnel, Coriolis, Duhamel, Saint-Venant, Poincaré, etc., in France and the disciples of F. Neumann (Kirchhoff, Clebsch, and Voigt) in Germany.

[2] The inequalities to be satisfied by viscosity coefficients in order to guarantee a non-negative dissipation, were in fact proved belatedly by Duhem toward the end of the nineteenth century.

contemporary view since only the fruitful avenues have been retained, those leading to dead ends being altogether ignored, perhaps unjustly because they also play a role in the evolution of ideas. However, this is moderated by the last two essays in which we have examined the appraisal by scientists Appell ([1]—but initially 1909); and Hellinger [6] of the early twentieth century. This does not necessarily coincide with our own appraisal one century later.

13.2 The Main General Themes and Breakthroughs

As shown by the scrutiny of original sources and the offered English translation of some crucial texts, the two most important lines of development exposed in the successive essays are the "continental" vision that emphasizes the consideration of a principle of virtual motion or, in modern terms, a "weak" formulation—and the more "Newtonian" approach based on the postulate of balance laws that follows Euler. The last viewpoint is the approach still favoured by disciples of the late Truesdell, an author who constantly expressed his (justified) admiration for Newton and his successors in the United Kingdom. This is best exemplified by the treatise of Truesdell and Toupin [19]. In contrast, under the influence of Leibniz and John Bernoulli the "variational" approach, in different guises, has been adopted by many scientists in France (d'Alembert, Lagrange, the Cosserats), Italy (Piola), and Germany (Kirchhoff, Helmholtz, Hellinger). This approach had a glorious destiny in mathematical physics, but also in engineering with the creation of numerical methods based on it (finite-element method) and in mathematical proofs based on weak formulations. Among the French exceptions who kept the Newtonian-Eulerian line, we find Cauchy who remained in favour of a postulate of balance laws as proved by his very argument concerning stresses—and also Saint-Venant, Boussinesq and Appell because this has remained the traditional expression of the principles of continuum mechanics in courses to students in engineering. This "postulational" approach of balance laws recently gained some additional favour with the implementation of the numerical method of finite volumes.

In the period extending from John Bernoulli to Hellinger—almost two centuries—breakthroughs have been numerous. They were listed at the end of Chap. 2 either in the form of realizations of the eighteenth century or as things to come in the nineteenth century. But if one has to select among these the few most important ones, then certainly one has to chose the formulation of the principles of linear and angular momenta (for a collection of particles or a global body) by Euler, the introduction of the general concept of stress by Cauchy, and the proposal of the first and second laws of thermodynamics by Sadi Carnot, Kelvin, Mayer, Helmholtz and Clausius. Those are all fundamental principles that still apply today in the fashionable combination known as *thermo-mechanics*. They have found natural extensions within relativistic physics (they even apply to black holes).

According to the already mentioned two possible avenues this led to the following two basic formulations for a deformable body made of a continuous material:

(1) along the "postulational" line: global statement of the two principles of momenta in the Eulerian form, and the two laws of thermodynamics;
(2) along the "variational" line: a global statement of the principle of virtual power, and the two laws of thermodynamics for real evolutions.

For the first line, see more particularly Truesdell and Toupin [19], Eringen [4], and Maugin [11]. For the second line, see the critical essay by the author [12] where it is emphasized that this line has to be preferred for theories of generalized continua where extra balance laws are automatically taken care of by the principle of virtual power. It also does not make use of the Cauchy postulate for stresses and its generalization to the notions of hyperstresses and couple stresses is straightforward. Of course, the reader may find that this line receives an exaggerated magnification in most of the previous chapters. But this emphasis is justified by what permeates from the considered works, mostly in the "Continental" works that were not written in English (see Chap. 1 for this deliberate choice and the initial purpose of this series of essays).

Still, Cauchy certainly is the most remarkable among the cited scientists because not only did he contribute the basic concept of continuum theories (see Chap. 3), but, as a mathematician, he also created some of the most efficient tools in the treatment of problems of continuum mechanics, such as in linear algebra and its geometric representations, elements of group representations, rigorous definitions of integrals and of limits, singular integrals and the notion of principal value, and an invaluable application of complex variables with the theory of residues (see [2]). This was particularly useful in two-dimensional problems of hydrodynamics (see [1 in Chap. 11, 7, 8, 15]) and of linear elasticity (see [9, 14, 16]), and more generally in potential theory.

