Chapter 12
A Successful Attempt at a Synthetic View of Continuum Mechanics on the Eve of WWI: Hellinger's Article in the German Encyclopaedia of Mathematics

Abstract This essay analyses the comprehensive nature of a remarkable synthesis published by Hellinger (Die allgemein ansätze der mechanik der kontinua. Springer, Berlin, pp. 602–694, 1914) in a German encyclopaedia. In this contribution Hellinger, a mathematician, succeeds in capturing the progress and subtleties of all what was achieved during the nineteenth century, accounting for most recent works and also pointing at forthcoming developments. On this occasion, the scientific environment of Hellinger is perused and the style of Hellinger and his excellent comprehending of continuum mechanics are evaluated from a document that is a true landmark in the field although often ignored.

12.1 Introduction

In the nineteenth and twentieth centuries German scientists and engineers have developed a special taste for the composition of impressive encyclopaedias and so-called "*Handbucher*" (Handbooks). A typical *Handbuch* (in fact a *Taschenbuch*) in mechanical engineering has been the very popular one by Hütte with many foreign translations, but this was more a catalogue of prescriptions, standards, and elementary formulas of mathematics and strength of materials. Famous collections of the "Handbuch der Physik" have been edited by Geiger and Scheel between 1926 and 1933 [18] and by Flügge between 1955 and 1988 [17]. Mathematicians did not escape this trend. In particular, renowned mathematicians such as Felix Klein (1849–1925) and Conrad H. Mueller (1857–1914) contributed their wide experience and many friendly connections to the creation and edition of a monumental encyclopaedia of mathematics under the German title "*Encyklopädie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen*"—in brief: Enz. Math. Wiss. (EmW)—published by B. G. Teubner (Verlag) in Leipzig

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between 1907 and 1914 [31].[1] Various mathematicians and physicists were called to contribute to this vast enterprise. Part Four of Volume Four was devoted to Mechanics (*Mechanik*).[2] In that Volume the burden of writing Article 30 on the General bases/formulation (principles) of continuum mechanics ("*Die allgemeinen Ansätze der Mechanik der Kontinua*") fell on Ernst Hellinger then in Marburg. The Encyclopaedia is well-documented with scholarly articles. It is aimed at the specialist. Concerning the whole EmW, it is salient to note the following appreciation of I. Grattan–Guiness (2009), the famous historian of sciences: "Many of the articles were the first of their kind on their topic, and several are still the last or the best. Some of them have excellent information on the deeper historical background. This is especially true of articles on applied mathematics, including engineering, which was stressed in its title". This particularly applies to Hellinger's contribution.

Ernst Hellinger (1883–1950) had been educated in Heidelberg, Breslau and Göttingen and was a doctoral student of David Hilbert. This indicates that he was a rather pure mathematician whose most famous mathematical accomplishments were in integral and spectral theories. He became a professor in Frankfurt am Main but he left Germany for the USA in 1939 and then taught at Evanston, Illinois. The writing of this contribution in continuum mechanics in 1913 [26][3] may have been a parenthetical episode in his career. Nonetheless, he was much interested in variational formulations (as shown by the forthcoming perusal of his contribution) and even introduced the notion of two-field variational principle now referred to as the Hellinger-Reissner variational principle in elasticity (cf. [53]). Nonetheless, we surmise that his formation with Hilbert led him to view continuum mechanics as one of the physical sciences to be formalised and given an axiomatic framework, an orientation that will be materialized later on by the Truesdellian school with Noll (cf. [48]). Although not a full time mechanician, Hellinger was able to capture in a rather concise contribution all recent and promising advances by keeping a

[1] This monumental work was translated into French [43] and edited under the direction of J. Molk—a mathematician specialist of elliptic functions—and P. Appell—a reputed mathematician himself the author of a magisterial treatise on rational mechanics (cf. [1]). A full facsimile reprint of this French translation was produced by Editions Gabay in Paris in the years (1991–1995). But only one volume (exactly IV/4, the presently examined one) was never translated into French, and therefore does not exist in the Gabay reprint. The reason for this phenomenon is not clear. Of course, its date of publication, 1914, was not the most appropriate one given the beginning of World War One. Another possible explanation given by J. Gabay is that P. Langevin, adviser for the translation of the Encyclopaedia after WWI, was not much in favour of phenomenological physics in the sense of Duhem et al. Together with Eleni Maugin, I produced a (non-published) partial translation from the German to English of Hellinger's contribution.

[2] Timoshenko [47, p. v] in his history of the strength of materials refers to this volume for an extended bibliography.

[3] Of course Hellinger's theoretical contribution was complemented by other more specific and applied ones such as those of Heun [29] on the general bases and methods of the mechanics of systems, Voss [52] on the general principles of mechanics, and von Kármán [30] on the physical bases of the mechanics of solids.

sufficiently high standpoint, a balanced neutrality, and an acute insight, and this, in our opinion, much more than some professional mechanicians who kept too much with well established subject matters. In order to help the reader not accustomed with reading in German, a partial translation in English of Hellinger's contribution is provided in an Appendix.

12.2 The Scientific Environment

Although Hellinger was essentially foreign to the engineering spirit, in writing his opus of 1914 he gathered a rich past and contemporary documentation and accounted for most of the recent works in the field of theoretical continuum mechanics. He was not building in a vacuum, but this voluntary embedding in a medium other than his own is altogether remarkable. Of course the influence of his mentor David Hilbert may have played a fundamental role in his clear interest for the general and somewhat axiomatic aspects, so that he must have been familiar with the then recent attempt of Hamel [24] to delineate the structure and principles of mechanics (as of the beginning of the twentieth century), and the recently published treatise on "energetics" by Duhem [12] with its pre-Truesdellian flavour which may have been to his taste. This is corroborated by his frequent citations of these two authors. But he also knows the impressive treatise of Appell [1] on the rational mechanics of deformable bodies and the German synthetic texts of Heun [29, Voss [52], and Voigt [51].

Being basically a mathematician and a great admirer of Lagrange, Hellinger is also very much concerned with variational formulations in the works of W. Thomson (Lord Kelvin), Kirchhoff and, above all, the Cosserat brothers [7, 8, 40]. The last connection may have been through his reading of the Third volume of Appell's treatise [1] in which there is a supplement written by the Cosserats. The appeal to group symmetries in the line of Sophus Lie and Henri Poincaré by the Cosserats may have been very attractive to him. But he also considered the possible occurrence of dissipation with the notion of dissipation potential introduced by Rayleigh, and even time-dependent (memory-like) behaviours in the manner of Boltzmann [2]. The recent work of Hadamard [23] on wave propagation has also left a strong print. Finally, in contrast with many other writers of the period who remain in the classical (Newtonian) framework, Hellinger has already integrated in his views the revolutionary ideas of Einstein in 1905 on relativity and Minkowski [42] on space-time. All these remarks are based on the citations of these authors by Hellinger as checked in the many footnotes to his contribution. This is the general background and favourable scientific environment in which this perspicacious author has framed his article.

12.3 The Contents of Hellinger's Article

12.3.1 Introductory Remark

Hellinger's article is only ninety two pages in print. Nonetheless, it succeeds in providing a rather complete survey of the field both with its established bases, its recent successes and some view of things to come. This friendly neutrality with which the author looks upon his assigned duty—in principle perusing a vast domain of knowledge with about a hundred fifty years of history and a vivid contemporary activity—is conducted with no a priori prejudice as a result—we surmise—of Hellinger being a somewhat outside observer. Hellinger is rather generous but also very accurate with citations. He cites many authors, whatever their nationality, but is clearly most influenced by works published within the thirty years before his synthesis, say in the period 1880–1910. Perhaps because of this "actuality", he does not confine himself to the well established fields (linear elasticity and Eulerian fluids), but he often venture in newly expanded fields of interest such as finite deformations, oriented (Cosserat) bodies, capillarity, formulation of thermo-mechanics, analogy with electrodynamics, and even relativistic continuum mechanics.

From a historical viewpoint, our perusal of this beautiful contribution should not be influenced by our own education in the field (rough period 1960s–1970s) and our knowledge accumulated over an active professional period of some forty five years that witnessed many developments. But it happened that many of these rich developments in a vivid period of research more or less coincide with many of the points touched upon by Hellinger. We do not think that this kind of resonance between Hellinger's approach to our field of interest and our own view is so much due to an influence that this author would have exerted on the generations that followed his own. Indeed, Hellinger's text may have been read by German scientists between the two World Wars. But we must notice that his article was published in an encyclopaedia of *mathematics*, in a style that is permeated by the rigorous thinking of a mathematician—far from engineering interests—and that the text was not translated in any foreign language. It just happened that a spirit close to that of Hellinger re-appeared in our period of activity, and this of course greatly facilitates our apprehending of his exposition.

12.3.2 The Layout and Articulation of the Contribution

Every synthetic work in a field has to respect a definite agenda. This particularly applies to an article in an encyclopaedia of which the readership is not so well delineated. In the present case a tradition has settled that the progression in the presentation of the subject matter follows an almost fixed order (as exemplified in many textbooks on continuum mechanics), geometric background being introduced first, followed by kinematics and the theory of deformations, then kinetics and the general laws of mechanics, general classes of mechanical behaviours and a few more specific examples, and finally (but not always) some more exotic extensions. Hellinger's approach is more difficult to grasp because he is ahead of his time while simultaneously following some masters such as Kirchhoff, Helmholtz, Clebsch and Barré de Saint-Venant, and he has thoroughly gone through the then recent works by W. Voigt, J. V. Boussinesq, E. and F. Cosserat, H. Poincaré, and P. Appell, authors who are very often accurately cited. In reason of the imposed exercise, Hellinger's text is extremely dense. Instead of perusing his contribution just in its order of presentation—the easy way—we have preferred to examine various points, that recur in the whole text and seem to emphasize Hellinger's repeated interest in some specific aspects as an exemplary mathematician (obviously not the point of view of an engineer).

12.4 The Identified Fields of Marked Interest of Hellinger

12.4.1 On General Principles of Mechanics and General Equations

This is not an original point of departure in Sect. 12.2. Hellinger builds on the commonly admitted bases of Newtonian mechanics in the tradition set forth by Euler, Lagrange, Cauchy, but with modern references to Brill [5]; Duhem [12]; Voigt [51], and other contributions to the same encyclopaedia by, e.g., Voss [52] and Heun [29]. He clearly indicates his favoured view of Hilbert and Hamel [24, 25] — later on formalized in Hamel's contribution to the Handbuch der Physik in 1927 — for axiomatization and the consideration of a general thermodynamic framework by Duhem [12]. He also heavily borrows from the treatise of Appell [1] and the recent works by the Cosserat brothers ([7, 8], and their numerous notes in the *Comptes-Rendus* of the Paris Academy of Sciences). But Hellinger does not hesitate to introduce the relativistic Einstein–Minkowski's vision in the last section of his contribution.

Formally, Hellinger is much more attached to the Lagrangian-Hamiltonian variational formulation than to the classical Newtonian type of approach that relies on a statement of laws of equilibrium or dynamics. This he shows even for the bases of statics where he readily implements the principle of virtual work (Sects. 12.3 and 12.4). This may be one of the reasons why this work is not so much cited in the "Anglo-Saxon" literature dominated by Newton's vision and made popular in continuum mechanics by the Truesdellian school in the 1960s. But Hellinger cannot avoid discussing the notion of force as a polar vector (p. 613) and the clever introduction of the concept of stress by Cauchy (*Cauchysche "Drucktheorem"*; p. 615). On this occasion, Hellinger, above all a mathematician, acknowledges the usefulness of the notions of vector analysis and dyadics—linear vector functions—in the line of

J. W. Gibbs (cf. [21]) and the matrix calculus of Cayley (p. 613). He also refers to "tensor components of a dyad" (*Tensorenkomponenten*) after Voigt's lectures on the physics of crystals (p. 624). This is to be contrasted with the rather shy attitude of contemporary authors (e.g., Appell [1]; see my own appraisal in Maugin [41]).

Hellinger's presentation of equilibrium equations in the Eulerian framework with the associated natural boundary conditions (reflecting Cauchy's postulate)—Eqs. (5a) and (5b), p. 617—is rather modern. But he also gives what may be considered the Piola-Kirchhoff format as Eqs. (9a) and (9b) in p. 618, after what looks like a Piola transform for the stress in Eq. (8). He indeed refers to the work of Piola [44] in p. 620. For the symmetry of the Cauchy stress, he refers (p. 619) to Hamel who calls this the "*Boltzmannsches Axiom*" for "*die Symmetrie der Spannungsdyade*". Reductions to the two-dimensional (e.g., plates) and one-dimensional cases (e.g., rods, filaments)—in Eqs. (18a) and (18b) in p. 622 for this last case—are given following the Cosserats.

