Chapter 11
A Course of Continuum Mechanics at the Dawn of the Twentieth Century (Volume III of Appell's Treatise on Rational Mechanics)

Abstract The treatise on *rational mechanics* published in French by Paul Appell starting in 1900 is a unique monument in the mathematical literature of the Pre-World War One period. Here we critically peruse the volume devoted to continuum mechanics (Volume III). This critical examination is performed in the light of what was known at the time, what were the fashionable themes in continuum mechanics in the early twentieth century, what mathematical techniques were preferred, and what was the naturally influential environment (especially among French mathematicians). All these gave a special tune and contents to a treatise that bears the print of its time, especially with an emphasis on subject matters such as potential theory, the consideration of complex variables, the interest for vortices, barotropic and "barocline" fluids, and new notions such as those put forward by J. Hadamard in wave propagation, by H. Villat and V. Bjerknes in fluid mechanics, and the many references to contemporary works by J.V. Boussinesq, A. Barré de Saint-Venant, H. Poincaré, P. Duhem, and the Cosserat brothers. In contrast, we note the few references to foreign works, the non-exploitation of the then recently proposed vectorial and tensorial concepts, and the lack of interest in dissipative behaviours, whether in fluids or in solids, this in accord with the bannered "rationality" of the treatise.

11.1 Prolegomena: On Paul Appell

The mathematician and mechanician Paul Appell (1855–1930) epitomizes the successful scientist of the French third Republic at the "Belle époque" (i.e. the 20 years before World War One): he presents two parallel careers, one as a true scientist and pedagogue and the other as a man of power in the educational system together with an involvement in some political and socio-cultural matters (see the Appendix). Along the first line, he is not as powerful a creative mathematician as his contemporaries Jacques Hadamard (1865–1963) and Paul Painlevé (1863–1933)—and of course Henri Poincaré (1854–1912), obviously an apart genius. He is more

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traditional in his choice of problems and implementation of solutions than Hadamard who later on became influential in the creation of the Bourbaki School. On the other hand, Painlevé went much farther than Appell in his political involvement going all the way as to become prime minister during World War One and even to be a candidate to the presidency of the Republic. From the point of view of applications Appell was closer to civil engineering while Painlevé, although not himself directly involved in such developments, demonstrated a true interest in the recent progress of aeronautics (he created a Ministry of Aviation). Despite and perhaps because of all these objective facts, Appell wrote the most advanced treatise (the object of this contribution) for the period in what was recognized as the utmost field of application of mathematical analysis, rational mechanics. No other treatise of this magnitude was ever published in France after this opus.

11.2 Setting the Stage

Our object of study here is the celebrated "Treatise on Rational Mechanics" of which the first edition was published in French in 1900 after teaching of this matter by Appell for fifteen years at the Sorbonne. A second edition was published in 1909, a third in 1921, a fourth in 1926, a fifth in 1932, and a sixth published in 1941 and edited by Georges Valiron (1884–1955).[1] A final volume V was published later on but essentially written by René Thiry (1886–1968), himself professor of mechanics at the Sorbonne. The present study refers to the third edition as published by Gauthier-Villars in Paris and reprinted in facsimile form by Editions Gabay, Paris, 1991. This edition, together with the supplement written by the Cosserat brothers for the second edition of 1909 and joined by Gabay to this reprint, is selected as being the most representative one, providing a snapshot of our field as of the relevant period, almost a hundred years ago. We have commented elsewhere [26, Chap. 2] about the state of general mechanics, and more particularly continuum mechanics at the time. This is characterized as a transition period between the contributions of the French, English and German "classics" (e.g., Cauchy, Navier, Duhamel, Lamé, Green, Gauss, Stokes, Maxwell, Kelvin, Kirchhoff, Neumann, Clebsch, Barré de Saint-Venant) and the twentieth century (post WWI). This period is permeated by a feeling of achievement and fulfilment in spite of the emerging quantum physics and relativistic mechanics and the very attractive notion of elementary particle. Furthermore, this is accompanied by a mature reflection on the bases of mechanics by such scientists-philosophers as Hertz, Mach, Poincaré, Duhem and Hamel. Here we focus attention on our specialty, continuum mechanics, as presented in Volume Three entitled "Equilibre et mouvement des milieux continus" ("Equilibrium and motion of continuous

[1] G. Valiron is a scientific "grand father" of the writer via Paul Germain according to Mathematical Genealogy.

media"), an opus of some six hundred seventy pages (without the Cosserats' supplement). These are Chaps. XXVIII–XXXIX of the total treatise. It can only be compared to the long article by C. A. Truesdell and R. A. Toupin published in 1960 in the *Handbuch der Physik* edited in Germany by S. Flügge [34]. These authors in fact often cite Appell's treatise.

