Chapter 6
The Relations of *Supremum* and *Mereological Sum* in Partially Ordered Sets

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6.1 Basic Axioms and Definitions

Let M be an arbitrary non-empty set and let $\sqsubseteq\ \subseteq M \times M$. We call $\sqsubseteq$ *the relation of being part of* and in case $x \sqsubseteq y$ we say that x *is part of* y, '$x \not\sqsubseteq y$' is to mean $\neg\, x \sqsubseteq y$. *Part of* is the only primitive concept of the theory we are going to present.

In the sequel we use standard logical constants: quantifiers $\exists$ and $\forall$, sentential operators $\neg$, $\wedge$, $\vee$, $\Rightarrow$ and $\Leftrightarrow$. For any set S, $\mathcal{P}(S)$ is its power set, while $\mathcal{P}_+(S) := \mathcal{P}(S) \setminus \{\varnothing\}$. Moreover, let $|S|$ be the cardinal number of S and id_S be the identity relation on S, i.e. $\mathrm{id}_S := \{\langle x, x\rangle : x \in S\}$.

A pair $\langle M, \sqsubseteq\rangle$ is a *degenerate structure* iff it consists of exactly one element, i.e. $|M| = 1$. We say that $\langle M, \sqsubseteq\rangle$ is a *partially ordered set* (*poset* for short) iff it satisfies the following three axioms of reflexivity, transitivity and antisymmetry:

$$\forall_{x\in M}\ x \sqsubseteq x\,, \tag{$\mathtt{r}_\sqsubseteq$}$$

$$\forall_{x,y,z\in M}(x \sqsubseteq y \wedge y \sqsubseteq z \Longrightarrow x \sqsubseteq y), \tag{$\mathtt{t}_\sqsubseteq$}$$

$$\forall_{x,y\in M}(x \sqsubseteq y \wedge y \sqsubseteq x \Longrightarrow x = y). \tag{$\mathtt{antis}_\sqsubseteq$}$$

$\langle M, \sqsubseteq\rangle$ is a *quasi-partially ordered set* (*quasi-poset* for short) iff satisfies ($\mathtt{r}_\sqsubseteq$) and ($\mathtt{t}_\sqsubseteq$). Let **POS** and **QPOS** be respectively the class of all posets and the class all quasi-poset.

We introduce some standard relations definable by means of the only primitive relation and the identity relation:

$$x \sqsubset y \ :\Longleftrightarrow\ x \sqsubseteq y \wedge x \neq y\,, \tag{df $\sqsubset$}$$

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$$x \bigcirc y \;:\Longleftrightarrow\; \exists_z (z \sqsubseteq x \wedge z \sqsubseteq y)\,, \tag{df $\bigcirc$}$$

$$x \wr y \;:\Longleftrightarrow\; \neg\exists_z (z \sqsubseteq x \wedge z \sqsubseteq y)\,. \tag{df $\wr$}$$

In case $x \sqsubset y$ (resp. $x \bigcirc y$, $x \wr y$) we say that x *is proper part of* y (resp. x *overlaps* y, x *is exterior to* y[1]). Only by definitions the relation $\sqsubset$ is irreflexive, the relations $\bigcirc$ and $\wr$ are symmetric, and $\wr$ is the (set theoretical) complement of $\bigcirc$, i.e.:

$$\forall_{x \in M}\; x \not\sqsubset x\,, \tag{irr$_\sqsubset$}$$

$$\forall_{x,y \in M} (x \bigcirc y \iff y \bigcirc x)\,, \tag{s$_\bigcirc$}$$

$$\forall_{x,y \in M} (x \wr y \iff y \wr x)\,, \tag{s$_\wr$}$$

$$\forall_{x,y \in M} (x \wr y \iff \neg\, x \bigcirc y). \tag{6.1}$$

If $\sqsubseteq$ satisfies (r$_\sqsubseteq$), then the relation $\bigcirc$ is reflexive and $\wr$ is irreflexive, i.e.:

$$\forall_{x \in M}\; x \bigcirc x\,, \tag{r$_\bigcirc$}$$

$$\forall_{x \in M}\; \neg\, x \wr x\,, \tag{irr$_\wr$}$$

If $\sqsubseteq$ satisfies (t$_\sqsubseteq$) and (antis$_\sqsubseteq$), then $\sqsubset$ is transitive, i.e.:

$$\forall_{x,y,z \in M} (x \sqsubset y \wedge y \sqsubset z \Longrightarrow x \sqsubset z). \tag{t$_\sqsubset$}$$

If $\sqsubseteq$ satisfies (antis$_\sqsubseteq$) then $\sqsubset$ is asymmetrical, i.e.:

$$\forall_{x,y \in M} (x \sqsubset y \Longrightarrow y \not\sqsubset x). \tag{as$_\sqsubset$}$$

Notice that from (df $\sqsubset$), (r$_\sqsubseteq$) and (antis$_\sqsubseteq$) we have that:

$$\forall_{x,y \in M} (x \sqsubset y \iff x \sqsubseteq y \wedge y \not\sqsubseteq x), \tag{6.2}$$

and from (df $\sqsubset$) and (r$_\sqsubseteq$) we get that:

$$\forall_{x,y \in M} (x \sqsubseteq y \iff x \sqsubset y \vee x = y). \tag{6.3}$$

To facilitate considerations in the sequel, we introduce three operations P, PP, O whose domain is M and co-domain $\mathcal{P}(M)$:

$$\mathsf{P}(x) := \{y \in M \mid y \sqsubseteq x\}\,, \tag{df P}$$

$$\mathsf{PP}(x) := \{y \in M \mid y \sqsubset x\}\,, \tag{df PP}$$

[1] Sometimes terms ‘incompatible’ or ‘disjoint from’ are used instead of the one used by us.

$$\mathsf{O}(x) := \{y \in M \mid y \circ x\}\,. \tag{df O}$$

Thus $\mathsf{P}(x)$ is the set of all parts of x, $\mathsf{PP}(x)$ the set of all its proper parts and $\mathsf{O}(x)$ the set of all these objects each of which has a common lower bound with x. Of course, the conjunction of ($\mathtt{r}_{\sqsubseteq}$) and ($\mathtt{t}_{\sqsubseteq}$) is equivalent to the following condition:

$$\forall_{x,y \in M}\big(x \sqsubseteq y \iff \mathsf{P}(x) \subseteq \mathsf{P}(y)\big). \tag{rt$_\sqsubseteq$}$$

Moreover, by ($\mathtt{r}_{\sqsubseteq}$) we obtain:

$$\forall_{x \in M}\ \mathsf{P}(x) \subseteq \mathsf{O}(x) \tag{6.4}$$

and by ($\mathtt{t}_{\sqsubseteq}$) we obtain:

$$\forall_{x,y \in M}\big(x \sqsubseteq y \Longrightarrow \mathsf{O}(x) \subseteq \mathsf{O}(y)\big), \tag{6.5}$$

$$\forall_{x,y \in M}\big(\mathsf{P}(x) \subseteq \mathsf{O}(y) \Longrightarrow \mathsf{O}(x) \subseteq \mathsf{O}(y)\big). \tag{6.6}$$

If **C** is a class of structures then any given sentence is said *to be true in this class* iff it is true in (satisfied by) every structure from this class. If φ is a formula expressing some property of the elements of **C**, then $\mathbf{C}+\varphi$ is the class of all these structures from **C** that satisfy φ.[2]

The symbol 'ι' is interpreted as the standard description operator, which we use to build the expression '$(\iota\, x)\ \varphi(x)$' being the individual constant 'the only object x such that $\varphi(x)$'. To use it, first we have to know that there exists exactly one object x such that $\varphi(x)$, i.e., the formula $\varphi(x)$ must fulfill the following two conditions:

$$\exists_x\ \varphi(x)\,,$$

$$\forall_{x,y}\big(\varphi(x) \wedge \varphi(y) \Rightarrow x = y\big).$$

In such case we also write: $\exists^1_x\ \varphi(x)$.