12.4.2 On Variational Formulations

For a mathematician like Hellinger the attraction to the beauty, economy of thought, and efficacy of variational formulations is inevitable. Hellinger, a follower of Lagrange, Piola, Hamilton, Kirchhoff, Helmholtz and the Cosserats, in fact starts by emphasizing the exploitation of the principle of virtual perturbations ("*virtuellen Verrückungen*"; p. 611 on)—virtual work (a weak formulation in the modern jargon). To the risk of creating an anomalous connection with modern standards, we perceive in these perturbations the notion of test functions (see Maugin [35, 37]). Note that Hellinger gives a mathematically correct definition of what is a material variation by considering an infinitesimal parameter noted σ (and not ε like in modern treatments; cf. pp. 607–608). As a matter of fact Hellinger's statement (7) in p. 612 is, but for different symbols, just the same as in a modern formulation where the principle of virtual powers (for statics) is written for a massive body as

$$P^*_{vol} + P^*_{int} + P^*_{surf} = 0, \tag{12.1}$$

where the three terms refer to volume, internal and surface forces, respectively. The Cauchy stress is introduced in the second term as a co-factor. A power of inertial (acceleration) forces is added in the right-hand side of Eq. (12.1) in the dynamical case. The second term is transformed with the help of Green's divergence theorem [22] to yield a divergence term in the bulk and Cauchy's natural boundary condition at the surface. Hellinger emphasizes the equivalence of the statement (1) with Newton's laws (cf. p. 630).

In dynamics we have D'Alembert's principle per se (*d'Alembertschen Prinzips*, p. 629) and this yields the looked for equations such as Eq. (2) in p. 630. On introducing the kinetic energy, Hellinger is led to the principle of least action (p. 633) of Maupertuis and Hamilton. Gauss' principle of least constraint is also evoked in the same page with the possibility to account for non-holonomic constraints. The general nature of such formulations is clearly acknowledged including with due reference to the Cosserats. With the assumed existence of a strain potential Hellinger touches upon the favourite subject matter of Kirchhoff, Boussinesq, Duhem [10], Poincaré and the Cosserats (pp. 643–651). This led him to examine some questions related to stability in agreement with Dirichet and above all Born [3], as also Italian authors such as Menabrea and Castigliano. He introduces appropriately the notions of canonical transformation (p. 657) and Legendre transformation (function H in p. 654). This leads him to say a few words about minimum principles and stability. Unknown multipliers (interpreted sometimes as stresses or "reaction forces") are introduced wherever a mathematical constraint is imposed (e.g., incompressibility), following ideas of the French mathematician J. Bertrand and also D. Hilbert (see pp. 661–663). Ideal fluids accept a characteristic equation $p = p(\rho)$ when, following Hadamard [23], the potential reduces to a function of the Jacobian of the deformation. In presence of some dissipation Hellinger follows an idea of Rayleigh to consider a potential of dissipation (p. 657). This will later be formalized even for plasticity (dissipation function homogeneous of order one only) in works of the 1970s–1980s (see, e.g., Maugin [36]).

Apart from the extensions to oriented media (his Paragraph 4b, and Paragraph 4.4 below), Hellinger touches two other extensions of the principle of virtual perturbations that were to bear fruits later on. One is the possibility of considering higher-order space derivatives of perturbations. This was envisaged early by Le Roux [33]—apparently unknown to Hellinger—to account for effects of spatially non-uniform strains (such as in torsion) in small-strain elasticity. This would later on be expanded in the so-called gradient theory of continua [Cf. works by R. D. Mindlin in the 1960s, and above all: Germain [19], for the second gradient, and Maugin [35] for a general framework, using the principle of virtual power without knowledge of Hellinger's contribution]. The other is the possibility to account for the existence of unilateral constraints during the variation (Cf. Paragraph 4c). This was to be expanded in the theory of variational inequalities in the mechanics of continua (Cf. e.g., Duvaut and Lions [13, 14]).

Finally, it is often said (cf. Washizu [53]) that Hellinger contributed to the variational formulation of continuum mechanics (elasticity) by introducing before Reissner [45] the notion of *two-field variational principles*. In these both displacement *and* stresses are varied, allowing a relatively easy accounting of boundary conditions of mixed type. Reissner—educated in Germany and himself the son of a reputed mathematical physicist—must have heard of, if not studied, Hellinger's contribution. However, he proudly told the present writer that "he did not see why Hellinger's name was attached to his own name for this notion". It is true that we could not locate where Hellinger introduced this notion. But the association may come from the fact that—as noted above—Hellinger duly considers Legendre transformations of the energy potential, introducing a kind of complementary energy.

12.4.3 On Finite Strains and Elasticity

Hellinger follows the tradition established by Piola, Kirchhoff, Boussinesq and the Cosserats by always considering the case of finite strains, linear elasticity being only an approximation. This is exemplified at many stages in his contribution. First both actual (noted x, y, z) and referential or material (Lagrangian) coordinates (noted a, b, c) are introduced. This later on allows for the introduction of the Piola transformation (8) in p. 618 with a clear algorithm in spite of the absence of tensor notation. The Piola-Kirchhoff form of the equilibrium equations follows at once as Eq. (9a, b). This also applies to Cosserats' media (p. 624–625). In the case of Green elasticity for which there exits a strain potential, the Cauchy-Green's finite strain is duly introduced (cf. Eq. (12.1) in p. 663). An example of higher order (than quadratic) strain energy function is given in p. 665. The exact constitutive equations for Cauchy's stress tensor in finite strains are given as Eq. (5) in p. 645 in Boussinesq's form while Piola's form is given in p. 654 together with Max Born's [4] equations in terms of a potential in stresses—Eqs. (22a, b) in p. 654—after introduction of the complementary energy by means of a Legendre transformation. The resulting compatibility condition for the finite deformation gradient is given in Eq. (24) in p. 655 in a form due to von Kármán. In the case of isotropic materials Hellinger rightfully calls for the invariance under orthogonal transformations (p. 664) and the resulting dependency of the strain energy on the basic invariants that are factors in the Cayley-Hamilton theorem. He evokes on this occasion the possible existence of self-stresses. Citations to Boussinesq, Duhem, Poincaré, the Cosserats, Helmholtz and J. Finger abound. All these now seems quite familiar to students who followed the masters of continuum mechanics in the 1960s–1980s—e.g., in the books of Truesdell and Toupin, Green and Zerna, Leigh, Eringen, etc., in the USA and those of Goldenblatt, Novozhilov, Lurie, Sedov, Ilyushin and others in the Soviet Union—this includes the present writer.

As a true mathematician, Hellinger views small-strain elasticity as a theory of perturbations introducing wherever necessary a small parameter (noted *sigma* and not *epsilon*) that indeed indicates the smallness of strains about an undeformed state [cf. Eq. (3') in p. 608].

[4] The name of Max Born (1882–1970) is most often associated with the matrix formulation of quantum mechanics (with P. Jordan and W. Heisenberg) and his statistical interpretation of the wave function in Schrödinger's equation for which Born received a belated Nobel Prize in 1954. But Born had defended a Ph.D. thesis (1906) on the "stability of the elastica in a plane or space" (to which Hellinger refers). He was also most active in studies related to relativity after 1905 (see here Paragraph 4.8). He was among the initial developers of the lattice dynamics of crystals and contributed much to optics. His friendship with Hellinger dated back to their undergraduate-student years in Breslau ("Wrocław" in Polish) in the early 1900s. He mentored many of the known theoretical physicists of the 1920s and 1930s while in Göttingen. Finally, he was instrumental in the publication by Caratheodory [6] of an axiomatics of thermodynamics (Born suggested a formulation of the second law, the so-called "inaccessibility of states").

12.4.4 On Oriented Media

From the very beginning of his exposition Hellinger envisages the possible existence of internal degrees of freedom of the type proposed by the Cosserats in 1909. For instance, introducing the basic physical parameters of a continuum, together with the notion of density (p. 609), he feels quite natural to consider the possible attachment to each material point (the "Quantum der Materie" with material coordinates in his own language; p. 606) of an oriented trihedron or triad of rigid vectors (*ein "rechtwinkliges Axenkreus"*) likely to represent the varying orientation of "molecules"—as proposed by Voigt [50] and possibly by S.D. Poisson much earlier in 1842 (cf. footnote in p. 609). This yields the notion of "*Medien mit orientierten Teilchen*" (pp. 609–610) in the manner of the Cosserat brothers (and perhaps Duhem [11], p. 206; see Maugin [39]).

Then in considering a variational formulation (principle of virtual work), Hellinger naturally generalizes it to the case including local orientational kinematic properties (pp. 623–627) with specialization to two-dimensional and one-dimensional cases. The concept of couple-stress tensor ["*Drehmoment*" (dyade)] then appears naturally. The author recurs to this framework of "generalized continua" on many occasions, in particular when considering the Green type of elasticity based on the existence of a potential for strains (pp. 646–651) with the application of the Cosserats' concept of "Euclidean action". He returns to the notion of "generalized continuum" while dealing with analogies with the equations of light propagation and electrodynamics (the MacCullagh "ether" of 1839 [34]—an elastic medium able to transmit only transverse waves (light) in agreement with Fresnel's observations—the deduction of Maxwell equations by identifying elastic displacement and electric field on the one hand and vorticity with magnetic induction on the other and as done by authors such as Kelvin or J. Larmor—see pp. 675–681). Of course this is now rather obsolete and was already evaporating in thin air at the time of Hellinger after the works of Lorentz, Poincaré and Einstein. But Hellinger's attitude is above all witness of a marked interest in the rich modelling potentiality offered by continuum mechanics—although sometimes along paths with dead-end—leaving the final choice to true physicists.

We must note that, just like most authors until 1966, Hellinger does not see that, similar to density with its conservation law (cf. Eq. (12.7) in p. 609 in the Lagrange-Piola format), there must exist a conservation law associated with the inertia of the new orientational degrees of freedom. This missed step was resolved much later by Eringen [15].

12.4.5 On One-Dimensional and Two-Dimensional Bodies

Hellinger always considers two-dimensional and one-dimensional material bodies ("Platten und Drähte") as special cases. In this he does not follow the Cosserat brothers who work more with an increase in spatial dimensions than with a successive reduction. Much more than that, in pp. 658–660, he shows his apprehension of the true mathematical problem at the basis of this reduction in dimensions by introducing small parameters (this time noted ε) that are representative of the slenderness in thickness or of two (small) lateral dimensions of the considered material structure. Equation (12.2) in p. 659 is typical of this "asymptotic" approach that will later on be the source of an efficient asymptotic derivation of equations for plates, shells and rods in the expert hands of Gold'denveizer, S.A. Ambartsumian, V.L. Berdichevsky, Ph. Ciarlet and others. Furthermore, Hellinger does not hesitate to introduce the Gaussian parametrization of curved surfaces to treat two-dimensional bodies [cf. pp. 620–621; in particular Eq. (14a, b)]. For one-dimensional elastic bodies, he is naturally led to mentioning the Bernoulli-Euler problem of the *elastica* (pp. 667–668) with the only surviving material coordinate taken as the arc-length along the curve. One had to await the remarkable work by A. E. H. Love (later perfected by R. D. Mindlin) to correctly deduce a quasi-one dimensional dynamical theory of rods with the strange lateral inertia term (the print left by the asymptotic procedure in passing from three dimensions to the rod-like picture).