Inevitably, Appell's exposition refers to the above cited "classic" scientists of the early and mid nineteenth century as definite contributors. These famous personalities are usually referred to simply by their family name or a name attached to a theorem or a typically solved or simply proposed problem. But Appell must also account for some of his contemporary fellow scientists. Among them we identify Joseph Bertrand (1822–1900; mathematician), Gaston Darboux (1842–1917, specialist of the theory of surfaces), François-Félix Tisserand (1845–1896; specialist of celestial mechanics), Joseph V. Boussinesq (1842–1929; all round mechanician of the continuum), François Cosserat (1852–1914, mathematician and astronomer), Eugène Cosserat (1866–1931; civil engineer), Léon Lecornu (1854–1940; professor of mechanics at *Ecole Polytechnique* and himself author of a treatise on mechanics published in three volumes in the period 1914–1918), Marcel Brillouin (1854–1948; mechanician and physicist), Henri Poincaré (1854–1912; the innovative great mathematician), Emile Picard (1856–1941; famous analyst), Gabriel Koenigs (1858–1931; also the author of a known course on mechanics), Henri Bénard (1874–1933; physicist of "convection" fame), Victor Robin (1855–1897; known for his boundary condition), Pierre Duhem (1861–1916; mathematical physicist, epistemologist and historian of science), Jacques Hadamard (1865–1963; the prototype of absentminded mathematician), and Henri Villat (1879–1972; specialist of fluid mechanics). Poincaré,[2] Duhem, Brillouin, Hadamard, Villat and the Cosserat brothers are the most frequently cited authors. Very few foreign authors of this period are cited, but we note the recurring names of Lord Rayleigh (1842–1919), Elwin B. Christoffel (1829–1900), Eugenio Beltrami (1835–1899), Enrico Betti (1823–1892), Tullio Levi-Civita (1873–1941), Alfred B. Basset (1854–1930), A. E. H. Love (1863–1940; the celebrated elastician), Kazimierz Zorawski (1866–1953), and V. Bjerknes (1862–1951; Norwegian fluid dynamicist, designer of climate models and meteorologist). Appell will welcome in his treatise lengthy contributions by the Cosserats, Villat and Bjerknes. All these male authors are usually referred to in the text by a respectful "M." (for *Monsieur*) and not mentioning their first names. There is no need to introduce a corresponding abbreviation for female authors who are here inexistent (although Sophie Germain and Sofia Kovaleskaya as past authors may have been included at some point).

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[2] The reader may be somewhat surprised to see so many references to works by Poincaré on continuum mechanics. In truth, Poincaré delivered at the Sorbonne many one-semester lecture courses on various aspects of continuum physics, renewing the contents every year and bringing each time a synthetic and critical view illustrated by worked out problems. These lectures were usually put in book form by some of the—obviously very few—auditors (see Poincaré [28–30]) while Poincaré was busy with his own deep mathematical researches.

11.3 The Contents of Appell's Volume on the Mechanics of Continua

11.3.1 Some Words of Introduction

It must be realized that Appell's treatise is a formidable enterprise with very few competitors in the world save for some volumes of the German Encyclopaedia of Mathematics (Enz. Math. Wiss.) edited by Felix Klein and Conrad H. Müller early in the twentieth century—but written by a large group of authors—, and obviously the chapters on mechanics in the *Handbuch der Physik* edited by S. Flügge in the 1950s–1960s, but also written by many contributors (among them: C. A. Truesdell, R. A. Toupin, W. Noll, R. Berker, etc.) [33, 34]. Volume III of Appell's is squeezed between Volume I (devoted to statics and dynamics of the point, about 600 pages) and Volume II (devoted to the dynamics of systems and analytical mechanics; about 570 pages) on the one hand, and Volume IV (concerning equilibrium figures of homogeneous and heterogeneous liquid masses, Figures of the Earth and planets; about 630 pages), on the other hand, all this complemented by a Volume V published later on (1933 and 1955; about 200 pages) and devoted to Elements of Tensor calculus, planned by Appell—for the edition of 1926—but effectively written by René Thiry (1886–1968), a student of H. Villat and a professor of mechanics at the Sorbonne. This last volume of great value includes elements of Riemann, Weyl, Eddington and Cartan geometries, all of interest in general relativity and other geometric generalizations of gravitation theory as also in the theory of structural defects (for Cartan's spaces with torsion). In all, this adds up to some two thousands and five hundred pages! But we analyze here only Volume III and its supplement by the Cosserats.

The overall redaction of this part of the treatise is lengthy, quite detailed, accompanied by exercises with given solutions, and historical comments and up dated references to contemporary works. It is clear that arguments used in derivations were polished in the course of many years of teaching. In the case of the mechanics of deformable solids problems are clearly related to civil engineering. In the case of fluid dynamics there is no allusion to the need of theoretical considerations in the emerging aerodynamics as a mathematical branch of hydrodynamics so that a critical work like that of Prandtl [31] on the boundary layer is completely missed. But there is a sure interest in the theory of barotropic fluids and in the propagation of discontinuity waves of which shock waves are the most well known examples. This is probably due to the influence of Hadamard for whom Appell clearly manifests a strong admiration. Dissipative processes are hardly mentioned (e.g., for viscous fluids or plasticity in the cased of solids). This correlates well with the marked interest of Appell for the theory of potential, the realm of nondissipative mechanical behaviours. All this seems to be the general background and framework in which Appell's treatise must be appreciated and sometimes contrasted with.