[2] All the notions such as *formula*, *sentence*, *satisfy*, *true* are imprecise here, since we do not present any formal theory – we have no alphabet, nor language specified. However this imprecision is intended here, since we do not want to get bogged down in formal details but rather would like to focus on semantical or model theoretical aspect of the problem. Yet it should be noticed, that with some effort the notions addressed in this footnote could be precised and formal theory could be built, similarly as it was for example done in Part B of Pietruszczak (2000). Then we would have some elementary language with suitable definitions of formulas and sentences for which the usual notion of model and satisfaction could be given. Then by a class of structures we would mean the class of all models of a given set of axioms. We use the notion of *class*, since the collections of structures considered are too big to be just sets.

6.2 The Supremum Relation for Posets

Let $\langle M, \sqsubseteq\rangle$ be a poset, $S \subseteq M$ and $x \in M$. Let us recall a couple of basic definitions.

We say that x is an *upper bound* (resp. a *lower bound*) of S iff $\forall_{y\in S}\ y \sqsubseteq x$ (resp. $\forall_{y\in S}\ x \sqsubseteq y$). We say that x is a *supremum* of the set S (with respect to $\sqsubseteq$) iff x is the least upper bound of S; formally:

$$x \mathsf{\ sup\ } S \ :\Longleftrightarrow\ \forall_{z\in S}\ z \sqsubseteq x \wedge \forall_{y\in M}(\forall_{z\in S}\ z \sqsubseteq y \Longrightarrow x \sqsubseteq y)\,. \qquad (\mathtt{df}\ \mathsf{sup})$$

Using the operation P we can give an alternative version of the definition:

$$x \mathsf{\ sup\ } S \ \Longleftrightarrow\ S \subseteq \mathsf{P}(x) \wedge \forall_{y\in M}(S \subseteq \mathsf{P}(y) \Longrightarrow x \sqsubseteq y)\,. \qquad (\mathtt{df}'\ \mathsf{sup})$$

The immediate consequences of ($\mathtt{df}$ sup) are stated in the following lemma.

Lemma 1. (i) *Only by its definition the relation* sup *is monotonic, i.e.*:

$$\forall_{S_1,S_2\in\mathcal{P}(M)}\forall_{x,y\in M}(S_1 \subseteq S_2 \wedge x \mathsf{\ sup\ } S_1 \wedge y \mathsf{\ sup\ } S_2 \Longrightarrow x \sqsubseteq y). \qquad (\mathrm{M}_{\mathsf{sup}})$$

(ii) *From* ($\mathtt{r}_{\sqsubseteq}$) *it follows that*:

$$\forall_{x\in M}\ x \mathsf{\ sup\ } \{x\}\,, \qquad (6.7)$$

$$\forall_{x\in M}\ x \mathsf{\ sup\ } \mathsf{P}(x)\,. \qquad (6.8)$$

(iii) *From* ($\mathtt{antis}_{\sqsubseteq}$) *it follows that if a set has a supremum, then it is unique, i.e.*:

$$\forall_{S\in\mathcal{P}(M)}\forall_{x,y\in M}(x \mathsf{\ sup\ } S \wedge y \mathsf{\ sup\ } S \Longrightarrow x = y). \qquad (\mathrm{U}_{\mathsf{sup}})$$

(iv) *From* ($\mathtt{r}_{\sqsubseteq}$) *and* ($\mathtt{antis}_{\sqsubseteq}$) *it follows that*:

$$\forall_{x,y\in M}(y \mathsf{\ sup\ } \{x\} \Longrightarrow x = y). \qquad (\mathrm{S}_{\mathsf{sup}})$$

6.3 Definition and Basic Properties of Mereological Sum

Let $\langle M, \sqsubseteq\rangle$ be a poset, $S \subseteq M$ and $x \in M$.

We say that x is a *mereological sum of all elements* of S iff x is an upper bound of S and every part of x overlaps some element of S; formally:

$$x \mathsf{\ sum\ } S \ :\Longleftrightarrow\ \forall_{z\in S}\ z \sqsubseteq x \wedge \forall_{y\in M}(y \sqsubseteq x \Longrightarrow \exists_{z\in S}\ z \bigcirc y). \qquad (\mathtt{df}\ \mathsf{sum})$$

Using the operations P and O we can give an alternative version of the definition:

$$x \ \mathsf{sum}\ S \iff S \subseteq \mathsf{P}(x) \subseteq \bigcup \mathsf{O}[S]\,, \qquad (\mathtt{df}'\,\mathsf{sum})$$

where $\mathsf{O}[S]$ is the image of the set X under the operation O, i.e.:

$$\mathsf{O}[S] := \{\mathsf{O}(z) \mid z \in S\} \quad \text{and} \quad \bigcup \mathsf{O}[S] = \{y \in M \mid \exists_{z \in S}\ z \circ y\}\,.$$

By the definition we obtain that:

$$\forall_{x \in M}(x \ \mathsf{sum}\ \varnothing \iff \mathsf{P}(x) = \varnothing). \tag{6.9}$$

Moreover, by ($\mathtt{r}_{\sqsubseteq}$), for any $x \in M$ we obtain that $\varnothing \neq \{x\} \subseteq \mathsf{P}(x) \subseteq \mathsf{O}(x) = \bigcup \mathsf{O}[\{x\}] \subseteq \bigcup \mathsf{O}[\mathsf{P}(x)]$ and if $\mathsf{PP}(x) \neq \varnothing$, then $\mathsf{P}(x) \subseteq \bigcup \mathsf{O}[\mathsf{PP}(x)]$. Hence:

Lemma 2. *The following conditions are consequences of* ($\mathtt{r}_{\sqsubseteq}$):

$$\neg\,\exists_{x \in M}\ x \ \mathsf{sum}\ \varnothing\,, \tag{6.10}$$

$$\forall_{x \in M}\ x \ \mathsf{sum}\ \{x\}\,, \tag{6.11}$$

$$\forall_{x \in M}\ x \ \mathsf{sum}\ \mathsf{P}(x)\,, \tag{6.12}$$

$$\forall_{x \in M}\big(\mathsf{PP}(x) \neq \varnothing \implies x \ \mathsf{sum}\ \mathsf{PP}(x)\big). \tag{6.13}$$

Now notice that:

Lemma 3 (Pietruszczak 2000). *The following condition is true in* **QPOS**:

$$\forall_{x,y \in M}(\mathsf{P}(x) \subseteq \mathsf{O}(y) \implies x \ \mathsf{sum}\ \mathsf{P}(x) \cap \mathsf{P}(y)).$$

Proof. Suppose that $\mathsf{P}(x) \subseteq \mathsf{O}(y)$. Since $\mathsf{P}(x) \cap \mathsf{P}(y) \subseteq \mathsf{P}(x)$, we only need to prove that $\mathsf{P}(x) \subseteq \bigcup \mathsf{O}[\mathsf{P}(x) \cap \mathsf{P}(y)]$. To see this, notice that from the assumption it follows that if $z \sqsubseteq x$, then $z \circ y$. So, by ($\mathtt{df}\circ$), for some z_0 we have that $z_0 \sqsubseteq z$ and $z_0 \sqsubseteq y$. By ($\mathtt{t}_{\sqsubseteq}$), $z_0 \sqsubseteq x$, so $z_0 \in \mathsf{P}(x) \cap \mathsf{P}(y) \cap \mathsf{P}(z)$. The more so $z_0 \circ z$, by ($\mathtt{r}_{\sqsubseteq}$), as required.