12.4.6 On Thermodynamics and Dissipative Behaviours

In his introduction (p. 604) Hellinger clearly expresses his opinion that the "mechanics of deformable media, as an autonomous discipline, comprises under formal statements, next to the usual theory of elasticity and hydrodynamics, all the related physical manifestations in the considered continuously extending bodies". The development of these ideas has certainly been influenced by the discipline of *thermodynamics* which, in principle, tries "to embrace the totality of physics" (my translation). Here Hellinger is obviously influenced by his recent reading of "energetists" such as Pierre Duhem with his magisterial treatise of 1911 [12]. The latter may have been read by a handful of happy few.[5] What Hellinger tries in his Sect. 15 (pp. 682–695) is to incorporate the dual notions of entropy and thermodynamic temperature in his fundamental variational formulation. Entropy is considered as an extensive quantity (i.e., proportional to the quantity of matter). Then a term δQ—Equation (12.1) in p. 683—representing the "Wärmezufuhr" with variation of the entropy and co-factor none other than the temperature is to be added to the purely mechanical variation mentioned above at point 4.2. With the introduction of a potential for thermoelastic processes this yields the thermal definition of the temperature (in modern terms: the derivative of internal energy with respect to entropy) and, more surprisingly for the period, Maxwell's

[5] When in 1992, during a one-year stay in Berlin, I borrowed Duhem's [12] opus from the library of the former Kaiser Wilhelm Institute in east Berlin, I discovered that this copy of the books had never been read (pages were not cut out but they were damaged by the water poured by firemen during the fire of the Institute that occurred during the Russian Army take over of Berlin in 1945).

compatibility condition for second-order derivatives of the energy in *thermo-elasticity in finite strains* (Eq. (5) in p. 684; in modern notation this reads

$$\frac{\partial \mathbf{T}}{\partial S} = \frac{\partial \theta}{\partial \mathbf{F}}, \tag{12.2}$$

where **F** is the deformation gradient and **T** is the first Piola-Kirchhoff stress).

In a more general context Hellinger comments on other coupled effects such as temperature and capillarity, pyro-electricity and thermo-chemical processes as considered by J. W. Gibbs in his original works of 1876–1878. He does not mention piezoelectricity although this is already more than thirty years old (experimental discovery by the Curie brothers in 1881) when he writes his contribution.

The above mentioned variational formulation that includes the notion of entropy and temperature is seldom considered. However, Sedov's [46] generalized variational principle—also discussed in Maugin [35]—is along the same line.

For truly dissipative phenomena such as viscosity, in spite of his familiarity with Duhem's treatise which does not propose yet a solution (the future "theory of irreversible processes"), Hellinger is reduced to invoking the notion of dissipation potential à la Rayleigh, as in the case of G.G. Stokes' viscous fluids (cf. p. 671). But he is aware of the existence of more sophisticated models of viscosity. Such a model is the one proposed by Boltzmann [2] in the form—"*elastischen Nachwirkung*"—of hereditary integrals [see p. 641 and Eq. (5) in p. 672] for which Hellinger also cites very recent works, in particular by Vito Volterra up to year 1913 (the year Hellinger completed his manuscript). This shows the concern of this author to be up to date until the last moment. Finally, he also mentions the possible occurrence of a plastic behaviour with a simple plasticity criterion in terms of principal stresses which recalls the Tresca criterion—Inequalities (7) in p. 673—although he refers for these to a work of 1909 by A. Haar and Th. von Kármán. More general or singular behaviours are simply referred to as "halb-platische" oder "vollplastische" Zustände (no need for translation).

What is strange is that Hellinger does not comment on the then recent Caratheodory [6] axiomatization of thermodynamics as suggested by his own friend M. Born, a contribution that is purely in the analytical line and would certainly had been to Hellinger's liking.

12.4.7 On Newly Studied Behaviours

This is just mentioned for the sake of completeness since hereditary materials, half plastic or fully plastic materials, are already evoked in the preceding paragraph. But Hellinger also pays some attention to the phenomenon of *capillarity* in pp. 674–675 for which a rather not commonly referenced work is by the mathematician of "relativity fame", Herrmann Minkowski (see below).

Hellinger, although not pursuing the line further, gives the exact mathematical definition of material inhomogeneity (dependence of material properties on the material coordinates; see top of p. 639). General anisotropic elastic materials (crystals) with at most twenty one independent elasticity coefficients are mentioned for the linear case. In the case of finite strains, like all authors since Cauchy he focuses on the case of isotropy with the resulting introduction of the principal invariants of strains (p. 664) in the strain-energy function. This is purely academic as Hellinger and all other authors of the period could not guess that only rubber-like materials and then finitely-deformable soft biological tissues would provide in time the realm of application of this material description (see Maugin [38]).

12.4.8 On Relativistic Continuum Mechanics

In his attempt at a large conspectus of the State of the Art in 1913, Hellinger included (Sect. 16, pp. 685–694) comments on the most recent developments concerning the relativistic mechanics of continua. This is rather exceptional for the period; in particular if we compare with other well established authors in mechanics (e.g., Appell). This may have aroused his sensibility of mathematician. He seems to be well aware of the original developments by Voigt, Lorentz, and Poincaré on the group structure of special-relativistic transformations. The Lorentz-Poincaré group was a good subject of interest with the works of Minkowski [42], A. Sommerfeld, and F. Klein. His friend Max Born may have had some influence on Hellinger's interest in the field since Born (especially, [4]) and Herglotz [28] seem to be his main sources for the basic definitions and the problem of the possibility of "rigid-body motion" in relativity.

Most of Hellinger's discussion is about the essential differences between the Lorentz-Poincaré group and the Galilean-Newtonian group of space-time transformations. But he is also particularly interested in two points. One is the possible re-formulation of the Cosserats' action principle in space-time in agreement with Minkowski and Herglotz (cf. Eq. (13a, b) in p. 693) with a space-time parametrization that combines material coordinates and a propertime (a parameter along the world line following Minkowski's description) and a total virtual variation for internal forces (components of the energy-momentum tensor). The second point is the possible definition of the notion of *rigid-body motion*, a much discussed matter being given the existing bound on velocities, with the possible local (i.e., differential) solution given by Born and Herglotz in space-time. Allusion to relativistic continuum mechanics will later be given in a bibliographic appendix by Truesdell and Toupin [49, pp. 790–793]. The present writer is one of the very few to have devoted a full albeit brief chapter to relativistic continuum mechanics in a treatise (cf. Eringen and Maugin [16], Vol. 2, Chap. 15; see also the historical perspective in Maugin [38], Chap. 15).

12.5 Conclusion

In his introduction—written in 1913—Hellinger claims that there exists no textbook or monograph in the literature on the specific subject treated in his contribution although there do exist textbooks and treatises of a general nature, but the latter do not emphasize the bases and various possibilities offered by the scheme of continuous matter. He does not intend to treat applications and specific problems. He confines himself to the essentials, "die allgemeinen Ansätze" in his own words. His viewpoint is that analytical mechanics (exploitation of variational formulations) is "the most uniform and efficient manner to approach the general problem of describing a large variety of descriptions of deformable media" in agreement with recent authors like the Cosserats and the initial standpoint of G. Green with an energy potential. This is the type of approach (principle of virtual work, d'Alembert's principle [9], Lagrange-Hamilton action principle [32], etc.) that suits best his essentially mathematical vision. The pregnant brevity of this approach possesses a "high heuristic value for the exploration of new areas. This is particularly stressed through the intimate relation of such variational principles with thermodynamics". Furthermore, this allows one to place in evidence the invariant theoretical nature of the considered problems with the notion of transformation groups. This gives a very "modern" print that helps us understand his exposition without too much effort. This "modernity" is striking in spite of the somewhat obsolete notation. Its opens up horizons to many models that will have to wait progress in some branches of pure and applied mathematics for a full blossom (e.g., large deformations, media with internal degrees of freedom, capillarity, hereditary processes, multi-field phenomena).

His mathematical inclination leads him to accept unhesitatingly all new mathematical tools of the period (vector and tensor analysis, matrix calculus, differential geometry, perturbations). The only part that is still missing is convex analysis to be much developed in the 1950s–1970s. But, overall, Hellinger is very successful in his endeavour. This is our appraisal one hundred years later. Unfortunately, we were not able to locate any substantial review or criticism of his contribution in the few years following its publication so that we have no precise idea of the quality and extent of its reception among professional circles, mechanicians and mathematicians. This may exist in some periodical bulletin of a mathematical society. It is therefore with modern eyes, perhaps themselves influenced by Hellinger's writing—a kind of feedback—that we evaluate it. This is an inevitable bias that we willingly acknowledge.

Following the Cosserat brothers, Hellinger's view of the domain of interest of continuum mechanics is essentially the mechanics of deformable bodies, by which must be understood the case of deformable solids. This is in contrast with treatises by famous authors such as Appell [1], where most contents rather deal with fluids. At the time fluid mechanics has become a rather autonomous field of study limited to perfect fluids and the Navier-Stokes equations, with specific mathematical techniques of which the use of complex variables has become endemic. But some recent developments of theoretical fluid mechanics such as the asymptotic method involved in the theory of the boundary layer by L. Prandtl could have been to the taste of an analyst like Hellinger. It is only with the birth of the science of *rheology* (concerning whatever can flow to a larger or smaller extent) and the notion of non-Newtonian fluids in the 1920s in the expert hands of E. C. Bingham (1878–1945) and M. Reiner (1888–1976) that fluids will return to the general stage of continuum mechanics. Liquid crystals, with a behaviour clearly classified in 1922 by Georges Friedel (1865–1933) and exhibiting mixed crystal (ordered state) and fluid (flow) characteristics with directional properties, will also enter this general framework with a natural connection with generalized continua of the Duhem-Cosserat type established in the 1960s–1970s. This could not be imagined by Hellinger who remains essentially an analyst, as shown by his other very successful contribution to the same Encyclopaedia of mathematics in co-operation with a friend of student days in Breslau and Göttingen, O. Toeplitz (cf. Hellinger and Toeplitz [27]).

In conclusion, we find in Hellinger's brilliant and very informative contribution all elements and remarks that we would like to deliver—even though superficially—to our mathematically oriented students in an introductory course of high level (e.g., something similar to what Germain tried to do in his course at Ecole Polytechnique, 1986 [20]); many students then thought that this was too much superficial, although all aspects of further developments in specialized short courses were outlined.

Appendix A

Partial translation from the German to English of Hellinger's contribution to the EmW (by Eleni and Gérard A. Maugin, © 2013).

Note: pages of original are indicated at the top left. Modern notations (cf. Maugin [38]) are sometimes given within squared brackets [..] along with Hellinger's notation. Footnotes are not given in full, being just replaced in the main text by a name and a year within brackets for a reference to an author. Some translator's remarks within square brackets are indicated by the initials GAM. Abbreviation EmW means this encyclopaedia.

The general basic laws of continuum mechanics
By E. Hellinger, Marburg A.I.

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Bibliography

For the time being [Hellinger's words, GAM] there are no textbooks or monographs in the literature on the specific subject treated here. In order to avoid repetition we have compiled a list of the most frequently cited works:

A. von Brill (1909).
E. and F. Cosserat (1909).
P. Duhem (1911).
G. Hamel (1912)
J.L. Lagrange (1788) and in Oeuvres complètes, Vol.11 and 12. edited by G. Darboux, Paris 1988/89.
W. Voigt (1895/96).
Cf. also Voss, Stäckel, Heun and Müller-Timpe in the EmW, Vol. IV.

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1. **Introduction**

The purpose of the present work is to give, from a uniform point of view, a comprehensive overview of the various *forms* taken by the different basic laws used in order to determine the evolution in time or even the state of equilibrium in an isolated spatial domain of "continuum mechanics" as a whole, i.e., the mechanics and physics of continuously extending media. Moreover, we shall always keep in mind only those types of continua that do not possess, thanks to restricting conditions, a particularly large number of continuous degrees of freedom. The possibility of expressing in a comparable form the basic equations of various disciplines has already been noticed in the past.

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The "mechanistic" theories of physics which would have reduced the physical existence to the manifestation in the form of motion have considered the quantity of matter from a formal-mathematical point of view, permitting thus to exhibit the equations of physics as special cases of the equations of a general system of varying masses in motion, as also of mass points. They must also make evident these analogies.

Next to the truly mechanical theories, which present more or less detailed pictures of the structure of matter, there has been an attempt, almost from the beginning, but more particularly from the middle of the nineteenth century, to adopt a specific method from analytical mechanics in the manner of J.L. Lagrange; in order to bring under the same general principles all the considered problems, there has been an effort to reduce the fundamental laws of an ever larger number of physical disciplines, to the form of those principles. From a purely phenomenological viewpoint, this could permit the identification of notions - energy, forces, etc. - entering them with certain physical entities. For systems with a finite large number of degrees of freedom, this development is mainly connected with research undertaken on cyclical systems and their applications in the reciprocal laws of mechanics by W. Thomson (Lord Kelvin), J.J. Thomson and H. von Helmholtz.