11.3.2 Vector Analysis and Potential Theory

The first two chapters of Volume III of the treatise (Chaps. XXVIII and XXIX)—viewed as some kind of prerequisites—bear a strong print of Newtonian and nineteenth-century mathematical physics. They naturally call for the proof of standard theorems of what is now called vector analysis: Green's and Stokes' theorems and Green function, and their application to potential theory, a favourite subject of Appell, with applications to Newtonian attraction, Ampère's theory of magnetism, Faraday's, Maxwell's and Gauss' works. In potential theory proper, emphasis is placed on harmonic functions, Dirichlet's principle, and Dirichlet and Neumann boundary conditions. An example of detailed treatment is that granted to the computation of a potential due to a homogeneous ellipsoid (after Dirichlet). References are often made to the courses on analysis by French contemporary mathematicians such as Poincaré, Boussinesq and Bertrand. Examples of applications are borrowed from Tisserand (in celestial mechanics), Poincaré (theory of the potential) and Kelvin, Maxwell (including from this author's treatise on magnetism and electricity), Lipschitz, Riemann, Sommerfeld, Levi-Civita and Betti (for applications in heat conduction, electromagnetism and mechanics).

These chapters deal with vector fields, but without the relatively new convenient intrinsic vector notation of Gibbs and Heaviside. As a matter of fact, Appell is conscious of this shortcoming as he refers (pp. 26–27) to this "special notation" for which he advises the reader to look at the recent French translation of the book by Coffin [12] (an ardent advocate of this "notation") and also the Italian authors Burali-Forti and Marcolongo.[3] This clearly indicates the little enthusiasm demonstrated by French physicists, engineers and mathematicians—since no true French reference on the subject can be cited by Appell—and the frequent difficulty met by French scientists to readily adopt in their lectures advances and notations proposed by foreigners (this also applies to Maxwell's electrodynamics and Einstein's special relativity, in spite of the works by Poincaré).

11.3.3 Equilibrium and Internal Motion of a Continuous Mass

The very title of this short Chap. XXX (20 pages) of primary importance indicates that the author will closely follow Cauchy's magisterial introduction of the concept of stress (cf. Cauchy [10] with a similar terminology). Only general equations are concerned. In accord with Cauchy—generalizing the case considered by Euler -, the "effort" per unit surface (called "traction" or "tension" by Cauchy, and "stress" in English by Rankine in elasticity) may be obliquely applied. Its linear

[3] For these see the remarkable book on the history of vector analysis by Crowe [15].

relation to the local unit normal yields (in modern terms) the notion of "stress tensor"—a word not yet used by Appell although introduced by W. Voigt some twenty years before. Appell could have used the expression of "linear vector function" in the sense of Gibbs. But he does refer to the expression of the "effort" in terms of the director cosines of the normal—his Eq. (4) in page 134—as a "linear and homogeneous function" of these (p. 133) after the 1828 proof of Cauchy exploiting the epoch-making "tetrahedron argument". Because of this lack of use of a condensed and mathematically justified wording, Appell is bound to repeatedly refer to these famous "six quantities" (in fact the six independent component of a symmetric second-order tensor). Here also we witness the shyness or reluctance with which French scientists adopt terms of foreign origin. The result indeed is the general equilibrium or dynamic equations of continua without further constitutive assumption. Also in the line of Cauchy, Appell considers the "ellipsoid of efforts" and possible changes of coordinates but not alluding to tensor transformations. More recent references are to the Cosserat brothers [13] and Brillouin [8].

11.3.4 Hydrostatics and Stability

The next much longer chapter (Chap. XXXI) with about ninety pages deals with hydrostatics, obviously a very special case already dealt with in the eighteenth century. The main notion here is that of "characteristic equation"—nowadays called "equation of state" or "constitutive relation"—relating pressure, density, and perhaps an additional argument such as temperature or a degree of salinity. Traditional considerations are those relating to level surfaces, barometric formulas, and Archimedes' principle. What are more impressive are the lengthy discussions about figures of equilibrium of masses of fluids and the related question of their stability. Here Appell necessarily refers to the buoyancy problem, the old geometric considerations of Bouguer [7] and Charles Dupin, and the (then) recent studies of Guyou [18], Duhem and Poincaré (in various memoirs for the last two authors).