Since we are interested in mutual dependencies between sum and supremum, let, for brevity and reference reasons, (†) denote the condition that every sum is a supremum:

$$\mathsf{sum} \subseteq \mathsf{sup}\,, \tag{†}$$

and (‡) the reversed condition:

$$\mathsf{sup} \subseteq \mathsf{sum}\,. \tag{‡}$$

Fig. 6.1 A poset in which supremum counterparts of (6.10), (6.13), ($\mathsf{S}_{\mathsf{sum}}$), ($\mathsf{U}_{\mathsf{sum}}$) and ($\mathsf{M}_{\mathsf{sum}}$) are not true

We will be interested as well in, weaker from (‡), the following sentence:

$$\forall_{x\in M}\forall_{S\in\mathcal{P}_+(M)}(x \;\mathsf{sup}\; S \Longrightarrow x \;\mathsf{sum}\; S) \tag{‡$_+$}$$

6.4 Basic Differences Between the Relations sup and sum

Firstly, notice that the supremum counterparts of (6.10) and (6.13), i.e. $\neg\exists_{x\in M}\; x \;\mathsf{sup}\; \varnothing$ and $\forall_{x\in M}(\mathsf{PP}(x) \neq \varnothing \Longrightarrow x \;\mathsf{sup}\; \mathsf{PP}(x))$, are not true in **POS**. Indeed, let us consider a two-element poset with $M = \{0, 1\}$ and $\sqsubseteq\, = \mathrm{id}_M \cup \{\langle 0, 1\rangle\}$ (see Fig. 6.1). We have that $0 \;\mathsf{sup}\; \varnothing$ and $\mathsf{PP}(1) = \{0\}$, but $\neg\, 1 \;\mathsf{sup}\; \{0\}$, since $0 \;\mathsf{sup}\; \{0\}$.

Secondly, notice that the mereological sum counterparts of ($\mathsf{S}_{\mathsf{sup}}$), ($\mathsf{U}_{\mathsf{sup}}$) and ($\mathsf{M}_{\mathsf{sup}}$), i.e.:

$$\forall_{x,y\in M}(y \;\mathsf{sum}\; \{x\} \Longrightarrow x = y), \tag{$\mathsf{S}_{\mathsf{sum}}$}$$

$$\forall_{S\in\mathcal{P}(M)}\forall_{x,y\in M}(x \;\mathsf{sum}\; S \wedge y \;\mathsf{sum}\; S \Longrightarrow x = y), \tag{$\mathsf{U}_{\mathsf{sum}}$}$$

$$\forall_{S_1,S_2\in\mathcal{P}(M)}\forall_{x,y\in M}(S_1 \subseteq S_2 \wedge x \;\mathsf{sum}\; S_1 \wedge y \;\mathsf{sum}\; S_2 \Longrightarrow x \sqsubseteq y) \tag{$\mathsf{M}_{\mathsf{sum}}$}$$

are not true in **POS**. Indeed, in the poset from Fig. 6.1, respectively by (6.11) and (6.13), we have that $0 \;\mathsf{sum}\; \{0\}$ and $1 \;\mathsf{sum}\; \{0\}$, but $1 \not\sqsubseteq 0$.

6.5 Basic Properties of ($\mathsf{S}_{\mathsf{sum}}$), ($\mathsf{U}_{\mathsf{sum}}$) and ($\mathsf{M}_{\mathsf{sum}}$)

The lemma below is obvious.

Lemma 4. (i) *From* ($\mathtt{r}_{\sqsubseteq}$) *and* ($\mathsf{U}_{\mathsf{sum}}$) *we obtain* ($\mathsf{S}_{\mathsf{sum}}$). *Consequently* **QPOS**+($\mathsf{U}_{\mathsf{sum}}$) $\subseteq$ **QPOS**+($\mathsf{S}_{\mathsf{sum}}$).

(ii) *From* ($\mathtt{antis}_{\sqsubseteq}$) *and* ($\mathsf{M}_{\mathsf{sum}}$) *we obtain* ($\mathsf{U}_{\mathsf{sum}}$). *Consequently* **POS**+($\mathsf{M}_{\mathsf{sum}}$) $\subseteq$ **POS**+($\mathsf{U}_{\mathsf{sum}}$) $\subseteq$ **POS**+($\mathsf{S}_{\mathsf{sum}}$).

Notice that enriching the axioms for posets with ($\mathsf{S}_{\mathsf{sum}}$) (resp. ($\mathsf{U}_{\mathsf{sum}}$)) does not entail uniqueness (resp. monotonicity) of sum. Indeed, we have:

Fact 1. (i) ($\mathsf{U}_{\mathsf{sum}}$) *is not true in* **POS**+($\mathsf{S}_{\mathsf{sum}}$). *Hence* **POS**+($\mathsf{U}_{\mathsf{sum}}$) $\subsetneq$ **POS**+($\mathsf{S}_{\mathsf{sum}}$).

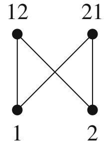

Fig. 6.2 A poset which satisfies ($\mathtt{S}_{\mathsf{sum}}$), but not ($\mathtt{U}_{\mathsf{sum}}$)

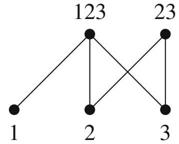

Fig. 6.3 A poset which satisfies ($\mathtt{U}_{\mathsf{sum}}$), but not ($\mathtt{M}_{\mathsf{sum}}$)

(ii) ($\mathtt{M}_{\mathsf{sum}}$) *is not true in* **POS**+($\mathtt{U}_{\mathsf{sum}}$). *Consequently* **POS**+($\mathtt{M}_{\mathsf{sum}}$) $\subsetneq$ **POS**+ ($\mathtt{U}_{\mathsf{sum}}$) $\subsetneq$ **POS**+($\mathtt{S}_{\mathsf{sum}}$).

Proof. *Ad* (i): The poset from Fig. 6.2 with $M = \{1, 2, 12, 21\}$ and $\sqsubseteq\, = \mathrm{id}_M \cup \{\langle 1, 12\rangle, \langle 1, 21\rangle, \langle 2, 12\rangle, \langle 2, 21\rangle\}$ belongs to **POS**+($\mathtt{S}_{\mathsf{sum}}$) and it shows that a set can have more than one mereological sum. Indeed, ($\mathtt{S}_{\mathsf{sum}}$) is satisfied in this poset, but $12 \;\mathsf{sum}\; \{1, 2\}$ and $21 \;\mathsf{sum}\; \{1, 2\}$.

Ad (ii): This time we consider the poset with $M = \{1, 2, 3, 12, 123\}$ and $\sqsubseteq\, = \mathrm{id}_M \cup \{\langle 1, 123\rangle, \langle 2, 123\rangle, \langle 3, 123\rangle, \langle 2, 23\rangle, \langle 3, 23\rangle\}$ (see Fig. 6.3). It satisfies ($\mathtt{U}_{\mathsf{sum}}$). On the other hand, we have that $23 \;\mathsf{sum}\; \{2, 3\}$ and $123 \;\mathsf{sum}\; \{1, 2, 3\}$, but $23 \not\sqsubseteq 123$.

Notice that:

Lemma 5. ($\mathtt{M}_{\mathsf{sum}}$) *entails* ($\mathtt{r}_{\sqsubseteq}$).