Eventually, even Lagrange applied his principles to some continuous systems (liquids, flexible strings and plates, etc.). After further elaboration of these approaches, particularly with the development of the theory of elasticity associated with A.L. Cauchy, as well as under the influence exerted by the development of other physical, particularly optical, theories, people became more and more accustomed to considering even continuous systems as autonomous objects of mechanics (with an infinite number of degrees of freedom), since although these systems stand in formal analogy to the old point mechanics, they can perfectly well be treated independently. The "mechanics of deformable continua", as an autonomous discipline, comprises under the formal statements, next to the usual theory of elasticity and hydrodynamics, all the related physical manifestations in the continuously extending media considered here.

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The development of these ideas has certainly been influenced by the discipline of *thermodynamics* which, in principle, tries to embrace the totality of physics and, this way, by putting forward everywhere the general energy function, hence a potential, it naturally yields analogous forms to the fundamental equations of various fields.

All these relations have been treated in the literature of mechanics and physics in many different ways. A lot of what was said in particular in the field of point mechanics, as also of systems with an infinite number of degrees of freedom, can be immediately extended to the continuous systems. Let us mention already the names of only a few authors who have paid special attention to the relations that we will discuss here and that we will often have the occasion to cite in the sequel: W. Voigt (1895–1896), P. Duhem (1911), and E. and F. Cosserat (1909) (For development of a similar kind, what follows has been influenced in many ways by some of the lectures given in Göttingen by D. Hilbert.)

The purpose of this work demands that, in what follows, the *pure* formal-mathematical factor stands in the foreground, by formulating the statements as well as their combinations in a homogeneous and in a, as simple and elegant, way as possible. The research of the mechanical and physical significance of the quantities and equations as well as the proper analytical-mathematical theory are included in various contributions to volumes IV and V—of the present encyclopaedia—where the various disciplines are discussed.

As a uniform mathematical formulation, which is the easiest to apply to the totality of all individual laws, we have used the *variational principle*. However, we find unsatisfactory the form that we observe as a rule in the calculus of variations, and where the unknown functions are determined in such a way that a certain defined integral containing them, acquires an extremal value. Here we find much more preferable the form that yields the variational computation as a necessary condition of the extremal and which has always been expressed by the principle of virtual work: "Let there be an ordered set of quantities $X, \ldots, X_a, \ldots$ dependent on the unknown function x of $a, \ldots, c$ and their derivatives; these functions should satisfy the condition that a determined integral of a linear form represented by these $X, \ldots, X_a, \ldots$ as coefficients, of the arbitrary functions $\delta x, \ldots$ of $a, \ldots, c$ and their derivatives

$$\int \ldots \int \left\{ X\delta x + \ldots + X_a \frac{\partial\ \delta x}{\delta a} + \ldots \right\} \quad da \ldots dc$$

or a sum of such integrals – vanishes identically for all $\delta x, \ldots$ (or else for all those satisfying certain auxiliary conditions)."

The advantage offered by the application of such a variational principle as a basis, as compared to other possible formulations, or even by taking into consideration the fundamental laws, is mainly that the variational principle is able to determine by a single formula the behaviour of the medium under consideration, in all places and at any instant of time, and especially to cover, besides the equations within the enclosed volume, both the boundary conditions and the initial conditions. Moreover, in its pregnant brevity, it is, in a way, much more transparent than the basic laws and, consequently, it possesses a substantial *heuristic* value for the exploration of new areas, for the expression of other generalisations, etc. This is particularly stressed through the intimate relation of the variational principle with thermodynamics. On the other hand, its claim to generalisation is of demonstrative value for the foundations of physical theories. But the variational principle, through the acceptance of coordinate transformations, has also another advantage against the explicit (field) equations; it often permits an easier understanding of the *invariant theoretical nature* of the considered problem, the question about the transformation groups which it leaves unaltered, with no need to introduce any special symbolism.

After an introductory discussion of the notion of continuum and its kinematics we shall present in the first chapter of this work the basic statements of *statics*, and in the second those of the *kinetics*, but regardless of the kind of the force effects that one of them exerts on the continuum. The nature of these force effects, and especially their dependence on the position and the motion of the continuum (*dynamics*), will be discussed in the third chapter, in which we classify the various disciplines; finally, in the same chapter we shall give a short draft of the relation with the laws of thermodynamics on the one hand, and, on the other, we shall stress the behaviour of some statements under transformations of the space and time coordinates and also the interpretation of the *relativistic theory of electrodynamics*.

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2. **The Notion of Continuum**
2a. **The continuum and its deformation**

The general three-dimensional extending continuous medium to which the following considerations apply means - abstraction made of specific properties of matter - a set of material "particles" which (a) are individually identifiable and (b) fill continuously the space within a regular bounded domain. The first property can be expressed by the fact that each particle is identified thanks to three parameters *a, b, c* (in modern terms, a labelling with material coordinates X^K, $K = 1, 2, 3$) so that under any condition that we may consider the medium, they always occupy a different place; the variable volume V_0 of these (particles labelled) *a, b, c* enclosed within the regular surface S_0, characterises the quantity of matter considered here. The second requirement means that the positions of all particles fills (after deformation and motion) a volume V bounded by the regular surface S. If the position of a particle is determined by its Cartesian coordinates $(x, y, z = \{x^i,\ i = 1, 2, 3\})$, then such a condition can be given analytically by the three following functions of *a, b, c*

$$x = x(a, b, c), y = y(a, b, c), z = z(a, b, c) \qquad \left[x^i = x^i\left(X^K\right)\right] \tag{1}$$

which map V_0 into V and whose functional (Jacobian) determinant

$$\varDelta = \frac{\partial\,(x, y, z)}{\partial(a, b, c)}\ \left[J = \det\left(\frac{\partial\, x^i}{\partial X^K}\right)\right] \tag{2}$$

inside V_0 does not vanish and is taken positive. We can take a fixed "final" (actual) position for *a, b, c*; then $x - a$, $y - b$, $z - c$, are the components of the translation suffered by each particle in its transition to position (1) and the functions (1) become continuous functions of *a, b, c* as long as we assume that the initially neighbouring particles always remain neighbours. Moreover, we can always suppose that the functions (1) possess enough derivatives with respect to their arguments; disruptions of continuity can be found only at singular points, lines and surfaces (Cf. Voss, Vol 4/1 of EmW, No.9). We shall generally not repeat similar assumptions about further physical occurrences of representative functions.

Each function system (1) fully describes a definite state of deformation of the continuum. Generally speaking, every deformation solution, i.e., every triplet of functions (1) that satisfies the just mentioned continuity conditions, is considered admissible. Restrictions in p. 607 the kind of possible functions will express specific properties of special materials. In any case, the partial derivatives of the functions (1) determine, as we know, the translations, rotations and form changes that suffer every small volume element during deformation (Cf. Abraham, in EmW, IV-14, no. 16).

The basis for the research of the equilibrium solution of any deformation process (1) is obtained by superimposing on it a so-called *infinitesimally small virtual perturbation*, called *virtual* to the extent that it enters arbitrarily in the real existing deformation case [cf. Voss EmW IV-1 No. 30; Voigt (1895–96) and C. Neumann (1879)]. In order to define this notion in a precise mathematical form, without giving up the usual convenient designation and use of the "*infinitesimally small*" quantity, we consider to begin with one of the deformation on which is superimposed another deformation depending upon a parameter σ, with vanishing deformation for $\sigma = 0$, which carries the particle from the original position (x, y, z) to the position

$$\bar{x} = x + \xi(x, y, z; \sigma),$$

[same with (x, y, z) and (ξ, η, ζ)]. This way (ξ, η, ζ) are functions of (x, y, z) and of the parameter σ, which can vary in any small neighbourhood of $\sigma = 0$. Thanks to (1), after elimination of (x, y, z), we can also write the newly introduced deformations in the other form

$$\bar{x} = \bar{x}(a, b, c; \sigma),\ where\, \bar{x}(a, b, c; 0) = x(x, y, z). \tag{3}$$

If f is any of the deformation functions (1) and we consider their derivatives as independent expressions, then we generally note as its "variation" the expression

$$\delta f(x, \ldots, x_a, \ldots) = \left\{ \frac{\partial}{\partial \sigma} f\left(\bar{x}, \ldots, \bar{x}_{a,\ldots}\right) \right\}_{\sigma=0},\ where\, x_a = \frac{\partial x}{\partial a}, \ldots;$$

yet, during the differentiation *a, b, c* remain constant; the operation δ commutes with the differentiation with respect to *a, b, c*:

$$\delta \frac{\partial f}{\partial a} = \frac{\partial(\delta f)}{\partial a}.$$

If the three functions

$$\left.\frac{\partial \bar{x}}{\partial \sigma}\right|_{\sigma=0} = \left.\frac{\partial \xi}{\partial \sigma}\right|_{\sigma=0} = \delta x(x, y, z),\ same\, for\ (x, y, z)$$

which, thanks to (1), can be considered as function of (x, y, z), do not vanish identically in (x, y, z), then, following the usual stability postulate, we can write

$$\bar{x} = x + \sigma\, \delta x(x, y, z), same\, for\ (x, y, z), \tag{3'}$$

if σ is chosen so small that σ^2 is sufficiently small compared to σ, the so given infinitesimally small virtual perturbation of the continuum is then determined up to the factor σ by the three functions $\delta x, \delta y, \delta z$ of x, y, z. We can immediately classify this perturbation under the notion of "infinitesimally small deformation", as studied in the kinetics of continua (Cf. Abraham, EmW IV-14, No. 18) and we also find that the "*virtual form changes*" ["strains", GAM] of these volume elements derived from it, are determined by the following six quantities

$$\frac{\partial \delta x}{\partial x}, \frac{\partial\, \delta y}{\partial y}, \frac{\partial\, \delta z}{\partial z}, \frac{\partial\, \delta y}{\partial z} + \frac{\partial\, \delta z}{\partial y}, \frac{\partial\, \delta x}{\partial z} + \frac{\partial\, \delta z}{\partial x} \frac{\partial\, \delta x}{\partial y} + \frac{\partial\, \delta y}{\partial x} \tag{4}$$

and their "*virtual rotations*" by

$$\frac{1}{2}\left(\frac{\partial\delta z}{\partial y}-\frac{\partial\delta y}{\partial z}\right),\frac{1}{2}\left(\frac{\partial\,\delta x}{\partial z}-\frac{\partial\,\delta z}{\partial x}\right),\frac{1}{2}\left(\frac{\partial\delta y}{\partial x}-\frac{\partial\delta x}{\partial y}\right), \tag{4'}$$

regardless of the σ factor.

A *motion of the continuum* will be interpreted as a consequence of a dependence of the deformation functions upon the time parameter t, and accordingly expressed through the three deformation functions

$$x = x(a,b,c;t),\ y = y(a,b,c;t),\ z = z(a,b,c;t) \quad \left[x^i = x^i\left(X^K,t\right)\right] \tag{5}$$

always depending upon t; these, as functions of all four variables in the necessary neighbourhood, are constant and differentiable. For fixed a, b, c (5) represents the trajectory of a certain specific particle.

Just as exposed above, by including in the formulas only the variable t, next to the motion (5) we also introduce the group of motions for $\sigma = 0$, that was omitted in (5),

$$\bar{x} = \bar{x}(a,b,c;t;\sigma) = x + \sigma\delta x(x,y,z;t),\ \textit{same for}\ (x,y,z)$$

for small values of the parameter σ and we note δx, δy, δzas the definitions of the *virtual perturbations* superimposed on the motion (5).

2b. Adjunction of Physical Parameters, Density and Orientation in Particular

Each physical property of a medium can be described by one or more functions of a, b, c; t which enter in the deformation functions.

In what follows we shall make general use of one such property, the presence of an *invariable mass m* for every volume element V_0 of the medium, which, as an integral over V_0, is expressed as a characteristic density function $\rho_0 = \rho_0(a,\ b,\ c)$ of the medium. By transition to the deformed location (1)

$$\rho = \frac{\rho_0}{\Delta} \qquad \left[\rho = J^{-1}\rho_0\right] \tag{7}$$

results as the true mass density ρ of the distribution of the medium, and the mass in the part V' of V is

$$m = \iiint_{(V')} \rho\, dx\, dy\, dz = \iiint_{\left(V'_0\right)} \rho_0 da\, db\, dc.$$

The variations of the continuum's location in relation to the behaviour of such an adjunction of a physical parameter are not yet firmly laid down. In the meantime, we always leave the mass of such an elementary quantity of matter, i.e., the function $\rho_0(a,b,c)$ unchanged by a virtual perturbation and we replace the density ρ by

$$\bar{\rho} = \bar{\rho}(x,y,z;\sigma) = \rho + \sigma\delta\rho(x,y,z), \tag{8}$$

so that regarding the continuity condition (cf. EmW IV-15, No.7 p.59 on, A.E.H. Love)

$$\delta\rho_0 = \delta(\rho\Delta)\, or\, \delta\rho + \rho\frac{\partial(\delta x)}{\partial x} + \rho\frac{\partial(\delta y)}{\partial y} + \rho\frac{\partial(\delta z)}{\delta z} = 0.$$

The same thing will be valid in the case of motion, i.e., $\rho_0(a,b,c)$ remains independent of t and ρ will be given as in (7).