11.3.5 Deformation of Continua

With this Chap. XXXII Appell returns to general notions based on geometry although the expression "differential geometry" is never used. He naturally bases his presentation on the original work of Cauchy [10, 11], but also frequently refers to recent works by Love, Darboux, Poincaré and the Cosserats [13]. Of course he meets with the "strain tensor" the same wording difficulty as with the stress tensor. Principal dilatation and stretches are casually introduced as well as the notion of homogeneous deformation. The continuity equation is practically proved in the

Lagrange-Piola format ($\rho_0 = \rho J$)—cf. Sect. 664. But in contrast he pays only lip service to the notion of compatibility condition, although dealt with before by Navier and Saint-Venant and more recently by E. Cesaro and V. Volterra—cf. the note in Sect. 673 simply referring to a paper by Riquier [32].

11.3.6 Kinematics of Continua

With the intervention of the time variable combined with the previously introduced sketchy theory of deformation comes the kinematics of continua in Chap. XXXIII. Here again Appell grants (p. 278) to Cauchy the role of founder of this kinematics, although he has to acknowledge the primary role played by Euler and Lagrange. Indeed, in what we think to be a rather modern approach, Appell deals at some length with both sets of Eulerian and Lagrangian variables. Like many Frenchmen, he does not seem to be much aware of the inclusive views expanded by Piola in Italy and Kirchhoff in Germany in the middle of the nineteenth century. In this line he compares the continuity condition in Eulerian and Lagrangian-Piola-an formats. It is also at this point that he introduces for the first time the notion of vorticity as a mean rotation (following Cauchy), but he also cites (Sect. 706) the definition of this vector by Boussinesq (true kinematic definition in terms of the instantaneous rotation of a triad of principal directions) and by Stokes (a mechanical definition related to internal friction in fluids). This manifests a specific interest in a notion that Appell will duly exploit in further chapters, an interest shared by many mechanicians of the period after the pioneering works of Helmholtz, Kelvin and others. Furthermore, irrotational motion brings him back to one of his favourite subjects, potential theory.

What remains most original in this chapter is the introduction of Hadamard's classification of propagating discontinuities in continua. This surprisingly occupies more than twenty pages and follows the remarkable work of Hadamard [19] on general properties of wave propagation. Hadamard cleverly distinguished, in a true three-dimensional framework, between dynamic and kinematic aspects of the phenomenon previously described by Riemann, Christoffel and Hugoniot for essentially one-dimensional motions in a fluid. He classifies moving discontinuities in terms of the order of the space and time derivatives of a field that suffer a true discontinuity. Thus a discontinuity surface across which the medium velocity and the strain are discontinuous is a discontinuity surface of the first order (a shock wave in usual jargon) while one across which the acceleration (second-order time derivative of the motion) is discontinuous is called a discontinuity surface of the second order or "acceleration wave". Jump conditions at the crossing of such surfaces replace the field equations across the surface while usual (continuous) field equations remain valid—but obviously with different field values—on either sides of the surface. As we know now irreversible thermodynamics is a necessary ingredient in a good appraisal of this singular phenomenon of propagation. Truesdell and co-workers will capitalize on this type approach in the 1960s–1970s in order to explain the formation of shock waves from weaker discontinuities such as acceleration waves, especially in deformable solids.[4]

11.3.7 General Theorems of the Dynamics of Perfect Fluids

General theorems that govern the dynamics of perfect fluids are the object of the rather long Chap. XXXIV. Only pressure appears as an internal force in the mass of fluid. Remarkably enough Appell gives the Lagrange format of the equations of motion, together with Lagrange's theorem when a velocity potential exists. He introduces the notion of sound speed by examining the propagation of a second-order discontinuity wave in the classification of Hadamard. As to the Eulerian format of these dynamic equations, it is exposed at length with a beautiful example (spherical layer around a solid spherical nucleus) borrowed from his contemporary Basset [5]. Changes of coordinates are discussed in both Lagrangian and Eulerian formats. So-called permanent motions (with velocity field constant in time) are examined along with fluid filaments and Bernoulli and Torricelli theorems. Irrotational such motions are considered in particular. The chapter concludes with the principle of images (e.g. that of a source with respect to a plane) in the manner of Lord Kelvin.

11.3.8 Theory of Vortices

As already remarked the theory of vortices is a subject of great interest to Appell. No wonder, therefore, that he devotes more than seventy pages to the subject (Chap. XXXV). Of course, Helmholtz is the main contributor with the original theorems that he proved in 1858. But works by Kirchhoff, Kelvin, Rayleigh and Poincaré (see the book on vortices by Poincaré [29]) are also often cited. The considered fluid is viewed as perfect. A primary notion is that of *circulation*. A fundamental result is that the flux of vorticity across a fluid surface is constant in time. The celebrated theorem of Helmholtz states the conservation of vorticity surfaces. Vortex lines play a fundamental role. Problems of connectivity of such lines are much relevant. Geometric proofs are favoured. What is more original (but perhaps in the air at the time of the publication of the treatise) is the strong analogy between vortex loops and electric-field lines (cf. p. 432; and again in p. 459 with the magnetic force). A long discussion about vortex loops follows occupying seventeen pages. The determination of the velocity field from vortices is examined at length. Altogether a rare mention—in the treatise—of experimental works is

[4] The present author will even formulate systematically the relevant kinetic and dynamic compatibility conditions in a covariant relativistic framework (cf. [24]).

given in a short section (Sect. 777) following studies by M. Brillouin. Mathematically oriented considerations on the analytic description of vortex surfaces by the German mathematician Clebsch are paid some attention, while, we admit, we do not know about the geometric problem posed by a certain Transon[5] to which six pages are devoted. In all, this sounds like rather classical material. The accompanying exercises are based on problems by Lagrange, Helmholtz, M. Lévy, H. Poincaré, Beltrami and Clebsch.