Proof. By ($\mathtt{M}_{\mathsf{sum}}$) we have (a): $\forall_{x\in M}(x \not\sqsubseteq x \Longrightarrow \neg\exists_{S\in\mathcal{P}(M)}\, x \;\mathsf{sum}\; S)$. By (a) we have (b): $\forall_{x\in M}(x \not\sqsubseteq x \Longrightarrow \exists_{y\in\mathsf{PP}(x)}\, y \not\sqsubseteq y)$. Indeed, if for any $y \in \mathsf{PP}(x)$ we have that $y \sqsubseteq y$, then $x \;\mathsf{sum}\; \mathsf{PP}(x)$. So $x \sqsubseteq x$ by (a).

By (b) we obtain (c): $\forall_{x\in M}(x \not\sqsubseteq x \Longrightarrow x \;\mathsf{sum}\; \mathsf{PP}(x))$. Indeed, let $x \not\sqsubseteq x$ and $y \sqsubseteq x$. Then $y \in \mathsf{PP}(x)$. Moreover, if $y \sqsubseteq y$, then $y \bigcirc y$. If $y \not\sqsubseteq y$, then by (b) there is $u \in \mathsf{PP}(y)$; so also $y \bigcirc y$. Hence in both cases there is $z \in \mathsf{PP}(x)$ such that $z \bigcirc y$.

By (a) and (c) we have that ($\mathtt{r}_{\sqsubseteq}$) holds.

We now point to some relationship between ($\mathtt{S}_{\mathsf{sum}}$) and the so-called *Weak Supplementation Principle*, used by Simons (1987):

$$\forall_{x,y\in M}\big(x \sqsubset y \Longrightarrow \exists_{z\in M}(z \sqsubset y \wedge z \wr x)\big), \qquad (\mathtt{WSP})$$

which will let us obtain a connection between the relations sup and sum (see Theorem 2).

Theorem 1. (i) *From* ($\mathtt{WSP}$) *we obtain* ($\mathtt{S}_{\mathsf{sum}}$).
(ii) *From* ($\mathtt{r}_{\sqsubseteq}$) *and* ($\mathtt{S}_{\mathsf{sum}}$) *we obtain* ($\mathtt{WSP}$).
(iii) *From* ($\mathtt{r}_{\sqsubseteq}$) *and* ($\mathtt{U}_{\mathsf{sum}}$) *we obtain* ($\mathtt{WSP}$).

In consequence, **QPOS**+($\mathtt{WSP}$) = **QPOS**+($\mathtt{S}_{\mathsf{sum}}$).

Proof. *Ad* (i): Suppose that y sum $\{x\}$ and $x \neq y$. Then $x \sqsubset y$. So, by (WSP), for some z we have: $z \sqsubset y$ and $z \wr x$. So we have a contradiction, since from $z \sqsubset y$ and y sum $\{x\}$ follows that $z \bigcirc x$.

Ad (ii): Suppose that $x \sqsubset y$ and $\mathsf{PP}(y) \subseteq \mathsf{O}(x)$. Since by ($\mathtt{r}_{\sqsubseteq}$) we have that $y \bigcirc x$, then y sum $\{x\}$. Thus $y = x$ by ($\mathsf{S}_{\mathsf{sum}}$), which is a contradiction.

(Hence y sum $\{x\}$ and, by ($\mathtt{r}_{\sqsubseteq}$), also $y \bigcirc x$. So, by ($\mathsf{S}_{\mathsf{sum}}$), we obtain a contradiction: $x = y$.)

Ad (iii): We use (1) and Lemma 4(i).

Corollary 1. *The sentence* (WSP) *is true in the class* **POS**+($\mathsf{M}_{\mathsf{sum}}$).

Proof. By Lemma 4(ii), from ($\mathtt{antis}_{\sqsubseteq}$) and ($\mathsf{M}_{\mathsf{sum}}$) follows ($\mathsf{U}_{\mathsf{sum}}$). Moreover, by Theorem 1(iii), ($\mathtt{r}_{\sqsubseteq}$) and ($\mathsf{U}_{\mathsf{sum}}$) entail (WSP).

Now we prove that in every structure from **QPOS**+($\mathsf{S}_{\mathsf{sum}}$), if both sum and supremum exists, then they are equal.

Theorem 2 (Pietruszczak 2000). ($\mathtt{t}_{\sqsubseteq}$) *and* (WSP) *entail the following sentence*:

$$\forall_{S \in \mathcal{P}(M)} \forall_{x,y \in M} (x \;\mathsf{sup}\; S \wedge y \;\mathsf{sum}\; S \Longrightarrow x = y). \tag{6.14}$$

Proof. Let x sup S and y sum S. Then $S \subseteq \mathsf{P}(y)$, so $x \sqsubseteq y$. Suppose that $x \neq y$. Then $x \sqsubset y$. Hence, by (WSP), for some $z \in M$ we have that $z \sqsubset y$ and $z \wr x$. Hence, by (df sum), there are $u \in S$ and $v \in M$ such that $v \sqsubseteq u$ and $v \sqsubseteq z$. By the assumption, $u \sqsubseteq x$. Hence, by ($\mathtt{t}_{\sqsubseteq}$), also $v \sqsubseteq x$. So we have a contradiction: $z \bigcirc x$.

Now we will prove an important lemma which will be useful a little bit further. Let us start with the following definition.

An object x is called *the zero element* of a poset $\langle M, \sqsubseteq \rangle$ iff every object from M is part of x, i.e. $\forall_{y \in M}\ x \sqsubseteq y$. The uniqueness of the zero element follows from antisymmetry of $\sqsubseteq$. Moreover, we immediately have that for any poset, if it has zero, then all objects overlap with each other:

$$\forall_{x,y \in M} (x \text{ is a zero} \wedge y \text{ is a zero} \Longrightarrow x = y), \tag{6.15}$$

$$\exists_{x \in M}\ x \text{ is a zero} \Longrightarrow \forall_{x,y \in M}\ x \bigcirc y\,. \tag{6.16}$$

Lemma 6 (Pietruszczak 2000).

(i) *From* (WSP) *we obtain the following implication*:

$$|M| > 1 \Longrightarrow \exists_{x,y \in M}\ x \wr y\,.$$

(ii) *From* ($\mathtt{r}_{\sqsubseteq}$) *we obtain the following implication*:

$$\exists_{x,y \in M}\ x \wr y \Longrightarrow |M| > 1\,.$$

(iii) *From* (WSP) *and* ($\mathtt{r}_{\sqsubseteq}$) *we obtain the following equivalence*:

$$\exists_{x\in M}\forall_{y\in M}\ x\sqsubseteq y \iff |M|=1\,.$$

Proof. *Ad* (i): Let $x_1, x_2 \in M$ be different: $x_1 \neq x_2$. Suppose that $\forall_{x,y\in M}\ x \bigcirc y$. So there is $u \in M$ such that $u \sqsubseteq x_1$ and $u \sqsubseteq x_2$. Moreover, either $u \sqsubset x_1$ or $u \sqsubset x_2$. In both cases, by (WSP) we obtain a contradiction: there is $z \in M$ such that $z \wr u$.

Ad (ii): By ($\mathtt{r}_{\sqsubseteq}$) we have ($\mathtt{irr}_{\wr}$); so if $x \wr y$, then $x \neq y$.

Ad (iii): "⇒" If $\exists_{x\in M}\forall_{y\in M}\ x \sqsubseteq y$ then $\forall_{x,y\in M}\ x \bigcirc y$, so we use (i). "⇐" Immediate, from ($\mathtt{r}_{\sqsubseteq}$).

By Corollary 1 we obtain:

Corollary 2. *The sentences from Lemma 6 are all true in the class* **POS**+($\mathtt{M}_{\mathsf{sum}}$).