We note that the most cited authors are, together with Cauchy and in chronological order: John Bernoulli, d'Alembert, Euler and Lagrange for the eighteenth century, and Navier, Lamé, Clebsch, Maxwell, Saint-Venant and Boussinesq for the nineteenth century. This is corroborated by Timoshenko [17] and Todhunter [18] with a bias toward the application of solid mechanics to the strength of materials. Indeed, while many of the perused works bear a strongly mathematical style, applications were not neglected by the same scientists as a result of professional obligations and a new interest in the mechanics of machines, mechanical and civil engineering, and then construction using metallic structural elements. For fluid mechanics which started as a "Swiss" specialty with the Bernoullis and Euler, the nineteenth century witnessed the take over of this field by the British school with stars such as Stokes and the Cambridgians in hydrodynamics. This has remained until now a remarkably fruitful field in the United Kingdom. This is illustrated by the lasting influence exerted by scientists like Lord Kelvin, Lamb [8] and Osborne Reynolds (1842–1912), and the enduring supremacy enjoyed by some journals such as the *Journal of Fluid Mechanics*. On the German side, we cannot overlook the influential works (in particular on vorticity) of Helmholtz—who of course also radiates in other fields of physical and medical science—and of Prandtl with the notion of boundary layer and the revolution it brought in the emerging science of aeronautical flight. The French school is more reduced but we particularly note Navier with his seminal works on viscous flows, Boussinesq with his innovative ideas (modelling and mathematically justified approximations), and Poincaré for his study of the equilibrium shapes of fluid masses. Many of these are described at length and mathematically in the imposing treatise on rational mechanics by Appell [1]. Much more on the history of hydrodynamics for the relevant period can be found in Darrigol [3].

13.3 The Breakthroughs of Second Rank

There is no pejorative or belittling consideration in this classification. It simply is that this is not so much related to principles, except for the laws of virtual work and virtual power and the analytic mechanics of Lagrange (which, probably, would not have existed without the pioneering work of Newton). We rank in this class the formulation of the laws of Eulerian fluids, the laws of linear elasticity by Cauchy and Navier, those of linear viscous fluids by Navier, Saint-Venant and Stokes, the thermo-elasticity of Duhamel and Neumann, and the initial studies on visco-elasticity by Kelvin, Maxwell, Voigt and Boltzmann, and those on plasticity by Tresca, and Saint-Venant. Still in a different class, because of much delayed recognition and applications only in the late twentieth century, we find the proposal of continua with microstructure by Duhem and the Cosserat brothers.

13.4 The Timid Steps in Coupled Fields

We have seen that both Duhamel in France and F. Neumann in Germany pioneered the theory of coupled fields in continuum mechanics by creating practically from nought an embryonic theory of thermo-mechanical interactions. This may have been premature as in fact in advance on the applications of well set laws of thermodynamics. Potential applications were only very few at the time, being limited to some problems posed by the then recent railway technology (overheating of metallic parts in motion). In so far as electro-magneto-mechanical interactions—a subject matter dear to the writer—are concerned, one must realize that very few such couplings had been identified when electromagnetic effects themselves were not yet fully exposed. Historically, the first coupling is *magnetostriction* discovered by James Prescott Joule in 1847 (the same Joule as the one of the Joule effect in electric conductors). Magnetostriction, an effect quadratic in the magnetic field, in principle exists, but to a rather small extent, in many materials (no specific material symmetry is required for its existence). The second coupling, of electro-mechanical nature, is linear *piezoelectricity* that was discovered in 1881 by the Curie brothers in Paris. This effect, linear in the electric field, requires a material symmetry with no centre of symmetry such as in quartz or Rochelle salt. Technical applications of this effect had to await the First World War with Paul Langevin and the conception of sonars in underwater acoustics for the detection of submarines.