11.3.9 Parallel Flows

Fluid motions parallel to a given plane are extensively considered in a Chap. (XXXVI) of almost ninety pages. It must be understood that these are motions that depend only on two planar coordinates, the whole picture being time invariant by translation in the orthogonal direction. The vorticity vector then is orthogonal to the family of parallel planes. The resulting essentially two-dimensional picture lends itself well to the introduction of a velocity potential and the exploitation of complex variables, obviously a technique with intricacies much enjoyed by Appell and his contemporaries (remember that France is the country of Cauchy, the inventor of the theory of residues). But here a short pose shows that Appell—short Sect. 789 in pp. 487–488—is aware of the notion of *solitary wave* as observed in 1834 and reported in 1844 by Scott-Russell, and whose mathematical solution owes much to Boussinesq and Lord Rayleigh.[6]

Of particular interest are the vortex motions of a liquid and the related problem of the evolution of vortex tubes. A remarkable canonical (in the sense of Hamilton's formulation) form is introduced here (pp. 497–503) for a function H that defines the coupling between two vortex tubes. This may look as supererogatory in such a treatise, but it probably reflects a personal interest of the author as also his proximity with Poincaré (cf. [29]). Some simple free-surface wave problems are studied (Gerstner trochoïdal waves for swell). However, in a move typical of Appell, the author then makes a gift of some forty pages to Henri Villat[7] in the third edition of the treatise for an exposition of this author's works on the

5 Probably this is Abel E. Transon (1805–1876) a brilliant student at Ecole Polytechnique and Ecole des Mines in Paris, who became mostly interested in problems of geometry.

6 Boussinesq and Rayleigh are dutifully cited, but Appell does not mention Korteweg and de Vries (1885) who are now given most of the credit for the typical solitary-wave solution, which was to gain an extraordinary renown in the 1960s–1970s with the use of computer simulations and the invention of the inverse-scattering method.

7 H. Villat (1879–1972) became first a professor of rational mechanics in Strasbourg, and then a professor of fluid mechanics at the Faculty of Sciences in Paris, while also teaching at the National School of Aeronautics, nicknamed "Sup' Aero". His course there gave rise to a classic book in fluid mechanics [35]. He was the founder of the Institute of Fluid Mechanics of the University of Paris. This was an ancestor of the Institut Jean Le Rond d'Alembert organized by the writer of this contribution.

irrotational motion of a liquid in contact with a fixed obstacle, and the ensuing problems posed by the notion of *wake*. The importance of such a problem is easily realized for the navigation of ships, and—as we known now—for the forthcoming developments of both theoretical and experimental aerodynamics.[8] Here, due to the essential two-dimensionality of the problem, this is but the realm of the exploitation of complex variables and complex-valued functions for which Villat shows an unmatched dexterity dealing with conformal mappings of various types in agreement with Kirchhoff, Riemann and Picard. The consideration of discontinuous plane motions of a liquid allows one to eliminate d'Alembert's paradox thanks to the introduction of stagnation points, attached streamlines and a rear wake of dead flow. This is magisterially dealt with by Villat who concludes with formulas for the pressure force and moment exerted by the flow on the obstacle. We cannot help but feel a great admiration for the cleverness of these evaluations in hydrodynamics. Most references are of course to Villat's own publications in a remarkably wide range of scientific journals (cf. pp. 559–560)—a rather seldom practice for the period.

11.3.10 Barocline Fluids

Barotropic fluids are those for which the state equation (characteristic equation) reads $f(p, \rho) = 0$. Accordingly level surfaces for pressure and density coincide. Barocline fluids are those for which this characteristic equation contains an additional variable, say, temperature or a degree of salinity. The interest for such a situation is obvious in atmospheric air—the domain of meteorology per se—with the influence of temperature and humidity in altitude and in oceanography with variation in the salinity with depth. This yields the notion of *stratified flows* where circulation and vortex formation are critical parameters. The Norwegian scientist Bjerknes[9] was responsible for a large part (34 pages) of the redaction of this Chap. XXXVII. This author pays a specific attention to analogies with electric and magnetic phenomena (e.g., magnetic induction replacing the fluid velocity).

[8] In this respect see Anderson [1].