6.6 The Inclusions (†) and (‡) in the Class POS+($\mathtt{U}_{\mathsf{sum}}$)

We show that neither (†) nor (‡) follows from the axioms for **POS** plus ($\mathtt{U}_{\mathsf{sum}}$). In consequence none of them follows from the axioms for **POS** plus ($\mathtt{S}_{\mathsf{sum}}$); see Lemma 4.

Fact 2. *None of the sentences* (†) *and* ($‡_+$) *is true in* **POS**+($\mathtt{U}_{\mathsf{sum}}$).

Proof. In the poset from Fig. 6.3 we have: 23 sum $\{2,3\}$, but $\neg$ 23 sup $\{2,3\}$. So $\mathsf{sum} \not\subseteq \mathsf{sup}$. In the same poset we have: 123 sup $\{1,2\}$, but $\neg$ 123 sum $\{1,2\}$. Thus $\mathsf{sup} \not\subseteq \mathsf{sum}$ as well.

6.7 The Inclusions (†) and (‡) in the Class POS+($\mathtt{M}_{\mathsf{sum}}$)

Firstly, we show that (‡) does not follow from the axioms for **POS** plus ($\mathtt{M}_{\mathsf{sum}}$).[3]

Fact 3. *The sentence* ($‡_+$) *is not true in* **POS**+($\mathtt{M}_{\mathsf{sum}}$).

Proof. We take the poset with $M = \{1,2,3,123\}$ and $\sqsubseteq\ = \mathrm{id}_M \cup \{\langle 1,123\rangle, \langle 2,123\rangle, \langle 3,123\rangle\}$ (see Fig. 6.4). Obviously, this poset satisfies ($\mathtt{M}_{\mathsf{sum}}$) but not ($‡_+$), since e.g. 123 sup $\{1,2\}$ while $\neg$ 123 sum $\{1,2\}$.

Secondly, we can show that (†) is true in the class **POS**+($\mathtt{M}_{\mathsf{sum}}$). Moreover we will demonstrate that for quasi-partially ordered sets the inclusion (†) is equivalent to the sentence ($\mathtt{M}_{\mathsf{sum}}$). But earlier we need to prove some interesting facts.

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[3]This, by Lemma 4, entails the case for (‡) in Fact 2.

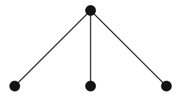

Fig. 6.4 A poset which satisfies ($\mathrm{M}_{\mathsf{sum}}$), but not (‡)

Firstly, notice that to examine properties of the relation sum we will make use of the following condition which is related to ($\mathtt{df}'\,\mathsf{sum}$) and ($\mathrm{M}_{\mathsf{sum}}$):

$$\forall_{S\in\mathcal{P}(M)}\forall_{x,y\in M}\big(\mathsf{P}(x)\subseteq\bigcup\mathsf{O}[S]\wedge S\subseteq\mathsf{P}(y)\Longrightarrow x\sqsubseteq y\big). \qquad (\mathrm{M}'_{\mathsf{sum}})$$

Lemma 7 (Pietruszczak 2000). ($\mathrm{M}'_{\mathsf{sum}}$) *entails* ($\mathrm{M}_{\mathsf{sum}}$).

Proof. If $S_1\subseteq S_2$, $x\ \mathsf{sum}\ S_1$ and $y\ \mathsf{sum}\ S_2$, then $\mathsf{P}(x)\subseteq\bigcup\mathsf{O}[S_1]\subseteq\bigcup\mathsf{O}[S_2]$ and $S_2\subseteq\mathsf{P}(y)$. So $x\sqsubseteq y$, by ($\mathrm{M}'_{\mathsf{sum}}$).

Lemma 8. *From* ($\mathrm{M}_{\mathsf{sum}}$) *and* ($\mathtt{t}_{\sqsubseteq}$) *we obtain* ($\mathrm{M}'_{\mathsf{sum}}$).

Proof. By Lemma 5 we have ($\mathtt{r}_{\sqsubseteq}$). If $\mathsf{P}(x)\subseteq\bigcup\mathsf{O}[S]$ and $S\subseteq\mathsf{P}(y)$, then $\mathsf{P}(x)\subseteq\bigcup_{z\in\mathsf{P}(y)}\mathsf{O}(z)$. Notice that by ($\mathtt{t}_{\sqsubseteq}$) we have (6.5), so we obtain that $\bigcup_{z\in\mathsf{P}(y)}\mathsf{O}(z)\subseteq\mathsf{O}(y)$. Thus, $\mathsf{P}(x)\subseteq\mathsf{O}(y)$. Hence, by Lemma 3, $x\ \mathsf{sum}\ \mathsf{P}(x)\cap\mathsf{P}(y)$. Moreover, $y\ \mathsf{sum}\ \mathsf{P}(y)$, by ($\mathtt{r}_{\sqsubseteq}$). Thus $x\sqsubseteq y$, by ($\mathrm{M}_{\mathsf{sum}}$).

From Lemmas 7 and 8 we obtain:

Theorem 3 (Pietruszczak 2000). *The following sentence is true in* **QPOS**:

$$(\mathrm{M}_{\mathsf{sum}})\iff(\mathrm{M}'_{\mathsf{sum}})\,.$$

Thus, **QPOS**+($\mathrm{M}_{\mathsf{sum}}$) = **QPOS**+($\mathrm{M}'_{\mathsf{sum}}$).

Now we prove that:

Theorem 4. *The following sentence is true in* **QPOS**:

$$(\mathrm{M}_{\mathsf{sum}})\iff(\dagger).$$

Thus, **QPOS**+($\mathrm{M}_{\mathsf{sum}}$) = **QPOS**+(†).

Proof. "⇒" Assume that $x\ \mathsf{sum}\ S$, i.e., $S\subseteq\mathsf{P}(x)\subseteq\bigcup\mathsf{O}[S]$. This gives us immediately the first conjunct of ($\mathtt{df}'\,\mathsf{sup}$). For the second one assume that $y\in M$ is such that $S\subseteq\mathsf{P}(y)$. Then $x\sqsubseteq y$, by ($\mathrm{M}'_{\mathsf{sum}}$) and Theorem 3. So $x\ \mathsf{sup}\ S$.

"⇐" If $S_1\subseteq S_2$, $x\ \mathsf{sum}\ S_1$ and $y\ \mathsf{sum}\ S_2$, then by (†) it is the case that $x\ \mathsf{sup}\ S_1$ and $y\ \mathsf{sum}\ S_2$; so $x\sqsubseteq y$, by ($\mathrm{M}_{\mathsf{sup}}$).

Remark 1. For structures from **POS**+($\mathrm{M}_{\mathsf{sum}}$) (= **POS**+(†), by Theorem 4) we have a simple proof of the sentence (6.14), i.e., if for a set has both sum and supremum, then they are equal. Indeed, if x sup S and y sum S, then also y sup S. Thus, since we can use ($\mathtt{antis}_{\sqsubseteq}$), we obtain $x = y$, by ($\mathrm{U_{sup}}$).

6.8 Separative Partially Order Sets

Any quasi-poset which satisfies the following sentence:

$$\forall_{x,y\in M}\big(x \not\sqsubseteq y \implies \exists_{z\in M}(z \sqsubseteq x \wedge z \wr y)\big), \tag{SSP}$$

will be called *separative*. Let **SPOS** be the class of all separative posets.

In Simons (1987) the sentence (SSP) is called *Strong Supplementation Principle*. According to (SSP) if one object is not a part of another, than they can be distinguished by some object from the domain, but not only in the sense that this object is part of one but not the other element of the domain – it is exterior to the latter.