There is another basic notion which belongs here and which we will use very often, that is, the idea *that for every particle of the continuum, the various directions attached to it possess different characteristic meanings, and that, for this reason, the specification of its orientation belongs essentially to the description of the situation of the continuum*. This kind of representations was developed in the molecular theory, where the bodies of crystalline structure were viewed as molecules; S.D. Poisson (1842) in particular has applied it in order to establish a better theory of elasticity. Recently, E. and F. Cosserat [1907; Théorie des corps déformables, 1909; Heun in EmW IV-11, Part II]) without any reference to molecular representations have treated extensively such continua equipped with a definite orientation in every particle.

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In a more general way, this notion of oriented particles of the continuum can be formulated analytically [Cf. a remark by P. Duhem 1893 p. 206], since we can think of each particle a, b, c of the continuum as equipped with a *trihedron* (triad; GAM) *of axes at right angles* and these three axes have each director cosines α_i, β_i, γ_i ($i = 1, 2, 3$) in order to describe fully the state of such a medium, next to the functions (1) we must also recognize as functions of a, b, c three independent parameters λ, μ, ν (e.g., the Eulerian angles) that define the orientation of such a medium in relation to the coordinate system x, y, z:

$$\lambda = \lambda(a, b, c), \quad \mu = \mu(a, b, c), \quad \nu = \nu(a, b, c). \tag{9}$$

Now, every virtual perturbation of the continuum shall be connected with a *virtual rotation* of this trihedron; this way, we get as a basis a group of rotations depending on a parameter σ and with vanishing $\sigma = 0$, starting from the position (9) and replace λ, μ, ν, being restricted to sufficiently small values of σ, by

$$\bar{\lambda} = \bar{\lambda}(a, b, c; \sigma) = \lambda + \sigma\delta\lambda(a, b, c) \quad same\, for\, (\lambda, \mu, \nu). \tag{10}$$

In this manner it is always possible to interpret λ, μ, ν as well as $\delta\lambda$, $\delta\mu$, $\delta\nu$ either as functions of a, b, c or, with the help of (1), as function of x, y, z. The variations themselves $\delta\alpha_1, \ldots, \delta\gamma_3$ of the director cosines of the three axes are linear homogeneous functions of $\delta\lambda$, $\delta\mu$, $\delta\nu$ obtained through the differentiation with respect to σ of the explicit expressions of $\alpha_1, \ldots, \gamma_3$; the components $\delta\pi$, $\delta\kappa$, $\delta\rho$ of the virtual rotation angle velocity in the three axes, are connected with $\delta\alpha_1, \ldots, \delta\gamma_3$ through the formulas

$$\delta\pi = \beta_1\delta\gamma_1 + \beta_2\delta\gamma_2 + \beta_3\delta\gamma_3 = -(\gamma_1\delta\beta_1 + \gamma_2\delta\beta_2 + \gamma_3\delta\beta_3)\, etc \tag{11}$$

$$\delta\alpha_i = \gamma_i\delta\kappa - \beta\delta\rho_i, i = 1,2,3, \quad etc; \tag{11'}$$

Incidentally, in contrast with the symbol δ used until now, these are not variations of certain definite functions of a, b, c, but become simultaneously linear homogeneous functions of $\delta\lambda$, $\delta\mu$, $\delta\nu$; we set

$$\delta\lambda = l_1\delta\pi + m_1\delta\kappa + n_1\delta\rho, \ etc. \tag{12}$$

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This way, $\delta\pi$, $\delta\kappa$, $\delta\rho$ (given as functions of a, b, c or x, y, z) define also the virtual rotation of the continuum [These are well known kinematic methods of the theory of surfaces (cf. also EmW, Vol. III D3 No.10; G. Darboux, Leçons sur la théorie générale des surfaces) that E. and F. Cosserat have applied (detailed exposition in their "Théorie des corps déformables", 1909)].

All these formulas can be extended immediately to the case of motion via the inclusion of the time parameter t.

2c. **Two- and one-dimensional continua**

By the suppression of one or two of the three parameters a, b, c, we also obtain immediately the statements for the treatment of two- and one-dimensional continua embedded in three-dimensional space [In a certain sense these problems are simpler than those we meet with in three-dimensional media; in fact some of them belong to the problems of continuum mechanics which have received early a very detailed treatment (cf. P. Stäckel in EmW IV-6, Nos. 22-24, also K. Heun in EmW IV-11, No.19, 20)]. In any case, their position is given by

$$x = x(a,b) \ or \ x = x(a) \quad [same\ for\ (x,y,z)]; \tag{13}$$

The parameters vary in an area S_0 (respectively, along a curve C_0) of the plane $a - b$ (respectively a line of arc length a) which through (13) is based upon a surface S (respectively a curve C). Here also we can assign to each particle a triplet of directions, orthogonal to each other [Cf. E. and F. Cosserat, Chapters II and III, 1909], defined by the functions

$$\lambda = \lambda(a,b), \ respectively\ \lambda = \lambda(a)\ [same\ for\ (\lambda,\mu,\nu)]. \tag{14}$$

12.8 The Basic Laws of Statics

3. **The principle of virtual perturbations**
3a. Forces and stresses

In order to construct the dynamic properties of the continuum upon this kinematic scheme, we shall rely upon the notion of *work*. The totality of the forces and stresses of all kinds which affect the continuum, because of its previous deformation conditions, of its position ["placement", GAM] in space or of some external circumstances - initially considered as a whole without regard to their origin—is in one expression, since they achieve, in every virtual perturbation, a "*virtual work*" δA; this is for us of primary importance and we define it as follows: *let* δA *be given as a linear homogeneous function of the totality of values of the perturbation components inside the continuum; and let it be a scalar quantity independent from the choice of the coordinate system*. The coefficients, with which each value of δx, δy, δz enters in δA, are the definition parts of the single active force system; the fact that p 612 these are independent from the virtual perturbations (i.e., the linearity of δA) makes us think that, due to their smallness, these perturbations do not modify the usual force effects exerted on each particle. In order to cover the totality of the laws of continuum mechanics, it is necessary to start from the most general expression of the already described types for δA, that consists of the sum of the linear functions of the quantities δx, δy, δz and their derivatives, in any single point of these expressions, on the line, surface and volume integrals which may compose such an expression. We rather consider, at the beginning, an expression - that we shall later elaborate—that consists of a volume integral extending over the whole region V of the continuum, and also an outer-surface integral extending over its surface S; this way, the first one contains a linear form of the nine derivatives of δx, δy, δz with respect to x, y, z [Such statements for the virtual work have been developed earlier, as obvious generalisations of the formulas of point mechanics, for many special problems.....]:

$$\delta A = \delta A_1 + \delta A_2 + \delta A_3, \tag{1}$$

with

$$\delta A_1 = \iiint_{(V)} \rho(X\delta x + Y\delta y + Z\delta z)dV \quad \left[\delta A_1 = \iiint_{(V)} \rho f_i \delta x_i dV\right]$$

$$\delta A_2 = -\iiint_{(V)} \left(X_x \frac{\partial \delta x}{\partial x} + X_y \frac{\partial \delta y}{\partial y} + \ldots + Z_z \frac{\partial \delta z}{\partial z}\right) dV \quad \left[\delta A_2 = -\iiint_{(V)} \sigma_{ij} (\delta x_i)_{,j} dV\right]$$

$$\delta A_3 = \iint_S (\bar{X}\delta x + \bar{Y}\delta y + \bar{Z}\delta z)dS \quad \left[\delta A_3 = \iint_S \bar{t}_i \delta x_i dS\right].$$

The fifteen coefficients present here, - factors of the already discussed perturbation quantities—will be, for every deformation of the considered medium, definite *finite continuous functions of* x, y, z *or* a, b, c, *along with their derivatives, everywhere, with the eventual exceptions of certain surfaces*. The obvious meaning of statement (1) then is that, in general, we will only take into consideration the continuously distributed *forces* over space as well over singular surfaces and the continuously distributed *stresses*.

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Initially, the first and last terms in δA are constructed in a very much analogous way with the well known work expressions of point mechanics, except that the factor present now is the mass of the volume element ρdV (respectively the surface element dS; so X, Y, Z are to be thought of as components of the acting forces on the mass unit of the medium, and $\bar{X}, \bar{Y}, \bar{Z}$ as components of the forces acting per unit surface on the outer surface, at the proper point. Since $\delta x, \delta y, \delta z$ are the Cartesian projections of a polar vector and since δA, as a scalar, remains invariant under coordinate transformations, these forces are also polar vectors.

Actually, the integral δA_2 is characteristic of continuum mechanics. The nine coefficients $X_x, \ldots, Z_z$ - in the known designation of Kirchhoff [1855, also works 1882, p.287] that measure the influence of the single determining parts of the virtual deformation by the performed work, will be understood as the *components of the stress state* at the point in question, calculated according to its influence upon the unit volume. Their behaviour, during the coordinate transformations, results from the remark that the nine derivatives $\partial\delta x/\partial x, \ldots, \partial\delta z/\delta z$ of the vector components behave during orthogonal coordinate transformations like the nine products of two vectors (a so-called *dyad*) [Here Hellinger refers to F. Klein, Abraham, Gibbs and Wilson, Heun, and to Cayley's matrix calculus; GAM]

$$X_1.X_2, \ldots, Y_1.Y_2, \ldots, Z_1.Z_2$$

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while the bilinear combination $X_x.\partial\delta x/\partial x + \ldots$ remains invariant. Therefore, if we want to speak of *stress dyads*, the stress components must be transformed again as dyad components. It is possible to decompose any dyad in a (symmetric) component consisting of six elements (a *tensor triple* [Cf. Voigt's terminology; Abraham in EmW IV-14, No.17])

$$X_x, Y_y, Z_z, \frac{1}{2}\left(Y_z + Z_y\right), \frac{1}{2}(Z_x + X_z), \frac{1}{2}\left(X_y + Y_x\right) \quad [\sigma_{(ij)} = \frac{1}{2}\left(\sigma_{ij} + \sigma_{ji}\right)] \qquad (2)$$

and as (skew symmetric) component of three elements

$$Z_y - Y_z, X_z - Z_x, Y_x - X_y \quad [\sigma_{[ij]} = \frac{1}{2}\left(\sigma_{ij} - \sigma_{ji}\right)] \qquad (2')$$

representing an *axial vector*. This splitting corresponds to the emphasis given in Section 2 to the two separate statements (4) and (4') of the virtual deformations of the continuum, and when the integrands of δA_2 are split in the same way

$$\sum_{(xyz,XYZ)} \left\{ X_x \frac{\partial\delta x}{\partial x} + \frac{1}{2}\left(Y_z + Z_y\right)\left(\frac{\partial\delta y}{\partial z} + \frac{\partial\delta z}{\partial y}\right) + \left(Z_y - Y_z\right)\frac{1}{2}\left(\frac{\partial\delta z}{\partial y} - \frac{\partial\delta y}{\partial z}\right)\right\}$$

[where the indication below the summation sign means that the summing expression consists of cyclical exchanges of x, y, z and X, Y, Z].

What follows here in particular is that the six quantities (2) determine that part of the stress that performs work in an infinitesimally small proper form change of the continuum [the strains. GAM] and therefore *the true elastic effects*, while the vector (2') makes possible the determination of the part (that performs work), by the virtual rotation of the volume elements, again without form change, and so the *rotation moment* determined by the stress condition. Moreover, from the negative sign in (1), it results that with positive X_x the performed work is positive even with negative $\partial\delta x/\delta x$, which is then measured as *positive pressure*.