[9] Vilhelm Bjerknes (1862–1951) is a Norwegian fluid dynamicist and meteorologist who was first an assistant to Heinrich Hertz in Bonn (1890–1891), and then specialized in hydrodynamics and created a true school of meteorology in Bergen and Oslo. He definitely influenced other fluid dynamicists such as Ekman and Rossby. He is largely responsible for the progress made in meteorology in the first half of the twentieth century. His early contact with Hertz probably kindled his life long side interest in electrodynamics. Together with other Norwegian scientists (cf. [6]), he wrote a splendid lengthy—in all, 864 pages—monograph on physical hydrodynamics and its applications to dynamic meteorology. His father Carl Anton Bjerknes was a precursor in the same field of fluid dynamics, already using hydrodynamic analogies to interpret Maxwell's electromagnetism.

Appell seems to be responsible for the writing of the remaining part of this chapter with extension of the theory of vortices and generalizations of formulas for the fluid circulation, this being complemented by bibliographic recommendations.

11.3.11 Elements of Elasticity Theory

With a chapter of 48 pages, we see that elasticity was not a field of utmost interest for Appell. He in fact first reports the bare essentials often referring the reader to more competent specialists such as Boussinesq, M. Brillouin and the Cosserat brothers for more recent works. The presentation of linear elasticity leans heavily on the original work of Cauchy for the general relationship between infinitesimal strains and stresses—exploiting the relationship between two quadrics. He naturally focuses on the case of homogeneous isotropic linear elastic media with only two elasticity coefficients, the Lamé coefficients. Still he mentions the possibility of anisotropy and shows the reduction to twenty one elasticity coefficients in a general case (cf. pp. 612–615). Duhem [17] are called upon for the proof of the inequalities to be satisfied for the positive definiteness of the quadratic form of the elastic energy. He refers to the Cosserats for the question of mixed boundary conditions and a uniqueness theorem. The only "energy theorem" mentioned is the one on reciprocity due to Betti. Three examples of problems are treated in detail referring to the lessons of Gabriel Lamé on the mathematical theory of elasticity.[10] These are exercises which are still given today to students, e.g., the equilibrium of a cylindrical tube.

The rest of the chapter deals with more advanced matter. This includes in brief form (but over some five pages) some of the considerations by the Cosserats and Poincaré on the "spectrum" of solutions of static equilibrium problems, following the rewriting of these equations for the displacement of Cartesian components u_i by the Cosserats as (in our notation)

$$\Delta u_i + \xi \nabla_i \theta = g_i, \tag{11.1}$$

where θ is the dilatation, $\theta = \nabla_i u_i$, and $\xi = (\lambda + \mu)/\mu$, and with prescribed values at a boundary. By an adequate formal expansion this can be compared to the solutions of Laplace and Helmholtz equations, including with orthogonality properties of two distinct solutions. This type of problems and formulations will wait some sixty-seventy years to be expanded by Russian mathematicians in St Petersburg. Another problem of importance for further research is that of the theory of slender bodies and the basic problem of Saint-Venant (for instance the torsion of a straight cylinder). Here again Appell refers to the works published by

[10] The book of Lamé [21] was unique in its class before the publication of a book by Clebsch in Germany. The latter was in fact translated into French, commented upon and much enriched (so as to triple in volume) by de Saint-Venant [16].

the Cosserats in various notes at the Comptes Rendus of the Paris Academy in the years 1907–1908. The chapter concludes with the proof of the formulas for the speed of longitudinal and transverse waves in isotropic linear elasticity, the "ether" (no longitudinal waves) and perfect fluids (no transverse waves) being extreme cases. In the listed problems we find again a classic (due to Lamé), that of the equilibrium of a thick spherical layer with different pressures applied on the outer and inner surfaces.

One remark concerns the relationship of elasticity with the more engineering like strength of materials. Appell apologizes (Sect. 834) for not treating this aspect, even superficially, explaining that in this volume he consumed the available space with the contributions of Villat and Bjerknes. What is more surprising is that, even cursorily, he does not mention the possible existence of an elasticity limit, nor does he allude to the existence of plasticity criteria although these were formulated some forty years before (by Tresca in 1872 and Barré de Saint-Venant in 1873), not to speak of the most recent criterion proposed by Huber in Poland in 1904 (but published in French) and von Mises in 1913 in Germany. This is all the more surprising that Maurice Lévy, one of the engineers-scientists who is otherwise often cited for other problems, was also responsible for the introduction of the notion of rigid-plastic behaviour (no elastic response at all) in 1871.[11] Perhaps that Appell considers that this matter does not enter (yet) the framework of rational (continuum) mechanics.