A true nonlinear combination of electrodynamics and continuum mechanics respecting the laws of thermodynamics will be achieved only in the second half of the twentieth century. We have given elsewhere [13, Chap. 12] elements of these developments in a concise historical perspective. Advanced technical treatises dealing at length with this rather complex but extremely rich theory are those of the author [10] and Eringen and Maugin ([5], reprinted in 2012). To be complete we should note that the interaction of light with deformable (transparent) matter was discovered by David Brewster (1781–1868) in 1814–1815 when he found that mechanical stresses induce temporarily in transparent solids directional properties with respect to polarized light. It is the French engineer-scientist Augustin Fresnel (1788–1827) who identified this property with the double refraction of crystals in 1822. This was readily applied by F. Neumann in his experiments on thermo-elasticity. We have here the basis of the technique of photo-elasticity. More along this line had to benefit from laser technology in the twentieth century.

Concerning another flourishing field of application of continuum thermo-mechanics in the last fifty years, biomechanics and mechano-biology, only very few hints at some early development in the perused period are the work of Poiseuille on the flow of blood in 1844 (laminar flow in a cylindrical tube) and the thesis work of John Bernoulli on the movement of muscles in 1694, although Galileo Galilei (in *Two new sciences*) in 1638 had previously pondered the mechanical strength of bones versus the size of animals. Again, one had to await the second half of the twentieth century to see a true blossom of mathematical studies and a realistic mechanical modelling in bio-thermo-mechanics (Cf. some historical remarks in [13]). This concludes our adventures in the realm of continuum thermo-mechanics, between John Bernoulli and Ernst Hellinger, on a more humane tune with the passing from the mechanics of inert matter to that of living matter.

References

1. Appell P (1921) Traité de mécanique rationnelle, 3rd edn. Gauthier-Villars, Paris (Fac simili reprint by Gabay, Paris, 1991)
2. Belhoste B (1991) Augustin-Louis Cauchy: a biography. Springer, New York (English translation from the French original “Cauchy, 1789–1857”, Belin, Paris)
3. Darrigol O (2005) Worlds of flows: a history of hydrodynamics from the Bernoullis to Prandtl. Oxford University Press, Oxford
4. Eringen AC (ed) (1971–1976) Continuum physics, Four volumes. Academic Press, New York

5. Eringen AC, Maugin GA (1990) Electrodynamics of continua, Two volumes. Springer, New York (Soft-cover reprint, Springer, New York, 2012)
6. Hellinger E (1914) Die allgemein Ansätze der Mechanik der Kontinua. In: Klein F, Wagner K (eds) Enz MathWiss, vol 4, Part 4. Springer, Berlin, pp 602–694
7. Jacob C (1959) Introduction mathématique à la mécanique des fluides. Gauthier-Villars, Paris/Ed. Acad. Sci. Romania, Bucarest (original in Romanian, 1952)
8. Lamb H (1879) Hydrodynamics, 1st edn. Cambridge University Press, Cambridge (6th edn, CUP, 1932; Dover reprint, New York, 1945)
9. Love AEH (1892) A Treatise on the mathematical theory of elasticity (1944, 4th edn, Dover reprint, New York; originally published in two volumes in 1892–1893)
10. Maugin GA (1988) Continuum mechanics of electromagnetic solids. North-Holland, Amsterdam
11. Maugin GA (1999) Thermomechanics of nonlinear irreversible behaviours. World Scientific, Singapore
12. Maugin GA (2012) The principle of virtual power: from eliminating metaphysical forces to providing an efficient modelling tool. Cont Mech Thermodynam 25:127–146
13. Maugin GA (2013) Continuum mechanics through the twentieth century: a concise historical perspective. Springer, Dordrecht
14. Mushkelishvili NI (1953) Some basic problems in the mathematical theory of elasticity. Noordhoff, Groningen
15. Sedov LI (1937) Two-dimensional problems in hydrodynamics and aeromechanics (in Russian) Moscow. (English translation: Wiley, New York, 1965)
16. Solomon L (1968) Elasticité linéaire. Masson, Paris
17. Timoshenko SP (1953) History of the strength of materials. McGraw Hill, New York (Dover reprint, New York, 1983)
18. Todhunter I (1886) A history of the theory of elasticity and the strength of materials from Galileo to the present time, vol 1. Cambridge University Press, UK (edited and published posthumously in 1886 by Karl Pearson)
19. Truesdell CA, Toupin RA (1960) The classical theory of fields. In: Flügge S (ed) Handbuch der Physik, vol III/1. Springer, Berlin, pp 226–858