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In order to obtain finally from the statement (1) the meaning of the stress component as surface forces [Cf. C.L. Navier, G. Green], we think of the part of the calculated virtual work reached by the stresses inside a part V_I of the continuum delimited by the closed surface S_I, i.e., the part of the integral δA_2 extended over V_I; if the stress components inside V_I are all, without exception, continuous, then by partial integration and application of the "*Gauss theorem*" (see EmW Chapter IV-14, p.12), this goes over to

$$\iiint_{V_I} \sum_{(xyz,XYZ)} \left(\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z} \right) \delta x dV$$
$$+ \iint_{(S_I)} \sum_{(xyz,XYZ)} \left(X_x \cos nx + X_y \cos ny + X_z \cos nz \right) \delta x dS_I,$$

where n means the rotated normal's direction of the surface S_I under V_I at the position of the element dS_I. By comparison with (1), it follows that the stress condition in V_I performs the same virtual work, i.e., it acts exactly as if, next to the volume forces in V_I, upon the surface element dS_I of S_I, computed per unit surface, we had in action the force

$$X_n = X_x \cos nx + X_y \cos ny + X_z \cos nz, (X, Y, Z) \quad [\bar{t}_i = \sigma_{ij} n_j]. \qquad (3)$$

This "*pressure theorem*" of Cauchy, by specialisation of the direction of n, yields, as we know, the meaning of the nine components [Cf. Müller-Timpe in EmW IV-23, No.3a; Helmholtz, 1902].

3b. Survey of the principle of virtual perturbations

Based on the constructions of the above notions, it is possible to transpose immediately the *Principle of virtual perturbations*, governing the statics of discrete mechanical systems to continuum mechanics: In a determined case of deformation, a continuous medium, in which there are present certain volume forces $X\ldots$ and outer surface forces $\bar{X}\ldots$ and a certain stress condition $X_x\ldots$, is then and only then in equilibrium when the total virtual work of these forces and stresses for each virtual perturbation which is compatible with the conditions somehow imposed on the continuum, vanish:

$$\iiint_{(V)}\left\{\rho\sum_{(xyz,X,Y,Z)}X\delta x-\sum_{(xyz,XYZ)}\left(X_x\frac{\partial\delta x}{\partial x}+X_y\frac{\partial\delta y}{\partial y}+X_z\frac{\partial\delta z}{\partial z}\right)\right\}dV+\iint_S\sum_{(xyz,XYZ)}\bar{X}\delta x dS=0. \tag{4}$$

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Actually, J.L. Lagrange had already conducted this transformation, when he established as the basis of his analytical mechanics the [John] *Bernoulli principle of virtual perturbations*; for him, an obvious consequence of the validity of this principle in the point mechanics, is its applicability in his available problems of continuum mechanics, where he always preferred to represent the work expression by a transformation of the limit of the discontinuous system out of or through direct intuition. Ever since, in the further development of the bounding areas of continuum mechanics people have shown a preference for the principle of virtual perturbations; often, they also have, just like Lagrange, relied on the idea that the continuum could be approached through a system of an infinite number of mass points, and that, at the same time, all physical effects in the continuum could be approached through equivalent effects in this approximate system; actually, it seems that the axiomatic specification of this relationship which, for the convertibility of these analogies, needs to postulate, above all, the necessary continuity requirements by strict deduction, does not seem as yet to have been obtained. In the meantime, for continuum mechanics, we prefer and place on top as the *highest axiom* the initially formulated principle itself. And we adopt this standpoint so much more willingly when we consider that the representation of the continuously extending media is much more natural than the abstract "mass points" of the point mechanics [Recently, this view had been particularly supported by G. Hamel, 1908, p.350 - also Hamel's textbook of 1912 where he gives a complete axiomatics of continuum mechanics, that resolves a basic principle like the one used here in a series of independent propositions]. The certainty of the correctness of this axiom is based on one hand on the fact that such a statement corresponds to our general ideas and thinking habits about physics, but mainly on the fact that it is appropriate enough to sufficiently represent the facts of experience.

3c. **Application to continuously deformable continua**

The well known formal operations of the calculus of variation show how easily we can, in many cases, transform the principle of virtual perturbations in a great number of equations between forces and stresses. As a start, if we consider only as typical the sufficiently continuous deformable medium, which is in no way restricted by side conditions, then the condition (4) for every system of continuous functions $\delta x, \delta y, \delta z$ is fulfilled. The transformation of (4) by partial integration, if the forces, stresses and their partial derivatives are always continuous in V, yields then the equations

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1) at every point in the domain V

$$\frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}+\frac{\partial X_z}{\partial z}+\rho X=0\,(X,Y,Z)\quad\left[\frac{\partial\sigma_{ij}}{\partial x_j}+\rho f_i=0\right]\tag{5a}$$

2) at every point of the bounding surface S with outer pointing normal directions n

$$X_x\cos\,nx+X_y\cos\,ny+X_z\cos\,nz=\bar{X}\quad\left[\sigma_{ij}n_j=\bar{t}_i\right].\tag{5b}$$

Therefore, along with the boundary surface condition, we obtain the so-called "stress equations", *that offer necessary and sufficient conditions, so that a determined system of forces and stresses acting at a certain position in a freely deformable continuum be in equilibrium* [These equations are similar to those of A.L. Cauchy, 1828.] Certainly, these conditions are by no means sufficient for us to determine the stress and force components: in order to do this we must introduce the relations that we will treat later, and which emphasize the dependence of the forces and stresses from the actually existing deformation or from other external sources (Cf. Stäkel in EmW IV-6, No.26, and Müller-Timpe in EmW IV-23, No.3b).

In (4) and (5) the independent variable coordinates are in the *deformed* configuration [Hellinger uses "condition". GAM] of the continuum, and the force and stress components find their evident meaning as effects upon mass units and with respect to the surface unit of the medium in a deformed configuration. In contrast to this, following S.D. Poisson's works [Poisson 1829, 1831; This difference has often been overlooked, since at closer examination of infinitesimally small deformations of a stressless quiet state, it actually vanishes so it has only been shown to advantage in the development of the theory of elasticity with finite deformations] people often use a, b, c, interpreted as coordinates at the initial site of the medium, as independent variables; it is true that this leads to components of lesser immediate physical importance, but from the analytical point of view it is more convenient for many purposes. This happens namely when we set [This is Nanson's formula in modern treatments. GAM]

$$kdS_0=dS,\tag{6}$$

and Equation (4) becomes

$$\iiint_{(V_0)}\left\{\rho_0\sum_{(xyz,X,Y,Z)}X\delta x-\sum_{(xyz,XYZ)}\left(X_a\frac{\partial\delta x}{\partial a}+X_b\frac{\partial\delta y}{\partial b}+X_c\frac{\partial\delta z}{\partial cz}\right)\right\}dV_0+\iint_{S_0}\sum_{(xyz,XYZ)}\bar{X}k\delta x dS_0=0\tag{7}$$

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and therefore

$$\Delta.X = X_a \frac{\partial x}{\partial a} + X_b \frac{\partial y}{\partial b} + X_c \frac{\partial z}{\partial c} \quad (X, Y, Z; x, y, z) \quad \left[\sigma_{ij} = J^{-1} T_i^K \frac{\partial x_j}{\partial X^K}\right]. \qquad (8)$$

Moreover, as it follows by resolution and comparison with (3), X_a, Y_a, Z_a, the components of the surface forces acting upon an element of the surface $a = const.$, thanks to the stress condition in the material lying to the side of increasing a, are calculated upon the unit surface in the actual position in the space $a - b - c$ [CF. Müller-Timpe in EmW IV-23, No.9, and also the elaborate presentation (predicting of course the symmetry of the stress dyad) by E. ad F. Cosserat, 1896]. Just like (5a) and (5b) result from (4), from (7) there results a new form of the equilibrium conditions:

$$\frac{\partial X_a}{\partial a} + \frac{\partial X_b}{\partial b} + \frac{\partial X_c}{\partial c} + \rho_0 X = 0 \, in \, V_0(X, Y, Z) \, [\frac{\partial}{\partial X^K} T^K_{.i} + \rho_0 f_i = 0] \qquad (9a)$$

and

$$X_a \, \cos n_0 a + X_b \cos n_0 b + X_c \cos n_0 c = k\bar{X} \, on \, S_0, (X, Y, Z) \quad [N_K T_i^K = k\bar{t}_i], \quad (9b)$$

where n_0 means the outer normal direction to the surface element dS_0 in the space $a - b - c$.

[In modern treatments, Equations (9a) and (9b) are referred to as the Piola-Kirchhoff format of the equilibrium equations. GAM].

3d. **Relations with rigid bodies**

It is also possible to derive the equilibrium conditions (5) in a somewhat different manner, from the principle (4). We obtain then the relationship with the "*Rigidification principle*" of A.L. Cauchy [cf. Cauchy, 1822 and 1828; Stäkel in EmW IV-6, No.26, Müller-Timpe in EmW, IV-23, No.3b], often used in the composition of his works. That is, each piece cut off the deformed continuum, under the influence of the intervening volume forces on its inside and of the intervening forces (3) on its outer surface, must be like a rigid body in equilibrium. To this purpose, we only need to consider certain discontinuous perturbations which, of course, will destroy the coherence of the continuously deformable continuum and which initially do not need to make δA vanish; but we can succeed if we approach it through a group of continuous virtual perturbations.

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So we approach a perturbation, which has in a domain V_1 of V constant values $\delta x = \alpha$, $\delta y = \beta$, $\delta z = \gamma$ with the boundary surface S_1, but outside V_1 it vanishes (i.e., a *translation* of the domain V_1) by steady virtual perturbations, while V_1 will be surrounded by any small domain V_2; inside this δx, δy, δz of α, β, γ decrease constantly to zero. For such a virtual perturbation it follows from (4):

$$\iiint_{(V_1)} \rho(X\alpha + Y\beta + Z\gamma)dV_1 + \iint_{(S_1)} (X_n\alpha + Y_n\beta + Z_n\gamma)dS_1 + \iiint_{(V_2)} \sum_{(xyz,XYZ)} \left(\rho X + \frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z}\right) \delta x \, dV_2 = 0$$

where n denotes a component in dS_1 of V_1. If we let V_2 become smaller and smaller, then the last integral will become sufficiently small as the X, X_x and their derivatives remain finite and since α, β, γ are whichever, there result the three equations

$$\iiint_{(V_1)} \rho X dV_1 + \iint_{(S_1)} X_n dS_1 = 0 \quad (X, Y, Z). \tag{10}$$

These are exactly the equations, in the above mentioned sense - through the application of the so-called *strong-point principle* ("Schwerpunktsatzes") - that govern the piece V_1 seen as rigid and cut out of the continuum. Because of the arbitrariness of the domain V_1, it is easy to obtain from (10) the equations (5a) (Cf. Müller-Timpe in EmW, IV-23m, p.23).

If we proceed in the same manner with a rigid rotation of a part of domain V_1 with the components $qz - ry$, $rx - pz$, $py - qx$, then we have the following equations:

$$\iiint_{(V_1)} \left(\rho(Zy - Yz) + Y_z - Z_y\right) dV_1 + \iint_{(S_1)} (Z_n y - Y_n z)\, dS_1 = 0,\ (X, Y, Z)) \tag{11}$$

This can only fully agree with the equilibrium of a domain V_1 as a rigid body, if we set opposite to the moments of the forces X, Y, Z, distributed in space, and to the surface forces X_n , Y_n, Z_n, another rotation moment affecting directly the volume element, calculated as the vector element (2') of the stress dyad. If then we postulate the surface part in the usual form, so that the sum of moments of the volume and surface forces vanishes, then we obtain immediately the symmetry of the stress dyad [Hamel has included this requirement in his axiomatics of the mechanics of volume elements under the expression "Boltzmann's axiom"].

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In close relationship with this fact, there is another interpretation of the principle of virtual rotations which, from the outset, considers as given only the real *force*, the mass forces X, Y, Z and the surface forces$\bar{X}, \bar{Y}, \bar{Z}$; it is the following easily improved formulation of G. Piola [Modena Mem., 1848]: For the equilibrium it is necessary that the virtual work of the specified forces

$$\iiint_{(V)} (X\delta x + Y\delta y + Z\delta z)dV + \iint_{(S)} (\bar{X}\delta x + \bar{Y}\delta y + \bar{Z}\delta z)dS$$

vanishes for all pure translational virtual perturbations of the entire domain V. These auxiliary conditions for the perturbations are mainly expressed by the nine partial differential equations

$$\frac{\partial \delta x}{\partial x} = 0, \frac{\partial \delta x}{\partial y} = 0, \ldots, \frac{\partial \delta z}{\partial z} = 0.$$

then, according to the well known calculation of variations, we can introduce nine necessary Lagrangian factors [multipliers, GAM] $-X_x, -X_y, \ldots, -Z_z$, and thus we obtain exactly the equations (4) of the former principle, proving this way the components of the stress dyad as Lagrange multipliers of certain rigidity conditions. Of course, they are not determined through this variational principle; they rather play exactly the same role as the internal stresses in the static undetermined problem of the mechanics of rigid bodies [Cf. also Stäckel in EmW IV-6, no.26, p. 550, and Müller-Timpe in EmW IV-23, no.3b, p.24].