11.3.12 On Viscous Fluids

This Chap. XXXIX on the equations of motion, of a viscous fluid is especially poor. It is extremely brief (less than 5 pages) and the names of Navier, Stokes and Barré de Saint-Venant are hardly cited. In analogy with the introduction of Cauchy's elasticity in the foregoing chapter, the linear (Newtonian) fluid constitutive equations are presented as an expansion limited to the first order in the six independent components of the symmetric part of the velocity gradient. Conditions of symmetry are evoked but not formulated to justify the reduction to two viscosity coefficients for isotropy. Perfect fluids governed by pure pressure (no shear) and Stokesian fluids are noted as a final point. The reader is referred to Boussinesq, Basset, Kirchhoff and Poincaré for further theoretical developments. Poiseuille is mentioned for the flow in capillaries. There is no mention of such notions as the Reynolds number (albeit introduced in 1883) and transition to turbulence. The situation is quite similar to the one of elasticity versus plasticity in the above examined chapter, but here the question resides in the transition between laminar and turbulent flows. In Appell's view, these subject matters did not yet belong to rational mechanics.

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[11] See historical remarks in the book of Maugin [25].

11.4 The Cosserats' Theory of Euclidean Action

Although this supplement written by the Cosserat brothers was not reprinted in the considered third edition of the treatise, we appreciate that the publisher of the facsimile reprint thought good to join a copy of this supplement to this reprint. This was a nice move as it offered an opportunity to the reader—who until recently had no access to the 1909 book of the Cosserats—to study in a shortened version the main revolutionary ideas expanded by the brothers in 1909. We do not enter the detail of this contribution since we analysed this famous but unread book in greater detail in a separate work [27]. In a nutshell, however, we remind the reader that the Cosserats propose a kind of analytical mechanics of finitely deformable bodies basing on the notion of Lagrangian-Hamiltonian action and the application to it of a group-theoretical reasoning, the invariance of the action under the group of Euclidean transformations (translations and rotations alike). It is this equal footing between translations and rotations that necessarily led the brothers to envisage the possibility of the existence of so-called couple stresses in parallel with the usual notion of stress, the latter then becoming non-symmetric. This will have a brilliant future and blossom in the second half of the twentieth century. If we mention this supplement here, it is because Appell himself, in the preface—written in 1908—to the second edition of his treatise in 1909, expressed an unbounded enthusiasm for the Cosserats' work on this Euclidean action, an enthusiasm matched by that of the American mathematician who reviewed their work, Wilson [36]. It is therefore strange that Appell, in his preface—written in 1920—to the third edition of his treatise, justifies the suppression of the Cosserat's supplement by claiming that limited available space forced him to this move as he wanted to offer some space to Villat and Bjerknes. We do not think that this is a sustainable argument in a global work of this extensive size. We rather surmise that Appell had realized that in spite of the intrinsic depth of the Cosserats' reasoning, the subject was not so much successful, too hard to be grasped by most mechanicians not trained in mathematical physics at the time, and so probably in advance of its time, while Villat's and Bjerknes' works were more in resonance with the fashionable scientific developments of the period. He may also have thought that there was no longer need for the Cosserats' supplement since practically the same text had appeared in two other media (Chwolson's encyclopaedia and separate book form with a new pagination) and in the same year of 1909 (see [14]), and the book version was still available.

11.5 Concluding Comments

Perusing our foregoing comments we recognize that Appell is very much representative of his own epoch, concerning both the selection of treated subjects and the choice of mathematical techniques. His basic reference in the general notions of continuum mechanics remains the great Cauchy. In comparison, Navier, also a great elastician and fluid dynamicist, receives much less attention. The genius of Cauchy is even more enhanced by the fact that the main contributing mathematical technique is the consideration of complex variables—and the ensuing residues theorem, a technique also due in great part to Cauchy. This is obviously related to the fundamental two-dimensional setting of many problems posed in both fluid dynamics and elasticity. Potential theory also appears as an essential tool in which Appell finds a field at his own level of interest (shared by others scientists of the period, e.g., Poincaré). In comparison, the techniques introduced by J.-B. Fourier, powerful as they were, are hardly mentioned due to the weak interest expressed for dynamics in the wave-frequency domain. Also pregnant in many parts of the volume is the great admiration expressed by Appell for some precursors, e.g., Lamé and Barré de Saint-Venant, but also for some of his fellow mathematicians, in particular, Boussinesq, Duhem, Poincaré and Hadamard. For closer contributors, we must account for the importance granted to the Cosserat brothers (in the foundations of continuum mechanics and many subtle aspects of deformable solids), Bjerknes for hydrodynamics, and Villat for the theory of wakes. He finds in Villat another great specialist of the use of complex-valued functions, while Poincaré, Villat and Bjerknes clearly satisfy his marked interest for the theory of vortices. Dealing with a book on "rational" mechanics we find here very few applications to, and mentions of, experiments. Whatever smells too much of this yet "non-rational" framework is discarded or not even mentioned. This applies to behaviours beyond pure elasticity and perfect fluidity. The contents of the next volume (Volume IV) published in 1921—and not examined in the present contribution—comfort this appraisal with an overemphasis—about 630 pages—on the study of the equilibrium figures of fluid ellipsoids and applications to celestial bodies.