The sentence (SSP) can as well be expressed in the following, definitionally equivalent, way:

$$\forall_{x,y\in M}\big(\mathsf{P}(x) \subseteq \mathsf{O}(y) \implies x \sqsubseteq y\big)\,. \tag{SSP°}$$

Hence, by (6.4)–(6.6), we also obtain:

Fact 4. *The following sentence is true in all separative quasi-posets*:

$$\forall_{x,y\in M}\big(x \sqsubseteq y \iff \mathsf{O}(x) \subseteq \mathsf{O}(y)\big).$$

Now we will show that **QPOS**+($\mathrm{M_{sum}}$) = **QPOS**+(SSP). In the proof of the equality in question we will use the equality **QPOS**+($\mathrm{M_{sum}}$) = **QPOS**+($\mathrm{M'_{sum}}$) from Theorem 3 together with the facts below.

Lemma 9 (Pietruszczak 2000, 2005). *From* (SSP) *and* ($\mathtt{t}_{\sqsubseteq}$) *we obtain* ($\mathrm{M'_{sum}}$) *and* ($\mathrm{M_{sum}}$).[4]

Proof. For ($\mathrm{M'_{sum}}$): If $\mathsf{P}(x) \subseteq \bigcup \mathsf{O}[S]$ and $S \subseteq \mathsf{P}(y)$, then $\mathsf{P}(x) \subseteq \bigcup_{z\in\mathsf{P}(y)} \mathsf{O}(z)$. By (6.5) we obtain that $\bigcup_{z\in\mathsf{P}(y)} \mathsf{O}(z) \subseteq \mathsf{O}(y)$. Therefore $\mathsf{P}(x) \subseteq \mathsf{O}(y)$. So $x \sqsubseteq y$, by (SSP°). For ($\mathrm{M_{sum}}$): Use Lemma 7.

Lemma 10. (i) ($\mathrm{M'_{sum}}$) *entails* (SSP).
(ii) ($\mathrm{M_{sum}}$) *and* ($\mathtt{t}_{\sqsubseteq}$) *entails* (SSP).

[4] Hence, by Lemma 5, we obtain that (SSP) and ($\mathtt{t}_{\sqsubseteq}$) entail ($\mathtt{r}_{\sqsubseteq}$). This fact was proven in Pietruszczak (2000, 2005). So separative posets can be defined by means of these three conditions: ($\mathtt{t}_{\sqsubseteq}$), ($\mathtt{antis}_{\sqsubseteq}$) and (SSP).

Proof. (i) Notice that, by Lemmas 5 and 7, we have ($\mathtt{r}_\sqsubseteq$). If $\mathsf{P}(x) \subseteq \mathsf{O}(y)$, then $\mathsf{P}(x) \subseteq \bigcup \mathsf{O}[\{y\}]$ and $\{y\} \subseteq \mathsf{P}(y)$ by ($\mathtt{r}_\sqsubseteq$). So $x \sqsubseteq y$, by ($\mathtt{M}'_{\mathsf{sum}}$).
(ii) By (i) and Lemma 8.

Thus, from the above lemmas and Theorem 4 we have:

Theorem 5. *The following sentence is true in* **QPOS**:

$$(\mathtt{SSP}) \iff (\mathtt{M}_{\mathsf{sum}}).$$

Thus, **QPOS**+($\mathtt{SSP}$) = **QPOS**+($\mathtt{M}'_{\mathsf{sum}}$) = **QPOS**+($\mathtt{M}_{\mathsf{sum}}$) = **QPOS**+(†).

Finally, we obtain:

Fact 5. (i) *The sentences* ($\mathtt{r}_\sqsubseteq$) *and* ($\mathtt{antis}_\sqsubseteq$) *entail the implication* ($\mathtt{SSP}$) $\Rightarrow$ ($\mathtt{WSP}$). *Consequently,* **SPOS** $\subseteq$ **POS**+($\mathtt{WSP}$).
(ii) **SPOS** $\subsetneq$ **POS**+($\mathtt{WSP}$).

Proof. (i): Let $x \sqsubset y$, i.e., $x \sqsubseteq y$ and $x \neq y$. Then $y \not\sqsubseteq x$, by ($\mathtt{antis}_\sqsubseteq$). Hence, by ($\mathtt{SSP}$), there is z such that $z \sqsubseteq y$ and $z \wr x$. We have that $z \neq y$, since $y \bigcirc x$, by ($\mathtt{r}_\sqsubseteq$). So $z \sqsubset y$.
(ii): The poset from Fig. 6.2 satisfies ($\mathtt{WSP}$). It is the case that $12 \not\sqsubseteq 21$, but there is no z such that $z \sqsubseteq 12$ and $z \wr 21$. So ($\mathtt{SSP}$) is not true in the structure considered.

6.9 Mereological Structures

We now take into account the following axiom of existence of mereological sum:

$$\forall_{S \in \mathcal{P}_+(M)} \exists_{x \in M}\ x \ \mathsf{sum}\ S\,. \qquad (\exists\mathsf{sum})$$

Any separative poset which satisfies ($\exists\mathsf{sum}$) is called a (*classical*) *mereological structure*.[5] Let **MS** and **MS**$^+$ be respectively the class of all mereological structures and the class all non-degenerate mereological structures. Of course, **MS**$^+ \subsetneq$ **MS**.

By Lemma 4 the formula ($\mathtt{U}_{\mathsf{sum}}$) is true in **MS**. So the following sentence is also true in **MS**:

$$\forall_{S \in \mathcal{P}_+(M)} \exists^1_{x \in M}\ x \ \mathsf{sum}\ S\,. \qquad (\exists^1\mathsf{sum})$$

[5]In Tarski (1956) we find an equivalent axiomatization of mereological structures consisted of the following sentences: ($\mathtt{t}_\sqsubseteq$) and given below ($\exists^1\mathsf{sum}$) (which is equivalent to: ($\mathtt{t}_\sqsubseteq$), ($\mathtt{U}_{\mathsf{sum}}$) and ($\exists\mathsf{sum}$)). Various equivalent axiomatizations of mereological structures are presented e.g. in Pietruszczak (2000, 2005).

Hence in any mereological structure $\langle M, \sqsubseteq\rangle$ there is exactly one object x such that $x \mathsf{\ sum\ } M$. By (df sum), x is *the unity* in the sense that: $\forall_{y \in M}\ y \sqsubseteq x$. So in any mereological structure we put:

$$\mathbf{1} := (\iota x)\ x \mathsf{\ sum\ } M\,, \tag{df 1}$$

and by ($\mathtt{antis}_{\sqsubseteq}$) we have:

$$\mathbf{1} = (\iota x)\ \forall_{y \in M}\ y \sqsubseteq x\,. \tag{6.17}$$

Theorem 6 (Pietruszczak 2000, 2005). *The following sentences are true in* **MS**[6]*:*

$$\forall_{x \in M} \forall_{S \in \mathcal{P}(M)} (x \mathsf{\ sum\ } S \iff S \neq \varnothing \wedge x \mathsf{\ sup\ } S), \tag{sum-sup}$$

$$|M| > 1 \iff \mathsf{sup} \subseteq \mathsf{sum}\,. \tag{$\star$}$$

Proof. Ad (sum-sup): By Theorems 4 and 5 we have (†). So if $x \mathsf{\ sum\ } S$, then $x \mathsf{\ sup\ } S$ and $S \neq \varnothing$, by ($\mathtt{r}_{\sqsubseteq}$). Let now $S \neq \varnothing$ and $x \mathsf{\ sup\ } S$. Then, by ($\exists$sum), there is y such that $y \mathsf{\ sum\ } S$. So $x = y$, by (6.14); see Remark 1. Therefore $x \mathsf{\ sum\ } S$.