If we actually impose the same requirement for all rigid motions of V (instead of for translations only), then we obtain exactly the Piola statement repeated in Vol. IV that according to the six auxiliary conditions it yields only six Lagrangian multipliers and so a symmetric stress dyad.

3e **Two- and one-dimensional continua in three-dimensional space**

All the foregoing statements can be immediately proved for the two- and one-dimensional continua embedded in a three-dimensional space, as it was mentioned at the end of Paragraph 2(32). The only modification is that the dimension of the integration domain changes, and that instead of the derivatives of the virtual perturbations along the three space coordinates, these enter along the two or one coordinates, respectively, inside the deformed medium.

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To begin with, let us consider in detail a two-dimensional continuum that, in the deformed configuration, forms a coherent surface-part S with a border curve C; let there be upon S - for the sake of simplicity – a system of orthogonal parameters u and v that define the length and surface elements given by

$$ds^2 = E\,du^2 + G\,dv^2, \quad dS = h\,du\,dv, \quad h = \sqrt{EG},$$

and ρ denotes the surface density of the mass over S. Then we consider the virtual work

$$\delta A = \iint_{(S)} \sum_{(xyz,XYZ)} \left\{ \rho X \delta x - \left(\frac{X_u}{\sqrt{E}} \frac{\partial \delta x}{\partial u} + \frac{X_v}{\sqrt{G}} \frac{\partial \delta x}{\partial v} \right) \right\} dS + \int_{(C)} \sum_{(xyz,XYZ)} \bar{X}\, \delta x\, ds. \tag{12}$$

Here X, Y, Z and $\bar{X}, \bar{Y}, \bar{Z}$ mean the components of the force attached to the mass unit over S, respectively to the length unit along C; over the surface $X_u, \ldots$ permit the development of expressions very analogous to the $X_x, \ldots$ On the one hand, they produce certain forces attached to the mass distributed over S, and on the other, a stress condition prevailing over S, so that, thanks to the stress condition, a force

$$X_\upsilon = X_u \cos(\upsilon, u) + X_v \cos(\upsilon, v) \tag{13}$$

is exerted on each line element lying along C on one side per unit length; here υ means the normals' orientation of the element.

For media allowing all kinds of continuous perturbations, it is possible to resolve the condition $\delta A = 0$ of the principle of virtual perturbations into six equilibrium conditions; we transform then δA by the well-known methods of partial integration:

$$\frac{1}{h}\left(\frac{\partial\sqrt{G X_u}}{\partial u} + \frac{\partial\sqrt{E X_v}}{\partial v}\right) + \rho X \ = 0 \, on \, S, (X, Y, Z) \tag{14a}$$

$$X_u \cos \upsilon u + X_v \cos \upsilon v = \bar{X} \quad along \, C, \quad (X, Y, Z). \tag{14b}$$

Here υ means the orientation standing normally to C in the surface S, and turned away from the surface-part under consideration. But it is also easy to transform these equations to the initial parameters a, b, when from the transformed equations of the virtual work we obtain

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$$\delta A = \iint_{(S_0)} \sum_{(xyz,XYZ)} \left\{\rho_0 X - \left(X_a \frac{\partial \delta x}{\partial a} + X_b \frac{\partial \delta x}{\partial b}\right)\right\} da\, db + \int_{(C_0)} \sum_{(xyz,XYZ)} \bar{X} \delta x \frac{ds}{ds_0} ds_0 \tag{15}$$

and so

$$h \, \frac{\partial(u, v)}{\partial(a, b)} X_u = X_a \frac{\partial u}{\partial a} + X_b \, \frac{\partial u}{\partial b}, \quad (X, Y, Z; u, v). \tag{16}$$

By comparing with (13) it follows that $X_a, \ldots$, thanks to the stress condition, means the forces acting on a line element $a = const.$, $b = const$ calculated over the length unit in the $a - b$ domain.

In one-dimensional continua things are presented in much the same way [CF. E. and F. Cosserat, Corps déformables, Chap. II, as well as K. Heun in EmW IV-11, No.19 and P. Stäckel in EmW IV-6, No. 23]. If $s (0 \leq \ s \ \leq l)$ is the length of the arc on the curve built in the deformed shape, then we get

$$\delta A = \int_0^l \sum_{(xyz,XYZ)} \left\{\rho X \delta x - X_s \frac{\partial \delta x}{\partial s}\right\} ds \ + \left[\sum_{(xyz,XYZ)} \bar{X} \delta s\right]\Bigg|_{s=0}^{s=l}, \tag{17}$$

where the meaning of the various quantities is given much as usual, and by arbitrary continuous variations the equilibrium conditions read as

$$\frac{dX_s}{ds} + \rho X = 0 \quad for\, 0 \leq s \leq l,\ (X, Y, Z) \tag{18a}$$

$$X_s = \bar{X}\, at\, s = 0,\ s = l, \quad (X, Y, Z). \tag{18b}$$

Here also, it is sometimes advisable to introduce the initial parameter a as independent, by using the formula

$$\delta A = \int_0^{l_0} \sum_{(xyz,XYZ)} \left\{ \rho_0 X \delta x - X_a \frac{\partial \delta x}{\partial s} \right\} da + \left[\sum_{(xyz,XYZ)} \bar{X} \delta s \right] \Bigg|_{a=0}^{a=l_0}, \quad X_s \frac{ds}{da} = X_a. \tag{19}$$

4. **Extensions of the principle of virtual perturbations**

4a. **Presence of higher perturbation derivatives** (partial translation only)

It is possible to add a whole series of extensions to the statement of the principle of virtual perturbations formulated in Section 3, which allows now, to the greatest extent, to include all the laws concerning continuum mechanics. The first thing consists in admitting in the virtual work the existence of a linear form of the eighteen [spatial] second-order derivatives of the virtual perturbations, e.g., $\partial^2 \delta x / \partial x^2$, per element of volume. In fact, we have introduced here some problems related to these expressions, where it would seem necessary to let the energy functions depend on the second derivatives of the deformation functions. To begin with, this applies to the one- and two-dimensional continua considered (strings and plates [Cf. the discussion of the statement of the potential in Paragraphs 7a, p.645 and also 8a, p.660].

[Here it seems that Hellinger was not aware of such developments by Le Roux in France in 1911–1913; Cf. Maugin [38], Chapter 13. GAM.].

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4b. **Media with oriented particles** (not translated here)

[In this section Hellinger generalizes the presentation of foregoing sections to the case including the Cosserats' trihedron. He essentially relies on the works of W. Voigt (complementing S.D. Poisson's original idea), the Cosserats, J. Larmor, and K. Heun in EmW, IV-11, Nos. 19 and 20. He also considers the special cases of two- and one- dimensional bodies. GAM].

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4c. **Presence of side conditions**

Until now the principle of virtual perturbations has been used mainly in those cases where the continuum was continuously deformable, in every possible way. But in the formulation of the principle there are immediately included such continua *whose mobility is restricted by all kinds of conditions*; actually, some of the first problems treated by Lagrange [Cf. his Mécanique Analytique, 1st part, Section V, Chapter III (non-extensible strings), Section VIII (incompressible fluids).] concern this very case. These conditions are expressed in the first place by equations for the functions (1) and (9) of Section 2, describing the deformations. In these, besides their functions as such, we can also have their derivatives with respect to a, b, c. The equation

$$\omega(a,\, b,\, c;\, x, y,\, z;\, x_a, \ldots, z_c;\, \lambda,\, \mu,\, \nu;\, \lambda_a, \ldots, \nu_c) \;= 0; \quad x_a = \frac{\partial x}{\partial a} \cdots \tag{13}$$

is then typical for every point in the body V_0. It is then possible to set similar expressions for parts of the body, bounding surfaces, etc. In any case, the possible deformations and the possible rotations (if needed) of the added [Cosserats'] trihedron restricted in this way, or are required to satisfy definite relations between rotations of the trihedron and deformation (for example, a certain orientation of the trihedron relative to space or the medium; see above p. 626). The presence of a, b, c in (13) means that the type of conditions may change from one particle to another. If then we apply to (13) the varied deformation, Section 2, (3) or (10), we obtain through differentiation with respect to σ

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$$\delta\omega = \sum_{(x,y,z)} \left(\frac{\partial \omega}{\partial x} \delta x + \frac{\partial \omega}{\partial x_a} \delta x_a + \ldots \right) + \sum_{(\lambda,\mu,\nu)} \left(\frac{\partial \omega}{\partial \lambda} \delta \lambda + \frac{\partial \omega_a}{\partial \lambda_a} \, \delta \lambda_a + \ldots \right) = 0 \tag{14}$$

and since according to Section 2, p.608, the $\delta x_a \ldots$ agree with the derivatives of $\delta x, \ldots$, thereexists here a *linear homogeneous condition for virtual perturbations*.

So the principle of virtual perturbations requires that δA vanishes for all functions $\delta x, \ldots$ satisfying (14). We can then if, by chance equations (14) do not allow the elimination of one of the perturbation components, replace it by the introduction of a Lagrange multiplier [This treatment was first introduced by Lagrange in his Mécanique Analytique] λ in such a way that

$$\delta A + \iiint_{(V)} \lambda \delta\omega \, dV \;=\; 0 \, for \, all \, \delta x, \ldots, \tag{15}$$

what corresponds exactly to the original principle. Eventually, when (13) applies only at isolated surfaces and curves, or actually the continuum fills only one surface or curve, instead of space integrals in (15) we have then surface or curve integrals. The interpretation of the multiplier λ as a "pressure" will be discussed later on (Paragraph 8b, p. 662).

Finally, we should also consider the possibility, which is also well-known from the mechanics of discrete systems, that there can occur "one-sided ["unilateral" in modern jargon. GAM] accompanying side conditions [constraints. GAM] in the form of inequalities - e.g., let the boundary surfaces of the continuum in their motion be restricted only on one side: let the inside (inner) deformation quantities be subjected to certain inequalities (somehow we think of bodies that allow no compression beyond a certain boundary – or some similar arrangement). Then the equilibrium will be given once more by Fourier's principle of virtual perturbations that, namely, for every system of virtual perturbations satisfying the side conditions, the virtual work is negative or zero:

$$\delta A \leq 0.$$

[CF. Voss in the EmW, IV-1, No. 54; formulation by Gauss in 1830 regarding from the start the extension of continua].

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12.9 The Basic Laws of Kinetics [Dynamics]

5a. **The equations of motion of the continuum**

The task of kinetics is to establish which are the motions of which the continuum is the object, as considered until now, when, somehow, certain force actions are exerted on it in time or, on the opposite, which are the actions necessary for the maintenance of a certain motion. At the same time, the action components are thought of, like in statics, as coefficients of the work expression δA, while the manner in which they depend on the function of motion will remain initially open.

At the beginning we will only be concerned with the ordinary media examined in Section 3. The transition from statics to kinetics can be made exactly as in the mechanics of discrete systems with the help of *d'Alembert's principle* (see Voss in the EmW IV-1, No.36); Passing to continuous systems is almost automatic if, as we did in statics (p. 616), we let ourselves be led by the idea of a limit transition to the continuum, by direct comparison, in the sense of what happens in the analogy between systems of points and continua. Lagrange (cf. Méc. Anal., 2nd part, Section XI, §1) also, when treating the problems of hydrodynamics, considered it form the same point of view. It is possible then to express in terms corresponding to d'Alembert's formulation (Traité de dynamique, Paris, 1743; Voss in the EmW IV-1, p.77) for the general mechanics of continua, the following principle: *If we consider the forces and stresses acting during the motion at a definite instant of time on the volume V_0 of the medium, then they are found to be in static equilibrium, in the earlier sense, in so far as we attach to them, at any time, additional forces whose components, calculated per unit mass of the continuum, are equal, by comparison, to the components of the acceleration*:

$$-\frac{\partial^2 x}{\partial t^2} = -x'', -\frac{\partial^2 y}{\partial t^2} = -y'', -\frac{\partial^2 z}{\partial t^2} = -z''.$$

Even if, in many ways, it proves advisable to place this principle at the summit of kinetics, still the question remains open, in what independent constituents it can be decomposed, and to what extent these are independent from the axioms of statics – a question we faced in exactly the same manner in the mechanics of discrete systems.