Finally, Appell as an active creative mathematician belongs to a period where vector and tensor analyses are not a subject of teaching in French universities, though there are local researches conducted along these lines following Ricci and Levi-Civita. Probably under the influence of the success met by Einstein's general relativity after the proof of validity of some of its consequences (in particular the deflection of light rays by heavy masses), Appell realized that something on tensor analysis should be added to his treatise if one wants to go further than classical rational mechanics. He was already old and sick when this came up in 1925 so that he commissioned René Thiry to write this additional part as a volume V devoted to "Elements of tensor calculus: geometric and mechanical applications".[12] [13] On perusing the contents of this volume we see the extent of the ambitious framework

[12] See Appell [4].

[13] In France it is only with a book (Brillouin [9] by Léon Brillouin (1889–1969)—a renowned physicist and son of M. Brillouin—that tensors and continuum mechanics were firmly associated. But the book is better known in the USA in its English translation than in France. A definite introduction to tensor analysis was successfully given by Lichnrowicz with a first edition in 1946, and a seventh in [23]. It is in this book that most French students of mathematical physics (including the present writer) were exposed to tensor analysis in the period 1950–1970.

provided by Appell for its redaction. Of course re-writing Volume III in that spirit would have been a formidable enterprise that no immediate successor of Appell would endeavour. One had to await such treatises as those of Truesdell, Toupin and Noll in the new *Handbuch der Physik* edited by S. Flügge after World War II [33, 34] for something more or less equivalent, with a nice complement on tensors by J. L. Ericksen in Truesdell and Toupin [34].

A.1 Appendix: Biography of Paul Appell

The following elements of the biography of Appell may be of enlightening interest. He came from a catholic family from Alsace. He left Alsace for Nancy when Alsace was occupied by the Prussian following the 1870–1871 Franco-Prussian war. He became friend with Henri Poincaré (1854–1912) in Nancy; they remained long life friends. He graduated in mathematics from the famous *Ecole Normale Supérieure* (ENS) in Paris, and obtained his Doctoral degree in this matter in 1876. He followed the teachings of several influential scientists: J.B. Serret, Gaston Darboux, Charles Hermitte, Joseph Bertrand, Maurice Lévy and Urbain Leverrier. After being lecturer at the ENS he obtained the Chair of Rational Mechanics at the Sorbonne in 1885, but also taught (we suppose) more applied matters at the engineering school known as the *Ecole Centrale des Arts et Manufactures* (for short, *Ecole Centrale de Paris* or ECP). His mathematical works (in the order of 200 publications; cf. [3]) were essentially in projective geometry, algebraic functions, differential equations and complex analysis. He occupied at some different or simultaneous times, such powerful positions as: Dean[14] of the Paris Faculty of Sciences (1903–1920), Rector[15] of the Academy of Paris (1920–1925), President of the council of the University of Paris, President of the French Society of Mathematics, and Member of the Paris Academy of Sciences. He was a visiting professor at Harvard and Rome. During WWI he created the "National Research Fund" to be considered the ancestor of the actual CNRS (National Centre for Scientific Research). He founded in 1920 the "Cité Internationale Universitaire de Paris" (CIUP) to accommodate foreign students and visiting scholars.[16] Philosophically he was an atheist while politically he would be

[14] The title of "Dean" practically corresponds to the actual title of President.

[15] The administrative title of "Rector of Academy" corresponds, for education, to that of "Prefect of Department" for other matters, but the "Academy" here refers to a larger region than a single department. He is the local representative in charge of implementing the views of the Minister of Education on the organization of education et al levels in State-own schools.

[16] The "Cité Internationale Universitaire de Paris" (CIUP) is a campus like condominium devised to accommodate foreign students. It comprises a general purpose building equipped with cafeterias, restaurants, concert and theatre halls, swimming pool and accommodations for visiting foreign professors. Most buildings for students were built in the course of time with the help of donations from philanthropists and industrialists or foreign governments. This was a unique

classified as belonging to the left of the political spectrum, having been a "Dreyfusard"[17] and an ardent defender of the separation between the State and the Church (in France this is called "radicalism" not to be mistaken for the meaning granted to this word in the USA). He had strong nationalist feelings (especially concerning his beloved Alsace) but proved also to be a true internationalist (being involved in the *League of Nations* in Geneva, and founding the CIUP). In all, Appell demonstrated an amazing level of activity in teaching, research, editing and public service.

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(Footnote 16 continued)
structure in France where the notion of campus did not exist. Exceptionally, a few French students were mixed with foreign students to the benefit of both groups. The writer benefited from such a status during three very rich academic years.

[17] The "Dreyfus affair" (1894–1906) was a political scandal involving false accusations of treason (to the benefit of Germany) against a French Jewish officer. This practically split the French population in two parts, "Dreyfusards" (defending Dreyfus) and "anti-Dreyfusards" (who claimed Dreyfus guilty in spite of proved evidence of the contrary).

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