Ad ($\star$): Firstly, let $|M| > 1$ and $x \mathsf{\ sup\ } S$. Then $S \neq \varnothing$, since by Corollary 2, in M there is no zero element. Hence, $x \mathsf{\ sum\ } S$, by (sum-sup). Secondly, assume that M has only one element x. Then $x \mathsf{\ sup\ } \varnothing$. But $\neg x \mathsf{\ sum\ } \varnothing$, by ($\mathtt{r}_{\sqsubseteq}$). So $\mathsf{sup} \not\subseteq \mathsf{sum}$.

By the above theorem we get:

Corollary 3. *The equality* $\mathsf{sum} = \mathsf{sup}$ *holds in* $\mathbf{MS}^+$.

6.10 Weakening and Replacing the Sum Existence Axiom

Consider the following weakened versions of ($\exists$sum):

$$\forall_{S \in \mathcal{P}_+(M)} (\exists_{u \in M}\ S \subseteq \mathsf{P}(u) \Longrightarrow \exists_{x \in M}\ x \mathsf{\ sum\ } S), \tag{$\mathtt{W}_1\exists$sum}$$

$$\forall_{S \in \mathcal{P}_+(M)} \big(\forall_{y,z \in S} \exists_{u \in M}\ \{y, z\} \subseteq \mathsf{P}(u) \Longrightarrow \exists_{x \in M}\ x \mathsf{\ sum\ } S\big). \tag{$\mathtt{W}_2\exists$sum}$$

The first one says that every non-empty set which is bounded from above has its mereological sum. The second (stronger than the first one) says that if every subset $\{y, z\}$ of S is bounded in M, then S has its sum.[7]

[6] The first one to prove (sum-sup), in original language of Leśniewski's mereology, was A. Tarski (see Leśniewski (1930), p. 87).

[7] This not exactly upward directedness of S. A subset S in a poset $\langle M, \sqsubseteq\rangle$ is *upward directed* iff $\forall_{y,z \in S} \exists_{u \in S} (y \sqsubseteq u \wedge z \sqsubseteq u)$, while we require the existence of upper bound in M. Consequently,

We have the following fact.

Fact 6. *The sentence* (sum-sup) *is true in both* **QPOS**+(SSP)+($W_1\exists$sum) *and in* **QPOS**+(SSP)+($W_2\exists$sum).

Proof. By Theorem 4 we have (†); so if x sum S, then x sup S and $S \neq \varnothing$, by ($r_\sqsubseteq$). Moreover, let $S \neq \varnothing$ and x sup S. Then $S \subseteq \mathsf{P}(x)$. Hence, there is y such that y sum S, by ($W_1\exists$sum) or ($W_2\exists$sum). So $x = y$, by (6.14); see Remark 1. Therefore x sum S.

The above fact shows that we can weaken the sum existence axiom to the forms presented yet keep the equality between sum and supremum. Of course, this does not solve the problem of characterization of structures from classes **SPOS**+($W_1\exists$sum) and **SPOS**+($W_2\exists$sum). In our opinion further study concerning their properties seems to be interesting from the following, a bit philosophical, point of view. The unrestricted sum axiom (∃sum) is often objected as counterintuitive in case of some so called ontological interpretations of mereology.[8] It is argued for example, that the Moon and a cup of coffee standing in front of you are parts of the world, yet it is hard to find anything that could be their sum. Axioms ($W_1\exists$sum) and ($W_2\exists$sum) could be interpreted (at least in a way) as saying that only these objects which have something in common (in the sense that they are both parts of something bigger) have their mereological sums.

No we consider the following principle, intimately connected with those analyzed by us in previous sections:

$$\forall_{x,y\in M}\big(x \not\sqsubseteq y \Longrightarrow \exists_z\big(z \sqsubseteq x \wedge z \wr y \wedge \forall_u(u \sqsubseteq x \wedge u \wr y \Longrightarrow u \sqsubseteq z)\big)\big) \quad \text{(SSP+)}$$

which we will call *the super strong supplementation principle* or "*SSP plus*". What it intuitively says is that if x is not part of y, then we can not only find some z being part of x which is external to y, but we can also find an element of the structure in question satisfying the aforementioned property and being the largest such object in the structure. The sentence (SSP+) is assumed as an axiom in Grzegorczyk's system of mereology from Grzegorczyk (1955).

Theorem 7. *The sentence* (SSP+) *is true in the class* **QPOS**+(SSP)+($W_1\exists$sum).[9]

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both axioms are equivalent in posets with the unity, since antecedents of ($W_1\exists$sum) and ($W_2\exists$sum) are both true in the presence of one.

[8]In our opinion these objections are not properly addressed and they result from a twisted perspective, as we can see it. Nothing is wrong with (∃sum) and no one should demand the world to behave according to it in all its aspects. Yet there are such applications of mereology in which it is very useful, as in building point-free systems of geometry for example, where elements of the domain are treated as regions of space. For details see Tarski (1956), Gruszczyński and Pietruszczak (2008, 2009, 2010), and Grzegorczyk (1960).

[9]Of course by this theorem (SSP+) is true in the class **MS** as well.

Proof. If $x \not\sqsubseteq y$, then by (SSP) the set $S_0 := \{z \in M \mid z \sqsubseteq x \wedge z \wr y\}$ is not empty and $S_0 \subseteq \mathsf{P}(x)$. Hence, by ($\mathtt{W}_1\exists\mathsf{sum}$), for some z_0 we have that z_0 sum S_0.

Firstly, notice that $z_0 \sqsubseteq x$. Indeed, suppose towards contradiction that $z_0 \not\sqsubseteq x$. Then, by (SSP), there is u such that $u \sqsubseteq z_0$ and $u \wr x$. Hence, by (df sum), there are $v \in S_0$ and $w \in M$ such that $w \sqsubseteq v \sqsubseteq x$ and $w \sqsubseteq u$. So $u \circ x$, by ($\mathtt{t}_{\sqsubseteq}$); a contradiction.

Secondly, notice that $z_0 \wr y$. Indeed, suppose towards contradiction that $z_0 \circ y$. Then there is u such that $u \sqsubseteq z_0$ and $u \sqsubseteq y$. Hence, by (df sum), there are $v \in S_0$ and $w \in M$ such that $w \sqsubseteq v \wr y$ and $w \sqsubseteq u \sqsubseteq y$. So $w \wr y$ and $w \sqsubseteq y$, by ($\mathtt{t}_{\sqsubseteq}$). Moreover, $w \circ y$, by ($\mathtt{r}_{\sqsubseteq}$); a contradiction again.

Thirdly, if $u \sqsubseteq x$ and $u \wr y$, then $u \in S_0$. So $u \sqsubseteq z_0$, by (df sum).

There is one more issue that can be addressed with respect to axioms and mutual relationship between sum and supremum – *in what effect results replacing* ($\exists$sum) *with the following version of completeness*:

$$\forall_{S \in \mathcal{P}_+(M)} \exists_{x \in M}\ x \ \mathsf{sup}\ S\,. \tag{$\exists$sup}$$

Algebraically speaking we consider the class **SPOS**+($\exists$sup) elements of which are separative posets being complete join-semilattices. The following fact answers the question.

Fact 7 (Pietruszczak 2005). *The sentence* ($\ddagger_+$) *is not true in* **SPOS**+($\exists$sup)*. Therefore the counterpart of Theorem 6 does not hold for* **SPOS**+($\exists$sup)*.*

Proof. The structure from Fig. 6.4 belongs to **SPOS**+($\exists$sup) and does not satisfy the sentence in question, since e.g. 123 sup $\{1, 2\}$, but $\neg$ 123 sum $\{1, 2\}$.