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Let us remark briefly that this D'Alembert principle contains essentially, on one hand, a statement equivalent to the second law of Newton, i.e., that the acceleration of a volume element considered as free, is the same as the sum of all the forces acting on it; but, on the other hand – something that Hamel (1908, p. 354; also his *Elementare Mechanik*, p. 306ff) has thoroughly proven – i.e., that one of these first constitutive elements contains, logically, perfectly independent expressions: if the forces acting on a continuum are such the ensuing accelerations on each particle, according to the second *Newtonian law*, are compatible with the kinematic conditions of the system, then these accelerations also really occur. If we cease to introduce the principle of virtual perturbations as an equilibrium condition in the D'Alembert principle, then we obtain the variational principle used by Lagrange (Méc. Anal., 2nd part, Section II) as the basic formulation of dynamics. We imagine the motion for every instant t in Section 3, (6), on which is superimposed an infinitesimally small virtual perturbation compatible with the somehow constituting kinematic conditions at the instant t for the continuum; *then the virtual work performed by the sustaining forces must always vanish*:

$$-\iiint_{(V)} \rho(x''\delta x + y''\delta y + z''\delta z)\, dV + \delta A = 0 \tag{1}$$

and this for every instant of time t in the course of the motion. In the case of a rather continuously deformable body, the equations

$$\rho x'' = \rho X + \frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z}, \quad (x, y, z; X, Y, Z) \tag{2}$$

follow; in the same way as in Paragraph 3c, as the equations of motion at any point of the continuum and every time, while the boundary conditions (5b) of Sect. 3 persist for every time t. On the other hand, these equations define the motion only when the relationship between the forces and stress components and the motion functions is established [i.e., the constitutive equations, GAM].

Concerning now the kinematic side conditions, we refer exclusively to the case of so-called *holonomic* conditions which contain *no time derivatives* of the motion functions [If we try to handle the problems with non-holonomic conditions by means of d'Alembert's principle, then we must foresee in continuum mechanics, just like in point mechanics, that the varied motion for small σ satisfies the condition – and even more, condition equations for perturbations will clearly be formally written by replacing the time differentiation by an operation; see below p. 633). Cf. Voss in the EmW, IV-1, Nos. 35 and 38, and bibliography there, particularly works by Hölder in 1896 and by Hamel in 1904]. Such a condition differs from the one considered in Paragraph 4c only through the explicit presence of t:

$$\omega(a,\, b,\, c;\, x,\, y, z;\, x_a, \ldots, z_c; t) = 0. \tag{3}$$

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For the virtual perturbations we shall consider no only the form of this condition in time t; the varied position (for any small σ) must satisfy the condition (3) for the considered fixed value of t, so that through differentiation with respect to σ ("variation of motion at fixed t") there follows

$$\sum_{(xyz)} \frac{\partial \omega}{\partial x} \delta x + \sum_{(xyz,abc)} \frac{\partial \omega}{\partial x_a} \delta x_a = 0 \quad \textit{for every}\, t. \tag{3'}$$

From this we obtain the equations of motion in the sense of Paragraph 4c

5b. **Transition to the so-called Hamiltonian principle**

Now we can also convert some very similar well known developments of point mechanics of the d'Alembert principle into variational principles determining the motion. The main object here is to transform the contributions due to the motion (the sustaining forces) in the variation of a unique determined expression for each motion path.

As with Lagrange [Méc. Anal., 2nd part, Section IV, art. 3], the basic identities are

$$x'' \delta x = \frac{d}{dt}(x'.\delta x) - \delta\left(\frac{1}{2} x'^2\right), (x, y, z)$$

which follow through repeated differentiation from Section 2, (6), with respect to the independent variables σ and t. If we carry this into (1), and considering that the operation symbols d/dt and δ can be taken out of the integrals, regardless of the factor ρ, since as according to the introduction of a, b, c as integration variables, the integration domain V_0 as well as the remaining factor ρ_0 are independent from σ and t, we obtain

$$-\frac{d}{dt} \iiint_{(V)} \rho \sum_{(xyz)} x' \delta x . dV + \delta T + \delta A = 0 \tag{4}$$

introducing in this way, by abbreviation, the *total kinetic energy*

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$$T = \frac{1}{2}\iiint_{(V_0)} \rho_0 \sum_{(xyz))} x'^2 dV_0 = \frac{1}{2}\iiint_{(V_0)} \rho \sum_{(xyz)} x'^2 dV. \tag{5}$$

Equation (4) is the equation used by G. Hamel [Zeit. Math. Phys. 50(1904), p.14] and K. Heun [Lehrbuch der Mechanik, Vol.1, Leipzig, 1906) and in EmW, IV-11, No. 11] under the name of *Lagrangian central equation*, as the basis of the mechanics of systems with a finite number of degrees of freedom, which is then valid in the same sense in continuum mechanics [Cf. Heun in the EmW IV-11, Nos. 19-21)], and is completely equivalent to (1): *The motion takes place so that for every virtual perturbation compatible with the somehow existing conditions at every instant, the time derivative of the virtual work of the quantities of motion ("impulses") x', y', z' per unit mass, is equal to the sum of the variations of the kinetic energy and of the virtual work of the totality of the actions of forces* [if in addition we also vary the time parameter, then it becomes possible to carry over the relation indicated by G. Hamel (Math. Ann., 59 (1904) p. 423, and K, Heun as general central equation to continuum mechanics; cf. Heun, in EmW, IV-11, Nos. 19-21].

If we consider now the motion in the interval of time $t_0 \leq t \leq t_1$, then (4) is valid for every instant, and through integration with respect to t with the assumption that the virtual perturbations vanish at the limits of the interval, it yields the so-called *Hamiltonian principle* [This principle, after it became typical of point mechanics, had been used very early for different specialized fields of continuum mechanics in many different manners (see Voss EmW IV-1, No.42); we can also compare, apart from the bibliography to be mentioned later for each discipline, A. Walter, Diss. Berlin, 1868, as well as the comprehensive presentations in Kirchhoff's Mechanik, p.117ff and W. Voigt's Kompendium, Vol. I, p. 227ff.]: *If over the motion of the continuum we superimpose some virtual perturbations compatible with the existing conditions, which vanish exactly at t_0 and t_1, then the time integral of the sum of virtual work and the variation of the kinetic energy over the interval t_0, t_1, vanishes also*:

$$\int_{t_0}^{t_1} (\delta T + \delta A)\, dt = 0. \tag{6}$$

Since in (6) the virtual perturbations for every time interval can be chosen arbitrarily, then it is all the more easy to conclude from (6), from (4) or from (1) that *these principles are fully equivalent.*

From this principle it is further possible to derive directly the *principle of least action* in its various forms [As an example, the considerations of O. Hölder in "Die Prinzipien von Hamilton und Maupertuis", Gött. Nach. Math.-Phys. Kl, 1996, p. 122ff, can be immediately extended to continua], but it seems that – regardless of those cases referring to systems with finitely many degrees of freedom – we have not as yet found any substantial application for it.
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5c. **The principle of least constraint**

It is also possible to transfer the inertial contribution of the d'Alembert principle, without integration in time, in the variation, that depends on an expression for each motion condition, determined only from the condition at instant t, where of course the occurrence of second-order time derivatives must be allowed. This way was created the Gauss principle of least constraint [CF. Gauss's Werke V, p. 23. The first analytic formulation of this principle, given only orally by Gauss, was published by R. Lipschitz, J. für Math., 82 (1877), p.321ff; and a little later by J.W. Gibbs in Amer. Journ. 2 (1879) p.49; for further bibliography see Voss in EmW IV-1, No. 39], that recently A. von Brill has chosen as the starting point for a systematic treatment of continuum mechanics (Cf. A. von Brill, 1909).

To reach this principle, we take the virtual perturbation of a group of varied motions Section 2, (6), in the following particular way: Each particle a, b, c will occupy at time t the same position and the same velocity as in the real motion, i.e., the following will be valid for each value of t:

$$\delta x(a, b, c; t) = 0, \delta x'(a, b, c; t) = 0, \quad (x, y, z) \tag{7}$$

while the variations $\delta x''$, $\delta y''$, $\delta z''$ of the accelerations are different from zero. It is now possible to use these three functions in every case as defining parts of the perturbations happening in (1). In the case of a freely deformable continuum, this is evident. But in the conditions of the form (3), this will yield through double differentiation with respect to time,

$$\sum_{(xyz)} \frac{\partial \omega}{\partial x} x'' + \sum_{(xyz,abc)} \frac{\partial \omega}{\partial x_a} x_a'' + \ldots = 0,$$

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where the known functions of $x, \ldots, x_a, \ldots$, and their first time derivatives are indicated by the ellipsis. By variation, i.e., differentiation with respect to σ, thanks to (7) at the chosen time t, there follows

$$\sum_{(xyz)} \frac{\partial \omega}{\partial x} \delta x'' + \sum_{(xyz,abc)} \frac{\partial \omega}{\partial x_a} \delta x_a'' + \ldots = 0,$$

and actually this is exactly the conditions represented above for δx. The introduction of the functions $\delta x'', \ldots$ in (1) is then permitted and it yields, with a light reformulation, the following new principle [Cf. Brill, op. cit.]: *If we alter the real motion of a continuum at a definite instant in such a way that the position and velocity of every one particle remain preserved save that the acceleration of the existing side conditions are modified accordingly, then the following integral sums always vanish*:

$$
\begin{aligned}
&-\delta \iiint_{(V)} \frac{1}{2}\rho \sum_{(xyz)} x''2 + \iiint_{(V)} \left(\rho \sum_{(XYZ)} X\delta x'' - \sum_{(XYZ)} X_x \frac{\partial \delta x''}{\partial x} \right) dV \\
&+ \iint_{(S)} \sum_{(XYZ)} \bar{X}\delta x'' dS \\
&= 0.
\end{aligned} \tag{8}
$$

This can be transformed to a Gaussian form

$$
\begin{aligned}
&-\delta \iiint_{(V)} \frac{1}{2}\rho \sum_{(xyz,XYZ)} (x'' - X)^2 dV - \iiint_{(V)} \left(\sum_{(XYZ,xyz)} X_x \frac{\partial \delta x''}{\partial x} \right) dV \\
&+ \iint_{(S)} \sum_{(XYZ)} \bar{X}\delta x'' dS = 0.
\end{aligned} \tag{8$'$}
$$

The main significance of this principle, just like in point mechanics, consists in the fact that it remains valid and fully unaltered also in systems with *non-holonomic* side conditions. If there exists such a condition, in which next to the motion functions and their spatial derivatives also occur the first time differential quotients:

$$\omega(a, b, c; x, y, z; x_a, \ldots, z_c; x', y'z'; x'_a, \ldots, z'_c; t) = 0$$

Then through single differentiation with respect to t, and by variation (differentiation with respect to σ) we obtain, thanks to (7)

$$\sum_{(xyz)} \frac{\partial \omega}{\partial x'} \delta x'' + \sum_{(xyz,abc)} \frac{\partial \omega}{\partial x'_a} \delta x''_a + \ldots = 0$$

which can be no more added as a side condition.

If ω is especially linear in the velocities $x', \ldots, x'_a, \ldots$, then the result is substantially identical with the form in which, often, one does not consider the d'Alembert principle with non-holonomic conditions and in so doing, instead of the simple virtual perturbations introduced only formally, there also occur the acceleration variations.

A further advantage of this principle as compared to the d'Alembertian one, which however does not seem to have been exploited until now in continuum mechanics, consists in that it offers an appropriate basis even for the treatment of dynamic problems with the inequality type of side conditions: all we need to do is to require that the expression (8) for all admissible variations of the acceleration in agreement with the side condition at instant t, with fixed position and velocity of the individual particles, *be smaller or equal to zero*, exactly like Gauss has already remarked in point mechanics [Cf. Gauss, Werke, Vol. V, p.27].

[The rest of Hellinger's contribution is not translated here].

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