6.11 Mereological Posets

Any structure from the class **SPOS**+($\ddagger_+$) will be called a *mereological poset* (*mereoposet* for short). Let **MPOS** be the class of all mereoposets. By Fact 3 and Theorem 5 we have that **MPOS** $\subsetneq$ **SPOS**. By Theorem 5, the sentences ($\mathtt{M}_{\mathsf{sum}}$), ($\mathtt{M}'_{\mathsf{sum}}$) and ($\dagger$) are true in **MPOS**. Moreover, by Fact 5 (or Corollary 1) the sentence (WSP) is true in **MPOS** as well.

We will also be interested in the class **MPOS**$^+$:= **SPOS**+($\ddagger$). By the definition, **MPOS**$^+$ $\subseteq$ **MPOS**. Below we show that **MPOS**$^+$ is the class of all non-degenerate mereoposets. So **MPOS**$^+$ $\subsetneq$ **MPOS**, which is a result of the following lemma.

Lemma 11. *No poset from* **POS**+($\ddagger$) *has a zero element. Consequently, it is a non-degenerate structure.*

Proof. If a poset $\langle M, \sqsubseteq\rangle$ has the zero element $\mathbf{0}$, then $\mathbf{0}$ sup $\varnothing$. But $\neg\exists_{x \in M}\ x$ sum $\varnothing$, by ($\mathtt{r}_{\sqsubseteq}$). So sup $\not\subseteq$ sum.

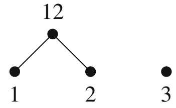

Fig. 6.5 The non-degenerate mereoposet without unity

From the above lemma, Corollary 2 and Theorem 5 we obtain:

Corollary 4. *No poset from* $\mathbf{MPOS}^{+}$ *has a zero element. Consequently, every structure from* $\mathbf{MPOS}^{+}$ *is non-degenerate and has at least two elements which are external two each other.*

Remark 2. Non-existence of zero element in the class $\mathbf{MPOS}^{+}$ and both supplementation principles are considered to be distinctive and fundamental features of structures that are examined within the field known as *mereology*.

Let $\langle M, \sqsubseteq\rangle$ be a mereoposet. We say that x is *isolated* in this structure iff x is a proper part of no element of M and no element of M is proper part of x. Let **is** be the set of all isolated elements, i.e.:

$$\mathbf{is} := \{x \in M \mid \neg \exists_{y \in M} (y \sqsubset x \vee x \sqsubset y\} . \qquad (\mathtt{df}\ \mathbf{is})$$

The simplest example of non-degenerate mereoposet is a pair $\langle M, \sqsubseteq\rangle$ with $M := \{1, 2\}$ and $\sqsubseteq := \mathrm{id}_M$. So this is a structure that consists of two isolated objects. Less trivial example is a four element structure $\langle\{1, 2, 12, 3\}, \sqsubseteq\rangle$, where $\sqsubseteq := \mathrm{id} \cup \{\langle 1, 12\rangle, \langle 2, 12\rangle\}$ and 3 is isolated (see Fig. 6.5).

The above model shows that the existence of unity is not a consequence of the axioms for mereological posets. However, neither is its non-existence, since every non-degenerate mereological structure is a mereoposet. So we have the following corollary.

Corollary 5. *Existence of unity is independent from axioms for mereoposets.*

Since the equality $\mathsf{sum} = \mathsf{sup}$ is true in $\mathbf{MS}^{+}$ (see Corollary 3) and by the structure from Fig. 6.5, we obtain:

Corollary 6. *Every non-degenerate mereological structure is a mereoposet, but not every mereposet is a mereological structure. So* $\mathbf{MS}^{+} \subsetneq \mathbf{MPOS}^{+}$.

On the other hand we have the following interesting result about mereoposets.

Fact 8. *The sentence* (SSP+) *is not true in the class* $\mathbf{MPOS}^{+}$.

Proof. We consider the following non-degenerate mereoposet $\langle M, \sqsubseteq\rangle$ with $M := \{-1, 1, 2, 3, -11, -12, 13, 23, -112, 123\}$ and for $x, y \in M$: $x \sqsubseteq y$ iff $\#x$ is part of $\#y$, where $\#x$ is the numeral of x (see Fig 6.6).

We have that $123 \not\sqsubseteq 3$, but only 1 and 2 are such that $1 \sqsubseteq 123$ and $1 \wr 3$, $2 \sqsubseteq 123$ and $2 \wr 3$. Notice that the set $\{1, 2\}$ does not have supremum, since $\{1, 2\} \subseteq \mathsf{P}(-112)$ and $\{1, 2\} \subseteq \mathsf{P}(123)$.

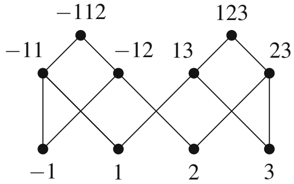

Fig. 6.6 The non-degenerate mereoposet without (SSP+)

Finally, we prove that:

Theorem 8 (Pietruszczak 2000).

(i) *The sentences* (sum-sup) *and* ($\star$) *are true in the class* **POS**+(SSP+)*. So* **POS**+(SSP+) $\subsetneq$ **MPOS**.

(ii) *The equality* sum $=$ sup *is true in all non-degenerate posets which satisfy* (SSP+).

Proof. Ad the part "$\Rightarrow$" of (sum-sup): By Lemma 2 we have (6.10). By Theorem 5 we have (†).

For ($\ddagger_+$): Suppose towards contradiction that (a) x sup S, $S \neq \emptyset$ and (b) $\neg\, x$ sum S. Hence there is u_0 such that (c) $u_0 \sqsubseteq x$ and (d) $\forall_{z \in S}\, z \wr u_0$.

We notice that $u_0 \neq x$. Indeed, if $u_0 = x$ then, by (a), (d) and ($\mathtt{r}_{\sqsubseteq}$), for some $z_0 \in S$ we have a contradiction: $z_0 \sqsubseteq x$ and $z_0 \wr x$,

Thus $x \not\sqsubseteq u_0$, by (c) and ($\mathtt{antis}_{\sqsubseteq}$). Hence, by (SSP+), there is y_0 such that (e) $y_0 \sqsubseteq x$, (f) $y_0 \wr u_0$ and (g) for any $v \in M$: if $v \sqsubseteq x$ and $v \wr u_0$, then $v \sqsubseteq y_0$. From (a) and (d) we obtain that $\forall_{z \in S}(z \sqsubseteq x \wedge z \wr u_0)$. Hence, by (g), we have that $\forall_{z \in S}\, z \sqsubseteq y_0$. So $x \sqsubseteq y_0$, by (a). Hence $x = y_0$, by (e) and ($\mathtt{antis}_{\sqsubseteq}$). Thus, by (c), (f) and ($\mathtt{r}_{\sqsubseteq}$), we obtain a contradiction: $u_0 \sqsubseteq y_0$ and $y_0 \wr u_0$.

Since ($\ddagger_+$) is true in **POS**+(SSP+), then by Fact 8 we have: **POS**+(SSP+) $\subsetneq$ **MPOS**.

Ad ($\star$): Firstly, let $|M| > 1$ and x sup S. Then $S \neq \emptyset$, since by Theorem 5 and Corollary 2, in M there is no zero element. Hence, x sum S by (sum-sup). Secondly, assume that M has only one element x. Then x sup $\emptyset$. But $\neg\, x$ sum $\emptyset$, by ($\mathtt{r}_{\sqsubseteq}$). So sup $\not\subseteq$ sum.

Ad (ii): By (i).

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